Theorem 0.1. *A solution of $ay" + by' + cy = 0$ is either everywhere zero or else can take on the value zero at most once if the roots of the characteristic equations are real.*

$$ay" + by' + cy = 0 \tag{1}$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2}$$

Case 1 $b^2 - 4ac = 0$

$$r_1 = \frac{-b}{2a},\ r_2 = \frac{-b}{2a}$$
$$y = c_1 e^{r_1 t} + c_2 t e^{r_2 t}$$
$$0 = e^{r_1 t}(c_1 + c_2 t)$$

An exponent can not equal zero.

$$0 \neq e^r r_1 t$$

Thus, this must be true.

$$0 = c_1 + c_2 t$$
$$-c_2 t = c_1$$

The solution can take on the value zero at most once if c_1 and c_2 are not zero. If c_1 and c_2 are zero, then the solution is zero everywhere.

Case 2 $b^2 - 4ac > 0$

$$r_1 = \frac{-b + sqrtb^2 - 4ac}{2a},\ r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
$0 = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
$-c_2 e^{r_2 t} = c_1 e^{r_1 t}$
$\frac{-c_2}{c_1} = e^{t(r_1 - r_2)}$
$ln(\frac{\frac{-c_2}{c_1}}{r_1 - r_2}) = t$
$r_1 - r_2 = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} - (\frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a})$
$r_1 - r_2 = \frac{-2\sqrt{b^2 - 4ac}}{2a}$
$r_1 - r_2 = \frac{b^2 - 4ac}{a}$
$\frac{ln(\frac{-c_1}{c_2})}{\frac{\sqrt{b^2 - 4ac}}{a}} = t$
$\frac{a * ln(\frac{-c_1}{c_2})}{\sqrt{b^2 - 4ac}} = t$
$ln(\frac{-c_1}{c_2}) = t$ when
$-c_1 = c_2$