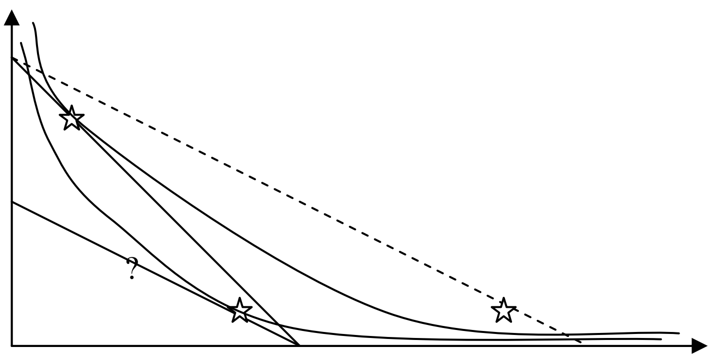
Economics 202
Exam Solutions 2003

1. Revealed Preference.

Consider a two-good economy, and a consumer with complete, transitive, continuous and strictly monotone preferences. Prices in this economy are always positive. Suppose the consumer has wealth 4. We make two observations of consumer choices:

1. At prices (1, 2) the choice is (3, ½).
2. At prices (1, 1) the choice is (1, 3).



Which of the following observations would be consistent with utility maximization?

(a) Choice (6, 1) at price (½, 1)

Consistent.

(b) Choice (1, ½) at price (2, 2)

Inconsistent: violates Walras law

(c) Choice (2, 1) at price (1.6, 0.8)

Inconsistent: by revealed preferences, (1, 3) > (2, 1) ≥ (1, 3)

Suppose that after making observations 1 and 2, we observe this consumer choosing consumption bundle (2, 1).

(d) What can we conclude about the prices the consumer faced choosing (2, 1)?

Answer: We must have
$2p_1 + p_2 < p_1 + 3p_2 \;\Rightarrow\; p_1 < 2p_2$
$2p_1 + p_2 < 3p_1 + \frac{1}{2}p_2 \Rightarrow p_1 > \frac{1}{2}p_2$
We conclude that prices must satisfy $p_1 \in [\frac{1}{2} p_2, 2p_2]$, with $2p_1 + p_2 = 4$.

2. Producer Theory Question

a) A standard indirect profit function is defined with prices on all inputs and outputs, as follows: $\hat{\pi}(p) = \max_{y \in Y} p \cdot y$. Our proof that such a function is homogeneous of degree 1 uses the fact that the objective is *linear in prices*. (**Note well**: Linearity of the objective in y is *irrelevant*.)

In the present problem, the function $\pi(p) = \max_x p \cdot x - C(x)$ generally is *not* homogeneous of degree one, because the objective is only affine—not linear—in prices. Formally, let $x(p)$ denote the firm's optimal choice.

Claim: If $C(x(\lambda p)) > 0$ and $\lambda > 1$, then $\pi(\lambda p) > \lambda \pi(p)$.

Proof of Claim: $\pi(\lambda p) = \max_x \lambda p \cdot x - C(x) \geq \lambda p \cdot x(\lambda p) - C(x(\lambda p)) >$ $\lambda p \cdot x(\lambda p) - \lambda C(x(\lambda p)) = \lambda \pi(p)$. The first inequality follows by maximization; the second (strict) inequality follows because $C(x(\lambda p)) > 0$ and $\lambda > 1$ imply $\lambda C(x(\lambda p)) > C(x(\lambda p))$.

b) Notice that $\pi(p) = \max_x p \cdot x - C(x)$ is a maximum of linear functions of p, so it is convex in p. Differentiating the given function twice with respect to any price p_i shows that $\partial^2 \pi / (\partial p_i)^2 = -1 < 0$, so the function is not convex, a contradiction.

(Many students answered this part by using the envelope theorem, to conclude that the derivative of the supply of the i^{th} good with respect to its price is $\partial^2 \pi / (\partial p_i)^2 = -1 < 0$, in contradiction to the law of supply.)

c) In a neighborhood of the prices (1,9,3), the maximum profit can be written as $\pi(p) = \alpha_2 (p_2 - 2 p_2^{.5})$ (the other terms are zero). The function π is differentiable on this neighborhood; in particular, $\pi_2(p) \equiv \dfrac{\partial}{\partial p_2} \pi(p) = \alpha_2 (1 - p_2^{-.5})$. By the envelope theorem, $x_2(p) = \pi_2(p) = \alpha_2 (1 - p_2^{-.5})$, so $x_2(1,9,3) = \alpha_2 (1 - \frac{1}{3}) = \frac{2}{3} \alpha_2$.

d) The conditions imply that the cost function is submodular, so the objective $p \cdot x - C(x)$ is supermodular in x and has isotone ("weakly increasing") differences in $(x; p)$, so by Topkis's Theorem $x(p)$ is isotone. (The strict inequality and interiority assumptions imply that the first-order conditions cannot be satisfied at a quantity $x_i^*(p')$ for any i, so the increase is actually a strict one.)

e) The perfect answer comes in two parts.

   i) The constraint $x_3 \leq x_3^*(p')$ is equivalent to $x_3 = x_3^*(p')$. Proof: By Topkis's Thm, $x^*(p') = \overline{x}(p') \leq \overline{x}(p'')$. So, $x_3(p') = \overline{x}_3(p') \leq \overline{x}_3(p'') \leq x_3(p')$. So $\overline{x}_3(p'') = x_3(p')$.

   ii) So, the LeChatelier principle for supermodular optimization applies. Hence, the new solution $\overline{x}(p'')$ satisfies $x^*(p'') \geq \overline{x}(p'') \geq x^*(p') = \overline{x}(p')$. (As in part (d) above, using the smoothness assumptions, one can show that the inequalities must actually be strict, because the first-order conditions cannot otherwise be satisfied.)

3. Choice under Uncertainty Problem

a) We know u is increasing, so we want conditions on q,y that ensure L_2 dominates L_3 in the first order stochastic dominance sense. This is guaranteed is either $y \leq 1$ or if both $y \leq x$ *and* $p \leq q$.

b) We know that u is increasing and concave, and we want conditions on p,x that ensure that $U(L_1) \geq U(L_2)$. Now, if $p(x-1) \leq 1$, then L_1 has higher expected return than L_2 and is less risky. In particular, L_2 can be obtained from L_1 by some combination of a mean-preserving spread and a first-order stochastic shift (adding a positive random variable). Therefore $U(L_1) \geq U(L_2)$. If the converse holds, so $p(x-1) > 1$, then L_2 has strictly higher expected return so some increasing concave u will prefer L_2.

c) First, observe that L_3 has expected return 3.5 and L_2 has expected return 2.5, but L_3 is not less risky. Therefore, agents who don't mind risk much like L_3 and those who are very risk averse will like L_2.

   Nevertheless, given any α,1-α mix of L_2 and L_1, there is a β,1-β mix of L_3 and L_1 that is FOSD better, so all agents with increasing u prefer a mix of L_3 and L_1. Proof: An α,1-α mix of L_2 and L_1 returns $4\alpha+2(1-\alpha)=2+2\alpha$ or $\alpha+2(1-\alpha)=2-\alpha$ with equal probability, while a β,1-β mix of L_3 and L_1 returns $7\beta+2(1-\beta)=2+5\beta$ or $2(1-\beta)=2-2\beta$ with equal probability. Setting $\beta=\alpha/2$ means that the latter portfolio returns $(2-\alpha,2+(5/2)\alpha)$ with equal probabilities, which is strictly better than $(2-\alpha,2+2\alpha)$ with equal probabilities provided that $\alpha>0$.

   By the way, this problem is based on a short paper by Paul Samuelson, who conjectures conjectures that the idea can be greatly generalized. So far as I know, though, no one has proved such a result.

d) Claim: $V(L_3) \geq V(L_2) \geq V(L_1)$. To see this, note that:

$$V(L')-V(L)= \quad U(L')-U(L) \quad + \quad H(L')-H(L),$$

   where $H(L)=\sum h(z)\Pr(z)$. Because $U(L_3) \geq U(L_2) \geq U(L_1)$ it suffices to check that $H(L_3) \geq H(L_2) \geq H(L_1)$. Note that moving from L_1 to L_2 entails a first order stochastic shift (the mean increases from 2 to 2.5) and a mean preserving spread; the same goes for moving from L_2 to L_3. A decision-maker with expected utility function $H(\cdot)$ likes both changes, so $H(L_3) \geq H(L_2) \geq H(L_1)$ and we are done.

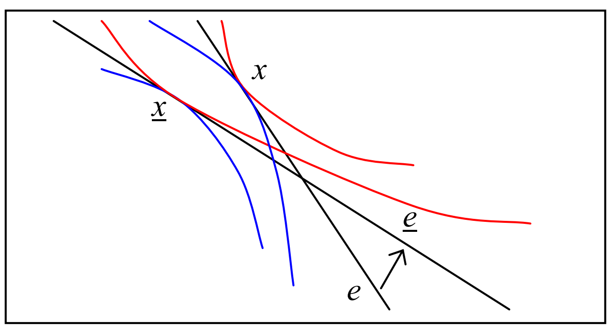
   Note: It is *not* quite enough in this problem to note that V has a higher Arrow-Pratt coefficient of risk aversion than U. That means that anytime U prefers some risky prospect to some sure thing, V will also (so it helps to compare L_2 and L_1). But is doesn't mean that anytime U prefers a riskier prospect to a less risky prospect, V will as well. The latter is a stronger concept of "more risk-averse".

4. General Equilibrium Problem.

Consider a pure exchange economy with two agents and two commodities. Assume that utility functions u^1 and u^2 are strictly increasing, strictly concave and continuously differentiable. In the first setting, endowments are e^1 and e^2, and there is a Walrasian equilibrium (p, x^1, x^2). In the second setting, endowments are $\underline{e}^1$ and $\underline{e}^2$, and there is a Walrasian equilibrium $(\underline{p}, \underline{x}^1, \underline{x}^2)$. Assume that all endowments are strictly positive, that total endowment is the same in both settings (i.e. $e^1 + e^2 = \underline{e}^1 + \underline{e}^2$), and that agent 1's endowment is strictly greater in the second setting (i.e. $\underline{e}^1 >> e^1$). (You may find it useful to argue your answers graphically, using a clearly labeled and clearly explained picture).

(a) Is it possible that in the second setting, agent 1 is worse off and agent 2 is better off than in the first setting? In other words, is it possible that $u^1(\underline{x}^1) < u^1(x^1)$ and $u^2(\underline{x}^2) > u^2(x^2)$?

Solution: Yes, this is possible. The only way that it is possible is shown in the figure below: the budget lines must intersect, and each equilibrium allocation and endowment must be on the opposite sides of the point of intersection.

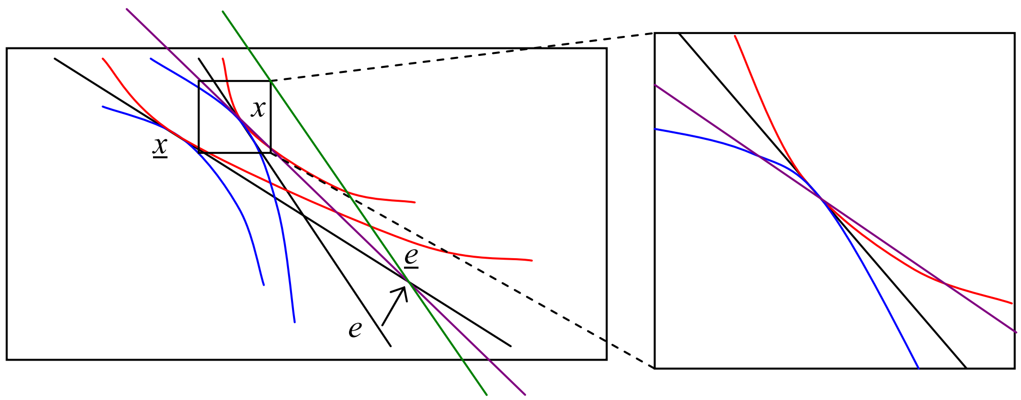


(b) Is it possible that both agents are worse off in the second setting? In other words, is it possible that $u^1(\underline{x}^1) < u^1(x^1)$ and $u^2(\underline{x}^2) < u^2(x^2)$?

Solution: No, this is impossible. Otherwise, because the total endowment is the same in both settings, the equilibrium allocation would not be Pareto efficient in the second setting. This would contradict the First Welfare Theorem.

(c) Suppose also that the Walrasian equilibrium is unique in the second setting. Is it possible that in the second setting, agent 1 is worse off and agent 2 is better off than in the first setting?

Solution: No, this is impossible. The only way this could happen would be as shown in the figure below. Let us show that the Walrasian equilibrium cannot be unique in this setting. Normalize all prices, so the price of good 1 is 1. Draw a purple line through points $\underline{e}$ and x. Let p^* be a price vector orthogonal to that line. Let us discuss the properties of excess demand at endowment level $\underline{e}$. Then, as we see in the figure below, there is excess demand for good 1 at price p^*. As we decrease the price of good 2 continuously from level p_2^* to 0, the excess demand for good 2 must converge to infinity (therefore, excess demand for good 1 becomes negative). From the continuity of excess demand functions and Walras law, there is a price $p_2' \in (0, p_2^*)$, at which excess demands are zero. This price gives a second Walrasian equilibrium (green line), contradiction.



(d) Suppose also that $u^1(c_1, c_2) = u^2(c_1, c_2) = \log c_1 + \log c_2$. Prove that the Walrasian equilibrium is unique for all endowment levels.

Solution: Indifference curves are defined by $c_1 c_2 = const$. By total differentiation

$$c_1\, dc_2 + dc_1\, c_2 = 0 \Rightarrow dc_2/dc_1 = -c_2/c_1$$

From this, we conclude that indifference curves are tangent on the line, connecting the bottom left and upper right corners of the Edgeworth box. For any fixed total endowment, equilibrium prices are the same in any equilibrium, and there is a unique equilibrium budget line passing through any point in the Edgeworth box.