

Show that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \quad (1)$$

Let

$$\begin{aligned} I &= \int_0^\infty e^{-x^2} dx \\ I^2 &= \int_0^\infty e^{-x^2} dx \left[ \int_0^\infty e^{-y^2} dy \right] \\ I^2 &= \int_0^\infty e^{-x^2} e^{-y^2} dx dy \end{aligned}$$

$$r = \sqrt{x^2 + y^2}$$

$$\begin{aligned} I^2 &= \int_0^\infty \int_0^\infty e^{-x^2-y^2} dx dy \\ I^2 &= \int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} r d\theta dr \\ I^2 &= \frac{\pi}{2} \int_0^\infty e^{-r^2} r dr \end{aligned}$$

$$\begin{aligned} u &= -r^2 \\ du &= -2r dr \\ \frac{du}{-2} &= r dr \end{aligned}$$

$$\begin{aligned} I^2 &= \frac{\pi}{-4} \int_0^\infty e^u du \\ I^2 &= \frac{\pi}{-4} [e^{-r^2} |_0^\infty] \\ I^2 &= \frac{\pi}{-4} [\lim_{r \rightarrow \infty} e^{-r^2} - 1] \\ I^2 &= \frac{\pi}{-4} [0 - 1] \\ I^2 &= \frac{\pi}{4} \\ \sqrt{I^2} &= \sqrt{\frac{\pi}{4}} \\ I &= \frac{\pi}{2} \\ \int_0^\infty e^{-x^2} dx &= \frac{\pi}{2} \end{aligned}$$