Assignment 1- Robot Motion Planning

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1.

Let's consider 3 cases for the bug algorithm-

The first case is that no obstacles are on the line segment from the start to the goal. This would mean our $d$tot would just equal $d$goal as the bug would just traverse the line segment that connects the start and the goal. Which is less than or equal to $d$goal $+\frac{1}{2}\sum_{1=1}^{M} n_i p_i$ where M=0 in this case.

The second case is where we have multiple objects that intersect the line segment from the start to the goal. In this case we can consider each object to intersect the line segment a minimum of 2 times- the edge case for this statement would be that the line segment is tangential but in this case we can consider the bug can traverse the line without hitting the obstacle or if the bug does come back to the original point of intersection then there is no path to the goal. Hence from this we can say any object that is intersected by the line segment is intersected a minimum of 2 times. We can also say that for every place a part of the object that obstructs the path of the line segment connecting the start and the goal – that part consists of 2 intersections on the line segment.

From the above statements we can conclude that for every place the object obstructs the line segment there are 2 points of intersection and no matter how close they may be there will always be a very small gap between them. Hence from the formula above we can say the bug traverses less than the perimeter for every pair of intersections. We can also say that the bug need not always come to the second point of intersection as the algorithm allows the bug to leave the surface of the obstacle if it encounters the line segment again and is closer to the goal than the first intersection, which will at worst case be the second point of intersection.

So we can say the worst case for a single pair of intersections the bug has to traverse slightly less than the perimeter and this can be expanded to multiple pairs of intersections and multiple objects in the path of the bug. This is always less than $\frac{1}{2}\sum_{1=1}^{M} n_i p_i$ where M is number of objects and n will be in multiples of 2 every time something crosses the line segment. In the worst case the bug will never traverse more than $d$goal $+\frac{1}{2}\sum_{1=1}^{M} n_i p_i$.

The third case will be when the bug is within a certain object. This can be proven exactly in the same way as the second case.

Hence from these three cases we can say the bug never travels more than $d$goal $+\frac{1}{2}\sum_{1=1}^{M} n_i p_i$

2.

Z-Y-X Euler angle representation- where C() and S() represent cos() and sin() respectively. α- angle wrt to Z axis, β- angle wrt to Y axis, γ- angle wrt to X axis,

$$\begin{matrix} C(\alpha) & -S(\alpha) & 0 \\ S(\alpha) & C(\alpha) & 0 \\ 0 & 0 & 1 \end{matrix} \quad X \quad \begin{matrix} C(\beta) & 0 & S(\beta) \\ 0 & 1 & 0 \\ -S(\beta) & 0 & C(\beta) \end{matrix} \quad X \quad \begin{matrix} 1 & 0 & 0 \\ 0 & C(\gamma) & -S(\gamma) \\ 0 & S(\gamma) & C(\gamma) \end{matrix}$$

If we consider β = pi/2- the above equation reduces to-

$$\begin{matrix} C(\alpha) & -S(\alpha) & 0 \\ S(\alpha) & C(\alpha) & 0 \\ 0 & 0 & 1 \end{matrix} \quad X \quad \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{matrix} \quad X \quad \begin{matrix} 1 & 0 & 0 \\ 0 & C(\gamma) & -S(\gamma) \\ 0 & S(\gamma) & C(\gamma) \end{matrix} =$$

$$\begin{matrix} 0 & C(\alpha)S(\gamma)-S(\alpha)C(\gamma) & C(\alpha)C(\gamma)+S(\alpha)S(\gamma) \\ 0 & S(\alpha)S(\gamma)+C(\alpha)C(\gamma) & S(\alpha)C(\gamma)-C(\alpha)S(\gamma) \\ -1 & 0 & 0 \end{matrix} \quad = \quad \begin{matrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{matrix}$$

$$\begin{matrix} C(\alpha)S(\gamma)-S(\alpha)C(\gamma) & C(\alpha)C(\gamma)+S(\alpha)S(\gamma) \\ S(\alpha)S(\gamma)+C(\alpha)C(\gamma) & S(\alpha)C(\gamma)-C(\alpha)S(\gamma) \end{matrix} \quad = \quad \begin{matrix} r_{12} & r_{13} \\ r_{22} & r_{23} \end{matrix}$$

$$\begin{matrix} r_{12} & r_{13} \\ r_{22} & r_{23} \end{matrix} \quad = \quad \begin{matrix} S(\gamma-\alpha) & C(\alpha-\gamma) \\ C(\gamma-\alpha) & S(\alpha-\gamma) \end{matrix}$$

From this we can say $\frac{r_{12}}{r_{22}} = Tan(\gamma-\alpha) \; = \; Tan^{-1}(\frac{r_{12}}{r_{22}}) = \gamma-\alpha$

This equation does not have a unique solution for γ and α hence we run into a situation known as gimbal lock.

The same can be done when β= -pi/2 , the central matrix will be $\begin{matrix} 0 & 0 & -1 \\ -0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}$

This is an issue because at this state we cannot determine the other 2 angles.

3.

Lets consider the point (0,1) and (x,y) on the circle S^1. We can define a mapping to (u,0) defined on U(x,y)-> R(u) represented by φ

Where U is all points on the circle S^1 –(0,1).

The line segment connecting these points (0,1), (x,y), (u,0) would be (x,y-1)= m(u,-1)- from this we can say-

φ (x,y)= x/1-y =u

We can see this function is continuous for all x,y where y$\neq$ 1. This condition is satisfied as we consider (x,y)= S^1 –(0,1).

Similarly if we consider points (0,-1) and (x,y) on the circle S^1. We can define a mapping to (v,0) defined from V(x,y)-> R(v) represented by ω

Where V is all points on the circle S^1 – (0,-1).

The line segment connecting these points (0,-1), (x,y), (v,0) would be (x,y+1)= m(v,1)- form this we can say-

ω (x,y)= x/1+y = v

We can see this function is continuous for all x,y where y$\neq$ -1. This condition is satisfied as we consider (x,y)= S^1 –(0,-1).

Bijectivity of both functions-

Onto-

φ (x,y)= x/1-y =u

When y->1 and x is positive this will cause the value of u-> +∞

When y->1 and x is negative this will cause the value of u-> -∞

From this we can say the function covers the entire range of R making this a onto function.

ω (x,y)= x/1+y = v

When y->-1 and x is positive this will cause the value of v-> +∞

When y->-1 and x is negative this will cause the value of v-> -∞

From this we can say the function covers the entire range of R making this a onto function

One to One-

Lets consider the function- ω (x,y)= x/1+y = v

Now we can substitute the value of x with $x^2 + y^2 = 1$ giving us- $\sqrt{1 - y^2}/(1 + y)$- squaring both numerator and denominator- $1 - y^2/(1 + y^2 + 2y)$

To prove one to one we consider f(y_1)=f(y_2) only if y_1= y_2 -

$1 - {y_1}^2/(1 + {y_1}^2 + 2y_1)$= $1 - {y_2}^2/(1 + {y_2}^2 + 2y_2)$

Simplifying this we get-

$({y_2}^2 - {y_1}^2) + (y_2 - y_1) + y_2 y_1 (y_2 - y_1)$=0

The above equation can only be true if y_1= y_2

Hence the function is one to one.

Similarly we can prove the same for φ (x,y)= x/1-y =u

Hence we can say both φ (x,y) and ω (x,y) are bijective.

To show inverse exists for both mapping-

The inverse of φ (x,y)= x/1-y can be found by using the constraint $x^2 + \ y^2 = 1$. It is defined as

φ^{-1}= (2u/u^2+1 , u^2-1/u^2+1)- this is the mapping from u in R->(x,y) in S^1

This function is continuous as u can take any value in R and φ^{-1} will be defined as the denominator never goes to zero.

The inverse of ω (x,y)= x/1+y can be found by using the constraint $x^2 + \ y^2 = 1$. It is defined as

ω^{-1}= (2v/v^2+1 , -v^2+1/v^2+1)- this is the mapping from v in R->(x,y) in S^1

This function is continuous as v can take any value in R and ω^{-1} will be defined as the denominator never goes to zero.

Differentiablity-

$\frac{\partial\varphi}{\partial x}$= 1/(1-y) – this is continuous for all values of x,y in S^1 –(0,1) as the denominator is never zero(y$\neq$ 1) and all higher order derivatives are constant.

$\frac{\partial\varphi}{\partial y}$= X/$(1-y)^2$- this is continuous for all values of x,y in S^1 –(0,1) as the denominator is never zero (y$\neq$ 1) and all higher order derivatives have the same denominator to a higher power making them also continuous.

$\frac{\partial\omega}{\partial x}$= 1/(1+y) – this is continuous for all values of x,y in S^1 –(0,-1) as the denominator is never zero (y$\neq$ -1) and all higher order derivatives are constant.

$\frac{\partial\omega}{\partial x}$= -X/$(1+y)^2$ - this is continuous for all values of x,y in S^1 –(0,-1) as the denominator is never zero (y$\neq -1$) and all higher order derivatives have the same denominator to a higher power making them also continuous.

$\frac{\partial\varphi^{-1}}{\partial u}$ = $(-\frac{2(u^2-1)}{(u^2+1)^2}, \frac{4u}{(u^2+1)^2})$

The x attribute of the function is continuous for all values of u in R as the denominator can never go to zero and all higher orders will have the denominator to higher powers. The y attribute is also continuous as it has the same denominator as x. Hence it is continuous for all values of u.

$\frac{\partial\omega^{-1}}{\partial u}$ = $(-\frac{2(v^2-1)}{(v^2+1)^2}, \frac{-4v}{(v^2+1)^2})$

The x attribute of the function is continuous for all values of v in R as the denominator can never go to zero and all higher orders will have the denominator to higher powers. The y attribute is also continuous as it has the same denominator as x. Hence it is continuous for all values of v.

Conclusion-

Chart φ (x,y) is a bijective mapping between U= (S^1 –(0,1)) to R and both φ and φ^{-1} are continuous .

Chart ω (x,y) is a bijective mapping between V=(S^1 – (0,-1)) to R and both ω and ω^{-1} are continuous.

The charts φ (x,y) and ω (x,y) cover S^1 and $\varphi\,.\,\omega^{-1}$ is a diffeomorphism.

4.

a) as each robot is not constrained by the other- each robot will have a configuration space of $R^2 * S^1$. Hence, they both will have a combined configuration space of $R^4 * T^2$.

b) when the robots are connected by a rope- we can consider one of the robots to be free and the other to be constrained by the length of the rope, the rope does not constrain the rotation of either robot. free robot configuration space- $R^2 * S^1$, constrained robot configuration space is the same as the free robot but cannot cover the whole R^2 space. Combined configuration space is $R^4 * T^2$ with constrain on 2 dimensions on R.

c) as the wheels of the train are directly related to the motion of train along the tracks we can disregard the rotation of the wheels. Configuration space is R^1

d)since our legs are fixed to the pedals our ankles don't have a part to play as they are always fixed. We can also say that both the pedals always differ by a fixed angle from each other, hence we only need to consider one of our legs as the other one will always be at the fixed angle ahead or behind. From this we can say the leg has 2 rotational joint and the configuration space is $S^1 * S^1 = T^2$

5.

X- set of all points in the workspace- open set- x is a subset of X

Config- set of all points in configuration space- open set- q is a subset of Config

A(x)- set of all points in X that are occupied by the robot

WO(i)= set of all points in the workspace where A(x)∩O(i)≠0 (object i) (mentioned in 4.3.1 in Planning algorithms by Lavalle)

Similarly for WO(j) for object j.

We can define the same for the configuration space

A(q)- set of all points in Config that are occupied by the robot

CO(i)= set of all points in the configuration space where A(q)∩O(i)≠0 (object i)

Similarly for CO(j) for object j

Both WO's and CO's are closed sets.

Using the forward kinematic transform- φ (mentioned in 4.7.1 in Choset) we can map the points in the configuration to the workspace-

φ (A(q))= A(x)

φ (CO(i))= WO(i)

φ (CO(j))= WO(j)

We can also define the inverse kinematic transform as φ^{-1} to map points from the workspace to the configuration space.

As object I and object j are independent of each other we can also say that the set defining them in either workspace or the configuration space are independent of each other.

Hence- *C*(*WO(i)* ∪ *WO(j)*) can be expanded to C(WO(i)) U C(WO(j))= φ^{-1}(WO(i)) U φ^{-1} (WO(j)).
From what we have concluded above we can say this is equal to CO(i) U CO(j).

We have proved that LHS = RHS.

6.

The wavefront planner on a discrete grid will find a path from start to goal if there exists one. This planner ensures that there always exists a grid block that is one step closer to the goal than the current grid position. We can say this due to the nature of the planner as it is built by moving one block outward in the specified directions(2D space)(4 point/ 8 point) from the goal position to the start position on the grid. The shortest path on the grid can be found by using the BFS algorithm on the grid from the start to the goal considering the distance from each grid block to the next is 1 unit distance.

The shortest path that is found is constrain by how the planner was built either 4 point or 8 point. If it was 4 point then we can say the planner finds the shortest Manhattan distance as it can't travel diagonal in the discrete grid. If the planner was built using 8 point we can say the shortest path would yield the shortest discrete Euclidean distance as you can move diagonally rather that move 2 times to reach the same position. We can also say that this diagonal path is the shortest distance to that grid block from the block diagonally next to it using the theorem that the sum of sides of a triangle is always greater than the third.