Assignment 1

Given the following probability functions:

$$\xi \sim Poisson(\lambda) \iff P(\xi = k) = \frac{\lambda^k}{k!} e^{-\lambda} \tag{1}$$

$$\eta \sim Bernoulli(p) \iff P(\eta|k) = \binom{k}{\eta} p^\eta (1-p)^{k-\eta} \tag{2}$$

where λ and $0 \leq p \leq 1$ are fixed parameters, proof that:

$$\eta \sim Poisson(p \cdot \lambda) \iff P(\eta) = \frac{(p\lambda)^\eta}{\eta!} e^{-p\lambda} \tag{3}$$

First, note that:

$$P(\eta) = \sum_k P(\eta, k) = \sum_k P(\eta|k) \cdot P(k) = \sum_k P(\eta|k) \cdot P(\xi = k) \tag{4}$$

Hence we fill in equations 1 and 2, execute the sum (over all values of $k = \{\eta, \eta+1, \eta+2, ..\}$) and simplify to proof equation 3.

$$P(\eta) = \sum_{k=\eta}^{\infty} \binom{k}{\eta} p^\eta (1-p)^{k-\eta} \frac{\lambda^k}{k!} e^{-\lambda} \tag{5}$$

$$= e^{-\lambda} \sum_{k=\eta}^{\infty} \frac{1}{\eta!(k-\eta)!} p^\eta \lambda^k (1-p)^{k-\eta} \tag{6}$$

Substitute $h = k - \eta$ to find:

$$P(\eta) = e^{-\lambda} \sum_{h=0}^{\infty} \frac{1}{\eta! h!} p^\eta \lambda^{h+\eta} (1-p)^h \tag{7}$$

$$= e^{-\lambda} \frac{(p\lambda)^\eta}{\eta!} \sum_{h=0}^{\infty} \frac{1}{h!} \left(\lambda(1-p)\right)^h \tag{8}$$

Finally, use the power series expansion of $e^x = \sum_{x=0}^{\infty} \frac{x^k}{k!}$:

$$P(\eta) = e^{-\lambda} \frac{(p\lambda)^\eta}{\eta!} e^{\lambda(1-p)} = \frac{(p\lambda)^\eta}{\eta!} e^{\lambda(1-p)-\lambda} = \frac{(p\lambda)^\eta}{\eta!} e^{-p\lambda} \tag{9}$$

which corresponds to equation 3 and finalizes the proof.

Assignment 2

There are two Gaussian distributions of review time t for the strict reviewer $R = s$ and the kind reviewer $R = k$:

$$P(t|R) = \frac{1}{\sqrt{2\pi\sigma_R^2}} \exp\left(\frac{-(t-\mu_R)^2}{2\sigma_R^2}\right) \tag{10}$$

where $\mu_s = 30, \mu_k = 20, \sigma_s = 10, \sigma_k = 5$. We are asked to assess the probability that $R = k$ given $t = 10$. Following Bayes we denote:

$$P(R=k|t=10) = \frac{P(t=10|R=k)\cdot P(R=k)}{P(t=10)} \tag{11}$$

$$= \frac{P(t=10|R=k)\cdot P(R=k)}{P(t=10|R=k)\cdot P(R=k) + P(t=10|R=s)\cdot P(R=s)} \tag{12}$$

because $R \in \{s,k\}$ and probabilities are normalized $\sum_R P(t|R) = 1$. Rewriting equation 11 yields:

$$P(R=k|t=10) = \frac{\frac{1}{\sqrt{2\pi\sigma_k^2}}\exp\left(\frac{-(t-\mu_k)^2}{2\sigma_k^2}\right)}{\frac{1}{\sqrt{2\pi\sigma_k^2}}\exp\left(\frac{-(t-\mu_k)^2}{2\sigma_k^2}\right) + \frac{1}{\sqrt{2\pi\sigma_s^2}}\exp\left(\frac{-(t-\mu_s)^2}{2\sigma_s^2}\right)} \tag{13}$$

$$= \frac{\frac{1}{\sigma_k}\exp\left(\frac{-(t-\mu_k)^2}{\sigma_k^2}\right)}{\frac{1}{\sigma_k}\exp\left(\frac{-(t-\mu_k)^2}{\sigma_k^2}\right) + \frac{1}{\sigma_s}\exp\left(\frac{-(t-\mu_s)^2}{\sigma_s^2}\right)} \tag{14}$$

$$= \frac{1}{1 + \frac{\sigma_k}{\sigma_s}\exp\left(\frac{-(t-\mu_s)^2}{\sigma_s^2} + \frac{(t-\mu_k)^2}{\sigma_k^2}\right)} \tag{15}$$

$$= \frac{1}{1 + \frac{\sigma_k}{\sigma_s}\exp\left(\frac{\sigma_s^2(t-\mu_k)^2 - \sigma_k^2(t-\mu_s)^2}{\sigma_k^2\sigma_s^2}\right)} \tag{16}$$

Solve numerically:

$$P(R=k|t=10) = \frac{1}{1 + \frac{1}{2}\exp\left(\frac{10^2(10)^2 - 5^2(20)^2}{5^2 10^2}\right)} \tag{17}$$

$$= \frac{1}{1 + \frac{1}{2}\exp\left(\frac{100^2 - 100^2}{5^2 10^2}\right)} = \frac{1}{1 + \frac{1}{2}\exp(0)} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3} \tag{18}$$