Applications of On-shell Techniques to The Standard Model and Beyond

Brad Bachu

A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE BY
THE DEPARTMENT OF
PHYSICS

ADVISER: Professor Nima Arkani-Hamed

May 2023

Abstract

Lagrangians, fields and Feynman diagrams are losing their monopoly on particle physics. A recent alternative, collectively referred to as 'on-shell techniques', has proven itself as not only a method to compute scattering amplitudes more efficiently, but also as a tool to unlock underlying structures of quantum theories previously obscured by traditional approaches. The Standard Model (SM) of particle physics, the best particle description of our world today, is a quantum field theory (QFT) based model. With on-shell methods on the rise, as well the recent ability to deal with massive particles, the time has come to see these techniques make contact with the SM. In this thesis, we use on-shell techniques to construct, describe, and push amplitudes across different scales, a journey which leads us to discover various on-shell analogs of QFT and string results. In particular, we explain the Higgs mechanism and spontaneous symmetry breaking, and derive all the relevant physics in generic theories as well as in the SM. In addition, we examine a class of string-inspired UV completions of the SM and place constraints on them. These results make no reference to Lagrangians or quantum fields, and so, present a new avenue to probe the SM and beyond.

Acknowledgements

My experience in physics has not only been shaped by lectures, textbooks and academic papers, but also by the people I've had the privilege of interacting with. Of course, I must first acknowledge and thank Nima Arkani-Hamed. Getting exposed to his way of thinking and questioning, as well as his ability to simplify problems, has given me a model of a scientist that I will continue to strive to emulate. I am grateful that he has provided topics for me to explore that align with my interests, and has also had the patience to answer my questions along the way.

Understanding and advancing the forefront of physics is a task better done together. As such, I would like to first acknowledge my collaborators Akshay Yelleshpur and Aaron Hillman, each of whom have contributed to my growth and progress as a physicists. In addition, I would like to recognize my officemates Nicholas Haubrich, Henry Lin, Wayne Zhao, and Wenli Zhao, for expanding my knowledge in physics by sharing their own research progress. I also thank Sebastian Mizera and Holmfridur Hannesdottir for welcoming me into the IAS community. Moreover, I would like to thank Yu-Tin Huan, for reading this thesis, as well as Herman Verlinde and James Olsen for being on my committee. Lastly, I would like to thank the Institute for Advanced Study for hosting me during this period and allowing me to interact with and learn from amazing community of scholars.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

Particle physics necessarily demands the combination of quantum mechanics and Poincarè invariance. The most successful mathematical framework that has been able to achieve this thus far, has been 'quantum field theory' (QFT). In QFT, the field is considered the fundamental object, particles are viewed as quantum fluctuations of the field, and the underlying physics is described by Lagrangians and path integrals. At the end of the day, observable quantities are our main concern, and these are encoded in the scattering matrix (S-Matrix). Elements of the S-Matrix are referred to as scattering amplitudes, which encode the probability that a set of particles will turn into another set of particles upon colliding. QFT is notorious for complicating S-Matrix calculations, and obscuring underlying structures in these mathematical objects. The past few decades has seen the emergence of an alternative approach, collectively referred to as 'on-shell' methods, which has gained popularity in its ability to add simplicity and uncover structures in scattering amplitudes. In this thesis, we harness 'on-shell' methods to do just that.

The source of most problems in QFT seem to emerge from the field, which, among other things, makes Lorentz covariance manifest at every stage of the calculation, an unnecessary goal, since all that matters is Lorentz invariance of the S-Matrix.

For example, the massless photon only has two degrees of freedom, and yet, it is packaged into a four component field A_μ, where μ makes the Lorentz convariance manifest. These extra degrees of freedom contribute to unnecessary complexity of calculations. The famous example is the computation of the gluonic $2 \to 4$ amplitude in [2], in which many lines of calculations simplified, and are now seen as various ways of rewriting zero. The simplicity of the final calculation was one of the signs that there may be an alternative, faster approach. This prompted the authors to use spinor-helicity variables, which only focus on the true degrees of freedom. The aforementioned amplitude could then be written directly, and the $2 \to 4$ amplitude could be generalized to n particles [3].

This ushered the rise of 'on-shell' methods. The result was more precise and quicker S-Matrix calculations, new structures, and alternative explanations of QFT based phenomena, much of which is summarized in [4, 5, 6, 7, 8, 9]. Most of the progress in the field, however, has been limited to massless particles. It was not until [10], that this conceptual hurdle was overcome. As such, on-shell methods are now ready to make contact with the world we live in, which has massive particles.

The Standard Model (SM) of particle physics is the best particle description of our world today. However, it is a QFT based model. With on-shell methods on the rise, as well as with the new ability to deal with massive particles, the time has come to see these techniques make contact with the SM. In this thesis, we use these techniques to describe and examine amplitudes and the SM across different scales. In the process, we uncover on-shell analogs QFT descriptions and results, such as the Higgs mechanism and spontaneous symmetry breaking, and possibly place constraints on UV competitions of theories.

In Chapter 2, we describe a new approach towards the development of an entirely on-shell description of the bosonic electroweak sector of the Standard Model and the Higgs mechanism. We write down on-shell three particle amplitudes consistent with

Poincaré invariance and little group covariance. Tree-level, four particle amplitudes are determined by demanding consistent factorization on all poles and correct UV behaviour. We present expressions for these $2 \to 2$ scattering amplitudes using massive spinor helicity variables. We show that on-shell consistency conditions suffice to derive relations between the masses of the $W^{\pm}, Z$, the Weinberg angle and the couplings. This provides a completely on-shell description of the Higgs mechanism without any reference to the vacuum expectation value of the Higgs field. This is based on the following paper,

- Brad Bachu and Akshay Yelleshpur. On-Shell Electroweak Sector and the Higgs Mechanism. *JHEP*, 08:039, 2020 [11]

In Chapter 3, we show how the well known patterns of masses and interactions that arise from spontaneous symmetry breaking can be determined from an entirely on-shell perspective, that is, without reference to Lagrangians, gauge symmetries, or fields acquiring a vacuum expectation value. To do this, we review how consistent factorization of $2 \to 2$ tree level scattering can lead to the familiar structures of Yang-Mills theories, and extend this to find structures of Yukawa theories. Considering only spins-0, 1/2 and 1 particles, we construct all the allowed on-shell UV amplitudes under a symmetry group G, and consider all the possible IR amplitudes. By demanding that on-shell IR amplitudes match onto on-shell UV amplitudes in the high energy limit, we reproduce the Higgs mechanism and generate masses for spins-1/2 and 1, find that there is a subgroup $H \subseteq G$ in the IR, and other interesting relations. To highlight the results, we show the breaking pattern of the Standard Model $U(1)_{EM} \subset SU(2)_L \times U(1)_Y$, along with the generation of the masses and interactions of the particles. This is based on a paper to be submitted to JHEP.

In Chapter 4, we study a class of tree-level ansätze for $2 \to 2$ scalar and gauge boson amplitudes inspired by stringy UV completions. These amplitudes manifest Regge boundedness and are exponentially soft for fixed-angle high energy scattering, but unitarity in the form of positive expandability of massive residues is a nontrivial consistency condition. In particular, unitarity forces these ansätze to include graviton exchange. In the context of gauge boson scattering, we study gauge groups $SO(N)$ and $SU(N)$. In four dimensions, the bound on the rank of the gauge group is 24 for both groups, and occurs at the maximum value of the gauge coupling $g_{YM}^2 = \frac{2M_s^2}{M_P^2}$. In integer dimensions $5 \leq D \leq 10$, we find evidence that the maximum allowed allowed rank r of the gauge group agrees with the swampland conjecture $r < 26 - D$. The bound is surprisingly identical for both $SU(N)$ and $SO(N)$ in integer spacetime dimensions. We also study the electroweak sector of the standard model via $2 \to 2$ Higgs scattering and find interesting constraints relating standard model couplings, the putative string scale, and the Planck scale. This is based on the following paper,

- Brad Bachu and Aaron Hillman. Stringy Completions of the Standard Model from the Bottom Up. 12 2022

Chapter 2

On-Shell Electroweak Sector and the Higgs Mechanism

2.1 Introduction

Quantum fields, path integrals and Lagrangians have been a cornerstone of 20th century theoretical physics. They have been used to describe a variety of natural phenomena accurately. Yet, it is becoming increasingly apparent that these mathematical tools are both inefficient and insufficient. They obscure the presence of a deeper, underlying structure, particularly of scattering amplitudes in quantum field theories. The field of scattering amplitudes has undergone a paradigm shift in the past three decades. This was sparked by the discovery of the stunning simplicity of the tree-level gluon scattering amplitudes in [13, 14]. The simplicity of these amplitudes was revealed due to the use of helicity spinors, $(\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}})$ which correspond to the physical degrees of freedom of massless particles - helicity. The forbidding complexity of the Feynman diagram based calculation of tree level gluon scattering amplitudes is now understood to be an artifact of the unphysical degrees of freedom introduced by gauge redundancy. These unphysical degrees of freedom are necessary to package the physical degrees of freedom into local quantum fields in a manner consistent with Poincaré invariance [15]. The simplicity of these amplitudes fueled the development of a variety of "on-shell" techniques for computing scattering amplitudes involving massless particles. These methods do not rely on Feynman diagrams, do not suffer from gauge redundancies and do not invoke virtual particles. For an overview of these methods, see [5, 6, 7, 8, 9] and the references therein. However, most of this progress was limited to amplitudes involving only massless particles.

Since helicity spinors correspond to the physical degrees of freedom of massless particles, it is natural to attempt to find variables akin to these for massive particles. The physical degrees of freedom of massive particles correspond to the little group $SU(2)$ [16]. Some early generalizations can be found in [17, 18, 19, 20, 21, 22, 23, 24]. However, the little group covariance was not manifest in these generalizations until the introduction of spin-spinors (or massive spinor-helicity variables) in [10]. These variables $(\lambda^I_\alpha, \tilde{\lambda}^I_{\dot{\alpha}})$ which carry both little group indices and Lorentz indices, make the little group structure of amplitudes manifest. Information about all the $(2S+1)$ spin components of each particle is packaged into compact, manifestly Lorentz invariant expressions. Amplitudes written in terms of these variables are directly relevant to physics. This is in contrast to a Feynman diagram based computation which involves an intermediate object with Lorentz indices which must then be contracted with polarization tensors which carry the little group indices. For some interesting applications of these variables, including black holes, supersymmetric theories and double copy constructions, see [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35].

One of the biggest successes of path integrals and the Lagrangian formulation is the development of effective field theory and the Higgs mechanism. Recently, there has been a lot of focus towards the development of on-shell methods for effective field theory and an on-shell understanding of the structure of the Standard Model (SM) [36, 37, 38, 39, 40, 41]. It is worth pointing out that the aims of [36] are similar to ours but differs in the strategy employed. Specifically, the authors aim to derive the constraints of electroweak symmetry breaking by specifying the IR structure of the SM and imposing perturbative unitarity. In contrast, we will derive these constraints by specifying both the ultraviolet (UV) and infrared (IR) behaviour and demanding that the low energy theory has a smooth high energy limit. A completely on-shell description of the Higgs mechanism was outlined in [10] for the abelian and non-abelian gauge theories. The conventional understanding of the Higgs mechanism involves a scalar field acquiring a vacuum expectation value and vector bosons becoming massive by "eating" the goldstone modes arising from spontaneously broken symmetry. However, the on-shell description has no mention of scalar fields, potentials and vacuum expectation values. Nevertheless, it reproduces all of the same physics. Additionally, well known results like the Goldstone Boson equivalence theorem become trivial consequences of the high energy limits of our expressions. From an amplitudes perspective, it is more natural to think of the Higgs mechanism as a unification of the massless amplitudes in the UV into massive amplitudes in the IR. In this paper, we will focus on computing scattering amplitudes in the bosonic electroweak sector of the SM and describing the Higgs mechanism and electroweak symmetry breaking using a completely on-shell language.

The paper is structured as follows. We begin with a brief review of the little group, spin-spinors and their properties in Section [2.2]. We focus on constructing three particle amplitudes in the IR in Section [2.3.1] and the UV in Section [2.3.2]. In Section [2.3.4], we compute the high energy limits of the three particle amplitudes in the IR and demand that they are consistent with the three point amplitudes in the UV. This gives us the all the standard relations between the coupling constants, the masses of the Z and $W^{\pm}$ and the Weinberg angle θ_w. We also see the emergence of the custodial SO(3) symmetry in the limit in which the hypercharge coupling vanishes. Finally, in Section [2.4], we construct 4 point amplitudes in the IR by gluing together the three point amplitudes found before. We elucidate the details involved in the gluing process. We will also discover that demanding that these amplitudes have a well defined high energy limit imposes constraints on the structure of the theory.

2.2 Scattering amplitudes and the little group

2.2.1 Helicity spinors and spin-spinors

In this section, we briefly review some aspects of the on-shell approach to constructing scattering amplitudes. We will review the formalism of spin-spinors introduced in [10] whilst highlighting some features important for this paper. One particle states are irreducible representations of the Poincare' group. They are labeled by their momentum and a representation of the little group. If the particle is charged under any global symmetry group, appropriate labels must be appended to these. In (3+1) spacetime dimensions, the little groups for massless and massive particles are $SO(2)$ and $SO(3)$ respectively.

Representations of the massless little group, $SO(2) = U(1)$ can be specified by an integer corresponding to the helicity of the massless particle. A massless one particle state is thus specified by its momentum and helicity. Under a Lorentz transformation Λ,

$$|p, h, \sigma\rangle \rightarrow w^{-2h}|\Lambda p, h, \sigma\rangle \, , \tag{2.1}$$

where σ are labels of any global symmetry group and w has the same meaning as in [15] and [10]. It is useful to introduce elementary objects $\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}$ which transform under the little group as

$$\lambda_\alpha \to w^{-1}\lambda_\alpha \qquad \text{and} \qquad \tilde{\lambda}_{\dot{\alpha}} \to w\tilde{\lambda}_{\dot{\alpha}}\,. \tag{2.2}$$

We can use these objects to build representations with any value of h. The natural candidates for these elementary objects are the spinors which decompose the null momentum $p_{\alpha\dot{\alpha}} \equiv p_\mu \sigma^\mu_{\alpha\dot{\alpha}}$. We have

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \equiv |\lambda\rangle_\alpha [\tilde{\lambda}|_{\dot{\alpha}}\,. \tag{2.3}$$

Throughout the paper we will find it convenient to make use of the following notation,

$$\lambda_\alpha \equiv |\lambda\rangle \qquad \tilde{\lambda}_{\dot{\alpha}} \equiv [\tilde{\lambda}| \qquad \lambda^\alpha \equiv \langle\lambda| \qquad \tilde{\lambda}^{\dot{\alpha}} \equiv |\tilde{\lambda}]\,. \tag{2.4}$$

For any two null momenta p_1, p_2, we can form two Lorentz invariant combinations of these spinors,

$$\langle 12\rangle \equiv \epsilon^{\alpha\beta}(\lambda_1)_\beta(\lambda_2)_\alpha \qquad\qquad [12] \equiv \epsilon^{\dot{\alpha}\dot{\beta}}(\tilde{\lambda}_1)_{\dot{\alpha}}(\tilde{\lambda}_2)_{\dot{\beta}}\,. \tag{2.5}$$

The massive little group is $SO(3) = SU(2)$. Its representations are well known and can be specified by the value of the Casimir operator which is restricted to values $S(S+1)$, where S is defined as the spin of the particle. The spin S representation is $2S+1$ dimensional. A massive one-particle state of spin S thus transforms as a tensor of rank 2S under $SU(2)$.

$$|p, I_1, \ldots I_{2S},\, \sigma\rangle \to W_{I_1 J_1} \ldots W_{I_{2S} J_{2S}} |\Lambda p, J_1, \ldots J_{2S},\, \sigma\rangle\,. \tag{2.6}$$

The elementary objects in this case are the spinors of $SU(2)$, which are referred to as spin-spinors. These transform as

$$\boldsymbol{\lambda}_\alpha^I \to \left(W^{-1}\right)^I_{\ J} \boldsymbol{\lambda}_\alpha^J \qquad \tilde{\boldsymbol{\lambda}}_\alpha^I \to W^I_J \, \tilde{\boldsymbol{\lambda}}_\alpha^J \,. \tag{2.7}$$

Higher representations can be built by taking tensor products of these. Decomposing the rank 2 momentum, similar to eq.(2.3), yields the requisite spinors,

$$p_{\alpha\dot{\alpha}} = \epsilon_{JI} |\boldsymbol{\lambda}\rangle^I [\tilde{\boldsymbol{\lambda}}|^J = \epsilon_{JI} \boldsymbol{\lambda}_\alpha^I \tilde{\boldsymbol{\lambda}}_{\dot{\beta}}^J \,. \tag{2.8}$$

From this we have

$$\det(p) = \det(\epsilon) \, \det(\lambda) \, \det(\tilde{\lambda}) \,. \tag{2.9}$$

Using the fact that $\det(\epsilon) = 1$ and $\det(p) = p^2 = m^2$, we have $\det(\lambda) \, \det(\tilde{\lambda}) = m^2$. For the rest of the paper, we will set $\det(\lambda) = \det(\tilde{\lambda}) = m$[1]. We will find it convenient to suppress the little group indices on the spin-spinors. We do this according to the convention in eq.(A.2). Finally, we can construct Lorentz invariants out of spin-spinors corresponding to two massive momenta p_1, p_2 similar to eq.(2.5).

$$\langle \mathbf{12} \rangle^{IJ} \equiv \epsilon^{\alpha\beta} (\boldsymbol{\lambda}_1)^I_\beta (\boldsymbol{\lambda}_2)^J_\alpha \qquad\qquad [\mathbf{12}]^{IJ} \equiv \epsilon^{\dot{\alpha}\dot{\beta}} (\tilde{\boldsymbol{\lambda}}_1)^I_{\dot{\alpha}} (\tilde{\boldsymbol{\lambda}}_2)^J_{\dot{\beta}} \,. \tag{2.10}$$

2.2.2 Scattering amplitudes as little group tensors

Scattering amplitudes are defined as the overlap of in and out states. We have

$$\mathcal{M}(p_1, \rho_1 \ldots p_n, \rho_n) = {}_{\text{out}}\langle p_1, \rho_1, \ldots p_n, \rho_n | 0 \rangle_{\text{in}} \,, \tag{2.11}$$

[1]There is more freedom to set $\det \lambda = M$ and $\det \tilde{\lambda} = \tilde{M}$ such that $M\tilde{M} = m^2$, but for our purposes $M = \tilde{M} = m$ suffices.

where we are assuming that all particles are outgoing. $\rho = (h, \sigma)$ for massless particles (eq.(2.1)) and $\rho = (\{I_1, \ldots I_{2S}\}, \sigma)$ for massive ones (eq.(2.6)). Translation invariance allows us to pull out a delta function which imposes momentum conservation

$$\mathcal{M}(p_1, \rho_1 \ldots p_n, \rho_n) = \delta^4(p_1 + \ldots p_n) M(p_1, \rho_1, \ldots, p_n, \rho_n)\,. \tag{2.12}$$

Assuming that the asymptotic multi-particle states transform under Lorentz transformations as the tensor products of one-particle states, we have the following transformation law for the function $M(p_1, \rho_1, \ldots, p_n, \rho_n)$ under a Lorentz transformation Λ,

$$M(p_a, \rho_a) \to \prod_a \left(D_{\rho_a \rho'_a}(W)\right) M((\Lambda p)_a, \rho'_a)\,, \tag{2.13}$$

where $D_{\rho_a \rho'_a}(W) = \delta_{\sigma, \sigma'_a} \delta_{h_a, h'_a} w^{-2h_a}$ for massless particles and $D_{\rho_a \rho'_a}(W) = \delta_{\sigma, \sigma'_a} W^{I_1}_{I'_1} \ldots W^{I_{2S}}_{I'_{2S}}$ for massive ones. As an example, we display the transformation law for a 4-particle amplitude where particle 1 is massive with spin 1, particle 2 is massless with helicity 5/2, particle 3 is massless with helicity -2 and particle 4 is massive with spin 0.

$$\begin{aligned} &M^{\{I_1, I_2\}, \{5/2\}, \{-2\}, \{0\}}(p_1, p_2, p_3, p_4) \to \\ &\qquad (W_1)^{I_1}_{I'_1}\ (W_1)^{I_2}_{I'_2}\ w_2^{-5}\ w_3^4\ M^{\{I'_1, I'_2\}, \{5/2\}, \{-2\}, \{0\}}(p_1, p_2, p_3, p_4)\,. \end{aligned} \tag{2.14}$$

Thus, objects constructed from helicity spinors and spin spinors can correspond to scattering amplitudes only if they are Lorentz invariant and have the above transformation law under the little group. This imposes restrictions on the functional forms of objects that make up scattering amplitudes. Indeed, three point amplitudes involving all massless particles are completely fixed by this restriction. At three points, we have,

$$2p_1.p_2 = \langle 12\rangle[12] = 0\,, \qquad 2p_2.p_3 = \langle 23\rangle[23] = 0\,, \qquad 2p_3.p_1 = \langle 31\rangle[31] = 0\,. \quad (2.15)$$

We must choose either the λ or the $\tilde{\lambda}$ to be proportional to each other. The two solutions are the MHV configuration

$$\tilde{\lambda}_1 = \langle 23\rangle\tilde{\zeta} \qquad \tilde{\lambda}_2 = \langle 31\rangle\tilde{\zeta} \qquad \tilde{\lambda}_3 = \langle 12\rangle\tilde{\zeta}\,, \tag{2.16}$$

and the anti-MHV configuration

$$\lambda_1 = [23]\zeta \qquad \lambda_2 = [31]\zeta \qquad \lambda_3 = [12]\zeta\,. \tag{2.17}$$

Using these along with locality (which constrains the mass dimension of the momentum dependence), one can show that the three point amplitudes can only take the following form,

$$\begin{aligned} M^{h_1h_2h_3} &= g\,\langle 12\rangle^{h_1+h_2-h_3}\langle 23\rangle^{h_2+h_3-h_1}\langle 31\rangle^{h_3+h_1-h_2}\,, \qquad \text{if} \quad h_1+h_2+h_2 > 0 \\ &= \tilde{g}\,[12]^{h_3-h_1-h_2}[23]^{h_1-h_2-h_3}[31]^{h_2-h_3-h_1}\,, \qquad \text{if} \quad h_1+h_2+h_2 < 0\,. \end{aligned} \tag{2.18}$$

In cases involving one or more massive particles, Lorentz invariance and little group covariance are not sufficient to completely fix the amplitude. However, they narrow down the form of the amplitude to a finite number of terms. For an exhaustive analysis, we refer the reader to [10] and [42, 43]. In this paper, we will discuss only the amplitudes relevant to us.

2.2.3 The high energy limit of spin-spinors

When particle energies are much higher than their masses, it is intuitive to treat them as massless. We can formalize this by expanding the spin-spinors in a convenient basis in little group space,

$$
\begin{aligned}
\lambda_\alpha^I &= \lambda_\alpha \zeta^{-I} + \eta_\alpha \zeta^{+I} && (2.19)\\
&= \sqrt{E+p}\,\zeta_\alpha^+(p)\,\zeta^{-I}(k) + \sqrt{E-p}\,\zeta_\alpha^-(p)\,\zeta^{+I}(k)\\
\tilde{\lambda}_{\dot{\alpha}I} &= \tilde{\lambda}_{\dot{\alpha}}\zeta_I^+ - \tilde{\eta}_{\dot{\alpha}}\zeta_I^-\\
&= \sqrt{E+p}\,\tilde{\zeta}_\alpha^-(p)\,\zeta_I^+(k) - \sqrt{E-p}\,\tilde{\zeta}_{\dot{\alpha}}^+(p)\,\zeta_I^-(k)\,, && (2.20)
\end{aligned}
$$

where $\lambda, \tilde{\lambda}$ are the helicity spinors, $\zeta^{\pm I}$ are eigenstates of spin 1/2 along the momentum. We give explicit expressions for all objects involved are in Appendix [A.1]. Here, we just note that $\eta_\alpha, \tilde{\eta}_{\dot{\alpha}} \propto \sqrt{E-m} = m + \mathcal{O}(m^2)$. Taking the high energy limit corresponds to taking $m/E \to 0$. In this limit, the spin-spinors reduce to massless helicity ones. Finally, it should be pointed out that we must take special care while taking the high energy limit of 3-point amplitudes. Owing to the special three point kinematics, factors like $\langle \mathbf{12} \rangle$ or $[\mathbf{12}]$ can tend to zero in the high energy limit.

2.3 Three particle amplitudes

3-particle amplitudes are the fundamental building blocks of scattering amplitudes. In this paper, we are interested in analyzing the bosonic content of the standard model both in the UV and IR. The spectrum in the UV is comprised solely of massless particles. The three point amplitudes are completely determined by Poincare' invariance and little group scaling as outlined in Section [2.2]. The form of these amplitudes was given in eq.(3.23). We will use this formula to write down all the relevant 3-particle amplitudes in Section [2.3.2]. All the amplitudes in the UV obey the $SU(2)_L \times U(1)_Y$ symmetry.

The spectrum in the IR consists of massive particles and a single massless vector. 3-particle amplitudes involving massive particles are not completely fixed. They can have several contributing structures. In Section [2.3.1], we will write down all the relevant amplitudes. The amplitudes in the IR obey only a $U(1)_{\text{EM}}$ symmetry.

Finally, we will demand that the high energy limit of the IR amplitudes is consistent with the amplitudes in the UV. We will find that this consistency is possible only if the masses of particles in the IR are related in a specific way. These turn out to be the usual relations involving the Weinberg angle.

2.3.1 The IR

The spectrum in the IR consists of the following particles.

- Three massive spin 1 bosons (W^+, W^-, Z) which have masses $(m_{\scriptscriptstyle W}, m_{\scriptscriptstyle W}, m_{\scriptscriptstyle Z})$ and charges $(+, -, 0)$ respectively under a global symmetry group $U(1)_{EM}$. Note that W^+ and W^- are eigenstates of the $U(1)_{\text{EM}}$ generator. They must have equal mass as they are related by charge conjugation.

- One massless spin 1 boson, the photon, γ which is not charged under the $U(1)_{EM}$.

- One massive scalar, the higgs, h which is also uncharged under $U(1)_{EM}$.

The only symmetry of the IR is the $U(1)_{\text{EM}}$. We will now discuss all the relevant three point amplitudes in the IR. Owing to the existence of various identities amongst the spin-spinors, each amplitude can be written in a multitude of different ways. In many of the cases below, we have chosen particularly convenient ways of writing them.

Different forms of the three point amplitudes lead to different expressions for four point amplitudes. The difference between these are contact terms that can be fixed by imposing other constraints on the amplitude. While the form of the contact term will depend on the form of the three point amplitudes used, the final amplitude will be the same. We will elaborate on these comments in the appropriate places below. In the rest of the paper we will follow the convention that the lines in bold face in the diagrams correspond to massive particles while the lines which are not in bold face correspond to massless ones.

$\mathbf{W^+W^-Z}$

./Figures/WWZ.pdf

$$\frac{e_W}{m_W^2 m_Z}\left[\langle\mathbf{12}\rangle[\mathbf{12}]\langle\mathbf{3}|p_1 - p_2|\mathbf{3}] + \text{cyc.}\right] \tag{2.21}$$

This is a form of the three point amplitude that is chosen to suit our needs. It should be noted that it can be reduced to a combination of $<>$ and $[\,]$. As an example, consider the first term in the above equation which can be re-written as follows,

$$\langle\mathbf{12}\rangle[\mathbf{12}]\langle\mathbf{3}|p_1 - p_2|\mathbf{3}\rangle = 2\left(m_1[\mathbf{12}]\langle\mathbf{23}\rangle[\mathbf{31}] - m_2[\mathbf{12}][\mathbf{23}]\langle\mathbf{31}\rangle + m_3[\mathbf{12}]\langle\mathbf{23}\rangle\langle\mathbf{31}\rangle\right) \tag{2.22}$$

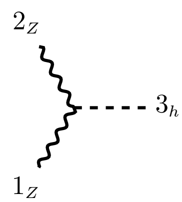
where we made use of the Schöuten identity $\langle\mathbf{12}\rangle\mathbf{3} + \langle\mathbf{23}\rangle\mathbf{1} + \langle\mathbf{31}\rangle\mathbf{2} = 0$.

$\mathbf{W^+W^-}\gamma$

$$\texttt{./Figures/WWA.pdf} = e\, x_{12}^{+} [\mathbf{12}]^2 \tag{2.23}$$

We discuss other forms of writing the same vertex in Appendix [A.2].

ZZh



$$= \frac{e_{HZZ}}{m_Z} \langle \mathbf{12} \rangle [\mathbf{12}] + \frac{\mathcal{N}_1}{m_Z} \left(\langle \mathbf{12} \rangle^2 + [\mathbf{12}]^2 \right) \tag{2.24}$$

$\mathbf{W^+W^-h}$

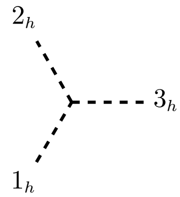
$$\texttt{./Figures/WWH.pdf} = \frac{e_{WWH}}{m_W} \langle \mathbf{12} \rangle [\mathbf{12}] + \frac{\mathcal{N}_2}{m_W} \left(\langle \mathbf{12} \rangle^2 + [\mathbf{12}]^2 \right) \tag{2.25}$$

We will set $\mathcal{N}_1 = \mathcal{N}_2 = 0$ in what follows since these terms do not have a well defined high energy limit as explained below eq.(2.36) [2]

---

hhh

[2] The factor of $\frac{\mathcal{N}_2}{m_W}$ or $\frac{\mathcal{N}_1}{m_Z}$ could be replaced by $\frac{\mathcal{N}_2}{\Lambda}$ or $\frac{\mathcal{N}_1}{\Lambda}$ where Λ is some energy scale much higher than m_W and m_Z. Such contributions might arise from loop level corrections and will be much smaller than the terms shown here.



$$= e_{HHH}\, m_h \tag{2.26}$$

2.3.2 The UV

The UV spectrum of the electroweak sector of the standard model consists of the following

- One massless spin-0 particle $\Phi = \{\phi_1, \phi_2, \phi_3, \phi_4\}$ with four real degrees of freedom in the fundamental representation of $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$.
- One massless spin-1 particle B with charge $\frac{1}{2}$ under a global $\mathrm{U}(1)_Y$ symmetry group.
- Three massless spin-1 particles (W_1, W_2, W_3), in the adjoint representation of $\mathrm{SU}(2)_L$. These are not charged under the group $U(1)_Y$. In order to facilitate easy comparison to the massive particles in the IR, we will work with particle states $W^{\pm} = \frac{1}{\sqrt{2}}(W^1 \pm iW^2)$ which are eigenstates of the $U(1)_{\mathrm{EM}}$ symmetry in the IR.

The electroweak sector has an $SU(2)_L \times U(1)_Y$ symmetry. The generators of these symmetries are related to the generators of $SO(4)$ listed in Appendix [A.5] as follows.

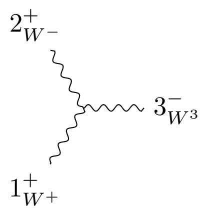
$$T^1 \equiv X^1 \qquad T^2 \equiv X^2 \qquad T^3 = X^3 \qquad T^B = Y^3\,. \tag{2.27}$$

The generator of $U(1)_{\mathrm{EM}}$, which we denote by Q, can be written as a linear combination of the generators of the UV

$$e\,Q = \alpha\, g\, T^3 + \beta\, g'\, T^B, \tag{2.28}$$

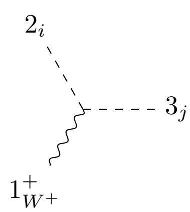
where e is $U(1)_{\mathrm{EM}}$ coupling. Since $T^{\pm} = \frac{1}{\sqrt{2}}(T^1 \pm iT^2)$ are eigenstates of Q, we are free to work with the states $W^{\pm}$ in the UV. We will now list all the relevant amplitudes in the UV. The superscripts on the particles indicate the corresponding helicities.

$\mathbf{W^+W^-W^3}$



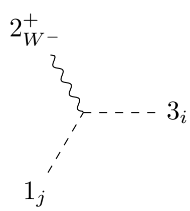
$$= g\,\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \tag{2.29}$$

$\mathbf{W^+\Phi\Phi}$



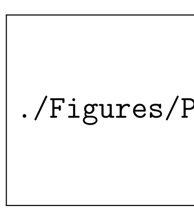
$$= g(T^-)_{ij}\frac{\langle 12\rangle\langle 31\rangle}{\langle 23\rangle} \tag{2.30}$$

$\mathbf{W^-\Phi\Phi}$



$$= g(T^+)_{ij}\frac{\langle 12\rangle\langle 23\rangle}{\langle 31\rangle} \tag{2.31}$$

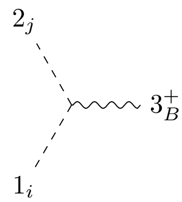
$\mathbf{W^3\Phi\Phi}$



$$= g(T^3)_{ij}\frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle} \tag{2.32}$$

These amplitudes must be proportional to a generator $T^{\{+,-,3\}}$ of the SU(2). For explicit forms of these generators, see Appendix [A.5].

BΦΦ



$$= g'(T^B)_{ij}\frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle} \tag{2.33}$$

The above list doesn't contain any WWB or WBB amplitudes since the W's are not charged under the $U(1)_Y$. Note that all the above amplitudes involve particles whose helicities, h_1, h_2, h_3 are such that $\sum h_i > 0$. The amplitudes with $\sum h_i < 0$ are given by flipping $<>\to [\,]$.

2.3.3 The HE Limit of the IR

All the amplitudes in the IR listed above have one or more factors of $\frac{1}{m}$. At first glance, this seems to suggest that they blow up in the UV and cannot be matched onto any 3-particle amplitude of massless particles. However, we will see that all these factors of inverse mass drop out when we take the special 3 particle kinematics into account and carefully take the high energy limit. Many of these high energy limits are worked out in [10] and [42]. For more details about these computations, please refer to Appendix[A.3]. We present the results here in a form compatible with our conventions. For each massive leg, in order to take the high energy limit we must first specify the component which we are interested in.

$\mathbf{W^+W^-Z}$

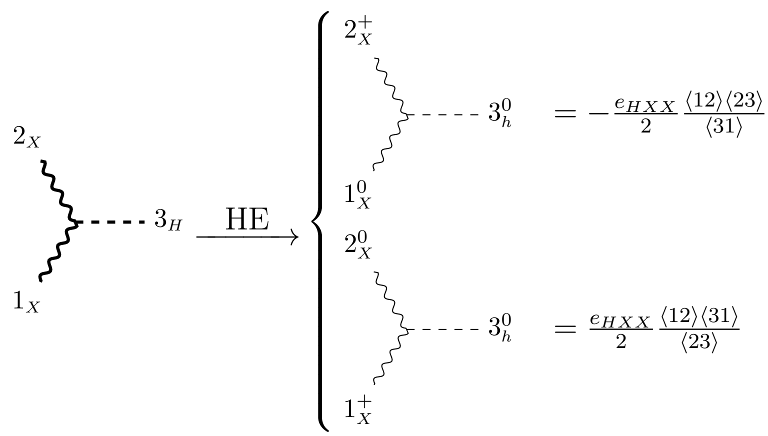
$$\text{./Figures/WWZHE.pdf} \longrightarrow \begin{cases} \left(2^{+}_{W^-},\ 1^{+}_{W^+},\ 3^{-}_{Z}\right) & = e_W \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \\ \left(2^{0}_{W^-},\ 1^{+}_{W^+},\ 3^{0}_{Z}\right) & = -e_W \frac{m_Z}{m_W} \frac{\langle 12\rangle\langle 31\rangle}{\langle 23\rangle} \\ \left(2^{-}_{W^-},\ 1^{0}_{W^+},\ 3^{0}_{Z}\right) & = -e_W \frac{m_Z}{m_W} \frac{\langle 12\rangle\langle 23\rangle}{\langle 31\rangle} \\ \left(2^{0}_{W^-},\ 1^{0}_{W^+},\ 3^{+}_{Z}\right) & = e_W \frac{m_Z^2 - 2m_W^2}{m_W^2} \frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle} \end{cases} \tag{2.34}$$

Amplitudes with one longitudinal mode and two transverse modes vanish in the high energy limit.

$\mathbf{W^+W^-}\gamma$

$$\text{./Figures/WWAHE.pdf} \longrightarrow \begin{cases} \left(2^{+}_{W^-},\ 1^{+}_{W^+},\ 3^{-}_{\gamma}\right) & = e \frac{\langle 23\rangle^3}{\langle 23\rangle\langle 31\rangle} \\ \left(2^{0}_{W^-},\ 1^{0}_{W^+},\ 3^{+}_{\gamma}\right) & = -2e \frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle} \end{cases} \tag{2.35}$$

$\mathbf{W^+W^-h}$ and ZZh

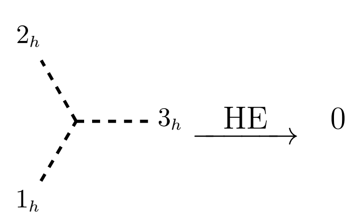


(2.36)

where $X = W, Z$. The high energy limit of amplitudes involving two transverse mode vanish. This is consistent with the fact that we have no $WW\Phi$ amplitudes in the UV. Here, we can also see that the HE limit involving two transverse modes would be ill defined if the terms in eq.(2.25) and eq.(2.24) involving $\mathcal{N}_1$ and $\mathcal{N}_2$ are included. This was the main motivation for setting $\mathcal{N}_1 = \mathcal{N}_2 = 0$. Furthermore, the high energy limit of the all longitudinal component of these amplitudes also vanish implying that there is no Φ^3 interaction in the UV. For more details, please see Appendix[A.3.2].

---

hhh



(2.37)

The hhh amplitude vanishes in the HE limit due to the explicit factor of m_h. This is again consistent with the fact that there is no Φ^3 amplitude in the UV.

2.3.4 UV-IR consistency

Thus far, we have specified the structure of the IR which consists of the interactions among the $W^{\pm}, Z, \gamma$ and h which preserve the $U(1)_{\text{EM}}$ symmetry and the structure of the UV which consists of the interactions among the W^a, B, Φ which preserve the $SU(2)_L \times U(1)_Y$ symmetry. We must now ensure that they are compatible with each other. We take the high energy limit of the IR amplitudes and demand that they are equal to the appropriate amplitudes in the UV. We refer to this process as 'UV-IR matching'. This imposes many constraints and determines the couplings in the IR in terms of those in the UV. Furthermore, it also imposes constraints on the masses of the particles in the IR. To begin with, we must relate the degrees of freedom in the IR to the ones in the UV. We assume that they are related by the following orthogonal transformation

$$\begin{pmatrix} W^+ \\ W^- \\ Z \\ \gamma \end{pmatrix} = \begin{pmatrix} \mathcal{O}_{++} & \mathcal{O}_{+-} & \mathcal{O}_{+3} & \mathcal{O}_{+B} \\ \mathcal{O}_{-+} & \mathcal{O}_{--} & \mathcal{O}_{-3} & \mathcal{O}_{-B} \\ \mathcal{O}_{Z+} & \mathcal{O}_{Z-} & \mathcal{O}_{Z3} & \mathcal{O}_{ZB} \\ \mathcal{O}_{\gamma+} & \mathcal{O}_{\gamma-} & \mathcal{O}_{\gamma3} & \mathcal{O}_{\gamma B} \end{pmatrix} \begin{pmatrix} W^+ \\ W^- \\ W^3 \\ B \end{pmatrix} \tag{2.38}$$

Clearly, we must have $\mathcal{O}_{+-} = \mathcal{O}_{+3} = \mathcal{O}_{+B} = \mathcal{O}_{-+} = \mathcal{O}_{-3} = \mathcal{O}_{-B} = 0$. This is a result of working with the same states in the UV and IR and of $U(1)_{\text{EM}}$ charge conservation. Orthogonality demands that the matrix be block diagonal, and so we have the simpler relation

$$\begin{pmatrix} Z \\ \gamma \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} W^3 \\ B \end{pmatrix} \tag{2.39}$$

for some unknown angle θ_w. All the massive particles in the IR have longitudinal components which must be generated by some linear combination of the scalars in the UV. We assume that

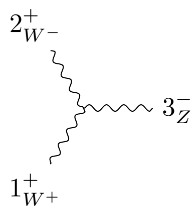
$$W^{+(0)} = U_{W^+i}\Phi_i \qquad W^{-(0)} = U_{W^-i}\Phi_i \qquad Z^{(0)} = U_{Zi}\Phi_i \tag{2.40}$$

The remaining linear combination of the components of Φ, $h = U_{hi}\Phi_i$ has an independent existence. Indeed, it is well known that its presence is crucial for the theory to have good UV behaviour. The high energy limit of each of the three point amplitudes in the IR must be equal to some combination of the amplitudes in the UV. This determines the masses in the IR in terms of the couplings in the UV. It also imposes some constraints on the couplings in the UV. All the constraints arising from eq.(2.34) - eq.(2.36) are determined below.

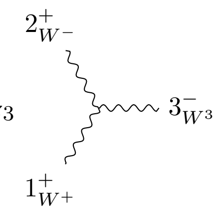
$\mathbf{W^+W^-Z}$

There are a total of 27 components to the W^+W^-Z amplitude corresponding to the $(+,-,0)$ spin component of each particle. Amplitudes with just one longitudinal mode all vanish in the high energy limit. This is consistent with the fact that there are no $WW\Phi$, $WB\Phi$, $BB\Phi$ amplitudes in the UV. The independent constraints arising from the remaining components are given below. Recall that the superscript on the particle is its helicity. These are also listed at the top of each diagram for particles 1, 2 and 3 respectively.

$$(++-)$$



$$\equiv \mathcal{O}_{Z3} \tag{2.41}$$

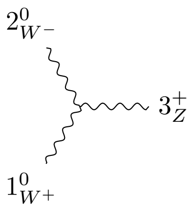


Using the expressions from eq.(2.34) and eq.(2.29), we get

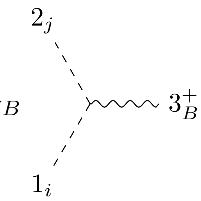
$$e_{\scriptscriptstyle W} \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} = g\,\mathcal{O}_{Z3}\,\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \implies e_{\scriptscriptstyle W} = g\cos\theta_w\,. \tag{2.42}$$

The absence of a W^+W^-B interaction in the UV means that there is no term proportional to $\mathcal{O}_{ZB}$ on the RHS.

$$(0\,0+)$$



$\equiv U_{W^+i}\, U_{W^-j}\, \mathcal{O}_{Z3}$ ./Figures/PhiPhiW3.pdf $+ U_{W^+i}\, U_{W^-j}\, \mathcal{O}_{ZB}$



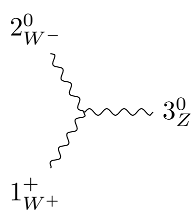
. (2.43)

Using eq.(2.34), eq.(2.32) and eq.(2.33) in the above gives,

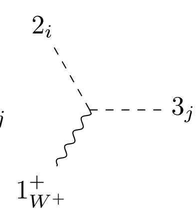
$$e_W \frac{m_Z^2 - 2m_W^2}{m_W^2} \frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle} = U_{W^+i}\, \left(g\, \mathcal{O}_{Z3}\, T^3_{ij} + g'\, \mathcal{O}_{ZB}\, T^B_{ij}\right) U_{W^-j} \frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle} \tag{2.44}$$

$$\Longrightarrow\; e_W \frac{m_Z^2 - 2m_W^2}{m_W^2} = U_{W^+i}\left(g \cos\,\theta_w\, T^3_{ij} - g' \sin\,\theta_w\, T^B_{ij}\right) U_{W^-j}\,. \tag{2.45}$$

$(+\,0\,0)$



$= U_{W^-i}\, U_{Zj}$



(2.46)

Again, eq.(2.34) and eq.(2.30) give

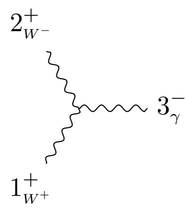
$$-e_W \frac{m_Z}{m_W} \frac{\langle 12\rangle\langle 31\rangle}{\langle 23\rangle} = g\, U_{W^-i}\, T^-_{ij}\, U_{Zj}\, \frac{\langle 12\rangle\langle 31\rangle}{\langle 23\rangle}$$

$$\Longrightarrow\; -e_W \frac{m_Z}{m_W} = g\, U_{W^-i}\, T^-_{ij}\, U_{Zj}\,. \tag{2.47}$$

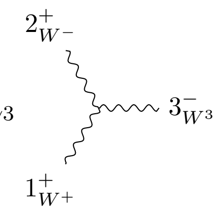
$\mathbf{W^+W^-}\gamma$

Since the photon is massless in the IR, the $W^+W^-\gamma$ amplitude only has 18 components. This leads to the following constraints.

$$(+\,+\,-)$$



$$\equiv \mathcal{O}_{\gamma 3}$$

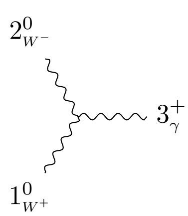


(2.48)

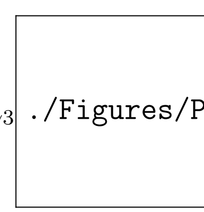
$$e_W \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} = g\,\mathcal{O}_{\gamma\,3}\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} \tag{2.49}$$

$$\implies e = g\sin\,\theta_w \tag{2.50}$$

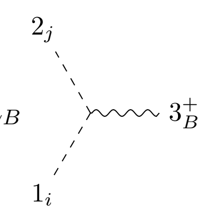
$$(0\,0\,+)$$



$$\equiv U_{W^+i}\,U_{W^-j}\,\mathcal{O}_{\gamma 3}$$



$$+\,U_{W^+i}\,U_{W^-j}\,\mathcal{O}_{\gamma B}$$



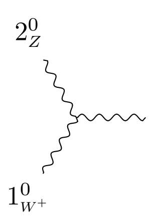
$$-2e\frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle} = U_{W^+i}\,\left(g\,\mathcal{O}_{\gamma 3}\,T^3_{ij}\;+\;g'\,\mathcal{O}_{\gamma B}\,T^B_{ij}\,\right)U_{W^-j}\frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle} \tag{2.51}$$

$$\implies\;-2e = U_{W^+i}\left(g\sin\,\theta_w T^3_{ij} + g'\cos\,\theta_w\,T^B_{ij}\right)U_{W^-j} \tag{2.52}$$

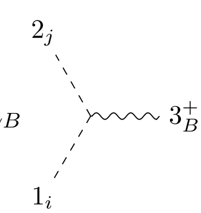
$\mathbf{W^+Z\gamma}$

Conservation of the $U(1)_{\mathrm{EM}}$ charge in the IR must be imposed. This is achieved by setting the $W^+Z\gamma$ amplitude to zero. A similar equation is given by setting the $W^-Z\gamma$ amplitude to zero.

$(0\,0\,+)$



$\equiv U_{W^+i}\,U_{Zj}\,\mathcal{O}_{\gamma 3}$ ./Figures/PhiPhiW3.pdf $+\,U_{W^+i}\,U_{Zj}\,\mathcal{O}_{\gamma B}$

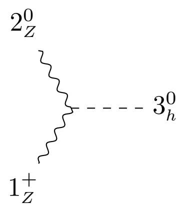


$$0 = U_{W^+i}\left(g\,\mathcal{O}_{\gamma 3}\,T^3_{ij}\;+\;g'\,\mathcal{O}_{\gamma B}\,T^B_{ij}\right)U_{Zj}\frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle} \tag{2.53}$$

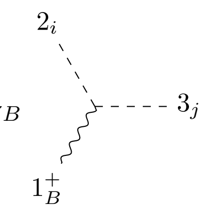
$$\implies\;0 = U_{W^+i}\left(g\sin\,\theta_w\,T^3_{ij}\;+\;g'\cos\,\theta_w\,T^B_{ij}\right)U_{Zj} \tag{2.54}$$

ZZh

$(+\,0\,0)$



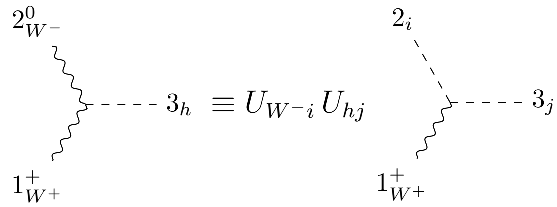
$\equiv U_{Zi}\, U_{Hj}\, \mathcal{O}_{Z3}$ ./Figures/W3PhiPhi.pdf $+\, U_{Zi}\, U_{Hj}\, \mathcal{O}_{ZB}$



$$
\begin{aligned}
e_{ZZH}\frac{\langle 12\rangle\langle 31\rangle}{\langle 23\rangle} &\equiv U_{Zi}\left(g\,\mathcal{O}_{Z3}\,T^3_{ij} \,+\, g'\;\mathcal{O}_{ZB}\,T^B_{ij}\right)U_{hj}\frac{\langle 12\rangle\langle 31\rangle}{\langle 23\rangle} \\
\Longrightarrow\; e_{ZZH} &= U_{Zi}\left(g\cos\,\theta_w\,T^3_{ij} - g'\sin\,\theta_w\,T^B_{ij}\right)U_{Hj} \qquad (2.55)
\end{aligned}
$$

$\mathbf{W^+W^-h}$

$(+\,0\,0)$



$$e_{WWH} \frac{\langle 12\rangle\langle 31\rangle}{\langle 23\rangle} \equiv g\, U_{W^- i}\, T^-_{ij}\, U_{hj} \frac{\langle 12\rangle\langle 31\rangle}{\langle 23\rangle}$$
$$\Longrightarrow\ e_{WWH} = g\, U_{W^- i} T^-_{ij}\, U_{hj} \tag{2.56}$$

As already highlighted, eq.(2.21) and eq.(2.50) yield

$$e = g \sin\theta_w \qquad \text{and} \qquad e_W = g\cos\theta_w \tag{2.57}$$

Further, the remaining set of constraints (2.42 - 2.56) can be solved by the ansatz

$$U_{W^+} = \frac{g}{m_W} T^- \cdot V \qquad U_{W^-} = \frac{g}{m_W} T^+ \cdot V$$
$$U_Z = \frac{1}{m_Z}\left(g\, \cos\theta_w\, T^3_{ij} - g'\, \sin\theta_w\, T^B_{ij}\right) \cdot V \tag{2.58}$$

where $V = \{v_1, v_2, v_3, v_4\}$. Note that despite the similarity of this equation with the usual Lagrangian based description of the Higgs mechanism, V does not have the interpretation as the vacuum expectation value of scalar field here. With the above anstaz, we find

$$v_1 = v_2 = 0\,, \qquad \tan\theta_w = \frac{g'}{g}$$
$$m_Z = g\sqrt{v_3^2 + v_4^2}\,, \qquad m_W = g\cos\theta_w \sqrt{v_3^2 + v_4^2}$$
$$\cos\theta_w = \frac{e_{WWH}}{e_{ZZH}} \tag{2.59}$$

We get the exact solutions as the Standard Model because we have restricted the form of the three point amplitude in eq.(2.21). Allowing for other structures will generalize the relation between m_z and m_W. Further note that when $g' \to 0$, we have $\theta_w = 0$ and $m_z = m_W$. Here, we see the emergence of the custodial $SU(2) = SO(3)$. The three particles $W^{\pm}, Z$ all have equal mass in the limit where the hypercharge coupling vanishes.

Note, the combination of generators in the anstaz (2.58) has the same form as the generators one would associate to the $W^{\pm}$ and Z gauge fields from a field theory perspective. The generator Q associated with the photon can be found in eq.(2.53). Substituting the values in eq.(2.59), we find the familiar $Q = T^3 + T^B$ such that $Q \cdot V = 0$.

2.4 Four point amplitudes in the Electroweak sector

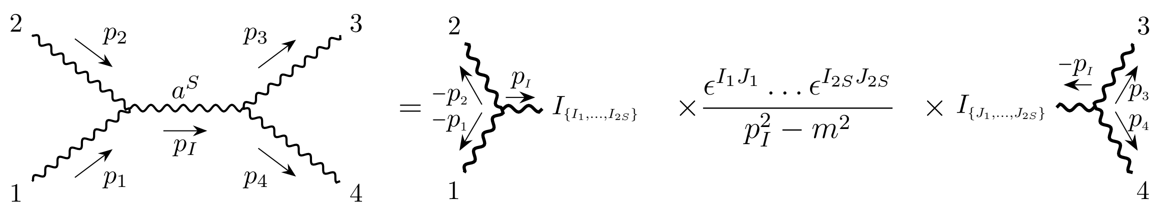
As we explained in the previous section, the structure of three point amplitudes is severely restricted by Poincare' invariance and little group constraints. The construction of four point amplitudes from the three point ones requires more work. Translation invariance is assured by the delta function in eq.(2.12) and Lorentz invariance is guaranteed if we build the amplitude from the invariants in eq.(2.5) and eq.(2.10). These amplitudes must be little group tensors of the appropriate rank (or in the case of massless particles have appropriate little group weights). This still leaves open a multitude of possibilities. But beyond three points, we have new constraints arising from causality. The amplitude must factorize consistently on all the poles, i.e. when some subset of the external momenta goes on shell, the residue on the corresponding pole must factorize into the product of appropriate lower point amplitudes. In particular, if the exchanged particle is massless, we must have

$$M \to \frac{M_L^{a\,h} M_R^{a\,-h}}{P^2}. \tag{2.60}$$

Here and below, a is an index for the intermediate particle. In cases where there are particles which may have identical helicity and mass, this index distinguishes between them. Similarly for the exchange of a particle with mass m and spin S, we have

$$M \to \frac{M_L^{a\{I_1,\ldots,I_{2S}\}} M^a_{R\{I_1,\ldots,I_{2S}\}}}{P^2 - m^2} = \frac{M^a_{L\{I_1,\ldots,I_{2S}\}} \epsilon^{I_1 J_1} \ldots \epsilon^{I_{2S} J_{2S}} M^a_{R\{J_1,\ldots,J_{2S}\}}}{P^2 - m^2}\,. \tag{2.61}$$

For the rest of this section, we will work with four particle amplitudes with particles 1 and 2 incoming and 3 and 4 outgoing. Diagrammatically,



(2.62)

At four points, there are only three possible factorization channels defined by

$$s = (p_1 + p_2)^2 \qquad u = (p_1 - p_3)^2 \qquad t = (p_1 - p_4)^2\,. \tag{2.63}$$

We must ensure that the four point amplitude factorizes into appropriate three point amplitudes on all these channels. We do this by computing the residues in the s, t and u channels and

$$\left(\frac{R_s}{s - m_s^2} + \frac{R_t}{t - m_t^2} + \frac{R_u}{u - m_u^2} \right)\,,$$

where m_s, m_t, m_u are the masses of the particles exchanged in the s, t, u channels respectively. This procedure will yield local amplitudes for almost all cases. Only in the case of the $W^+W^-\gamma$ amplitude, which has one massless particle and two particles of equal mass, this yields a four point amplitude with x factors which must be eliminated to get a local expression. We will go into more details in the corresponding section.

This represents only the factorizable part of the four point amplitude. We will find that these need to be supplemented by contact terms which depend on the specific form of the three point vertices. We can determine these by specifying the UV behaviour of the four point amplitudes. For the case of the Standard Model, we demand that they do not have any terms which grow with energy. This ensures that the theory doesn't violate unitarity. This lets us determine the required contact terms. The complete four point amplitude is then written as

$$M_4 = \left(\frac{R_s}{s - m_s^2} + \frac{R_t}{t - m_t^2} + \frac{R_u}{u - m_u^2}\right) + P(\lambda_i, \tilde{\lambda}_i)\,,$$

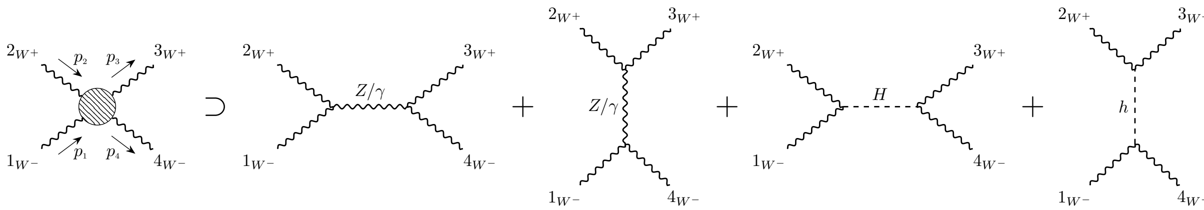
where P is a Lorentz invariant polynomial in the spin spinors corresponding to the four particles with the appropriate number of little group indices.

2.4.1 $W^+W^- \to W^+W^-$

In this section, we analyze the scattering of $W^+W^- \to W^+W^-$. For the sake of explicit calculations, we make the following choice for the 4 particle kinematics (with particles 1, 2, 3 and 4 corresponding to W^-, W^+, W^+, W^- respectively.

$$\begin{aligned} p_1 &= (E, 0, 0, p) & p_2 &= (E, 0, 0, -p) \\ p_3 &= (E, p\sin\theta, 0, p\cos\theta) & p_4 &= (E, -p\cos\theta, 0, -p\cos\theta) \end{aligned} \tag{2.64}$$

We will see that, based on the three point amplitudes listed in Section [2.3.1], the scattering can occur in the s and t channels via the exchange the Z, A or h.



(2.65)

s **- Channel**

- Z exchange

We can glue together two W^+W^-Z three point amplitudes and construct the residue in the s - channel.

$$(M_L^Z)^{\{I_1 I_2\}} = \frac{e_W}{m_W^2 m_Z} (\langle\mathbf{12}\rangle[\mathbf{12}]\langle\mathbf{I}|(-p_1) - (-p_2)|\mathbf{I}] + \text{cyc.})^{\{I_1 I_2\}} \tag{2.66}$$

$$(M_R^Z)_{\{I_1 I_2\}} = \frac{e_W}{m_W^2 m_Z} (-\langle\mathbf{34}\rangle[\mathbf{34}]\langle\mathbf{I}|p_3 - p_4|\mathbf{I}] + \text{cyc.})_{\{I_1 I_2\}} \tag{2.67}$$

Here $I = p_1 + p_2$ is the momentum exchanged and we have suppressed the little group indices corresponding to the external particles. The residue on the s - channel is

$$R_s^Z = (M_L^Z)^{\{I_1 I_2\}} (M_R^Z)_{\{I_1 I_2\}}$$

Evaluating this expression yields

$$R_s^Z = \frac{e_W^2}{m_W^4}\Bigg\{ 2\,\langle\mathbf{12}\rangle\,[\mathbf{12}]\,\langle\mathbf{34}\rangle\,[\mathbf{34}]\,(p_1-p_2).(p_3-p_4) \tag{2.68}$$

$$+\,4\Big(\langle\mathbf{42}\rangle\,[\mathbf{24}]\,\langle\mathbf{1}|p_2|\mathbf{1}]\,\langle\mathbf{3}|p_4|\mathbf{3}]+\langle\mathbf{31}\rangle\,[\mathbf{13}]\,\langle\mathbf{2}|p_1|\mathbf{2}]\,\langle\mathbf{4}|p_3|\mathbf{4}]-(1\leftrightarrow 2)\Big)$$

$$+\,2\,\Big(\langle\mathbf{12}\rangle[\mathbf{12}]\,\langle\mathbf{4}|p_3|\mathbf{4}]\,\langle\mathbf{3}|p_1-p_2|\mathbf{3}]\;+\;\langle\mathbf{12}\rangle\,[\mathbf{12}]\,\langle\mathbf{3}|p_4|\mathbf{3}]\,\langle\mathbf{4}|p_1-p_2|\mathbf{4}]$$

$$-\,(1,2\leftrightarrow 3,4)\Big)\Bigg\}$$

The full details of the calculation are presented in Appendix [A.4].

- Photon exchange

This corresponds to gluing together the two $W^+W^-\gamma$ vertices. There are two possibilities

$$M_L^- = \frac{e}{m_W}x_{12}^-\,\langle\mathbf{12}\rangle^2 \qquad M_R^+ = \frac{e}{m_W}x_{34}^+\,[\mathbf{34}]^2 \tag{2.69}$$

$$M_L^+ = \frac{e}{m_W}x_{12}^+\,[\mathbf{12}]^2 \qquad M_R^- = \frac{e}{m_W}x_{34}^-\,\langle\mathbf{34}\rangle^2$$

where the superscripts indicate the helicity of the photon. Note that the definition of x-factors differs slightly from Appendix [A.2] due to the fact that p_1 and p_2 are now incoming momenta. The appropriate definitions are

$$\frac{(-p_1+p_2)_{\alpha\dot\alpha}}{2m}\lambda_I^\alpha = x_{12}^+\,\tilde\lambda_{I\dot\alpha} \qquad \frac{(p_3-p_4)_{\alpha\dot\alpha}}{2m}\lambda_I^\alpha = x_{34}^+\,(-\tilde\lambda_{I\dot\alpha})$$

$$\frac{(-p_1+p_2)_{\alpha\dot\alpha}}{2m}\tilde\lambda_I^{\dot\alpha} = x_{12}^-\,\lambda_{I\alpha} \qquad \frac{(p_3-p_4)_{\alpha\dot\alpha}}{2m}(-\tilde\lambda_I^{\dot\alpha}) = x_{34}^-\,\lambda_{I\alpha}$$

The extra minus sign that accompanies $\tilde\lambda_{I\dot\alpha}$ in the equations defining $x_{34}^\pm$ is because the momentum I is incoming. The residue corresponding to the photon exchange is a sum over both the possibilities in eq.(2.69).

$$R_s^A = \frac{e^2}{m_W^2}\left(x_{12}^- x_{34}^+ \langle\mathbf{12}\rangle^2[\mathbf{34}]^2 + x_{12}^+ x_{34}^- [\mathbf{12}]^2\langle\mathbf{34}\rangle^2\right) \tag{2.70}$$

We must now eliminate the x - factors in order to obtain a local expression for this residue. There are multiple ways to achieve this and they generally result in different expressions for the residue. It is important to emphasize that while these forms are precisely equal on the factorization channel, they all lead to different expressions away from the pole. Since the physical amplitude must be the same, they yield different contact terms. The complete details of the calculation are delegated to Appendix [A.4]. Here, we present two different expressions for the residue on the $s-$channel.

$$\begin{aligned} R_s^\gamma = \frac{e^2}{2m_W^4}\Bigg\{ & (p_1-p_2).(p_3-p_4)\,\langle\mathbf{12}\rangle\,[\mathbf{12}]\,\langle\mathbf{34}\rangle\,[\mathbf{34}] \qquad (2.71)\\ & + \Bigg(\langle\mathbf{12}\rangle[\mathbf{12}]\Big[\langle\mathbf{3}|\,(p_1-p_2)\,(p_1+p_2)\,|\mathbf{4}\rangle\,[\mathbf{34}] - \langle\mathbf{34}\rangle\,[\mathbf{3}|\,(p_1+p_2)\,(p_1-p_2)\,|\mathbf{4}]\Big] \\ & - \langle\mathbf{1}|\,p_1+p_2\,|\mathbf{4}]\,[\mathbf{3}|\,p_1+p_2\,|\mathbf{2}\rangle\,[\mathbf{12}]\,\langle\mathbf{34}\rangle + (1,\,2 \leftrightarrow 3,4)\Bigg)\Bigg\} \end{aligned}$$

where $(1,2 \leftrightarrow 3,4)$ means $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$ simultaneously. This expression can be manipulated to look identical to eq.(2.68). This requires the use of the following Schöuten identities

$$\begin{aligned} & \langle\mathbf{3}|\,(p_1-p_2)\,I\,|\mathbf{4}\rangle\,[\mathbf{34}] - \langle\mathbf{34}\rangle\,[\mathbf{3}|\,I\,(p_1-p_2)\,|\mathbf{4}] \\ & = 2\Big(\langle\mathbf{4}|\,p_3\,|\mathbf{4}]\,\langle\mathbf{3}|\,(p_1-p_2)\,|\mathbf{3}] - (3\leftrightarrow 4)\Big)\,, \end{aligned} \tag{2.72}$$

and

$$\begin{aligned}&\langle\mathbf{1}|\,I\,|\mathbf{4}]\,[\mathbf{3}|\,I\,|\mathbf{2}\rangle\,[\mathbf{12}]\,\langle\mathbf{34}\rangle+(1,\,2\leftrightarrow 3,4)\\&=4\Big(\langle\mathbf{42}\rangle\,[\mathbf{24}]\,\langle\mathbf{1}|\,p_2\,|\mathbf{1}]\,\langle\mathbf{3}|\,p_4\,|\mathbf{3}]\;+\;\langle\mathbf{31}\rangle\,[\mathbf{13}]\,\langle\mathbf{2}|\,p_1\,|\mathbf{2}]\,\langle\mathbf{4}|\,p_3\,|\mathbf{4}]\;-\;(1\leftrightarrow 2)\Big),\end{aligned} \tag{2.73}$$

where $I=p_1+p_2$. These identities are true only on the factorization channel $I^2=0$ on which we can write $R_s^\gamma=R_s^Z\,(m_Z=0)$. This is not true away from the factorization channel. Consequently the contact terms that must be added to achieve the correct UV behaviour differ. This explicitly demonstrates the dependence of contact terms on the specific form of the three point amplitudes.

- Higgs exchange

This is the simplest to compute. We just glue together the following amplitudes.

$$M_L=\frac{e_{WWH}}{m_W}\langle\mathbf{12}\rangle[\mathbf{12}]\qquad M_R=\frac{e_{WWH}}{m_W}\langle\mathbf{34}\rangle[\mathbf{34}]\,, \tag{2.74}$$

which directly yields

$$R_s^h=\frac{e_{WWH}^2}{m_W^2}\langle\mathbf{12}\rangle[\mathbf{12}]\langle\mathbf{34}\rangle[\mathbf{34}]\,. \tag{2.75}$$

The complete contribution of the $s-$ channel is

$$M_s=\left(\frac{R_s^Z}{s-m_Z^2}+\frac{R_s^\gamma}{s}+\frac{R_s^h}{s-m_h^2}\right)\,.$$

t - channel

The computation of the t - channel residues is very similar to that of the s - channel. In fact, we can obtain them from the $s-$ channel ones by the replacement $p_2\leftrightarrow -p_4$. The results are presented below with $I=p_1-p_4$.

- Z exchange

$$R_t^Z = \frac{e_W^2}{m_W^4}\Bigg\{2\langle\mathbf{14}\rangle[\mathbf{14}]\langle\mathbf{32}\rangle[\mathbf{32}](p_1+p_4).(p_3+p_2) \tag{2.76}$$
$$+4\Big(\langle\mathbf{31}\rangle[\mathbf{13}]\langle\mathbf{4}|p_1|\mathbf{4}]\langle\mathbf{2}|p_3|\mathbf{2}] - \langle\mathbf{42}\rangle[\mathbf{24}]\langle\mathbf{1}|p_4|\mathbf{1}]\langle\mathbf{3}|p_2|\mathbf{3}] - (1\leftrightarrow 4)\Big)$$
$$+2\Big(\langle\mathbf{12}\rangle[\mathbf{14}]\langle\mathbf{2}|p_3|\mathbf{2}]\langle\mathbf{3}|p_1+p_2|\mathbf{3}] + \langle\mathbf{14}\rangle[\mathbf{14}]\langle\mathbf{3}|p_2|\mathbf{3}]\langle\mathbf{2}|p_1+p_4|\mathbf{2}] - (1\leftrightarrow 4)\Big)\Bigg\}$$

- Photon exchange

The residue on the $t-$ channel resulting from gluing together two $W^+W^-\gamma$ amplitudes is

$$R_t^\gamma = \frac{e^2}{2m_W^4}\Bigg\{-(p_1+p_4).(p_3+p_2)\langle\mathbf{14}\rangle[\mathbf{14}]\langle\mathbf{32}\rangle[\mathbf{32}] \tag{2.77}$$
$$+\Big(\langle\mathbf{14}\rangle[\mathbf{41}]\Big[\langle\mathbf{3}|(p_1+p_4)(p_1-p_4)\mathbf{2}\rangle[\mathbf{32}] - \langle\mathbf{32}\rangle[\mathbf{3}|(p_1-p_4)(p_1+p_4)|\mathbf{2}]\Big]$$
$$+\langle\mathbf{1}|p_1-p_4|\mathbf{2}][\mathbf{3}|p_1-p_4|\mathbf{4}\rangle[\mathbf{14}]\langle\mathbf{32}\rangle + (1,\,4\leftrightarrow 3,2)\Big)\Bigg\}$$

- Higgs exchange

$$R_t^h = \frac{e_{WWH}^2}{m_W^2}\langle\mathbf{14}\rangle[\mathbf{14}]\langle\mathbf{23}\rangle[\mathbf{23}] \tag{2.78}$$

The total contribution from the $t-$channel is

$$M_t = \left(\frac{R_t^Z}{t-m_Z^2} + \frac{R_t^\gamma}{t} + \frac{R_t^h}{t-m_h^2}\right)$$

Contact terms

The quantity $M \equiv M_s + M_t$ has been constructed to have the correct factorization properties. As explained before, the behaviour away from the factorization channels depends on the specific forms of the three point amplitudes. We can impose further constraints on the amplitude to fix it completely. It is evident that the high energy limit of the amplitude is ill defined due to the presence of the $\frac{1}{m_W^4}$ poles which leads to amplitudes which grow with energy as E^4. This violates perturbative unitarity. If we insist that the theory has a well defined high energy limit, we must add contact terms (which by definition have 0 residue on the factorization poles) to cancel this E^4 growth [3]. The form of the contact terms can be deduced by figuring out which components of the amplitude grow in the UV. Plugging in the 4-particle kinematics in 2.64, we find that only the all longitudinal component grows as E^4,

$$M \to \frac{4E^4}{m_W^4}\left(e^2+e_W^2\right)\left(-5-12\cos\theta+\cos 2\theta\right) . \tag{2.79}$$

The following contact term serves to kill these high energy growths

$$c_{WWWW} = \frac{e^2+e_W^2}{m_W^4}\left(-\langle\mathbf{12}\rangle[\mathbf{12}]\langle\mathbf{34}\rangle[\mathbf{34}]+2\langle\mathbf{13}\rangle[\mathbf{13}]\langle\mathbf{24}\rangle[\mathbf{24}]-\langle\mathbf{14}\rangle[\mathbf{14}]\langle\mathbf{23}\rangle[\mathbf{23}]\right) \tag{2.80}$$

Adding these contact terms, we find that the amplitude still grows as E^2/m_W^2. Demanding that the coefficient of this growing term vanishes enforces $e_{WWH}^2 = 2(e^2+e_W^2)$.

2.4.2 $W^+Z \to W^+Z$

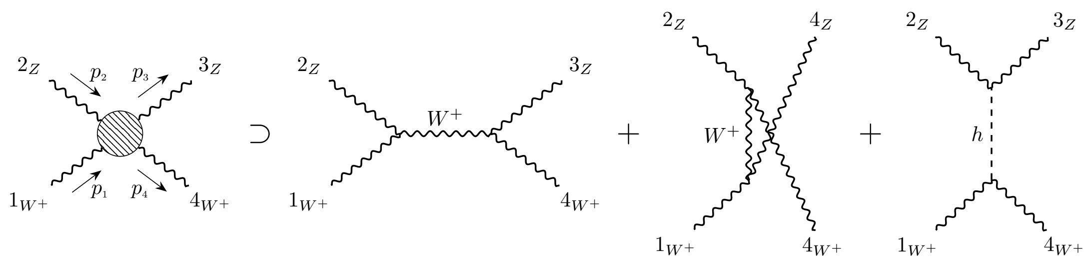
The 4 particle kinematics appropriate to this situation is

$$\begin{aligned} p_1 &= (E_1, 0, 0, p) & p_2 &= (E_2, 0, 0, -p) \\ p_3 &= (E_2, p\sin\theta, 0, p\cos\theta) & p_4 &= (E_1, -p\sin\theta, 0, -p\cos\theta) \end{aligned} \tag{2.81}$$

This configuration automatically satisfies momentum conservation. We can rewrite E_2 in terms of E_1 by using the on-shell constraint as $E_2 = \sqrt{E_1^2 - m_W^2 + m_Z^2}$. We can build this amplitude by gluing together two W^+W^-Z amplitudes in two ways and

[3]We thank the authors of [36] for pointing out to us that this contact term can also be derived by UV-IR matching as in Section[2.3.4]

by gluing a W^+W^-h and a ZZh amplitude.



(2.82)

We present the final expressions below. The calculations are very similar to those involved in $W^+W^- \to W^+W^-$.

- $s-$channel W - exchange

$$
\begin{aligned}
R_s^W = \frac{e_W}{m_W^4 m_Z^2}\Bigg\{ & 2(m_W^2 - m_Z^2)^2 \langle \mathbf{12}\rangle[\mathbf{12}]\langle\mathbf{34}\rangle[\mathbf{34}] \\
& + m_W^2 \Bigg(2\langle\mathbf{12}\rangle[\mathbf{12}]\langle\mathbf{34}\rangle[\mathbf{34}](p_1 - p_2).(p_3 - p_4) \\
& + 4\Big[\langle\mathbf{42}\rangle[\mathbf{24}]\langle\mathbf{1}|p_2|\mathbf{1}]\langle\mathbf{3}|p_4|\mathbf{3}] - \langle\mathbf{32}\rangle[\mathbf{23}]\langle\mathbf{1}|p_2|\mathbf{1}]\langle\mathbf{4}|p_3|\mathbf{4}] + (1,3\leftrightarrow 2,4)\Big] \\
& + 2\Big[\langle\mathbf{12}\rangle[\mathbf{12}]\Big(\langle\mathbf{4}|p_3|\mathbf{4}]\langle\mathbf{3}|p_1 - p_2|\mathbf{3}] - (3\leftrightarrow 4)\Big) + (1,2\leftrightarrow 3,4)\Big]\Bigg)\Bigg\}
\end{aligned}
\tag{2.83}
$$

- $u-$channel W - exchange

$$
\begin{aligned}
R_u^W = \frac{e_W}{m_W^4 m_Z^2}\Bigg\{ & 2(m_W^2 - m_Z^2)^2 \langle \mathbf{13}\rangle[\mathbf{13}]\langle\mathbf{24}\rangle[\mathbf{24}] \\
& + m_W^2 \Bigg(-2\langle\mathbf{13}\rangle[\mathbf{13}]\langle\mathbf{24}\rangle[\mathbf{24}](p_1 + p_3).(p_2 + p_4) \\
& + 4\Big[-\langle\mathbf{43}\rangle[\mathbf{34}]\langle\mathbf{1}|p_3|\mathbf{1}]\langle\mathbf{2}|p_4|\mathbf{2}] - \langle\mathbf{32}\rangle[\mathbf{23}]\langle\mathbf{1}|p_3|\mathbf{1}]\langle\mathbf{4}|p_2|\mathbf{4}] + (1,2\leftrightarrow 3,4)\Big] \\
& + 2\Big[-\langle\mathbf{13}\rangle[\mathbf{13}]\Big(\langle\mathbf{4}|p_2|\mathbf{4}]\langle\mathbf{2}|p_1 + p_3|\mathbf{3}] + (2\leftrightarrow 4)\Big) + (1,3\leftrightarrow 2,4)\Big]\Bigg)\Bigg\}
\end{aligned}
\tag{2.84}
$$

- $t-$channel Higgs - exchange

$$R_t^h = \frac{e_{WWH}}{m_W}\frac{e_{ZZH}}{m_Z}\langle \mathbf{23}\,\rangle[\mathbf{23}]\,\langle \mathbf{14}\,\rangle[\mathbf{14}]\,. \tag{2.85}$$

- Contact terms

We are again in the familiar situation where the quantity

$$\left(\frac{R_s^W}{s-m_W^2}+\frac{R_u^W}{u-m_W^2}+\frac{R_t^h}{t-m_h^2}\right)\,,$$

factorizes correctly on all the factorization channels. However, the all longitudinal component again grows with energy as can be seen by evaluating this using the kinematics in eq.(2.81). We find that the following contact term is needed to fix this and have a well behaved theory in the UV,

$$c_{WZWZ} = \frac{e_W^2}{m_W^4}(\langle\mathbf{12}\rangle\,[\mathbf{12}]\,\langle\mathbf{34}\rangle\,[\mathbf{34}] + \langle\mathbf{23}\rangle\,[\mathbf{23}]\,\langle\mathbf{14}\rangle\,[\mathbf{14}] - \mathbf{2}\langle\mathbf{24}\rangle\,[\mathbf{24}]\,\langle\mathbf{13}\rangle\,[\mathbf{13}])\,. \tag{2.86}$$

Furthermore, to kill growth at $\mathcal{O}(E^2)$, we must also have $e_{WWH}\,e_{ZZH}\,m_W^3 = e_W^2\,m_Z^3$.

2.4.3 $W^+W^- \to Zh$

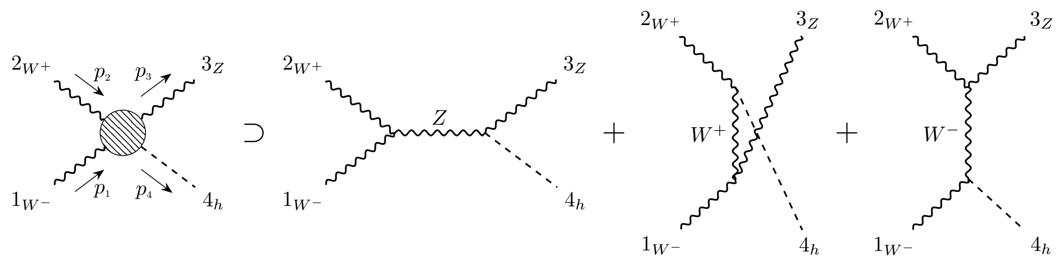
We next consider the scattering $W^+W^- \to Zh$ with the following kinematics

$$\begin{aligned} p_1^\mu &= (E_1, 0, 0, p_1) & p_2^\mu &= (E_1, 0, 0, -p_1) \\ p_3^\mu &= (E_3, p_2\,\sin\,\theta, 0, p_2\,\cos\,\theta) & p_4^\mu &= (E_4, -p_2\,\sin\,\theta, 0, -p_2\,\cos\,\theta) \end{aligned} \tag{2.87}$$

Using on-shell constraints, we can eliminate p_1, p_2, E_3, E_4 in favor of E_1,

$$p_1 = \sqrt{E_1^2 - m_W^2} \quad p_2 = \sqrt{E_2^2 - m_Z^2} \quad E_3 = \frac{m_Z^2 - m_h^2 + 4E_1^2}{4E_1}, E_4 = \frac{4E_1^2 - m_Z^2 + m_h^2}{4E_1}\,.$$

We can build this amplitude by gluing together (W^+W^-Z, Zhh) on the $s-$channel and by gluing together (W^+W^-h, ZZh) in the u and t channels as shown



(2.88)

Using the familiar procedure, we get

- $s-$ channel Z exchange

$$R_s^Z = \frac{e_{ZZH}\, e_W}{m_W^2}\left(-\langle\mathbf{12}\rangle\,[\mathbf{12}]\,\langle\mathbf{3}|p_1 - p_2|\mathbf{3}] - 2\langle\mathbf{23}\rangle\,[\mathbf{23}]\,\langle\mathbf{1}|p_2|\mathbf{1}] + 2\langle\mathbf{13}\rangle\,[\mathbf{13}]\langle\,\mathbf{2}|p_1|\mathbf{2}]\right) \tag{2.89}$$

- $u-$ channel W exchange

$$R_u^W = \frac{e_W\ e_{WWH}}{m_W m_Z}\left(-\langle\mathbf{13}\rangle\,[\mathbf{13}]\langle\mathbf{2}|p_1 + p_3|\mathbf{2}] + 2\langle\mathbf{23}\rangle\,[\mathbf{23}]\langle\mathbf{1}|p_3|\mathbf{1}] - 2\langle\mathbf{12}\rangle\,[\mathbf{12}]\langle\mathbf{3}|p_1|\mathbf{3}]\right) \tag{2.90}$$

- $t-$ channel W exchange

$$R_t^W = \frac{e_W\ e_{WWH}}{m_W m_Z}\left(-\langle\mathbf{23}\rangle\,[\mathbf{23}]\,\langle\mathbf{1}|p_2 + p_3|\mathbf{1}] + 2\langle\mathbf{13}\rangle\,[\mathbf{13}]\langle\mathbf{2}|p_3|\mathbf{2}] - 2\langle\mathbf{12}\rangle\,[\mathbf{12}]\,\langle\mathbf{3}|p_2|\mathbf{3}]\right) \tag{2.91}$$

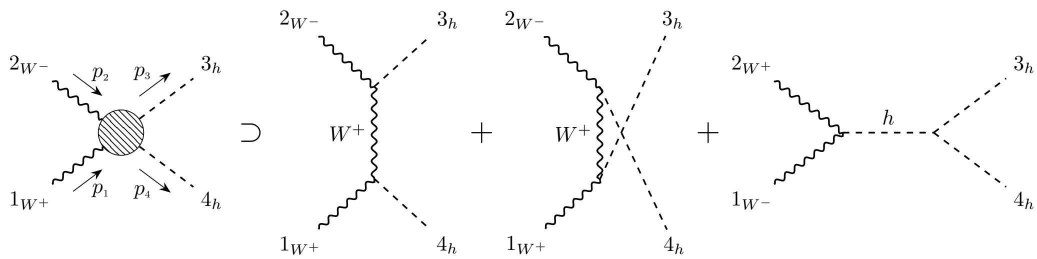
- Contact terms

In this case, the component of the amplitude with W^+, W^-, Z all being longitudinal grows with energy. However, there are no possible contact terms that are compatible with Lorentz invariance and the little group. The vanishing of the growing term imposes a constraint on the couplings e_{WWH}, e_{ZZH},

$$\frac{e_{ZZH}}{e_{WWH}} = \frac{m_W}{m_Z} \tag{2.92}$$

2.4.4 $W^+W^- \to hh$

To compute this amplitude, we can glue together two W^+W^-h amplitudes in the t and u channels.



(2.93)

- t-channel W exchange

$$R_t^W = \frac{e_{WWH}^2}{m_W^2}\left(-m_W^2 \langle \mathbf{12}\rangle\,[\mathbf{12}] - \frac{1}{2}\langle \mathbf{1}|\,p_1 - p_4|\mathbf{1}]\,[\mathbf{2}|p_1 - p_4\,|\mathbf{2}\rangle\right) \tag{2.94}$$

- $u-$ channel W exchange

$$R_u^W = \frac{e_{WWH}^2}{m_W^2}\left(-m_W^2 \langle \mathbf{12}\rangle\,[\mathbf{12}] - \frac{1}{2}\langle \mathbf{1}|\,p_1 - p_3|\mathbf{1}]\,[\mathbf{2}|p_1 - p_3\,|\mathbf{2}\rangle\right) \tag{2.95}$$

- $s-$ channel h exchange

$$R_s^h = \frac{e_{WWH}\,e_{HHH}\,m_H}{m_W}\langle \mathbf{12}\rangle\,[\mathbf{12}] \tag{2.96}$$

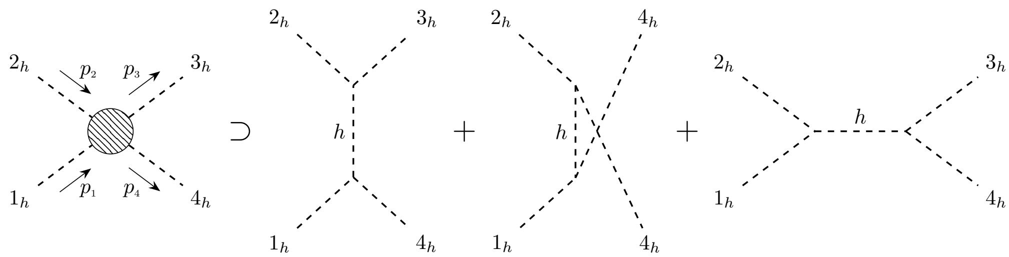
- Contact terms

The contact term necessary to kill the $\mathcal{O}(E^2)$ growth is

$$c_{WWHH} = \frac{e_{WWH}^2}{2\,m_W^2}\langle \mathbf{12}\rangle\,[\mathbf{21}] \tag{2.97}$$

2.4.5 $hh \to hh$

The only three point amplitudes which contribute to this are the hhh vertices eq.(2.26). As opposed to the usual, Lagrangian based approach to the Higgs mechanism, where we discover the new triple higgs vertex in the IR, here we must include its contribution simply because it is non zero and contributes to this scattering process.



(2.98)

The complete amplitude is

$$M_{hhhh} = e^2_{HHH} m_h^2 \left(\frac{1}{s - m_h^2} + \frac{1}{t - m_h^2} + \frac{1}{u - m_h^2} \right) + \lambda \tag{2.99}$$

where λ is a contact term.

2.5 Conclusions and Outlook

We have presented a completely on-shell description of the higgs mechanism within the Standard Model. We see that all the physics is reproduced by demanding consistent factorization, correct ultraviolet behaviour and consistency of the UV and IR. The precise relations between the masses of the $W^\pm, Z$ and θ_w depend on the structures that have been included in the three point W^+W^-Z amplitude. Our choice of the three point amplitude in eq.(2.21) ensured that we reproduced the usual result. We have constructed four particle, tree-level amplitudes from three particle amplitudes. The construction of higher point amplitudes and extensions to loop amplitudes are the obvious next questions. While the construction of higher point, tree level amplitudes is in principle, straightforward, the computation of loop amplitudes is more challenging. In particular, an on-shell version of the renormalization group is completely unknown and is a open problem. We have also restricted the particle content of the scalar sector to a single, real scalar transforming under an $SO(4)$ global symmetry. We have studied the Higgs mechanism for $SU(2)_L \times U(1)_Y$ breaking to $U(1)_{\text{EM}}$, relevant to electroweak symmetry breaking. It would be interesting to extend this analysis to completely general theories.

This work is a preliminary step in connecting modern methods in scattering amplitudes to the real world. There have been many developments in new ways of thinking about scattering amplitudes. It has proved useful to think of them as differential forms on kinematic space [44]. These differential forms are associated to geometric structures in many cases. The physics of scattering amplitudes emerges from simple properties of the underlying geometry as seen in the few known cases [45, 46, 47, 48, 49]. It would therefore be useful to rewrite amplitudes in the Standard Model as differential forms. This would lay the groundwork for an attempt to look for hidden geometric structure within these amplitudes.

Chapter 3

Spontaneous Symmetry Breaking from an On-Shell Perspective

3.1 Introduction

On-shell techniques have proven themselves as not only an alternative method to compute scattering amplitudes more efficiently, but also as a tool to unlock underling structures of quantum theories. Most of the progress of on-shell techniques has been limited to amplitudes containing only massless particles. Though there has been progress with on-shell massive amplitudes [50, 51, 52, 53], the techniques changed in a fundamental way when [54] introduced spin-spinors for massive particles, natural counterparts to helicity-spinors for massless particles. This has since opened the doors for a variety of massive on-shell works, including the computation of massive amplitudes [55, 56, 57, 58, 59], effective field theories [60, 61, 62, 63], gauge invariance [64] and supersymmetry [65]. In this paper, we continue these efforts and generalize the on-shell Higgs mechanism and spontaneous symmetry breaking, while finding interesting results along the way.

The Higgs mechanism and spontaneous symmetry breaking are traditionally described from a field theoretic perspective. Again, it was not until [54] that it was highlighted how to achieve it on-shell. Further, the works of [11, 62] showed how this could be applied to the electroweak symmetry breaking of the Standard Model (SM). In this work, we generalize some aspects of the Higgs mechanism and spontaneous symmetry breaking. In particular, we show how masses for spins 1/2 and 1 transforming under a general group G, can be generated from an entirely on-shell perspective, that is, without reference to Lagrangians, gauge symmetries, or fields acquiring a vacuum expectation value.

It is by now well known that the structure of Yang-Mill theories can be derived on-shell. To give mass to fermions without reference to Lagrangians, we use consistent factorization of $2 \to 2$ tree level amplitudes to derive the structure of Yukawa theories under a general group G. Our strategy for generating masses for spins-1/2 and 1 will be to demand consistency between a high energy theory (UV) and a low energy theory (IR). The consistency is implemented by imposing that the high energy limit of the IR matches onto the UV. As such we construct all the allowed on-shell UV amplitudes under a symmetry group G, and consider all the possible IR amplitudes. After we succeed in generating masses for a general theory, we will show how these results can be used to describe the process of electroweak symmetry breaking in the Standard Model. The nature of these results provide an opportunity for a pedagogical introduction.

This paper is structured as follows. We begin with a brief review of the little group and the construction of scattering amplitudes in Section 3.2, while deferring explicit details of helicity and spin spinors to Appendix B.2. We then use these ideas to construct three and four particle amplitudes in Section 3.3, which are necessary for the structures of Yang-Mills and Yukawa theories. These two sections set the foundations for the UV-IR matching that we will perform. In Section **??**, which experts start at, we show how demanding an IR theory be consistent with a UV theory in the high energy limit allows us to discover the on-shell version of the Higgs Mechanism for spins-1/2 and 1 particles. Lastly, in Section 3.5 we show how to interpret these results within the Standard Model.

3.2 Scattering Amplitudes and the Little Group

In this section, we review the on-shell construction of amplitudes with helicity and spin spinors, and defer to [66, 54, 64] for more details and related discussions. We start the discussion with the little group for massless and massive particles, along with their representations. We then review the properties of scattering amplitudes, and note that they must be Lorentz invariant and little group covariant, which motivates the introduction of helicity and spin spinors.

3.2.1 The Little Group

Following Wigner [67], we can think of particles as irreducible representations of the Poincare' group. We can diagonalize the translation operator with their momentum p^μ, and label any other quantum numbers with σ. Using the reference momentum trick, we can write any momentum p as a Lorentz transformation $L(p;k)$ acting on a reference momentum k i.e., $p = L(p;k)k$. Assuming that we have unitary representations U of elements of the Lorentz group Λ, we define one-particle states as

$$|p,\sigma\rangle = U(L(p;k))|k,\sigma\rangle \, . \tag{3.1}$$

Now, $L(p;k)$ is not unique, as there are many Lorentz transformations that also leave k invariant. The subgroup of the Lorentz group that leaves momentum invariant is referred to as the little group i.e., for W an element of the little group, $Wk = k$. Then, the non-uniqueness of $L(p;k)$ is seen as the freedom to add little group transformations via $p = L(p;k)k = L'(p;k)k$, for $L'(p;k) = L(p;k)W$.

The little group plays an important role in characterizing how single particle states transform, so let us make the above statement more precise. First, consider a general Lorentz transformation Λ acting on p. Using the reference momentum trick, we can write this as $\Lambda p = \Lambda L(p;k)k = L(\Lambda p;k)k$. Next, consider the identity $1 = L(\Lambda p;k)L^{-1}(\Lambda p;k)$ acting on Λp,

$$\begin{aligned} 1\,\Lambda p = L(\Lambda p;k)L^{-1}(\Lambda p;k)\Lambda L(p;k)k = L(\Lambda p;k)L^{-1}(\Lambda p;k)L(\Lambda p;k)k \\ L(\Lambda p;k)W(\Lambda,p;k)k = L(\Lambda p;k)k\,, \end{aligned} \tag{3.2}$$

where we have identified $W(\Lambda,p;k) = L^{-1}(\Lambda p;k)\Lambda L(p;k)$ as an element of the little group! To act on the state $|k,\sigma\rangle$, we simply need unitary representations,

$$U(W(\Lambda,p;k))|k,\sigma\rangle = D_{\sigma\sigma'}(W(\Lambda,p;k))|k,\sigma'\rangle\,, \tag{3.3}$$

where $D_{\sigma\sigma'}(W)$ is a unitary representation of the little group.

Finally, consider the action of a general Lorentz transformation on the state $|p,\sigma\rangle$. From the discussion above, we see that,

$$U(\Lambda)|p,\sigma\rangle = D_{\sigma\sigma'}(W(\Lambda,p;k))|\Lambda p,\sigma'\rangle\,, \tag{3.4}$$

and so, we conclude that a single particle state is labelled by its momentum and transforms under some representation of the little group.

We end this subsection by elucidating the form of $D_{\sigma\sigma'}(W)$. There is a clear distinction of the little group for massless and massive particles. For massless particles in four dimensions, the little group is $SO(2) = U(1)$, and representations of $U(1)$ are labeled by the integers $n = 2h$, where h is referred to as the particle's helicity. For massive particles in four dimensions, the little group is $SO(3) = SU(2)$, and finite dimensional irreducible representations of $SU(2)$ are labeled by non-negative integers

$n = 2S + 1$, where S is referred to as spin. The simplicity introduced in [54], was to use symmetric tensor representations of $SU(2)$. That is, an irreducible spin-S representation of $SU(2)$ corresponds to a fully symmetric rank $2S$ tensor.

With this in mind, we can revisit eq. (3.4) for massless and massive particles. Under a general Lorentz transformations, a massless particle with helicity-h transforms as,

$$U(\Lambda)|p, h, \sigma\rangle = D_{\sigma\sigma'}(W(\Lambda, p; k))|\Lambda p, h, \sigma'\rangle \,, \tag{3.5}$$

where,

$$D_{\sigma\sigma'}(W(\Lambda, p; k)) = w_h \delta_{\sigma\sigma'} \,, \tag{3.6}$$

for $w_h \in U(1)$. Under general Lorentz transformations, a massive particle with spin-s transforms as

$$U(\Lambda)|p, \{I_1, \ldots, I_{2s}\}, \sigma\rangle = D_{\sigma\sigma'}(W(\Lambda, p; k))|\Lambda p, \{J_1, \ldots, J_{2s}\}, \sigma'\rangle \,, \tag{3.7}$$

where

$$D_{\sigma\sigma'}(W(\Lambda, p; k)) = \delta_{\sigma\sigma'} W_{I_1}{}^{J_1} \ldots W_{I_{2s}}{}^{J_{2s}} \,, \tag{3.8}$$

for $W \in SU(2)$.

3.2.2 Scattering Amplitudes

Next, lets consider what this means when scattering n particles, labelled by $|p_a, \rho_a\rangle$, where ρ represents additional quantum numbers needed to specify the particle. As mentioned above, for massless particles of helicity-h, $\rho = (h, \sigma)$, and for massive particles of spin-s, $\rho = (\{I_1, \ldots, I_{2s}\}, \sigma)$. Since we consider all particles outgoing, the scattering amplitude is defined as

$$\mathcal{M}(p_1, \rho_1; \ldots; p_n, \rho_n) =_{\text{out}} \langle p_1, \rho_1; \ldots; p_n, \rho_n | 0 \rangle_{\text{in}} \,. \tag{3.9}$$

Poincare' invariance of the S-matrix implies,

$$\mathcal{M}(p_1, \sigma_1; \ldots; p_n, \sigma_n) = \delta^4 \left(\sum p_a \right) \mathcal{A}(p_1, \sigma_1; \ldots; p_n, \sigma_n) \,, \tag{3.10}$$

where we have used translation invariance to pull out a momentum conserving delta function.

To see the action of the little group in a scattering amplitude, let us consider the transformation law of $M(p_1, \sigma_1; \ldots; p_n, \sigma_n)$ under a general Lorentz transformation Λ. Assuming that the asymptotic multi-particle states transform as a tensor product of one-particle states i.e. $|p_1, \rho_1; \ldots; p_n, \rho_n\rangle = |p_1, \rho_1\rangle \otimes \cdots \otimes |p_n, \rho_n\rangle$, so that $U(\Lambda)|p_1, \rho_1; \ldots; p_n, \rho_n\rangle = U(\Lambda)|p_1, \rho_1\rangle \otimes \cdots \otimes U(\Lambda)|p_n, \rho_n\rangle$, then

$$U(\Lambda)\mathcal{A}(p_1, \sigma_1; \ldots; p_n, \sigma_n) = \left(\prod_a D_{\sigma_a \sigma'_a}(W) \right) \mathcal{A}(\Lambda p_1, \sigma_1; \ldots; \Lambda p_n, \sigma_n) \,, \tag{3.11}$$

where $D_{\sigma_a \sigma'_a}(W)$ takes the form of eq. (3.6) or eq. (3.8) if a is a massless or massive particle respectively.

So, the $\mathcal{A}(p_1, \sigma_1, \ldots, p_n, \sigma_n)$ must be Lorentz invariant and covariant under the little group. With this in mind, we can further determine what form $\mathcal{A}(p_a, \rho_a)$ must take. For example, consider a the scattering amplitude of three particles where particles 1, 2, and 3 have spin-s_1, helicity-h_2 and helicity-h_3 respectively . We would represent this object as,

$$M^{\{I_1, \ldots, I_{2s_1}\}, \{h_2\}, \{h_3\}}(p_1, p_2, p_3) \,, \tag{3.12}$$

where the indices $\{I_1, \ldots, I_{2s_1}\}$ are fully symmetrized. Under a little group transformation, we have,

$$\mathcal{A}^{\{I_1,\ldots,I_{2s_1}\},\{h_2\},\{h_3\}} = (W^{I_1}_{1L_1} \ldots W^{I_{2s_1}}_{1L_{2s_1}})(w^{2h_2})(w^{2h_3})M^{\{L_1,\ldots,L_{2s_1}\},\{h_2\},\{h_3\}}\,, \tag{3.13}$$

where W are $SU(2)$ transformations in the spin-$\frac{1}{2}$ representation and $w = e^{i\theta}$ are $U(1)$ transformations in the helicity-$\frac{1}{2}$ representation.

3.2.3 Helicity and Spin Spinors

Given the Lorentz invariant and little group covariant structure of the amplitude, it would be great if we had variables that transformed under both the Lorentz group and little group. This is in fact the usefulness of 'helicity-spinors' and 'spin-spinors'. Helicity-spinors are introduced as objects that transform under both the $SL(2, C)$ Lorentz group and $U(1)$ little group, while spin-spinors are objects that transform under both the $SL(2, C)$ Lorentz group and $SU(2)$ little group.

Thus, the amplitude can be written as a function of these variables,

$$\mathcal{A}^{\{I_1,\ldots,I_{2s_1}\},\{h_2\},\{h_3\}}(p_1, p_2, p_3) = \mathcal{A}(\{\boldsymbol{\lambda}_1^{\{I_1}, \ldots, \boldsymbol{\lambda}_1^{I_{2s_1}\}}\}, \{\lambda_2, \tilde{\lambda}_2, h_2\}, \{\lambda_3, \tilde{\lambda}_3, h_3\}) \tag{3.14}$$

where the helicity spinors λ are unbolded and the spin spinors are bolded by convention (introduced in [54]). Since the bolded spin spinors are always symmetrized, we can remove the explicit indices and infer the spin by keeping in mind that we need $2S$ spin spinors for a particle of spin S. Thus, we will often write instead,

$$\mathcal{A}^{\{I_1,\ldots,I_{2s_1}\},\{h_2\},\{h_3\}}(p_1, p_2, p_3) = \mathcal{A}(\{\overbrace{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_1}^{2S\,\text{times}}\}, \{\lambda_2, \tilde{\lambda}_2, h_2\}, \{\lambda_3, \tilde{\lambda}_3, h_3\})\,, \tag{3.15}$$

In appendix B.2, we demonstrate how to construct helicity and spin spinors, that have the correct transformation properties. Specifically, helicity spinors are defined to transform under the $h = \pm\frac{1}{2}$ representation of $U(1)$,

$$\begin{aligned}
\lambda_\alpha &\to e^{i2h\theta}\lambda_\alpha \;\; = e^{i\theta}\lambda_\alpha \;\; = w\lambda_\alpha \,, \\
\tilde{\lambda}_{\dot\alpha} &\to e^{-i2h\theta}\tilde{\lambda}_{\dot\alpha} = e^{-i\theta}\tilde{\lambda}_{\dot\alpha} = w^{-1}\tilde{\lambda}_{\dot\alpha} \,.
\end{aligned} \tag{3.16}$$

Spin-spinors are defined to transform under the $S = \frac{1}{2}$ representation, or a rank-1 (trivially) symmetric tensor,

$$\lambda_\alpha{}^I \to W_J{}^I \lambda_\alpha{}^J \qquad \text{and} \qquad \tilde{\lambda}_{\dot\alpha I} \to (W^{-1})_I{}^J \tilde{\lambda}_{\dot\alpha J} \,, \tag{3.17}$$

with W an $SU(2)$ transformation in the spin-$\frac{1}{2}$ representation, where I, J are the $SU(2)$ little group indices. The choice of λ or $\tilde{\lambda}$ is irrelevant as one can use the Dirac equation (see Appendix B.2) to freely convert from one to another, which is why in (3.14) we have only chosen one.

Transformations under the Lorentz group are encoded in the $SL(2, C)$ indices α and $\dot\alpha$. With these spinors available, one can contract the Lorentz indices to form Lorentz invariant objects, that are still covariant with respect to the little group, which are exactly the type of objects that correspond to scattering amplitudes!

For three-particle amplitudes, building these objects to have the proper little group covariance, is one of the central constraints, and a systematic analysis of all possible three particle amplitudes has been carried out in [54], and applied to the SM in [56].

3.3 Massless Three and Four Particle Amplitudes

In this section, we review the construction of massless three and four particle amplitudes, and show how these naturally lead to the structures of Yang-Mills and Yukawa theories. We begin with the standard three particle amplitude to assist the reader with our conventions. Then, we construct tree level four particle amplitudes by gluing three particle amplitudes. By demanding unitary in the form of consistent factorization, we derive constraints on couplings which are identical to those one would attain from a gauge symmetry. We defer the reader to [68, 54] for more details.

3.3.1 Three Particle Amplitudes

Massless three particle kinematics forces either $\lambda_1 \propto \lambda_2 \propto \lambda_3$ or $\tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3$. Lets consider the first case, so a massless three particle amplitude will only be a function of $\lambda_1, \lambda_2, \lambda_3$, which, to match the notation in eq. (3.14),

$$\mathcal{A}^{\{h_1\},\{h_2\},\{h_3\}}(p_1, p_2, p_3) = \mathcal{A}(\{\lambda_1, h_1\}, \{\lambda_2, h_2\}, \{\lambda_3, h_3\})\,. \tag{3.18}$$

The most general Lorentz invariant object we can construct is given by making all possible contractions on the $SL(2, C)$ indices,

$$\mathcal{A}(\{\lambda_1, h_1\}, \{\lambda_2, h_2\}, \{\lambda_3, h_3\}) = \langle 12\rangle^a \langle 23\rangle^b \langle 31\rangle^c\,, \tag{3.19}$$

where a, b, c are unspecified. To fix a, b, c we must impose that the amplitude has the correct little group covariance as in eq. (3.13). That is, under a little group transformation, we must have, $\mathcal{A}^{\{h_1\},\{h_2\},\{h_3\}} \to (w_1^{2h_1})(w_2^{2h_2})(w_3^{2h_3})\mathcal{A}^{\{h_1\},\{h_2\},\{h_3\}}$. Our anstaz, under a little group transformation, takes the form

$$\langle 12\rangle^a \langle 23\rangle^b \langle 31\rangle^c \to w_1^{a+c} w_2^{a+b} w_3^{b+c} \langle 12\rangle^a \langle 23\rangle^b \langle 31\rangle^c\,. \tag{3.20}$$

Solving these constraints for a, b, c yields,

$$\mathcal{A}(\{\lambda_1, h_1\}, \{\lambda_2, h_2\}, \{\lambda_3, h_3\}) = \langle 12\rangle^{h_1+h_2-h_3} \langle 23\rangle^{h_2+h_3-h_1} \langle 31\rangle^{h_3+h_1-h_2}\,. \tag{3.21}$$

An identical analysis can be performed assuming $\tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3$, and yields

$$\mathcal{A}(\{\tilde{\lambda}_1, h_1\}, \{\tilde{\lambda}_2, h_2\}, \{\tilde{\lambda}_3, h_3\}) = [12]^{-h_1-h_2+h_3}[23]^{-h_2-h_3+h_1}[31]^{-h_3-h_1+h_2}\,. \tag{3.22}$$

For a given set of helicities h_1, h_2, h_3, to distinguish which of these to use, we must impose locality, which constrains the mass dimension of the momentum dependence (a three particle amplitude in four dimensions must have mass dimension one). This yields

$$\begin{aligned}\mathcal{A}^{\{h_1\},\{h_2\},\{h_3\}} &= g\,\langle 12\rangle^{h_1+h_2-h_3}\langle 23\rangle^{h_2+h_3-h_1}\langle 31\rangle^{h_3+h_1-h_2}\,, \qquad \text{if} \quad h_1+h_2+h_2>0\\ &= \tilde{g}\,[12]^{h_3-h_1-h_2}[23]^{h_1-h_2-h_3}[31]^{h_2-h_3-h_1}\,, \qquad \text{if} \quad h_1+h_2+h_2<0\,.\end{aligned} \tag{3.23}$$

Now that we have fixed the kinematic form of the amplitude, we can return to the ignored σ_a labels. As mentioned before, these encode any additional quantum numbers that the particle can take. The only contractions we have left are of the form ${}_{\text{out}}\langle \sigma_1, \ldots, \sigma_n | 0\rangle_{\text{in}}$, which, in the simplest sense, say that the sum of 'charges' is zero, assuming all outgoing. For a three particle vertex, between possibly different particles with labels σ_a, σ_b and σ_c, we attach a simple function $f(\sigma_a, \sigma_b, \sigma_c)$ to the vertex. In the next section, we will see how the properties of these functions can be connected to kinematics.

3.3.2 Four Particle Amplitudes

Let us move on to four particle amplitudes. The additional constraint on the structure of the amplitude comes from unitarity in form of consistent factorization. That is, when some internal momentum goes on-shell, the residue must be factorizable as a product of lower point amplitudes. In this section, we only consider massless particles, so the condition takes the form,

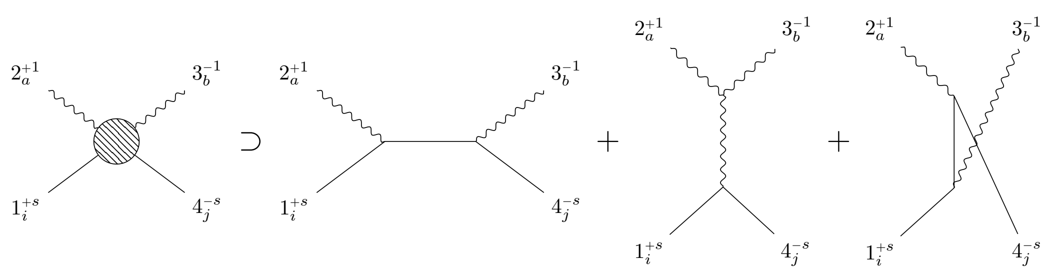
$$\mathcal{A} = \frac{\mathcal{A}_L^{a\,h} \mathcal{A}_R^{a\,-h}}{P^2}\,, \tag{3.24}$$

where a is the index of some intermediate particle.

It is by now well known that consistent factorization on the amplitude $\mathcal{A}(1_i^{+s}, 2_a^{+1}, 3_b^{-1}, 4_j^{-s})$ leads to the discovery of the familiar Yang-Mills structure of spin 0, 1/2 and 1 particles [54, 69]. We follow similar logic, and demand consistent factorization on the amplitude $\mathcal{A}(1_j^{+s}, 2_I^0, 3_a^{-1}, 4_k^{-s})$. In doing so, we discover the familiar structures of Yukawa theories. These structures are then used in Section 3.4.4 to show the Higgs mechanism is necessary for UV-IR consistency.

First, we will present the analysis in [54] for the amplitude $\mathcal{A}(1_i^{+s}, 2_a^{+1}, 3_b^{-1}, 4_j^{-s})$ to set notation and conventions. Then, we will repeat the analysis on the amplitude $\mathcal{A}(1_j^{+1/2}, 2_I^0, 3_a^{-1}, 4_k^{-1/2})$ to derive similarly interesting constraints.

We begin by noting that the amplitude $\mathcal{A}(1_i^{+s}, 2_a^{+1}, 3_b^{-1}, 4_j^{-s})$ has three channels given by,



(3.25)

The residues in every channel can be obtained by gluing M_L and M_R. When we have several spin-s particles i, j coupling to a spin-1 particle a, we attach the vertex $(T_a^R)_{ij}$. Note that $R = s$ for the representation the scalars ($s = 0$) transform under, $R = f$ for the representation the fermions ($s = 1/2$) transform under, and lastly $R = \mathrm{ad}$ for the (adjoint) representation the bosons ($s = 1$) transform under. The residues in every channel takes the form $(\langle 12\rangle[34])^{2s}\langle 2|1|3]^{2-2s}\times r$ where,

$$\begin{aligned}
r_s &= (T^R_a T^R_b)_{ij}\times\frac{1}{u}\,,\\
r_u &= (T^R_b T^R_a)_{ij}\times\frac{1}{s}\,,\\
r_t &= (T^R_c)_{ij}(T^{\mathrm{Ad}}_c)_{ab}\times\frac{1}{2}\left(\frac{1}{s}-\frac{1}{u}\right)\,.
\end{aligned}\tag{3.26}$$

Given the structure of the residues, for $s=\{0,1/2,1\}$, the ansatz for the four particle amplitude has the form,

$$\mathcal{A}(1_i^{+s},2_a^{+1},3_b^{-1},4_j^{-s}) = (\langle 12\rangle[34])^{2s}\langle 2|1|3]^{2-2s}\left(\frac{A_{ijab}}{su}+\frac{B_{ijab}}{ut}+\frac{C_{ijab}}{ts}\right)\,.\tag{3.27}$$

Matching the residues in the ansatz above to those computed by gluing in eq. (3.26) yields,

$$\begin{aligned}
r_s &= (T^R_a T^R_b)_{ij}\times\frac{1}{u} &&= (A_{ijab}-C_{ijab})\,\frac{1}{u}\,,\\
r_u &= (T^R_b T^R_a)_{ij}\times\frac{1}{s} &&= (A_{ijab}-B_{ijab})\,\frac{1}{s}\,,\\
r_t &= (T^R_c)_{ij}(T^{\mathrm{Ad}}_c)_{ab}\times\frac{1}{s} &&= (-B_{ijab}+C_{ijab})\,\frac{1}{s}\,,
\end{aligned}\tag{3.28}$$

which has solutions only if $[T^R_b,T^R_a]_{ij}=(T^R_c)_{ij}(T^{\mathrm{Ad}}_c)_{ab}$. Identifying $(T^{\mathrm{Ad}}_c)_{ab}=-f_{abc}$ yields the familiar

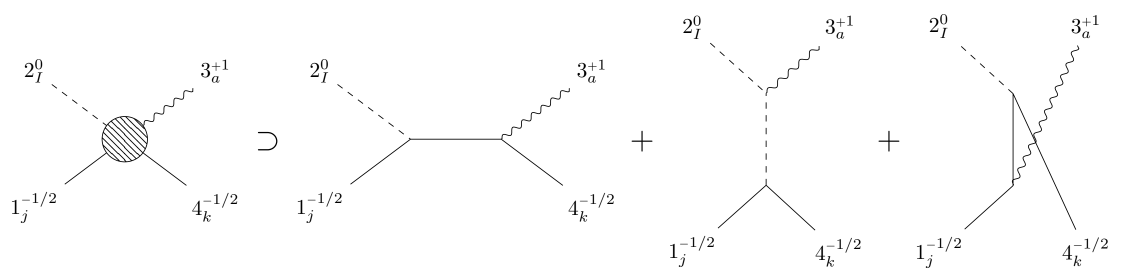
$$[T^R_a,T^R_b]_{ij}=f_{ab}{}^{c}\,(T^R_c)_{ij}\,.\tag{3.29}$$

When we consider the case of $s=1$, then $i,j=d,e$, and $R=\mathrm{Ad}$. Along with $(T^{\mathrm{Ad}}_c)_{ab}=-f_{abc}$, we discover that the coefficients f_{abc} must satisfy the Jacobi identity,

$$f_{ade}f_{ebc}+f_{bde}f_{eac}+f_{abe}f_{ecd}=0\,.\tag{3.30}$$

We defer to [54] for further exploration and discussion on these amplitudes.

Next, we move on to the amplitude $\mathcal{A}(1_j^{-1/2}, 2_I^0, 3_a^{+1}, 4_k^{-1/2})$. By demanding consistent factorization identical to the above, we will derive constraints between the couplings and the matrices T_a^R that are identical to those one would obtain by assuming the existence of a gauge symmetry. The amplitude $\mathcal{A}(1_j^{-1/2}, 2_I^0, 3_a^{+1}, 4_k^{-1/2})$ has three channels given by



(3.31)

When we have several spin-1/2 particles i, j coupling to a spin-0 particle I, we attach the vertex $(Y_I)_{ij}$. The residues in every channel take the form $\langle 13\rangle\langle 34\rangle[14]^2 \times r$ where,

$$
\begin{aligned}
r_s &= +(Y_I)_{jl}(T_a^f)_{lk} \times \frac{1}{t}\,, \\
r_u &= -(Y_I)_{kl}(T_a^f)_{lj} \times \frac{1}{t}\,, \\
r_t &= +(Y_L)_{jk}(T_a^s)_{LI} \times \frac{1}{u}\,.
\end{aligned}
\tag{3.32}
$$

Given the structure of the residues, the ansatz for the four particle amplitude has the form,

$$
\mathcal{A}(1_j^{+1/2}, 2_I^0, 3_a^{-1}, 4_k^{-1/2}) = \langle 13\rangle\langle 34\rangle[14]^2 \left(\frac{A_{ijab}}{su} + \frac{B_{ijab}}{ut} + \frac{C_{ijab}}{ts} \right). \tag{3.33}
$$

Again, matching the residues in the ansatz, to those computed by gluing in eq. (3.33) yields the condition,

$$
\begin{aligned}
r_s &= +(Y_I)_{jl}(T_a^f)_{lk} \times \frac{1}{t} &= (-A_{jIak} + C_{jIak})\,\frac{1}{t}\,, \\
r_u &= -(Y_I)_{kl}(T_a^f)_{lj} \times \frac{1}{t} &= (-A_{jIak} + B_{jIak})\,\frac{1}{t}\,, \\
r_t &= +(Y_L)_{jk}(T_a^s)_{LI} \times \frac{1}{u} &= (B_{jIak} - C_{jIak})\,\frac{1}{u}\,.
\end{aligned} \tag{3.34}
$$

which has solutions only if,

$$
(Y_I)_{jl}(T_a^f)_{lk} + (Y_I)_{lk}(T_a^f)_{lj} + (Y_L)_{jk}(T_a^s)_{LI} = 0\,. \tag{3.35}
$$

This is exactly the constraint we would get assuming a gauge theory. To illustrate, we know Y_{Ijk} picks out a singlet as $Y_{Ijk}\phi^I\psi^j\psi^k$. Applying the gauge transformations $\phi^{I'} = (e^{iT_a^s})^{I'}{}_I\phi^I$ and $\psi^{I'} = (e^{iT_a^f})^{i'}{}_i\psi^i$ to the transformation

$$
\begin{aligned}
Y_{Ijk}\phi^I\psi^j\psi^k &\to Y_{I'j'k'}\phi^{I'}\psi^{j'}\psi^{k'} \\
&= Y^f_{I'j'k'}\;(e^{iT_a^s})^{I'}{}_I\phi^I\;(e^{iT_a^f})^{j'}{}_j\psi^j\;(e^{iT_a^f})^{k'}{}_k\psi^k\,,
\end{aligned} \tag{3.36}
$$

at first order yields eq. (3.35).

Lastly, lets consider the special case of eq. (3.35), where $(Y_L)_{jk}(T_a^s)_{LI} = 0$. Since $(T_a^s)_{LI}$ only acts on the L scalar degrees of freedom, we can separate its action as $(Y_L)_{jk} = n_L\tilde{m}_{jk}$. Eq. (3.35) now reads,

$$
\tilde{m}_{jl}(T_a^f)_{lk} + \tilde{m}_{lk}(T_a^f)_{lj} = 0\,, \tag{3.37}
$$

which is the condition needed for background scalars to contribute to fermion masses.

We can see this by applying the gauge transformations $\psi^{I'} = (e^{iT_a^f})^{i'}{}_i \psi^i$ to

$$
\begin{aligned}
\tilde{m}_{jk}\psi^j\psi^k &\to \tilde{m}_{j'k'}\psi^{j'}\psi^{k'} \\
&= \tilde{m}_{j'k'}\, (e^{iT_a^f})^{j'}{}_j \psi^j\, (e^{iT_a^f})^{k'}{}_k \psi^k\,,
\end{aligned}
\tag{3.38}
$$

at first order yields eq. (3.37).

3.4 Non-Abelian Higgs: $G \to H$

Much of the simplicity will be gained by considering the action of the full symmetry group G at once, rather than working with its subgroups $G_1, \ldots, G_n$. In the real world, and many toy examples, we usually specify the subgroups and assign different coupling constants to them. For example, in the Standard Model, the internal symmetry group is $G = SU(3) \times SU(2) \times U(1)$, to which we can assign a strong, weak and hypercharge coupling constant. Although this distinction is useful, we find that by reliving ourselves of the subgroups, in the appropriate manner, can elucidate additional structures in these amplitudes.

The strategy we will take is to define a UV and IR theory, through the three particle scattering amplitudes, and demand that the high energy limit of the IR theory can match onto the UV theory. In doing this, we will rediscover many familiar results about spontaneous symmetry breaking.

We will use hatted indexes in the UV and unhatted indexes in the IR. Since the spinors for massive particle momenta are bolded, and massless are unbolded, we will bold any IR particles in our diagrams for visual consistency and aid.

3.4.1 The UV

In the UV, all the particle labels are hatted. We have a symmetry group G of dimension d_G. In general, G can be expressed as a direct product of simple compact Lie groups and $U(1)$,

$$G = G_1 \times \cdots \times G_i \times \cdots \times G_n \,. \tag{3.39}$$

The associated Lie algebra $\tilde{\mathfrak{g}}$ (we will drop all the tildes soon) is spanned by d_G generators

$$\tilde{\mathfrak{g}} = \{\tilde{T}_1, \ldots, \tilde{T}_{\hat{a}}, \ldots, \tilde{T}_{d_G}\} \,. \tag{3.40}$$

There is a separate coupling g_a for each simple group or $U(1)$ factor of the group G. The generators, $\tilde{T}_{\hat{a}}$, thus separate out into different classes, and $g_{\hat{a}} = g_{\hat{b}}$ if $\tilde{T}_{\hat{a}}$ and $\tilde{T}_{\hat{b}}$ are in the same class. Of course, by standard definitions, $[\tilde{T}_{\hat{a}}, \tilde{T}_{\hat{b}}] = \tilde{f}_{\hat{a}\hat{b}\hat{c}}\tilde{T}_{\hat{c}}$. As mentioned earlier, we find it convenient to re-scale the basis of our Lie algebra as $T_{\hat{a}} = g_{\hat{a}}\tilde{T}_{\hat{a}}$ (no sum over $\hat{a}$, and tildes now removed),

$$\begin{aligned}\mathfrak{g} &= \left\{\{g_i\tilde{T}_{1_i}, g_i\tilde{T}_{2_i}, \ldots g_i\tilde{T}_{n_i}\}, \{g_j\tilde{T}_{1_j}, g_j\tilde{T}_{2_j}, \ldots, g_j\tilde{T}_{n_j}\}, \ldots, \{g_n\tilde{T}_{1_n}\}\right\} \,, \\ &= \{T_1, T_2, \ldots, T_{d_G}\} \,. \end{aligned} \tag{3.41}$$

Note that this statement is representation independent. These (rescaled) generators will obey the commutation relation in eq. (3.29), with hatted indices,

$$[T_{\hat{a}}, T_{\hat{b}}] = f_{\hat{a}\hat{b}}{}^{\hat{c}}T_{\hat{c}} \,, \tag{3.42}$$

where $f_{\hat{a}\hat{b}\hat{c}} = \frac{g_{\hat{a}} g_{\hat{b}}}{g_{\hat{c}}} \tilde{f}_{\hat{a}\hat{b}\hat{c}}$. Given a general set of generators and couplings, one can easily compute the new structure constants via,

$$f_{\hat{a}\hat{b}\hat{c}} = \frac{1}{\mathrm{Tr}\{T_{\hat{a}} T_{\hat{a}}\}} \mathrm{Tr}\{[T_{\hat{a}}, T_{\hat{b}}] T_{\hat{c}}\} \,. \tag{3.43}$$

The spectrum consist of N_s spin-0, N_f spin-1/2 and N_{ad} spin-1 particles, which transform under (possibly reducible) representations $R = s$, $R = f$ and $R = \mathrm{ad}$ of the symmetry group G respectively. So, the number of particles is equal to the dimension of the representation, that is, $N_s = \dim(R = s)$, $N_f = \dim(R = f)$, and $N_{\mathrm{ad}} = \dim(R = \mathrm{ad}) = d_G$. The scalars are labelled by $\{\hat{I}, \hat{J}, \hat{K}\}$, the fermions by $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, and the bosons by $\{\hat{a}, \hat{b}, \hat{c}\}$. We summarize the notation in Table 3.1.

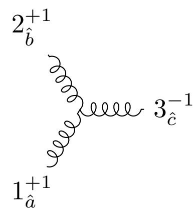
We consider all particles of the same helicity to be identical, and assume all our particles are real. This may seem like a over simplicity, however, since we are working with (possibly) reducible representations, this will not be an obstruction in considering more complicated models. Hence, we allow $SO(\dim(R))$ symmetries within the same helicity. To elaborate, under free propagation, the spin-0, spin-1/2 and spin-1 particles have an $SO(N_s)$, $SO(N_f)$ and $SO(d_G)$ symmetry that we will represent with U, Ω and $\mathcal{O}$ respectively. We will exploit these in the matching process.

Spin	Rep., R	$\dim(R)$	Labels	$SO(\dim(R))$
0	s	N_s	$\hat{I}, \hat{J}, \hat{K}$	U
$\frac{1}{2}$	f	N_f	$\hat{\imath}, \hat{\jmath}, \hat{k}$	Ω
1	ad	N_{ad}	$\hat{a}, \hat{b}, \hat{c}$	$\mathcal{O}$

Table 3.1: Summary of UV spectrum.

Now that we have defined the spectrum and symmetry group, we can move on to interactions. We will consider only the 3-particle amplitudes. To each of these amplitudes, we assign a coupling. These couplings must obey the constraints from consistent factorization, notably eqns. (3.29), (3.30) and (3.35). We only consider amplitudes with $\sum h = \pm 1$, as these correspond to amplitudes that have dimensionless coupling constants (see [70] for general considerations). We list the amplitudes considered and elaborate on any properties that will be useful:

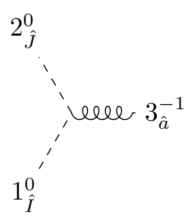
- Three spin one particles:



$$= \mathcal{A}(1^{+1}_{\hat{a}}, 2^{+1}_{\hat{b}}, 3^{-1}_{\hat{c}}) = f_{\hat{a}\hat{b}\hat{c}}\, \mathcal{A}(1^{+1}, 2^{+1}, 3^{-1})\,, \tag{3.44}$$

  where $\mathcal{A}(1^{+1}, 2^{+1}, 3^{-1}) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$. Note that, since particles 1 and 2 have the same helicity, this amplitude should be even under the exchange of the bosons $1^{+1}_{\hat{a}} \leftrightarrow 2^{+1}_{\hat{b}}$, so that $\mathcal{A}(1^{+1}_{\hat{a}}, 2^{+1}_{\hat{b}}, 3^{-1}_{\hat{c}}) = \mathcal{A}(2^{+1}_{\hat{b}}, 1^{+1}_{\hat{a}}, 3^{-1}_{\hat{c}})$. Since $\mathcal{A}(1^{+1}, 2^{+1}, 3^{-1}) = -\mathcal{A}(2^{+1}, 1^{+1}, 3^{-1})$, then we must have the antisymmetric property $f_{\hat{a}\hat{b}\hat{c}} = -f_{\hat{b}\hat{a}\hat{c}}$.

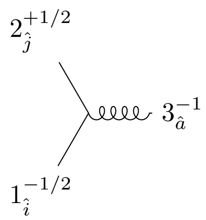
- Two spin zero and one spin one:



$$= \mathcal{A}(1^{0}_{\hat{I}}, 2^{0}_{\hat{J}}, 3^{-1}_{\hat{a}}) = (T^s_{\hat{a}})_{\hat{I}\hat{J}}\, \mathcal{A}(1^{0}, 2^{0}, 3^{-1})\,, \tag{3.45}$$

  with $\mathcal{A}(1^0, 2^0, 3^{-1}) = \frac{[23][31]}{[12]}$. By demanding that the amplitude is even under the exchange of the bosons $1^0_{\hat{I}} \leftrightarrow 2^0_{\hat{J}}$, and following the same logic as previously, we find the antisymmetric property $(T^s_{\hat{a}})_{\hat{I}\hat{J}} = -(T^s_{\hat{a}})_{\hat{J}\hat{I}}$.

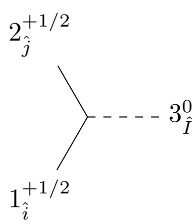
- Two spin half and one spin one:



$$= \mathcal{A}(1^{-1/2}_{\hat{i}}, 2^{+1/2}_{\hat{j}}, 3^{+1}_{\hat{a}}) = (T^f_{\hat{a}})_{\hat{i}\hat{j}}\ \mathcal{A}(1^{-1/2}, 2^{+1/2}, 3^{-1})\,, \tag{3.46}$$

with $\mathcal{A}(1^{-1/2}, 2^{+1/2}, 3^{-1}) = \frac{[31]^2}{[12]}$. In this case, all the particles are distinguishable, and so we have no further constraints on $(T^f_{\hat{a}})_{\hat{j}\hat{i}}$.

- Two spin half and one spin zero:



$$= \mathcal{A}(1^{+1/2}_{\hat{i}}, 2^{+1/2}_{\hat{j}}, 3^{0}_{\hat{I}}) = Y^s_{\hat{I}\hat{i}\hat{j}}\ \mathcal{A}(1^{+1/2}, 2^{+1/2}, 3^{0})\,, \tag{3.47}$$

where $\mathcal{A}(1^{+1/2}, 2^{+1/2}, 3^0) = \langle 12\rangle$. Demanding the amplitude is odd under the exchange of the fermions $1^{+1/2}_{\hat{i}} \leftrightarrow 2^{+1/2}_{\hat{j}}$, tells us that the Yukawa coupling is symmetric in $Y_{\hat{I}\hat{i}\hat{j}} \leftrightarrow Y^s_{\hat{I}\hat{j}\hat{i}}$.

The four amplitudes above define our UV theory. Next, we will move on to define our IR theory.

3.4.2 The IR

In the IR, all particle labels are unhatted. There is a symmetry group, which the particles must respect when interacting, but instead of imposing a group in advance, we will try to discover what it is allowed, if it is to be consistent with the UV theory defined previously.

The spectrum consist of massless and massive particles. Again, we will only consider spins 0, 1/2 and 1. Since the spectrum is slightly more complicated, we will continue to use the un-bolded and bolded notation for massless and massive particles respectively. For spin-0 particles, the massless particles are labeled by $\{I, J, K\}$ and the massive particles by $\{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$. For spin-1/2 particles, the massless particles are labeled by $\{i, j, k\}$ and the massive by $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Lastly, for spin-1 particles, the massless particles are labelled by $\{a, b, c\}$ and the massive by $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. Keep in mind, all labels are unhatted. We summarize the labels in Table 3.2.

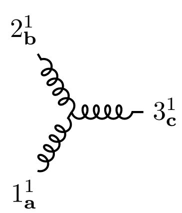
Spin	Massless Labels	Massive Labels	Collectively
0	I, J, K	$\mathbf{I}, \mathbf{J}, \mathbf{K}$	$\underline{I}, \underline{J}, \underline{K}$
$\frac{1}{2}$	i, j, k	$\mathbf{i}, \mathbf{j}, \mathbf{k}$	$\underline{i}, \underline{j}, \underline{k}$
1	a, b, c	$\mathbf{a}, \mathbf{b}, \mathbf{c}$	$\underline{a}, \underline{b}, \underline{c}$

Table 3.2: Summary of IR spectrum.

Now that we have defined the spectrum, we can move onto the interactions. The presence of massless and massive particles in the IR makes constructing an exhaustive list quite cumbersome for our purposes. We will instead take the approach of defining only the amplitudes with all massive external legs. Then, any other combination of massless and massive particles in the IR can be determined by taking the appropriate high energy limit of that leg, and un-bolding the particle label. For a more exhaustive list of these combinations amplitudes, see [56, 11, 54].

The relevant amplitudes we consider are:

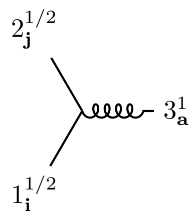
- Three massive spin one particles:



$$= \mathcal{A}(\mathbf{1}^1_{\mathbf{a}}, \mathbf{2}^1_{\mathbf{b}}, \mathbf{3}^1_{\mathbf{c}}) = h_{\mathbf{abc}}\, \mathcal{A}(\mathbf{1}^1, \mathbf{2}^1, \mathbf{3}^1)\,, \tag{3.48}$$

where $\mathcal{A}(\mathbf{1}^1, \mathbf{2}^1, \mathbf{3}^1) = \frac{1}{m_{\mathbf{a}} m_{\mathbf{b}} m_{\mathbf{c}}}(\langle \mathbf{12}\rangle[\mathbf{12}]\langle \mathbf{3}|p_1 - p_2|\mathbf{3}] + \text{cyc.})$.

- Two massive spin half and one massive spin one:



$$= \mathcal{A}(\mathbf{1}_{\mathbf{i}}^{1/2}, \mathbf{2}_{\mathbf{j}}^{1/2}, \mathbf{3}_{\mathbf{c}}^{1}) = (X^f_{1\,\mathbf{a}})_{\mathbf{ij}}\langle\mathbf{13}\rangle[\mathbf{32}] + (X^f_{2\,\mathbf{a}})_{\mathbf{ij}}[\mathbf{13}]\langle\mathbf{32}\rangle$$
$$+ (X^f_{3\,\mathbf{a}})_{\mathbf{ij}}\langle\mathbf{13}\rangle\langle\mathbf{32}\rangle + (X^f_{4\,\mathbf{a}})_{\mathbf{ij}}[\mathbf{13}][\mathbf{32}]\,. \tag{3.49}$$

Note that in both eq. (3.48) and eq. (3.49), we have not specified any properties on the coefficients. We expect that they should conserve charge for some symmetry group, but we will discover their properties in the next section.

Before we move on, we illustrate how the notation adapts to massless particles in the IR. For example, suppose we wanted to consider one massless helicity one particle a, and two massive spin one particles $\mathbf{b}$ and $\mathbf{c}$. The amplitude, given by $\mathcal{A}(1_a^{+1}, \mathbf{2_b}, \mathbf{3_c})$, can be determined by taking the high energy limit of particle $\mathbf{1}_a^1$ of $\mathcal{A}(\mathbf{1}_{\mathbf{a}}^1, \mathbf{2}_{\mathbf{b}}^1, \mathbf{3}_{\mathbf{c}}^1)$, i.e.

$$\mathcal{A}(\mathbf{1}_{\mathbf{a}}^1, \mathbf{2}_{\mathbf{b}}^1, \mathbf{3}_{\mathbf{c}}^1) \xrightarrow{\text{HE limit of } 1_{\mathbf{a}}} \mathcal{A}(1_a^{+1}, \mathbf{2}_{\mathbf{b}}^1, \mathbf{3}_{\mathbf{c}}^1)\,, \tag{3.50}$$

where $\mathcal{A}(1_a^{+1}, \mathbf{2}_{\mathbf{b}}^1, \mathbf{3}_{\mathbf{c}}^1) \propto h_{a\mathbf{bc}}$. Although we have taken the high energy limit, we have remained in the IR, as our goal was simply to obtain the form of an amplitude with a massless leg in the IR. In the next section, we will consider taking high energy limits to map us onto the UV.

3.4.3 UV IR Matching

Now, we demand that the high energy limit of amplitudes in the IR match onto some combination of amplitudes in the UV.

The process of matching can be summarized as:

1. Select an amplitude in the IR.

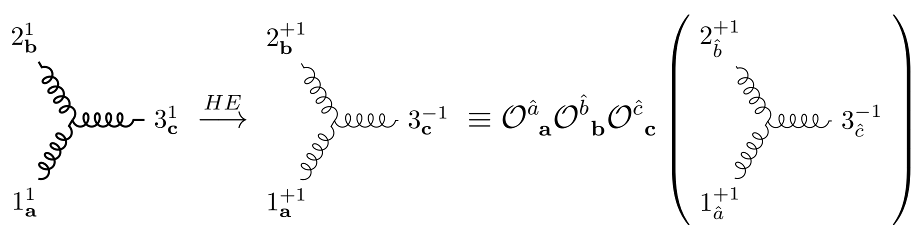
2. Project some spin configuration, and take the high energy limit of all particles.

3. Demand that it is equivalent to some linear combination of amplitudes in the UV.

For Step 1, our menu of IR amplitudes are eq. (3.48), eq. (3.49) and any variation with massless legs (which we mentioned how to construct). For Step 2, we will defer explicit calculations to Appendix B.3, and point the reader to additional calculations in [56, 11, 54]. For Step 3, our menu of UV amplitudes comprise of eq. (3.44), eq. (3.45), eq. (3.46) and eq. (3.47), and we will use the $SO(\dim(R))$ symmetries to construct the linear combinations.

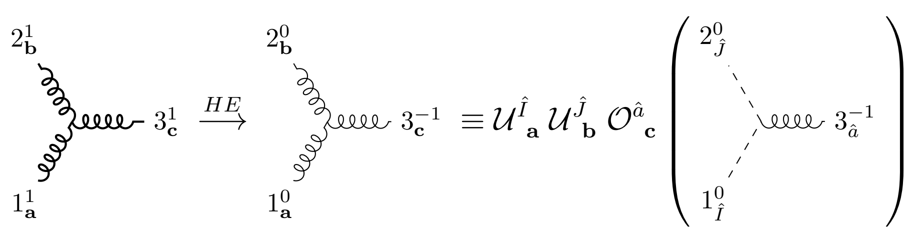
We summarize the relevant results of this process below, and refer the reader to Tables 3.1 and 3.2 for the index notation.

- $\mathcal{A}(\mathbf{1}_{\mathbf{a}}^{+1}, \mathbf{2}_{\mathbf{b}}^{+1}, \mathbf{1}_{\mathbf{c}}^{-1})$



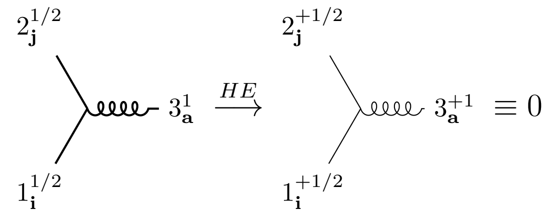
$$h_{\mathbf{abc}} \mathcal{A}(1^{+1}, 2^{+1}, 3^{-1}) \equiv \mathcal{O}^{\hat{a}}{}_{\mathbf{a}} \mathcal{O}^{\hat{b}}{}_{\mathbf{b}} \mathcal{O}^{\hat{c}}{}_{\mathbf{c}} \, f_{\hat{a}\hat{b}\hat{c}} \mathcal{A}(1^{+1}, 2^{+1}, 3^{-1}) \tag{3.51}$$

- $\mathcal{A}(\mathbf{1}_{\mathbf{a}}^{0}, \mathbf{2}_{\mathbf{b}}^{0}, \mathbf{1}_{\mathbf{c}}^{-1})$



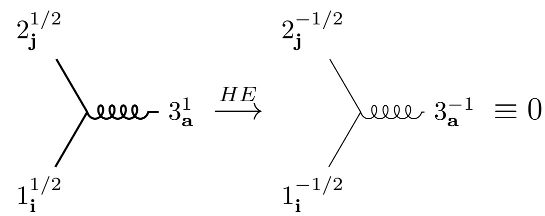
$$h_{\mathbf{abc}} \frac{(m_{\mathbf{c}}^2 - m_{\mathbf{a}}^2 - m_{\mathbf{b}}^2)}{2\, m_{\mathbf{a}} m_{\mathbf{b}}} \mathcal{A}(1^{0}, 2^{0}, 3^{-1}) \equiv \mathcal{U}^{\hat{I}}{}_{\mathbf{a}} \, \mathcal{U}^{\hat{J}}{}_{\mathbf{b}} \, \mathcal{O}^{\hat{a}}{}_{\mathbf{c}} \, (T^{s}_{\hat{a}})_{\hat{I}\hat{J}} \mathcal{A}(1^{0}, 2^{0}, 3^{-1}) \tag{3.52}$$

- $\mathcal{A}(\mathbf{1}_{\mathbf{i}}^{+1/2}, \mathbf{2}_{\mathbf{j}}^{+1/2}, \mathbf{1}_{\mathbf{a}}^{+1})$



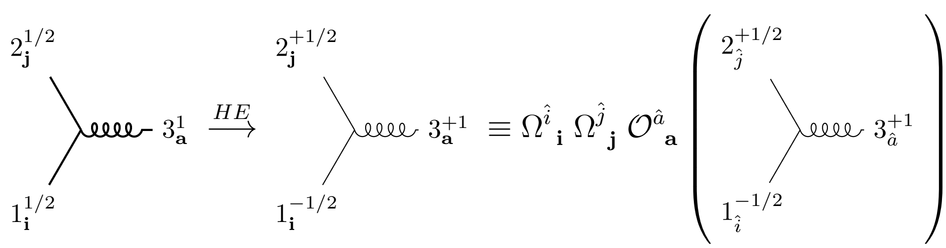
$$(X^f_{3\,\mathbf{a}})_{\mathbf{ij}}\mathcal{A}(1^{+1/2}, 2^{+1/2}, 3^{+1}) \equiv 0_{\mathbf{aij}} \tag{3.53}$$

- $\mathcal{A}(\mathbf{1}_{\mathbf{i}}^{-1/2}, \mathbf{2}_{\mathbf{j}}^{-1/2}, \mathbf{1}_{\mathbf{a}}^{-1})$



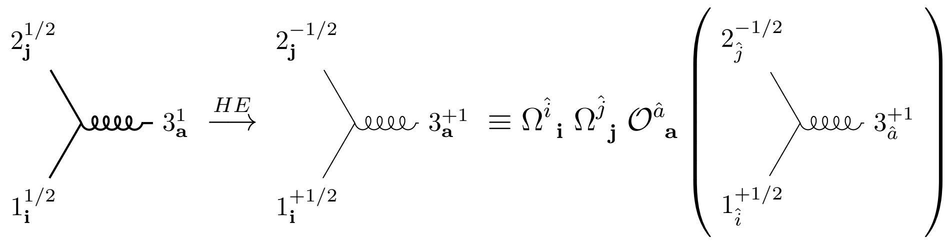
$$(X^f_{4\,\mathbf{a}})_{\mathbf{ij}}\mathcal{A}(1^{-1/2}, 2^{-1/2}, 3^{-1}) \equiv 0_{\mathbf{aij}} \tag{3.54}$$

- $\mathcal{A}(\mathbf{1}_{\mathbf{i}}^{-1/2}, \mathbf{2}_{\mathbf{j}}^{+1/2}, \mathbf{1}_{\mathbf{a}}^{+1})$



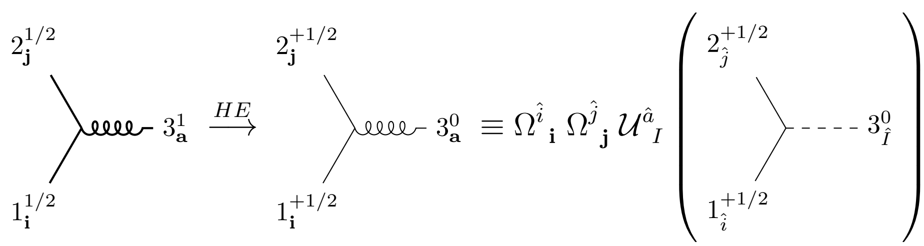
$$m_{\mathbf{a}}(X^f_{2\,\mathbf{a}} + X^f_{3\,\mathbf{a}})_{\mathbf{ij}}\mathcal{A}(1^{-1/2}, 2^{+1/2}, 3^{+1}) \equiv \Omega^{\hat{i}}{}_{\mathbf{i}}\, \Omega^{\hat{j}}{}_{\mathbf{j}}\, \mathcal{O}^{\hat{a}}{}_{a}(T^f_{\hat{a}})_{\hat{i}\hat{j}}\mathcal{A}(1^{-1/2}, 2^{+1/2}, 3^{+1}) \tag{3.55}$$

- $\mathcal{A}(\mathbf{1}_{\mathbf{i}}^{+1/2}, \mathbf{2}_{\mathbf{j}}^{-1/2}, \mathbf{1}_{\mathbf{a}}^{+1})$



$$-m_{\mathbf{a}}(X^{f}_{1\,\mathbf{a}})_{\mathbf{ij}}\mathcal{A}(1^{+1/2}, 2^{-1/2}, 3^{+1}) \equiv \Omega^{\hat{i}}{}_{\mathbf{i}}\, \Omega^{\hat{j}}{}_{\mathbf{j}}\, \mathcal{O}^{\hat{a}}{}_{\mathbf{a}}\, (T^{f}_{\hat{a}})_{\hat{j}\hat{i}}\mathcal{A}(1^{+1/2}, 2^{-1/2}, 3^{+1}) \tag{3.56}$$

- $\mathcal{A}(\mathbf{1}_{\mathbf{i}}^{+1/2}, \mathbf{2}_{\mathbf{j}}^{+1/2}, \mathbf{1}_{\mathbf{a}}^{0})$



$$\left(m_{\mathbf{j}}X^{f}_{1\,\mathbf{a}} - m_{\mathbf{i}}X^{f}_{2\,\mathbf{a}} + m_{\mathbf{a}}X^{f}_{3\,\mathbf{a}}\right)_{\mathbf{ij}} \mathcal{A}(1^{+1/2}, 2^{+1/2}, 3^{0}) \equiv \Omega^{\hat{i}}{}_{\mathbf{i}}\, \Omega^{\hat{j}}{}_{\mathbf{j}}\, \mathcal{U}^{\hat{I}}{}_{\mathbf{a}}\, Y_{\hat{I}\hat{i}\hat{j}}\, \mathcal{A}(1^{+1/2}, 2^{+1/2}, 3^{0}) \tag{3.57}$$

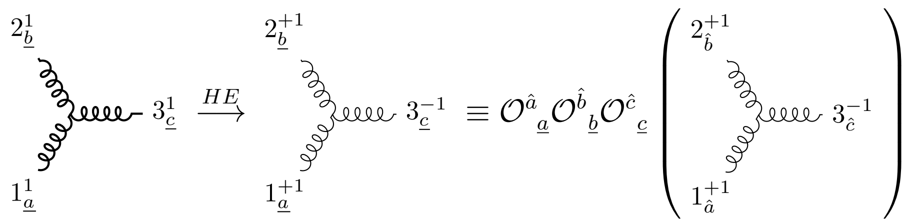
Note that the kinematic part of the amplitudes on each side of the matching conditions are the same, as they should be. Thus, we are left with relations between the masses and color structures or charges only.

Of course, these only represent a small subset of the possible matching conditions we can obtain. Considering every possible spin configuration leads to redundant constraints, and so we omit those. However, we must address the possibility of IR amplitudes with massless particles.

For the case of possibly massless particles in the IR, its easier to make the simultaneous replacements $\mathbf{a} \to a, m_{\mathbf{a}} \to m_a = 0$ and/or $\mathbf{i} \to i, m_{\mathbf{i}} \to m_i = 0$, than compute the actual IR amplitudes from first principles. For example, eq. (3.51) reads

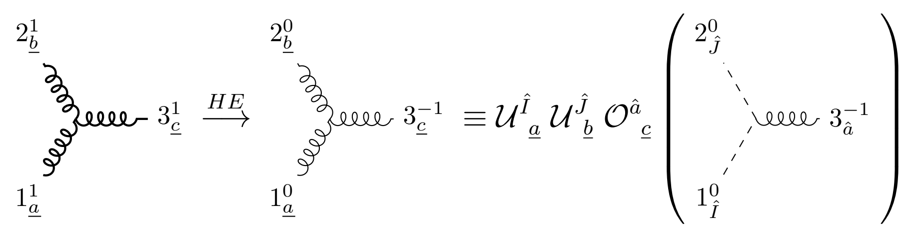
$h_{a\mathbf{bc}} = f_{\hat{a}\hat{b}\hat{c}}\, \mathcal{O}^{\hat{a}}{}_{a}\, \mathcal{O}^{\hat{b}}{}_{\mathbf{b}}\, \mathcal{O}^{\hat{c}}{}_{\mathbf{c}}$ for $\mathbf{a} \to a$. Considering every possible case is not necessary and obscures the simplicity of the constraints and solutions. As such, we use we introduce an underlined notation, that can represent a massless or massive particle, summarized in Table 3.2. With this notation, we can make the replacements to consider all combinations of massless and massive by $\mathbf{a} \to \underline{a}$, $\mathbf{b} \to \underline{b}$ and $\mathbf{c} \to \underline{c}$, where $\underline{a} = \{a, \mathbf{a}\}$, $\underline{b} = \{b, \mathbf{b}\}$ and $\underline{c} = \{c, \mathbf{c}\}$. The matching now takes the form,

- $\mathcal{A}(\mathbf{1}_{\underline{a}}^{+1}, \mathbf{2}_{\underline{b}}^{+1}, \mathbf{1}_{\underline{c}}^{-1})$



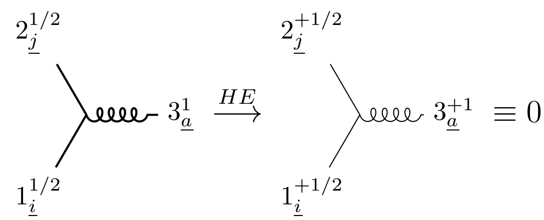
$$h_{\underline{abc}}\mathcal{A}(1^{+1}, 2^{+1}, 3^{-1}) \equiv \mathcal{O}^{\hat{a}}{}_{\underline{a}}\mathcal{O}^{\hat{b}}{}_{\underline{b}}\mathcal{O}^{\hat{c}}{}_{\underline{c}}\; f_{\hat{a}\hat{b}\hat{c}}\mathcal{A}(1^{+1}, 2^{+1}, 3^{-1}) \quad (3.58)$$

- $\mathcal{A}(\mathbf{1}_{\underline{a}}^{0}, \mathbf{2}_{\underline{b}}^{0}, \mathbf{1}_{\underline{c}}^{-1})$



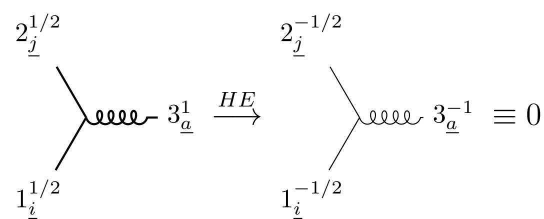
$$h_{\underline{abc}}\frac{(m_{\underline{c}}^2 - m_{\underline{a}}^2 - m_{\underline{b}}^2)}{2\, m_{\underline{a}} m_{\underline{b}}}\mathcal{A}(1^0, 2^0, 3^{-1}) \equiv \mathcal{U}^{\hat{I}}{}_{\underline{a}}\,\mathcal{U}^{\hat{J}}{}_{\underline{b}}\,\mathcal{O}^{\hat{a}}{}_{\underline{c}}\,(T^s_{\hat{a}})_{\hat{I}\hat{J}}\mathcal{A}(1^0, 2^0, 3^{-1}) \quad (3.59)$$

- $\mathcal{A}(\mathbf{1}_{\underline{i}}^{+1/2}, \mathbf{2}_{\underline{j}}^{+1/2}, \mathbf{1}_{\underline{a}}^{+1})$



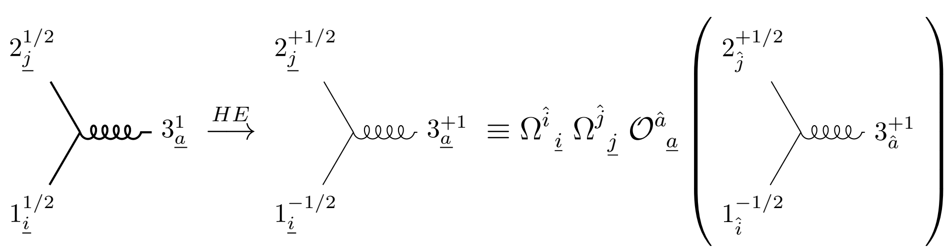
$$(X^f_{3\ \underline{a}})_{\underline{ij}}\mathcal{A}(1^{+1/2}, 2^{+1/2}, 3^{+1}) \equiv 0_{\underline{aij}} \tag{3.60}$$

- $\mathcal{A}(\mathbf{1}_{\underline{i}}^{-1/2}, \mathbf{2}_{\underline{j}}^{-1/2}, \mathbf{1}_{\underline{a}}^{-1})$



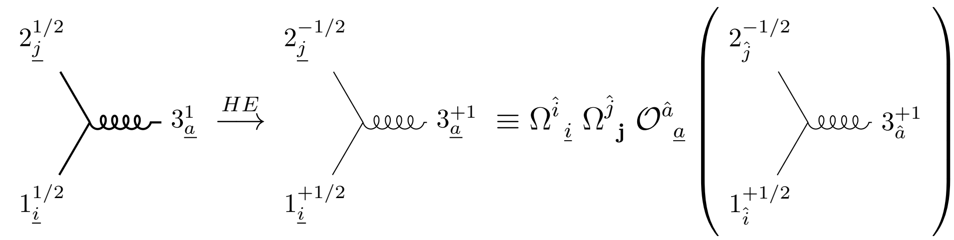
$$(X^f_{4\ \underline{a}})_{\underline{ij}}\mathcal{A}(1^{-1/2}, 2^{-1/2}, 3^{-1}) \equiv 0_{\underline{aij}} \tag{3.61}$$

- $\mathcal{A}(\mathbf{1}_{\underline{i}}^{-1/2}, \mathbf{2}_{\underline{j}}^{+1/2}, \mathbf{1}_{\underline{a}}^{+1})$



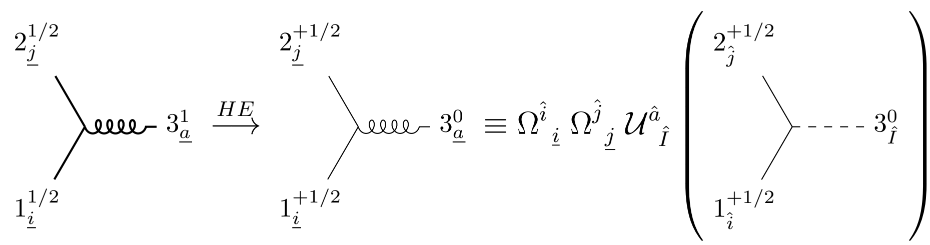
$$m_{\underline{a}}(X^f_{2\ \underline{a}} + X^f_{3\ \underline{a}})_{\underline{ij}}\mathcal{A}(1^{-1/2}, 2^{+1/2}, 3^{+1}) \equiv \Omega^{\hat{i}}{}_{\underline{i}}\, \Omega^{\hat{j}}{}_{\underline{j}}\, \mathcal{O}^{\hat{a}}{}_{a}(T^f_{\hat{a}})_{\hat{i}\hat{j}}\mathcal{A}(1^{-1/2}, 2^{+1/2}, 3^{+1}) \tag{3.62}$$

- $\mathcal{A}(\mathbf{1}_{\underline{i}}^{+1/2}, \mathbf{2}_{\underline{j}}^{-1/2}, \mathbf{1}_{\underline{a}}^{+1})$



$$-m_{\underline{a}}(X^f_{1\,\underline{a}})_{\underline{i}\underline{j}}\mathcal{A}(1^{+1/2}, 2^{-1/2}, 3^{+1}) \equiv \Omega^{\hat{i}}{}_{\underline{i}}\, \Omega^{\hat{j}}{}_{\underline{j}}\, \mathcal{O}^{\hat{a}}{}_{\underline{a}}\, (T^f_{\hat{a}})_{\hat{j}\hat{i}}\mathcal{A}(1^{+1/2}, 2^{-1/2}, 3^{+1}) \tag{3.63}$$

- $\mathcal{A}(\mathbf{1}_{\underline{i}}^{+1/2}, \mathbf{2}_{\underline{j}}^{+1/2}, \mathbf{1}_{\underline{a}}^{0})$



$$\left(m_{\underline{j}} X^f_{1\,\underline{a}} - m_{\underline{i}} X^f_{2\,\underline{a}} + m_{\underline{a}} X^f_{3\,\underline{a}}\right)_{\underline{i}\underline{j}} \mathcal{A}(1^{+1/2}, 2^{+1/2}, 3^{0}) \equiv \Omega^{\hat{i}}{}_{\underline{i}}\, \Omega^{\hat{j}}{}_{\underline{j}}\, \mathcal{U}^{\hat{I}}{}_{\underline{a}}\, Y_{\hat{I}\hat{i}\hat{j}}\, \mathcal{A}(1^{+1/2}, 2^{+1/2}, 3^{0}) \tag{3.64}$$

We summarize ALL the constraints from matching below,

$$h_{\underline{abc}} = f_{\hat{a}\hat{b}\hat{c}}\, \mathcal{O}^{\hat{a}}{}_{\underline{a}}\, \mathcal{O}^{\hat{b}}{}_{\underline{b}}\, \mathcal{O}^{\hat{c}}{}_{\underline{c}}\,, \tag{3.65}$$

$$h_{\underline{abc}} \frac{(m_{\underline{c}}^2 - m_{\underline{a}}^2 - m_{\underline{b}}^2)}{2\, m_{\underline{a}} m_{\underline{b}}} = \mathcal{U}^{\hat{I}}{}_{\underline{a}}\, \mathcal{U}^{\hat{J}}{}_{\underline{b}}\, \mathcal{O}^{\hat{c}}{}_{\underline{c}}\, (T^s_{\hat{c}})_{\hat{I}\hat{J}}\,, \tag{3.66}$$

$$m_{\underline{a}}(X^f_{2\,\underline{a}})_{\underline{i}\underline{j}} = \Omega^{\hat{i}}{}_{\underline{i}}\, \Omega^{\hat{j}}{}_{\underline{j}}\, \mathcal{O}^{\hat{a}}{}_{\underline{a}} (T^f_{\hat{a}})_{\hat{i}\hat{j}}\,, \tag{3.67}$$

$$-m_{\underline{a}}(X^f_{1\,\underline{a}})_{\underline{i}\underline{j}} = \Omega^{\hat{i}}{}_{\underline{i}}\, \Omega^{\hat{j}}{}_{\underline{j}}\, \mathcal{O}^{\hat{a}}{}_{\underline{a}}\, (T^f_{\hat{a}})_{\hat{j}\hat{i}}\,, \tag{3.68}$$

$$m_{\underline{j}}(X^f_{1\,\underline{a}})_{\underline{i}\underline{j}} - m_{\underline{i}}(X^f_{2\,\underline{a}})_{\underline{i}\underline{j}} = \Omega^{\hat{i}}{}_{\underline{i}}\, \Omega^{\hat{j}}{}_{\underline{j}}\, \mathcal{U}^{\hat{I}}{}_{\underline{a}}\, Y_{\hat{I}\hat{i}\hat{j}}\,. \tag{3.69}$$

3.4.4 Matching Solutions

We will first present the solutions to the above constraints, and then show why they work. The solutions are,

- The structure constants in the IR are simply a change of basis or rotation of the structure constants in the UV,

$$h_{\underline{abc}} = f_{\hat{a}\hat{b}\hat{c}} \mathcal{O}^{\hat{a}}{}_{\underline{a}} \mathcal{O}^{\hat{b}}{}_{\underline{b}} \mathcal{O}^{\hat{c}}{}_{\underline{c}} \,, \tag{3.70}$$

  for $\underline{a} = \{a, \mathbf{a}\}$, $\underline{b} = \{b, \mathbf{b}\}$ and $\underline{c} = \{c, \mathbf{c}\}$. In other words, the couplings of any three massive or massless spin-1 particles in the IR can be determined from their massless counterparts in the UV.

- The 'generators' in the IR, in the scalar and adjoint representation, are simply a change of basis of the generators in the UV,

$$X^s_{\underline{a}} = T^s_{\hat{a}} \mathcal{O}^{\hat{a}}{}_{\underline{a}} \quad \text{and} \quad X^{\mathrm{ad}}_{\underline{a}} = T^{\mathrm{ad}}_{\hat{a}} \mathcal{O}^{\hat{a}}{}_{\underline{a}} \,, \tag{3.71}$$

  for $\underline{a} = \{a, \mathbf{a}\}$. So the symmetry group, or more specifically the algebra, in the UV is inherited by the IR.

- The masses of the bosons are generated by

$$m_{\underline{a}} U^{\hat{I}}{}_{\underline{a}} = (X^s_{\underline{a}})^{\hat{I}\hat{J}} v_{\hat{J}} = (T^s_{\hat{a}} \mathcal{O}^{\hat{a}}{}_{\underline{a}})^{\hat{I}\hat{J}} v_{\hat{J}} \,, \tag{3.72}$$

  for $\underline{a} = \{a, \mathbf{a}\}$ and for some vector $\boldsymbol{v}$. Note that $\boldsymbol{v}$ here plays a similar role to that of the vacuum expectation value in the traditional Higgs mechanism, however, in our case, we have no mention of quantum fields. Further, since U

is an element of $SO(N_s)$, then

$$m_a^2 \delta_{ab} = \boldsymbol{v}^T X_a^{sT} X_b^s \boldsymbol{v} \,. \tag{3.73}$$

- The masses of the fermions are generated by

$$\mathrm{diag}(m_1, \ldots, m_{N_f}) = m_{\underline{k}} \delta_{\underline{k}}^{\,\underline{i}} = [\Omega^T (v^{\hat{I}} Y_{\hat{I}\hat{j}\hat{k}} + \tilde{m}_{\hat{j}\hat{k}}) \Omega]_{\underline{k}}^{\,\underline{i}} \,, \tag{3.74}$$

  for m_i real and non-negative, and $\tilde{m}_{jk}$ contributions from a background scalar, eq. (3.37). More succinctly, $\mathrm{diag}(m_1, \ldots, m_{N_f}) = \Omega^T(\boldsymbol{v}.Y + \tilde{m})\Omega$. See [71] for discussions on Takagi diagonalization. Note that non-zero gauge invariant fermion masses are allowed, with no contributions from the Yukawa couplings or background scalars, which we highlight in Section 3.4.5.

For the rest of this subsection, we will show why these are the solutions. The approach will be to substitute our solutions into our constraints and use our conditions from consistent factorization.

The first solution, eq. (3.70) is the easiest, as it is a consequence of considering all the combinations of massive and massless particles in the IR from eq. (3.65).

The second and third solution, eq. (3.71) and eq. (3.72), solve the constraint eq. (3.66). To illustrate, starting with eq. (3.66),

$$\frac{1}{2}\, h_{\underline{abc}} (m_{\underline{c}}^2 - m_{\underline{a}}^2 - m_{\underline{b}}^2) = m_{\underline{a}} \mathcal{U}^{\hat{I}}{}_{\underline{a}}\, m_{\underline{b}} \mathcal{U}^{\hat{J}}{}_{\underline{b}}\, \mathcal{O}^{\hat{c}}{}_{\underline{c}}\, (T^s_{\hat{c}})_{\hat{I}\hat{J}} \,, \tag{3.75}$$

we will consider the right hand side of the equation, substitute the solution, and show that it is equal to the left hand side. Let us proceed. Substituting the solution on the right hand side, and using the anti-symmetry of $(T^s_{\hat{a}})_{\hat{I}\hat{J}} = -(T^s_{\hat{a}})_{\hat{J}\hat{I}}$,

$$\begin{aligned}
m_{\underline{a}}\mathcal{U}^{\hat{I}}{}_{\underline{a}}\, m_{\underline{b}}\mathcal{U}^{\hat{J}}{}_{\underline{b}}\, \mathcal{O}^{\hat{c}}{}_{\underline{c}}\, (T^s_{\hat{c}})_{\hat{I}\hat{J}} &= (X^s_{\underline{a}})^{\hat{I}\hat{K}} v_{\hat{K}}\, (T^s_{\hat{c}}\mathcal{O}^{\hat{c}}{}_{\underline{c}})_{\hat{I}\hat{J}}\, (X^s_{\underline{b}})^{\hat{J}\hat{L}} v_{\hat{L}} \\
&= \mathcal{O}^{\hat{a}}{}_{\underline{a}}\, \mathcal{O}^{\hat{b}}{}_{\underline{b}}\, \mathcal{O}^{\hat{c}}{}_{\underline{b}}\, v_{\hat{K}}\, [(T^s_{\hat{a}})^{\hat{I}\hat{K}}\, (T^s_{\hat{c}})_{\hat{I}\hat{J}}\, (T^s_{\hat{b}})^{\hat{J}\hat{L}}]\, v_{\hat{L}} \\
&= \mathcal{O}^{\hat{a}}{}_{\underline{a}}\, \mathcal{O}^{\hat{b}}{}_{\underline{b}}\, \mathcal{O}^{\hat{c}}{}_{\underline{c}}\, v_{\hat{K}}\, [-(T^s_{\hat{a}})^{\hat{K}\hat{I}}\, (T^s_{\hat{c}})_{\hat{I}\hat{J}}\, (T^s_{\hat{b}})^{\hat{J}\hat{L}}]\, v_{\hat{L}} \\
&= \mathcal{O}^{\hat{a}}{}_{\underline{a}}\, \mathcal{O}^{\hat{b}}{}_{\underline{b}}\, \mathcal{O}^{\hat{c}}{}_{\underline{c}}\, v_{\hat{L}}\, [(T^s_{\hat{a}})^{\hat{L}\hat{J}}\, (T^s_{\hat{c}})_{\hat{J}\hat{I}}\, (T^s_{\hat{b}})^{\hat{I}\hat{K}}]\, v_{\hat{K}} \\
&= \frac{1}{2}\mathcal{O}^{\hat{a}}{}_{\underline{a}}\, \mathcal{O}^{\hat{b}}{}_{\underline{b}}\, \mathcal{O}^{\hat{c}}{}_{\underline{c}}\, v_{\hat{K}}\, [T^s_{\hat{b}}T^s_{\hat{c}}T^s_{\hat{a}} - T^s_{\hat{a}}T^s_{\hat{b}}T^s_{\hat{c}}]^{\hat{K}\hat{L}}\, v_{\hat{L}} \\
&= \frac{1}{2} v_{\hat{K}}\, [X^s_{\underline{b}}X^s_{\underline{c}}X^s_{\underline{a}} - X^s_{\underline{a}}X^s_{\underline{b}}X^s_{\underline{c}}]^{\hat{K}\hat{L}}\, v_{\hat{L}}\,. \qquad (3.76)
\end{aligned}$$

Next, by using the commutation relation from consistent factorization in eq. (3.29), $[T_{\hat{a}}, T_{\hat{b}}] = f_{\hat{a}\hat{b}}{}^{\hat{c}}\, T_{\hat{c}}$ repeatedly, we have,

$$\begin{aligned}
T_{\hat{b}}T_{\hat{c}}T_{\hat{a}} &= (T_{\hat{c}}T_{\hat{b}} + f_{\hat{b}\hat{c}}{}^{\hat{d}}\, T_{\hat{d}})\, T_{\hat{a}} \\
&= T_{\hat{c}}(T_{\hat{a}}T_{\hat{b}} + f_{\hat{b}\hat{a}}{}^{\hat{d}}\, T_{\hat{d}}) + f_{\hat{b}\hat{c}}{}^{\hat{d}}\, T_{\hat{d}}T_{\hat{a}} \\
&= (T_{\hat{a}}T_{\hat{c}} + f_{\hat{c}\hat{a}}{}^{\hat{d}}\, T_{\hat{d}})\, T_{\hat{b}} + f_{\hat{b}\hat{a}}{}^{\hat{d}}\, T_{\hat{c}}T_{\hat{d}} + f_{\hat{b}\hat{c}}{}^{\hat{d}}\, T_{\hat{d}}T_{\hat{a}} \\
&= T_{\hat{b}}T_{\hat{c}}T_{\hat{a}} + f_{\hat{c}\hat{a}}{}^{\hat{d}}\, T_{\hat{d}}T_{\hat{b}} + f_{\hat{b}\hat{a}}{}^{\hat{d}}\, T_{\hat{c}}T_{\hat{d}} + f_{\hat{b}\hat{c}}{}^{\hat{d}}\, T_{\hat{d}}T_{\hat{a}} \\
T_{\hat{b}}T_{\hat{c}}T_{\hat{a}} - T_{\hat{b}}T_{\hat{c}}T_{\hat{a}} &= f_{\hat{c}\hat{a}}{}^{\hat{d}}\, T_{\hat{d}}T_{\hat{b}} + f_{\hat{b}\hat{a}}{}^{\hat{d}}\, T_{\hat{c}}T_{\hat{d}} + f_{\hat{b}\hat{c}}{}^{\hat{d}}\, T_{\hat{d}}T_{\hat{a}}\,. \qquad (3.77)
\end{aligned}$$

Acting on the above with the rotations $\mathcal{O}^{\hat{a}}{}_{\underline{a}}\, \mathcal{O}^{\hat{b}}{}_{\underline{b}}\, \mathcal{O}^{\hat{c}}{}_{\underline{c}}$

$$X_{\underline{b}}\, X_{\underline{c}}\, X_{\underline{a}} - X_{\underline{a}}\, X_{\underline{c}}\, X_{\underline{b}} = h_{\underline{ca}}{}^{\underline{d}}\, X_{\underline{d}}\, X_{\underline{b}} + h_{\underline{ba}}{}^{\underline{d}}\, X_{\underline{c}}\, X_{\underline{d}} + h_{\underline{bc}}{}^{\underline{d}}\, X_{\underline{d}}\, X_{\underline{a}}\,, \qquad (3.78)$$

and contracting with v, eq. (3.76) now reads,

$$\frac{1}{2}\boldsymbol{v}[X^s_{\underline{b}}\, X^s_{\underline{c}}\, X^s_{\underline{a}} - X^s_{\underline{a}}\, X^s_{\underline{c}}\, X^s_{\underline{b}}]\boldsymbol{v} = \frac{1}{2} v_{\hat{K}}[h_{\underline{ca}}{}^{\underline{d}}\, X^s_{\underline{d}}\, X^s_{\underline{b}} + h_{\underline{ba}}{}^{\underline{d}}\, X^s_{\underline{c}}\, X^s_{\underline{d}} + h_{\underline{bc}}{}^{\underline{d}}\, X^s_{\underline{d}}\, X^s_{\underline{a}}]^{\hat{K}\hat{L}} v_{\hat{L}}\,. \qquad (3.79)$$

Now, consider the term $h_{\underline{ca}}{}^{\underline{d}}\, v_{\hat{K}}[X_{\underline{d}}\, X_{\underline{b}}]^{\hat{K}\hat{L}} v_{\hat{L}}$ in the eq above,

$$
\begin{aligned}
h_{\underline{ca}}{}^{\underline{d}}\, v_{\hat{K}}[X_{\underline{d}}\, X_{\underline{d}}]^{\hat{K}\hat{L}} v_{\hat{L}} &= h_{\underline{ac}}{}^{\underline{dc}}\, V_{\hat{K}}\,(X_{\underline{d}})^{\hat{K}}{}_{\hat{J}}\,(X_{\underline{b}})^{\hat{J}\hat{L}}\, V_{\hat{L}} \\
&= -h_{\underline{ca}}{}^{\underline{d}}\,(X_{\underline{d}})_{\hat{J}}{}^{\hat{K}} v_{\hat{K}}\,(X_{\underline{b}})^{\hat{J}\hat{L}}\, v_{\hat{L}}\,.
\end{aligned}
\tag{3.80}
$$

This can be simplified again by inserting our solution,

$$
\begin{aligned}
h_{\underline{ca}}{}^{\underline{d}}\, v_{\hat{K}}[X_{\underline{d}}\, X_{\underline{b}}]^{\hat{K}\hat{L}} v_{\hat{L}} &= -h_{\underline{ca}}{}^{\underline{d}}\, m_{\underline{d}}\, U_{\underline{d}\hat{J}}\, U_{\underline{b}}{}^{\hat{J}}\, m_{\underline{b}} \\
&= -h_{\underline{ca}}{}^{\underline{d}}\, m_{\underline{d}}\,(U^T U)_{\underline{db}}\, m_{\underline{b}} \\
&= -h_{\underline{cab}}\, m_{\underline{b}}^2\,.
\end{aligned}
\tag{3.81}
$$

So, making the replacements above in eq. (3.79),

$$
\begin{aligned}
\frac{1}{2}\boldsymbol{v}[X^s_{\underline{b}}\, X^s_{\underline{c}}\, X^s_{\underline{a}} - X^s_{\underline{a}}\, X^s_{\underline{c}}\, X^s_{\underline{b}}]\boldsymbol{v} &= \frac{1}{2}(-h_{\underline{cab}}\, m_{\underline{b}}^2 - h_{\underline{bac}}\, m_{\underline{c}}^2 - h_{\underline{bac}}\, m_{\underline{a}}^2) \\
&= \frac{1}{2} h_{\underline{abc}}\,(m_{\underline{c}}^2 - m_{\underline{b}}^2 - m_{\underline{a}}^2)\,,
\end{aligned}
\tag{3.82}
$$

which is the left hand side of eq. (3.75), which we set out to show.

Finally, we show that fermion mass in eq. (3.74) is a solution. To do this, we will take our constraints, substitute our solutions, and then show that the resulting equation is that one would expect from consistent factorization from eq. (3.35).

We begin by combining our constraints eq. (3.67) and eq. (3.68) in eq. (3.69), and making the replacements to consider all combinations of massive and massless by $\mathbf{a} \to \underline{a}$, $\mathbf{b} \to \underline{b}$ and $\mathbf{c} \to \underline{c}$, where $\underline{a} = \{a, \mathbf{a}\}$, $\underline{b} = \{b, \mathbf{b}\}$ and $\underline{c} = \{c, \mathbf{c}\}$, we have,

$$
(X^f_{1\,\underline{a}})_{\underline{ij}}\, m_{\underline{j}} - (X^f_{2\,\underline{a}})_{\underline{ij}}\, m_{\underline{i}} = \Omega^{\hat{i}}{}_{\underline{i}}\, \Omega^{\hat{j}}{}_{\underline{j}}\, \mathcal{U}^{\hat{I}}{}_{\underline{a}}\, Y^f_{\hat{I}\hat{i}\hat{j}}\,,
\tag{3.83}
$$

which, after substitution of eq. (3.67) and eq. (3.68) give,

$$m_{\underline{j}}\,(\Omega^T)_{\underline{i}}{}^{\hat{i}}\,(T^f_{\hat{a}}\,\mathcal{O}^{\hat{a}}{}_{\underline{a}})_{\hat{j}\hat{i}}\,\Omega^{\hat{j}}{}_{\underline{j}} + m_{\underline{i}}\,(\Omega^T)_{\underline{j}}{}^{\hat{j}}\,(T^f_{\hat{a}}\mathcal{O}^{\hat{a}}{}_{\underline{a}})_{\hat{i}\hat{j}}\,\Omega^{\hat{i}}{}_{\underline{i}} + (\Omega^T)_{\underline{j}}{}^{\hat{j}}\,Y^f_{\hat{I}\hat{j}\hat{k}}\,U^{\hat{I}}{}_{\underline{a}}\,m_{\underline{a}}\,\Omega^{\hat{i}}{}_{\underline{i}} = 0\,. \tag{3.84}$$

Substituting for the Higgs mechanism eq. (3.72), and removing some index contractions, we have,

$$m_{\underline{j}}\,\delta_{\underline{j}}^{\underline{k}}[\Omega^T(T^f\mathcal{O})_{\underline{a}}\Omega]_{\underline{ki}} + m_{\underline{i}}\,\delta_{\underline{i}}^{\underline{k}}[\Omega^T(T^f\mathcal{O})_{\underline{a}}\Omega]_{\underline{kj}} + [\Omega^T(Y^f.X^s.\boldsymbol{v})_{\underline{a}}\Omega]_{\underline{ij}} = 0\,. \tag{3.85}$$

(no sum over i and j). Substituting the solution for fermion masses eq. (3.74), and the relation between the UV and IR generators in the representation of the scalars,

$$[\Omega^T(\boldsymbol{v}.Y+\tilde{m})\Omega\Omega^T(T^f\mathcal{O})_{\underline{a}}\Omega\ ]_{\underline{ij}} + [\Omega^T(\boldsymbol{v}.Y+\tilde{m})\Omega\Omega^T(T^f\mathcal{O})_{\underline{a}}\Omega]_{\underline{ji}} + [\Omega^T(Y.T^s\mathcal{O}.\boldsymbol{v})_{\underline{a}}\Omega]_{\underline{ij}} = 0\,. \tag{3.86}$$

Multiplying from the left by Ω and the right by Ω^T and $\mathcal{O}^{-1}$, then making the obvious cancellations $\Omega\Omega^T = 1$ yields,

$$(\boldsymbol{v}.Y+\tilde{m})T^f + (\boldsymbol{v}.Y+\tilde{m})T^f + (Y.T^s.\boldsymbol{v}) = 0\,, \tag{3.87}$$

which is simply the sum of the Yukawa constraint eq. (3.35)

$$(\boldsymbol{v}.Y)T^f + (\boldsymbol{v}.Y)T^f + (Y.T^s.\boldsymbol{v}) = 0\,, \tag{3.88}$$

and the background scalar mass eq. (3.37)

$$\tilde{m}T^f + \tilde{m}T^f = 0\,. \tag{3.89}$$

Hence, we have shown that these are the solutions to our constraints.

3.4.5 Discussion

Consider eq. (3.72) for massless particles in the IR $\underline{a} = a$ and $m_{\underline{a}} = 0$,

$$0_{\hat{J}} = v^{\hat{I}}(X^s_a)_{\hat{I}\hat{J}} = v^{\hat{I}}(T^s_{\hat{a}}\mathcal{O}^{\hat{a}}{}_a)_{\hat{I}\hat{J}} \,. \tag{3.90}$$

This is the statement that for every massless spin-1 particle in the IR, a, there is a generator X^s_a that annihilates $\boldsymbol{v}$. This places constraints on the form of $\boldsymbol{v}$.

Next, consider two massless spin-1 particles in the IR, a and b. By the above, $(X^s_a)\boldsymbol{v} = \mathbf{0}$ and $(X^s_b)\boldsymbol{v} = \mathbf{0}$. So, $[X^s_a, X^s_b]\boldsymbol{v} = \mathbf{0}$. But our solutions eq. (3.70) and (3.71) also tell us that $[X^s_a, X^s_b] = h_{ab}{}^{\underline{c}} X^s_{\underline{c}}$. Acting on $\boldsymbol{v}$,

$$\begin{aligned}
[X^s_a, X^s_b]\boldsymbol{v} &= h_{ab}{}^{\underline{c}} X^s_{\underline{c}}\boldsymbol{v} \\
\mathbf{0} &= h_{ab}{}^{c} X^s_c\boldsymbol{v} + h_{ab}{}^{\mathbf{c}} X^s_{\mathbf{c}}\boldsymbol{v} \\
\mathbf{0} &= \mathbf{0} + h_{ab}{}^{\mathbf{c}} X^s_{\mathbf{c}}\boldsymbol{v} \,.
\end{aligned} \tag{3.91}$$

However, for massive particles in the IR, $\mathbf{c}$, $X^s_{\mathbf{c}}\boldsymbol{v} \neq \mathbf{0}$ in general, and so we conclude that $h_{ab}{}^{\mathbf{c}} X^s_{\mathbf{c}} = 0_{ab}$. This means that coupling constant between two massless spin-1 particles and one massive spin-1 particle is zero, or, a massive spin-1 particle cannot decay to two massless spin-1 particles. Of course, we already independently knew this from Yang's theorem, which is a statement from a kinematics. It was amusing to see it here from pure group theory.

With this in mind, we can revisit the commutator,

$$\begin{aligned}
[X^s_a, X^s_b] &= h_{ab}{}^{\underline{c}} X^s_{\underline{c}} \\
&= h_{ab}{}^{c} X^s_c + h_{ab}{}^{\mathbf{c}} X^s_{\mathbf{c}} \\
&= h_{ab}{}^{c} X^s_c \,,
\end{aligned} \tag{3.92}$$

which says that the generators associated with massless particles form a subalgebra $\mathfrak{h}$. So we have discovered that, if there are massless particles in the IR, then there must be a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Hence, there must be a subgroup $H \subset G$.

Lastly, we mention that the constraints eq. (3.75) and eq. (3.85) provide additional information connecting the charges (or more generally the generators) to the masses. For the simplest examples, consider an unbroken generator in the IR, which corresponds to a massless spin-1 particle a. Then, with $m_a = 0$, eq. (3.75) reads,

$$\frac{1}{2}\, h_{a\underline{b}\underline{c}}(m_{\underline{c}}^2 - m_{\underline{b}}^2) = 0\,. \tag{3.93}$$

For $\underline{b} = b$ massless, $m_{\underline{b}} = m_b = 0$, and the equation is almost spoiled, except for the fact that $h_{ab\underline{c}} = 0$. For $\underline{b} = \mathbf{b}$ and $\underline{c} = \mathbf{c}$ massive, then the equation reads,

$$\frac{1}{2}\, h_{a\mathbf{b}\mathbf{c}}(m_{\mathbf{c}}^2 - m_{\mathbf{b}}^2) = 0\,. \tag{3.94}$$

and so we have $m_{\mathbf{b}} = m_{\mathbf{c}}$. In the SM, we know this as the masses of the $W^{\pm}$ being the same.

With respect to the fermion masses, for a unbroken generator $\underline{a} = a$ in the IR, eq. (3.85) reads,

$$m_{\underline{j}}[\Omega^T(T^f\mathcal{O})_a\Omega]_{\underline{j}\underline{i}} + m_{\underline{i}}[\Omega^T(T^f\mathcal{O})_a\Omega]_{\underline{i}\underline{j}} = 0\,, \tag{3.95}$$

which shows that there is a possibility for fermions to have gauge invariant masses. Furthermore, since these masses do not come from Yukawa couplings or background scalars, they can exist in the UV.

3.5 Standard Model

3.5.1 The UV

The symmetry group in the UV, is $G = SU(2)_L \times U(1)_Y$ (we will be working with singlets of $SU(3)$). The Lie algebra associated with this group is spanned by four generators, three from $SU(2)_L$ and one from $U(1)_Y$,

$$\tilde{\mathfrak{g}} = \left\{ \tilde{T}_{1_1}, \tilde{T}_{2_1}, \tilde{T}_{3_1}, \tilde{T}_{1_2} \right\} . \tag{3.96}$$

The particle labels are usually denoted by $1_1 = W_1$, $1_2 = W_2$, $1_3 = W_3$ and $2_1 = B$. As previously mentioned, we find it convenient to rescale our Lie algebra by the coupling constants,

$$\begin{aligned} \mathfrak{g} &= \left\{ g_1\tilde{T}_{1_1}, g_1\tilde{T}_{2_1}, g_1\tilde{T}_{3_1}, g_2\tilde{T}_{2_1} \right\} , \\ &= \{T_1, T_2, T_3, T_4\} , \\ &= \{T_{W_1}, T_{W_2}, T_{W_3}, T_B\} . \end{aligned} \tag{3.97}$$

Now that we have fixed the group structure, we can move on to defining the representations which the spin-0, spin-1/2 and spin-1 transform under.

Starting with the scalar sector, we need to define T^s_a. The Higgs is defined as a hypercharge $Y = y_\phi$ (usually $y_\phi = 1$ with our convention, though we will leave it unspecified for now), $SU(2)$ complex doublet. The Higgs has four real degrees of freedom, and, although we can work in a basis that makes these manifest, for now we choose to work in a basis that has the maximal set of diagonal generators, which is a reducible complex representation. From Appendix B.4, we have $(T^s_a)_{\hat{I}\hat{J}}$,

$$T^s_1 = \frac{g_1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad T^s_2 = \frac{g_1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$T^s_3 = \frac{g_1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T^s_4 = \frac{g_2 y_\phi}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3.98}$$

where $\hat{I}, \hat{J} = \{\phi^+, \phi^0, \phi^-, \phi^{0*}\}$ (note that the labels above will become clear in the next section). We summarize the eigenvalues in Table 3.3 and Figure 3.1.

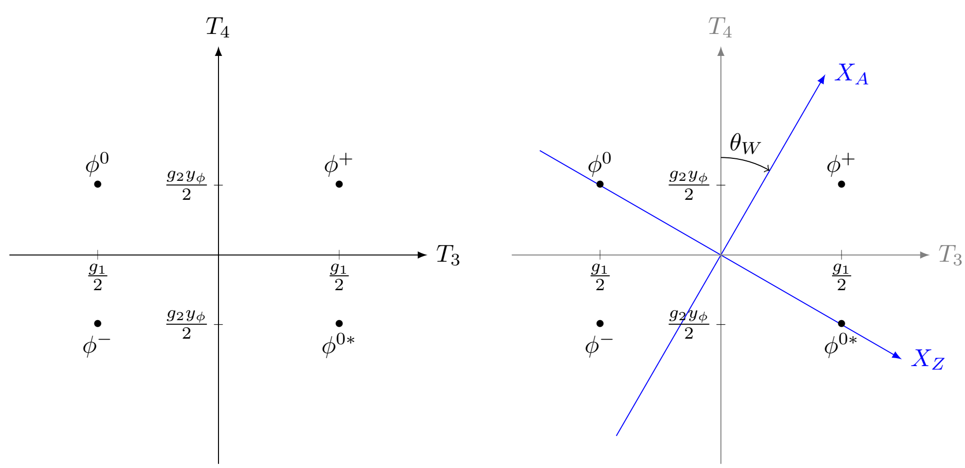


Figure 3.1: The weight diagram of the scalar sector of the Standard Model.

Next, we move onto the spin-1/2 sector. The L is a doublet under $SU(2)_L$ and has hypercharge $Y = y_l$, e_R is a singlet under $SU(2)$ and has hypercharge $Y = y_e$, and ν_R is a singlet under $SU(2)_L$ and has hypercharge $Y = y_\nu$. Since L is a doublet, we have two extra degrees of freedom which we label as $L = \nu_L, e_L$. Hence, we can construct the reducible representation under which the fermions trasnsform, using

Particle Label	$T_3 = T_{W_3}$	$T_4 = T_B$	$X_4 = Q$	$X_4 = Q$
ϕ^+	$\frac{1}{2}\,g_1$	$\frac{1}{2}\,g_2\,y_\phi$	$\frac{g_1 g_2}{\sqrt{g_1^2+g_2^2 y_\phi^2}} y_\phi$	e
ϕ^0	$-\frac{1}{2}\,g_1$	$\frac{1}{2}\,g_2\,y_\phi$	0	0
ϕ^-	$-\frac{1}{2}\,g_1$	$-\frac{1}{2}\,g_2\,y_\phi$	$-\frac{g_1 g_2}{\sqrt{g_1^2+g_2^2 y_\phi^2}} y_\phi$	$-e$
ϕ^{0*}	$\frac{1}{2}\,g_1$	$-\frac{1}{2}\,g_2\,y_\phi$	0	0
ν_L	$\frac{1}{2}\,g_1$	$\frac{1}{2}\,g_2\,y_L$	$\frac{g_1 g_2}{\sqrt{g_1^2+g_2^2 y_\phi^2}} \frac{y_\phi+y_L}{2}$	0
e_L	$-\frac{1}{2}\,g_1$	$\frac{1}{2} g_2\,y_L$	$\frac{g_1 g_2}{\sqrt{g_1^2+g_2^2 y_\phi^2}} \frac{-y_\phi+y_L}{2}$	$-e$
e_R	0	$\frac{1}{2}\,g_2\,y_e$	$\frac{g_1 g_2}{\sqrt{g_1^2+g_2^2 y_\phi^2}} \frac{y_e}{2}$	$-e$
ν_R	0	$\frac{1}{2}\,g_2\,y_\nu$	$\frac{g_1 g_2}{\sqrt{g_1^2+g_2^2 y_\phi^2}} \frac{y_\nu}{2}$	0
ν_L^*	$-\frac{1}{2}\,g_1$	$-\frac{1}{2}\,g_2\,y_L$	$-\frac{g_1 g_2}{\sqrt{g_1^2+g_2^2 y_\phi^2}} \frac{y_\phi+y_L}{2}$	0
e_L^*	$\frac{1}{2}\,g_1$	$-\frac{1}{2} g_2\,y_L$	$\frac{g_1 g_2}{\sqrt{g_1^2+g_2^2 y_\phi^2}} \frac{y_\phi-y_L}{2}$	e
e_R^*	0	$-\frac{1}{2}\,g_2\,y_e$	$\frac{g_1 g_2}{\sqrt{g_1^2+g_2^2 y_\phi^2}} \frac{-y_e}{2}$	e
ν_R^*	0	$-\frac{1}{2}\,g_2\,y_\nu$	$\frac{g_1 g_2}{\sqrt{g_1^2+g_2^2 y_\phi^2}} \frac{-y_\nu}{2}$	0
W^+	g_1	0	$\frac{g_1 g_2}{\sqrt{g_1^2+g_2^2 y_\phi^2}} y_\phi$	e
W_3	0	0	0	0
W^-	$-g_1$	0	$-\frac{g_1 g_2}{\sqrt{g_1^2+g_2^2 y_\phi^2}} y_\phi$	$-e$
B	0	0	0	0

Table 3.3: Weight vectors for the SM spectrum. In the last column, we set $y_\phi = -y_L = -y_e/2$, and $y_\nu = 0$.

$i, j = \{\nu_L, e_L, e_R, \nu_R\}$,

$$\tau_1^f = \frac{g_1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tau_2^f = \frac{g_1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tau_3^f = \frac{g_1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tau_4^f = \frac{g_2}{2}\begin{pmatrix} y_L & 0 & 0 & 0 \\ 0 & y_L & 0 & 0 \\ 0 & 0 & y_{e_R} & 0 \\ 0 & 0 & 0 & y_{\nu_R} \end{pmatrix}. \tag{3.99}$$

Since this representation is in a complex basis, and we would like to consider all the real degrees of freedom explicitly, we follow the exact procedure as we did with the scalars (see Appendix B.4). Notably, we upgrade our representation matrices above to,

$$T_a^f = \begin{pmatrix} \tau_a^f & \\ & -(\tau_a^f)^* \end{pmatrix} . \tag{3.100}$$

where the indexes $i, j = \{\nu_L, e_L, e_R, \nu_R, \nu_L^*, e_L^*, e_R^*, \nu_R^*\}$ now run over the conjugate representation as well. Note that the asterisks are just part of the label names, and only represent the basis that the conjugate representation of the group G is defined over. Further, this basis is still complex, but we are simply a change of basis away from the real degrees of freedom. We summarize the eigenvalues in Table 3.3 and Figure 3.2.

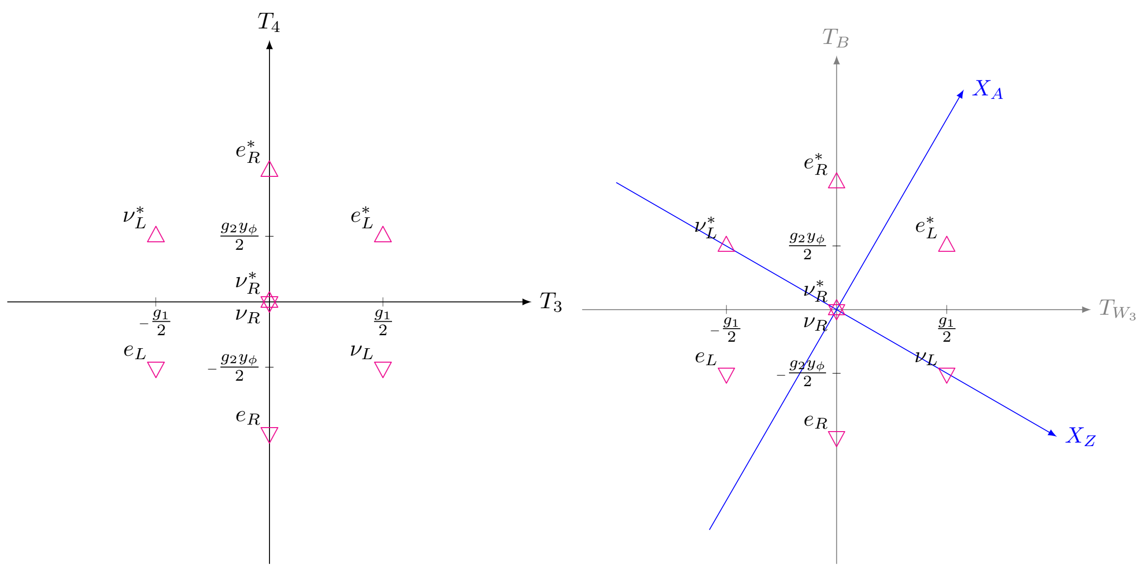


Figure 3.2: The weight diagram for the spin-1/2 sector of the Standard Model.

The spin-1 particles transform under the adjoint representation. Since this is defined through the group G, we have no freedom to select charges. Using eq. (3.43), we can use the representations $R = s, f$, or simply the definitions of $SU(2) \times U(1)$, to first get the structure constant in the UV,

$$f_{\hat{a}\hat{b}\hat{c}} = \begin{pmatrix} \{0,0,0,0\} & \{0,0,ig_1,0\} & \{0,-ig_1,0,0\} & \{0,0,0,0\} \\ \{0,0,-ig_1,0\} & \{0,0,0,0\} & \{ig_1,0,0,0\} & \{0,0,0,0\} \\ \{0,ig_1,0,0\} & \{-ig_1,0,0,0\} & \{0,0,0,0\} & \{0,0,0,0\} \\ \{0,0,0,0\} & \{0,0,0,0\} & \{0,0,0,0\} & \{0,0,0,0\} \end{pmatrix} \tag{3.101}$$

which summarizes that the only interactions in the UV are permutations of $\{W_1, W_2, W_3\}$. The B does not interact with any of the other bosons. Then, we can determine the adjoint representaion via $f_{\hat{a}\hat{b}\hat{c}} = -(T^{\text{ad}}_{\hat{a}})_{\hat{b}\hat{c}}$,

$$T_1^{\text{ad}} = g_1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, T_2^{\text{ad}} = g_1 \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.102}$$

$$T_3^{\text{ad}} = g_1 \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, T_4^{\text{ad}} = g_2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.103}$$

The usual strategy is to diagonalize these, however, instead of changing our basis explicitly, we can determine their values under measurement of T_3^{ad} via commutation, which is how the adjoint representation acts. Defining $T^R_{W^\pm} \equiv \frac{1}{\sqrt{2}}(T_1^R \pm iT_2^R)$,

$$\begin{aligned} T_3^R|T^R_{W^+}\rangle &= |[T_3^R, T^R_{W^+}]\rangle = g_1|T^R_{W^+}\rangle\,, \\ T_3^R|T^R_{W^-}\rangle &= |[T_3^R, T^R_{W^-}]\rangle = -g_1|T^R_{W^-}\rangle\,, \\ T_3^R|T_3^R\rangle &= |[T_3^R, T_3^R]\rangle = 0\,, \\ T_3^R|T_4^R\rangle &= |[T_3^R, T_4^R]\rangle = 0\,, \end{aligned} \tag{3.104}$$

and $T_B^R|T_a^R\rangle = 0$ for all a, where the same results carry though for any representation R. Its worth mentioning here that the $W^{\pm}$ labels are purely associated with the fact that adjoint representation for $SU(2)$ is the spin-1 representation, and so the states T_{W^-},T_3, and T_{W^+} form a triplet. Indeed, $T_{W^+}|T_{W^-}\rangle = |[T_{W^+}, T_{W^-}]\rangle = g_1|T_3\rangle$, and $T_{W^+}|T_3\rangle = |[T_{W^+}, T_3]\rangle = -g_1|T_{W^+}\rangle$. So T_{W^+} raises the states in units of g_1. We have not mentioned anything about $U(1)_{EM}$. We summarize the spin-1 spectrum in Figure 3.3. The summary of Table 3.3 for all particles is shown in

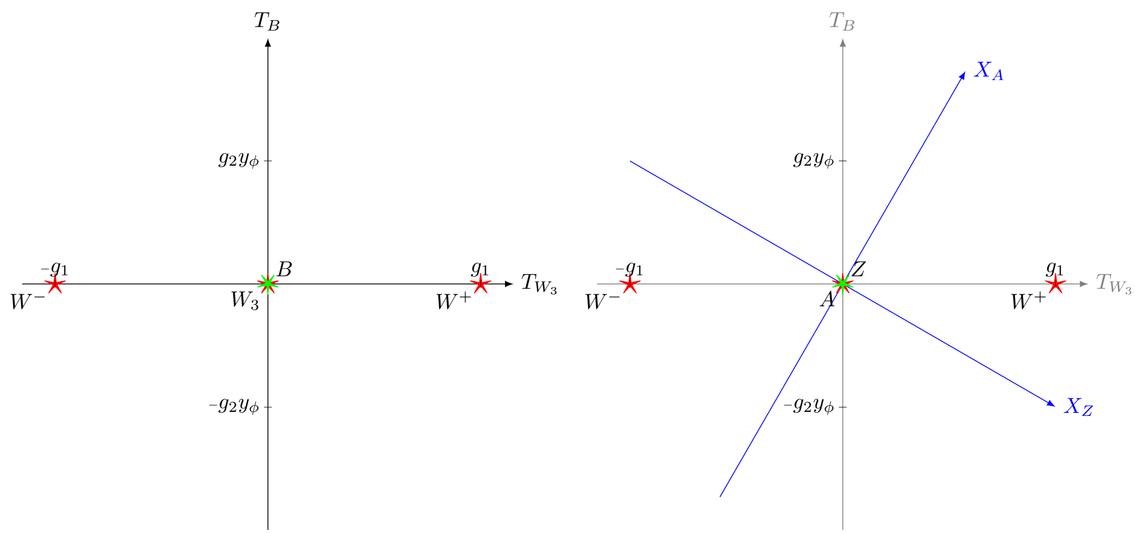


Figure 3.3: The weight diagram of the spin-1 particles of the Standard Model.

The Yukawa couplings $Y_{\hat{I}\hat{j}\hat{k}}$ must satisfy eq. (3.35). Setting $y_\phi = -y_L = -y_e/2$, and $y_\nu = 0$ allows us to determine the non-zero Yukawa couplings and gauge invariant mass terms.

3.5.2 IR

The first thing to address it the presence of a massless particle $a = A$ in the IR, known as the photon. This means we must have a vector $\boldsymbol{v}$ that satisfies eq. (3.90), notably, $X_A^s \boldsymbol{v} = \mathcal{O}_A{}^{\hat{a}} T_{\hat{a}}^s \boldsymbol{v} = \mathbf{0}$. In other words, $\boldsymbol{v}$ must be an eigenvector of X_A^s with eigenvalue 0. Posed in this way, we can use the weight diagrams in Figure 3.1, which represent the eigenvalues of the Cartan subalgebra of $\mathfrak{g}$, in the representation of the

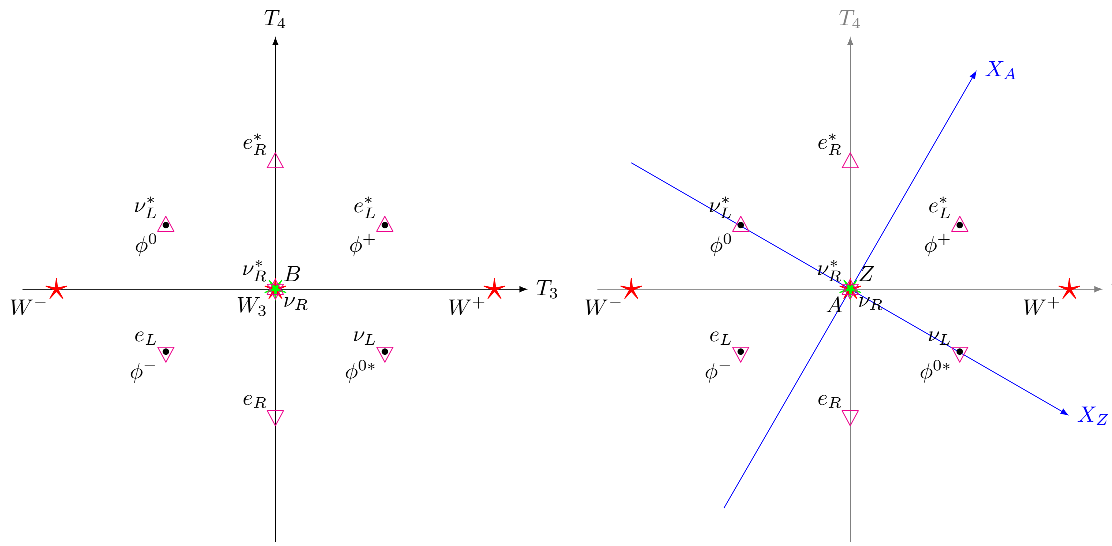


Figure 3.4: The weight diagram for the spin-0, 1/2, and 1 particles of the Standard Model.

scalars. Thus, our only hope of getting a zero eigenvalue is to change the basis of our algebra. The Weinberg angle θ_w [72] was introduced to do just that,

$$\mathcal{O}^{\hat{a}}{}_{\underline{a}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_w & \sin\theta_w \\ 0 & 0 & -\sin\theta_w & \cos\theta_w \end{pmatrix} . \tag{3.105}$$

Note that nothing has required us to make rotations in W_1 and W_2. Now, one can easily solve for θ_w,

$$\tan\theta_w = \frac{g_2 y_\phi}{g_1} \Rightarrow \cos\theta_w = \frac{g_1}{\sqrt{g_1^2 + g_2^2 y_\phi^2}} , \quad \text{and} \quad \sin\theta_w = \frac{g_2 y_\phi}{\sqrt{g_1^2 + g_2^2 y_\phi^2}} . \tag{3.106}$$

To make contact with usual descriptions, we can write the generators in the IR by implementing the rotation,

$$X_1^R = T_1^R , \quad X_2^R = T_2^R , \quad X_3^R = c_w T_3^R - s_w T_4^R , \quad X_4^R = s_w T_3^R + c_w T_4^R . \tag{3.107}$$

The alignment of X_3^s and X_4^s is shown in Figure 3.1. Using the identification $\underline{a}, \underline{b}, \underline{c} \in \{1,2,3,4\} = \{W^1, W^2, Z, A\}$, it is clear that the 'un-broken' generator $X_4^s = X_A^s = Q^s$. Putting back in the coupling constants, we have the familiar,

$$Q^s = \frac{g_1 g_2 y_\phi}{\sqrt{g_1^2 + g_2^2 y_\phi^2}} \left(\tilde{T}_3 + \tilde{T}_4\right) = e \left(\tilde{T}_3 + \tilde{T}_4\right) , \tag{3.108}$$

where we have identified the $e = \frac{g_1 g_2 y_\phi}{\sqrt{g_1^2 + g_2^2 y_\phi^2}}$. Explicitly,

$$X_4^s = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -e & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{3.109}$$

which now makes the notation $\hat{I}, \hat{J} = \{\phi^+, \phi^0, \phi^-, \phi^{0*}\}$ hopefully clear. Since there is only one generator for which $X_a^s \boldsymbol{v} = \mathbf{0}$, our subalgebra $\mathfrak{h} = Q$ and generates a $U(1)$ group, referred to as $U(1)_{EM}$.

With the rotations in eq. (3.107), we can re-measure IR spin-1 particles with X_4. Since we have kept $X_1 = T_1$ and $X_2 = T_2$, then $X_{W^\pm}^R = T_{W^\pm}^R \equiv \frac{1}{\sqrt{2}}(T_1^R \pm i T_2^R)$,

$$\begin{aligned} X_4^R | X_{W^+}^R \rangle &= | [X_4^R, X_{W^+}^R] \rangle = e | X_{W^+}^R \rangle \, , \\ X_4^R | X_{W^-}^R \rangle &= | [X_4^R, X_{W^-}^R] \rangle = -e | X_{W^-}^R \rangle \, , \\ X_4^R | X_3^R \rangle &= | [X_4^R, X_3^R] \rangle = 0 \, , \\ X_4^R | X_4^R \rangle &= | [X_4^R, X_4^R] \rangle = 0 \, , \end{aligned} \tag{3.110}$$

which tells us that the Z and photon A are neutral, and the $W^\pm$ have charges $\pm e$ under $U(1)_{EM}$. Thus, the $\pm$ labels on $W^\pm$ serve a dual purpose, of labeling the charges under T_3 as well as X_4.

Further, with θ_w and $\mathcal{O}$, we can easily compute the couplings between all massless and massive gauge bosons (or the 'structure constant' in the IR). Using $\underline{a}, \underline{b}, \underline{c} \in \{1,2,3,4\} = \{W^1, W^2, Z, A\}$,

$$
\begin{aligned}
h_{\underline{abc}} &= f_{\hat{a}\hat{b}\hat{c}} \mathcal{O}^{\hat{a}}{}_{\underline{a}} \mathcal{O}^{\hat{b}}{}_{\underline{b}} \mathcal{O}^{\hat{c}}{}_{\underline{c}} \\
&= \begin{pmatrix}
\{0,0,0,0\} & \{0,0,ig_1c_w, ig_1s_w\} & \{0,-ig_1c_w,0,0\} & \{0,-ig_1s_w,0,0\} \\
\{0,0,-ig_1c_w,-ig_1s_w\} & \{0,0,0,0\} & \{ig_1c_w,0,0,0\} & \{ig_1s_w,0,0,0\} \\
\{0,ig_1c_w,0,0\} & \{-ig_1c_w,0,0,0\} & \{0,0,0,0\} & \{0,0,0,0\} \\
\{0,ig_1s_w,0,0\} & \{-ig_1s_w,0,0,0\} & \{0,0,0,0\} & \{0,0,0,0\}
\end{pmatrix}
\end{aligned}
\tag{3.111}
$$

From this, it is clear that the $Z = 3$ only interacts with $W_1 = 1$ and $W_2 = 2$, and similarly, the photon $A = 4$ only interacts with $W_1 = 1$ and $W_2 = 2$. Identical bosons do not interact, as seen by the diagonal of zeros. The $W_1 = 1$ can interact with the W_2, Z and A. Not only have we found all the allowed interactions (for massless and massive bosons), but we have also listed all of their couplings!

Now, let us consider the eigenvector $\boldsymbol{v}$. Note that we have yet to say anything about $\boldsymbol{v}$ itself. We have only demanded the existence of a non-trivial solution to $X^s_A \boldsymbol{v} = \boldsymbol{0}$, which was a statement about eigenvalues. In the current reducible complex basis, the most general $\boldsymbol{v}$ takes the form $\boldsymbol{v} = (0, v_3, 0, v_4)$. From Appendix B.4, in the real basis,

$$
\boldsymbol{v} \to A^{-1}\boldsymbol{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_3 + v_4 \\ 0 \\ -v_3 i + v_4 i \end{pmatrix} \tag{3.112}
$$

which tells us that $v_3 = v_4 = v \in \mathbb{R}$. So, in the complex reducible basis $\boldsymbol{v} = v(0,1,0,1)$

and in the real basis $\boldsymbol{v} = v\sqrt{2}(0,1,0,0)$. In the real basis, it is clear that there is an $SO(3)$ symmetry of $\boldsymbol{v}$, known as the 'custodial $SU(2)$' [73].

With the structure of $\boldsymbol{v}$ fixed, we can compute the masses of the spin-1 particles. Of course, the answer must be basis independent, and using either the complex reducible basis or real basis, with $m^2_{\underline{ab}} = \boldsymbol{v}^T X^{sT}_{\underline{a}} X^s_{\underline{b}} \boldsymbol{v}$

$$m^2 = \frac{v^2}{2}\begin{pmatrix} g_1^2 & 0 & 0 & 0 \\ 0 & g_1^2 & 0 & 0 \\ 0 & 0 & g_1^2 + g_2^2\, y_\phi^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} m^2_{W_1} & 0 & 0 & 0 \\ 0 & m^2_{W_2} & 0 & 0 \\ 0 & 0 & m_Z^2 & 0 \\ 0 & 0 & 0 & m_A^2 \end{pmatrix}. \tag{3.113}$$

Next, we can consider the masses of the fermions. As mentioned in eq. (3.74), the masses are given by $m_{\underline{k}}\delta^{\underline{i}}_{\underline{k}} = [\Omega^T(v^{\hat{I}}Y_{\hat{I}\hat{j}\hat{k}})\Omega]^{\underline{i}}_{\underline{k}}$. In the complex basis with $\hat{I} = \{\phi^+, \phi^0, \phi^-, \phi^{0*}\}$ and $\boldsymbol{v} = v(0,1,0,1)$, the non-zero contributors to this are

$$v^{\hat{I}} Y_{\hat{I}\hat{j}\hat{k}} + \tilde{m} = v\left(Y_{\phi^0\hat{j}\hat{k}} + Y_{\phi^{0*}\hat{j}\hat{k}}\right), \tag{3.114}$$

which, restricting to the electron degrees of freedom, yields the singular values,

$$\Omega^T v^{\hat{I}} Y_{\hat{I}\hat{j}\hat{k}} \Omega = v|Y_e| \mathrm{diag}(1,1,1,1)\,. \tag{3.115}$$

So the masses are degenerate with $m_e = v|Y_e|$. In addition, the most general mass term for the neutrinos can be similarly determined, and gives the well known see-saw mechanism.

Chapter 4

Stringy Completions of the Standard Model from the Bottom Up

4.1 Introduction

Already at tree level one can diagnose the need for the UV completion of amplitudes with graviton exchange. Such amplitudes grow with center-of-mass energy and need to be unitarized below the Planck scale. This problem is of course not unique to gravitational amplitudes; scattering of longitudinally polarized W bosons exhibits the same growth with energy, a violation of unitarity which we now understand to be remedied by a weakly coupled Higgs. What is more surprising is that the force we have known the longest has proven the hardest to UV complete.

This state of affairs can be understood in the context of unitarity constraints on massless scattering amplitudes. From this point of view, the graviton mediating a universally attractive force is understood as a consequence of consistent factorization of massless scattering amplitudes with a helicity two particle. By dimensional analysis, the helicity demands gravity has an irrelevant coupling and an associated growth in energy for amplitudes exchanging gravitons. So, consistency of massless scattering both make sense of gravity's privileged position as our oldest force on the books and implies it is the most immediately problematic long-range force at high energies.

But why is it harder to UV complete than W scattering? *Because* it is a long-range force. High energy growth of the amplitude is not the only way to violate unitarity; it can be violated at low energies as well, by not having a positively expandable imaginary part. Contact interactions produce neither singularities nor imaginary parts for tree level amplitudes and therefore do not directly impose additional *a priori* constraints on the amplitude. This affords a certain freedom in engineering UV-completions by resolving a contact interaction into particle production. This is not the case for amplitudes with massless exchanges. The same consistency conditions which imply gravity's universal attraction are positivity constraints which must be respected by any putative UV-completion. This proves to be a surprisingly stringent constraint.

But we also know gravity need not merely be a fly in our unitarity ointment; in the context of string theory, gravity is *required* for unitarity. And by studying consistency conditions on amplitudes from the right point of view, we can see in an analogous way that gravity is not an obstruction, but rather a facilitator of certain means of unitarization. That is what we find in particular for the class of ansätze considered in this work. And by considering stringy ansätze motivated by amplitudes with graviton exchange, we can initiate a program of building UV completions of the standard model (SM) from the bottom up.

The outline of this work is as follows. In Sec. 4.2 we review unitarity constraints on amplitudes and motivate a certain class of stringy ansätze for $2 \to 2$ scattering which unitarize field theory amplitudes. These amplitudes resemble closed string amplitudes, but are strictly studied from the point of view of unitarity constraints on $2 \to 2$ scattering, not any explicit realization in some string background. In Sec. 4.3, with the goal of studying standard model and beyond the standard model (BSM) amplitudes, we study $SO(N)$ and $SU(N)$ gauge boson amplitudes. While in this neighborhood, we also probe unitarity constraints in general spacetime dimensions, finding evidence for example that the maximum allowed N for $SO(N)$ is 32 as realized by the $SO(32)$ heterotic string. Finally, in Sec. 4.4 we study Higgs scattering and comment on the compatibility of our amplitudes with the SM.

4.2 Unitarity and UV Completion

Before moving on to specific ansätze, it is useful to review the constraints of Lorentz invariance and unitarity on the amplitude and some well-known instances of tree-level UV-completion in the absence of gravity. This will illustrate the fundamental challenge with even tree-level UV-completion in the presence of gravity and motivate the stringy form factor which will dress all the amplitudes considered herein. We always consider amplitudes with massless external states and employ spinor helicity with mandelstams

$$s = 2p_1 \cdot p_2 \qquad t = 2p_2 \cdot p_3 \qquad u = 2p_1 \cdot p_3 \tag{4.1}$$

and $s + t + u = 0$. For on-shell kinematics the spinor-brackets obey $\langle ij \rangle = \pm [ij]^\star$ with the sign depending whether the states are ingoing or outgoing. When imposing positive expandability on a basis of orthogonal polynomials, the argument of the polynomials is $\cos\theta$ where

$$t = -\frac{s}{2}(1 - \cos\theta) \qquad u = -\frac{s}{2}(1 + \cos\theta) \tag{4.2}$$

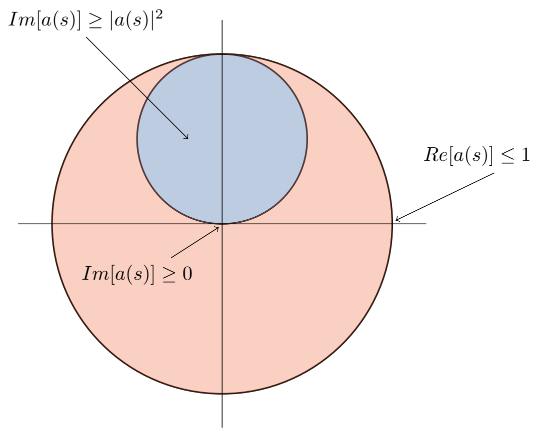


Figure 4.1: Above we depict in the complex $a(s)$ the bounds relevant for our analysis. The non-perturbative bound on $a(s)$ is shaded in blue. At weak coupling, one can diagnose unitarity violation at large s using the weaker, pink shaded condition. We then impose positivity of the imaginary part which is the weak-coupling portion of the blue shading (where $|a(s)|^2$ is parametrically suppressed.)

4.2.1 Review of Unitarity Constraints

Lorentz invariance allows us to expand the elastic two-to-two amplitude for massless particles as

$$\mathcal{A}(1^{h_1}_{r_1}, 2^{h_2}_{r_2}, 3^{h_3}_{r_3}, 4^{h_4}_{r_4}) = 16\pi \sum_J (2J+1) a^{\{h_i\}}_{J,R}(s) \mathbb{G}^{\{h_i\}}_{D,J}(\cos\theta) \mathbb{P}_R(\{r_i\}) \tag{4.3}$$

The $\mathbb{G}^{\{h_i\}}_{D,J}$ are the relevant basis of orthogonal (in general spinning) polynomials for the scattering process in question, corresponding to spin-J exchange in D space-time dimensions with external helicities $\{h_i\}$. We will state the particular polynomial basis for each process we consider. The $\mathbb{P}_R(\{r_i\})$ are projectors in the internal symmetry space for representations r_i exchanging representation R (in the s-channel here). The weights $a^{\{h_i\}}_{J,R}(s)$ in front of these two sets of orthogonal projectors are partial wave coefficients, which depend on Mandelstam s and the constants in the amplitude. Unitarity then imposes the bound on partial waves

$$\text{Im}[a(s)] \geq |a(s)|^2 \tag{4.4}$$

where the constraint holds for each $a_{J,R}^{\{h_i\}}(s)$ individually and we suppress the labels on $a(s)$.[1] Noting that this equation is identical to

$$(\mathrm{Im}[a(s)] - 1/2)^2 + \mathrm{Re}[a(s)]^2 \leq \frac{1}{4} \tag{4.5}$$

the unitarity bound clearly forces each $a(s)$ to lie in the Argand circle depicted in figure 4.1. Two limits of this non-perturbative constraint will be useful for our weak coupling analysis, each simplifying either the left or right-hand side of equation (4.4). These limits are

$$\begin{cases} \mathrm{Im}[a(s)] \geq 0 & \text{positive expandability at singular loci} \\ |a(s)| \leq 1 & \text{boundedness at large } s \end{cases} \tag{4.6}$$

The tension between these two conditions and causality imposed via Regge boundedness in the complex s plane, places stringent constraints on the form of weakly-coupled gravitational UV completions. We will be considering processes with graviton exchange, which exhibit unitarity violation at large s and need to be UV completed below the Planck scale. Rather than give up on weak coupling, we will posit an ansatz for a weakly coupled completion which softens the high-energy behavior. This will be done by the introduction of new heavy states and positive expandability at the associated singularities must be checked. This constrains the low-energy data in the context of the specific ansatz to which we have committed. In this work we will manifest all necessary conditions on the amplitude other than positive expandability of residues, leaving it as the non-trivial condition which must be checked.

In addition to boundedness at fixed angle, positive expandibility of the imaginary part, and Regge boundedness, we impose additional reasonable constraints on the

[1] The S-matrix is required to be a positive operator in general, therefore in general this constraint is a statement about positivity of a matrix, but we will be studying amplitudes for which each individual $a(s)$ obeys (4.4).

spectrum. In particular we require that there are a fixed number of spins at a given mass i.e. that the Chew-Frautschi plot has no spikes. We summarize the full set of constraints below

- Boundedness at fixed angle and high energy via

$$|a(s)| \leq 1 \tag{4.7}$$

- Regge boundedness for complex s:

$$\lim_{|s|\to\infty} \mathcal{A}/s^2 \to 0 \qquad\qquad \text{fixed } t < 0 \tag{4.8}$$

- Fixed number of spins at a given mass by imposing that any residue in s is a polynomial in t.

- Positive expandability of the imaginary part, the only condition which we will not manifest.

4.2.2 Completions

Before moving on to constructing gravitational completions, it is useful to recapitulate a well-known example of a non-gravitational tree level amplitude requiring UV completion: the non-linear sigma model. This will help frame the unique challenge and opportunities afforded in studying gravitational completions.

Non-linear Sigma Model: We consider the amplitude for four Goldstones

$$\mathcal{A}_{\text{NLSM}} = -\frac{1}{f^2}\Big[(s+t)\left(\text{Tr}(T^aT^bT^cT^d) + \text{Tr}(T^dT^cT^bT^a)\right)$$
$$+ (s+u)(\text{Tr}(T^bT^aT^dT^c) + \text{Tr}(T^cT^dT^aT^b))$$
$$+ (t+u)(\text{Tr}(T^aT^cT^bT^d) + \text{Tr}(T^dT^bT^cT^a))\Big]$$

With the T^a standard generators of the defining representation of $SU(N)$. We have scalar partial wave for any exchanged representation

$$a_0(s) \sim \frac{s}{f^2} \tag{4.9}$$

which means the amplitude needs to be unitarized before $s \sim f^2$. Already in this case we can illustrate how the tensions between boundedness at fixed-angle and Regge boundedness in the complex plane don't allow the most naive softenings of the amplitude. For example, we could try the exponential form factor

$$\hat{\mathcal{A}} = e^{-\frac{s^2+t^2+u^2}{M^4}}\mathcal{A}_{\text{NLSM}} \tag{4.10}$$

which is exponentially soft at fixed angle, but diverges exponentially for imaginary s at fixed t. If we settle on more humble expectations for softening, the most obvious way to soften the amplitude is to divide it by something i.e. introduce a massive exchange, which should in particular have positive mass-squared. Therefore the simplest ansatz is

$$\hat{\mathcal{A}}_{\text{NLSM}} = -\frac{1}{f^2}\left[\frac{s}{1-\frac{s}{M^2}}\left(\mathbb{P}_1^s + \mathbb{P}_2^s\right) + \frac{t}{1-\frac{t}{M^2}}\left(\mathbb{P}_1^t + \mathbb{P}_2^t\right) + \frac{u}{1-\frac{u}{M^2}}\left(\mathbb{P}_1^u + \mathbb{P}_2^u\right)\right] \tag{4.11}$$

where we decomposed the usual flavor-ordered form into the form manifesting exchanges in respective channels, with

$$
\begin{cases} \mathbb{P}_1^s = \frac{2}{N}\delta_{ab}\delta_{cd} \\ \mathbb{P}_2^s = d_{abe}d_{cde} \end{cases} \tag{4.12}
$$

which correspond to the singlet and symmetric adjoint irreducible flavor exchanges. Now, the partial waves are clearly bounded so long as $M^2 < f^2$ i.e. the UV completion scale is below the UV cutoff f^2. But we are not done, the amplitude now has a singularity at M^2 in each channel and the residue must be positive. This additionally gives the condition $f^2 > 0$. The low-energy amplitude alone was agnostic about the sign of f^2, but once committed to a certain UV completion, it is possible to constrain low-energy data by imposing positive expandability of residues. We will find in the case of gravitational amplitudes that constructing a UV completion is more difficult, the upshot being that the constraints on low energy data are far more interesting.

Graviton Exchange: Our objective is to study UV completions in the presence of graviton exchange, so let's see if we can apply the lessons from the non-linear sigma model to the case of four identical scalars exchanging gravitons. The amplitude is

$$
\mathcal{A}_{\text{grav}} = -\frac{1}{M_P^2}\left(\frac{tu}{s} + \frac{su}{t} + \frac{st}{u}\right) \tag{4.13}
$$

Where M_P is the reduced Planck mass $M_P^{-2} = 8\pi G$. Again we see that the scalar partial wave grows with energy

$$
a_0(s) \sim \frac{s}{M_P^2} \tag{4.14}
$$

As with the NSLM amplitude, we do not want to give up on perturbativity; we will pursue a tree level UV completion. It was easy to engineer a unitarization of the NLSM amplitude, what could be the difficulty with gravity? Particle production at low energies. More explicitly, unlike the NLSM amplitude, the graviton exchanges already impose a sign condition on the coupling squared. This fixed sign in the low-energy amplitude is something our putative completion will have to respect, and it has surprisingly drastic consequences. The issue has everything to do with these low energy poles, and can already be seen in the context of ϕ^3 amplitudes. Suppose we wanted to UV-improve the behavior of the ϕ^3 amplitude; we can try to mimic the strategy employed in the NLSM. Focusing on a single channel

$$\frac{g^2}{s} \to \frac{g^2}{s} \times \frac{1}{\prod\limits_{i=1}^{n} \left(1 - \frac{s}{M_{s,i}^2}\right)} \tag{4.15}$$

where we have introduced n new massive poles to soften the high-energy behavior to $\frac{1}{s^{n+1}}$. As with the initial $s = 0$ pole, we require the residue on each of these massive poles to be positive. But notice that because there is no pole at infinity, a residue theorem guarantees that some of the residues must have the wrong sign. We have arrived at the need for an infinite number of poles to have a hope of softening this amplitude in a way consistent with unitarity. We can make an ansatz for a factor with an infinite number of poles in each channel which will dress the amplitude:

$$\mathcal{D}(s,t,u) = \frac{N(s,t,u)}{\prod\limits_{i} \left(s - M_{s,i}^2\right) \prod\limits_{j} \left(t - M_{t,j}^2\right) \prod\limits_{k} \left(u - M_{u,k}^2\right)} \tag{4.16}$$

In order to further constraint this putative dressing factor, we will impose the well-motivated condition that there is a finite number of spins at given mass-level. Analytically, this means that the residue at each massive pole must be a polynomial in the remaining Mandelstam invariant, in particular the numerator must cancel the poles in the other channel upon taking such a residue. The denominator is a product of polynomials so it's natural to have the same ansatz for the numerator

$$N(s,t,u)=\prod_i (s+r_{s,i})\prod_j (t+r_{t,j})\prod_k (u+r_{u,k}) \tag{4.17}$$

The residue is then

$$\mathrm{Res}_{s\to M_{s,i}^2}\mathcal{D}\propto \frac{\prod\limits_j (t+r_{t,j})\prod\limits_k \left(t+M_{s,i}^2-r_{u,k}\right)}{\prod\limits_j \left(t-M_{t,j}^2\right)\prod\limits_k \left(t+M_{u,k}^2+M_{s,i}^2\right)} \tag{4.18}$$

requiring cancellation of the remaining poles in all channels yields the condition:

$$M_{s,i}^2+M_{t,j}^2\in\{r_{u,k}\} \tag{4.19}$$

plus its cyclic rotations. We will proceed with the most obvious way of solving this constraint, which is to make all six of these putative sets one mass scale times the positive integers. We will call this mass scale M_s in which case we find

$$\mathcal{D}(s,t,u)=\frac{\prod\limits_{i=1}^{\infty}(s+M_s^2 i)\prod\limits_{j=1}^{\infty}(t+M_s^2 j)\prod\limits_{k=1}^{\infty}(u+M_s^2 k)}{\prod\limits_{i=1}^{\infty}(s-M_s^2 i)\prod\limits_{j=1}^{\infty}(t-M_s^2 j)\prod\limits_{k=1}^{\infty}(u-M_s^2 k)} \tag{4.20}$$

Which we recognize as a famous function

$$\Gamma^{\mathrm{str}}=-\frac{\Gamma\left(-\alpha' s\right)\Gamma\left(-\alpha' t\right)\Gamma\left(-\alpha' u\right)}{\Gamma\left(\alpha' s\right)\Gamma\left(\alpha' t\right)\Gamma\left(\alpha' u\right)} \tag{4.21}$$

with $\alpha'=\frac{1}{M_s^2}$. Though we did not impose these conditions, it is readily verified from the form in (4.21) that the amplitude satisfies Regge boundedness and boundedness at fixed angle and high energy. We summarize the properties of this amplitude below:

- Exponentially soft at high-energy, fixed angle

- Regge limit:

$$\lim_{|s|\to\infty} \Gamma^{\rm str} = s^{\alpha' t} \qquad\qquad \text{fixed } t<0 \tag{4.22}$$

- A residue in s at level n i.e. mass $M^2 = nM_s^2$ is a polynomial of degree $2n$ in t and is in particular

$$\text{Res}_{s\to n/\alpha'}\Gamma^{\rm str} = \frac{1}{n!(n-1)!}\left(\prod_{i=1}^{n-1}(i+\alpha' t)^2\right) t(n+\alpha' t) \tag{4.23}$$

If we dress our graviton exchange amplitude (4.13) with $\Gamma^{\rm str}$ this is in fact the four dilaton amplitude in type IIB. This quasi-derivation of the Virasoro-Shapiro amplitude is far from new; Virasoro remarked on it in his original paper after all [74], but it is surprising. Though we by no means claim to be proving that this is the unique way of unitarizing graviton exchange, it is remarkable that with very little input and at each stage making the simplest of moves, we arrive at a form factor with Regge boundedness and *exponential* softness at fixed angle. These two conditions are extremely non-trivial to engineer, but we were able to build an amplitude with both properties by seeking a UV completion in the presence of gravity. We will find that the arrow goes both ways. Gravity does not merely necessitate the discovery of this somewhat sophisticated completion, it is in fact necessary for the use of such a completion.

Partial Wave Unitarity: With an ansatz in hand for gravitational amplitudes, we can now impose positive expandability of the residues. The amplitude is by construction positive on the massless poles, so we only need to check the positive expandability on the new poles in the completion. These come from $\Gamma^{\rm str}$ and the residue of $\Gamma^{\rm str}$ at general mass level n was stated above in (4.23).

Now that we have $\Gamma^{\rm str}$ and all of its wonderful properties at our disposal, one might hope that with this unitarizing hammer, any field theory amplitude looks like a nail. We can test it on something simple such as ϕ^4 theory, in which case we merely check the partial wave expansion of Γ^{str}. At the first mass-level one can readily verify that

$$\text{Res}_{s\to 1/\alpha'}\Gamma^{\text{str}} \propto \Big(P_2(\cos\theta) - P_0(\cos\theta)\Big) \tag{4.24}$$

so we find a negative residue. We could try coupling it to gravity; the amplitude is just augmenting the four-dilaton amplitude in type IIB by a contact interaction:

$$\mathcal{A}^\lambda = \Gamma^{\text{str}}\left[-\frac{1}{M_P^2}\left(\frac{tu}{s}+\frac{su}{t}+\frac{st}{u}\right)+\lambda\right] \tag{4.25}$$

We have

$$\text{Res}_{s\to 1/\alpha'}\mathcal{A}^\lambda \propto \frac{1}{70}P_4(\cos\theta) + \left(\frac{2}{7}+\frac{\alpha' M_P^2\lambda}{6}\right)P_2(\cos\theta) + \left(\frac{7}{10}-\frac{\alpha' M_P^2\lambda}{6}\right)P_0(\cos\theta) \tag{4.26}$$

And so we find the two-sided bound on λ:

$$\frac{-12}{7} \leq \lambda\frac{M_P^2}{M_s^2} \leq \frac{21}{5} \tag{4.27}$$

So we can have a quartic coupling so long as it does not overwhelm the graviton exchange piece of the amplitude. In this sense, we are able to UV-improve $\lambda\phi^4$ theory at high energies so long as we couple it to gravity. In the context of these ansätze, gravity is not an obstruction to unitary UV softening, but a necessity. One could imagine that perhaps this was merely the absence of long-distance physics at low energies that was the problem, not gravity in particular. We can test this on pure gauge boson scattering. Checking unitarity in this case requires an additional layer of calculation: 6-j coefficients.

6-j Coefficients: Since we are working at tree level, taking the imaginary part merely amounts to taking the residue (4.18). Moreover, in practice we find that the strongest constraints come at low mass levels, stabilizing at the latest by mass level three for the amplitudes considered in this analysis. The most non-trivial aspect of testing positive expandability is in re-expanding the color structures of (4.36), which as such are not expressed in the basis of s-channel projectors. The full set of projectors in any one of the three channels, full set meaning the projectors for all representations that can be exchanged in this channel, are an orthogonal basis of projectors depending on the external indices. Therefore, each projector in the t and u channels can be uniquely expanded in terms of the projectors in the s-channel:

$$\mathbb{P}^t_R = \sum_{R'} C^{t,s}_{R,R'} \mathbb{P}^s_{R'} \tag{4.28}$$

where we have suppressed the external indices which these projectors are functions of. This is merely the crossing the equation, and it is solved trivially by contraction given that the basis of projectors corresponding to the exchange of irreps R are orthogonal. That is

$$C^{t,s}_{R,R'} = \frac{\mathbb{P}^t_R \cdot \mathbb{P}^s_{R'}}{\mathbb{P}^s_{R'} \cdot \mathbb{P}^s_{R'}} \tag{4.29}$$

where the dot denotes contraction with the relevant invariants on the external indices which have been suppressed. This coefficient $C^{t,s}_{R,R'}$ is a 6-j symbol, depending on the four external representations in addition to the two exchanged representations R and R'. These must be calculated for the group and representations in question. Once this is done, the projection onto irreducible exchanged states is straightforward. A useful resource for the determination of these projectors is [75].

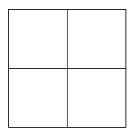
We can return to analyzing our four gauge boson amplitude

$$\mathcal{A} = \frac{g^2_{YM}}{3} \langle 12\rangle^2 [34]^2 \Gamma^{\text{str}} \left(\frac{\mathbb{P}^s_{\text{Adj}} - \mathbb{P}^t_{\text{Adj}}}{st} + \frac{\mathbb{P}^t_{\text{Adj}} - \mathbb{P}^u_{\text{Adj}}}{tu} + \frac{\mathbb{P}^u_{\text{Adj}} - \mathbb{P}^s_{\text{Adj}}}{su} \right) \tag{4.30}$$

where we have the adjoint projectors

$$\mathbb{P}^s_{\text{Adj}} = f^{abe} f^{edc} \tag{4.31}$$

and similarly for the t and u channels. We can focus on the explicit case of $SO(N)$. Positive expandability is already violated at level one for the most subleading Regge trajectory in the largest representation exchanged between the adjoints. The exchanged state and associated coefficient in the partial wave expansion are:



$$m^2 = 1,\ J = 0 \qquad : \qquad -g^2_{YM} \tag{4.32}$$

Where the left is the Young tableaux for the exchanged representation of $SO(N)$, the mass-squared in units of M_s^2, and the spin. The coefficient is identical for the analogous representation of $SU(N)$. Perhaps the most salient feature of this violation is the indication that the actual heterotic string amplitude becomes unitarity violating once gravity becomes *too weak* relative to the gauge interactions; we cannot have the gauge interactions all on their own. Yet again we find that unitarity of these amplitudes requires graviton exchange. String theory of course necessitates gravity, so unitarity violation in the complete absence of gravity, is perhaps not so surprising from that point of view. But what is interesting about this analysis is that we can move around in the space of parameters untethered to any choice of background or class of compactifications and make what appear to be robust observations about constraints on this class of amplitudes with graviton exchange. For instance, extending the above to our full ansätze will reveal that the generalization of (4.32) is in tension, but compatible, with the weak gravity conjecture (WGC).

Summary of Gravitational Ansätze: Now that we have motivated both the prefactor Γ^{str} and that it must be used for completions along with graviton exchange, we can discuss more completely the rules bounding our ansätze. The basic game is clear: multiply a massless field theory amplitude by the stringy form-factor Γ^{str} and check positive expandability. Now we discuss the restrictions we impose on the massless field theory amplitudes.

In motivating Γ^{str} we already took as assumptions that the residues of the amplitude in s were polynomials in t. Additionally, we impose Regge boundedness of the amplitude, which in the case of our gravitational amplitudes means bounded by s^2. Seeing as Γ^{str} goes as $s^{\alpha' t}$ in the Regge limit, the field theory amplitude which this factor multiplies can scale at most as s^2 in the Regge limit, which the graviton exchange piece indeed will. This bounds the dimension of interactions coming from operators that we may put in by hand. As for denominators, one can consider the heterotic-type deformation

$$\frac{1}{s} \to \frac{1}{s(1+\alpha' s)} \tag{4.33}$$

It is crucial that this potential tachyon pole has vanishing residue

$$\mathrm{Res}_{s\to -1/\alpha'} \frac{1}{s(1+\alpha' s)} \Gamma^{\text{str}} = 0 \tag{4.34}$$

Indeed any product of $(n + \alpha' s)$ for distinct integers n can be added to the denominator and will obey the vanishing residue condition above. But they will violate our condition for a finite number of spins at a given mass level. Recall from (4.18) that at the first mass level we have

$$\mathrm{Res}_{s\to 1/\alpha'} \Gamma^{\text{str}} \propto t(1+\alpha' t) = u(1+\alpha' u) \tag{4.35}$$

which means, considering the t and u channel terms in our amplitude, that we can have our massless poles and the $(1+\alpha' t)$ or $(1+\alpha' u)$ factors as well, but no more. Additional factors would fail to cancel upon taking the residue at the first mass level and introduce an infinite number of spins. For simplicity, we only consider these poles on the graviton exchange part of the amplitude.

Therefore, we can compute a massless scattering amplitude, the form of which is itself fixed by consistent factorization into three-particle amplitudes on the massless poles. This amplitude can have contact terms put in by hand up to scaling as s^2 in the Regge limit. The amplitude is then dressed by Γ^{str} which ensures that the amplitude satisfies all necessary criteria, except potentially positive expandability on the new massive poles. Checking this condition produces constraints on the low energy data.

4.3 Gauge Boson Scattering

First we take up massless gauge boson scattering. At tree-level, the non-trivial $2 \to 2$ gauge boson amplitudes are those of W's and gluons. There is a $2 \to 2$ amplitude for B's (gauge boson of $U(1)_Y$), but it is purely mediated by graviton exchange and therefore not further constrained by unitarity. In addition to these, in many GUT models we have gauge bosons of $SO(N)$ or $SU(N)$, and we will carry out the analysis for general N in both cases. The massless scattering amplitude is

$$\begin{aligned}\mathcal{A}(1^{-a}, 2^{-b}, 3^{+c}, 4^{+d}) = \langle 12\rangle^2[34]^2 \Bigg[&\frac{1}{M_P^2}\left(\frac{\mathbb{P}_1^s}{s} + \frac{\mathbb{P}_1^t}{t} + \frac{\mathbb{P}_1^u}{u}\right) + \\ &\frac{g_{\text{YM}}^2}{3}\left(\frac{\mathbb{P}_{\text{Adj}}^s - \mathbb{P}_{\text{Adj}}^t}{st} + \frac{\mathbb{P}_{\text{Adj}}^t - \mathbb{P}_{\text{Adj}}^u}{tu} + \frac{\mathbb{P}_{\text{Adj}}^u - \mathbb{P}_{\text{Adj}}^s}{su}\right)\Bigg] \end{aligned} \tag{4.36}$$

we write the color structures in this way as they are the color structures to be expanded in on a factorization channel. In particular, these are

$$\begin{cases} \mathbb{P}_1^s & \delta^{ab}\delta^{cd} \\ \mathbb{P}_{\text{Adj}}^s & f^{abe}f^{edc} \end{cases} \tag{4.37}$$

Note that in the Regge limit, this amplitude already scales as s^2. Any additional contribution consistent with Regge behavior would require new poles, which we do not have at low energies. Our ansatz is then

$$\mathcal{A}^{\text{UV}}(1^{-a}, 2^{-b}, 3^{+c}, 4^{+d}) = \Gamma^{\text{str}}\mathcal{A}(1^{-a}, 2^{-b}, 3^{+c}, 4^{+d}) \tag{4.38}$$

or $\mathcal{A}^{\text{UV}}_{\text{Het}}$ which deforms the graviton poles to look heterotic

$$\begin{aligned} \mathcal{A}^{\text{UV}}_{\text{Het}}(1^{-a}, 2^{-b}, 3^{+c}, 4^{+d}) = \langle 12\rangle^2[34]^2 \Bigg[\frac{1}{M_P^2}\left(\frac{\mathbb{P}_1^s}{s(1+\alpha' s)} + \frac{\mathbb{P}_1^t}{t(1+\alpha' t)} + \frac{\mathbb{P}_1^u}{u(1+\alpha' u)}\right) + \\ \frac{g_{\text{YM}}^2}{3}\left(\frac{\mathbb{P}_{\text{Adj}}^s - \mathbb{P}_{\text{Adj}}^t}{st} + \frac{\mathbb{P}_{\text{Adj}}^t - \mathbb{P}_{\text{Adj}}^u}{tu} + \frac{\mathbb{P}_{\text{Adj}}^u - \mathbb{P}_{\text{Adj}}^s}{su}\right)\Bigg] \end{aligned} \tag{4.39}$$

We introduce g_s via

$$g_s^2 = \frac{M_s^2}{M_P^2} \tag{4.40}$$

which is the dimensionless strength characterizing how far below the Planck scale the UV completion scale M_s is i.e. how weakly coupled a completion of gravity it is. The strongest constraints come from the pole in the s-channel with the helicity configuration in (4.36). Furthermore, we note that this ansatz is perfectly good in any number of spacetime dimensions, and we will also consider the necessary but insufficient condition of positive expandability on the *scalar* D-dimensional Gegenbauer polynomials.

4.3.1 Constraints in Four Spacetime Dimensions

In four dimensions, the polynomials are the spinning Gegenbauer's

$$\mathbb{G}^{\{h_i\}}_{D=4,J}(\cos\theta) = d^J_{h_{12},h_{34}}(\cos\theta) \tag{4.41}$$

where $h_{ij} = h_i - h_j$. In the s-channel for (4.36) these polynomials actually correspond to the scalar Legendre's, and we find the strongest constraints.

SO(N): For $SO(N)$ the adjoints exchange six representations. Positivity of g_{YM}^2 is already a consistency condition imposed at massless level. As can be seen in the left plot of figure 4.2, both the heterotic and non-heterotic cases have an N-independent upper bound on the ratio $\frac{g_{YM}^2}{g_s^2}$ and an upper bounding curve depending on both the coupling and N. In the non-heterotic case this upper curve is only piecewise smooth, the result of two smooth constraints. In particular, we find for the non-heterotic case, the constraints and the associated mass-level, spin, and $SO(N)$ representation are

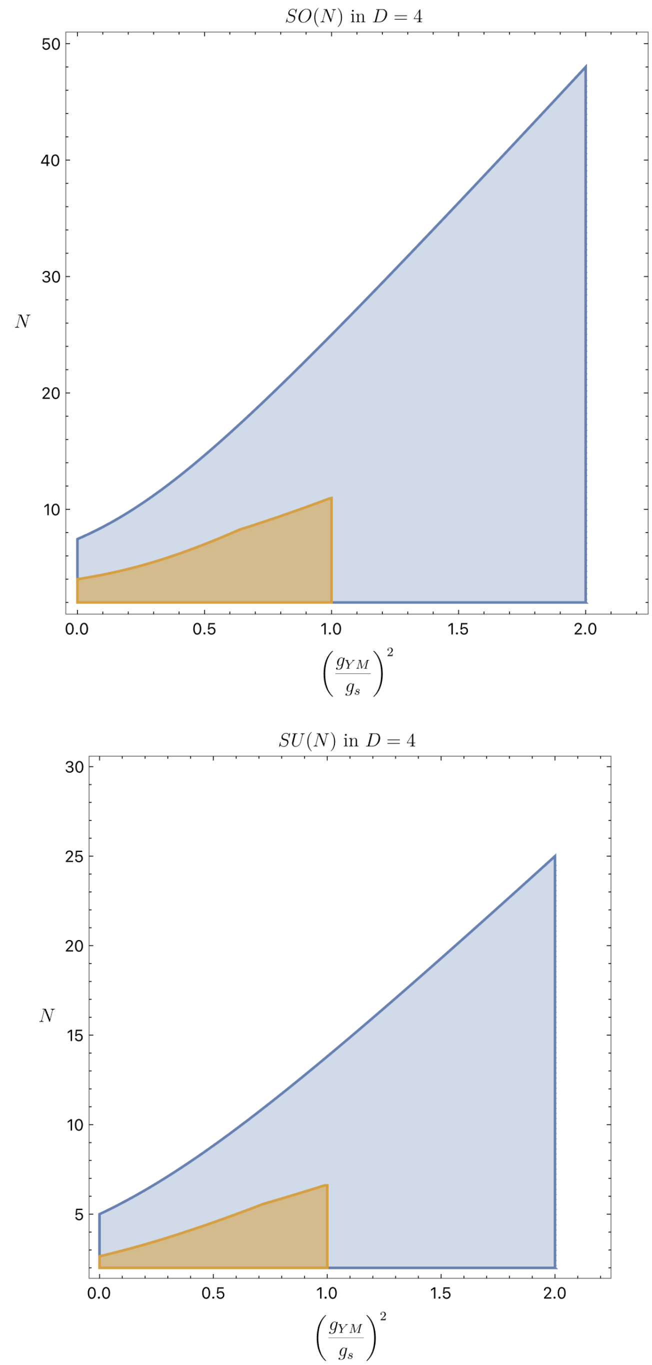


Figure 4.2: Allowed space of N and g_{YM}/g_s in four spacetime dimensions, for $SO(N)$ (left) and $SU(N)$ (right). In both cases, the larger allowed region (blue) has heterotic poles and contains the region without heterotic poles (orange). In the heterotic case, the maximum rank is 24 for both groups and occurs at the coupling $g_{YM}^2 = 2g_s^2$, the value fixed in the heterotic string.

the following:

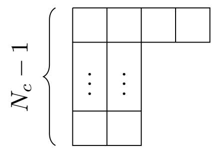
$$\begin{cases} \underset{m^2=1,\,J=0}{\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}} & g_{YM}^2 \leq g_s^2 \\[2em] \underset{m^2=1,\,J=0}{\bullet} & 1+\left(\frac{g_{YM}}{g_s}\right)^2(N-2)-\frac{N(N-1)}{12}\geq 0 \\[2em] \underset{m^2=2,\,J=0}{\bullet} & 4+2\left(\frac{g_{YM}}{g_s}\right)^2(N-2)-\frac{N(N-1)}{5}\geq 0 \end{cases} \tag{4.42}$$

And in the heterotic case simply the two constraints:

$$\begin{cases} \underset{m^2=1,\,J=0}{\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}} & g_{YM}^2 \leq 2g_s^2 \\[2em] \underset{m^2=1,\,J=0}{\bullet} & 2+\left(\frac{g_{YM}}{g_s}\right)^2(N-2)-\frac{N(N-1)}{24}\geq 0 \end{cases} \tag{4.43}$$

Where the masses m^2 are measured in units of M_s^2. The two-by-two tableaux is the largest representation exchanged between the gauge bosons, having dimension $\frac{N(N+1)(N+2)(N-3)}{12}$, and is the exchanged state enforcing the anti-weak-gravity bound. We note that all constraints come from the most subleading Regge trajectory.

SU(N): For $SU(N)$ the adjoints exchange seven representations, bu the situation is analogous, with analogous representations imposing similar bounds. We find for the non-heterotic case:

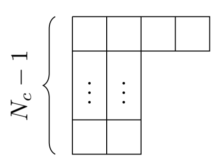


$$m^2 = 1,\ J = 0 \qquad g_{YM}^2 \leq g_s^2$$

$$\bullet_{\ m^2 = 1,\ J = 0} \qquad 5 + 4\left(\frac{g_{YM}}{g_s}\right)^2 N - \frac{1+2N^2}{3} \geq 0 \tag{4.44}$$

$$\bullet_{\ m^2 = 2,\ J = 0} \qquad 11 + 5\left(\frac{g_{YM}}{g_s}\right)^2 N - N^2 \geq 0$$

and in the heterotic case:



$$m^2 = 1,\ J = 0 \qquad g_{YM}^2 \leq 2g_s^2 \tag{4.45}$$

$$\bullet_{\ m^2 = 1,\ J = 0} \qquad \left(\frac{g_{YM}}{g_s}\right)^2 N - \frac{1}{12}(N^2 - 25) \geq 0$$

We see that for $SU(N)$ again the largest representation enforces an identical bound on the coupling to the $SO(N)$ case, this time he dimension of the representation is $\frac{N^2(N+3)(N-1)}{4}$. These bounds are not the same function of rank r for each group, but the bound on the rank is identical at the maximum value of g_{YM}^2 in the heterotic case, which is $2g_s^2 = \frac{2M_s^2}{M_P^2}$. This is the condition relating the Yang-Mills coupling, the string scale, and the Planck scale in the heterotic string, a condition which survives compactification as both ten-dimensional couplings get the same volume dilution [76].

Phenomenology Here we comment on the implications for four-dimensional phenomenology. First we note the obvious, which is that only the combination of constraints from singlets of the gauge group, massless adjoints, and the largest exchanged representation both for $SO(N)$ and $SU(N)$ combine to produce a bounded region. The largest representation furnishes the bound relating the UV completion scale, Planck scale, and the coupling:

$$g_{YM}^2 M_P^2 \leq 2M_s^2 \tag{4.46}$$

or with no factor of two on the right-hand side when we do not have heterotic denominators. This is an anti-weak-gravity type of bound: consistent with weak-gravity, but pointing in the opposite direction. But for sufficiently large ranks of the gauge-group, we cannot afford for the gauge-coupling to be too-weak with a low string scale, either. In particular, no global symmetry is allowed for these massless adjoint vectors for $N > 7$ for $SO(N)$ and $N > 5$ for $SU(N)$.

We can discuss these constraints in the context of W bosons and gluons. For the case of heterotic poles, the constraint is the same for both W's and gluons and is (4.46). For couplings around the GUT scale, we have g_2^2, $g_3^2 \sim \frac{1}{2}$, in which case the putative string scale is bound by

$$\frac{M_P}{2} \lesssim M_s \tag{4.47}$$

meaning the lowest putative string scale is around 10^{18} GeV. For the case without heterotic poles we see that the constraint on M_s^2 changes by a factor of two. Though there is also a lower bound for $SU(3)$ in the non-heterotic case, the bound is not interesting unless the Yang-Mills coupling is sufficiently weak. Otherwise, the constraint $g_s < 1$ justifying the perturbative analysis is much stronger anyway. Nonetheless, for sufficiently weak g_{YM}^2 (of order a tenth) we see that we get a two-sided bound on M_s. It is also amusing to note that in the case of non-heterotic poles, $SU(5)$ and $SO(10)$ gauge bosons are essentially pegged to have $g_{YM}^2 = 2g_s^2$.

4.3.2 Constraints in General Spacetime Dimensions

In general dimensions, the full constraints require D-dimensional spinning polynomials. The Yang-Mills amplitude in D spacetime dimensions is of the form

$$\mathcal{A} = \mathcal{F}^4 \mathcal{A}^{\text{scalar}} \tag{4.48}$$

where $\mathcal{A}^{\text{scalar}}$ has no dependence on polarization vectors and $\mathcal{F}^4$ is the famous polynomial permutation-invariant in field strengths which sits in front of the field theory Yang-Mills amplitude. In [77], positivity of $\mathcal{F}^4$ alone was remarkably shown to contain the critical dimension constraint $D \leq 10$ and information about the spectrum of 11-dimensional supergravity. This is the only condition for positive expandability of $\mathcal{F}^4$. For our analysis, what is relevant is that so long as we satisfy the critical dimension constraint $D \leq 10$, positive expandability of $\mathcal{A}^{\text{scalar}}$ on the scalar D-dimensional Gegenbauer polynomials then implies positive expandability of the full amplitude's residues, as we merely have a product of two positively expandable functions, which in turn must be positively expandable. So for general dimensions equal to or below ten, we will consider the sufficient though not strictly necessary condition of positive expandability on the *scalar* D-dimensional Gegenbauer polynomials. This means that the true constraints could in principle be weaker, but merely expanding on the scalar polynomial basis already provides interesting constraints. So we have[2]

$$\mathbb{G}_{D,J}^{\{h_i\}}(\cos\theta) = G_J^{(D)}(\cos\theta) \tag{4.49}$$

It is worth commenting that in four spacetime dimensions, where we did the full spinning analysis, the strongest constraints for residues in the s-channel come from the process with $h_1 = h_2 = -1$ and $h_3 = h_4 = +1$ for which the spinning polynomials reduce to the Legendre polynomials. This might hint at the constraints on $\mathcal{A}^{\text{scalar}}$ constituting the full set of constraints, but we will leave such analysis to future work.

Heterotic $SO(N)$ in Ten Dimensions Given that our ansatz is essentially the heterotic string amplitude (the gauge group and gauge-coupling are not fixed), it is natural to check the constraints of the previous section but in ten dimensions, which can be found in figure 4.3. Similar exchanged states produce the analogous curves in the $D = 10$ case, the only difference being that the strongest upper bounding curve occurs at mass level three rather than one. In particular, the maximum allowed value of the coupling ratio $\frac{g_{YM}^2}{g_s^2}$ is 2, enforced by the same exchanged state as in four dimensions, and this meets the upper bounding curve from mass-level three singlet exchange at the corner of the allowed region with the maximum allowed

$$N \leq 32 + \frac{16}{19} \tag{4.50}$$

which of course means the only theory allowed by this corner of the allowed region is the $SO(32)$ heterotic string.

[2]These would be denoted in the math literature as $C_J^{(\frac{D-3}{2})}(\cos\theta)$

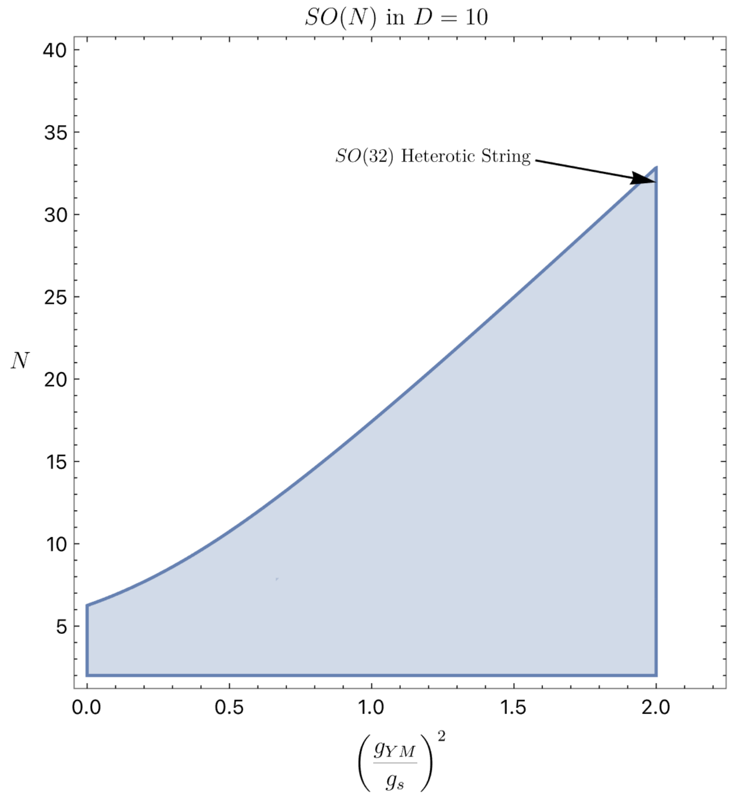


Figure 4.3: Imposing positive expandability of the heterotic form of (4.36) on scalar Gegenbauer's in $D = 10$, we find the allowed region shaded in blue. The value of coupling and N fixed in $SO(32)$ heterotic string theory is indicated by the arrow, $g_{YM}^2 = 2g_s^2$ and $N = 32$.

Dimension vs. Rank In general dimensions we observe that the maximum allowed rank of the gauge group always occurs at $g_{YM}^2 = 2g_s^2$, the maximum allowed value of the coupling and also the value in the $SO(32)$ heterotic string. We can then track how the maximum allowed rank varies with the spacetime dimension, which is presented in figure 4.4. We look at bounds between four and ten dimensions, over which three distinct smooth bounding curves comprise the full upper bounding curve. The bounding curve always corresponds to a spin-0 singlet of the gauge group, the only quantum number varying being the mass. For dimensions $4 \leq D \leq 7$ the constraint comes from the first mass level. In higher dimensions the strongest constraint comes from the second mass level until in the vicinity of $D = 10$, at which a yet stronger

• $\left(\frac{g_{YM}}{g_s}\right)^2 N - \frac{1}{12}(N^2 - 25) \geq 0$

$m^2 = 1, J = 0$

Figure 4.4: Imposing positive expandability of the heterotic form of (4.36) on scalar D-dimensional Gegenbauer's with $g_{YM}^2 = 2g_s^2$, we find the allowed region in rank r vs. dimension D shaded in blue. There are three-piecewise segments, the two leftmost are identical for $SO(N)$ and $SU(N)$. The zoomed portion shows that in the vicinity of $D = 10$, the third curve is not the same curve in r and D for $SO(N)$ and $SU(N)$ but agrees in $D = 10$, the constraints agreeing in all physical dimensions.

constraint comes in from the third mass level. This is seen in the zoomed portion of figure 4.4. In $D = 10$ this constraint from mass level three is necessary to disallow $N = 33$ in the case of $SO(N)$. The dashed line on figure 4.4 represents the swampland conjecture $r < 26-D$, derived assuming BPS completeness with 16 supercharges [78] . We emphasize that the first two of these upper bounding curves is identical for $SO(N)$ and $SU(N)$, the agreement being not at all manifest even at the level of the analytic expressions in r and D which are required to be positive; the expressions merely share a common factor enforcing positivity. Even more dramatically, the zoomed portion reveals the constraint at the third mass level is not even the same curve for $SU(N)$ and $SO(N)$, but the two curves are such that they meet at $D = 10$ constraining the rank to be $r \leq 16 + \frac{8}{19}$, so the rank constraint in any physical dimension is identical. This non-trivial conspiracy of the Gegenbauer polynomials and recoupling coefficients suggests a universality to these constraints when centered on the correct data which could perhaps be exploited in greater generality.

Swampland Conjectures: In the context of these ansätze we are able to make contact with some swampland conjectures, in particular weak gravity and consequences of completeness, such as the conjecture about the maximum allowed rank of the gauge group. In the context of gravitational completions, we also see the mechanism for generating something like completeness. Placing the field theory amplitude in front of a common stringy form-factor means that the u and t channel projectors for the gauge-group must be re-expanded in terms of the s-channel projectors when we take a residue in s. This is done via the group theory crossing equation which is solved via 6-j symbols. In this way, we see that scattering some specified representations builds up the need for other representations in their tensor product. This kind of mechanism for completeness has everything to do with gravity, as one can note that with the non-gravitational NLSM completion (4.11) the poles merely produce the singlet and adjoint exchanges. Even for a stringy completion of the NLSM amplitude via Lovelace-Shapiro, we still only generate either anti-symmetric adjoint exchange or symmetric adjoint and singlet exchange. But in the context of our gravitational amplitudes, we find every possible representation that can be exchanged between the external states is indeed exchanged. This provides a mechanism to bootstrap the completeness hypothesis in the context of these amplitudes, one which crucially relies on the presence of gravity.

4.4 Standard Model Electroweak Sector

Now we direct our attention to the electroweak sector. We already studied ansätze for the scattering of $SU(N)$ gauge bosons, and found a constraint on the relation between the gauge-coupling, the UV completion scale, and the Planck scale. The only constraint coming from scattering $SU(N)$ gauge bosons in the heterotic case with $N \leq 5$ was

$$g_{YM}^2 M_P^2 \leq 2M_s^2 \tag{4.51}$$

Then the lowest string scale we can obtain is

$$M_s^2 \sim 10^{18}\ \text{GeV} \tag{4.52}$$

and the bound pushes M_s^2 up by a factor of two in the non-heterotic case. We can further probe the electroweak sector by studying the scattering of four Higgs's. The amplitude in this case is

$$\begin{aligned}\mathcal{A}(1,\bar{2},3,\bar{4}) = \Gamma^{\text{str}}\Bigg(- \frac{1}{M_P^2}\left(\frac{tu}{s}\mathbb{P}^s_{1,1} + \frac{su}{t}\mathbb{P}^t_{1,1}\right) + \frac{t-u}{2s}\left(\frac{g_1^2}{4}\mathbb{P}^s_{1,1} + g_2^2\mathbb{P}^s_{\text{Adj},1}\right) \\ + \frac{s-u}{2t}\left(\frac{g_1^2}{4}\mathbb{P}^t_{1,1} + g_2^2\mathbb{P}^t_{\text{Adj},1}\right) + 2\lambda(\mathbb{P}^s_{1,1} + \mathbb{P}^t_{1,1})\Bigg) \quad (4.53)\end{aligned}$$

and similarly for the configuration with massless t and u channel poles only. The subscript labels on the projector denote the exchanged representation of the corresponding factor of $SU(2)\times U(1)_Y$ with 1 denoting singlet exchange (with zero charge in the $U(1)_Y$ case). The projectors are normalized such that these are the conventional normalizations for the gauge-couplings and Higgs quartic coupling in the standard model. If we fix the gauge-couplings such that $\alpha_S^{-1} = \alpha_W^{-1} = 25$ we can produce a plot relating the Higgs quartic coupling and the putative string scale in Planck units. The kink in the allowed region minimizing M_s is just outside of the bounds of the running of λ predicted by the standard model [1]. The running predicted has λ become negative at $\sim 10^{10}$ GeV and asymptote to within the gray shaded region of figure 4.5 below the Planck scale. The lowest scale allowed by this bound is slightly below that allowed by the four gauge boson scattering, constituting a weaker constraint.

It is crucial to note though, that we find this bound on the Higgs quartic coupling in the absence of heterotic gravitational denominators. If instead we have the heterotic denominators, then we find that λ is strictly positive, with the explicit lower bound varying with the gauge couplings. It is also worth noting that in analogy with these poles being associated with gauge bosons' and graviton's non-minimal couplings to the dilaton via $F^2\phi$ and $R^2\phi$ in the heterotic string, we would expect a coupling $(H^\dagger H)\phi$ for these amplitudes. In the context of many BSM models, the Higgs quartic running is modified by new states below $\sim 10^{10}$GeV, and could be consistent with

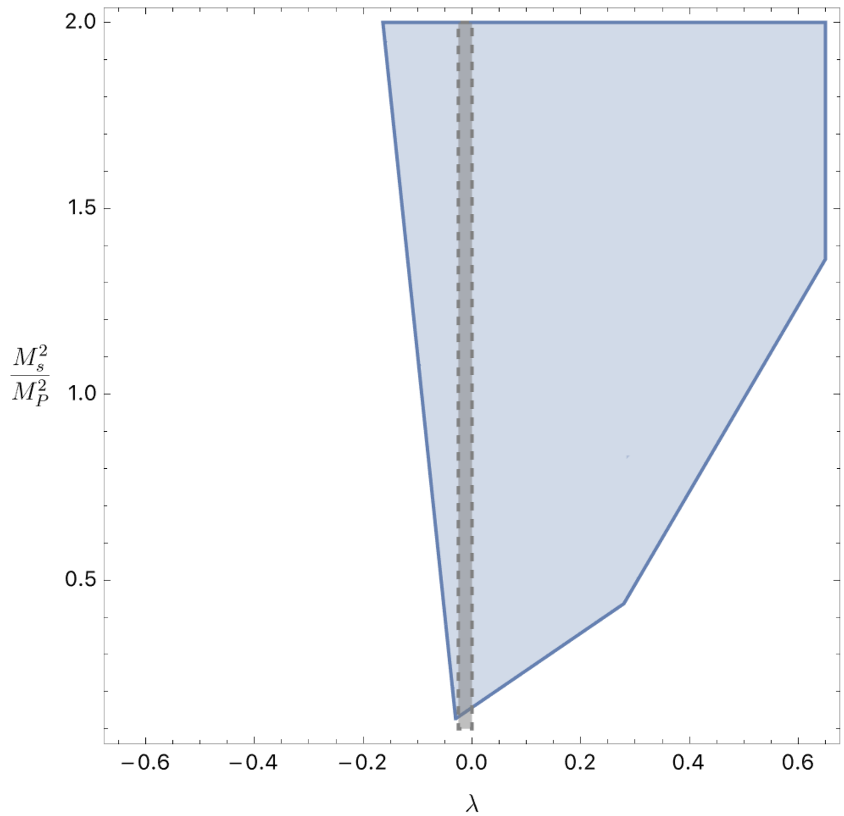


Figure 4.5: Plot of allowed region (blue) for Higgs quartic coupling λ versus the string scale squared M_s^2 in reduced Planck units. The dashed lines bounding the shaded green band are roughly 3σ bounds for the SM running of λ with uncertainty coming from the uncertainty in the mass of the top quark (taken from [1]). The bounds come from imposing unitarity of (4.53). With heterotic gravitational denominators, λ is strictly positive.

the small positive λ's allowed by this heterotic analogue of (4.53). But this is at least naively in tension with the only new states coming in at this putative string scale, a scale which gauge-boson scattering required be only slightly below the Planck scale for α's $\sim \frac{1}{25}$. This basic tension in the $2 \to 2$ Higgs amplitude already poses an interesting puzzle in this particular bottom up approach to building stringy completions of the standard model.

4.5 Conclusions Outlook

In this work we considered a class of stringy ansätze for $2 \to 2$ amplitudes and constrained them via unitarity. The stringy nature of these amplitudes flipped gravity's role from an obstruction to UV softening to a necessary condition for UV softening consistent with positive expandability of the imaginary part, with constraints bounding the strength of other interactions relative to gravity from above. We found that gauge boson scattering in particular furnished stringent constraints, especially in higher dimensions, where we found that the $SO(32)$ heterotic string is barely consistent with perturbative unitarity. Perhaps most compelling is the agreement between the suggested constraints on rank versus spacetime dimension for both $SO(N)$ and $SU(N)$ gauge boson scattering not only with each other but with the swampland conjecture $r < 26 - D$ in greater than four dimensions. These results indicate that perturbative unitarity constraints can become particularly potent in the context of gravity. There are a variety of further questions to pursue.

These amplitudes morally resolved the particles into closed strings. It would be interesting to pursue the open string analogue of this analysis which would correspond to braneworld scenarios. In this case the leading unitarizing interactions are the gauge-boson exchanges, not gravity, and a genus 0 analysis will not furnish constraints on gauge or global symmetry groups, though one can still expect constraints on couplings. In order to get gravity in the low-energy limit, which makes symmetry constraints more likely, we will need control over ansätze at genus 1.

Another interesting extension of this work would be considering deformations like the Coon amplitude or the more general deformations considered in [79]. Again here, deformations of closed string amplitudes will provide more readily amenable to this analysis as they will provide generalizations of the above constraints already at genus 0.

Finally, in order to continue to probe the consistency of the amplitudes considered herein, two obvious avenues are to develop the analysis both for higher points and for massive external legs. The analysis above not only obtains bounds from requiring the exchange of positive norm states, but tells you the quantum numbers of such states. One can iteratively build up amplitudes with these massive states as external legs and impose further consistency.

Acknowledgments We would like thank Wayne Zhao for initial collaboration. We would also like to thank Sebastian Mizera, Lorenz Eberhardt, and Yu-tin Huang for useful discussions. And we would like to especially thank Nima Arkani-Hamed for many valuable discussions and comments on the draft.

Appendix A

Appendix to Chapter 2

A.1 Conventions

In this appendix, we explicitly state all our conventions. We work with the metric signature $(+,-,-,-)$. $\mathrm{SU}(2)_L$ (of the bosons in the UV) and $\mathrm{SL}(2,\mathbf{C})$ spinor indices are raised and lowered using

$$\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{A.1}$$

We raise and lower the $\mathrm{SL}(2,\mathbf{C})$ as follows

$$\lambda^{\alpha} = \epsilon^{\alpha\beta}\lambda_{\beta} \qquad \lambda_{\beta} = \epsilon_{\beta\alpha}\lambda^{\alpha} \qquad \tilde{\lambda}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{\dot{\beta}} \qquad \tilde{\lambda}_{\dot{\beta}} = \epsilon_{\dot{\beta}\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}\,.$$

We use the same tensor for lowering and raising the indices of the massive little group $\mathrm{SU}(2)_L$ with the convention

$$\lambda_{\alpha I} = \lambda_{\alpha}^{J}\,\epsilon_{JI} \qquad \lambda_{\alpha}^{I} = \lambda_{\alpha J}\,\epsilon^{JI}.$$

Note that the Greek SL$(2, \mathbf{C})$ spinor indices are raised and lowered on the left while the Latin little group indices are raised and lowered on the right. We also make use of angle and square spinors which are defined as

$$
\begin{aligned}
|\mathbf{i}\rangle &:= |\mathbf{i}\rangle^I_\alpha = \lambda^I_\alpha, & \langle\mathbf{i}| &:= \langle\mathbf{i}|^{\alpha I} = \lambda^{\alpha I}, \\
|\mathbf{i}] &:= |\mathbf{i}]^{\dot\alpha I} = \tilde\lambda^{\dot\alpha I}, & [\mathbf{i}| &:= [\mathbf{i}|^I_{\dot\alpha} = \tilde\lambda^I_{\dot\alpha}.
\end{aligned} \tag{A.2}
$$

With this the momentum can be written as,

$$
p_{\alpha\dot\alpha} = \epsilon_{JI}|\mathbf{i}\rangle^I[\mathbf{i}|^J = \epsilon_{JI}\lambda^I_\alpha\tilde\lambda^J_{\dot\alpha}, \qquad p^{\dot\alpha\alpha} = \epsilon_{JI}\,\langle\mathbf{i}|^I\,|\mathbf{i}]^J = \epsilon_{JI}\lambda^{\alpha I}\tilde\lambda^{\dot\alpha J}\,.
$$

We follow the convention that undotted indices are contracted from top to bottom while dotted indices are contracted from the bottom to the top.

$$
\begin{aligned}
\langle\mathbf{ij}\rangle = \langle\mathbf{i}|^{\alpha I}|\mathbf{j}\rangle^J_\alpha, \qquad [\mathbf{ij}] = [\mathbf{i}|^I_{\dot\alpha}|\mathbf{j}]^{\dot\alpha J} \\
\langle\mathbf{i}|p_k|\mathbf{j}\rangle = \langle\mathbf{i}|^{\alpha I}p_{k\alpha\dot\beta}|\mathbf{j}\rangle^{\dot\beta J}, \quad [\mathbf{i}|p_k|\mathbf{j}\rangle = [\mathbf{i}|^I_{\dot\alpha}p_k^{\dot\alpha\beta}|\mathbf{j}\rangle^J_\beta
\end{aligned} \tag{A.3}
$$

In this notation, the Dirac equation reads

$$
\begin{aligned}
\langle\mathbf{i}|p_i = m_i[\mathbf{i}| \qquad p_i|\mathbf{i}] = -m_i|\mathbf{i}\rangle \\
p_i|\mathbf{i}\rangle = -m_i|\mathbf{i}] \qquad [\mathbf{i}|p_i = m_i\langle\mathbf{i}|
\end{aligned} \tag{A.4}
$$

We can expand λ^I_α and $\tilde\lambda_{\dot\alpha I}$ as explained in eq.(2.19)

$$
\begin{aligned}
\lambda^I_\alpha &= \lambda_\alpha\,\zeta^{-I} + \eta_\alpha\,\zeta^{+I} \\
&= \sqrt{E+p}\,\zeta^+_\alpha(p)\,\zeta^{-I}(k) + \sqrt{E-p}\,\zeta^-_\alpha(p)\,\zeta^{+I}(k)
\end{aligned} \tag{A.5}
$$

$$
\begin{aligned}
\tilde\lambda_{\dot\alpha I} &= \tilde\lambda_{\dot\alpha}\,\zeta^+_I - \tilde\eta_{\dot\alpha}\,\zeta^-_I \\
&= \sqrt{E+p}\,\tilde\zeta^-_\alpha(p)\,\zeta^+_I(k) - \sqrt{E-p}\,\tilde\zeta_{\dot\alpha}(p)\,\zeta^-_I(k)
\end{aligned} \tag{A.6}
$$

where

$$\zeta_I^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \zeta_I^- = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \zeta^{+I} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \zeta^{-I} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{A.7}$$

and

$$\zeta_\alpha^+(p) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}, \quad \zeta_\alpha^-(p) = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix} \tag{A.8}$$

$$\zeta_{\dot\alpha}^+(p) = \begin{pmatrix} -\sin\frac{\theta}{2}e^{i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix}, \quad \zeta_{\dot\alpha}^-(p) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{-i\phi} \end{pmatrix}. \tag{A.9}$$

With this choice, we have the following contractions

$$\begin{aligned} &\lambda_\alpha^I \zeta_I^+ = \lambda_\alpha \qquad \lambda_\alpha^I \zeta_I^- = -\eta_\alpha \\ &\tilde\lambda_{\dot\alpha}^I \zeta_I^+ = -\tilde\eta_{\dot\alpha} \qquad \tilde\lambda_{\dot\alpha}^I \zeta_I^- = -\tilde\lambda_{\dot\alpha}\,. \end{aligned} \tag{A.10}$$

Furthermore, using A.4 we can deduce following relations between λ and η

$$\begin{aligned} &p_{\alpha\dot\alpha}\lambda^\alpha = -m\,\tilde\eta_{\dot\alpha} \qquad p_{\alpha\dot\alpha}\eta^\alpha = m\,\tilde\lambda_{\dot\alpha} \\ &p_{\alpha\dot\alpha}\tilde\eta^{\dot\alpha} = m\,\lambda_\alpha \qquad p_{\alpha\dot\alpha}\tilde\lambda^{\dot\alpha} = -m\,\eta_\alpha \end{aligned} \tag{A.11}$$

A.2 Amplitudes with one massless particle and 2 equal mass particles

Three particle amplitudes involving one massless particle and two massive particles of equal mass present a difficulty. Consider the three particle amplitude with both particles 1 and 2 having mass m, spins S_1 and S_2 and a third massless particle of helicity h. In order to construct such amplitudes, it is useful to have a Lorentz invariant object which has the correct helicity weight for particle three and is invariant under the little groups for particles 1 and 2. Unfortunately, the obvious candidates vanish,

$$[3|\, p_1\, |3\rangle = 2\, p_1.p_3 = 0 \qquad [3|\, p_2\, |3\rangle = 2\, p_2.p_3 = 0\,. \tag{A.12}$$

The x-factors defined in [10] solve this problem. In our paper, we adopt a slightly different definition and notation which we explain below. For all outgoing momenta, we define

$$\frac{(p_1 - p_2)_{\alpha\dot{\alpha}}}{2m}\lambda_3^{\alpha} = x_{12}^{+}\ \tilde{\lambda}_{3\dot{\alpha}} \qquad \frac{(p_1 - p_2)_{\alpha\dot{\alpha}}}{2m}\tilde{\lambda}_3^{\dot{\alpha}} = x_{12}^{-}\ \lambda_{3\alpha}\,. \tag{A.13}$$

Under little group scaling of particle 3, the helicity spinors scale as $\lambda_3 \to t^{-1}\lambda_3$ and $\tilde{\lambda}_3 \to t\,\tilde{\lambda}_3$. It follows that $x_{12}^{+} \to t^{-2}\, x_{12}^{+}$ and $x_{12}^{-} \to t^{2}\, x_{12}^{-}$. An object with helicity h transforms as t^{-2h} under a little group scaling. This justifies the $\pm$ signs on the x-factors.

We can obtain explicit expressions for the x-factors by contracting eq.(A.13) with reference spinors ξ^{α} or $\tilde{\xi}^{\dot{\alpha}}$.

$$x_{12}^{+} = \frac{\langle 3|p_1 - p_2|\xi]}{2\, m[3\, \xi]} \qquad x_{12}^{-} = \frac{\langle \xi|p_1 - p_2|3]}{2\, m\langle \xi\, 3\rangle}\,. \tag{A.14}$$

These are the same as the conventional expressions for polarization vectors of massless particles upto a factor of $\frac{1}{m}$. It is crucial that the x-factors are independent of the reference spinor. To see this, consider two different definitions of x_{12}^{+} with reference spinors ξ_1 and ξ_2. Their difference,

$$\frac{\langle 3|p_1-p_2|\xi_1]}{2\,m[3\,\xi_1]}-\frac{\langle 3|p_1-p_2|\xi_2]}{2\,m[3\,\xi_2]}=-\frac{\langle 3|p_1-p_2|3][\xi_1\,\xi_2]}{2\,m[3\,\xi_1][3\,\xi_2]}=0\,, \tag{A.15}$$

where the first equality follows from a Schöuten identity and the second from eq.(A.12).

We can build three point amplitudes using the x-factors. Here, we will focus on the amplitude involving two spin 1 particles of mass m and a massless particle of helicity ± 1. The contributing structures are

$$\langle \mathbf{12}\rangle^2\,x_{12}^{\pm} \qquad [\mathbf{12}]^2\,x_{12}^{\pm} \qquad \langle \mathbf{1}3\rangle\,[\mathbf{2}3]\langle \mathbf{12}\rangle \qquad \langle \mathbf{1}3\rangle\,\langle \mathbf{2}3\rangle\langle \mathbf{12}\rangle \qquad \ldots \tag{A.16}$$

We pick our amplitudes to be $\langle \mathbf{12}\rangle^2\,x_{12}^{-}$ and $[\mathbf{12}]^2\,x_{12}^{+}$. This corresponds to minimal coupling. For more details about this and amplitudes corresponding to multipole moments, see [28]. We can also compare these with the vertices that we get from the usual Feynman rules (for a photon with positive helicity)

$$\epsilon_3^{+}\cdot(p_1-p_2)\,\epsilon_1\cdot\epsilon_2\;+\;\epsilon_1\cdot(p_2-p_3)\,\epsilon_2\cdot\epsilon_3^{+}\;+\;\epsilon_2\cdot(p_3-p_1)\,\epsilon_3^{+}\cdot\epsilon_1\,, \tag{A.17}$$

where

$$(\epsilon_3^{+})_{\alpha\dot\alpha}\equiv\frac{\lambda_{3\alpha}\,\xi_{\dot\alpha}}{[3\,\xi]} \qquad (\epsilon_1)_{\alpha\dot\alpha}^{I_1\,I_2}\equiv\frac{1}{m}\,\lambda_{1\alpha}^{\{I_1}\tilde\lambda_{1\dot\alpha}^{I_2\}} \qquad (\epsilon_2)_{\beta\dot\beta}^{J_1\,J_2}\equiv\frac{1}{m}\,\lambda_{2\beta}^{\{J_1}\tilde\lambda_{2\dot\beta}^{J_2\}}\,. \tag{A.18}$$

Using these definitions in eq.(A.17) and applying Schöuten identities to eliminate the reference spinors, it reduces to

$$\frac{x_{12}^{+}}{2m}[\mathbf{12}]\left(\langle \mathbf{21}\rangle+\frac{\langle \mathbf{1}3\rangle\langle 3\mathbf{1}\rangle}{m\,x_{12}^{+}}\right)=-\frac{x_{12}^{+}}{2m}[\mathbf{12}]^2\,. \tag{A.19}$$

The following identities are useful in showing this equality

$$
\begin{aligned}
[\mathbf{12}] \quad &= \langle \mathbf{12} \rangle - \frac{\langle \mathbf{1}|P|\mathbf{2}]}{m} \\
&= \langle \mathbf{12} \rangle + \frac{1}{2m} \left([\mathbf{1}|P|\mathbf{2}\rangle - \langle \mathbf{1}|P|\mathbf{2}] \right) ,
\end{aligned}
\tag{A.20}
$$

$$
[\mathbf{21}] = \langle \mathbf{21} \rangle + \frac{\langle \mathbf{2}3\rangle\langle 3\mathbf{1}\rangle}{mx^+} = \langle \mathbf{21} \rangle + \frac{[\mathbf{2}3][3\mathbf{1}]}{mx^-} \, . \tag{A.21}
$$

Similarly, for a negative helicity photon we have,

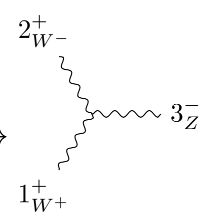
$$
\frac{-\,x_{12}^-}{2m} \langle \mathbf{21} \rangle \left([\mathbf{12}] + \frac{[\mathbf{2}3][3\mathbf{1}]}{mx^-} \right) = \frac{x_{12}^-}{2m} \langle \mathbf{21} \rangle^2 \, . \tag{A.22}
$$

A.3 High energy limits of 3 point amplitudes

In this section, we will evaluate the high energy limits of select three point amplitudes for illustrative purposes.

A.3.1 HE limits of WWZ

$$
\boxed{\texttt{./Figures/WWZ.pdf}} \xrightarrow{\text{HE}\,(++-)} \tag{A.23}
$$



We will compute the HE limit of the WWZ amplitude with legs 1 and 2 having plus helicity and leg 3 having minus.

$$
\mathbf{M}_{WWZ} = \frac{e_W}{m_W^2 m_Z} \left[\langle \mathbf{12} \rangle [\mathbf{12}] \langle \mathbf{3}|p_1 - p_2|\mathbf{3}] + \text{cyc.} \right] \xrightarrow{\text{HE}\,(++-)} M_{WWZ}^{++-} . \tag{A.24}
$$

We can get this component by contracting $\mathbf{M}_{WWZ}$ with $\zeta^+_{I_1}\zeta^+_{I_2}\zeta^+_{J_1}\zeta^+_{J_1}\zeta^-_{K_1}\zeta^-_{K_2}$ where $\{I_1, I_2\}$, $\{J_1, J_2\}$ and $\{K_1, K_2\}$ are the little group indices corresponding to legs 1,2 and 3 respectively.

This gives

$$M^{++-}_{WWZ} = \frac{e_W}{m_W^2 m_Z}\Big(\quad \langle 12\rangle[\eta_1\,\eta_2]\langle\eta_3|p_1 - p_2|3] + \langle 2\eta_3\rangle[\eta_2 3]\langle 1|p_2 - p_3|\eta_1] \\ +\langle\eta_3 1\rangle[3\eta_1]\langle 2|p_3 - p_1|\eta_2]\Big). \tag{A.25}$$

The leading term should be $\mathcal{O}(1)$ for the HE limit to be well defined. Remembering that for massless 3-particle kinematics, with helicity $(++-)$, we have $\tilde{\lambda}_1 = \langle 23\rangle\xi$, $\tilde{\lambda}_2 = \langle 31\rangle\xi$ and $\tilde{\lambda}_3 = \langle 12\rangle\xi$, we find that all square brackets involving $\tilde{\lambda}_i$ vanish. Further, we have $\langle i\eta_i\rangle = m_i$, $[\eta_i i] = m_i$ and $p_i = \lambda_i\tilde{\lambda}_i + \eta_i\tilde{\eta}_i$. This allows us to simplify the above expression to

$$M^{++-}_{WWZ} = \frac{2\,e_W}{m_Z}\Big(-\frac{\langle 2\eta_3\rangle}{\langle 23\rangle} + \frac{\langle\eta_3 1\rangle}{\langle 31\rangle}\Big) = 2\,e_W\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\,, \tag{A.26}$$

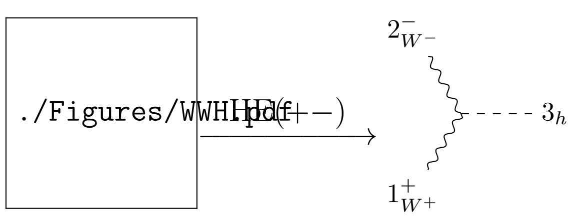
where we have made use of the Schöuten identity,

$$\langle 12\rangle\langle\eta_3 3\rangle + \langle 23\rangle\langle\eta_3 1\rangle + \langle 31\rangle\langle\eta_3 2\rangle = 0\,. \tag{A.27}$$

The other HE limits may be obtained in a similar manner.

A.3.2 HE limits of WWh and ZZh

In this section, we will illustrate all the HE limits of different components of the WWh amplitude. The computations for ZZh are identical. We begin with both legs being transverse.



$$= M^{+-}_{WWH} \tag{A.28}$$

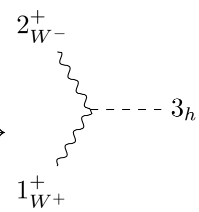
$$\mathbf{M}_{WWH} = \frac{e_{WWH}}{m_W}\langle\mathbf{12}\rangle[\mathbf{12}] + \frac{\mathcal{N}_2}{m_W}\left(\langle\mathbf{12}\rangle^2 + [\mathbf{12}]^2\right) \xrightarrow{\text{HE}(+-)} M^{+-}_{WWH} \tag{A.29}$$

As before, we must project with the appropriate little group tensors to get the required components. This gives

$$M^{+-}_{WWH} = \frac{e_{WWH}}{m_W}\langle 1\eta_2\rangle[\eta_1 2] + \frac{\mathcal{N}_2}{m_W}\left(\langle 1\eta_2\rangle^2 + [\eta_1 2]^2\right) \propto m_W \to 0\,. \tag{A.30}$$

This the HE limit involving two transverse components of opposite helicity vanishes. Next consider,

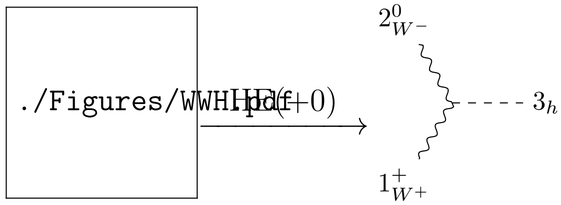
$$\boxed{\texttt{./Figures/WWH.pdf}} \xrightarrow{\text{HE}(++)}$$



$$= M^{++}_{WWH} \tag{A.31}$$

$$M^{++}_{WWH} = \frac{e_{WWH}}{m_W}\langle 12\rangle[\eta_1\eta_2] + \frac{\mathcal{N}_2}{m_W}\left(\langle 12\rangle^2 + [\eta_1\eta_2]^2\right). \tag{A.32}$$

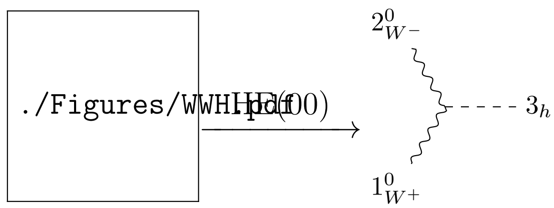
The first term is proportional to m_W and vanishes. However, $\langle 12\rangle^2$ is finite in the HE limit while $\frac{1}{m_W}$ diverges. This forces us to set $\mathcal{N}_2 = 0$. A similar analysis shows that we must have $\mathcal{N}_1 = 0$ for the WWZ amplitude to have a well defined HE limit. For the rest of the computations in this appendix, we will work with $\mathcal{N}_2 = 0$. We move on to the components of the amplitude which contain one transverse and one longitudinal mode.



$$= M^{+0}_{WWH}. \tag{A.33}$$

$$M^{+0}_{WWH} = \frac{e_{WWH}}{2m_W}\left(\langle 12\rangle[\eta_1 2] - \langle 1\eta_2\rangle[\eta_1\eta_2]\right) = \frac{e_{WWH}}{2m_W}\frac{\langle 12\rangle\langle 31\rangle}{\langle 23\rangle}, \tag{A.34}$$

where we have used $[\eta_1 2] = m_W \frac{\langle 31\rangle}{\langle 23\rangle}$. Finally, the all longitudinal component



$$= M^{00}_{WWH} \tag{A.35}$$

$$M^{00}_{WWH} = \frac{e_{WWH}}{m_W}\left(\langle 12\rangle[12] - \langle 1\eta_2\rangle[1\eta_2] - \langle\eta_1 2\rangle[\eta_1 2] + [\eta_1\eta_2]\langle\eta_1\eta_2\rangle\right). \tag{A.36}$$

Recalling that $\langle 12\rangle[12] = p_3^2 = m_h^2$ and $\eta_i \propto m_W$, we see that leading term in M^{00}_{WWH} is $\mathcal{O}(m_W)$ and vanishes in the HE limit (here we are assuming that $m_h^2/m_W \to 0$).

A.4 Computation of 4-particle amplitudes

In Sections [2.4.1-2.4.2], we glue together two three point amplitudes to construct the four point amplitude. In cases in which the exchanged particle has spin 1, the following identities are useful

$$\epsilon_{I_1 J_1}\epsilon_{I_2 J_2}\mathbf{I}^{\alpha\{I_1}\tilde{\mathbf{I}}^{\dot{\alpha}I_2\}}\mathbf{I}^{\beta\{J_1}\tilde{\mathbf{I}}^{\dot{\beta}j_2\}} = \frac{1}{2}\left(\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}m_I^2 - I^{\dot{\alpha}\beta}I^{\dot{\beta}\alpha}\right) = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}m_I^2 - \frac{1}{2}I^{\dot{\alpha}\alpha}I^{\dot{\beta}\beta}\,. \tag{A.37}$$

Note that the second equality can be obtained by using a property of the two dimensional Levi Civita tensor

$$\epsilon_{I_1 J_1}\epsilon_{I_2 J_2} + \epsilon_{I_1 I_2}\epsilon_{J_2 J_1} + \epsilon_{I_1 J_2}\epsilon_{J_1 I_2} = 0\,. \tag{A.38}$$

The following identities are useful in the computation of the 4 point amplitude in Section [2.4.1]

$$[\mathbf{12}] = \langle\mathbf{12}\rangle - \frac{\langle\mathbf{1}I\rangle\langle I\mathbf{2}\rangle}{mx_{12}^+} = \langle\mathbf{12}\rangle - \frac{[\mathbf{1}I][I\mathbf{2}]}{mx_{12}^-}, \tag{A.39}$$

$$\langle\mathbf{34}\rangle = [\mathbf{34}] + \frac{\langle\mathbf{3}I\rangle\langle I\mathbf{4}\rangle}{m_{34}^+} = [\mathbf{34}] + \frac{[\mathbf{3}I][I\mathbf{4}]}{mx_{34}^-}\,, \tag{A.40}$$

and

$$x_{12}^+x_{34}^- + x_{12}^-x_{34}^+ = \frac{1}{2m^2}(p_1-p_2)\cdot(p_3-p_4)\,. \tag{A.41}$$

A.5 Generators of $SO(4)$ and the embedding of $SU(2)\times U(1)_Y$

The representation of the generators of SU(2)× U(1) as 4×4 matrices was also introduced in [80]. These are derived by identifying the appropriate subgroups of SO(4) whose generators are

$$A_1 = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_3 = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B_1 = i \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B_3 = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The following combinations can be used to identify the $SU(2)$ and $U(1)$ subgroups

$$\begin{aligned} X^+ &= \frac{1}{2\sqrt{2}}(A_1 + iA_2 + B_1 + iB_2)\,, & Y^+ &= \frac{1}{2\sqrt{2}}(A_1 + iA_2 - B_1 - iB_2)\,, \\ X^- &= \frac{1}{2\sqrt{2}}(A_1 - iA_2 + B_1 - iB_2)\,, & Y^- &= \frac{1}{2\sqrt{2}}(A_1 - iA_2 - B_1 + iB_2)\,, \\ X^3 &= \frac{1}{2}(A_3 + B_3)\,, & Y^3 &= \frac{1}{2}(A_3 - B_3)\,. \end{aligned} \tag{A.42}$$

It is easy to see that they satisfy two copies of the $SU(2)$ algebra

$$\begin{aligned} &\left[X^+, X^-\right] = X^3, \quad \left[X^3, X^+\right] = X^+, \quad \left[X^3, X^-\right] = -X^-, \\ &\left[Y^+, Y^-\right] = Y^3, \quad \left[Y^3, Y^+\right] = Y^+, \quad \left[Y^3, Y^-\right] = -Y^-\,. \end{aligned} \tag{A.43}$$

We will associate the generators $X^\pm, X^3$ with the symmetry $SU(2)_L$. These are referred to as $T^\pm, T^3$ in the paper, such that $T^\pm \equiv 2X^\pm$. The $U(1)_Y$ is a subgroup of the $SU(2)$ formed by $Y^\pm, Y^3$ and we will set $T^B \equiv 2Y^3$.

Appendix B

Appendix to Chapter 3

B.1 Conventions

We use the metric signature $(+,-,-,-)$. The $SL(2,C)$ spinor indices α, β, $\dot{\alpha}$ and $\dot{\beta}$ are raised and lowered using

$$\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = -\epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{B.1}$$

The $SU(2)$ little group indices I, J are raised and lowered using

$$\epsilon_{IJ} = -\epsilon^{IJ} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{B.2}$$

We also use the Weyl/Chiral representation of the Dirac algebra,

$$\gamma^{\mu} = \begin{pmatrix} 0 & (\sigma^{\mu})_{\alpha\dot{\beta}} \\ (\bar{\sigma}^{\mu})^{\dot{\alpha}\beta} & 0 \end{pmatrix} , \tag{B.3}$$

with $\sigma^{\mu} = (1, \sigma^{i})$ and $\bar{\sigma}^{\mu} = (1, -\sigma^{i})$ with $\sigma^{1,2,3}$.

B.2 Kinematics

We can rewrite p_μ as a Weyl bi-spinor via,

$$p_\mu \gamma^\mu = \begin{pmatrix} 0 & p_\mu(\sigma^\mu)_{\alpha\dot{\beta}} \\ p_\mu(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} & 0 \end{pmatrix} = \begin{pmatrix} 0 & p_{\alpha\dot{\beta}} \\ p^{\dot{\alpha}\beta} & 0 \end{pmatrix}, \tag{B.4}$$

which implements the isomorphism between the Lorentz group $SO(3,1)$ and $SL(2,C)\times SL(2,C)$. Explicitly,

$$p_{\alpha\dot{\beta}} = \begin{pmatrix} p_0+p_3 & p_1-ip_2 \\ p_1+ip_2 & p_0-p_3 \end{pmatrix}, \quad \text{and} \quad p^{\dot{\alpha}\beta} = \begin{pmatrix} p_0-p_3 & -p_1+ip_2 \\ -p_1-ip_2 & p_0+p_3 \end{pmatrix}. \tag{B.5}$$

Raising and lowering of the $SL(2,C)$ indices are defined by $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma^\mu)_{\beta\dot{\beta}}$. For example, contracting with p_μ gives $p^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}p_{\beta\dot{\beta}}$. Lastly, we note that

$$\det(p_{\alpha\dot{\alpha}}) = \det(p^{\dot{\alpha}\alpha}) = p^2 = m^2\,, \tag{B.6}$$

which can be seen explicitly from the matrices above, or using the fact that $(\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\nu)^{\dot{\alpha}\alpha} = 2\eta^{\mu\nu}$, to simplify $\det(p_{\alpha\dot{\alpha}}) = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}p_{\alpha\dot{\alpha}}p_{\beta\dot{\beta}} = \frac{1}{2}p_{\alpha\dot{\alpha}}p^{\dot{\alpha}\alpha} = p^2$. Hence, we see a clear distinction between massless and massive particles, which we will discuss next.

B.2.1 Massless Particles and Helicity-Spinors

For massless particles, $\det(p_{\alpha\dot{\alpha}}) = \det(p^{\dot{\alpha}\alpha}) = 0$, and so the matrices $p_{\alpha\dot{\alpha}}$ and $p^{\dot{\alpha}\alpha}$ have rank 1. Hence, we can write it as a direct product of two spinors,

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha\tilde{\lambda}_{\dot{\alpha}} \equiv |\lambda\rangle_\alpha[\tilde{\lambda}|_{\dot{\alpha}} \qquad \text{and} \qquad p^{\dot{\alpha}\alpha} = \tilde{\lambda}^{\dot{\alpha}}\lambda^\alpha \equiv |\tilde{\lambda}]^{\dot{\alpha}}\langle\lambda|^\alpha\,, \tag{B.7}$$

where we have introduced the Dirac bra-ket notation. Keeping in mind $p^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}p_{\beta\dot{\beta}}$, it is easy to see that the spinors are related in the way we expect,

$$\lambda^{\alpha} = \epsilon^{\alpha\beta}\lambda_{\beta} \qquad \text{and} \qquad \tilde{\lambda}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{\dot{\beta}}\,, \tag{B.8}$$

or equivalently

$$\langle\lambda|^{\alpha} = \epsilon^{\alpha\beta}|\lambda\rangle_{\beta} \qquad \text{and} \qquad |\tilde{\lambda}]^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}[\tilde{\lambda}|_{\dot{\beta}}\,. \tag{B.9}$$

For real momenta $(p_{\alpha\dot{\alpha}})^* = (p_{\alpha\dot{\alpha}})^T$, which implies $(\lambda_\alpha)^* = \tilde{\lambda}_{\dot{\alpha}}$.

As mentioned before, spinors are especially useful because they transform under both the Lorentz group and little group. The action of the little group can be observed by the scaling

$$\lambda_\alpha \to \lambda'_\alpha = w\lambda_\alpha \qquad \text{and} \qquad \tilde{\lambda}_{\dot{\alpha}} \to \tilde{\lambda}'_{\dot{\alpha}} = w^{-1}\tilde{\lambda}_{\dot{\alpha}}\,, \tag{B.10}$$

which leaves $p_{\alpha\dot{\alpha}}$ invariant. For real momenta, to preserve $(\lambda_\alpha)^* = \tilde{\lambda}_{\dot{\alpha}}$ under the transformation, $(\lambda'_\alpha)^* = \tilde{\lambda}'_{\dot{\alpha}}$, we must have $w^{-1} = w^*$ i.e. w must be an element of $U(1)$.

The massless Dirac equation follows immediately

$$p_{\alpha\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}} = 0 \quad \text{and} \quad p^{\dot{\alpha}\alpha}\lambda_\alpha = 0\,, \tag{B.11}$$

since $\lambda^\alpha\lambda_\alpha = \tilde{\lambda}_{\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}} = 0$.

The above conditions are enough to find explicit representations for the spinors.

Picking a general momentum $p^\mu = (E, |\boldsymbol{p}|\sin\theta\cos\phi, |\boldsymbol{p}|\sin\theta\cos\phi, |\boldsymbol{p}|\cos\theta)$, yields

$$\lambda_\alpha = \sqrt{2E}\begin{pmatrix} c \\ s \end{pmatrix} \quad \text{and} \quad \tilde{\lambda}^{\dot{\alpha}} = \sqrt{2E}\begin{pmatrix} s^* \\ -c \end{pmatrix}, \tag{B.12}$$

with $c = \cos\theta/2$ and $s = e^{i\phi}\sin\theta/2$.

Finally, we note that these spinors are also eignstates of the helicity operator $\Sigma = S.\boldsymbol{p}/|\boldsymbol{p}|$ where spin $S_i = \frac{1}{2}\epsilon_{ijk}S^{jk}$ is defined by the spin matrix $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$,

$$\Sigma\lambda_\alpha = +\frac{1}{2}\lambda_\alpha \quad \text{and} \quad \Sigma\tilde{\lambda}^{\dot{\alpha}} = -\frac{1}{2}\tilde{\lambda}^{\dot{\alpha}}. \tag{B.13}$$

B.2.2 Massive Particles and Spin-Spinors

For massive particles, $\det(p_{\alpha\dot{\alpha}}) = \det(p^{\dot{\alpha}\alpha}) = m^2$, and so the matrices $p_{\alpha\dot{\alpha}}$ and $p^{\dot{\alpha}\alpha}$ have rank 2, and can be written as a sum of two rank 1 matrices,

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha{}^I\tilde{\lambda}_{\dot{\alpha}I} \equiv |\lambda\rangle_\alpha{}^I[\tilde{\lambda}|_{\dot{\alpha}I} \qquad \text{and} \qquad p^{\dot{\alpha}\alpha} = \tilde{\lambda}^{\dot{\alpha}}{}_I\lambda^{\alpha I} \equiv |\tilde{\lambda}]^{\dot{\alpha}}{}_I\langle\lambda|^{\alpha I} \tag{B.14}$$

where $I = 1, 2$, and again, we have introduced the Dirac bra-ket notation. As in the massless case, the spinors are related by

$$\lambda^{\alpha I} = \epsilon^{\alpha\beta}\lambda_\beta{}^I \qquad \text{and} \qquad \tilde{\lambda}^{\dot{\alpha}I} = \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{\dot{\beta}}{}^I, \quad \text{or} \tag{B.15}$$

$$\langle\lambda|^{\alpha I} = \epsilon^{\alpha\beta}|\lambda\rangle_\beta{}^I \qquad \text{and} \qquad |\tilde{\lambda}]^{\dot{\alpha}I} = \epsilon^{\dot{\alpha}\dot{\beta}}[\tilde{\lambda}|_{\dot{\beta}}{}^I. \tag{B.16}$$

Again, for real momenta, $(p_{\alpha\dot{\alpha}})^* = (p_{\alpha\dot{\alpha}})^T$, which implies $(\lambda_\alpha{}^I)^* = (\tilde{\lambda}_{\dot{\alpha}I})^T$.

We can see the action of the little group by the $SL(2)$ transformations W,

$$\lambda_\alpha{}^I \to \lambda_\alpha{}^{I'} = W_{I'}{}^I\lambda_\alpha{}^{I'} \qquad \text{and} \qquad \tilde{\lambda}_{\dot{\alpha}I} \to \tilde{\lambda}_{\dot{\alpha}I'} = (W^{-1})_I{}^{I'}\tilde{\lambda}_{\dot{\alpha}I'}. \tag{B.17}$$

For real momenta, to preserve $(\lambda_\alpha{}^I)^* = \tilde{\lambda}_{\dot{\alpha}I}$ under the transformation,$(\lambda_\alpha{}^{I'})^* = \tilde{\lambda}_{\dot{\alpha}I'}$, we must have $W^{-1} = W^\dagger$, i.e. W must be an element of $SU(2)$. Hence, the little group indices I, J can be raised and lowered as,

$$\lambda^{\alpha I} = \epsilon^{IJ}\lambda^\alpha{}_J \qquad \text{and} \qquad \tilde{\lambda}^{\dot{\alpha}I} = \epsilon^{IJ}\tilde{\lambda}^{\dot{\alpha}}{}_J\,, \qquad \text{or} \tag{B.18}$$

$$\langle\lambda|^{\alpha I} = \epsilon^{IJ}\langle\lambda|^\alpha{}_J \qquad \text{and} \qquad |\tilde{\lambda}]^{\dot{\alpha}I} = \epsilon^{IJ}|\tilde{\lambda}]^{\dot{\alpha}}{}_J,. \tag{B.19}$$

This allows us to express the momentum as

$$p_{\alpha\dot{\alpha}} = \epsilon_{IJ}\lambda_\alpha{}^I\tilde{\lambda}_{\dot{\alpha}}{}^J \equiv \epsilon_{IJ}|\lambda\rangle_\alpha{}^I[\tilde{\lambda}|_{\dot{\alpha}}{}^J \quad \text{and} \quad p^{\dot{\alpha}\alpha} = \epsilon_{IJ}\tilde{\lambda}^{\dot{\alpha}J}\lambda^{\alpha I} \equiv \epsilon_{IJ}|\tilde{\lambda}]^{\dot{\alpha}J}\langle\lambda|^{\alpha I} \tag{B.20}$$

Note that $\det(p_{\alpha\dot{\alpha}}) = \det(\lambda_\alpha{}^I)\det(\tilde{\lambda}_{\dot{\alpha}I}) = m^2$, which, without loss of generality, can be used to set $\det(\lambda_\alpha{}^I) = \det(\tilde{\lambda}_{\dot{\alpha}I}) = m$. This choice imposes the following identities,

$$\tilde{\lambda}_{\dot{\alpha}I}\tilde{\lambda}^{\dot{\alpha}}{}_J = -m\epsilon_{IJ} \quad \text{and} \quad \lambda^{\alpha I}\lambda_\alpha{}^J = -m\epsilon^{IJ}\,, \tag{B.21}$$

which are necessary to maintain the condition $m^2 = \det(p_{\alpha\dot{\alpha}})$. This can be made clearer by noting that $\det(p_{\alpha\dot{\alpha}}) = \frac{1}{2}\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}p_{\alpha\dot{\alpha}}p_{\beta\dot{\beta}}$, and after substituting the expansions of momentum in terms of spin-spinors, $m^2 = \frac{1}{2}\lambda^{\alpha J}\lambda_\alpha{}^I\epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{\dot{\alpha}I}\tilde{\lambda}_{\dot{\beta}J} = m\det(\tilde{\lambda}_{\dot{\alpha}I})$, where demanding the last equality imposes the identity above.

The above definitions yield the Dirac equation,

$$p_{\alpha\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}{}_I = m\lambda_{\alpha I} \quad \text{and} \quad p^{\dot{\alpha}\alpha}\lambda_\alpha{}^I = m\tilde{\lambda}^{\dot{\alpha}I}, \tag{B.22}$$

which is easily seen by, for example, $p_{\alpha\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}{}_J = \lambda_\alpha{}^I\tilde{\lambda}_{\dot{\alpha}I}\tilde{\lambda}^{\dot{\alpha}}{}_J$ and then applying eq. (B.21).

We often project spin components from our massive particles. As such, it would be useful to expand our spin-spinors in a basis of two-dimensional vectors $\zeta^{\pm I}$ in the

$SU(2)$ little group space,

$$\lambda_\alpha{}^I = \lambda_\alpha \zeta^{-I} + \eta_\alpha \zeta^{+I}$$
$$\tilde{\lambda}_{\dot{\alpha} I} = \tilde{\lambda}_{\dot{\alpha}} \zeta_I^+ + \tilde{\eta}_{\dot{\alpha}} \zeta_I^- \,, \tag{B.23}$$

Now, the massive $p_{\alpha\dot{\alpha}}$, from eq. (B.14) takes the form,

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \zeta^{-I} \zeta_I^+ + \eta_\alpha \tilde{\eta}_{\dot{\alpha}} \zeta^{+I} \zeta_I^-$$
$$= \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} - \eta_\alpha \tilde{\eta}_{\dot{\alpha}} \tag{B.24}$$

where we have chosen $\zeta^{-I}\zeta_I^+ = -\zeta^{+I}\zeta_I^- = 1$, or explicity,

$$\zeta_I^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \zeta_I^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{B.25}$$

Demanding the on-shell condition $\det(p_{\alpha\dot{\alpha}}) = m^2$ imposes further constraints on our coefficients λ, $\tilde{\lambda}$, η and $\tilde{\eta}$ in the little group space. To see this, we can again use $\det(p_{\alpha\dot{\alpha}}) = \frac{1}{2}\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} p_{\alpha\dot{\alpha}} p_{\beta\dot{\beta}}$, substitute eq. (B.24), and get $m^2 = \langle\lambda\eta\rangle[\tilde{\lambda}\tilde{\eta}]$. Thus, our components in the little group space must satisfy

$$m = \langle\lambda\eta\rangle = [\tilde{\lambda}\tilde{\eta}] \,. \tag{B.26}$$

With these definitions, we have the following identities,

$$\lambda^{\alpha I}\zeta_I^+ = \lambda^\alpha \,, \quad \tilde{\lambda}^{\dot{\alpha} I}\zeta_I^+ = \tilde{\eta}^{\dot{\alpha}} \,, \quad \lambda^{\alpha I}\zeta_I^- = -\eta^\alpha \,, \quad \tilde{\lambda}^{\dot{\alpha} I}\zeta_I^- = -\tilde{\lambda}^{\dot{\alpha}} \,, \tag{B.27}$$

which will be useful in taking the high energy limits. Comparing the above to the helicity-spinors in eq. (B.13), we see that the spin components in eq. (B.23) were constructed to be proportional their massless helicity counterparts. We can make the explicit connection to the massless spinors in eq. (B.12) by

$$\begin{aligned}
\lambda_\alpha{}^I &= \lambda_\alpha \zeta^{-I} + \eta_\alpha \zeta^{+I} \\
&= \sqrt{E+p}\,\zeta^+_\alpha \zeta^{-I} + \sqrt{E-p}\,\zeta^-_\alpha \zeta^{+I}\,, \\
\tilde{\lambda}_{\dot\alpha I} &= \tilde{\lambda}_{\dot\alpha} \zeta^+_I + \tilde{\eta}_{\dot\alpha} \zeta^-_I \\
&= \sqrt{E+p}\,\tilde{\zeta}^-_{\dot\alpha} \zeta^+_I + \sqrt{E-p}\,\tilde{\zeta}^+_{\dot\alpha} \zeta^-_I\,,
\end{aligned} \tag{B.28}$$

where η_α and $\tilde{\eta}_{\dot\alpha}$ are also eigenstates of helicity and,

$$\zeta^+_\alpha = \begin{pmatrix} c \\ s \end{pmatrix}, \quad \zeta^-_\alpha = \begin{pmatrix} s^* \\ -c \end{pmatrix}, \quad \tilde{\zeta}^+_{\dot\alpha} = \begin{pmatrix} -s \\ c \end{pmatrix}, \quad \tilde{\zeta}^-_{\dot\alpha} = \begin{pmatrix} c \\ s^* \end{pmatrix}. \tag{B.29}$$

Note that in the high energy limit, $E \gg m$, we have that $\sqrt{E+p} \to \sqrt{2E}$ and $\sqrt{E-p} \to m/\sqrt{2E}$. Thus, in this sense, the massive spin-spinors are constructed to coincide with their massless helicity spinors in the high energy limit.

B.3 High Energy Limits

In this section, we illustrate how to take the high energy limits of amplitudes with our conventions. We use the IR amplitude in eq. (3.49). The kinematic part of the amplitude is given by,

$$\mathcal{A}(\mathbf{1}^{1/2}, \mathbf{2}^{1/2}, \mathbf{3}^1) = X_1\langle\mathbf{13}\rangle[\mathbf{32}] + X_2[\mathbf{13}]\langle\mathbf{32}\rangle + X_3\langle\mathbf{13}\rangle\langle\mathbf{32}\rangle + X_4[\mathbf{13}][\mathbf{32}] \tag{B.30}$$

which, after restoring the explicit $SU(2)$ little group indices takes the form,

$$\begin{aligned}
\mathcal{A}(\mathbf{1}^I, \mathbf{2}^J, \mathbf{3}^{\{K_1,K_2\}}) &= X_1\langle\mathbf{1}^I\mathbf{3}^{\{K_1}\rangle[\mathbf{3}^{K_2\}}\mathbf{2}^J] + X_2[\mathbf{1}^I\mathbf{3}^{\{K_1}]\langle\mathbf{3}^{K_2\}}\mathbf{2}^J\rangle \\
&\quad + X_3\langle\mathbf{1}^I\mathbf{3}^{\{K_1}\rangle\langle\mathbf{3}^{K_2\}}\mathbf{2}^J\rangle + X_4[\mathbf{1}^I\mathbf{3}^{\{K_1}][\mathbf{3}^{K_2\}}\mathbf{2}^J]
\end{aligned} \tag{B.31}$$

As previsouly mentioned, the momentum of a particle with spin-s is described by a rank-$2s$ symmetric tensor, and has $2s+1$ dimensions. So the amplitude $\mathcal{A}(\mathbf{1}^I, \mathbf{2}^J, \mathbf{3}^{\{K_1,K_2\}})$ has dimension $2\times 2\times 3 = 12$, corresponding to the different spin components of each particle. One can either take the high energy limit of the amplitude first, and attain all the 16 spin projections, or project a particular spin configuration first, and then take the high energy limit. We will adopt the latter strategy.

Our goal is to determine the coefficient of the spin components $\mathcal{A}(1^{+1/2}, 2^{+1/2}, 3^0)$ in the high energy limit of $\mathcal{A}(\mathbf{1}^I, \mathbf{2}^J, \mathbf{3}^{\{K_1,K_2\}})$. We begin by projecting the spin components, which is a straightfoward application of the identites in eq. (B.27),

$$\begin{aligned}\mathcal{A}(\mathbf{1}^{+1/2}, \mathbf{2}^{+1/2}, \mathbf{3}^0) &= \mathcal{A}(\mathbf{1}^I, \mathbf{2}^J, \mathbf{3}^{\{K_1,K_2\}}) \times \zeta_I^+\zeta_J^+(\zeta_{K_1}^+\zeta_{K_2}^- + \zeta_{K_2}^+\zeta_{K_1}^-) \\ &= -X_1\left(\langle 13\rangle[3\tilde{\eta}_2] + \langle 1\eta_3\rangle[\tilde{\eta}_3\tilde{\eta}_2]\right) - X_2\left([\tilde{\eta}_1 3]\langle 32\rangle + [\tilde{\eta}_1\tilde{\eta}_3]\langle \eta_3 2\rangle\right) \\ &\quad - X_3\left(\langle 13\rangle\langle \eta_3 2\rangle + \langle 1\eta_3\rangle\langle 32\rangle\right) - X_4\left([\tilde{\eta}_1 3][\tilde{\eta}_3\tilde{\eta}_2] + [\tilde{\eta}_1\tilde{\eta}_3][3\tilde{\eta}_2]\right) .\end{aligned} \tag{B.32}$$

The second step is to take the high energy limit. This can be simplified by recalling that a three particle amplitude in four dimensions must have mass dimension 0, and so the X's must have mass dimension -1. As such, terms $\mathcal{O}(m^2)$ or higher will vanish in the high energy limit. In addition, $\eta, \tilde{\eta} \propto \sqrt{E-p} = m\sqrt{2E} + \mathcal{O}(m^2)$ in the high energy limit. Thus, the only relevant terms are those that have at most one $\eta, \tilde{\eta}$. Thus, our next task is to convert terms involving η or $\tilde{\eta}$ into a form that is suitable for taking a high energy limit.

Any square brackets with $\tilde{\eta}$ can be converted to angle brackets through a Schouten identity $|1]\langle 1| - |\tilde{\eta}_1]\langle \eta_1| + |2]\langle 2| - |\tilde{\eta}_2]\langle \eta_2| + |3]\langle 3| - |\tilde{\eta}_3]\langle \eta_3| = 0$. For example, contracting with $[\tilde{\eta}_1|$, omitting terms of $\mathcal{O}(m^2)$, and using the on-shell conditions in eq. (B.26), we get $-m_1\langle 1| + [\tilde{\eta}_1 2]\langle 2| + [\tilde{\eta}_2 3]\langle 3| = 0$, from which, either contracting with $|2\rangle$ or $|3\rangle$, we get $[\tilde{\eta}_1 2] = m_1\frac{\langle 13\rangle}{\langle 23\rangle}$ and $[\tilde{\eta}_1 3] = m_1\frac{\langle 12\rangle}{\langle 32\rangle}$. Repeated use of the Schouten identity, or applying a cyclic permutation on the set of identities

$\left\{[\tilde\eta_1 1] = -m_1, [\tilde\eta_1 2] = m_1\frac{\langle 13\rangle}{\langle 23\rangle}, [\tilde\eta_1 3] = m_1\frac{\langle 12\rangle}{\langle 32\rangle}\right\}$ gives, for $\sum h > 0$,

$$\begin{aligned}
&[\tilde\eta_1 1] = -m_1\,, \quad [\tilde\eta_1 2] = m_1\frac{\langle 13\rangle}{\langle 23\rangle}\,, \quad [\tilde\eta_1 3] = m_1\frac{\langle 12\rangle}{\langle 32\rangle}\,,\\
&[\tilde\eta_2 2] = -m_2\,, \quad [\tilde\eta_2 3] = m_2\frac{\langle 21\rangle}{\langle 31\rangle}\,, \quad [\tilde\eta_2 1] = m_2\frac{\langle 23\rangle}{\langle 13\rangle}\,,\\
&[\tilde\eta_3 3] = -m_3\,, \quad [\tilde\eta_3 1] = m_3\frac{\langle 32\rangle}{\langle 12\rangle}\,, \quad [\tilde\eta_3 2] = m_3\frac{\langle 31\rangle}{\langle 21\rangle}\,.
\end{aligned} \tag{B.33}$$

Next, any angle brackets with η can be written in a form more useful for a high energy limit by use of the on-shell condition eq. (B.26). For example, if all that matters is $\langle\eta_1 1\rangle = -m_1$, then we can write $\langle\eta_1| = -m_1\frac{\langle 2|}{\langle 21\rangle}$, from which we obtain $\langle\eta_1 2\rangle = 0$ and $\langle\eta_1 3\rangle = -m_1\frac{\langle 32\rangle}{\langle 21\rangle}$. Again, either by repeatedly using this, or applying a cyclic permutation on the set of identities $\left\{\langle\eta_1 1\rangle = -m_1, \langle\eta_1 2\rangle = 0, \langle\eta_1 3\rangle = -m_1\frac{\langle 32\rangle}{\langle 21\rangle}\right\}$, gives, for $\sum h > 0$,

$$\begin{aligned}
&\langle\eta_1 1\rangle = -m_1, \quad \langle\eta_1 2\rangle = 0, \quad \langle\eta_1 3\rangle = -m_1\frac{\langle 32\rangle}{\langle 21\rangle}\,,\\
&\langle\eta_2 2\rangle = -m_2, \quad \langle\eta_2 3\rangle = 0, \quad \langle\eta_2 1\rangle = -m_2\frac{\langle 13\rangle}{\langle 32\rangle}\,,\\
&\langle\eta_3 3\rangle = -m_3, \quad \langle\eta_3 1\rangle = 0, \quad \langle\eta_3 2\rangle = -m_3\frac{\langle 21\rangle}{\langle 13\rangle}\,.
\end{aligned} \tag{B.34}$$

For $\sum h < 0$, simply take eq. (B.33) and eq. (B.34) and replace all angle with square brackets and vice-versa.

Finally, we can take the high energy limit of eq. (B.32) by substituting eq. (B.33) and eq. (B.34), and dropping all terms $\mathcal{O}(m^2)$ and higher,

$$\mathcal{A}(\mathbf{1}^{+1/2}, \mathbf{2}^{+1/2}, \mathbf{3}^0) \xrightarrow{HE} \mathcal{A}(1^{+1/2}, 2^{+1/2}, 3^0) = \langle 12\rangle\,(X_1 m_2 - X_2 m_1 + X_3 m_3) \tag{B.35}$$

We summarize the results of this process on all spin components of the amplitude in eq. (B.30) in Table B.1.

1	2	3	$\sum h$	$\langle\mathbf{13}\rangle[\mathbf{32}]$	$[\mathbf{13}]\langle\mathbf{32}\rangle$	$\langle\mathbf{13}\rangle\langle\mathbf{32}\rangle$	$[\mathbf{13}][\mathbf{32}]$
				X_1	X_2	X_3	X_4
$+1/2$	$+1/2$	$+1$	$+2$	0	0	$\langle 13\rangle\langle 32\rangle$	0
$-1/2$	$+1/2$	$+1$	$+1$	0	$m_3\frac{\langle 32\rangle^2}{\langle 12\rangle}$	$m_3\frac{\langle 32\rangle^2}{\langle 12\rangle}$	0
$+1/2$	$-1/2$	$+1$	$+1$	$-m_3\frac{\langle 13\rangle^2}{\langle 12\rangle}$	0	0	0
$-1/2$	$-1/2$	$+1$	0	0	0	0	0
$+1/2$	$+1/2$	-1	0	0	0	0	0
$-1/2$	$+1/2$	-1	-1	0	$-m_3\frac{[13]^2}{[12]}$	0	0
$+1/2$	$-1/2$	-1	-1	$m_3\frac{[23]^2}{[12]}$	0	0	$m_3\frac{[23]^2}{[12]}$
$-1/2$	$-1/2$	-1	-2	0	0	0	$[13][32]$
$+1/2$	$+1/2$	0	$+1$	$m_2\langle 12\rangle$	$-m_1\langle 12\rangle$	$m_3\langle 12\rangle$	0
$-1/2$	$+1/2$	0	0	0	0	0	0
$+1/2$	$-1/2$	0	0	0	0	0	0
$-1/2$	$-1/2$	0	-1	$-m_1[12]$	$m_2[12]$	0	$m_3[12]$

Table B.1: High energy limits of two massive spin-1/2, one massive spin-1

B.4 Representation of the Higgs

We would like to construct a representation that makes the four degrees of freedom clear. This can be done, for example, by considering a four dimensional real representation of $SO(4)$ and then extracting an $SU(2)\times U(1)$ subgroup [11, 81]. We will instead take the approach [82].

The scalar field has four real degrees of freedom, ϕ_1, ϕ_2, ϕ_3 and ϕ_4, and are usually packaged in the form

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{pmatrix} \tag{B.36}$$

To make contact with a Lagrangian formalism, the $SU(2)_L\times U(1)_Y$ covariant derivative acting on ϕ is given by,

$$D_\mu\phi_i = \partial_\mu\phi_i + W^k_\mu(\mathcal{L}^k)_i^{\,j}\phi_j + B_\mu(\mathcal{L}^4)_i^{\,j}\phi_j \tag{B.37}$$

where $i, j = 1, 2,\ k = 1, 2, 3$ and $(\mathcal{L}^k)_i^{\ j} = \frac{1}{2} i g_1 (\tau^k)_i^{\ j}$ and $(\mathcal{L}^4)_i^{\ j} = \frac{1}{2} i g_2 Y \delta_i^{\ j}$. To see how the real degrees of freedom transform, we can first consider a reducible scalar representation Φ, a complex 4-component vector,

$$\Phi = \begin{pmatrix} \phi \\ \phi^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \\ \phi_1 - i\phi_2 \\ \phi_3 - i\phi_4 \end{pmatrix} = \begin{pmatrix} \phi^+ \\ \phi^0 \\ \phi^- \\ \phi^{0*} \end{pmatrix} \tag{B.38}$$

with half the components redundant. Now, Φ transforms under the reducible representation,

$$\mathcal{L}_\Phi^k = \begin{pmatrix} \mathcal{L}_\phi^k & \\ & -\mathcal{L}_\phi^{k*} \end{pmatrix} . \tag{B.39}$$

Since $\mathcal{L}_\Phi^3$ and $\mathcal{L}_\Phi^4$ are diagonal, that is,

$$\mathcal{L}_\Phi^3 = \frac{1}{2} i g_1 \begin{pmatrix} \sigma^3 & \\ & -\sigma^3 \end{pmatrix} \quad \text{and} \quad \mathcal{L}_\Phi^4 = \frac{1}{2} i g_2 Y \begin{pmatrix} \delta & \\ & -\delta \end{pmatrix} \tag{B.40}$$

we can read of the z-component of the $SU(2)_L$ spin, as well as the $U(1)_Y$ charge of every component of the scalar field.

Component	$\mathcal{L}^3$	$\mathcal{L}^4$	Q
ϕ^+	$+\frac{1}{2}g_1$	$+\frac{1}{2}g_2 Y$	$+e$
ϕ^0	$-\frac{1}{2}g_1$	$+\frac{1}{2}g_2 Y$	0
ϕ^-	$+\frac{1}{2}g_1$	$-\frac{1}{2}g_2 Y$	$-e$
ϕ^{0*}	$-\frac{1}{2}g_1$	$-\frac{1}{2}g_2 Y$	0

Table B.2: Weights of the Higgs components

The usefulness of this reducible representation is that we can now perform a simple change of basis to act on the real degrees of freedom ϕ_h. The two representations are related by $\Phi = A\phi_h$, with

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & i\mathbf{1} \\ \mathbf{1} & -i\mathbf{1} \end{pmatrix} \quad \text{and} \quad \phi_h = \begin{pmatrix} \phi_1 \\ \phi_3 \\ \phi_2 \\ \phi_4 \end{pmatrix}. \tag{B.41}$$

Thus, in this basis, the representation matrices are given by $\mathcal{L}^i_{\phi_h} = A^\dagger \mathcal{L}^i_\Phi A$.

Explicitly, we have

$$\mathcal{L}^1_\Phi = \frac{g_1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{L}^2_\Phi = \frac{g_1}{2} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$\mathcal{L}^3_\Phi = \frac{g_1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{L}^4_\Phi = \frac{g_2}{2} Y \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{B.42}$$

in the basis $\phi^+, \phi^0, \phi^-, \phi^{0*}$ and

$$\mathcal{L}^1_{\phi_h} = \frac{g_1}{2}\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{L}^2_{\phi_h} = \frac{g_1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$
$$\mathcal{L}^3_{\phi_h} = \frac{g_1}{2}\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \mathcal{L}^4_{\phi_h} = \frac{g_1}{2}Y\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}. \tag{B.43}$$

in the basis $\phi_1, \phi_2, \phi_3, \phi_4$. Eq. (B.43) corresponds to the subset of $SO(4)$ generators usually used to describe the real degrees of freedom of the Higgs.

Bibliography

[1] Giuseppe Degrassi, Stefano Di Vita, Joan Elias-Miro, Jose R. Espinosa, Gian F. Giudice, Gino Isidori, and Alessandro Strumia. Higgs mass and vacuum stability in the Standard Model at NNLO. *JHEP*, 08:098, 2012.

[2] Stephen J. Parke and T. R. Taylor. Gluonic Two Goes to Four. *Nucl. Phys. B*, 269:410–420, 1986.

[3] Stephen J. Parke and T. R. Taylor. An Amplitude for n Gluon Scattering. *Phys. Rev. Lett.*, 56:2459, 1986.

[4] Henriette Elvang and Yu-tin Huang. *Scattering Amplitudes in Gauge Theory and Gravity*. Cambridge University Press, 2015.

[5] Lance J. Dixon. A brief introduction to modern amplitude methods. In *Proceedings, 2012 European School of High-Energy Physics (ESHEP 2012): La Pommeraye, Anjou, France, June 06-19, 2012*, pages 31–67, 2014.

[6] Lance J. Dixon. Calculating scattering amplitudes efficiently. In *QCD and beyond. Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics, TASI-95, Boulder, USA, June 4-30, 1995*, pages 539–584, 1996.

[7] Johannes M. Henn and Jan C. Plefka. Scattering Amplitudes in Gauge Theories. *Lect. Notes Phys.*, 883:pp.1–195, 2014.

[8] Ruth Britto. Loop amplitudes in gauge theories: modern analytic approaches. *Journal of Physics A: Mathematical and Theoretical*, 44(45):454006, Oct 2011.

[9] Clifford Cheung. TASI Lectures on Scattering Amplitudes. In *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics : Anticipating the Next Discoveries in Particle Physics (TASI 2016): Boulder, CO, USA, June 6-July 1, 2016*, pages 571–623, 2018.

[10] Nima Arkani-Hamed, Tzu-Chen Huang, and Yu-tin Huang. Scattering Amplitudes For All Masses and Spins. 2017.

[11] Brad Bachu and Akshay Yelleshpur. On-Shell Electroweak Sector and the Higgs Mechanism. *JHEP*, 08:039, 2020.

[12] Brad Bachu and Aaron Hillman. Stringy Completions of the Standard Model from the Bottom Up. 12 2022.

[13] Stephen J. Parke and T. R. Taylor. Gluonic Two Goes to Four. *Nucl. Phys.*, B269:410–420, 1986.

[14] Stephen J. Parke and T. R. Taylor. An Amplitude for n Gluon Scattering. *Phys. Rev. Lett.*, 56:2459, 1986.

[15] Steven Weinberg. *The Quantum theory of fields. Vol. 1: Foundations*. Cambridge University Press, 2005.

[16] Eugene P. Wigner. On Unitary Representations of the Inhomogeneous Lorentz Group. *Annals Math.*, 40:149–204, 1939. [Reprint: Nucl. Phys. Proc. Suppl.6,9(1989)].

[17] R. Kleiss and W. James Stirling. Spinor Techniques for Calculating p anti-p —> W+- / Z0 + Jets. *Nucl. Phys.*, B262:235–262, 1985.

[18] R. Kleiss and W. James Stirling. TOP QUARK PRODUCTION AT HADRON COLLIDERS: SOME USEFUL FORMULAE. *Z. Phys.*, C40:419–423, 1988.

[19] Kaoru Hagiwara and D. Zeppenfeld. Helicity Amplitudes for Heavy Lepton Production in e+ e- Annihilation. *Nucl. Phys.*, B274:1–32, 1986.

[20] Christian Schwinn and Stefan Weinzierl. Scalar diagrammatic rules for Born amplitudes in QCD. *JHEP*, 05:006, 2005.

[21] Christian Schwinn and Stefan Weinzierl. SUSY ward identities for multi-gluon helicity amplitudes with massive quarks. *JHEP*, 03:030, 2006.

[22] S. D. Badger, E. W. Nigel Glover, V. V. Khoze, and P. Svrcek. Recursion relations for gauge theory amplitudes with massive particles. *JHEP*, 07:025, 2005.

[23] S. D. Badger, E. W. Nigel Glover, and Valentin V. Khoze. Recursion relations for gauge theory amplitudes with massive vector bosons and fermions. *JHEP*, 01:066, 2006.

[24] K. J. Ozeren and W. J. Stirling. Scattering amplitudes with massive fermions using BCFW recursion. *Eur. Phys. J.*, C48:159–168, 2006.

[25] Timothy Cohen, Henriette Elvang, and Michael Kiermaier. On-shell constructibility of tree amplitudes in general field theories. *Journal of High Energy Physics*, 2011(4), Apr 2011.

[26] Alfredo Guevara, Alexander Ochirov, and Justin Vines. Scattering of spinning black holes from exponentiated soft factors. *Journal of High Energy Physics*, 2019(9), Sep 2019.

[27] Alfredo Guevara, Alexander Ochirov, and Justin Vines. Black-hole scattering with general spin directions from minimal-coupling amplitudes. *Physical Review D*, 100(10), Nov 2019.

[28] Ming-Zhi Chung, Yu-tin Huang, and Jung-Wook Kim. Kerr-Newman stress-tensor from minimal coupling to all orders in spin. 2019.

[29] Daniel J. Burger, William T. Emond, and Nathan Moynihan. Rotating Black Holes in Cubic Gravity. 2019.

[30] Ming-Zhi Chung, Yu-tin Huang, Jung-Wook Kim, and Sangmin Lee. The simplest massive s-matrix: from minimal coupling to black holes. *Journal of High Energy Physics*, 2019(4), Apr 2019.

[31] Yilber Fabian Bautista and Alfredo Guevara. From Scattering Amplitudes to Classical Physics: Universality, Double Copy and Soft Theorems. 2019.

[32] Henrik Johansson and Alexander Ochirov. Double copy for massive quantum particles with spin. *Journal of High Energy Physics*, 2019(9), Sep 2019.

[33] Yilber Fabian Bautista and Alfredo Guevara. On the Double Copy for Spinning Matter. 2019.

[34] Aidan Herderschee, Seth Koren, and Timothy Trott. Massive On-Shell Supersymmetric Scattering Amplitudes. *JHEP*, 10:092, 2019.

[35] Aidan Herderschee, Seth Koren, and Timothy Trott. Constructing $\mathcal{N} = 4$ Coulomb branch superamplitudes. *JHEP*, 08:107, 2019.

[36] Yael Shadmi and Yaniv Weiss. Effective Field Theory Amplitudes the On-Shell Way: Scalar and Vector Couplings to Gluons. *JHEP*, 02:165, 2019.

[37] Rafael Aoude and Camila S. Machado. The Rise of SMEFT On-shell Amplitudes. 2019.

[38] Teng Ma, Jing Shu, and Ming-Lei Xiao. Standard Model Effective Field Theory from On-shell Amplitudes. 2019.

[39] Clifford Cheung, Karol Kampf, Jiri Novotny, Chia-Hsien Shen, Jaroslav Trnka, and Congkao Wen. Vector Effective Field Theories from Soft Limits. *Phys. Rev. Lett.*, 120(26):261602, 2018.

[40] Clifford Cheung, Chia-Hsien Shen, and Jaroslav Trnka. Simple Recursion Relations for General Field Theories. *JHEP*, 06:118, 2015.

[41] Gauthier Durieux, Teppei Kitahara, Yael Shadmi, and Yaniv Weiss. The electroweak effective field theory from on-shell amplitudes. 2019.

[42] Neil Christensen and Bryan Field. Constructive standard model. *Phys. Rev.*, D98(1):016014, 2018.

[43] Neil Christensen, Bryan Field, Annie Moore, and Santiago Pinto. 2-, 3- and 4-Body Decays in the Constructive Standard Model. 2019.

[44] Song He and Chi Zhang. Notes on Scattering Amplitudes as Differential Forms. *JHEP*, 10:054, 2018.

[45] Nima Arkani-Hamed and Jaroslav Trnka. The Amplituhedron. *JHEP*, 10:030, 2014.

[46] Nima Arkani-Hamed, Yuntao Bai, Song He, and Gongwang Yan. Scattering forms and the positive geometry of kinematics, color and the worldsheet. *Journal of High Energy Physics*, 2018(5), May 2018.

[47] Pinaki Banerjee, Alok Laddha, and Prashanth Raman. Stokes polytopes: the positive geometry for ϕ^4 interactions. *JHEP*, 08:067, 2019.

[48] Giulio Salvatori. 1-loop Amplitudes from the Halohedron. 2018.

[49] P. B. Aneesh, Mrunmay Jagadale, and Nikhil Kalyanapuram. Accordiohedra as positive geometries for generic scalar field theories. *Phys. Rev.*, D100(10):106013, 2019.

[50] Nathaniel Craig, Henriette Elvang, Michael Kiermaier, and Tracy Slatyer. Massive amplitudes on the Coulomb branch of N=4 SYM. *JHEP*, 12:097, 2011.

[51] Michael Kiermaier. The Coulomb-branch S-matrix from massless amplitudes. 5 2011.

[52] Eduardo Conde and Andrea Marzolla. Lorentz Constraints on Massive Three-Point Amplitudes. *JHEP*, 09:041, 2016.

[53] Eduardo Conde, Euihun Joung, and Karapet Mkrtchyan. Spinor-Helicity Three-Point Amplitudes from Local Cubic Interactions. *JHEP*, 08:040, 2016.

[54] Nima Arkani-Hamed, Tzu-Chen Huang, and Yu-tin Huang. Scattering Amplitudes For All Masses and Spins. 2017.

[55] Gauthier Durieux, Teppei Kitahara, Camila S. Machado, Yael Shadmi, and Yaniv Weiss. Constructing massive on-shell contact terms. *JHEP*, 12:175, 2020.

[56] Neil Christensen and Bryan Field. Constructive standard model. *Phys. Rev.*, D98(1):016014, 2018.

[57] Alexander Ochirov. Helicity amplitudes for QCD with massive quarks. *JHEP*, 04:089, 2018.

[58] Chao Wu and Shou-Hua Zhu. Massive on-shell recursion relations for n-point amplitudes. *JHEP*, 06:117, 2022.

[59] Marco Chiodaroli, Murat Gunaydin, Henrik Johansson, and Radu Roiban. Spinor-helicity formalism for massive and massless amplitudes in five dimensions. *JHEP*, 02:040, 2023.

[60] Hongkai Liu, Teng Ma, Yael Shadmi, and Michael Waterbury. An EFT hunter's guide to two-to-two scattering: HEFT and SMEFT on-shell amplitudes. 1 2023.

[61] Reuven Balkin, Gauthier Durieux, Teppei Kitahara, Yael Shadmi, and Yaniv Weiss. On-shell Higgsing for EFTs. *JHEP*, 03:129, 2022.

[62] Gauthier Durieux, Teppei Kitahara, Yael Shadmi, and Yaniv Weiss. The electroweak effective field theory from on-shell amplitudes. *JHEP*, 01:119, 2020.

[63] Rafael Aoude and Camila S. Machado. The Rise of SMEFT On-shell Amplitudes. *JHEP*, 12:058, 2019.

[64] Da Liu and Zhewei Yin. Gauge invariance from on-shell massive amplitudes and tree-level unitarity. *Phys. Rev. D*, 106(7):076003, 2022.

[65] Aidan Herderschee, Seth Koren, and Timothy Trott. Massive On-Shell Supersymmetric Scattering Amplitudes. *JHEP*, 10:092, 2019.

[66] Steven Weinberg. *The Quantum theory of fields. Vol. 1: Foundations*. Cambridge University Press, 6 2005.

[67] Eugene P. Wigner. On Unitary Representations of the Inhomogeneous Lorentz Group. *Annals Math.*, 40:149–204, 1939.

[68] Henriette Elvang and Yu-tin Huang. *Scattering Amplitudes in Gauge Theory and Gravity*. Cambridge University Press, 4 2015.

[69] Paolo Benincasa and Freddy Cachazo. Consistency Conditions on the S-Matrix of Massless Particles. 5 2007.

[70] David A. McGady and Laurentiu Rodina. Higher-spin massless S-matrices in four-dimensions. *Phys. Rev. D*, 90(8):084048, 2014.

[71] Herbi K. Dreiner, Howard E. Haber, and Stephen P. Martin. Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry. *Phys. Rept.*, 494:1–196, 2010.

[72] S. L. Glashow. Partial Symmetries of Weak Interactions. *Nucl. Phys.*, 22:579–588, 1961.

[73] P. Sikivie, Leonard Susskind, Mikhail B. Voloshin, and Valentin I. Zakharov. Isospin Breaking in Technicolor Models. *Nucl. Phys. B*, 173:189–207, 1980.

[74] M. A. Virasoro. Alternative constructions of crossing-symmetric amplitudes with regge behavior. *Phys. Rev.*, 177:2309–2311, 1969.

[75] Predrag Cvitanovic. *Group theory: Birdtracks, Lie's and exceptional groups*. 2008.

[76] David J. Gross, Jeffrey A. Harvey, Emil J. Martinec, and Ryan Rohm. Heterotic String Theory. 2. The Interacting Heterotic String. *Nucl. Phys. B*, 267:75–124, 1986.

[77] Nima Arkani-Hamed, Lorenz Eberhardt, Yu-tin Huang, and Sebastian Mizera. On unitarity of tree-level string amplitudes. *JHEP*, 02:197, 2022.

[78] Hee-Cheol Kim, Houri-Christina Tarazi, and Cumrun Vafa. Four-dimensional $\mathcal{N} = \mathbf{4}$ SYM theory and the swampland. *Phys. Rev. D*, 102(2):026003, 2020.

[79] Clifford Cheung and Grant N. Remmen. Veneziano Variations: How Unique are String Amplitudes? 10 2022.

[80] Andreas Helset, Michael Paraskevas, and Michael Trott. Gauge fixing the Standard Model Effective Field Theory. *Phys. Rev. Lett.*, 120(25):251801, 2018.

[81] Andreas Helset, Michael Paraskevas, and Michael Trott. Gauge fixing the Standard Model Effective Field Theory. *Phys. Rev. Lett.*, 120(25):251801, 2018.

[82] Paul Langacker. *The Standard Model and Beyond.* Taylor & Francis, 2017.