Vivian Duong

Derivatn of Poisson Distn from Binomial Distn

Thm The Poisson (μ) distn is the limit of the binomial (n,p) distn, w/ $E(\text{binomial}(n,p)) = \mu = np$ as $n \to \infty$

Pf. Let $X \sim \text{binomial}(n,p)$

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

As stated above, $\mu = np$ so we can replace p w/ $\frac{\mu}{n}$ (which will be btwn $\emptyset$ & 1 for large n)

$$P(X=x) = \binom{n}{x}\left(\frac{\mu}{n}\right)^x\left(1-\frac{\mu}{n}\right)^n\left(1-\frac{\mu}{n}\right)^{-x}$$

$$\lim_{n\to\infty} P(X=x) = \lim_{n\to\infty}\left[\binom{n}{x}\left(\frac{\mu}{n}\right)^x\left(1-\frac{\mu}{n}\right)^{n-x}\right]$$

$$= \lim_{n\to\infty}\left[\frac{n!}{(n-x)!\,x!}\cdot\frac{\mu^x}{n^x}\left(1-\frac{\mu}{n}\right)^{n-x}\right]$$

$$= \lim_{n\to\infty}\left[\frac{n(n-1)\ldots(n-x-2)(n-x-1)}{n^x}\right]\lim_{n\to\infty}\left[\frac{\mu^x}{x!}\left(1-\frac{\mu}{n}\right)^{n-x}\right]$$

$$= 1\cdot\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n\cdot\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^x\cdot\lim_{n\to\infty}\frac{\mu^x}{x!}$$

$$= 1\cdot e^{-\mu}\cdot 1\cdot\frac{\mu^x}{x!}$$

$$\lim_{n\to\infty} P(X=x) = \frac{\mu^x}{x!}e^{-\mu}$$

$\lim_{n\to\infty} P(X=x) =$ pmf for the Poisson (μ) distn. $\square$

Why do we use the Poisson distn? (from Khan Academy). Say $X =$ # cars that pass /hr

$X \sim$ binomial

$E(X) = \lambda = np$

λ cars/hr $\quad \frac{1\text{ hr}}{60\text{ mins}} \times \frac{\lambda\text{ cars}}{\text{hr}} = \frac{\lambda}{60}$ cars/min

$P(X=k) = \binom{60}{k}\left(\frac{\lambda}{60}\right)^k\left(1-\frac{\lambda}{60}\right)^k$

problem is that if 2 cars pass in one minute, it would only count as one "success" so we can look & how many cars pass /sec

$P(X=k) = \binom{3600}{k}\left(\frac{\lambda}{3600}\right)^k\left(1-\frac{\lambda}{60^2}\right)^k$ but 2 cars can still pass in one sec. ~~If~~ ~~look for an~~ so we use Poisson

Vivian Duong

Derivatn of Poisson Distn from Binomial Distn

Thm The Poisson(μ) distn is the limit of the binomial(n,p) distn, w/ $E(\text{binomial}(n,p)) = \mu = np$ as $n \to \infty$

Pf. Let $X \sim \text{binomial}(n,p)$

$$p(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

As stated above, $\mu = np$ so we can replace p w/ $\frac{\mu}{n}$ (which will be btwn $\emptyset$ & 1 for large n)

$$p(X=x) = \binom{n}{x} \left(\frac{\mu}{n}\right)^x \left(1-\frac{\mu}{n}\right)^n \left(1-\frac{\mu}{n}\right)^{-x}$$

$$\lim_{n\to\infty} p(X=x) = \lim_{n\to\infty} \left[\binom{n}{x} \left(\frac{\mu}{n}\right)^x \left(1-\frac{\mu}{n}\right)^{n-x} \right]$$

$$= \lim_{n\to\infty} \left[\frac{n!}{(n-x)!\,x!} \cdot \frac{\mu^x}{n^x} \left(1-\frac{\mu}{n}\right)^{n-x} \right]$$

$$= \lim_{n\to\infty} \left[\frac{n(n-1)\dots(n-x-2)(n-x-1)}{n^x} \right] \lim_{n\to\infty} \left[\frac{\mu^x}{x!} \left(1-\frac{\mu}{n}\right)^{n-x} \right]$$

$$= 1 \cdot \lim_{n\to\infty} \left(1-\frac{\mu}{n}\right)^n \cdot \lim_{n\to\infty} \left(1-\frac{\mu}{n}\right)^x \cdot \lim_{n\to\infty} \frac{\mu^x}{x!}$$

$$= 1 \cdot e^{-\mu} \cdot 1 \cdot \frac{\mu^x}{x!}$$

$$\lim_{n\to\infty} P(X=x) = \frac{\mu^x}{x!} e^{-\mu}$$

$\lim_{n\to\infty} P(X=x) =$ pmf for the Poisson(μ) distn.

$\square$

Why do we use the Poisson distn? (from Khan Academy). Say $X =$ # cars that pass /hr

$X \sim$ binomial

$E(X) = \lambda = np$

λ cars/hr $\frac{1\text{hr}}{60\text{mins}} \times \frac{\lambda\text{cars}}{\text{hr}} = \frac{\lambda}{60}$ cars/min

$P(X=k) = \binom{60}{k} \left(\frac{\lambda}{60}\right)^k \left(1-\frac{\lambda}{60}\right)^k$

problem is that if 2 cars pass in one minute, it would only count as one "success" so we can look @ how many cars pass /sec

$P(X=k) = \binom{3600}{k} \left(\frac{\lambda}{3600}\right)^k \left(1-\frac{\lambda}{60^2}\right)^k$ but 2 cars can still pass in one sec. ~~If~~ ~~look for an~~ so we use Poisson

Derivatn of the Negative Binomial Distn from the Binomial Distn

Defn Binomial Distn: The binomial distn gives the probability of r successes out of N trials.

$$p(r|N) = \binom{N}{r} p^r (1-p)^{N-r}$$

Defn Negative Binomial Distn: The neg. binomial distn gives the probability of (r-1) successes & x failures in x+r-1 trials, & success on the (x+r)th trial.

Take 1st part of defn: "probability of (r-1) successes out of (x+r-1) trials"

By defn of Binomial Distn

$$p(r-1|x+r-1) = \binom{x+r-1}{r-1} p^{r-1} (1-p)^x$$

The p(success on last trial, the (x+r)th trial) is p.

$\Rightarrow$ by the multiplicatn rule of probability

$$NB(x|r,p) = \binom{x+r-1}{r-1} p^{r-1} (1-p)^x \cdot p$$

$$\Rightarrow NB(x|r,p) = \binom{x+r-1}{r-1} p^r (1-p)^x$$

$$E(NB) = \frac{r(1-p)}{p}$$

$$Var(NB) = \frac{r(1-p)}{p^2}$$

Vivian Duong

Derivatn of the Negative Binomial Distn from the Binomial Distn

Defn Binomial Distn: The binomial distn gives the probability of r successes out of N trials.

$$p(r|N) = \binom{N}{r} p^r (1-p)^{N-r}$$

Defn Negative Binomial Distn: The neg. binomial distn gives the probability of (r-1) successes & x failures in x+r-1 trials, & success on the (x+r)th trial.

Take 1st part of defn: "probability of (r-1) successes out of (x+r-1) trials"

By defn of Binomial Distn

$$p(r-1|x+r-1) = \binom{x+r-1}{r-1} p^{r-1} (1-p)^x$$

The p(success on last trial, the $(x+r)^{th}$ trial) is p.

=> by the multiplicatn rule of probability

$$NB(x|r,p) = \binom{x+r-1}{r-1} p^{r-1} (1-p)^x \cdot p$$

$$\Rightarrow NB(x|r,p) = \binom{x+r-1}{r-1} p^r (1-p)^x$$

$$E(NB) = \frac{r(1-p)}{p}$$

$$Var(NB) = \frac{r(1-p)}{p^2}$$