June 2006

## Question 2

Consider an economy with two agents and two goods: water and diamonds. Agent $i = 1, 2$ is endowed with $e_i$ units of water, which we will treat as the numeraire; assume that $e_1 \geq e_2$. There is a single diamond and the agents initially own equal shares of it. The diamond must be owned in full, however, to be consumed. The agents have identical preferences. Agent $i$'s utility from consuming $c_i \geq 0$ units of water and $s_i \in \{0, 1\}$ diamonds is $u(c_i, s_i)$, where $u_c > 0$, $u_{cc} < 0$, and $u(c, 1) - u(c, 0)$ is strictly positive for all $c$ and also strictly increasing in $c$.

(a) Prove that with these preferences, diamonds are a normal good.

(b) Who will own the diamond in a Walrasian equilibrium?

(c) Characterize one Walrasian equilibrium of this economy.

(d) Prove that there are multiple Walrasian equilibria.

Parts (e)-(h) ask you to consider Pareto optimal allocations in this economy.

(e) Graph the utility possibility frontier for this economy (i.e. the utility pairs arising from Pareto efficient allocations). What does the frontier look like where it intersects the 45-degree line? dimple

(f) Suppose the planner could propose a symmmetric random allocation, in which each agent had an equal chance to obtain the bundle $(c, 1)$ or its complement $(e_1 + e_2 - c, 0)$, where $c \in [0, e_1 + e_2]$. From an expected utility standpoint, would the agents ever prefer such an allocation to a deterministic Pareto efficient allocation?

(g) Characterize the best symmetric random allocation. Compare the two bundles the agents might receive.

(h) Is there a way to decentralize this allocation through competitive markets?

2. General equilibrium.

(a) Let $y$ denote wealth. Then the consumer's problem is:

$$\max_{c,s} u(c,s) \text{ s.t. } c + ps \leq y.$$

We can substitute the budget constraint because $u_c > 0$ and obtain

$$\max_{s \in \{0,1\}} u(y - ps, s).$$

The returns to choosing $s = 1$ rather than $s = 0$ are:

$$u(y-p,1) - u(y,0) = [u(y,1) - u(y,0)] + [u(y-p,1) - u(y,1)].$$

The second term is increasing in $y$ by $u_{cc} < 0$, the first term by supermodularity.

(b) Because the diamond is a normal good, the consumer with more wealth must consume it in equilibrium.

(c) To characterize all equilibria, consider any price $p$ of diamonds that satisfies:

$$\begin{aligned} u(e_1 - p/2, 1) &\geq u(e_1 + p/2, 0), \\ u(e_2 - p/2) &\leq u(e_2 + p/2, 0). \end{aligned}$$

At least one such price always exists. For any such price, there is a Walrasian equilibrium with consumption $(e_1 - p/2, 1)$ for agent 1 and $(e_2 + p/2, 0)$ for agent 2. If $e_1 = e_2$, there is also an equilibrium where agent 2 consumes the diamond and the inequalities are equalities.

(d) If $e_1 = e_2$ either agent can own the diamond in equilibrium. If $e_1 > e_2$, then if agent 2's optimaization constraint holds, agent 1's will be slack and vice-versa, so there will be a range of prices $p$ (with corresponding optimal consumptions) between the low price $\underline{p}$ that leaves agent 2 indifferent to consuming the diamond and the high price $\overline{p}$ that leaves agent 1 indifferent.

(e) The picture below shows the utility possibility frontier: the curve that intersects the y-axis higher up shows consumption possibilities if agent 2 consumes the diamond; the other curve represents possibilities if agent 1 consumes the diamond.

The frontier is not convex at the intersection with the $45^0$ line.

(f) This can be seen graphically. One possible lottery leaves the agents at either point $A_1$ or $A_2$ rather than at point $B$. From an ex ante standpoint, the lottery puts the agents at point $C$ which strictly Pareto dominates $B$.

(g) The best symmetric random allocation shown in the picture solves:

$$\max_c \frac{1}{2}u(c,1) + \frac{1}{2}(e_1 + e_2 - c, 0),$$

so the first order condition at the optimum $c^*$ is:

$$u_c(c^*,0) < u_c(c^*,1) = u_c(e_1 + e_2 - c^*, 0).$$

Because $u$ is supermodular, $c^* > e_1 + e_2 - c^*$.

(h) We could start the agents with equal endowments, then introduce a new product which is a lottery. For a given price (equal to their whole endowment) they agents could buy a ticket that gives them a half chance to win $(c^*, 1)$ and its complement. Alternatively, we could have two stages of markets. A first-stage where agents buy lottery tickets that pay out in water and a second stage where they trade water and diamonds to get to $A_1$ or $A_2$.

# June 2006 Micro Comp — Question 2

a) CP: $\max_{c,s} \; u(c,s)$

s.t. $c + ps \leq w$

Locally non-~~satiable~~ satiated ($u_c > 0$), so Walras holds, and we get

CP': $\max_{s \in \{0,1\}} \; u(w-ps, s)$

Is the objective supermodular in $(s,w)$?

$\Delta s$: $u(w-p, 1) - u(w, 0)$

How do we make this look like the SM condition we have already been given?

$$[u(w,1) - u(w,0)] + [u(w-p,1) - u(w,1)]$$

$\nearrow$ increasing in $w$, by SM in $(c,s)$ $\quad$ $\nearrow$ increasing since $u_{cc} < 0$

So the objective is s.m. in $(w,s)$ and we apply Topkis to conclude

$\boxed{s^*(w,p) \text{ is weakly increasing in } w\text{, i.e. diamonds are a normal good}}$

b) In equilibrium, only one agent can demand the diamond. Our conclusion in (a) tells us that if the less wealthy agent demands it, so must the more wealthy agent.

Hence the more wealthy agent, agent 1, gets the diamond in equilibrium.

c,d) An equilibrium will be determined by any price for diamonds, $p$, that satisfies:

$$u(e_1 - \tfrac{1}{2}p, 1) \geq u(e_1 + \tfrac{1}{2}p, 0) \qquad \text{(1 demands the diamond)} \quad A$$
$$u(e_2 - \tfrac{1}{2}p, 1) \leq u(e_2 + \tfrac{1}{2}p, 0) \qquad \text{(2 sells his shares)} \quad B$$

Say that B holds w/ equality at $\hat{p}$, so that

$$u(e_2 - \tfrac{1}{2}\hat{p}, 1) - u(e_2 + \tfrac{1}{2}\hat{p}, 0) = 0$$

Then,

$$\left[u(e_2 + \tfrac{1}{2}\hat{p}, 1) - u(e_2 + \tfrac{1}{2}\hat{p}, 0)\right] + \left[u(e_2 - \tfrac{1}{2}\hat{p}, 1) - u(e_2 + \tfrac{1}{2}\hat{p}, 1)\right] = 0$$

But the 1st term is increasing in $e_2$ by s.m. in $(c, s)$ and the 2nd by $u_{cc} < 0$.

So, increasing to $e_1$, yields

$$\left[u(e_1 + \tfrac{1}{2}\hat{p}, 1) - u(e_1 + \tfrac{1}{2}\hat{p}, 0)\right] + \left[u(e_1 - \tfrac{1}{2}\hat{p}, 1) - u(e_1 + \tfrac{1}{2}\hat{p}, 1)\right] > 0$$

$$u(e_1 - \tfrac{1}{2}\hat{p}, 1) > u(e_1 + \tfrac{1}{2}\hat{p}, 0)$$

So at $\hat{p}$, 1 strictly prefers to demand the diamond. We can then lower $p$ in some range $p \in [p, \hat{p}]$ and still obey both A and B. Hence $\exists$ multiple Walrasian equilibrium.

But, all of this hinges on B binding w/ equality. Must this always happen?

If $B$ ~~does not~~ never holds w/ equality, then we must ~~have~~ make the diamond ~~un~~ affordable for agent 1. Clearly $p = 2e_2$ is the most 2 can pay. Hence, $B$ not holding w/ equality for any $p \in [0, 2e_2] \Rightarrow$

$$u(0,1) > u(2e_2, 0)$$

Any price above $p = 2e_2$ forces agent 2 to sell. But will agent 1 buy? Not if

$$u(e_1 - e_2, 1) < u(e_1 + e_2, 0)$$

Can both of these ~~on~~ inequalities hold? Combining them yields

$$(\star\star) \qquad u(e_1 - e_2, 1) - u(0,1) < u(e_1 + e_2, 0) - u(2e_2, 0)$$

But, supermodularity tells us

$$u(e_1 - e_2, 1) - u(e_1 - e_2, 0) > u(0,1) - u(0,0) \qquad \text{or}$$

$$(\star) \qquad u(e_1 - e_2, 1) - u(0,1) > u(e_1 - e_2, 0) - u(0,0)$$

Connecting $(\star)$ w/ $(\star\star)$, we find

$$u(e_1 - e_2, 0) - u(0,0) < u(e_1 + e_2, 0) - u(2e_2, 0)$$

which can be rewritten

$$u(e_1 - e_2, 0) - u(0,0) < u(2e_2 + (e_1 - e_2), 0) - u(2e_2, 0)$$

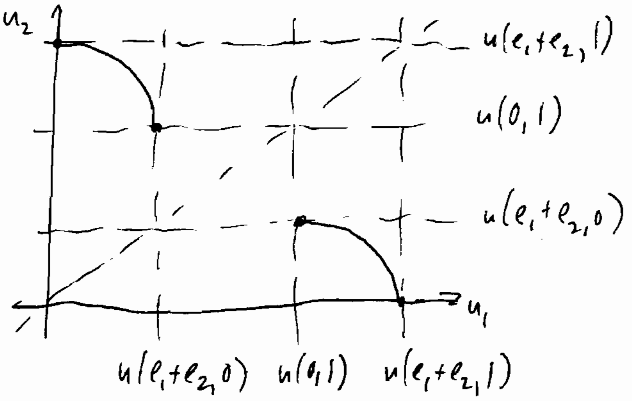
a clear violation of $u_{cc} < 0$.

Hence if B holds w/equality for some $\hat{p} \in [0, 2e_2]$, we have a range of solutions from at least $p \in [\hat{p}, \hat{p}+\varepsilon]$.

If B never holds w/equality for some $\hat{p} \in [0, 2e_2]$, then we force 1 to sell and must have a range of solutions from $\hat{p} \in [2e_2, 2e_2+\varepsilon]$.

So we have characterized solutions and demonstrated their multiplicity.

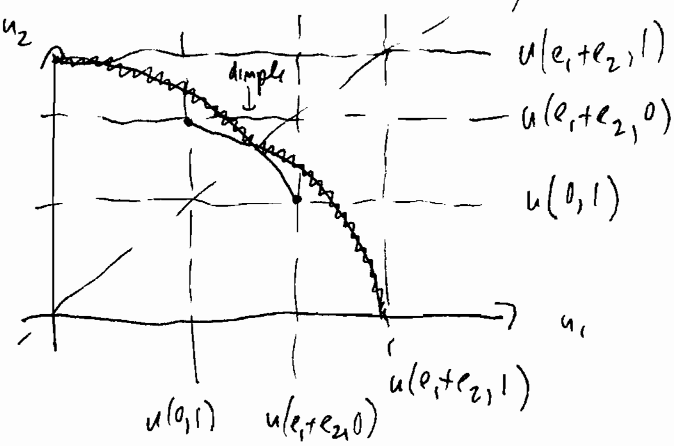
e) So, we can think of two situations. Say $u(0,1) > u(e_1+e_2, 0)$ (diamond $\succsim$ all water in the eco
Then,




norm $u(0,0) = 0$

where the UPF bows outwards because $u_{cc} < 0$ and $u_c(c,1) > u_c(c,0)$.

If $u(0,1) < u(e_1+e_2, 0)$ (there is enough water in the econ to compensate for 1 diamond).



We know that there is a dimple and not a cusp, ~~there~~ by looking at the slope of the UPF there.

$$\frac{dy}{dx} = \frac{dy/dc}{dx/dc} \qquad \text{where} \quad (x,y) = \left(u(e_1+e_2-c,1),\, u(c,0)\right)$$

Let $c^*$ be ~~defined~~ defined by $\quad u(e_1+e_2-c^*,1) = u(c^*,0)$

$$\frac{dy}{dx} = \frac{u_c(c^*,0)}{-u_c(e_1+e_2-c^*,1)}$$

We demonstrate that $\frac{dy}{dx} \in [-1,0]$ by noting

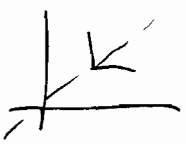
$$u_c(c^*,1) > u_c(c^*,0) \qquad \text{by s.m.}$$

Since the diamond is a "good" and not a "bad", we must have $e_1+e_2-c < c$. So, by $u_{cc}$ we get

$$u_c(e_1+e_2-c^*,1) > u_c(c^*,1) > u_c(c^*,0)$$

demonstrating $\frac{dy}{dx} \in [-1,0]$.

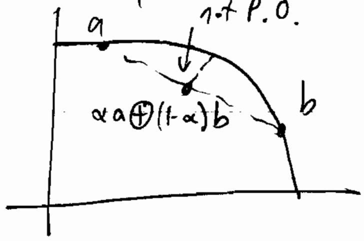
This is the slope ~~of~~ of the lower portion of the UPF as it hits the 45° line. Symmetrically, the upper portion will hit the 45° line at a slope of less than $-1$. So, we have (locally)



i.e. a dimple.

Either case yields a non-convex set: either  or 

f) Expected utility maximizers will see lotteries as equivalent to points on lines connecting two points. In a standard convex UPF case, lotteries of points on the boundary yield points in the interior that are not Pareto optimal




But in our case, we can use lotteries to get outside of the non-convex UPF set, essentially ~~[illegible]~~ improving upon Pareto optima by convexifying the set.


g) The best symmetric random allocation is given by

$$\max_{c} \; \tfrac{1}{2} u(e_1+e_2-c, 1) + \tfrac{1}{2} u(c, 0)$$

whose FOCs yield

$$[c]: \quad u_c(c_{best}, 0) = u_c(e_1+e_2-c_{best}, 1) > u_c(e_1+e_2-c_{best}, 0)$$

$$u_{cc} < 0, \text{ so } \quad c_{best} < e_1+e_2-c_{best}$$

$$c_{best} < \frac{e_1+e_2}{2}$$

So the ~~allocations~~ lotteried bundles would be $(e_1+e_2-c_{best}, 1)$ and $u(c_{best}, 0)$ — so the lottery is w/ no diamond and a bad payoff and diamond and a good p[ayoff] which makes sense, since $u$ is S.M., meaning diamonds and water are <u>complements</u> and not substitutes..

h b) Basically, we could introduce a lottery, which isn't really decentralized, but is the obvious answer (I guess it is more decentralized than just handing out lottery tickets at whim.