8.11

a) $\phi(63) = \phi(9)\,\phi(7)$ $\leftarrow$ since $\gcd(9,7)=1$.
$= \phi(3^2)\,\phi(7)$
$= (3^2-3)(7-1)$
$= 6 \cdot 6 = 36$

b) $\phi(243) = \phi(3^5)$
$= (3^5 - 3^4)$
$= 162$

c) $\phi(519) = \phi(173)\,\phi(3)$ $\leftarrow$ $\gcd(173,3)=1$
$= (173-1)(3-1)$
$= 344$

$2 \cdot 10$, $5 \cdot 2$

$2\,100 \to 10 \cdot 10 \to 5 \cdot 2 \quad 5 \cdot 2$

d) $\phi(4200) = \phi(7)\,\phi(600)$ $\gcd(7,600)=1$
$= \phi(7)\,\phi(3)\,\phi(200)$ $\gcd(3,200)=1$
$= \phi(7)\,\phi(3)\,\phi(25)\,\phi(8)$ $\gcd(25,8)=1$
$= \phi(7)\,\phi(3)\,\phi(5^2)\,\phi(2^3)$
$= (7-1)(3-1)(5^2-5)(2^3-2^2)$
$= (6)(2)(20)(4)$
$= 960$

all 3 are relative prime to each other

2. Let $m = 1001$, then since $13 \cdot 7 \cdot 11 = 1001$ then 13, 7, 11 can be used as the 3-tuple representation of integer 452, 937.

Then to find the residuals of 452 we have

$452 \bmod 13 \equiv 10 = m_1$
$452 \bmod 7 \equiv 4 = m_2$ $\quad (10, 4, 1)$
$452 \bmod 11 \equiv 1 = m_3$

And for 937 we have

$937 \bmod 13 \equiv 1$ $\quad (1, 6, 2)$
$937 \bmod 7 \equiv 6$
$937 \bmod 11 \equiv 2$

Thus we need to compute $452^{937} \bmod 1001$

First note that we are working mod p, thus the exponents are computed mod $\phi(p)$ (ie 13, 7, 11 are all primes)

also notice 937 is prime

$$\left([452 \bmod 13]^{937 \bmod 12}, [452 \bmod 7]^{937 \bmod 6}, [452 \bmod 11]^{937 \bmod 10} \right)$$

$$= (10^1, 4^1, 1^1) = (10, 4, 1)$$

Now to find $M_i = \prod m_j \quad j \neq i$ we get

$M_1 = 7 \cdot 11 = 77 \Rightarrow 77 \bmod 13 = 12 \Rightarrow 12^{-1} \bmod 13 = 12$
$M_2 = 13 \cdot 11 = 143 \Rightarrow 143 \bmod 7 = 3 \Rightarrow 3^{-1} \bmod 7 = 5$
$M_3 = 13 \cdot 7 = 91 \Rightarrow 91 \bmod 11 = 3 \Rightarrow 3^{-1} \bmod 11 = 4$

Thus

Thus we have

$C_1 = 77 \cdot 12 = 924$
$C_2 = 143 \cdot 5 = 715$
$C_3 = 91 \cdot 4 = 364$

$(10, 4, 1) \rightarrow [(10 \cdot 924) + (4 \cdot 715) + (1 \cdot 364)] \bmod 1001$

$= [9240 + 2860 + 364) \bmod 1001$

$= (12\,464) \bmod 1001 = 452$

So $452^{937} \bmod 1001 = 452$

* confirmed with calculator.