Fall 2012
Economics 202 Final Exam
Solutions

Question 1.

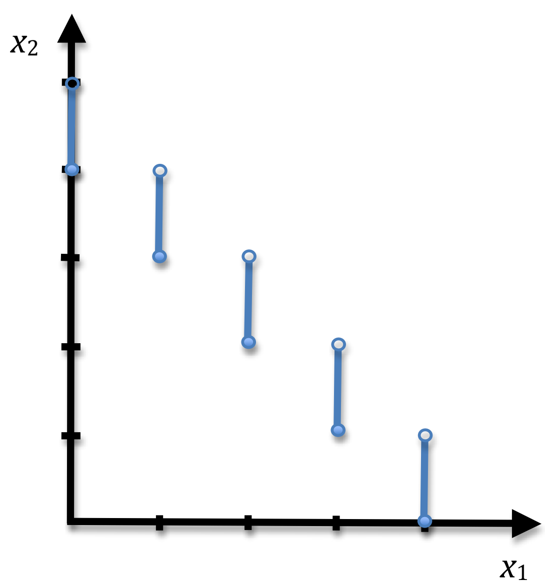
(a) (i) Yes. The preferences are represented by a utility function u that has no local maxima. They are therefore locally non-satiated.

(ii) No. The consumer prefers both $(1,0)$ and $(0,1)$ to $(\frac{2}{3},0)$. But the consumer does not prefer $\frac{1}{2}(1,0)+\frac{1}{2}(0,1)=(\frac{1}{2},\frac{1}{2})$ to $(\frac{2}{3},0)$.
Note: Since a function can be quasiconcave without being concave, it is not enough to show that u is not concave.

(iii) No. The consumer prefers $(\frac{1}{2},0)$ to $(0,\frac{n}{n+1})$ for every $n \in \mathbb{N}$. However $(0,\frac{n}{n+1}) \to (0,1)$, and the consumer does not prefer $(\frac{1}{2},0)$ to $(0,1)$.
Note: if g is any monotonically increasing function, then $g \circ u$ and u represent the same preferences. Continuous preferences can therefore be represented by a discontinuous utility function, and so it is not enough to show that u is discontinuous.

(iv) No. The consumer prefers $(\frac{1}{2},0)$ to $(\frac{1}{3},\frac{1}{3})$, but does not prefer $3\cdot(\frac{1}{2},0)=(\frac{3}{2},0)$ to $3\cdot(\frac{1}{3},\frac{1}{3})=(1,1)$.
Note: if g is any monotonically increasing function, then $g \circ u$ and u represent the same preferences. Homothetic preferences can therefore be represented by a function that is not homogeneous of degree one, and so it is not enough to show that u is not homogeneous of degree one.

(b) The indifference curve is plotted below:



(c) Since $p_1 < p_2$, given any wealth w, it is optimal for the consumer to exhaust her budget on good x_1. Therefore Marshallian demand and indirect utility are

$$\begin{aligned} x(p, w) &= \left(\frac{w}{p_1}, 0\right) \\ v(p, w) &= \frac{w}{p_1} \end{aligned}$$

Similarly, the least expensive way to reach any utility level u is through consumption of good x_1. Therefore Hicksian demand and expenditure are

$$\begin{aligned} h(p, u) &= (u, 0) \\ e(p, u) &= p_1 u \end{aligned}$$

(d) Since $p_1 > p_2$, the agent will consume as much of good 2 as she can afford, before spending the rest of her budget on good 1. Therefore Marshallian demand and indirect utility are

$$x(p, w) = \left(\frac{w}{p_1} - \frac{p_2}{p_1}\left\lfloor \frac{w}{p_2} \right\rfloor, \left\lfloor \frac{w}{p_2} \right\rfloor\right)$$

$$\begin{aligned} v(p, w) &= \frac{w}{p_1} - \frac{p_2}{p_1}\left\lfloor \frac{w}{p_2} \right\rfloor + \left\lfloor \frac{w}{p_2} \right\rfloor \\ &= \frac{w}{p_1} + \frac{p_1 - p_2}{p_1}\left\lfloor \frac{w}{p_2} \right\rfloor \end{aligned}$$

There are two natural candidates for how to attain a target utility level u in the least expensive way. The first is to consume good x_2 until u has been reached or exceeded. The second is to consume good x_2 until the next unit would push utility over u, and then consume good x_1 until u is reached. If $u - \lfloor u \rfloor \leq \frac{p_2}{p_1}$, then the latter is optimal, and if $u - \lfloor u \rfloor \geq \frac{p_2}{p_1}$, then the former is optimal. Therefore Hicksian demand and expenditure are

$$h(p, u) = \begin{cases} (u - \lfloor u \rfloor, \lfloor u \rfloor) & \text{if } u - \lfloor u \rfloor \leq \frac{p_2}{p_1} \\ (0, \lceil u \rceil) & \text{if } u - \lfloor u \rfloor \geq \frac{p_2}{p_1} \end{cases}$$

$$e(p, u) = \begin{cases} p_1(u - \lfloor u \rfloor) + p_2 \lfloor u \rfloor & \text{if } u - \lfloor u \rfloor \leq \frac{p_2}{p_1} \\ p_2 \lceil u \rceil & \text{if } u - \lfloor u \rfloor \geq \frac{p_2}{p_1} \end{cases}$$

(e) The answer to part (c) would not change. The ability to consume negative amounts of good x_1 is not used.

On the other hand the answers to part (d) would change. Given any wealth w, the agent could achieve arbitrarily high amounts of utility by consuming negative amounts of x_1 and positive amounts of x_2. Indirect utility is infinite:

$$v(p, w) = \infty.$$

Marshallian demand is defined as the argmax of a function that is not bounded above, and so it does not exist.

$$x(p, w) = \emptyset.$$

Similarly, any utility level u could be reached at an arbitrarily small cost by consuming negative amounts of x_1 and positive amounts of x_2. Expenditure is negative infinity:

$$e(p, u) = -\infty.$$

Hicksian demand is defined as the argmin of a function that is not bounded below, and so it does not exist:

$$h(p, u) = \emptyset.$$

Question 2.

(a) This is a quasilinear economy. Pareto optimal allocations are those that maximize total surplus. The numeraire can be allocated in any way, so long as it is not wasted. The allocation of the non-numeraire goods must solve

$$(x_2^A, x_2^B) = \underset{x_2^A, x_2^B \geq 0}{\arg\max}\, \gamma x_2^B - \frac{(x_2^B)^2}{2} + \lfloor x_2^A \rfloor \quad \text{subject to} \quad x_2^A + x_2^B = 10.$$

Plugging in $x_2^A = 10 - x_2^B$,

$$x_2^2 = \underset{0 \leq x_2^B \leq 10}{\arg\max}\, \gamma x_2^B - \frac{(x_2^B)^2}{2} + \lfloor 10 - x_2^B \rfloor.$$

Ignoring both the floor function and the constraints for the time being, we take the first order condition to obtain

$$x_2^B = \gamma - 1.$$

With the floor function, it is optimal to choose $x_2^B = [\gamma - 1]$, the nearest integer to $\gamma - 1$.[1] Since $\gamma \in (2, 9)$, this choice will not violate the constraints $0 \leq x_2^B \leq 10$.

Therefore the Pareto optimal allocations are:

$$\left\{(x_1^A, x_2^B) = ((t, [11 - \gamma]), (10 - t, [\gamma - 1])) : t \in \mathbb{R}\right\}.$$

(b) Neither u^A nor u^B has any local maxima. Consequently the preferences of both consumers are locally non-satiated, and so the assumptions of the First Welfare Theorem are met for all values of γ. Therefore the conclusions of the First Welfare Theorem hold for all values of γ.

[1] We must choose an integer, and by concavity of the version of the objective without the floor function, the optimal integer will be either $\lfloor \gamma - 1 \rfloor$ or $\lceil \gamma - 1 \rceil$. By symmetry, whichever is closest to $\gamma - 1$ maximizes the objective.

(c) First observe that since consumer B's utility function is quasilinear, his demand for the second good at prices $(p_1, p_2) = (1, p)$ is

$$x_2^B(p) = \arg\max_{x_2^B \geq 0} \gamma x_2^B - \frac{(x_2^B)^2}{2} - p x_2^B$$

Taking the first order condition, we find $x_2^B(p) = \gamma - p$. After consuming this amount of good 2, the consumer will then exhaust the rest of his budget on good 1.

If $\gamma \in \mathbb{Z}$, then any Pareto optimal allocation can be supported as a Walrasian equilibrium by prices $(p_1, p_2) = (1, 1)$. If consumer A is endowed with $(t, [11 - \gamma]) = (t, 11 - \gamma)$, then at these prices, she can do no better than to consume it. Similarly, if consumer B is endowed with $(10 - t, [\gamma - 1]) = (10 - t, \gamma - 1)$, then, as computed above, at those prices he will optimally demand that bundle.

Now assume that $\gamma \notin \mathbb{Z}$. If $p_1 < p_2$, then as in 1(d), consumer A's Marshallian demand will not exist. If $p_2 > p_1$, then as in 1(b), consumer A will not demand any of good 2. However, in every Pareto optimal allocation, consumer A receives some of good 2. Therefore if any Pareto optimal allocation is to be supported by prices as a Walrasian equilibrium, those prices must be proportional to $(p_1, p_2) = (1, 1)$. However at these prices, consumer B will demand $\gamma - 1 \neq [\gamma - 1]$ units of good 2, and so supporting prices do not exist.

Therefore, the conclusions of the Second Welfare Theorem hold if and only if $\gamma \in \mathbb{Z}$.

Question 3.

(a) By the Law of Supply, a necessary condition for profit maximization is that for all i, j

$$(y_i - y_j)(p_i - p_j) \geq 0.$$

Since $p_1 < p_2 < \cdots < p_K$, a necessary condition for consistency with profit maximization by a price-taking firm is

$$y_1 \leq y_2 \leq \cdots \leq y_K.$$

To show that this is also sufficient, we show that for any output choices satisfying this condition, there exists a price-taking firm for which these are the profit-maximizing choices at $p_1, \ldots, p_K$. Assume $Y = \{y_1, \ldots, y_K\}$. Also assume that at the given input prices, the cost of producing each of these output levels is $c(y_k) = \sum_{j=1}^{k-1} p_{j+1}(y_{j+1} - y_j)$. If the the firm produces y_k at output price p, then it makes profit

$$\begin{aligned} p y_k - c(y_k) &= p y_k - \sum_{j=1}^{k-1} p_{j+1}(y_{j+1} - y_j) \\ &= \sum_{j=1}^{k-1} (p - p_{j+1})(y_{j+1} - y_j). \end{aligned}$$

Since $y_{j+1} \geq y_j$ for every j, it is easy to see that if if $p_k \leq p < p_{k+1}$, then y_k is a profit-maximizing choice of output. Therefore, this firm optimally produces y_k at price p_k for every k.

(b) For notational ease, define $p_0 = 0$. If the output price is zero, then the firm cannot make positive profits, and by the shutdown property, it makes zero profit, so $\pi(0) = 0$.

Let $Y^*(p)$ be the optimal output correspondence, where p denotes the output price. Let $y(p)$ be a selection from $Y^*(p)$ such that $y(p_k) = y_k$ for all $k = 1, \ldots, K$. As in part (a), the Law of Supply tells us that $y(p)$ must be nondecreasing. Therefore $\underline{y}(p) \leq y(p) \leq \overline{y}(p)$ for all $p \in [0, p_K]$, where

$$\underline{y}(p) = \begin{cases} 0 & \text{if } p \in [0, p_1) \\ y_1 & \text{if } p \in [p_1, p_2) \\ & \vdots \\ y_{K-1} & \text{if } p \in [p_{K-1}, p_K) \\ y_K & \text{if } p = p_K \end{cases}$$

and

$$\overline{y}(p) = \begin{cases} y_1 & \text{if } p \in [0, p_1] \\ y_2 & \text{if } p \in (p_1, p_2] \\ & \vdots \\ y_{K-1} & \text{if } p \in (p_{K-2}, p_{K-1}] \\ y_K & \text{if } p \in (p_{K-1}, p_K] \end{cases}$$

Using the Producer Surplus Formula,

$$\begin{aligned} \pi(p_K) &= \pi(0) + \int_0^{p_K} y(p)\, dp \\ &\geq \int_0^{p_K} \underline{y}(p)\, dp \\ &= \sum_{j=1}^{k} y_{j-1}(p_j - p_{j-1}), \end{aligned}$$

where for notational simplicity we define $y_0 = 0$. Similarly,

$$\begin{aligned} \pi(p_K) &= \pi(0) + \int_0^{p_K} y(p)\, dp \\ &\leq \int_0^{p_K} \overline{y}(p)\, dp \\ &= \sum_{j=1}^{k} y_j(p_j - p_{j-1}). \end{aligned}$$

In conclusion, we have formulated upper and lower bounds on $\pi(p_K)$:

$$\sum_{j=1}^{k} y_{j-1}(p_j - p_{j-1}) \leq \pi(p_K) \leq \sum_{j=1}^{k} y_j(p_j - p_{j-1}).$$

In order to show that these bounds are as tight as possible, it suffices to demonstrate by example that both $\underline{y}(p)$ and $\overline{y}(p)$ can arise as supply functions of profit-maximizing firms. The firm described in the example constructed in part (a) has supply function $\underline{y}(p)$. If in that example we had instead defined $c(y_k) = \sum_{j=1}^{k-1} p_j(y_{j+1} - y_j)$, then we would obtain a firm with supply function $\overline{y}(p)$.

(c) The size of the interval of uncertainty is the distance between the upper and lower bounds:

$$\sum_{j=1}^{k}(y_j - y_{j-1})(p_j - p_{j-1}),$$

where, as before, we define $y_0 = 0$ for notational simplicity. The outputs for which we seek are

$$(y_1, \ldots, y_{K-1}) = \underset{0 \le y_1 \le \cdots \le y_{K-1} \le y_K}{\arg\max} \sum_{j=1}^{k}(y_j - y_{j-1})(p_j - p_{j-1}).$$

The size of this interval is maximized if the $\{y_k\}$ are chosen so as to maximize the "weight" put on the largest gaps between the $\{p_k\}$. More formally, let $J = \arg\max_{1 \le j \le K}(p_j - p_{j-1})$. Then $(y_1, \ldots, y_{K-1})$ belongs to the argmax if and only if

$$\sum_{j \in J}(y_j - y_{j-1}) = y_K.$$

The maximum size of the interval of uncertainty is then

$$y_K \cdot \max_{1 \le j \le K}(p_j - p_{j-1}).$$

(d) You should take these $K - 1$ observations at prices below p. The maximum size of the interval of uncertainty is then by (c) equal to

$$y \cdot \max_{1 \le j \le K}(p_j - p_{j-1}),$$

where we define $p_K = p$. In order to minimize this quantity, the prices should be equally spaced along the interval $[0, p]$. That is to say the prices should be

$$p_k = \frac{k}{K} \cdot p.$$

Question 4.

(a) I claim that the agent has von Neumann-Morgenstern preferences if and only if $\gamma = 1$. If $\gamma = 1$, then $U(X) = \mathbb{E}X$, which has the expected utility form, so it represents von Neumann-Morgenstern preferences.

Now assume $\gamma \neq 1$. It remains to show that U does not represent von Neumann-Morgenstern preferences in this case. This can be shown by constructing an example in which the indifference curves are not parallel straight lines. Assume that X is a random variable that is equal to zero with probability $1-p_1-p_2$, is equal to one with probability p_1, and is equal to two with probability p_2. Then $\mathbb{E}X = p_1 + 2p_2$. Consider a neighborhood $\mathcal{P}$ in probability space such that $p_1 + 2p_2 \in (0,1)$ on $\mathcal{P}$. Then

$$U(p_1,p_2) = p_1 + 2p_2 + \gamma[(-p_1-2p_2)(1-p_1-p_2) + (1-p_1-2p_2)p_1] + (2-p_1-2p_2)p_2.$$

The slope of an indifference curve at a point $(p_1,p_2) \in \mathcal{P}$ is

$$-\frac{\partial U/\partial p_1}{\partial U/\partial p_2} = -\frac{p_2(\gamma-1)+1}{4p_2(\gamma-1)+p_1(\gamma-1)+4-2\gamma},$$

which is not constant in the probabilities if $\gamma \neq 1$.

(b) The risk premium of a lottery X is

$$\begin{aligned} U(\mathbb{E}X) - U(X) &= -\gamma\mathbb{E}[X-\mathbb{E}X|X<\mathbb{E}X]\cdot\Pr(X<\mathbb{E}X) \\ &\quad - \mathbb{E}[X-\mathbb{E}X|X\geq\mathbb{E}X]\cdot\Pr(X\geq\mathbb{E}X) \\ &= (1-\gamma)\mathbb{E}[X-\mathbb{E}X|X<\mathbb{E}X]\cdot\Pr(X<\mathbb{E}X). \end{aligned}$$

The agent is risk averse if the risk premium is nonnegative for every lottery X. If $\gamma \geq 1$, then this is the case. If, on the other hand, $\gamma < 1$, then the risk premium is negative for every non-degenerate lottery X. Therefore the agent is risk averse if and only if $\gamma \geq 1$.

(c) From (b), the risk premium of a lottery $X+w$ is

$$\begin{aligned} &(1-\gamma)\mathbb{E}\left[X+w-\mathbb{E}[X+w]|X+w<\mathbb{E}[X+w]\right]\cdot\Pr\left(X+w<\mathbb{E}[X+w]\right) \\ &\quad = (1-\gamma)\mathbb{E}[X-\mathbb{E}X|X<\mathbb{E}X]\cdot\Pr(X<\mathbb{E}X) \end{aligned}$$

Since the risk premium does not change with wealth, the agent's preferences exhibit constant absolute risk aversion.

(d) Denote the agent's initial wealth by w. If she rejects the gamble, then she receives utility $U(w) = w$. We now compute utility from the gamble. Let X be the random variable that is $w+40$ with probability $\frac{3}{4}$ and is $w-80$ with probability $\frac{1}{4}$. Notice that $\mathbb{E}X = w+10$.

$$\begin{aligned} U(X) &= w + 10 + 2(-90)\cdot\frac{1}{4} + 30\cdot\frac{3}{4} \\ &= w - 12.5 \end{aligned}$$

The agent would therefore reject the gamble.

(e) Denote the agent's initial wealth by w. If she rejects the gamble, then she receives utility $U(w) = w$. We now compute utility from the gamble. Let X be the random variable that is $w+160$ with probability $\frac{81}{256}$, is $w+40$ with probability $\frac{108}{256}$, is $w-80$ with probability $\frac{54}{256}$, is $w-200$ with probability $\frac{12}{256}$, and is $w-320$ with probability $\frac{1}{256}$. Notice that $\mathbb{E}X = w+40$.

$$\begin{aligned} U(X) = w + 40 + 2&\left[(-360)\cdot\frac{1}{256} + (-240)\cdot\frac{12}{256} + (-120)\cdot\frac{54}{256}\right] \\ &+ 0\cdot\frac{108}{256} + 120\cdot\frac{81}{256} \\ = w + &\frac{65}{32} \end{aligned}$$

The agent would therefore accept this gamble.

(f) This agent finds it optimal to reject a single bet at any wealth level, but also to accept a bundle of four bets. Notice that by part (a), this agent is not an expected utility maximizer.

As we saw in the Samuleson's bet question on Problem Set 4, this behavior could occur from an expected utility maximizer. The reasoning is that if the agent always turns down the single bet at any wealth level, then she will always turn down the last independent single bet, since nowhere is it beneficial after any realized outcomes of the first three bets. Therefore, backward induction says that it is not rational for an expected utility maximizer to take the bundle of four bets either.