2nd ODE Repeated Roots

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Prove that the 2nd root of a characteristic equation of repeated roots is te^{r_2t}. Suppose that r_1 and r_2 are roots of $ar^2 + br + c = 0$ and that $r_1 \neq r_2$; then e^{r_1t} and e^{r_2t} are solutions of the differential equation $ay" + by' + cy = 0$. Show that $\phi(t; r_1, r_2) = \frac{e^{r_2t} - e^{r_1t}}{r_2 - r_1}$ is also a solution for $r_2 \neq r_1$. Then think of r_1 as fixed and use $L'H\hat{o}pital'sRule$ to evaluate the limit of $\phi(t : r_1, r_2)$ as $r_2 \to r_1$, thereby obtaining the 2nd solution in the case of equal roots.

Suppose the 2nd Order Differential Equation (ODE)

$$ay" + by' + c = 0 \tag{1}$$

and ϕ is a solution.

Then its characteristic equation is

$$ar" + br' + c = o \tag{2}$$

Now suppose

$$r_1 \neq r_2 \tag{3}$$

Therefore,

$$\phi_1 = e^{r_1t} \tag{4}$$

and

$$\phi_2 = e^{r_2t} \tag{5}$$

are solutions of (1)

To show that

$$\phi = \frac{e^{r_2t} - e^{r_1t}}{r_2 - r_1} \tag{6}$$

is a solution, plug in ϕ:

$$a\phi" + b\phi' + c\phi = 0 \tag{7}$$

We know ϕ from (6)

$$\phi' = \frac{r_2e^{r_2t} - r1e^{r_1t}}{r_2 - r_1} \tag{8}$$

$$\phi" = \frac{r_2^2e^{r_2t} - r_1^2e^{r_1t}}{r_2 - r_1} \tag{9}$$

Actually plugging in ϕ

$$a(\frac{r_2^2e^{r_2t} - r_1^2e^{r_1t}}{r_2 - r_1}) + b(\frac{r_2^2e^{r_2t} - r_1^2e^{r_1t}}{r_2 - r_1}) + c(\frac{r_2^2e^{r_2t} - r_1^2e^{r_1t}}{r_2 - r_1}) = 0$$

$$(\frac{1}{r_2 - r_1})[a(r_2^2e^{r_2t} - r_1^2e^{r_1t}) + b(r_2e^{r_2t} - r1e^{r_1t}) + c(e^{r_2t} - e^{r_1t}) = 0$$

$$(\frac{1}{r_2 - r_1})[e^{r_1t}(ar_1^2 + br_1 + c) - e^{r_2t}(ar_2^2 + br_2 + c)] = 0$$

From (2), we know that

$$(\frac{1}{r_2 - r_1})[e^{r_1 t}(0) - e^{r_2 t}(0)] = 0$$
$$(\frac{1}{r_2 - r_1})[0] = 0$$
$$0 = 0$$

Therefore, we know for sure that ϕ is a solution of (1)

$$\lim_{r_2 \to r_1} \phi$$
$$\lim_{r_2 \to r_1} \frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1}$$

Using $L'h/hatopital'sRule$ by integrating ϕ with respect to r_2

$$\lim_{r_2 \to r_1} \frac{te^{r_2 t}}{1}$$
$$te^{r_1 t}$$

Therefore, when $r_1 = r_2$, the second solution of $(1), \phi_2 = te^{rt}$

Prove that the second solution for a characteristic equation, ay" + by' + cy = 0, with repeated roots is cte^{at}.

One way to prove this is with using two pieces of information:

1. The characteristic equation for repeated roots is of the form
$$y" + 2ay' + a^2y = 0 \tag{10}$$

2.

Definition 1 (Abel's Formula)**.** *Suppose* $y" + p(t)y' + q(t)y = 0$

$$Wronskian(y_1, y_2) = ce^{-\int p(t)\mathrm{d}t} \tag{11}$$

$$det\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = ce^{-\int p(t)\mathrm{d}t}$$

To show that (10) is a characteristic equation with repeated roots.

$$y" + 2ay' + a^2y = 0 \; r^2 + 2ar + a^2 = 0 \; (r + a)^2 = 0 \; r_1 = r_2 = -a$$

Therefore, the Wronskian of the characteristic equation with repeated roots (10) is $ce^{-int2adt}$.

$$W(y_1, y_2) = ce^{-\int 2a\mathrm{d}t}$$
$$W(y_1, y_2) = ce^{-2at}$$
$$W(y_1, y_2) = ce^{-2at}$$
$$\det\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = ce^{-2at}$$
$$\begin{vmatrix} e^{-at} & y_2 \\ -ae^{-at} & y_2' \end{vmatrix} = ce^{-2at}$$
$$y_2'e^{-at} - ae^{at}y_2 = ce^{-2at}$$
$$(e^{-at}y_2)' = ce^{-2at}$$
$$e^{-at}y_2 = c_1te^{-2at} + c_2e^{-2at}$$
$$y_2 = c_1te^{-at} + c_2e^{-at}$$
$$c_2e^{-at} \text{ absorbs into } y_1 = ce^{-at}$$
$$y_2 = cte^{-at}$$