Solutions to Econ 202 final exam
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1 Question 1

1.1 Part a

Note that at prices $p_1 = p_2 = 1$, the bundle y was affordable (because $2.1+3.1 < 6 = w$). By revealed preferences, $(4,2) \succ (2,3)$ and therefore this bundle is not affordable at the new prices, i.e. $4p_1 + 2p_2 > 6$ (In addition, by local non-satiation we know that $2p_1 + 3p_2 = 6$).

1.2 Part b

From local non-satiation,

$$2p_1 + 3p_2 = 6$$

If $p_1 = 2$, $4 + 3p_2 = 6$ implies that

$$p_2 = \frac{2}{3}$$

We know (check part a) that $(4,2) \succ (2,3)$. Moreover, since $(2,3)$ was chosen at prices $\left(2, \frac{2}{3}\right)$ we know that

$$(2,3) \succeq (z_1, z_2) \text{ for all } \{(z_1, z_2) \in \Re_+^2 : 2z_1 + \frac{2}{3}z_2 \leq 6\}$$

By transitivity,

$$(4,2) \succ (z_1, z_2) \text{ for all } (z_1, z_2) \in \{(z_1', z_2') \in \Re_+^2 : 2z_1' + \frac{2}{3}z_2' \leq 6\}$$

Moreover $(4,2)$ was chosen at prices $p_1 = p_2 = 1$. There fore, we must also have

$$(4,2) \succ (y_1, y_2), \forall (y_1, y_2) \in \{(y_1', y_2') \in \Re_+^2 : y_1' + y'_2 \leq 6\}.$$

(insert graph here)

[1] I am grateful to Felix Reichling for typing up the handwritten version of this solution set. All errors are mine.

1.3 Part c

Convexity and strict monotonicity implies that any point in the shaded region is preferred to $(2, 3)$

(insert graph)

2 Question 2

2.1 Part a

I claim that k will always decrease with θ. By defining $L(x, k)$ implicitly by $f\left(k, L\left(x, k\right)\right) = x$, the problem

$$\begin{aligned}&\min_{k,l} rk + wl\\&s.t.\\&f(k, \theta l) = x\end{aligned}$$

can be re-written as

$$\min_{k,l} rk + w\frac{L(x, k)}{\theta}$$

(that is, I am just plugging the constraint into the objective function!).

Noting that this is a single variable minimization problem which has the same solution as

$$\max_{k,l} -rk - \frac{w}{\theta}L(x, k)$$

By Topkis' theorem, in order to show that the optimal level of capital will always decrease with θ,we just need to check that objective function has decreasing differences in (k, θ) (i.e. increasing differences in $(k, -\theta)$)

Noting that

$$\frac{\partial(-rk - \frac{w}{\theta}L(x, k))}{\partial\theta} = \frac{w}{\theta^2}L(x, k)$$

which is decreasing in k (by the assumptions of the problem). The result follows.

2.2 Part b

Yes, as long as $L(l, x)$ is decreasing in k (and this **does not** depend on the capital being "lumpy" or not).

2.3 Part c

By exactly the same reasoning as in part a), the solution to

$$\begin{aligned}&\min_{k,l} rk + wl\\&s.t.\\&f(k, \theta l) = x\end{aligned}$$

is the same as the solution to the program below:

$$\max_{l} -rk(x,\theta l)-wl$$

By Topkis', the solution to this problem will be increasing in θ if the objective function has increasing differences in (l,θ). Noting that

$$\frac{\partial(-rk(x,\theta l)-wl)}{\partial\theta}=-r\frac{\partial k(x,\theta l)}{\partial(\theta l)}l$$

This is increasing in l iff

$$\frac{\partial k(x,\theta l)}{\partial(\theta l)}l$$

is increasing in l iff

$$\frac{f_{\theta l}(k,\theta l)}{f_k(k,\theta l)}l$$

is increasing in l iff

$$\ln\left[\frac{f_{\theta l}(k,\theta l)}{f_k(k,\theta l)}\right]+\ln l$$

is increasing in $\ln l$. Taking derivatives with respect to $\ln l$ the condition follows.

2.4 Part d

Define

$$\begin{aligned}\hat{l} &\equiv \theta l\\ \hat{w} &\equiv \frac{w}{\theta}\end{aligned}$$

The long run maximization problem can be re-written as

$$\max_{k,\hat{l}} pf(k,\hat{l})-\hat{w}\hat{l}-rk$$

Then, an increase in θ can be seen as a reduction in $\hat{w}$. By the complementarity assumption and by the Le Chatelier Principle, we know that the response of $\hat{l}$ (and therefore l, under the condition stated in (b)) to an increase in θ will be higher in the long run (under the condition stated in (b), demand for labor will increase more in the long run than in the short run).

3 Question 3

3.1 Part a

It is easy to see that $\forall x\in[0,10]$, $H(x)\leq F(x) \implies H\succ_{FOSD}F$. Note that

$$H(4)=0<\frac{1}{6}=G(4)$$

and

$$H(5) = \frac{1}{3} > \frac{1}{6} = G(5)$$

Thus, we cannot rank $H(\cdot)$ and $G(\cdot)$ in terms of first order stochastic dominance.

As for $F(\cdot)$ and $G(\cdot)$, $G(x) \leq F(x) \quad \forall x$ so that $G \succ_{FOSD} F$.

As for the second order stochastic dominance, I claim that:

1.) $H \succ_{SOSD} G$

2.) $G \succ_{SOSD} F$ and

3.) $H \succ_{SOSD} F$

To prove the first claim, note that

$$\begin{aligned} \frac{1}{6}[u(6) - u(3)] &\geq \frac{1}{3}[u(6) - u(5)] \\ \Longrightarrow \quad [u(6) - u(5)] + [u(6) - u(3)] &\geq 2[u(6) - u(5)] \\ \Longleftrightarrow u(5) - u(3) &\geq u(6) - u(5) \end{aligned}$$

and this holds for all $u(\cdot)$ concave and increasing. [note that $u(5) - u(3) = u(5) - u(4) + u(4) - u(3)$ and $u(5) - u(4) \geq u(6) - u(5)$ by concavity].

By the same token, $G \succ_{SOSD} F$ iff $\forall\ u(\cdot)$ concave and increasing we have

$$\frac{1}{6}u(3) - \frac{1}{3}u(6) + \frac{1}{4}u(8) + \frac{1}{4}u(9) \geq \frac{1}{3}u(3) - \frac{1}{6}u(5) + \frac{1}{4}u(8) + \frac{1}{4}u(9)$$

which implies (after playing with the terms a little bit)

$$\frac{1}{4}u(8) - u(7) + \frac{1}{3}[u(6) - u(5)] + \frac{1}{3}[u(5) - u(3)] \geq \frac{1}{6}[u(5) - u(3)]$$

which holds trivially.

Since $U^5 \subset U^2$, the second order stochastic ranking of the lotteries above is preserved under the fifth order stochastic dominance criterion.

3.2 Part b

Let $x \in (a, b)$ so that $G(x) = 0$ and $F(x) > 0$. Take

$$\tilde{u}(y) = \begin{cases} y & y \leq x \\ x & y > x \end{cases}$$

Note that

$$\int_0^{10} \tilde{u}(y) dF(y) = \int_0^x y dF(y) + x \Pr_F(y > x) < x = \int_0^{10} \tilde{u}(y) dG(y)$$

However, $\tilde{u} \notin U^h$ for any u as it is not differentiable at x. However, take an $\varepsilon > 0$ and let

$$u(y) = \tilde{u}(y) \forall y \in [0, 10] \backslash (x - \varepsilon, x + \varepsilon)$$

and let $u(y)$ be a ∞ differentiable function on $(x-\varepsilon, x+\varepsilon)$ so that

$$u^{(i)}(y)(-1)^{(i+1)} \geq 0 \text{ for } i = 1, ... \forall y \in (x-\varepsilon, x+\varepsilon)$$

. Such a $u \in U^h$ $\forall h$ and for ε sufficiently small

$$\int u(y) dF(\cdot) < \int u(y) dG(\cdot)$$

proving the claim.

4 Question 4

4.1 Part a

Yes, it does since the utility is strictly increasing (and concave). Check the sloppy existence proof in your notes!

4.2 Part b

Suppose that, say, $p_1 \geq p_2$. I want first prove the following remark:

If $p_1 \geq p_2$, we must have $c_1^h \leq c_2^h$ $\forall h \in \{1,2,3\}$.

Suppose we had $c_1^h > c_2^h$ at an optimum. Consider a new bundle

$$\left(c^{h'}\right) = \left(c_1^{h'}, c_2^{h'}, c_3^{h'}\right) = \left(\frac{1}{2}\left(c_1^h + c_2^h\right), \frac{1}{2}\left(c_1^h + c_2^h\right), c_3^h\right)$$

Note that if $p_1 \geq p_2$, this new bundle would be affordable. Moreover,

$$\begin{aligned}
U^h(c') &= v^h\left(c_1^{h'}\right) + v^h\left(c_2^{h'}\right) + v^h\left(c_3^{h'}\right) \\
&= 2v^h\left(\frac{1}{2}\left(c_1^h + c_2^h\right)\right) + v^h\left(c_3^h\right) \\
&>_{(strict\ concavity)} 2\left[\frac{1}{2}v^h\left(c_1^h\right) + \frac{1}{2}v^h\left(c_2^h\right)\right] + v^h\left(c_3^h\right) \\
&= U^h(c)
\end{aligned}$$

which is contradicts optimality.

But then, if $c_1^h \leq c_2^h$ $\forall h \in \{1,2,3\}$,

$$\sum_h c_1^h \leq \sum_h c_2^h$$

Market clearing calls for

$$\sum_h e_1^h = \sum_h c_1^h \leq \sum_h c_2^h = \sum_h e_2^h$$

and we have a contradiction. Thus we must have $p_1 < p_2$ in equilibrium. The other cases are analogous.

4.3 Part c

We know that for "log utility functions" demands are given by

$$c_i^h = \frac{1}{3p_i}\left[\sum_i p_i e_i^h\right];\ i = 1,2,3; \quad h = 1,2,3 \tag{(*)}$$

Market clearing calls for

$$\begin{aligned}\sum_h c_1^h &= \sum_h e_1^h \\ \sum_h c_2^h &= \sum_h e_2^h\end{aligned}$$

(Recall that by Walras' Law we don't nee to check the market clearing condition for the third market.)

Using $(*)$ one gets that

$$\begin{aligned}\sum_h c_1^h &= \frac{1}{3p_1}\left[\sum_i p_i\left(\sum_h e_i^h\right)\right] = \sum_h e_1^h \\ \sum_h c_2^h &= \frac{1}{3p_2}\left[\sum_i p_i\left(\sum_h e_i^h\right)\right] = \sum_h e_2^h\end{aligned}$$

Normalize $p_3 = 1$, $(***)$ is a system of two equations in two unknowns and the solution to the system only depends on

$$\sum_h e_i^h \quad \text{for } i = 1,2,3$$

This proves the claim.

No, it is not the case that this holds true whenever preferences are the same. For an example, let

$$U^h(c) = c_1^h - \frac{1}{2}\left(c_1^h\right)^2 + c_2^h - \frac{1}{2}\left(c_2^h\right)^2 \quad \text{for } h = 1,2$$

with total endowments being $(0.5; 0.7)$. For

$$\begin{aligned}\left(e_1^1, e_2^1\right) &= (0.5, 0.7) \\ \left(e_1^2, e_2^2\right) &= (0, 0)\end{aligned}$$

and

$$\begin{aligned}\left(e_1^1, e_2^1\right) &= (0.5, 0.65) \\ \left(e_1^2, e_2^2\right) &= (0, 0.05)\end{aligned}$$

prices will be different in equilibrium (check yourself!)