Show that

$$\int_0^\infty e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2} \tag{1}$$

Let

$$\mathrm{I} = \int_0^\infty e^{-x^2}\mathrm{d}x$$
$$\mathrm{I}^2 = \int_0^\infty e^{-x^2}\mathrm{d}x[\int_0^\infty e^{-y^2}\mathrm{d}y]$$
$$\mathrm{I}^2 = \int_0^\infty e^{-x^2}e^{-y^2}\mathrm{d}x\mathrm{d}y$$

$$r = \sqrt{x^2 + y^2}$$

$$\mathrm{I}^2 = \int_0^\infty \int_0^\infty e^{-x^2-y^2}\mathrm{d}x\mathrm{d}y$$
$$\mathrm{I}^2 = \int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2}r\mathrm{d}\theta\mathrm{d}r$$
$$\mathrm{I}^2 = \frac{\pi}{2}\int_0^\infty e^{-r^2}r\mathrm{d}r$$

$$u = -r^2$$
$$\mathrm{d}u = -2r\mathrm{d}r$$
$$\frac{\mathrm{d}u}{-2} = r\mathrm{d}r$$

$$\mathrm{I}^2 = \frac{\pi}{-4}\int_0^\infty e^u\mathrm{d}u$$
$$\mathrm{I}^2 = \frac{\pi}{-4}[e^{-r^2}|_0^\infty]$$
$$\mathrm{I}^2 = \frac{\pi}{-4}[\lim_{r\to\infty} e^{-r^2} - 1]$$
$$\mathrm{I}^2 = \frac{\pi}{-4}[0-1]$$
$$\mathrm{I}^2 = \frac{\pi}{4}$$
$$\sqrt{\mathrm{I}^2} = \sqrt{\frac{\pi}{4}}$$
$$\mathrm{I} = \frac{\pi}{2}$$
$$\int_0^\infty e^{-x^2}\mathrm{d}x = \frac{\pi}{2}$$