Econ 202 Exam Answer Key

Let $\mathcal{N} = \{1, \ldots, n\}$ be a "society" with n members and let X be the set of choices or allocations available to society. In the three problems below, each $i \in \mathcal{N}$ has a *complete*, but not necessarily *transitive*, preference relation $\succcurlyeq_i$. As usual, we define $x \succ_i y$ to mean $y \not\succcurlyeq_i x$. The assumed properties of $\succcurlyeq_i$ may vary among the three parts below.

Question 1. (15 points) Suppose that each individual *i*'s preference relation $\succcurlyeq_i$ is complete and transitive and represented by a utility function $u_i: X \to \mathbb{R}$. Someone has proposed that $W(x) \stackrel{\text{def}}{=} \sum_{i=1}^{n} u_i(x)$ be used as a welfare criterion to rank society's preferences over alternatives $x \in X$. Criticize this proposal.

The problem is that for any utility function u_i that represents preferences $\succcurlyeq_i$ and any increasing function $f_i: \mathbb{R} \to \mathbb{R}$, the composite function $f_i \circ u_i$ is another representation of $\succcurlyeq_i$, and each different representation can lead to a different welfare criterion.

Question 2. Suppose that each individual *i*'s preferences over consumption bundles in $\mathbb{R}^k_{++} = \{x \in \mathbb{R}^k | x_i > 0 \; for \; i = 1, \ldots, k\}$ satisfies $x \succcurlyeq_i y \Leftrightarrow u_i(x) > u_i(y) - 1$ for some continuous, increasing, quasi-concave function $u_i: \mathbb{R}^k_{++} \to \mathbb{R}$.

a. (5 points) Show that, with these preferences, $\succcurlyeq_i$ may not be transitive.

For example, let $k = 1$ and $u_i(x) = x$. Then, $0 \succcurlyeq_i 0.9$ and $0.9 \succcurlyeq_i 1.8$ but $0 \not\succcurlyeq_i 1.8$.

b. (15 points) Someone has proposed that $\sum_{i=1}^{n} u_i(x_i)$ be used as a welfare criterion to rank society's preferences over alternative allocations $x \in \mathbb{R}^{nk}_{++}$, in which $x = (x_i)_{i=1}^{n}$ and x_i is the bundle allocated to individual i. Is this proposal subject to the same criticism as in part (1) above? Explain.

No, in this case, the representation-preserving transformations are more limited, which enables some valid welfare comparisons. (In particular, for any two utility representations u_i, v_i, and for all , y $|(u_i(x) - u_i(y)) - ((v_i(x) - v_i(y))| < 1.$)

c. (10 points) Let $k = 2$. Draw a preferred set $P_i(x) = \{y | y \succcurlyeq_i x\}$. What are its properties?

Since u_i is continuous, increasing, and quasi-concave, the preferred set $P_i(x)$ is open, upward comprehensive, and convex.

d. (5 points) Is Walras' Law satisfied for these preferences?

No. Since u_i is continuous, the consumer is indifferent between x and y ($x \succcurlyeq_i y$ and $y \succcurlyeq_i x$) for any y in a small neighborhood x.

Question 3. Suppose $\succcurlyeq_i$ is complete and that $\succ_i$ is transitive: ($x \succ_i y$ and $y \succ_i z$) $\Rightarrow$ $x \succ_i z$.

e. (10 points) Show that these two conditions do not imply that $\succcurlyeq_i$ is transitive.

For example, the preferences in (2) above satisfy these conditions (see d below) and we have already shown in (2a) that $\succcurlyeq_i$ is not transitive.

f. (10 points) Show that $x \succcurlyeq_i y$ and $y \succ_i z$ implies $x \succcurlyeq_i z$.

If not, then there is some x, y, z such that $x \succcurlyeq_i y$, $y \succ_i z$, and $z \succ_i x$. Then, transitivity of $\succ_i$ implies that $y \succ_i x$, which contradicts $x \succcurlyeq_i y$.

g. (10 points) Show that if X is finite, then there exists a "least preferred" element $z \in X$ such that for all $y \in X$, $y \succcurlyeq z$. (Hint: use induction on the size of X.)

This is trivially true when $|X| = 1$. Suppose inductively that $k \geq 1$ and that the statement is true for all sets with no more than k members. Let $X = \{x_1, \dots, x_{k+1}\}$ and let $z \in \{x_1, \dots, x_k\}$ satisfy $x_j \succcurlyeq z$ for $j = 1, \dots, k$ be the least preferred element in $\{x_1, \dots, x_k\}$. If $x_{k+1} \succcurlyeq z$, then z is worst in X. Otherwise, for $j = 1, \dots, k$, $x_j \succcurlyeq z \succ x_{k+1}$, so (using part (a)) $x_j \succcurlyeq x_{k+1}$, which implies that x_{k+1} is a worst element.

h. (10 points) Using the preferences defined in (2), show that $\succcurlyeq_i$ is complete and $\succ_i$ is transitive.

For any x and y, either $u_i(x) > u_i(y) - 1$ or $u_i(y) \geq u_i(x) + 1 > u_i(x)$, and those alternatives correspond to $x \succcurlyeq_i y$ or $y \succcurlyeq_i x$, which proves completeness.

For any x, y, z, notice that $x \succ_i y$ and $y \succ_i z$ mean that $u_i(x) \geq u_i(y) + 1$ and $u_i(y) \geq u_i(z) + 1$. These imply that $u_i(x) \geq u_i(z) + 1$ and hence that $x \succ_i z$.

Question 4:

(a) No, because they do not an expected utility representation. To see this, consider preferences over three-outcome lotteries $p \in \Delta(\{x_1, x_2, x_3\})$ with $x_1 < x_2 < x_3$ and see that utility function $U(p_1, p_2, 1 - p_1 - p_2)$ has a non-constant marginal rate of substitution between p_1 and p_2.

(b) No. E.g. consider $Y = X + \varepsilon$, where $\varepsilon = 0$ w. prob. 1 when $X < \mathbb{E}[X]$ and $\varepsilon > 0$ w. prob. 1 when $X > \mathbb{E}[X]$. Then Y FOSD X, but $\mathrm{Var}[Y] > \mathrm{Var}[X]$, hence X will be strictly preferred to Y if $\rho > 0$ is low enough. Also, for any fixed $\rho > 0$, kX will be strictly preferred to kY for $k > 0$ large enough, even though kY FOSD kX for any $k > 0$.

(c) Yes. If $\mathbb{E}[\varepsilon|X] = 0$, then $\mathrm{Cov}[\varepsilon, X] = 0$ and therefore $\mathrm{Var}[X + \varepsilon] = \mathrm{Var}[X] + \mathrm{Var}[\varepsilon] \geq \mathrm{Var}[X]$, with the inequality being strict unless $\varepsilon = 0$ with prob. 1.

Question 5:

(a) $\max_{\alpha \in \mathbb{R}^J, \alpha_0 \in \mathbb{R}} \alpha'\mu + \alpha_0 - \frac{1}{2\rho^i}\alpha'\Omega\alpha$ subject to $\alpha_0 + q'\alpha \leq e_i^0 + q'e^i$.

(b) Expressing α_0 from the binding budget constraint, plugging it into the objective function, and differentiating the resulting quadratic concave function with respect to α yields the FOC $\mu - q - \frac{1}{\rho^i}\Omega\alpha^i = 0$, i.e., $\alpha^i = \rho^i\Omega^{-1}(\mu - q)$.

Thus, consumers hold different stakes in the same portfolio $\Omega^{-1}(\mu - q)$ of risky assets, with their stakes determined by their risk tolerance. This is known as the "Mutual Fund Theorem."

(c) From the market-clearing condition $\sum_i \alpha^i = \sum_i e^i$ we obtain $\rho\Omega^{-1}(\mu - q) = \bar{e}$, where $\rho = \sum_i \rho^i$ Is the economy's aggregate risk-tolerance and $\bar{e} = \sum_i e^i$ is the aggregate endowment of risky assets. Hence in equilibrium each consumer i holds portfolio $\frac{\rho^i}{\rho}\bar{e}$ of risky assets. The holdings of the safe asset follow from consumers' budget constraints, and the market for the safe asset clears by Walras' Law. From consumers' FOC, the risky asset prices must be $q = \mu - \frac{1}{\rho}\Omega\bar{e}$.

(d) No. Consumer demand derived in part (b) exhibits GS if and only if the off-diagonal entries of Ω^{-1} are all negative. For *J* = 2, this holds if and only if the off-diagonal entries of Ω are positive, i.e., the two assets are positively correlated. Intuitively, when assets are negatively correlated, they are complements: when one asset becomes cheaper and agent buy more of it as a result, they find it optimal to buy more of the other asset to reduce the portfolio's variance.

(e) Since utilities are quasilinear in the safe asset, the economy admits a representative consumer, therefore tatonnement is globally stable.

(f) The net expected payout on the market portfolio $\bar{e}$ is $(q-\mu)'\bar{e} = \frac{1}{\rho}\bar{e}'\Omega\bar{e} = \frac{1}{\rho}Var[\bar{e}]$. The net expected payout on portfolio α is $(q-\mu)'\alpha = \frac{1}{\rho}\alpha'\Omega\bar{e} = \frac{1}{\rho}Cov[\alpha,\bar{e}] = (q-\mu)'\bar{e}\frac{Cov[\alpha,\bar{e}]}{Var[\bar{e}]}$.

(g) Since the economy is quasilinear in the safe asset, all P.O. allocations maximize the sum of utilities. Furthermore, by the Second Welfare Theorem, we can sustain any such allocation as W.E. for some endowments. We can view W.E. as special case of asset-market equilibrium when we have complete markets (e.g., when the assets are the *S* state-contingent goods), and so we can appeal to the result in part (b) that in equilibrium each agent i should hold share ρ^i/ρ of the aggregate endowment, which is the same as the asset market allocation in part (c). Hence the asset market allocation actually happens to be unconstrained Pareto optimal, even though markets may be incomplete. (Clearly, this conclusion depends a lot on the assumption that agents' endowments are given entirely in terms of traded assets, and also, more subtly, it depends on the assumption of mean-variance preferences, which makes it optimal to divide the market portfolio in fixed proportions.) All Pareto optimal allocations have this allocation of risk and arbitrary allocation of the safe asset.