Theorem 0.1. *Euler equations, equations in the form*

$$t^2 y\text{"} + \alpha t y' + \beta y = 0 \tag{1}$$

where $t > 0$, α and β are real constants, can be transformed into an equation with constant coefficients by transforming x into lnt.

Proof. $t^2 y\text{"} + \alpha t y' + \beta y = 0$

$$t^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \alpha t \frac{\mathrm{d}y}{\mathrm{d}x} + \beta y = 0$$

$$x = ln(t) \tag{2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{t}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$t^2 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \alpha t \frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}x} + \beta y = 0$$
$$t^2 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \alpha \frac{\mathrm{d}y}{\mathrm{d}x} + \beta y = 0$$

$$\begin{aligned}
\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}t} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}[\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}x}] \\
&= \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}x}{\mathrm{d}t}.\frac{\mathrm{d}}{\mathrm{d}t}[\frac{\mathrm{d}y}{\mathrm{d}x}] \\
&+ \frac{\mathrm{d}x}{\mathrm{d}t}.\frac{\mathrm{d}}{\mathrm{d}t}[\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}y}] \\
&+ \frac{\mathrm{d}x}{\mathrm{d}t}.\frac{\mathrm{d}}{\mathrm{d}t}[\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}y}] \\
&+ \frac{\mathrm{d}x}{\mathrm{d}t}.\frac{\mathrm{d}}{\mathrm{d}t}[\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}y}] \\
&+ \frac{\mathrm{d}x}{\mathrm{d}t}.\frac{\mathrm{d}}{\mathrm{d}t}[.\frac{\mathrm{d}x}{\mathrm{d}y}] \\
&+ \frac{\mathrm{d}x}{\mathrm{d}t}.[\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}y}.\frac{\mathrm{d}x}{\mathrm{d}}] \\
&+ \frac{\mathrm{d}x}{\mathrm{d}t}.[\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\frac{\mathrm{d}x}{\mathrm{d}t}] \\
\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\frac{\mathrm{d}y}{\mathrm{d}x} + (\frac{\mathrm{d}x}{\mathrm{d}t})^2.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \\
\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= \frac{-1}{t^2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{t^2.}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}
\end{aligned}$$

Going back to (1),

$$t^2 y\text{"} + \alpha t y' + \beta y = 0$$
$$t^2 \frac{-1}{t^2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{t^2.}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} 0 + \alpha t \frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}x} + \beta y = 0$$
$$\underbrace{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (\alpha - 1)\frac{\mathrm{d}y}{x} + \beta y = 0}$$
$$\uparrow$$

An equation with constant coefficients

$\square$