ECON 202: Suggested Solutions to 2014 Final Exam

Alex Bloedel

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1. Problem 1

1.1. Part (a)

The firm's problem is

$$\pi(\mathbf{p}) := \max_{q_0 \geqslant 0,\ \mathbf{q} \in [0,q_0]^K} \underbrace{\mathbf{p} \cdot \mathbf{q} - \sum_{i=1}^{K} C_i(q_i) - C_0(q_0)}_{=: F(q_0, \mathbf{q}, \mathbf{p})}$$

Denote the argmax correspondence by $Q^* : \mathbb{R}_+^K \to \mathbb{R}^{K+1}$, i.e.,

$$Q^*(\mathbf{p}) := \{(q_0, \mathbf{q}) : \pi(\mathbf{p}) = F(q_0, \mathbf{q}, \mathbf{p}),\ \ q_k \leqslant q_0\ \forall k\}$$

Denote the set of quantity maximizers by

$$Q_{\mathbf{q}}^*(\mathbf{p}) := \{\mathbf{q} : \exists\ \ q_0\ \ s.t.\ \ (q_0, \mathbf{q}) \in Q^*(\mathbf{p})\}$$

1.2. Part (b)

Assume henceforth that the argmax correspondences are singletons, as suggested in the problem. We'll abuse notation and view these singleton-valued correspondences as functions.

We want to show, for each fixed k, that each component i of $Q_{\mathbf{q}}^*(\cdot, \mathbf{p}_{-k})$ is non-decreasing. This is a MCS question, so your first thought should be that we want to employ Topkis in some way. The challenge, relative to more "standard" Topkis problems, is that one of the choice variables – namely, q_0 – affects the feasible set for the other choice variables. We'll get around this by using a nested optimization approach.

For fixed $(q_0, \mathbf{p}) \in \mathbb{R}_+^{K+1}$, define the inner optimization problem

$$G(q_0, \mathbf{p}) := \max_{\mathbf{q} \in [0,q_0]^K} \mathbf{p} \cdot \mathbf{q} - \sum_{i=1}^{K} C_i(q_i)$$

Let $\mathbf{q}^*(q_0, \mathbf{p})$ denote the argmax correspondence for the inner problem (assume that it's also singleton-valued, and use the same notation for the correspondence and function). Note that for each i, $q_i^*(q_0, \mathbf{p})$ is trivially non-decreasing in q_0. (An increase of q_0 does nothing but enlarge the feasible set in the subproblem.) Moreover, using the Envelope Theorem, we can compute that

$$\frac{\partial G(q_0, \mathbf{p})}{\partial p_k} = q_k^*(q_0, \mathbf{p})$$

which is non-negative because $q_i \geqslant 0$ for all i. Moreover, it is increasing in q_0 from the above discussion.

The full optimization problem from part (a) can then be written as

$$\pi(\mathbf{p}) = \max_{q_0 \geqslant 0} G(q_0, \mathbf{p}) - C_0(q_0)$$

Note that

$$\frac{\partial \pi(\mathbf{p})}{\partial p_k} = \frac{\partial G(q_0, \mathbf{p})}{\partial p_k} = q_k^*(q_0, \mathbf{p})$$

from above. It follows that the objective function $G(q_0, \mathbf{p}) - C_0(q_0)$ has ID in (q_0, p_k). The univariate Topkis Theorem then implies that $q_0^*(\mathbf{p})$ is non-decreasing in p_k. (Recall we're assuming unique maximizers.)

To finish, we define

$$q_i^*(\mathbf{p}) := q_i^*(\mathbf{p}, q_0^*(\mathbf{p}))$$

which, from the above arguments, must be non-decreasing in p_k.

1.3. Part (c)

The Lagrangian saddle-point problem is

$$\min_{\lambda \in \mathbb{R}_+^K} \max_{q_0 \geqslant 0, \quad \mathbf{q} \in [0, q_0]^K} \mathscr{L}(q_0, \mathbf{q}, \lambda, \mathbf{p})$$

where

$$\mathscr{L}(q_0, \mathbf{q}, \lambda, \mathbf{p}) := \mathbf{p} \cdot \mathbf{q} - \sum_{i=1}^{K} C_i(q_i) - C_0(q_0) + \sum_{i=1}^{K} \lambda_i (q_0 - q_i)$$

As suggested in the problem, we ignore the $K + 1$ non-negativity constraints. The KKT conditions consist of the FOCs for quantities

$$0 = p_i - C_i'(q_i) - \lambda_i \qquad \text{for all } i = 1, \ldots, K$$

the FOC for capacity

$$0 = -C_0'(q_0) + \sum_{i=1}^{K} \lambda_i$$

and the CS conditions

$$\lambda_i(q_0 - q_i) = 0 \qquad \text{for all } i = 1, \ldots, K$$

(You also need to keep track of the "dual feasibility" constraints $\lambda_i \geqslant 0$ for all $i \geqslant 1$.)

1.4. Part (d)

Given smoothness and strict convexity,[1] we can use the Implicit Function Theorem. (As emphasized throughout the class, this is just about the *only* setting in which the IFT can be used!) Let $k \in \{1, \ldots, K\}$ be given and fix $\mathbf{p}_{-k}$. Define the function

$$h(q_i(p_k), p_k) := p_i - C_i'(q_i^*(p_k)) - \lambda_i^*(p_k)$$

where the RHS is just the FOC (now necessary and sufficient for finding the unique saddle point) for good i, with all dependence on $\mathbf{p}_{-k}$ suppressed.

Consider those i such that $q_i^* < q_0^*$, i.e., the capacity constraint does not bind for those goods. By CS, it must be that $\lambda_i^* = 0$ (and indeed, the multiplier must be zero in a neighborhood of these prices). Invoking the IFT, we have

$$\frac{\partial q_i^*(p_k)}{\partial p_k} = -\frac{\partial h(q_i(p_k), p_k)/\partial p_k}{\partial h(q_i(p_k), p_k)/\partial q_i} = \frac{\mathbb{1}(i = k)}{C''(q_i^*(p_k))}$$

Thus, if the constraint is slack (in the above sense) for good i, then $q_i^*(p_k)$ is locally constant for $i \neq k$. Thus, a minimal necessary condition for the supply of good i to strictly increase with p_k is that the capacity constraint for good i binds (i.e., the constraint holds with an equality, though I think it's necessary that the actual multiplier is positive).

(1) I think there is a typo in the problem. We want the maximization problem to be concave, which means that the cost functions should be convex.