Solutions to the 2007 202/202N Exam

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1 Subsistence

(a) Local non-satiation means that Walra's Law holds. Hence, we solve

$$c_1 = \frac{1}{p_1}(W - p_2 c_2) \tag{1}$$

After some algebra, and a bit of rearrangement, we find that

$$c_1 = \gamma + \frac{1-\alpha}{p_1}[W - \gamma(p_1 + p_2)] \tag{2}$$

in the case that $W \geq \gamma(p_1 + p_2)$ and

$$c_1 = \frac{W}{p_1} \tag{3}$$

in the case that $W < \gamma(p_1 + p_2)$.

(b) The most straightforward interpretation is that $W = \gamma(p_1 + p_2)$ is the wealth you need to afford to consume $(c_1, c_2) = (\gamma, \gamma)$. This interpretation, along with the demand behavior paints a picture of subsistence consumption. If the consumer can't get at least γ in both periods, then he spends everything on c_1, since he will die in the second period anyways.

(c) Yes, this is consistent with utility maximization. $u(c_1, c_2) = c_1$ readily rationalizes the behavior.

(d) This is also consistent with utility maximization. The easiest way to see this is to guess and check. The guess requires a bit of ingenuity. This problem concerns an agent with a subsistence income of γ in both periods. If he can't afford to purchase γ in both periods, then he consumes all he can in the first period, since he knows that he will die in the second. The $W > \gamma(p_1+p_2)$ case is the case that he can afford to consume γ in both periods. Hence, we will restrict the domain of our utility to $\{(c_1, c_2)|c_1, c_2 \geq \gamma\}$. Based on our answer to (a), the agent never chooses anything outside of this domain when $W > \gamma(p_1 + p_2)$, so we can just set utility there to $-\infty$. Rearranging the expression for c_2, we see

$$p_2(c_2 - \gamma) = \alpha[W - \gamma(p_1 + p_2)] \tag{4}$$

What does this say? After setting aside γ consumption in both periods, the agent devotes the constant fraction α of the remainder of his income to c_2. What does this sound like? Cobb-Douglas. On our domain, we can readily rationalize the observed behavior with

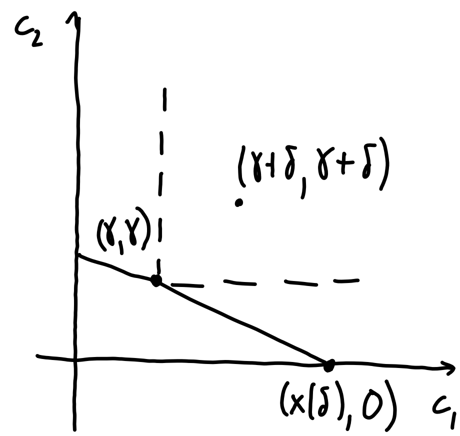
$$u(c_1, c_2) = (c_1 - \gamma)^{1-\alpha}(c_2 - \gamma)^{\alpha} \tag{5}$$

Without this clever observation, the problem is markedly harder. The easiest way to show rationalizability without guessing the utility is to use the Slutsky equation to construct the Slutsky matrix, and then to show that it is negative semidefinite, symmetric, and that it's determinant is zero (the last condition is the equivalent of the requirement that the expenditure function is homogeneous of degree one in prices). The algebra here is rote, so I won't bother with it.

The problem can also be solved by solving for the expenditure function. One of the Slutsky equations can be written as $\partial e(p, u)/\partial p_2 = \gamma + \alpha/p_2[e(p, u) - \gamma p_2 - \gamma p_1]$. This is a strong clue that the expenditure function has a $(p_2)^{\alpha}$ in it as well as a γp_2 Looking at the symmetric Slutsky equation for p_1 yields the symmetric conclusion about $(p_1)^{1-\alpha}$ and γp_2 The expenditure function $e(p, u)=(p_2)^{\alpha}(p_1)^{1-\alpha}u + \gamma p_2 + \gamma p_1$ follows immediately. It is not hard to invert this expenditure function in the region above (γ, γ).

(e) To prove this, we need the property that $(c_1+\delta, c_2+\varepsilon) \succ (c_1, c_2), \forall \varepsilon, \delta > 0$. Now, assume by way of contradiction that we have some point $(\gamma + \delta, \gamma + \delta) \sim (x(\delta), 0)$. Consider the budget set that runs from

$(x(\delta), 0)$ to (γ, γ). It shows that $(\gamma, \gamma) \succsim (x(\delta), 0)$. But monotonicity tells us that $(\gamma + \delta, \gamma + \delta) \succ (\gamma, \gamma)$. So, we have $(x(\delta), 0) \sim (\gamma, \gamma) \prec (\gamma + \delta, \gamma + \delta) \sim (x(\delta), 0)$, a contradiction. So, we have proven that our contradiction hypothesis must be wrong, i.e. that it cannot be the case that $\exists x(\delta)$ such that $(\gamma + \delta, \gamma + \delta) \sim (x(\delta), 0)$ for any $\delta > 0$.



(f) So, essentially, we have two utility functions that both go to infinity that have to be tacked onto each other end to end. Say that we stick with

$$u(c_1, c_2) = (c_1 - \gamma)^{1-\alpha}(c_2 - \gamma)^{\alpha} \tag{6}$$

when $(c_1, c_2) \geq (\gamma, \gamma)$. How can we still rationalize the behavior in the $W < \gamma(p_1 + p_2)$ case? $u(\gamma, \gamma) = 0$, so we need all consumption bundles outside of the domain $(c_1, c_2) \geq (\gamma, \gamma)$ to have negative utility, even $(c_1, c_2) = (\infty, 0)$. One way to do this is to set the utility outside of our domain equal to $u(c_1, c_2) = -\frac{1}{c_1}$. Outside of the domain then, we would try to set c_1 as high as possible, which is exactly the behavior we are trying to rationalize. Hence, the one utility that rationalizes our behavior is

$$u(c_1, c_2) = \begin{cases} (c_1 - \gamma)^{1-\alpha}(c_2 - \gamma)^{\alpha} & (c_1, c_2) \geq (\gamma, \gamma) \\ -\frac{1}{c_1} & \text{o.w.} \end{cases} \tag{7}$$

2 Risky assets

(a) The agent solves

$$\max_{1 \geq \alpha \geq 0} \int u(\alpha w x + (1 - \alpha) w r) dG(x) \tag{8}$$

or cloistering the α's,

$$\max_{1 \geq \alpha \geq 0} \int u(wr + \alpha w(x - r)) dG(x) \tag{9}$$

(b) A sufficient condition to make an optimum non-decreasing should send visions of Topkis dancing through your head. We can combine these problems with the standard Ω trick, with $\Omega = 1 \Rightarrow F$.

$$\max_{1 \geq \alpha \geq 0} \left[\Omega \int u(wr + \alpha w(x - r)) dF(x) + (1 - \Omega) \int u(wr + \alpha w(x - r)) dG(x) \right] \tag{10}$$

Now, calling this objective φ, we take the mixed partial, arriving at

$$\frac{\partial^2 \varphi}{\partial \Omega \partial \alpha} = \int w(x-r) u'(wr+\alpha w(x-r)) dF(x) - \int w(x-r) u'(wr+\alpha w(x-r)) dG(x) \tag{11}$$

We want α to be increasing in Ω here, so our condition becomes, $\forall 0 \leq \alpha \leq 1$,

$$\int w(x-r) u'(wr+\alpha w(x-r)) dF(x) \geq \int w(x-r) u'(wr+\alpha w(x-r)) dG(x) \tag{12}$$

(c) So, the definition of first-order stochastic dominance is exactly the condition in (b) with one important caveat – the integrands must be increasing in x. So, we calculated what is needed to ensure that the integrands are increasing in x. Call the integrand ψ. We need $\partial\psi/\partial x \geq 0$

$$\frac{\partial\psi}{\partial x} = \alpha w^2(x-r)u''(wr+\alpha w(x-r)) + wu'(wr+\alpha w(x-r)) \geq 0 \quad (13)$$

After some manipulation, this becomes

$$-\alpha w(x-r)\frac{u''(wr+\alpha w(x-r))}{u'(wr+\alpha w(x-r))} \leq 1 \quad (14)$$

Now, if $\alpha = 0$, this is always true, since u is increasing. So, we focus on $\alpha > 0$. Now, to get the relative risk aversion to pop out, we will need to multiply both sides of the inequality by $\frac{wr+\alpha w(x-r)}{\alpha w(x-r)} = 1 + \frac{r}{\alpha(x-r)}$. If $x \geq r$, then this quantity is positive, and we do not need a sign flip. If it is, we do. Hence, we consider two cases

- $x \geq r$, so no sign flip is needed. Thus, we arrive at

$$R(u''(wr+\alpha w(x-r))) \leq 1 + \frac{r}{\alpha(x-r)} \quad (15)$$

  The right-hand side of the inequality is always ≥ 1, so for it to hold for all $x > r$ and α, we need the condition

$$R(u''(wr+\alpha w(x-r))) \leq 1 \quad (16)$$

- $x < r$, so we have to flip the sign, arriving at

$$R(u''(wr+\alpha w(x-r))) \geq 1 + \frac{r}{\alpha(x-r)} \quad (17)$$

  How big is the right-hand side? Since $x \in [0, r)$ and $\alpha \in (0, 1]$, we must conclude that $\frac{r}{\alpha(x-r)} = \frac{1}{\alpha(\frac{x}{r}-1)} \leq -1$. So, for the condition to hold for all $x \in [0, r)$ and $\alpha \in (0, 1]$, we need

$$R(u''(wr+\alpha w(x-r))) \geq 0 \quad (18)$$

  which is always true, since the agent is risk averse

Hence, if we restrict

$$R(x) \leq 1, \forall x > 0 \tag{19}$$

the we have our sufficient condition

(d) We will follow a very similar approach to the previous part. Looking at (12), we see that it is the defition of second-order stochastic domination, without the caveat that the integrand in concave. Hence, we will solve for when the integrand is concave. For this, we will need $\frac{\partial^2 \psi}{\partial x^2} \leq 0$. Calculating

$$\frac{\partial^2 \psi}{\partial x^2} = \alpha^2 w^3 (x-r) u'''(wr + \alpha w(x-r)) + 2\alpha w^2 u''(wr + \alpha w(x-r)) \leq 0 \tag{20}$$

We want the coefficient of absolute prudence, so we will need to do the same thing we did in the previous problem. Forgoing the details, the condition we arrive at is that

$$P(x) \geq 2, \forall x > 0 \tag{21}$$

3 Cars, externality, and general equilibrium

(a) Her maximization problem is

$$\max_{(i,x)\in\{c,s\}\times\mathbb{R}_+} (1-\pi_i)u_i(x) \text{ such that } x \leq \theta_i \frac{w-p_i}{\gamma}.$$

Since $u_i(\cdot)$ is strictly increasing, we know the constraint will bind, so the consumer's problem is equivalently

$$\max_{i\in\{c,s\}} (1-\pi_i)u_i\left(\theta_i \frac{w-p_i}{\gamma}\right).$$

(b) Let $\Omega = 1$ if the consumer chooses a car, and $\Omega = 0$ if she chooses a scooter. Then the consumer maximizes $F(\Omega; \dots)$ over $\Omega \in \{0,1\}$ where

$$F(\Omega; \dots) \equiv \Omega(1-\pi_c)u_c\left(\theta_c \frac{w-p_c}{\gamma}\right) + (1-\Omega)(1-\pi_s)u_s\left(\theta_s \frac{w-p_s}{\gamma}\right).$$

Consider

$$\begin{aligned}
\frac{\partial F}{\partial \Omega} &= (1-\pi_c)u_c\left(\theta_c\frac{w-p_c}{\gamma}\right) - (1-\pi_s)u_s\left(\theta_s\frac{w-p_s}{\gamma}\right) \\
\frac{\partial^2 F}{\partial \Omega\,\partial p_c} &= -\left(\frac{\theta_c}{\gamma}\right)(1-\pi_c)u_c'\left(\theta_c\frac{w-p_c}{\gamma}\right) \leq 0 \\
\frac{\partial^2 F}{\partial \Omega\,\partial p_s} &= \quad \left(\frac{\theta_s}{\gamma}\right)(1-\pi_s)u_s'\left(\theta_s\frac{w-p_s}{\gamma}\right) \geq 0
\end{aligned}$$

Thus F has increasing differences in $(\Omega, -p_c)$ and in (Ω, p_s). By Topkis' Theorem, the optimal Ω^* is nondecreasing in p_s and nonincreasing in p_c. As p_s increases, the consumer buys weakly more cars. As p_c increases, the consumer buys weakly fewer cars, and hence weakly more scooters.

(c)

$$\begin{aligned}
\frac{\partial^2 F}{\partial \Omega\,\partial w} &= \left(\frac{\theta_c}{\gamma}\right)(1-\pi_c)u_c'\left(\theta_c\frac{w-p_c}{\gamma}\right) - \left(\frac{\theta_s}{\gamma}\right)(1-\pi_s)u_s'\left(\theta_s\frac{w-p_s}{\gamma}\right) \\
&= \left(\frac{\theta}{\gamma}\right)\left[(1-\pi_c)u_c'\left(\theta\frac{w-p_c}{\gamma}\right) - (1-\pi_s)u_s'\left(\theta\frac{w-p_s}{\gamma}\right)\right]
\end{aligned}$$

By assumptions, $(\theta\frac{w-p_s}{\gamma}) \geq (\theta\frac{w-p_c}{\gamma})$. Therefore $u_s'(\theta\frac{w-p_s}{\gamma}) \leq u_s'(\theta\frac{w-p_c}{\gamma}) \leq u_c'(\theta\frac{w-p_c}{\gamma})$, where the first inequality obtains by concavity of $u_s(\cdot)$ and the second by our assumptions. Thus since $\pi_c \leq \pi_s$,

$$\geq 0.$$

Thus F has increasing differences in (Ω, w); by Topkis' Theorem, the optimal Ω^* is nondecreasing in w; i.e., c is a normal good.

(d) The consumer prefers a scooter exactly when

$$(1-\pi_c)u_c\left(\theta_c\frac{w-p_c}{\gamma}\right) \geq (1-\pi_s)u_s\left(\theta_s\frac{w-p_s}{\gamma}\right)$$

Firms price at marginal cost, which gives $\gamma = 1$, $p_s = 800$, and $p_c = 955$. Plugging in,

$$\begin{aligned}
(1-\pi_c)\left(5\frac{1000-955}{1}\right)^{1/2} &\geq (1-\tfrac{1}{2})\left(40\frac{1000-800}{1}\right)^{1/3} \\
\pi_c &\leq \tfrac{1}{3}
\end{aligned}$$

Thus the consumer strictly prefers a scooter when $\pi_c > \frac{1}{3}$, strictly prefers a car when $\pi_c < \frac{1}{3}$, and is indifferent when $\pi_c = \frac{1}{3}$.

(e) As above, $\gamma = 1$, $p_s = 800$, and $p_c = 955$. If any prices were above the corresponding marginal cost, firms would scale up production of that good and earn arbitrarily high profits. If prices were any lower, firms would earn negative profits and shut down. Since $N_s > 0$ and $N_c > 0$, we know the vehicle manufacturers have not shut down. No matter how high the price of gasoline, consumers would demand it (marginal utility goes to infinity as miles driven goes to zero), so gasoline manufacturers couldn't have shut down at equilibrium.

(f) Since all consumers are identical, if some choose scooters and others choose cars it must be that consumers are indifferent between one and the other. We showed that this requires $\pi_c = \frac{1}{3}$. Thus $N_c/N = \frac{2}{3}$ and $N_s/N = \frac{1}{3}$.

(g) Normalize the total number of consumers to $N = 1$. The Samuelson-Bergson social welfare function is

$$\begin{aligned} N_c(1-\pi_c)u_c\left(\theta_c \frac{w-p_c}{\gamma}\right) &+ (1-N_c)(1-\pi_s)u_s\left(\theta_s \frac{w-p_s}{\gamma}\right) \\ &= N_c\left(1-\frac{N_c}{2}\right)\cdot 15 + (1-N_c)\cdot 10 = -\frac{15}{2}N_c^2 + 5N_c + 10. \end{aligned}$$

Taking a first-order condition shows that average utility is maximized at $N_c = \frac{1}{3}$. Thus at the planner's optimum, $N_c/N = \frac{1}{3}$ and $N_s/N = \frac{2}{3}$.

(h) Moving from $\frac{2}{3}$ of the population driving cars to $\frac{1}{3}$ is a Pareto improvement: Consumers who remain on scooters are indifferent, those who have been moved from a car to a scooter are indifferent, while those who remain on cars have become better off since their cars have gotten safer with fewer cars on the road. The FWT fails because there is a positive externality of driving a car, which is not priced in the market.

(i) At the planner's optimum, $\pi_c = \frac{1}{6}$. For some consumers to choose cars and others scooters at equilibrium, they must be indifferent between the two. That is,

$$
\begin{aligned}
(1-\pi_c)u_c\left(\theta_c\frac{w-p_c}{\gamma'}\right) &= (1-\pi_s)u_s\left(\theta_s\frac{w-p_s}{\gamma'}\right)\\
(1-\tfrac{1}{6})\left(5\frac{1000-955}{\gamma'}\right)^{1/2} &= (1-\tfrac{1}{2})\left(40\frac{1000-800}{\gamma'}\right)^{1/3}\\
\gamma' &= (\tfrac{5}{4})^6.
\end{aligned}
$$

The government should tax gasoline so that its price is $(\frac{5}{4})^6$.

(j) Not really. Taxing big cars directly could also implement the optimum. It would likely waste less money, since no one is unnecessarily taxed.