Final Exam Solutions

Econ 202

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1 Consumer theory

(a) So, for rationality, we require that the Slutsky matrix is symmetric and negative semidefinite and that the demand is homogeneous of degree zero in prices. Since we normalize p_3 to one, we don't have to worry about the homogeneity requirement. This also keeps us from having to worry about the third row and column of the Slutsky matrix. Since the demands for goods one and two do not depend on wealth, the Slutsky equation tells us that

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial x_i}{\partial p_j} \tag{1}$$

So, calculating the relevant Slutsky matrix S, we find

$$S = -B = \begin{pmatrix} -b_{11} & -b_{12} \\ -b_{21} & -b_{22} \end{pmatrix} \tag{2}$$

So, the symmetry requirement tells us that

$$\boxed{b_{12} = b_{21} \equiv b} \tag{3}$$

To satisfy the negative semidefiniteness requirement we will use Sylvester's criterion which states that the determinants of all the upper left submatrices must be equal to $(-1)^n$ where n is the dimension of the submatrix. So, we then must have

$$\boxed{\begin{aligned} b_{11} &\geq 0 \\ b_{11}b_{22} - b^2 &\geq 0 \end{aligned}} \tag{4}$$

These are all the requirements for rationality.

(b) Since the demands for goods one and two do not depend on wealth, the concepts of gross and compensated substitutes or complements are equivalent. Hence, the derivatives of interest are

$$\begin{aligned} \tfrac{\partial x_1}{\partial p_2} &= -b \\ \tfrac{\partial x_2}{\partial p_1} &= -b \end{aligned} \tag{5}$$

$$\boxed{\text{So if } b \geq 0, \text{ then goods one and two are } \textbf{complements} \text{ and if } b \leq 0, \text{ then goods one and two are } \textbf{substitutes}.}$$

(c) Using Roy's identity, we know that

$$x_i = -\frac{\partial v/\partial p_i}{\partial v/\partial w} \tag{6}$$

So, let's assume that $\frac{\partial v}{\partial w} = 1$ and use Roy's identity as a partial differential equation to solve for the indirect utility. The solution to this is given by the two equations

$$v = -\int x_i p_i = \begin{cases} -a_1p_1 + \frac{1}{2}b_{11}p_1^2 + bp_1p_2 + f_1(p_2, w) \\ -a_2p_2 + bp_1p_2 + \frac{1}{2}b_{22}p_2^2 + f_2(p_1, w) \end{cases} \tag{7}$$

where the f functions are simply constants of integration. If we allow the f's to be given by

$$\begin{aligned} f_1(p_2, w) &= -a_2p_2 + \tfrac{1}{2}b_{22}p_2^2 + w \\ f_2(p_1, w) &= -a_1p_1 + \tfrac{1}{2}b_{11}p_1^2 + w \end{aligned} \tag{8}$$

then our integrals agree and we end up with the indirect utility

$$\boxed{v(p, w) = w + \frac{1}{2}b_{11}p_1^2 + bp_1p_2 + \frac{1}{2}b_{22}p_2^2 - a_1p_1 - a_2p_2} \tag{9}$$

(d) So, we have learned that a quasilinear utility function will lead to demands that are independent of wealth. Since our given demands are independent of wealth, this is a good place to start. So, we will try a utility of form

$$u(x) = x_3 + \varphi(x_1, x_2) \tag{10}$$

Now, we can write the utility maximization problem as (noting that the constraint will bind yielding $x_3 = w - p_1 x_2 - p_2 x_2$)

$$\max_{x_1, x_2 \geq 0} w - p_1 x_1 - p_2 x_2 + \varphi(x_1, x_2) \tag{11}$$

The first order conditions for this problem can be written as

$$\nabla_x \varphi(x_1^*(p), x_2^*(p)) = p \tag{12}$$

But, since we know that $x^*(p) = a - B \cdot p$, we can just as easily invert, yielding $p^*(x) = B^{-1} \cdot (a - x)$. Re-expressing the first-order conditions then gives us

$$\nabla_x \varphi(x_1, x_2) = p^*(x) = B^{-1} \cdot (a - x) \tag{13}$$

Let's define matrix elements to make life easier. Let

$$-B^{-1} = \frac{1}{\Delta} \begin{pmatrix} b_{22} & -b \\ -b & b_{11} \end{pmatrix} \equiv \begin{pmatrix} c & d \\ d & g \end{pmatrix} \tag{14}$$

(where $\Delta = b^2 - b_{11} b_{22}$) and

$$B^{-1} \cdot a = \frac{1}{\Delta} \begin{pmatrix} b a_2 - b_{22} a_1 \\ b a_1 - b_{11} a_2 \end{pmatrix} \equiv \begin{pmatrix} j \\ k \end{pmatrix} \tag{15}$$

Now we can express our first-order condition as

$$\begin{pmatrix} \partial\varphi / \partial x_1 \\ \partial\varphi / \partial x_2 \end{pmatrix} = \begin{pmatrix} j + c x_1 + d x_2 \\ k + d x_1 + g x_2 \end{pmatrix} \tag{16}$$

Integrating, we then find

$$\varphi(x_1, x_2) = \begin{cases} j x_1 + \frac{1}{2} c x_1^2 + d x_1 x_2 + f_1(x_2) \\ k x_2 + \frac{1}{2} g x_2^2 + d x_1 x_2 + f_2(x_1) \end{cases} \tag{17}$$

So, if we let the f's be given by

$$\begin{aligned} f_1(x_2) &= k x_2 + \tfrac{1}{2} g x_2^2 \\ f_2(x_1) &= j x_1 + \tfrac{1}{2} c x_1^2 \end{aligned} \tag{18}$$

then we arrive at

$$\varphi(x_1, x_2) = jx_1 + kx_2 + \frac{1}{2}cx_1^2 + dx_1x_2 + \frac{1}{2}gx_2^2 \tag{19}$$

This can be written more readily as a quadratic form, yielding the final answer

$$\boxed{u(x_1, x_2, x_3) = x_3 + (x'_{-3} \cdot B^{-1} \cdot a) - (x'_{-3} \cdot B^{-1} \cdot x_{-3})} \tag{20}$$

(e) So, again the budget constraint will bind, yielding $x_3 = w - p_1x_1 - p_2x_2$. Then, the short term utility maximization problem is given by

$$\max_{x_2 \geq 0} w - p_1\overline{x_1} - p_2x_2 + j\overline{x_1} + kx_2 + \frac{1}{2}c\overline{x_1}^2 + d\overline{x_1}x_2 + \frac{1}{2}gx_2^2 \tag{21}$$

The first order condition for the short run UMP is then

$$k + d\overline{x_1} + gx_2^{SR} = p_2 \tag{22}$$

So the short run demand is

$$x_2^{SR} = \frac{1}{g}(p_2 - k - d\overline{x_1}) \tag{23}$$

and its derivative is

$$\boxed{\frac{\partial x_2^{SR}}{\partial p_2} = \frac{1}{g} = \frac{b^2 - b_{11}b_{22}}{b_{11}} \leq 0} \tag{24}$$

where the last inequality comes from our rationality criteria that were derived in (a). For comparison, let's calculate the derivative of the long term demand.

$$\frac{\partial x_2^{LR}}{\partial p_2} = -b_{22} \leq 0 \tag{25}$$

The inequality comes from the fact that the diagonal entries of a negative semidefinite matrix must themselves be weakly negative. Finally, note that if $\partial x_2^{SR}/\partial p_2 \geq \partial x_2^{LR}/\partial p_2$, then we have shown that the LeChâtelier principle holds (since both derivatives are negative, this inequality is equivalent to saying that the agent will respond more strongly in the long run (adjusting both inputs) than in the short run (adjusting only one input)). So, is this inequality true? Calculating

$$\begin{aligned}
\tfrac{b^2-b_{11}b_{22}}{b_{11}} &\geq -b_{22} \\
b^2 - b_{11}b_{22} &\geq -b_{11}b_{22} \\
b^2 &\geq 0
\end{aligned} \tag{26}$$

which is unconditionally true.

So, the LeChâtelier principle holds regardless of the complementarity/substitutability of the two goods - $\partial x_2^{SR}/\partial p_2 \geq \partial x_2^{LR}/\partial p_2$.

2 Profit Maximization under Uncertainty

(a) i. We want to show that $-C(q,\theta)$ has increasing differences in (q,θ). Using the hint, it is sufficient to show that $-C_q(q,\theta)$ is weakly increasing in θ. Choose $\theta' > \theta$ and note that, by definition,

$$-C_q(q,\theta') = \int_{w_i} [-c_q(q,w_i)]\, dF(w_i|\theta'). \tag{27}$$

Since $c_{qw_i} \leq 0$ by assumption, we know that $-c_q(q,w_i)$ is weakly increasing in w_i. But then by first order stochastic dominance, it follows immediately that

$$\boxed{-C_q(q,\theta') = \int_{w_i} [-c_q(q,w_i)]\, dF(w_i|\theta') \geq \int_{w_i} [-c_q(q,w_i)]\, dF(w_i|\theta) = -C_q(q,\theta)} \tag{28}$$

which is exactly what we needed to show.

ii. The firm's objective function $pq - C(q,\theta)$ has increasing differences in (q,θ). To verify this (using the hint from part i.), we need to show that $p - C_q(q,\theta)$ is weakly increasing in θ. Since p does not depend on θ, this amounts to showing that $-C_q(q,\theta)$ is weakly increasing in θ. But note that we already showed this in part i.

Thus, we may conclude from Topkis' Theorem that q^* is weakly increasing in θ (in the sense discussed in class). Intuitively, the assumption $c_{qw_i} \leq 0$ implies that the firm's marginal cost decreases as w_i increases. Higher θ makes high realizations of w_i more likely (in the first order stochastic dominance sense), and hence lowers expected marginal cost C_q. With lower marginal cost, the firm optimally produces more output.

iii. Recall that $c_{w_i} \geq 0$ for any cost function. That is, as the price of an input goes up, the total cost of producing any given quantity q must at least weakly increase. Thus, since $c(q, w_i)$ is a weakly increasing function of w_i, it follows from first order stochastic dominance that, for all q,

$$C(q, \theta') = \int_{w_i} c(q, w_i)\, dF(w_i|\theta') \geq \int_{w_i} c(q, w_i)\, dF(w_i|\theta) = C(q, \theta). \tag{29}$$

But this implies that

$$\pi(\theta') \equiv pq^*(\theta') - C(q^*(\theta'), \theta') \leq pq^*(\theta') - C(q^*(\theta'), \theta) \leq pq(\theta) - C(q(\theta), \theta) \equiv \pi(\theta) \tag{30}$$

where the first inequality comes from submodularity of the cost function and the second comes from optimality.

$\boxed{\text{Hence, profits are weakly decreasing in } \theta.}$

(b) By a logic symmetric to that used in part a), we need to ensure that $C(q, \theta)$ has *increasing* differences in (q, θ). This would allow us to apply Topkis' Theorem and conclude that q^* is weakly decreasing in θ. Using again the hint from part a.i., we need to impose a condition ensuring that

$$C(q, \theta') = \int_{w_i} c_q(q, w_i)\, dF(w_i|\theta') \geq \int_{w_i} c_q(q, w_i)\, dF(w_i|\theta) = C(q, \theta). \tag{31}$$

We know that $F(w_i|\theta')$ second order stochastically dominates $F(w_i|\theta)$, so we conclude that (31) will be true if c_q is concave in w_i, or alternatively if $c_{qw_iw_i} \leq 0$.

(c) We need to compare the firm's (expected) profits under uncertainty, which are given by $\max\limits_{q \geq 0}\ pq - C(q, \bar{\theta})$ and its profits under certainty given by $\max\limits_{q \geq 0}\ pq - c(q, \overline{w}_i)$.

Recall that cost functions are concave in input prices, that is, $c(q, w_i)$ is concave in w_i. Therefore, by Jensen's Inequality, we know that, for all q,

$$C(q, \bar{\theta}) = \int_{w_i} c(q, w_i)\, dF(w_i|\bar{\theta}) \leq c(q, \overline{w}_i). \tag{32}$$

Thus,

$$\max_{q\geq 0}\ pq - C(q,\bar{\theta}) \geq \max_{q\geq 0}\ pq - c(q,\overline{w}_i) \tag{33}$$

implying that the firm prefers to operate under *uncertainty*. Another way to see this is that since $c(q, w_i)$ is concave in w_i, the firm's Bernoulli utility function $pq - c(q, w_i)$ is convex in w_i, so it is risk loving with respect to fluctuations in w_i.

This result may seem surprising at first. However, recall that, in general, the firm can produce any given quantity of output q in a variety of ways (using different input bundles). Now, if w_i is uncertain, it can exploit fluctuations in w_i by using a combination of inputs that is optimal given the realization of w_i. For example, if w_i is high, it may use less of input i, whereas if w_i is low, it will probably want use more of it. This option to act opportunistically given the uncertain environment is valuable – thus it lowers expected production costs for any output level q. Equivalently, note that $c(q, \overline{w}_i)$ would be the expected cost of producing q if w_i were random, but the firm had to pick its mix of inputs *before* observing the realization of w_i. In contrast, $C(q, \bar{\theta})$ is the expected cost if w_i is random, but the firm can choose inputs *after* observing the realization of w_i. It should be relatively intuitive that the firm prefers the latter scenario, since it leaves more flexibility to minimize costs and thus maximize profits.

3 General Equilibrium

(a) At a price p, an agent demands a license if and only if $v \geq p$ and $w \geq \frac{1}{2}p$, so demand is:

$$D(p) = (1-p)\left(1 - \frac{1}{2}p\right) = 1 - \frac{3}{2}p + \frac{1}{2}p^2$$

For market clearing, we need $D(p) = 1/2$, so the market clearing price solves:

$$p^2 - 3p + 1 = 0$$

or

$$p^c = \frac{3-\sqrt{5}}{2} \approx 0.38.$$

(b) The average license value is $\frac{1}{2}(1+p^c)$.

(c) At a price $p < p^c$, no license owner will want to sell because they have $v \geq p^c > p$. For $p \geq p^c$, no non-owner will want to buy because they either have $v < p^c \leq p$, or else they have $w + \frac{1}{2}p^c < p^c \leq p$. So no trade is possible.

(d) From above, an agent demands a license if and only if $v \geq p^a$ and $w \geq \frac{1}{2}p^a$. So demand is $D(p^a)$, and the probability a demander gets a license is $\frac{1}{2D(p^a)}$.

(e) The average license value is $\frac{1}{2}(1+p^a) < \frac{1}{2}(1+p^c)$.

(f) Anticipating a resale price $p^R > p^a$, all agents with $w \geq \frac{1}{2}p^a$ demand a license.

(g) Initial demand is therefore $D(p^a) = 1 - \frac{1}{2}p^a$. Supply is $1/2$, so a demander gets a license with probability $\frac{1}{2-p^a}$ and doesn't get a license with probability $\frac{1-p^a}{2-p^a}$. Let p denote the price in the resale market. Any interim license holder with $v < p$ will be a seller in the resale market, so the mass of sellers is $s(p) = \frac{1}{2}p$. Any interim non-holder with $v \geq p$ and $w + \frac{1}{2}p^a > p$ will be a buyer in the resale market. Because $p \geq p^a$, these buyers also demanded but were rationed in the initial market, so their mass is:

$$d(p) = (1-p)\left(1 - p + \frac{1}{2}p^a\right)\left(\frac{1-p^a}{2-p^a}\right).$$

Market clearing in the resale market means that $d(p) = s(p)$ or

$$d(p^R) = \left(1-p^R\right)\left(1 - p^R + \frac{1}{2}p^a\right)\left(\frac{1-p^a}{2-p^a}\right) = \frac{1}{2}p^R = s(p^R)$$

(h) The average license value is $\frac{1}{2}\left(1+p^R\right)$, so the trick is to compare p^R to p^c and p^a. If $p^a = 0$, then you can check that $p^R = p^c$. Similarly, solving for the case where $p^a = p^c$ yields $p^R = p^c$. So this leaves the case where $0 < p^a < p^c$, which is hard. I found it useful to re-arrange the market clearing condition so that p^R solves:

$$p^2 - 3p + 1 = -\frac{1}{2}\frac{p^a}{1-p^a}\left[pp^a - (2+p^a) - 1\right] = -\frac{1}{2}\frac{p^a}{1-p^a}\Delta\left(p, p^a\right).$$

The left hand side is decreasing in p over $[0, 1]$ and equal to zero at $p = p^c$. So $p^R > p^c$ if and only if the right hand side is negative, i.e. if and only if the bracketed term $\Delta(p, p^a) > 0$ at $p = p^R$. The Δ term is equal to zero if $p^a = p = p^c$ and is decreasing in both p and p^a. Therefore if $0 < p^a < p^c$, it must be the case that $p^R > p^c$.