Economics 202

Final Exam and Solutions

Professor Paul Milgrom

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Read and follow the following instructions carefully:

1. Please fill out the acknowledgment and acceptance of the honor code on the cover of each bluebook, *writing your Stanford Student ID number in place of your name*.

2. You have three hours to complete this exam.

3. Please answer each question in a separate bluebook, writing the question number on the cover of the bluebook.

4. You may not use any aids (e.g., notes, books, calculators, etc.)

5. Answering these questions may require you to make reasonable assumptions that are not explicitly stated in the problem; please try to be explicit whenever you do so.

Good luck!

Home Production. (75 minutes)

A family unit that consumes (x,y,z) of three goods can produce any non-negative quantities (x,y) of the first two goods subject to $x + y \le 2$. In addition, it is endowed with z =1 units of the third good. The family's consumption preferences are described by the utility function $U(x,y,z) = x^{2/3} + y^{2/3}z^{1/3}$.

a) If there are no markets for any of the goods, what is the family's optimal production and consumption decision?

$\max_{x,y,z\in\Re_+} x^{2/3} + y^{2/3}z^{1/3}$ *subject to two constraints:* $x + y \le 2$ *and* $z \le 1$. *Substituting* $z = 1$, *the Lagrangian is* $\max_{x,y\ge 0} x^{2/3} + y^{2/3} - \lambda(x + y - 2)$. *The preferences are locally non-satiated, concave and symmetric, so the solution is* $x = y = 1$ *and* $\lambda = 2/3$. *Hence, the family's optimal consumption is* $(x,y,z) = (1,1,1)$.

b) Next, suppose that the family can sell goods in the market. Let $\hat{x},\hat{y}$ denote the family's production choices while x and y denote its consumption choices. Market prices of goods are given by $p = (p_x, p_y, p_z)$. Write the family's decision problem as an optimization problem deciding how much to produce and consume. Write the corresponding Lagrangian.

The problem is $\max_{(x,y,z,\hat{x},\hat{y})\in\Re_+^5} x^{2/3} + y^{2/3}z^{1/3}$ *subject to* $p_x x + p_y y + p_z z \le p_x\hat{x} + p_y\hat{y} + p_z$ *and* $\hat{x} + \hat{y} \le 2$. *The Lagrangian problem is*

$$\max_{(x,y,z,\hat{x},\hat{y})\in\Re_+^5} x^{2/3} + y^{2/3}z^{1/3} - \lambda(p_x x + p_y y + p_z z - p_x\hat{x} - p_y\hat{y} - p_z) - \mu(\hat{x} + \hat{y} - 2)$$

c) Assume that the prices are all distinct and positive. Find the family's optimal production.

Analyzing the Lagrangian and focusing first on the production decision, we observe that with positive prices, at least one good is produced with positive quantity. The first-order conditions are $p_x - \mu \le 0$ *and* $p_y - \mu \le 0$, *with at least one equality, so* $\mu = \max(p_x, p_y)$ *and* $(\hat{x},\hat{y}) = (2,0)$ *or* (0,2) *as* $p_x = \mu > p_y$ *or* $p_y = \mu > p_x$.

(Notice that the family's optimal production choice does not depend on its utility function: it simply maximizes the value of its production.)

d) Write the optimization problem that determines the family's expenditure function in the form $e(p,u) = ...$ Write the Lagrangian for this problem.

$$e(p,u) = \min_{(x,y,z)\in\Re_+^3} p_x x + p_y y + p_z z \text{ subject to } x^{2/3} + y^{2/3}z^{1/3} \ge u$$

$$\textit{Lagrangian} = p_x x + p_y y + p_z z - \gamma(x^{2/3} + y^{2/3}z^{1/3} - u)$$

Observe that if prices are all positive and u is positive, then any solution must have $x > 0$ and either $y = z = 0$ or both y and z strictly positive.

e) Characterize the optimal quantity x in the expenditure minimization problem when the consumer optimum entails $y = z = 0$.

   In this case, since the minimum utility constraint is binding, we have $x = u^{3/2}$.

f) The other possibility is that all three consumptions are strictly positive: $x, y, z > 0$. Characterize the Kuhn-Tucker conditions for the optimal consumption quantities in the expenditure minimization problem for this case.

   The Lagrangian problem is:

$$\min_{(x,y,z)\in\Re_+^3} p_x x + p_y y + p_z z - \gamma(x^{2/3} + y^{2/3}z^{1/3} - u)$$

   With the assumptions that $x, y, z > 0$*, the Kuhn-Tucker conditions can be written as:*

$$\begin{aligned} p_x - \gamma \tfrac{2}{3} x^{-1/3} &= 0 \\ p_y - \gamma \tfrac{2}{3} y^{-1/3} z^{1/3} &= 0 \\ p_z - \gamma \tfrac{1}{3} y^{2/3} z^{-2/3} &= 0 \\ u - x^{2/3} - y^{2/3} z^{1/3} &= 0 \end{aligned}$$

Banks and Risky Lending (45 minutes)
A firm has applied to borrow an amount L from a bank. The loan agreement stipulates that the firm is to repay the loan with interest, the total amount due being $L(1+r)$, but if the firm goes bankrupt, it instead pays whatever resources it has, which may be less than the amount due.

Suppose that the firm with equity resources E borrows and invests the amount L (where $L > E$) in a project with *gross* return x per unit invested, so the total gross return is Lx, where x is a non-negative random variable with probability distribution F and corresponding probability density f. The firm repays the loan to the extent that its resources allow, so its realized *net* profit for any realization of x is $-E + \max\left(0, Lx + E - L(1+r)\right)$.

a) Write an integral expression to show the borrower's expected net profit $\pi(L,E)$ as a function of the loan amount and equity resources. Show that π is strictly convex in L, so if the borrower can choose any L in a closed interval $[0,\bar{L}]$, then for any distribution F, the choice that maximizes its expected net profit is either 0 or $\bar{L}$.

We compute as follows:

$$\pi(L,E) = -E + \int_0^{\infty} \max\left(0, Lx + E - (1+r)L\right) f(x)\,dx$$

$$= -E + \int_{1+r-E/L}^{\infty} \left(Lx + E - (1+r)L\right) f(x)\,dx$$

$$\frac{\partial \pi}{\partial L} = \int_{1+r-E/L}^{\infty} \left(x - (1+r)\right) f(x)\,dx$$

$$\frac{\partial^2 \pi}{\partial L^2} = E^2 L^{-3} > 0$$

b) Show that for any fixed loan terms, the firm is more likely to take the loan if E is smaller.

It suffices to show that the $\pi(L,E)$ *is decreasing in E. And,*

$$\pi(L,E) = -E + \int_{1+r-E/L}^{\infty} \left(Lx + E - (1+r)L\right) f(x)\,dx$$

$$\frac{\partial \pi}{\partial E} = -1 + \int_{1+r-E/L}^{\infty} f(x)\,dx = \int_0^{1+r-E/L} f(x)\,dx = -F(1+r-E/L) < 0$$

c) Suppose the interest charge $r(L)$ depends on the size L of the loan, with $r'(L) > 0$. In this revised model, for any realization of x, the borrower's net return is: $\max(0, Lx + E - (1 + r(L))L) - E$. Show that if the borrower can choose any loan amount L in $[0, \bar{L}]$, then the borrower's optimal choice is non-increasing in E.

The borrower's problem is:

$$\pi(L,E) = -E + \int_{1+r(L)-E/L}^{\infty} \left[Lx + E - \left(1 + r(L)\right)L\right] f(x)\,dx$$

$$\frac{\partial \pi}{\partial E} = -1 + \int_{1+r(L)-E/L}^{\infty} f(x)\,dx$$

$$\frac{\partial^2 \pi}{\partial E \partial L} = -f\left(1 + r(L) - E/L\right)\left(r'(L) + EL^{-2}\right) < 0$$

So π is submodular and hence, by the Topkis Theorem, the optimal choice of L is non-increasing in E.

d) Next let us replace the distribution F by a parameterized family of distributions $\left\{F_\theta(\cdot) : \theta \in \Re_+\right\}$ and study how the borrower's optimal choice depends on the parameter. Show that if increases in θ are increases in the sense of first-order stochastic dominance, then for any E, the borrower's optimal choice of L is a non-decreasing function of θ.

Let us write the borrower's objective and check its mixed partial derivative:

$$\pi(L,\theta) = -E + \int_0^{\infty} \max\left(0, Lx + E - \left(1 + r(L)\right)L\right) f_\theta(x)\,dx$$

$$\frac{\partial \pi}{\partial L} = \int_0^{\infty} \max\left(0, x - \frac{d}{dL}\left(\left(1 + r(L)\right)L\right)\right) f_\theta(x)\,dx$$

$$\frac{\partial^2 \pi}{\partial L \partial \theta} > 0$$

The inequality follows from the definition of first-order stochastic dominance, because the integral is a non-decreasing function of x. Since π is supermodular, the optimal loan amount is non-decreasing in L.

Ambiguity Aversion and General Equilibrium (60 minutes)

Consider an economy with a finite number S of states-of-the-world s and a single physical consumption good. To interpret what follows, you should think informally about the Ellsberg paradox and consider a consumer j who is unsure about the probabilities of the various states and supposes the "true" probability distribution is an element of a certain compact set: $\pi \in P^j$. Assume that all components of each $\pi \in P^j$ are strictly positive ($\pi \gg 0$) and that $\sum_{s=1}^{S} \pi_s = 1$.

Suppose that consumer j's preferences over state-contingent consumption vectors are represented by the utility function: $U^j(z_1^j,...,z_S^j) = \min_{\pi \in P^j} \sum_{s=1}^{S} \pi_s u^j(z_s^j)$. Each consumer j is endowed with a consumption lottery $\overline{z}^j$ that, in any state s, pays $\overline{z}_s^j > 0$ units of the physical good (with probability 1).

For ease of reading, we suppress the superscript j below.

a) Given our assumptions about u and P, is the utility function $U(z_1,...,z_S) = \min_{\pi \in P} \sum_{s=1}^{S} \pi_s u(z_s)$ continuous? Concave? Strictly increasing? Justify your answers. (**Hint**: for the strictly increasing conclusion, one approach uses the envelope theorem.)

   Yes. For each fixed $\pi \in P$, the expression is increasing, continuous and concave by assumption. By Berge's theorem, treating z as the parameter, the minimum is continuous in z. It is well-known that the minimum of concave functions is concave. To show that U is increasing, fix any z and let $\pi^*(z) \in \arg\min_{\pi \in P} \sum_s \pi_s u(z_s)$. Apply the envelope theorem as follows:
   $\frac{\partial}{\partial z_s} U(z_1,...,z_S) = \pi_s^*(z) u'(z_s) > 0$*, so U is increasing in each argument.*

b) Imagine an Arrow-Debreu economy with complete markets. All trading happens before the state is realized ("at time zero") and it provides for consumption in each state s. Write the consumer problem for this model, assuming that the consumer gets income only be selling his endowment.

   Given prices p for the securities, the problem is:

   $\max_{z \in \Re_+^S} \min_{\pi \in P^j} \sum_{s=1}^{S} \pi_s u^j(z_s^j)$ subject to

   $p \cdot z^j \le p \cdot \overline{z}^j$

c) Do the First and Second Welfare Theorems apply to this economy? If not, identify any conditions of the theorem(s) that are violated.

Yes, the theorems apply because by part (a), $U^j(z)$ is continuous, increasing, and concave for each consumer j. Also, endowments are strictly positive in every state by assumption.

Assume now that there is a single representative consumer, allowing us to suppress the superscript *j*. Suppose that $S=2$ and $P=\{(\pi,1-\pi):.40\le\pi\le.60\}$. Let $u(x)=\ln(x)$. In each part, normalize so that the total state price is $p_1+p_2=1$.

d) Suppose that the consumer's endowment is 1 unit in each state *s*. Find an equilibrium price vector expressing the price of consumption in the two states.

By symmetry, the set of market equilibrium prices includes $(\frac{1}{2},\frac{1}{2})$.

e) Suppose that the consumer's endowment is 1 in state 1 and $\alpha>1$ in state 2. Find an equilibrium price vector expressing the price of consumption in the two states? (Hint: you may find it helpful to use the envelope theorem).

$$\max_{z\in\Re^S_+}\min_{\pi\in P}\sum_{s=1}^{S}\pi_s\ln(z_s)\text{ subject to}$$

$$p\cdot z\le p\cdot\bar{z}$$

The Lagrangian is:

$$\max_{z\in\Re^S_+}\min_{\pi\in P}\sum_{s=1}^{S}\pi_s\ln(z_s)-\lambda\sum_{s=1}^{S}p_s(z_s-\bar{z}_s)$$

Let $\pi^*(z)\in\arg\min_{\pi\in P}\sum_{s=1}^{S}\pi_s\ln(z_s)$. By the envelope theorem, for almost all z, $\frac{\partial}{\partial z_s}\min_{\pi\in P}\sum_{\hat{s}=1}^{S}\pi_{\hat{s}}\ln(z_{\hat{s}})=\pi^*_s(z)/z_s$. In the example, since the high endowment state is state 2 and the consumer must, in equilibrium, consume his endowment, we must have that $\pi^*(\bar{z})=(.60,.40)$. So,

$$0=\left.\frac{\partial}{\partial z_s}\right|_{z=\bar{z}}\min_{\pi\in P}\sum_{\hat{s}=1}^{S}\pi_{\hat{s}}\ln(z_{\hat{s}})-\lambda\sum_{\hat{s}=1}^{S}p_{\hat{s}}(z_{\hat{s}}-\bar{z}_{\hat{s}})=\frac{\pi^*_s(\bar{z})}{\bar{z}_s}-\lambda p_s$$

$$\therefore\frac{p_2}{p_1}=\frac{\pi^*_2}{\pi^*_1}\frac{\bar{z}_1}{\bar{z}_2}=\frac{.4}{.6}\frac{1}{\alpha}=\frac{2}{3\alpha}.$$

$$1=p_1+p_2=p_1\left(1+\frac{2}{3\alpha}\right)$$

$$\therefore p_1=\frac{1}{1+\frac{2}{3\alpha}}=\frac{3\alpha}{3\alpha+2}\text{ and so }p=\left(\frac{3\alpha}{3\alpha+2},\frac{2}{3\alpha+2}\right)$$