Econ 202 final examination (Fall 2011) Solutions*

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Question 1. [*15 points*]

A friend tells you his method of choosing wine from restaurants' wine lists. Is this method consistent with making a uniquely optimal choice according to complete and transitive preferences? Is it consistent with the preferences satisfying the von Neumann-Morgenstern's Independence Axiom (in the case of (f))? Justify your answers. Assume that there are no tie in prices on any wine list.

(a) [*3 points*] Choose the cheapest wine on the list.

Answer. Yes. This choice rule is consistent with the preferences

$$a \succsim b \Leftrightarrow p(a) \leq p(b).$$

Completeness and transitivity of $\leq$ on $\mathbb{R}$ ensure completeness and transitivity of $\succsim$. Furthermore, since there are no ties in prices, there will be a uniquely optimal choice.

(b) [*3 points*] Choose the second-cheapest wine on the list.

Answer. No. Suppose $p(a) > p(b) > p(c)$. Then he chooses a from $\{a, b\}$ and b from $\{a, b, c\}$. So if his choices are consistent with preferences $\succsim$, then we must have both $a \succsim b$ and $b \succsim a$. But then $a \sim b$, and so a is not the uniquely optimal choice from $\{a, b\}$.

(d) [*3 points*] If French wine is available, choose the cheapest French wine, otherwise choose the cheapest wine.

Answer. Yes. Consider a utility representation

$$u(a) = \begin{cases} 1/p(a) & \text{if } a \text{ is French,} \\ -p(a) & \text{otherwise.} \end{cases}$$

These choices are consistent with the preferences represented by this utility function. Since the preferences have a utility representation, they are complete and transitive. Furthermore, since there are no ties in prices, there will be a uniquely optimal choice.

*To a certain extent these solutions remain a work in progress. They may contain errors. Take everything with a grain of salt.

(e) [*3 points*] If both French and New Zealand wine are available, choose the cheapest French wine; otherwise choose the cheapest wine.

Answer. No. Suppose $p(a) > p(b) > p(c)$, where a is French, b is from New Zealand, and c is neither. Then he chooses c from $\{a, c\}$ and a from $\{a, b, c\}$. So if his choices are consistent with preferences $\succsim$, then we must have both $c \succsim a$ and $a \succsim c$. But then $a \sim c$, and so c is not the uniquely optimal choice from $\{a, c\}$.

(f) [*3 points*] Toss a coin. If it falls heads, choose the cheapest French wine, otherwise choose the cheapest California wine.

Answer.

(i) No. Suppose a is French, and b is from California. Then we will observe the agent occasionally choosing a from $\{a, b\}$, and occasionally agent choosing b from $\{a, b\}$. So his preferences cannot be consistent with making any sort of uniquely optimal choice.

(ii) Yes. His choices are not inconsistent with indifference among all wines. Assuming indifference among all wines, his preferences over lotteries of wines have the rather trivial expected utility representation $U(p) \equiv 0$. Any preferences that admit an expected utility representation satisfy von Neumann-Morgenstern's Independence Axiom.

Question 2. [35 *points*]

A von-Neumann-Morgenstern (vNM) decision maker chooses between two lotteries, p and q, which put positive probabilities on three monetary payoffs, \$1, \$2, and \$3. Let $p_i \geq 0$ and $q_i \geq 0$ denote the probabilities of payoff $\$i$ $(i = 1, 2, 3)$ in the two respective lotteries, with $\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} q_i = 1$.

(a) [*10 points*] State *the simplest possible* necessary and sufficient condition for any vNM decision maker who prefers higher certain monetary payoffs to weakly prefer lottery p to lottery q.

Answer. "Who prefers higher certain monetary payoffs" just means that his Bernoulli utility function is weakly increasing. So the necessary and sufficient condition is that p First Order Stochastically Dominates q. Using the characterization that G FOSD F means $G(x) \leq F(x)$ for all x, this translates to $p_1 \leq q_1$ and $p_1 + p_2 \leq q_1 + q_2$.

Note: a high number of students answered this question using utility function. The question includes the word "any," so you can't use utility function.

(b) [*5 points*] State the condition that lotteries p and q have the same expected value.

Answer. $p_1 + 2p_2 + 3p_3 = q_1 + 2q_2 + 3q_3$.

(c) [*10 points*] Assuming that the condition in part (b) holds, state *the simplest possible* necessary and sufficient condition for any risk-averse vNM decision maker to weakly prefer lottery p to lottery q.

Answer. The condition is p Second Order Stochastically Dominates q. In terms of the p_i and q_i, the characterization $\int_{-\infty}^{x} G(y)dy \leq \int_{-\infty}^{x} F(y)dy$ for all x should come to mind, and yields

$$\begin{aligned} \lambda p_1 &\leq \lambda q_1 \quad \forall \lambda \in [0,1] \\ \lambda p_1 + \lambda(p_1 + p_2) &\leq \lambda q_1 + \lambda(q1 + q2) \quad \forall \lambda \in [0,1] \end{aligned}$$

The first condition is satisfied if and only if $p_1 \leq q_1$. By linearity, the second condition holds for all $\lambda \in [0,1]$ if and only if it holds at the end points. For $\lambda = 0$, the condition is equivalent to $p_1 \leq q_1$. For $\lambda = 1$, the condition is equivalent to $2p_1 + p_2 \leq 2q_1 + q2$. However, this inequality holds automatically, since it is implied by the 2 distributions having the same mean. To wit,

$$\begin{aligned} p_1 + 2p_2 + 3p_3 &= q_1 + 2q_2 + 3q_3 \\ \Rightarrow p_1 + 2p_2 + 3(1 - p_1 - p_2) &= q_1 + 2q_2 + 3(1 - q_1 - q_2) \\ \Rightarrow 2p_1 + p_2 &= 2q_1 + q_2 \end{aligned}$$

In conclusion, a necessary and sufficient condition is $p_1 \leq q_1$.

(d) [*10 points*] Assuming that the condition in part (b) holds, state *the simplest possible* necessary and sufficient condition for any vNM decision maker with constant and nonnegative absolute risk aversion to weakly prefer lottery p to lottery q.

Answer. The CARA utility function is $u(x) = -e^{-rx}$. So for a decision maker with this utility function to prefer p to q we need

$$p_1(-e^{-r}) + p_2(-e^{-2r}) + p_3(-e^{-3r}) \geq q_1(-e^{-r}) + q_2(-e^{-2r}) + q_3(-e^{-3r})$$

For *any* decision maker with CARA preferences to prefer p to q, the above inequality must hold for all r. For large r, the first term dominates, i.e., $-e^{-r} >> -e^{-2r} >> -e^{-3r}$, so essentially we must have

$$p_1(-e^{-r}) \geq q_1(-e^{-r})$$

which holds if we have $p_1 \leq q_1$.

We have shown that $p_1 \leq q_1$ is necessary. Since CARA decision makers are risk averse, this condition is also sufficient, by part (c).

Question 3. [*60 points*]

Consider an economy with two goods, x and y. Consumers can consume nonnegative amounts of good x, but arbitrary (positive or negative) amounts of good y. There are N consumers, and the utility of each consumer $i = 1, ..., N$ from consumption bundle (x, y) takes the form $u^i(x, y) = v^i(x) + y$, where $v^i : \mathbb{R}^+ \to \mathbb{R}$ is an increasing function. Each consumer starts out with a zero endowment of good x and a sufficiently large endowment of good y. In addition, there is a firm that can produce any number $q \geq 0$ of units of good x, using up $c(q)$ units of good y.

(a) [*15 points*] Write the conditions characterizing a Walrasian equilibrium of the economy. Characterize the equilibrium output of the firm.

Answer. Let the price of x be p, and let y be the numeraire (normalizing its price to 1). A Walrasian equilibrium is prices and quantities such that

- Agents $i \in \{1, \dots, N\}$ optimize: Given $(p, q, \alpha^i, \bar{y}^i)$, the quantities $(x^i, y^i) \geq \mathbf{0}$ maximize $v^i(x^i) + y^i$ subject to $px^i + y^i = \bar{y}^i + \alpha^i\big[pq - c(q)\big]$.
- Firm: Given p, the quantity $q \geq 0$ maximizes $pq - c(q)$.
- Markets clear: $q = \sum x^i$ and $\sum \bar{y}^i = \sum y^i + c(q)$ (though the former ensures the latter by Walras' law).

Suppose now that the firm does not take prices as given, but recognizes its effect on prices. Namely, after the firm chooses the output q of good x, the prices emerge to clear the market for good x when consumers take the prices as given. (The market for good y will then also clear by Walras' Law.) Let $P(q)$ denote the resulting price of good x in terms of good y (i.e., with the price of good y normalized to 1).

(b) [*15 points*] Suppose that the firm's objective is to maximize its profit expressed in terms of good y. Characterize the firm's optimal output and compare it to the Walrasian equilibrium output characterized in part (a).

Answer. I assume that $P(q)$ is weakly decreasing. I also assume that the Walrasian equilibrium is unique. The firm's output in Walrasian equilibrium satisfies

$$q^{\text{WE}} = \arg\max_{q \geq 0} P(q^{\text{WE}})q - c(q).$$

The firm's output in the current environment belongs to the set

$$Q^* = \arg\max_{q \geq 0} P(q)q - c(q). \tag{1}$$

Consider the following objective function

$$\pi(q, \Omega) = q\left(P(q^{\text{WE}}) + \Omega\left[P(q) - P(q^{\text{WE}})\right]\right) - c(q).$$

Observe that

$$q^{\text{WE}} = \arg\max_{q \geq q^{\text{WE}}} \pi(q, 0).$$

I also define the set

$$\hat{Q}^* = \arg\max_{q \geq q^{\text{WE}}} \pi(q, 1).$$

Since $Q^* = \arg\max_{q \geq 0} \pi(q, 1)$, it must be the case that $Q^* \leq \hat{Q}^*$ (in the strong set order).

Differentiating the objective, we obtain $\pi_\Omega = q\left[P(q) - P(q^{\text{WE}})\right]$, which is weakly decreasing in q on $[q^{\text{WE}}, \infty)$. The objective π therefore has decreasing differences in (q, Ω), and so by Topkis' Theorem, $\hat{q}^* \leq q^{\text{WE}}$ for all $\hat{q}^* \in \hat{Q}^*$. Therefore $q^* \leq q^{\text{WE}}$ for all $q^* \in Q^*$.

Note: it is also interesting to point out that the firm's profit must increase by "revealed preference," since the Walrasian Equilibrium quantity from part (a) is still available to the firm.

(c) [*15 points*] Suppose now instead that the firm sets the price p to maximize the utility of its controlling shareholder, who is consumer k holding share $\alpha_k \in [0, 1]$ of the firm's profits. Characterize the firm's optimal price and compare it to that in part (b).

Answer. We first study agent k's consumption problem for every possible choice of q. Let $B(q) = \bar{y}^k + \alpha_k\big[P(q) \cdot q - c(q)\big]$ represent the agent's budget as a function of q. The agent then solves

$$\max_{x \geq 0, y} v^k(x) + y \quad \text{subject to} \quad P(q)x + y \leq B(q).$$

By Walras' Law, the budget constraint will hold, and so if the agent consumes optimally, she receives utility

$$\begin{aligned} U^k(q) &= \max_{x \geq 0} v^k(x) - P(q)x + B(q) \\ &= B(q) + V^k(q), \end{aligned}$$

where $V^k(q) = \max_{x \geq 0} v^k(x) - P(q)x$. Since $P(q)$ is weakly decreasing, $V^k(q)$ is weakly increasing.

We now study the agent's optimal choice of q. This corresponds to maximizing

$$F(q, \Omega) = \alpha_k\big[P(q) \cdot q - c(q)\big] + \Omega\big[V^k(q) + \bar{y}^k\big].$$

with $\Omega = 1$, while part (b) corresponds to maximizing the same objective with $\Omega = 0$. Differentiating the objective, we obtain $F_\Omega = V^k(q) + \bar{y}^k$, which is weakly increasing in q. The objective F therefore has increasing differences in (q, Ω), and so by Topkis' Theorem, the firm's optimal output choice is higher when it seeks to maximize its controlling shareholder's utility rather than its profits.

(d) [*15 points*] Suppose that all the consumers have the same utility function. Compare the firm's output in part (c) to that in part (a).

Answer. As in part (c), all agents choose their consumption of the first good by solving $x^*(q) \in \max_{x\geq 0} v(x) - P(q)x$. Assuming that this maximization problem always has a unique solution, then the fact that all agents have identical utility functions implies $x^*(q) = \frac{q}{N}$.

If $\alpha_k = 0$, the agent from part (c) chooses q as large as possible. Production therefore exceeds q^{WE}. Henceforth we focus on the case where $\alpha_k > 0$.

Consider the objective function

$$G(q, \Omega) = P(q^{\text{WE}})q - c(q) + \Omega\left[P(q)q + \frac{1}{\alpha_k}V(q) - P(q^{\text{WE}})q\right],$$

where $V(q) = \max_{x\geq 0} v(x) - P(q)x$. When $\Omega = 0$, this is the objective maximized by the firm in part (a). When $\Omega = 1$, this is the objective maximized by the firm in part (c). I now consider two cases.

<u>Case 1:</u> $\alpha_k \leq \frac{1}{N}$. Observe that

$$q^{\text{WE}} = \underset{0\leq q\leq q^{\text{WE}}}{\arg\max}\, G(q, 0).$$

I also define the set

$$\hat{Q}^* = \underset{0\leq q\leq q^{\text{WE}}}{\arg\max}\, G(q, 1).$$

Since $Q^* = \arg\max_{q\geq 0} G(q, 1)$, it must be the case that $Q^* \geq \hat{Q}^*$ (in the strong set order). Differentiating,

$$\begin{aligned} G_\Omega &= P(q)q + \frac{1}{\alpha_k}V(q) - P(q^{\text{WE}})q \\ G_{\Omega q} &= P(q) - P(q^{\text{WE}}) + P'(q)\left[q - \frac{1}{\alpha_k}x^*(q)\right] \\ &= P(q) - P(q^{\text{WE}}) + P'(q)q\left[\frac{N\alpha_k - 1}{N\alpha_k}\right], \end{aligned}$$

which is weakly positive on $q \in [0, q^{\text{WE}}]$. The objective G therefore has increasing differences in (q, Ω), and so by Topkis' Theorem, $\hat{q}^* \geq q^{\text{WE}}$ for all $\hat{q}^* \in \hat{Q}^*$. Therefore $q^* \geq q^{\text{WE}}$ for all $q^* \in Q^*$.

<u>Case 2:</u> $\alpha_k \geq \frac{1}{N}$. Observe that

$$q^{\text{WE}} = \underset{0\leq q\leq q^{\text{WE}}}{\arg\max}\, G(q, 0).$$

I also define the set

$$\hat{Q}^* = \arg\max_{q \geq q^{\mathrm{WE}}} G(q, 1).$$

Since $Q^* = \arg\max_{q\geq 0} G(q,1)$, it must be the case that $Q^* \leq \hat{Q}^*$ (in the strong set order). Differentiating,

$$\begin{aligned} G_\Omega &= P(q)q + \frac{1}{\alpha_k}V(q) - P(q^{\mathrm{WE}})q \\ G_{\Omega q} &= P(q) - P(q^{\mathrm{WE}}) + P'(q)\left[q - \frac{1}{\alpha_k}x^*(q)\right] \\ &= P(q) - P(q^{\mathrm{WE}}) + P'(q)q\left[\frac{N\alpha_k - 1}{N\alpha_k}\right], \end{aligned}$$

which is weakly negative on $q \in [q^{\mathrm{WE}}, \infty)$. The objective G therefore has decreasing differences in (q, Ω), and so by Topkis' Theorem, $\hat{q}^* \leq q^{\mathrm{WE}}$ for all $\hat{q}^* \in \hat{Q}^*$. Therefore $q^* \leq q^{\mathrm{WE}}$ for all $q^* \in Q^*$.

In conclusion, the answer depends upon the sign of $\alpha_k - \frac{1}{N}$. If $\alpha_k \geq \frac{1}{N}$, then the firm produces weakly less in part (c) than in part (a). If on the other hand $\alpha_k \leq \frac{1}{N}$, then the firm produces weakly more in part (c) than in part (a).

The intuition for this is as follows. The owner faces two forces. The owner wants to increase supply and drive down prices so that he can afford more goods. But this also hurts the owner, who now receives slightly less profit from his ownership stake. As α_k decreases, the second effect becomes less important, and the first effect dominates.

Note: I think it should be possible to do this problem without assuming differentiability of $P(q)$, but I was stuck so I assumed it. Notice, however, that we never had to assume differentiability of $c(q)$.

Question 4. [*70 points*]

Consider an exchange economy with N consumers and L goods. Each consumer i has continuous, strictly monotone and strictly convex preferences $\succeq_i$ and initial endowment $e^i \gg 0$. Given an allocation $(x^1, ..., x^N)$, say that i *envies* j in x if $x^j \succ_i x^i$. An allocation x is *envy-free* if there exists no pair of consumers i and j such that i envies j in x.

(a) [*10 points*] Is every Walrasian equilibrium allocation envy-free? Prove or disprove with a counter example.

Answer. No. Counter example: Economy with 2 agents, 1 good. Utility for both agent is $u(x) = x$. Endowments are $e_1 = 1, e_2 = 2$. Initial endowment is Walrasian allocation, but this allocation is not envy-free since agent 1 envies agent 2.

An allocation contains an *envy cycle* if there exists a set of consumers $\{i_1, ..., i_k\}$ such that $i_1 = i_k$ and $x^{i_{n+1}} \succ_{i_n} x^{i_{n+1}}$ for each $n = 1, ..., k$.

(b) [*10 points*] Can an equilibrium allocation contain an envy cycle? Prove or disprove.

Answer. No. Suppose in negation that such a cycle exists. Then by revealed preference we must have that $p \cdot x^{i_{n+1}} > p \cdot x^{i_n}$ for all agents i_n in the cycle. We can rewrite this as $w^{i_{n+1}} > w^{i_n} > w^{i_{n-1}} > ... > w^{i_{n+1}}$, so $w^{i_{n+1}} > w^{i_{n+1}}$, a contradiction.

Consider a social planner who has to make sure that all allocations are envy-free. Suppose that the social planner can change consumers' initial endowments.

(c) [*10 points*] Can the social planner change the initial endowments such that after the change, every Walrasian equilibrium allocation is envy-free?

Answer. Yes. He can make endowment the same across agents. This ensures that in any Walrasian equilibrium every agent's wealth is the same, which in turn means that each agent's allocation is affordable by any other agent. Revealed preference then implies no envy.

Now suppose that the social planner cannot change initial endowments, but can instead impose a progressive tax that is proportional to the difference between a consumer's wealth and the average wealth in the economy. More specifically, the tax takes the following form

$$T^i(p) = \alpha \left(p \cdot e^i - \frac{p \cdot \sum_i e^i}{N} \right),$$

where $\alpha \in [0, 1]$.

(d) [*10 points*] Define a Walrasian equilibrium with progressive tax.

Answer. A Walrasian equilibrium with progressive tax α consists of prices and allocations $(p, (x^i)_{i=1,...,N})$ satisfying

(i) For every $i = 1, \ldots, N$, every consumer is maximizes subject to their budget constraint, i.e.
$$x^i \succeq x'^i$$
for all x'^i such that $p \cdot x'^i \leq p \cdot e^i - T^i(p)$.

(ii) Markets clear.
$$\sum_{i=1}^{N} x^i = \sum_{i=1}^{N} e^i.$$

Note that the total tax is zero, so there's no need to distribute the tax revenue. This taxation scheme can be viewed as a redistribution of initial endowments from the relatively rich to the relatively poor (rich and poor *at* the equilibrium prices).

(e) [*10 points*] Is every Walrasian equilibrium with progressive tax Pareto Optimal? Prove or disprove.

Answer. Yes. Any Walrasian equilibrium with progressive tax can be viewed as a regular Walrasian equilibrium resulting from a modified initial endowment, and FWT applies. An alternative approach would be to show Pareto Optimality from first principles.

(f) [*10 points*] Can every interior Pareto Optimal allocation be supported as an equilibrium with some progressive tax $\alpha \in [0, 1]$?

Answer. Yes. The SWT states that any interior Pareto Optimal allocation can be supported as equilibrium after some redistribution. Therefore, any interior Pareto Optimal allocation can be supported as equilibrium with tax $\alpha = 0$ after some redistribution.

Note: This question is a little unclear and should have clarified that in addition to the progressive tax, the social planner also has the ability to redistribute.

(g) [*10 points*] Can the social planner choose some $\alpha \in [0, 1]$ such that every equilibrium with progressive tax is envy-free?

Answer. Yes. Choose $\alpha = 1$. This results in each agent having the same wealth in any equilibrium with tax. Revealed preference then ensures that the allocation is envy-free.