Vivian Duong

Derivatn of Poisson Distn from Binomial Distn

Thm The Poisson(μ) distn is the limit of the binomial(n,p) distn, w/ $E(\text{binomial}(n,p)) = \mu = np$ as $n \to \infty$

Pf. Let $X \sim \text{binomial}(n,p)$

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

As stated above, $\mu = np$ so we can replace p w/ $\frac{\mu}{n}$ (which will be btwn 0 & 1 for large n)

$$P(X=x) = \binom{n}{x} \left(\frac{\mu}{n}\right)^x \left(1-\frac{\mu}{n}\right)^n \left(1-\frac{\mu}{n}\right)^{-x}$$

$$\lim_{n\to\infty} P(X=x) = \lim_{n\to\infty} \left[\binom{n}{x} \left(\frac{\mu}{n}\right)^x \left(1-\frac{\mu}{n}\right)^{n-x} \right]$$

$$= \lim_{n\to\infty} \left[\frac{n!}{(n-x)!\,x!} \cdot \frac{\mu^x}{n^x} \left(1-\frac{\mu}{n}\right)^{n-x} \right]$$

$$= \lim_{n\to\infty} \left[\frac{n(n-1)\cdots(n-x-2)(n-x-1)}{n^x} \right] \lim_{n\to\infty} \left[\frac{\mu^x}{x!} \left(1-\frac{\mu}{n}\right)^{n-x} \right]$$

$$= 1 \cdot \lim_{n\to\infty} \left(1-\frac{\mu}{n}\right)^n \cdot \lim_{n\to\infty} \left(1-\frac{\mu}{n}\right)^x \cdot \lim_{n\to\infty} \frac{\mu^x}{x!}$$

$$= 1 \cdot e^{-\mu} \cdot 1 \cdot \frac{\mu^x}{x!}$$

$$\lim_{n\to\infty} P(X=x) = \frac{\mu^x}{x!} e^{-\mu}$$

$\lim_{n\to\infty} P(X=x) =$ pmf for the Poisson(μ) distn. □

Why do we use the Poisson distn? (from Khan Academy). Say X = # cars that pass /hr

$X \sim$ binomial

$E(X) = \lambda = np$

$\lambda \text{ cars/hr} \cdot \frac{1\text{ hr}}{60\text{ mins}} \times \frac{\lambda \text{ cars}}{\text{hr}} = \frac{\lambda}{60} \text{ cars/min}$

$P(X=k) = \binom{60}{k} \left(\frac{\lambda}{60}\right)^k \left(1-\frac{\lambda}{60}\right)^k$

problem is that if 2 cars pass in one minute, it would only count as one "success" SO we can look @ how many cars pass /sec

$P(X=k) = \binom{3600}{k} \left(\frac{\lambda}{3600}\right)^k \left(1-\frac{\lambda}{60^2}\right)^k$ but 2 cars can still pass in one sec. I look for another so we use Poisson

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