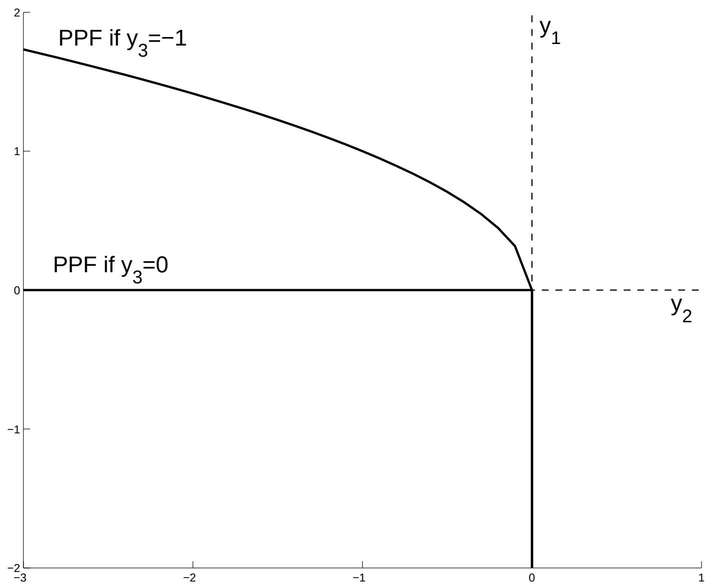
Economics 202/202n: Solutions for the 2008 Final

1 Production in the short-run and long-run

The production set looks a bit strange, but staring at it you can realize that y_3 is only a fixed input that allows the firm to operate. If the firm wants to produce anything at all, she must pay an upfront cost of p_3.

(a) The figure shows the Production Possibility Frontier in y_1, y_2 space for $y_3 = -1$ and $y_3 = 0$. The vertical axis corresponds to the unique output, y_1. The firm has free-disposal in (y_1, y_2), so the lower-left region is always feasible. In the short-run, shutdown may not be available to the firm (because it might be forced to buy y_3).



(b) The choice variable y_3 represents whether the firm decides to operate or not, with $y_3 = -1$ meaning she does. The price p_3 is a fixed-cost the firm has to pay to operate.

Let $y^*(p) = (y_1^*, y_2^*, y_3^*)$ denote the optimal demand at prices p.

(c) **In the short-run.** If $y_3 = 0$ then the firm cannot gain by producing. The optimal supply for y_1 is zero if $p_1 > 0$ and anything feasible (i.e., negative) otherwise. The demand for y_2 is zero if $p_2 > 0$ and anything (negative) otherwise.

If $y_3 = -1$ then the cost p_3 is a sunk-cost and doesn't enter the firm's decision. First asume $(p_1, p_2) \gg 0$. The firm then solves

$$\max_{y_2} p_1 \sqrt{-y_2} - p_2 y_2.$$

The F.O.C. is:

$$\frac{p_1}{2y_2^{1/2}} = p_2$$
$$\Rightarrow y_2^* = \left(\frac{p_1}{2p_2}\right)^2.$$

The optimal demand is

$$y^*(p) = \left(\frac{p_1}{2p_2}, \left(\frac{p_1}{2p_2}\right)^2, -1\right).$$

If $p_1 = 0$ and $p_2 > 0$ the optimal production plan is $(0, 0, -1)$. If $p_2 = 0$ and $p_1 > 0$ the optimal production plan is empty, as the firm can get unlimited profits by scaling up production arbitrarily. If both are zero then the firm optimally produces anything feasible.

(d) **In the long-run** the firm has the choice to operate or not. She choses to operate if her maximal profit (gross of the sunk cost) is larger than p_3. Assume $p_2 > 0$. Then

$$y^*(p) = \begin{cases} \{(0,0,0)\} & \text{if } \frac{(p_1)^2}{4p_2} < p_3; \\ \big\{(0,0,0), \big(\frac{p_1}{2p_2}, (\frac{p_1}{2p_2})^2, -1\big)\big\} & \text{if } \frac{(p_1)^2}{4p_2} = p_3; \\ \big\{\big(\frac{p_1}{2p_2}, (\frac{p_1}{2p_2})^2, -1\big)\big\} & \text{if } \frac{(p_1)^2}{4p_2} > p_3. \end{cases}$$

If $p_2 = 0$ and $p_1 > 0$ the firm would want to produce arbitrarily large amounts, so the optimal supply correspondence is empty. If $p_2 = 0$ and $p_1 = 0$ the firm would shut-down. That covers all the cases as we always assume at least one price is strictly positive.

2 Thanksgiving complements

(a) Let $\lambda \geqslant 0$. Then

$$u(\lambda x_1, \lambda x_2) = \min\{\lambda x_1, \lambda x_2\} = \lambda \min\{x_1, x_2\} = \lambda u(x_1, x_2).$$

(b) (i) Found by substituting Marshallian demand (see below) into utility function:

$$v(p, y) = \frac{y}{p_1 + p_2}.$$

(ii) Found by solving $u = v(p, w)$ for w:

$$e(p, u) = u(p_1 + p_2).$$

(iii) Found by taking the derivative of the expenditure function:

$$h(p, u) = (u, u).$$

(iv) Found using the fact that the optimal solution the individual must always buy equal amounts of x_1 and x_2, then substituting in the budget restriction:

$$x(p,y) = \left(\frac{y}{p_1+p_2}, \frac{y}{p_1+p_2}\right).$$

(c) The Slutsky equation states that

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - \frac{\partial x_i}{\partial y} x_j.$$

Below are the values for our problem expressed in matrix notation:

$$\begin{aligned}
\begin{pmatrix} -\frac{y}{(p_1+p_2)^2} & -\frac{y}{(p_1+p_2)^2} \\ -\frac{y}{(p_1+p_2)^2} & -\frac{y}{(p_1+p_2)^2} \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{p_1+p_2} \\ \frac{1}{p_1+p_2} \end{pmatrix} \begin{pmatrix} \frac{y}{p_1+p_2} & \frac{y}{p_1+p_2} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{y}{(p_1+p_2)^2} & -\frac{y}{(p_1+p_2)^2} \\ -\frac{y}{(p_1+p_2)^2} & -\frac{y}{(p_1+p_2)^2} \end{pmatrix}.
\end{aligned}$$

Notice that the Slutsky matrix is all zeros. This occurs because the Leontieff goods have no substitution effect. All the effect of price on the Marshallian demand comes from the wealth effect.

(d) (i) The equivalent variation is the change in wealth at the initial price that gives an "equivalent" effect to the change of price. In this question the price increased so we should expect the EV to be a negative number.

$$e(p^0, u^1) - w = e(p^0, u^1) - e(p^1, u^1) = -2u^1 = -100.$$

(ii) The compensating variation is the amount of money we have to give the consumer to keep him at his old utility level under the new prices.

$$w - e(p^1, u^0) = e(p^0, u^0) - e(p^1, u^0) = -2u^0 = -50.$$

(iii) The consumer surplus is (-1 times) the integral of the Marshallian demand over the relative range:

$$-\int_{p_1^0}^{p_1^1} \frac{y}{p+p_2}\, dp = -100\big(\log(p_1^1) - \log(p_1^0)\big) = -100\log(2).$$

Note that as expected, this lies between the answers to the previous two problems.

3 Limited liability

(a) The agent is Risk-Loving since $v(x)$ is a convex function. Note that if we restricted attention on lottery realizations that are either always positive or always negative, the agent would be risk-neutral in these regions. However, for more general lotteries that have both positive and negative realizations, he is risk-loving.

(b) We can't apply the derivatives-based definition of RA using the curvature of the utility function since the function has a kink at one point, and this kink makes a big difference! Here is a definition that doesn't use derivatives: The agent has DARA/CARA/IARA if he is more/less/equally likely to take a given lottery X resulting in payout $X + w$ rather than w when his initial wealth w is higher.

(c) In our case, the gain in expected utility from taking lottery X at wealth w equals

$$\int \max\{x + w, 0\}\, dF(x) - \max\{w, 0\}.$$

When we focus on nonnegative wealth $w \geq 0$, this expression equals $\int \max\{x, -w\}\, dF(x)$. Since the integrand is nonincreasing in w, we have IARA for nonnegative wealth. Note that since the agent is risk loving it is more appropriate to call this "Decreasing Absolute Risk Loving." Intuitively, an agent with very low but positive wealth will have a high incentive to gamble on negative-expected-value lotteries since he has little to lose and the richer he becomes the more he has to lose, in particular he becomes risk-neutral when his wealth exceeds any possible loss in the lottery.

On the other hand, when we consider negative wealth $w < 0$, the above displayed difference is simply $\int \max\{x + w, 0\}\, dF(x)$. Since the integrand is nondecreasing w, we have DARA. Intuitively, an agent with a very low negative wealth has nothing to gain from a bounded-stakes lottery, so he can be viewed as risk-neutral. But as his wealth becomes less negative, he may gets something to gain, so he is "Increasing Absolute Risk Loving" as his wealth increases to zero.

(d) The agent has DRRA/CRRA/IRRA if he is more/less/equally likely to take a given "proportionate" lottery X resulting in payout Xw rather than w when his initial wealth w is higher. Let's focus on positive wealth $w > 0$. In our case, the gain in expected utility from proportionate lottery X is

$$\int \max\{xw, 0\}\, dF(x) - w = w\left(\int \max\{x, 0\}\, dF(x) - 1\right),$$

and the sign of this expression does not depend on w, thus the agent has Constant Relative Risk Aversion. (As for negative wealth $w < 0$, the gain is $\int \max\{xw, 0\}\, dF(x) - 0 = w \int \min\{x, 0\}\, dF(x)$, and so again we have CRRA.)

(e) Let's start with (ii). Since $v(x)$ is a convex function, the function

$$U(\gamma) = \int v(w + \gamma R)\, dF(x)$$

is convex in γ (it is a weighted combination of convex functions, and you can check by definition of convexity). Hence, $U(\gamma)$ is optimized at an extreme point of the interval $[0, \bar{\gamma}]$. (Indeed, for any $\gamma \in [0, \bar{\gamma}]$ by convexity we have $U(\gamma) \leq \frac{\gamma}{\bar{\gamma}} U(0) + \left(1 - \frac{\gamma}{\bar{\gamma}}\right) U(\bar{\gamma}) \leq \max\{U(0), U(\bar{\gamma})\}$.)

Now, for (i) it suffices to consider the choice between $\gamma = 0$ (not taking a lottery) and $\gamma = \bar{\gamma}$ (taking the lottery). By part (c), we have IARA hence taking the lottery is less attractive when w is higher. Thus, there is a cutoff wealth $\hat{w}$ such that the agent chooses $\gamma = 0$ when $w > \hat{w}$ and chooses $\gamma = \bar{\gamma}$ when $w < \hat{w}$.

4 The Sneetches on the beaches

For more on consumption with status see Bernheim's *A Theory of Conformity* (JPE 1994) or Rayo's *Monopolistic Signal Provision* (BE Journal of Theoretical Economics 2005). For general lessons on societies with status go directly to the source, Dr. Seuss' *The Sneetches and other stories.*

(a) The question asks us to assume a positive fraction of the Sneetches is buying the star. From the firm's cost we know the price of stars has to be positive in this economy. If π_0 is weakly-higher than π_1 then buying a star gives less status and less food, so the Sneetches who are buying them would not be optimizing.

(b) We can't apply Topkis' theorem directly because the choice structure is not a lattice. This is always a problem with consumer maximization problems. Nevertheless we can transform this particular problem into an equivalent one where we can apply Topkis' theorem. By local non-satiation we know that in equilibrium $y = 1 - px$. Therefore we can rewrite the problem as:

$$\max_{x \in \{0,1\}} x(\pi_1 \theta + 1 - p) + (1 - x)(\pi_0 \theta + 1),$$

where the new choice set is obviously a lattice. To apply Topkis' theorem we only need to check that the objective function has increasing differences in (θ, x).

$$\begin{aligned} u(1, 1-p; \theta) - u(0, 1; \theta) &= \pi_1 \theta + 1 - p - (\pi_0 \theta + 1) \\ &= (\pi_1 - \pi_0)\theta - p, \end{aligned}$$

which is strictly increasing in θ whenever $\pi_1 > \pi_0$. We know by part (a) this must happen in equilibrium. If a type is indifferent between buying or not, it must be that all higher types strictly prefer to buy and all lower types strictly prefer not to buy. In other words, we have a monotone selection.

(c) Using parts (a) and (b) we know that in equilibrium we will have a threshold type θ who is indifferent between buying or not. Using the fact the distribution is uniform we calculate the average type for those above and below the threshold.

$$\pi_0 = \int_0^\theta t f(t|t < \theta)\, dt = \int_0^\theta \frac{t}{\theta}\, dt = \frac{\theta^2}{2\theta} = \frac{\theta}{2};$$

$$\pi_1 = \int_\theta^1 t f(t|t > \theta)\, dt = \int_\theta^1 \frac{t}{1-\theta}\, dt = \frac{1-\theta^2}{2(1-\theta)} = \frac{1+\theta}{2}.$$

The indifferent type solves:

$$\begin{aligned} u(1, 1-p; \theta) &= u(0, 1; \theta) \\ 2\theta\left(\frac{1+\theta}{2}\right) + 1 - p &= 2\theta\left(\frac{\theta}{2}\right) + 1 \\ \theta^* &= p. \end{aligned}$$

(d) Because the firm has constant returns to scale the equilibrium price must be $p = c$. The equilibrium allocation has $1 - c$ stars produced. All types above θ^* buy and all types below do not.

(e) Suppose without loss that the tax is charged on the buyers of stars. Because preferences are quasi-linear in food, the lump-sum tax does not change the incentives to buy the star. Therefore the threshold type must be $\theta = p + t$. For the firm to operate it must still be that $p = c$. Therefore the equilibrium with tax has a fraction $1 - (c + t)$ of individuals buying stars. The lump-sum transfer is $T = \big(1 - (c + t)\big)t$.

(f) The lump-sum transfer is just a redistribution from some agents to others. Because preferences are quasi-linear, the redistribution does not affect the average utility. Therefore to maximize the average utility we can solve a planner's problem, where the planner choses a cut-off θ such that higher types buy the star and lower types do not. After finding the cut-off the planner choses the corresponding tax. The planner's problem is as follows:

$$\max_\theta 2\frac{1+\theta}{2}\int_\theta^1 t\,dt - c\int_\theta^1 dt + 2\frac{\theta}{2}\int_0^\theta t\,dt$$

$$\max_\theta \int_\theta^1 t\,dt - c(1-\theta) + \theta\int_0^1 t\,dt$$

$$\max_\theta \frac{1-\theta^2}{2} - c(1-\theta) + \frac{\theta}{2}.$$

The First Order Condition is then:

$$\begin{aligned} {}^1\!/_2 + c - \theta^* &= 0 \\ \Rightarrow \theta^* &= c + {}^1\!/_2 \\ \Rightarrow t^* &= {}^1\!/_2. \end{aligned}$$

(g) The First Welfare Theorem doesn't hold because adding a tax is a Pareto improvement. The theorem doesn't hold because lower types impose a negative externality by consuming the good with more status and higher types impose a positive externality when consuming the good with less status. Without the tax too many people get a star. To see why the tax is a Pareto improvement consider three cases:

Case 1: Types that were not buying a star before the tax improve their utility because of they receive a lump-sum transfer and because the average utility of those not getting stars increases, which gives them a positive externality.

Case 2: Types that used to buy a star and no longer do so lose status but gain the lump-sum transfer. Let's calculate their net utility.

$$\begin{aligned} u(0, 1+T; \theta) - u(1, 1-c; \theta) &= \big(c + {}^1\!/_2\big)\theta + 1 + T - \big((1+c)\theta + 1 - c\big) \\ &= T - \frac{\theta}{2} + c \geqslant T - \frac{\theta^*}{2} + c = \frac{{}^1\!/_2 - c}{2} - \frac{c + {}^1\!/_2}{2} + c = 0. \end{aligned}$$

Case 3: Finally types who still buy the star gain additional status after the tax but have to pay a higher price. To see they are better off, we solve the difference in utilities.

$$\begin{aligned} u(1, 1-c-t+T; \theta) - u(1, 1-c; \theta) &= \big(1+c+{}^1\!/\!_2\big)\theta + 1 - c - {}^1\!/\!_2 + T - \big((1+c)\theta + 1 - c\big) \\ &= \frac{\theta}{2} + T - {}^1\!/\!_2 = T - \frac{1-\theta}{2} \geqslant T - \frac{1-\theta^*}{2} = \frac{{}^1\!/\!_2 - c}{2} - \frac{1-c-{}^1\!/\!_2}{2} = 0. \end{aligned}$$