Vivian Duong

Derivatn of Poisson Distn from Binomial Distn

Thm The Poisson(μ) distn is the limit of the binomial(n,p) distn, w/ $\mathcal{E}(\text{binomial}(n,p)) = \mu = np$ as $n \to \infty$

Pf. Let $X \sim \text{binomial}(n,p)$

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

As stated above, $\mu = np$ so we can replace p w/ $\frac{\mu}{n}$ (which will be btwn $\emptyset$ & 1 for large n)

$$P(X=x) = \binom{n}{x} \left(\frac{\mu}{n}\right)^x \left(1-\frac{\mu}{n}\right)^n \left(1-\frac{\mu}{n}\right)^{-x}$$

$$\lim_{n\to\infty} P(X=x) = \lim_{n\to\infty} \left[\binom{n}{x} \left(\frac{\mu}{n}\right)^x \left(1-\frac{\mu}{n}\right)^{n-x} \right]$$

$$= \lim_{n\to\infty} \left[\frac{n!}{(n-x)!\,x!} \cdot \frac{\mu^x}{n^x} \left(1-\frac{\mu}{n}\right)^{n-x} \right]$$

$$= \lim_{n\to\infty} \left[\frac{n(n-1)\dots(n-x-2)(n-x-1)}{n^x} \right] \lim_{n\to\infty} \left[\frac{\mu^x}{x!} \left(1-\frac{\mu}{n}\right)^{n-x} \right]$$

$$= 1 \cdot \lim_{n\to\infty} \left(1-\frac{\mu}{n}\right)^n \cdot \lim_{n\to\infty} \left(1-\frac{\mu}{n}\right)^x \cdot \lim_{n\to\infty} \frac{\mu^x}{x!}$$

$$= 1 \cdot e^{-\mu} \cdot 1 \cdot \frac{\mu^x}{x!}$$

$$\lim_{n\to\infty} P(X=x) = \frac{\mu^x}{x!} e^{-\mu}$$

$\lim_{n\to\infty} P(X=x) =$ pmf for the Poisson(μ) distn. $\square$

Why do we use the Poisson distn? (from Khan Academy). Say $X =$ # cars that pass /hr

$X \sim$ binomial

$\mathcal{E}(X) = \lambda = np$

λ cars/hr $\quad \frac{1\text{ hr}}{60\text{ mins}} \times \frac{\lambda \text{ cars}}{\text{hr}} = \frac{\lambda}{60}$ cars/min

$P(X=k) = \binom{60}{k} \left(\frac{\lambda}{60}\right)^k \left(1-\frac{\lambda}{60}\right)^k$

problem is that if 2 cars pass in one minute, it would only count as one "success" so we can look & how many cars pass /sec

$P(X=k) = \binom{3600}{k} \left(\frac{\lambda}{3600}\right)^k \left(1-\frac{\lambda}{60^2}\right)^k$ but 2 cars can still pass in one sec. ~~I~~ ~~look for~~ so we use Poisson

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$$= 1\cdot\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n\cdot\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^x\cdot\lim_{n\to\infty}\frac{\mu^x}{x!}$$

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Derivatn of the Negative Binomial Distn from the Binomial Distn

Defn Binomial Distn: The binomial distn gives the probability of r successes out of N trials.

$$p(r|N) = \binom{N}{r} p^r (1-p)^{N-r}$$

Defn Negative Binomial Distn: The neg. binomial distn gives the probability of (r-1) successes & x failures in x+r-1 trials, & success on the (x+r)th trial.

Take 1st part of defn: "probability of (r-1) successes out of (x+r-1) trials"

By defn of Binomial Distn

$$p(r-1|x+r-1) = \binom{x+r-1}{r-1} p^{r-1} (1-p)^x$$

The p(success on last trial, the (x+r)th trial) is p.

⇒ by the multiplicatn rule of probability

$$NB(x|r,p) = \binom{x+r-1}{r-1} p^{r-1} (1-p)^x \cdot p$$

$$\Rightarrow NB(x|r,p) = \binom{x+r-1}{r-1} p^r (1-p)^x$$

$$E(NB) = \frac{r(1-p)}{p}$$

$$Var(NB) = \frac{r(1-p)}{p^2}$$

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Derivatn of Gamma from Poisson Distn

If $X \sim$ Poisson (λ), the time until k arrivals is $\Gamma(k \frac{1}{\lambda})$

T = time until k arrivals of event x.

$$F(t) = P(T \leq t)$$
$$= 1 - P(T \geq t)$$
$$= 1 - P(X < k)$$
$$= 1 - P(X \leq k-1)$$

X w/ parameter λ has parameter λt over $t = (0, t)$

$$F(t) = 1 - \sum_{x=0}^{k-1} \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

$$f(t) = \frac{t^{k-1} \lambda^k e^{-\lambda t}}{\Gamma(k)}$$

Compare w/ gamma pdf

$$f(x) = \frac{x^{\alpha-1} e^{-x/\theta}}{\theta^\alpha \Gamma(\alpha)}$$

$$T \sim \text{Gamma}(\alpha = k, \theta = \lambda)$$