The gamma function is denoted by $\Gamma(p)$ and is defined by the integral

$$\Gamma(p+1) = \int_o^\infty e^{-x} x^p \mathrm{d}x \tag{1}$$

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper because the integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge at $x = 0$ for $p > -1$.

I Show that , for $p > 0$,

$$\Gamma(p+1) = p\Gamma(p) \tag{2}$$

To begin,

$$\Gamma(p+1) = \textstyle\int_0^\infty e^{-x} x^p \mathrm{d}x$$

$$\begin{aligned}\textstyle\int u\mathrm{d}v = uv - \int v\mathrm{d}u\\ u = x^p, v = e^{-x}\\ du = px^{p-1}, dv = -e^{-x}\end{aligned}$$

$$\begin{gathered}= \lim_{A\to\infty}[\frac{x^p}{e^x}|_0^A] + p\textstyle\int_0^\infty e^x x^{p-1}\mathrm{d}x\\ = \lim_{A\to\infty}\frac{A^p}{e^A} - \frac{0^p}{e^0} + p\textstyle\int_0^\infty e^x x^{p-1}\mathrm{d}x\\ = p\textstyle\int_0^\infty e^x x^{p-1}\mathrm{d}x\\ = p\Gamma(p)\end{gathered}$$

II Show that $\Gamma(1) = 1$

$$\begin{gathered}\Gamma(p) = \textstyle\int_0^\infty e^x x^{p-1}\mathrm{d}x\\ \Gamma(1) = \textstyle\int_0^\infty e^x x^0\mathrm{d}x\\ \Gamma(1) = \textstyle\int_0^\infty e^x\mathrm{d}x\\ \Gamma(1) = lim_{A\to\infty}[-e^{-x}|_0^A]\\ \Gamma(1) = 0 - (-1)\\ \Gamma(1) = 1\end{gathered}$$

III Show that if p is a positive integer n,

$$\begin{gathered}\Gamma(n+1) = n!\\ \Gamma(p+1) = p\Gamma(p)\\ \Gamma(n+1) = n\Gamma(n)\\ = n(n-1)\Gamma(n-1)\\ = n(n-1)(n-2)\Gamma(n-2)\\ = n(n-1)(n-2)...\Gamma(1)\\ = n(n-1)(n-2)...3.2.1\end{gathered}$$

$$\Gamma(1) = 1$$

$$\Gamma(n+1) = n!$$

IV Show that, for p¿0,

$$p(p+1)(p+2)...(p+n-1) = \frac{\Gamma(p+n)}{\Gamma(p)} \tag{3}$$

Recall (2) $\Gamma(p+1) = p\Gamma(p)$ for $p > 0$

$$
\begin{gathered}
\text{Let } \Gamma(p+n) = p\Gamma(p+n) \text{ for } (p+n) > 0 \\
= (p+n-1)\Gamma(p+n-1) \\
= (p+n-2)\Gamma(p+n-2) \\
\vdots \\
\Gamma(p+n) = (p+n-1)(p+n-2)...(p+2)(p+1)p\Gamma(p) \\
\frac{\Gamma(p+n)}{\Gamma(p)} = (p+n-1)(p+n-2)...(p+2)(p+1)p
\end{gathered}
$$