Introduction to Riemannian Geometry

Course module NWI-WM310

Radboud University Nijmegen
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Annegret Burtscher

Contents

Preface

These lecture notes have been composed for the 3rd-year Bachelor course "Riemannian Geometry" (NWI-WB045B) at Radboud University Nijmegen.

The lecture notes closely follow the structure of the book on Riemannian Geometry by John Lee [36], which builds upon his earlier book [35] on smooth manifolds. The book [35] was also used for the course "Manifolds" (NWI-WB079C) which is a prerequisite for this course on Riemannian Geometry. Besides, there are many other excellent introductory books (and lecture notes) about Riemannian Geometry that can be used as well, such as the classic book by do Carmo [23], the comprehensive book by Petersen [45], the book by O'Neill [43] with a broader perspective also on semi-Riemannian Geometry and the vast panorama by Berger [7].

If you continue with a Master, a good follow-up course will be the MasterMath course on Differential Geometry and certain courses in Mathematical Physics.

If you encounter any mistakes or typos please send me an email to burtscher@math.ru.nl. Comments and suggestions are also welcome.

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Nijmegen, January 2022
Annegret Burtscher

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I am grateful to Max van Horssen for pointing out several typos.

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Nijmegen, July 2022
Annegret Burtscher

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In September 2024 "Riemannian Geometry" has been moved to the Master progam. We'll still be using these notes as primary material but mildly shift some emphasis from Chapter 2 to Chapter 6.

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Nijmegen, September 2024
Annegret Burtscher

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Thanks to Luuk Venbrux for sending me a list of typos.

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Nijmegen, February 2025
Annegret Burtscher

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CHAPTER 1

Introduction

Riemannian geometry is the branch of differential geometry, where the ancient geometric objects such as length, angles, areas, volumes and curvature come back to life in a modern reincarnation on smooth manifolds.

Read [**36**, Ch. 1] for an intuitive understanding of curvature and to understanding where we are aiming at with this course. To catch a glimpse of typical results and properties we will encounter in this course it helps to recall what we have learned about curves and surfaces in the Euclidean plane and space (this is covered in the "Curves and surfaces" course).

1.1. Classfication and local-to-global theorems

The machinery used in smooth manifold theory is very technical (as you may know from the "Manifolds" course) but Riemannian geometry is also about intuition. If you have attended the course "Curves and Surfaces" you have a good idea of where we are going. In short, we are extending the notion of *curvature* from two-dimensional surfaces in $\mathbb{R}^3$ to arbitrary dimensions. This requires a great deal of machinery that we need to develop, but at the end we will be rewarded with a deep understanding between local and global concepts (*local-to-global theorems*) such as the relation of curvature to topology in the Gauss–Bonnet Theorem, as well as a characterization of rigid shapes via curvature (classification theorems).

1.2. Intrinsic vs. extrinsic properties

When dealing with submanifolds it is crucial to destinguish between *intrinsic* and *extrinsic* properties. Intrinsic properties do not depend on the ambient space (Gauss's Theorema Egregium tells us that the Gaussian curvature of surfaces in $\mathbb{R}^3$ is intrinsic) while extrinsic ones do (such as the second fundamental form or mean curvature). See also Figure 1.1.

1.3. Abstract Riemannian manifolds in higher dimensions

Abstract Riemannian manifolds as we will encounter in this course are not per se embedded in a higher-dimensional Euclidean space (although they can be, via Nash's embedding theorem). Instead, the geometry has to be described entirely by intrinsic tools. The most important tool here are curves, and particularly geodesics, along which one can move effortlessly. Curves help us to "feel" curvature, more precisely, sectional curvature which is derived from 2-planes cutting through a point, similar to what one does for surfaces in $\mathbb{R}^3$. The manifolds of constant sectional curvature are the usual Euclidean space, sphere, and hyperbolic space, see Figure 1.2. Comparing these model spaces to other manifolds with lower or upper (sectional or Ricci) curvature bounds will again reveal a lot about their global geometry and topology.
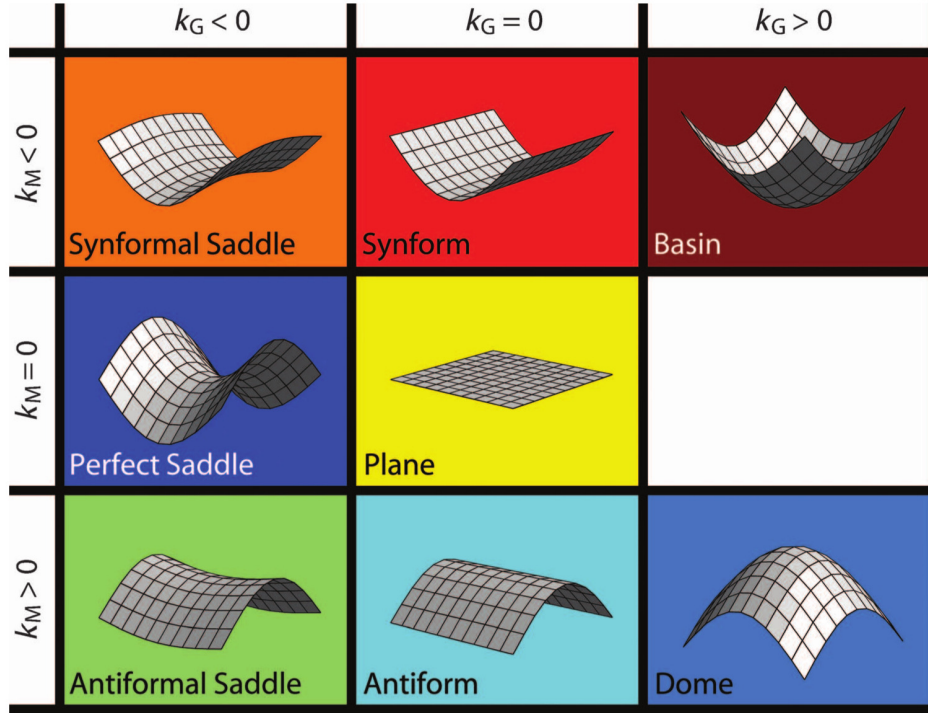
Figure 1.1. Plot of intrinsic Gaussian curvature k_G versus extrinsic mean curvature k_M for surfaces in space (taken from [**12**, Fig. 1], where it is used for the classification of folds in geology). Note that, for instance, a basin/valley is intrinsically the same as a dome/mountain but from an extrinsic perspective they are fundamentally different.

1.4. Applications

Riemannian geometry appears in many areas of pure and applied mathematics (e.g., minimal surfaces, curvature flows optimizing shapes, learning) as well as in mathematical physics (e.g., Einstein's theory of relativity, noncommutative geometry). On the other hand, it also makes use of knowledge from various fields, not only geometry but also calculus, ordinary/partial differential equations and so on. It is beyond the scope of this introductory course to deal with applications.

See Figure 1.3 that displays Eddington's experiment carried in 1919 to verify Einstein's general theory of relativity which models the universe as a Lorentzian manifold, where matter induces curvature and curvature determines how light and particles travel. Lorentzian geometry is the indefinite twin of Riemannian geometry, where the scalar product is no longer positive definite (a bit more about it in Section 2.1.3 below).
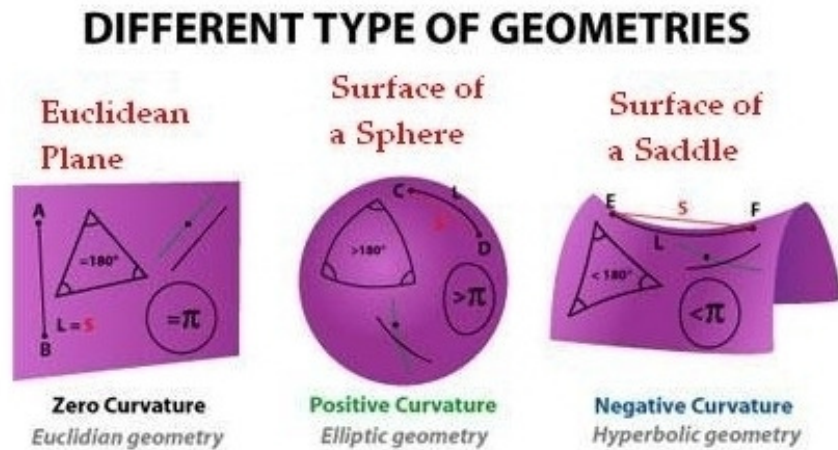
Figure 1.2. Model spaces of zero curvature (plane), constant positive (sphere), and constant negative (hyperbolic space) curvature (taken from `https://www.mu6.com/riemann_space.html`).
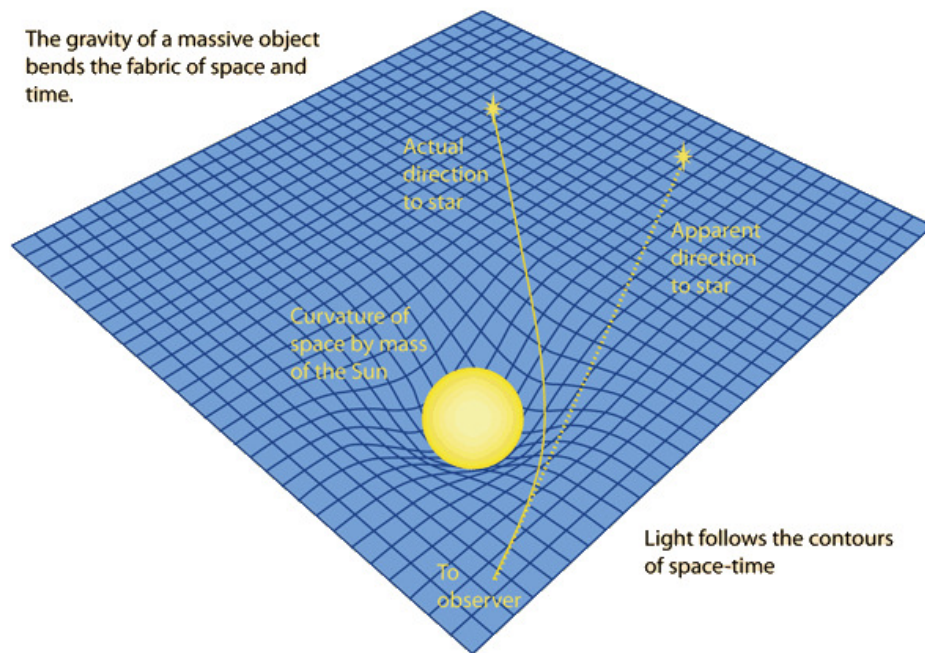
Figure 1.3. A sketch of Eddington's experiment that, roughly put, measured that light bends in the universe. It was spectacular news at the time (in 1919) and one of the first experimental proofs of the general theory of relativity. Read more at `https://en.wikipedia.org/wiki/Eddington_experiment`.

CHAPTER 2

Riemannian manifolds

2.1. Basic definitions

We always consider manifolds are smooth, Hausdorff, second countable and connected. Riemannian manifolds are smooth manifolds which are equipped with inner products on their tangent spaces that vary smoothly. Tangent spaces are vector spaces. In this Section we introduce the basic notions of inner product, Riemannian and semi-Riemannian metrics, and discuss their local repentations, isometries and submanifolds.

2.1.1. Inner products. Inner products are crucial for Euclidean Geometry. On $\mathbb{R}^n$ the *dot product* is given by

$$v \cdot w = \sum_{i=1}^{n} v^i w^i, \qquad v = (v^1, \dots, v^n), w = (w^1, \dots, w^n) \in \mathbb{R}^n, \tag{2.1}$$

More generally, in Linear Algebra one considers inner products on vector spaces.

DEFINITION 2.1. An *inner product on V* is a map $\langle .,. \rangle \colon V \times V \to \mathbb{R}$ satisfying the following properties for all $v, w, x \in V$ and $\alpha, \beta \in \mathbb{R}$:
   (i) symmetry: $\langle v, w \rangle = \langle w, v \rangle$,
   (ii) bilinearity: $\langle \alpha v + \beta w, x \rangle = \alpha \langle v, x \rangle + \beta \langle w, x \rangle$,
   (iii) positive definiteness: $\langle v, v \rangle \geq 0$, and $\langle v, v \rangle = 0 \Leftrightarrow v = 0$.
A vector space V endowed with an inner product $\langle .,. \rangle$ is called an *inner product space.*

With an inner product at hand, one can define the *length* (or *norm*) of vector v by

$$|v| := \langle v, v \rangle^{\frac{1}{2}}.$$

The norm completely determines the inner product via the *polarization identity*

$$\langle v, w \rangle = \frac{1}{4} \left(|v + w|^2 - |v - w|^2 \right), \qquad v, w \in V. \tag{2.2}$$

EXERCISE 2.2. Prove the polarisation identity (2.2).

The *angle* between two nonzero vectors $v, w \in V$ is the unique $\theta \in [0, \pi]$ for which

$$\cos \theta = \frac{\langle v, w \rangle}{|v||w|}.$$

Two vectors $v, w \in V$ are called *orthogonal* if

$$\langle v, w \rangle = 0,$$

that is, if $\theta = \frac{\pi}{2}$ or one of them is zero. If S is a linear subspace of V then

$$S^{\perp} := \{ v \in V; \langle v, s \rangle = 0 \text{ for all } s \in S \}$$

is the *orthogonal complement* of S.

EXERCISE 2.3. Show that $S^\perp$ is also a linear subspace of V.

The vectors $v_1, \ldots, v_k$ are *orthonormal* if they are of unit length and pairwise orthogonal, that is, if

$$\langle v_i, v_j \rangle = \delta_{ij},$$

where δ_{ij} denotes the Kronecker delta. Recall that one can always construct an *orthonormal basis* using the Gram–Schmidt orthogonalization.

PROPOSITION 2.4 (Gram–Schmidt orthogonalization). *Let V be an n-dimensional vector space with inner product $\langle ., . \rangle$ and with ordered basis $(v_1, \ldots, v_n)$. Then there exists an orthonormal basis $(b_1, \ldots, b_n)$ of V such that*

$$\mathrm{span}(v_1, \ldots, v_k) = \mathrm{span}(b_1, \ldots, b_k), \qquad k = 1, \ldots, n.$$

PROOF. It is easy to see that the recursively defined vectors

$$b_1 := \frac{v_1}{|v_1|},$$

$$b_j := \frac{v_j - \sum_{i=1}^{j-1} \langle v_j, b_i \rangle b_i}{|v_j - \sum_{i=1}^{j-1} \langle v_j, b_i \rangle b_i|}, \qquad 2 \le j \le n,$$

have the desired properties. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

DEFINITION 2.5. Let V and W be two vector spaces equipped with inner products $\langle ., . \rangle_V$ and $\langle ., . \rangle_W$. A map $F \colon V \to W$ is called a *linear isometry* if it preserves the inner product, that is,

$$\langle F(v), F(w) \rangle_W = \langle v, w \rangle_V, \qquad v, w \in V.$$

EXERCISE 2.6. Show that a linear isometry $F \colon V \to W$ is a linear map.

By mapping one orthonormal basis to another, we immediately obtain the following.

PROPOSITION 2.7. *All inner product spaces of the same finite dimension are linearly isometric.* $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

2.1.2. Riemannian metrics. On a manifold we define inner products in each tangent space. We assume that a manifold is always Hausdorff and second countable.

DEFINITION 2.8. Let M be a smooth manifold. A *Riemannian metrics* on M is a smooth covariant 2-tensor field $g \in \mathcal{T}^2(M)$ whose value g_p at every $p \in M$ is an inner product on $T_p M$.
The pair (M, g) is called a *Riemannian manifold*.

EXAMPLE 2.9 (Euclidean space). The simplest and most important Riemannian manifold is Euclidean space $\mathbb{E}^n = (\mathbb{R}^n, \bar{g})$. Due to the tangent bundle identification $\bigsqcup_{p \in \mathbb{R}^n} T_p \mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ the standard dot product (2.1) induces a Riemannian metric on $\mathbb{R}^n$ pointwise by

$$\bar{g}_p(v, w) = v \cdot w, \qquad v, w \in T_p \mathbb{R}^n \cong \mathbb{R}^n, p \in \mathbb{R}^n.$$

The following fundamental result distinguishes Riemannian structures from semi-Riemannian ones.

PROPOSITION 2.10. *Every smooth manifold admits a Riemannian metric.*

PROOF. Cover M with chart neighborhoods and let $\{\chi_\alpha\}_{\alpha \in A}$ be a subordinate partition of unity. For each $\alpha \in A$ chose some chart $((x^1, \ldots, x^n), U)$ such that $\operatorname{supp} \chi_\alpha \subseteq U$. On U define

$$g_\alpha := \sum_i dx^i \otimes dx^i.$$

Since a positive linear combination of inner products is again a (positive definite) inner product, the expression

$$g := \sum_{\alpha \in A} \chi_\alpha \, g_\alpha$$

is a Riemannian metric on M. $\qquad \square$

If M is a smooth manifold with boundary then (M, g) is called a *Riemannian manifold with boundary*. Most proofs will work as well also for manifolds with boundary, but occassional difficulties may arise in the proof. In this course we are not so much interested in this case, so we will not specifically look at these situations.

As in Section 2.1.1 we can define the lengths of tangent vectors $v \in T_p M$ via $|v|_g := \langle v, v \rangle_g^{1/2}$, and also angles and orthogonality in the same way. If the Riemannian metric g is fixed we occasionally also just write $\langle v, w \rangle$ and $|v|$ without the index g.

Isometries. A Riemannian structure is preserved by isometries.

DEFINITION 2.11. Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be Riemannian manifolds. An *isometry* from (M, g) to $(\widetilde{M}, \widetilde{g})$ is a diffeomorphism $\varphi \colon M \to \widetilde{M}$ such that[1] $\varphi^* \widetilde{g} = g$, that is,

$$g_p(v, w) = \widetilde{g}_{\varphi(p)}(d\varphi_p(v), d\varphi_p(w)), \qquad v, w \in T_p M, p \in M.$$

We say that (M, g) and $(\widetilde{M}, \widetilde{g})$ are *isometric* if there exists an isometry between them.

Being an isometry $\varphi \colon (M, g) \to (\widetilde{M}, \widetilde{g})$ is equivalent to φ being a smooth bijection and each differential $d\varphi_p \colon T_p M \to T_{\varphi(p)} \widetilde{M}$ being a linear isometry. Global isometries are crucial for Riemannian Geometry, but also local isometries can provide important information.

DEFINITION 2.12. A map $\varphi \colon M \to \widetilde{M}$ is a *local isometry* between Riemannian manifold (M, g) and $(\widetilde{M}, \widetilde{g})$ if for each point $p \in M$ there exists a neighborhood U such that $\varphi|_U$ is an isometry onto an open subset in $\widetilde{M}$.

EXERCISE 2.13. Prove that if (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds of the same dimension, then a smooth map $\varphi \colon M \to \widetilde{M}$ is a local isometry if and only if $\varphi^* \widetilde{g} = g$.

A Riemannian manifold is said to be *flat* if it is locally isometric to Euclidean space.

PROBLEM 2.14. Show that every Riemannian 1-manifold is flat.

An important result in connection with isometries is the Nash Embedding Theorem from 1954 [**40, 41**] which states that every Riemannian manifold can be *isometrically* embedded into some (higher dimensional) Euclidean space. Recall that every n-dimensional smooth manifold can be embedded into $\mathbb{R}^{2n}$ already by the Whitney Embedding Theorem, but this result tells us nothing about Riemannian structures.

The set of isometries of a Riemannian manifold to itself forms a group with respect to composition. More about this group and important examples in Section 2.4.1.

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[1] Recall that $\varphi^* \widetilde{g} = g$ denotes the pullback of a tensor field by φ.

2.1.3. Semi-Riemannian metrics. Semi-Riemannian (in particular, Lorentzian) manifolds are crucial in Mathematical Physics, specifically in General Relativity. While we will not study semi-Riemannian manifolds in this course explicitely, it is good to know that many basic constructions, in fact, carry over verbatim to this setting. For a short overview of the basics see the book of Lee [**36**, p. 40–45], for an extensive treatment look at the books of O'Neill [**43**] and Beem, Ehrlich and Easley [**6**].

In place of inner products we use certain scalar products.

DEFINITION 2.15. Let V be a finite-dimensional vector space over $\mathbb{R}$. A symmetric bilinear form $b\colon V \times V \to \mathbb{R}$ that is *nondegenerate*, i.e.,

$$b(v, w) = 0 \text{ for all } w \Rightarrow v = 0,$$

is called a *scalar product*.

A symmetric bilinear form b is called *positive* (or negative) *semidefinite* if $b(v, v) \geq 0$ (or ≤ 0) for all $v \in V$. The bilinear form b is neither definite or semidefinite (positive or negative), it is called *indefinite*.

EXERCISE 2.16. Show that a symmetric bilinear form is positive definite if and only if it is positive semidefinite and nondegenerate.

EXERCISE 2.17. Show that $g(v, w) = -v^1 w^1 + v^2 w^2$ is a scalar product on $\mathbb{R}^2$.

Clearly, if b is symmetric bilinear form on V and W is a linear subspace of V, then also $b|_W$ is a symmetric bilinear form. Similarly, if b is (semi-)definite, then so is $b|_W$. This gives rise to the following definition.

DEFINITION 2.18. The *index* of a symmetric bilinear form b on V is

$$\operatorname{ind} V := \nu := \max\{\dim S; S \text{ is a subspace of } V \text{ such that } b|_S \text{ is negative definite}\}.$$

Thus $0 \leq \nu \leq \dim V$, where $\nu = 0$ if and only if b is positive semidefinite.

Orthogonality is defined in the same way, but note that for indefinite scalar products orthogonal vectors no longer need to have right angles. Moreover, the orthogonal subspace $S^\perp$ need not be a complement of S since, in general, $S + S^\perp \neq V$.

EXERCISE 2.19. Show that in Exercise 2.17 we have for $w = (1, 1)$ and $S = \operatorname{span}(\{w\})$ that $S = S^\perp \subsetneq \mathbb{R}^2$.

EXERCISE 2.20. Let S be a subspace of V. Then
  (i) $\dim S + \dim S^\perp = \dim V$,
  (ii) $(S^\perp)^\perp = S$.

For indefinite scalar products there *always* exist degenerate subspaces (e.g., the span of a null vector as in Exercise 2.19), while for inner products *every* subspace is again an inner product space. Using basic Linear Algebra one can prove the following result.

LEMMA 2.21. *Let g a scalar product on V and let S be a subspace of V. The following are equivalent:*
  (i) *S is nondegenerate, meaning that $g|_S$ is nondegenerate.*
  (ii) *$S^\perp$ is nondegenerate.*

(iii) $V = S \oplus S^{\perp}$ *(direct sum).*

PROBLEM 2.22. Prove Lemma 2.21.

As in Section 2.1.1 one can show that a nontrivial vector space with scalar product possesses an orthonormal basis. This implies that for a nondegenerate subspace S of V we have

$$\operatorname{ind} V = \operatorname{ind} S + \operatorname{ind} S^{\perp}.$$

Note that in order to have a linear isometry between vector spaces with scalar products both dimension *and* index have to agree!

A semi-Riemannian manifold is defined in analogy to a Riemannian manifold with scalar products on the tangent spaces.

DEFINITION 2.23. Let M be a smooth manifold. A *semi-Riemannian metric* on M is a symmetric nondegenerate covariant 2-tensor field $g \in \mathcal{T}^2(M)$ with constant index ν (called the *index of M*).

A *semi-Riemannian manifold* (or pseudo-Riemannian manifold) (M, g) is a smooth manifold M endowed with a semi-Riemannian metric g.

We have $0 \leq \nu \leq n = \dim M$. If $\nu = 0$ then (M, g) is a Riemannian manifold. If $\nu = 1$ and $n \geq 2$ then (M, g) is called a *Lorentzian*[2] *manifold* and g is a *Lorentzian metric*. Lorentzian Geometry is the geometric framework needed to understand General Relativity.

EXAMPLE 2.24 (Minkowski space). Let $0 \leq \nu \leq n$. Then

$$\bar{q}(v, w) := \langle v, w \rangle := -\sum_{i=1}^{\nu} v^i w^i + \sum_{i=\nu+1}^{n} v^i w^i$$

defines a metric tensor of index ν on $\mathbb{R}^n$, often denoted by $\mathbb{R}^n_{\nu}$ or $\mathbb{R}^{\nu, n-\nu}$. If $\nu = 1$ then $\mathbb{R}^{1, n-1}$ is called the *Lorentzian vector space* (or Minkowski space) of dimension n and the scalar product denoted by $\bar{\eta}$. If $n = 4$, $\mathbb{R}^{1,3}$ it is the physically relevant *Minkowski space*, which is also a *spacetime* (time-oriented Lorentzian manifold) and the simplest solution of the vacuum Einstein equations in General Relativity.

One of the most notable differences between Riemannian and Lorentzian metrics is that while every manifold admits a Riemannian metric (see Proposition 2.10) *not* every manifold admits a semi-Riemannian or even Lorentzian metric. A problem occurs, in particular, for compact manifolds.

THEOREM 2.25 ([**43**, p. 149]). *For a smooth manifold M the following are equivalent:*
  (i) *existence of a Lorentzian metric,*
  (ii) *existence of a continuous line field,*
  (iii) *existence of a nonvanishing continuous vector field,*
  (iv) *existence of a spacetime structure,*
  (v) *either M is noncompact, or M is compact and has zero Euler characteristic.*

We will not prove this entire result, but you can look at the following part.

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[2]Hendrik Lorentz (1853–1928, born in Arnhem) was an eminent and influental Dutch physicist who won the nobel prize in physics in 1902 with Pieter Zeeman for the Zeemann effect. He also worked extensively in Special and General Relativity.

PROBLEM 2.26. A smooth manifold M admits a Lorentzian metric if and only if it admits a rank-1 distribution, i.e., a rank-1 subbundle of TM (see hints in the book of Lee [**36**, p. 54, Problem 2-34]).

2.1.4. Local representation of metrics. We discuss the expression of a Riemannian metric in coordinates and frames. Let (M, g) be a Riemannian manifold (with or without boundary).

Coordinates. If $(x^1, \dots, x^n)$ are coordinates on an open set $U \subseteq M$, then we can write g locally in U as[3]

$$g = g_{ij} dx^i \otimes dx^j,$$

where the g_{ij} are n^2 smooth functions given by

$$g_{ij}(p) = \langle \partial_i|_p, \partial_j|_p \rangle, \qquad p \in U,$$

where $\partial_i = \partial / \partial x^i$ is the i-th coordinate vector field. The matrix $(g_{ij}(p))_{ij}$ is nonsingular and symmetric in i and j, which is why we can also drop the $\otimes$ and simply write

$$g = g_{ij} dx^i dx^j.$$

EXAMPLE 2.27. The Euclidean metric on $\mathbb{R}^n$ in standard coordinates can be expressed in the following ways

$$\bar{g} = \sum_i dx^i dx^i = \sum (dx^i)^2 = \delta_{ij} dx^i dx^j.$$

Frames. A *local frame for* an n-dimensional smooth manifold M is an ordered n-tuple of vector fields $(E_1, \dots, E_n)$ defined on an open set $U \subseteq M$ such that for each $p \in U$ the vectors $(E_1|_p, \dots, E_n|_p)$ are linearly independent and span $T_p M$. For example, the coordinate vector fields $(\frac{\partial}{\partial x^i})$ define a smooth frame on a coordinate nieghborhood U, but often frames that are adapted to a particular problem at hand are more useful. On Riemannian manifolds we often use orthonormal frames.

Generally, if $(E_1, \dots, E_n)$ is any smooth local frame for TM on an open set $U \subseteq M$ and $(\varepsilon^1, \dots, \varepsilon^n)$ is its dual coframe, then we can locally write the metric tensor g as

$$g = g_{ij} \varepsilon^i \varepsilon^j,$$

where $g_{ij}(p) = \langle E_i|_p, E_j|_p \rangle$. Again, the functions (g_{ij}) are smooth and symmetric.

For smooth vector fields $X, Y \in \mathfrak{X}(M)$, we can thus locally write $\langle X, Y \rangle = g_{ij} X^i Y^j$ and $|X| = \langle X, X \rangle^{1/2}$, which is continuous and smooth on the open set $\{X \neq 0\}$.

A particularly useful frame is an *orthonormal frame*, which is a local frame (E_i) on $U \subseteq M$ such that $E_1|_p, \dots E_n|_p$ form an orthonormal basis on $T_p M$ at each $p \in U$, that is,

$$\langle E_i, E_j \rangle = \delta_{ij}.$$

In this frame g is locally

$$g = (\varepsilon^1)^2 + \dots (\varepsilon^n)^2,$$

where $(\varepsilon^i)^2 = \varepsilon^i \varepsilon^i = \varepsilon^i \otimes \varepsilon^i$ as before.

Thanks to the Gram–Schmidt algorithm there exists an orthonormal frame locally around every point [**36**, Prop. 2.8]. Note that while one can use any coordinate frame (∂_i) to construct

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[3]Using the Einstein summation convention, which drops the $\sum$ sign and replaces it by upper and lower indices, automatically indicating the summation. Here, $g_{ij} dx^i \otimes dx^j$ actually means $\sum_{i=1}^n \sum_{j=1}^n g_{ij} dx^i \otimes dx^j$.

an orthonormal frame, one can find an orthonormal coordinate frame only if the metric is flat, i.e., locally isometric to Euclidean space (see [**36**, Ch. 7]).

2.1.5. Riemannian submanifolds. Generally, submanifolds can be obtained via immersions[4] (or embeddings). For Riemannian manifolds we also want to preserve the Riemannian structure.

LEMMA 2.28. *Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold, M be a smooth manifold and $F\colon M \to \widetilde{M}$ be a smooth map. The smooth 2-tensor field $g := F^*\widetilde{g}$, i.e.,*

$$g_p(v, w) := (F^*\widetilde{g})_p(v, w) = \widetilde{g}_{F(p)}(dF_p(v), dF_p(w)), \qquad v, w \in T_p M, p \in M$$

is a Riemannian metric on M if and only if F is an immersion.

EXERCISE 2.29. Prove Lemma 2.28.

An immersion (or embedding) $F\colon M \to \widetilde{M}$ between two Riemannian manifolds (M, g) and $(\widetilde{M}, \widetilde{g})$ that satisfies $g = F^*\widetilde{g}$ is called an *isometric immersion* (or *isometric embedding*).

Because of Lemma 2.28 we can equip submanifolds with a natural Riemannian structure coming from the ambient Riemannian manifold.

DEFINITION 2.30. Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold. Suppose $M \subseteq \widetilde{M}$ is an (immersed or embedded) submanifold. The *induced Riemannian metric on M* is the pullback metric $g = \iota^*\widetilde{g}$ induced by the inclusion $\iota\colon M \hookrightarrow \widetilde{M}$.

We automatically assume that submanifolds are endowed with the induced metrics. Many examples are of this kind.

EXAMPLE 2.31 (Sphere). An important example of a submanifold is the n-dimensional sphere of radius $R > 0$, defined by

$$\mathbb{S}^n(R) := \{x \in \mathbb{R}^{n+1}; |x| = R\}$$

The metric induced from the embedding $\mathbb{S}^n(R) \hookrightarrow (\mathbb{R}^{n+1}, \bar{g})$ is the canonical metric on $\mathbb{S}^n(R)$. It is called the *round metric of radius R* and denoted by $\mathring{g}_R$. The unit sphere is $\mathbb{S}^n := \mathbb{S}^n(1)$ with round metric $\mathring{g}$.

By Hilbert's Theorem from 1901 the hyperbolic plane $\mathbb{H}^2$ cannot be isometrically immersed in $\mathbb{E}^3$, but one could do it in $\mathbb{E}^4$. We can, however, in analogy to the sphere introduce hyperbolic space as a codimension-1 submanifold of Minkowski space $\mathbb{R}^{1,n}$.

EXAMPLE 2.32 (Hyperbolic space). Let $n > 1$ and fix $R > 0$. The *hyperbolic space*[5] $\mathbb{H}^n(R)$ *of radius R* is the submanifold of the Minkowski space $(\mathbb{R}^{1,n}, \bar{\eta})$ (see Example 2.24), defined as the "upper sheet" (by using the standard unit timelike vector $e_0 = (1, 0, \ldots, 0)$, or in standard

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[4]Recall that a map $F\colon M \to \widetilde{M}$ is an *immersion* if its differential $dF_p\colon T_pM \to T_{F(p)}\widetilde{M}$ is injective at every point $p \in M$ (equivalently, rank $F = \dim M$). An *embedding* $F\colon M \hookrightarrow \widetilde{M}$ is an immersion that is homeomorphic to its image. Therefore, immersed submanifolds may have selfintersections but embedded submanifolds don't.

[5]This definition of hyperbolic space, called the "hyperboloidal model", is particularly useful to understand its symmetries, see Section 2.4.1. There are several isometric models of hyperbolic space [**36**, Thm. 3.7] which are used in other situations.

coordinates $(\tau, \xi^1, \ldots, \xi^n)$ in terms of the sign of τ) of the two-sheeted hyperboloid in $\mathbb{R}^{1,n}$, that is,

$$\mathbb{H}^n(R) := \{x \in \mathbb{R}^{1,n}; \bar\eta(x,x) = -R^2 \text{ and } \bar\eta(x,e_0) < 0\}$$
$$= \{(\tau, \xi) \in \mathbb{R}^{1,n}; \tau^2 - |\xi|^2 = R^2\} \cap \{\tau > 0\},$$

with the pullback metric $\breve{g}_R$ induced by Minkowski space $(\mathbb{R}^{1,n}, \bar\eta)$, called the *hyperbolic metric*. *Hyperbolic space* is $\mathbb{H}^n := \mathbb{H}^n(1)$ with metric $\breve{g}$. If $n = 2$ we call $\mathbb{H}^2$ the hyperbolic plane.

Many computations on n-dimensional submanifolds $M \hookrightarrow \widetilde{M}$ are carried out using an *adapted orthonormal frame* (an orthonormal frame $(E_1, \ldots, E_n, E_{n+1}, \ldots, E_m)$ such that the first n vector fields are tangent to M, which exists by [**36**, Prop. 2.14]) or *smooth local parametrizations*[6] (the inverse of charts).

If M is an immersed n-dimensional submanifold of $\mathbb{R}^m$ and $X \colon U \to \mathbb{R}^m$ is a local parametrization of M, then the induced metric on U is

$$g = X^*\bar{g} = \sum_{i=1}^m (dX^i)^2 = \sum_{i=1}^m \Big(\sum_{j=1}^n \frac{\partial X^i}{\partial u^j} du^j\Big)^2 = \sum_{i=1}^m \sum_{j,k=1}^n \frac{\partial X^i}{\partial u^j} \frac{\partial X^i}{\partial u^k} du^j du^k. \qquad (2.3)$$
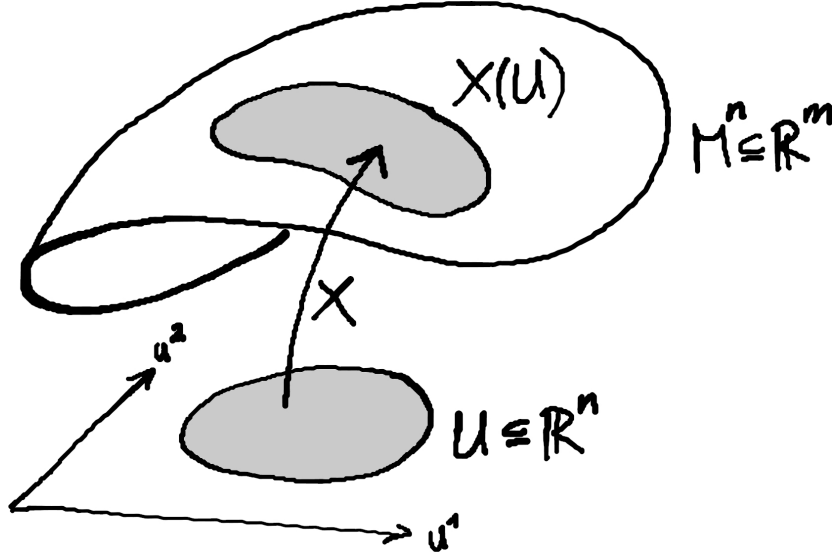
Figure 2.1. An immersed 2-dimensional submanifold M of $\mathbb{R}^3$, given by as local parametrization $X \colon U \to \mathbb{R}^3$. In fact, M is a graph, and X could be the graph parametrization $X = (u^1, u^2, f)$.

EXAMPLE 2.33 (Surfaces in $\mathbb{R}^3$). For surfaces in $\mathbb{R}^3$ the expression (2.3) is the *first fundamental form* (or *line element*), usually given in terms of the coefficients $E = |\frac{\partial X}{\partial u^1}|^2$, $F = \frac{\partial X}{\partial u^1} \cdot \frac{\partial X}{\partial u^2}$, and $G = |\frac{\partial X}{\partial u^2}|^2$ (studied in the "Curves and Surfaces" course).

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[6]A smooth local parametrization is a smooth map $X \colon U \to \widetilde{M}$, where U is an open subset of $\mathbb{R}^n$ and X is a diffeomorphism onto its image $X(U) \subseteq M$. Note that $(V = X(U), \varphi = X^{-1})$ is a smooth coordinate chart on M.

We can use the expression (2.3) to compute metrics induced on submanifolds of $\mathbb{R}^m$ in various coordinates. Important examples are graphs and surfaces of revolution for which special parametrizations and coordinates adapted to the setting in hand to are used.

EXAMPLE 2.34 (Graphs). For U an open subset of $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}$ a smooth function the *graph of* f,
$$\Gamma(f) := \{(y, f(y)); y \in U\} \subseteq \mathbb{R}^{n+1}$$
is an embedded submanifold of dimension n. The global parametrization $X\colon U \to \mathbb{R}^{n+1}$ is called a *graph parametrization* and the corresponding coordinates $(u^1, \ldots, u^n)$ on $\Gamma(f)$ are the *graph coordinates*. In graph coordinates, the induced metric on $\Gamma(f)$ is
$$X^*\bar{g} = X^*((dx^1)^2 + (dx^{n+1})^2) = (du^1)^2 + \ldots + (du^n)^2 + df^2.$$

EXERCISE 2.35. Compute the round metric of the upper hemisphere of $\mathbb{S}^2$ with respect to the graph parametrization $X\colon \mathbb{B}^2 \to \mathbb{S}^2$, given by $X(u,v) = (u, v, \sqrt{1 - u^2 - v^2})$.

EXAMPLE 2.36 (Surfaces of revolution). Let C be an embedded 1-dimensional submanifold in the half plane $H = \{(r, z); r > 0\}$. The set C is used as *generating curve* for the *surface of revolution*
$$S_C := \{(x, y, z) \in \mathbb{R}^3; (\sqrt{x^2 + y^2}, z) \in C\}.$$
Every smooth local parametrization $\gamma(t) = (a(t), b(t))$ of C yields a smooth local parametrization
$$X(t, \theta) = (a(t)\sin\theta, a(t)\cos\theta, b(t)).$$
for S_C. The induced metric on S_C is given by (2.3) and thus
$$\begin{aligned} X^*\bar{g} &= d(a(t)\cos\theta)^2 + d(a(t)\sin\theta)^2 + d(b(t))^2 \\ &= (a'(t)^2 + b'(t)^2)dt^2 + a(t)^2 d\theta^2. \end{aligned}$$
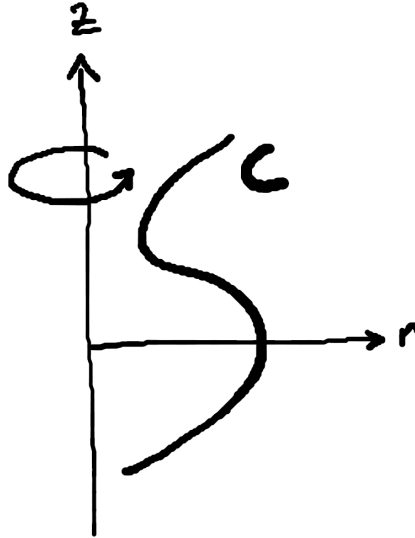

Figure 2.2. Generating curve C in the half plane H for a surface of revolution in $\mathbb{R}^3$.

EXERCISE 2.37. Consider a torus $\mathbb{T}^2$ of revolution obtained by rotating the circle $C = \{(r, z); (r-2)^2 + z^2 = 1\}$. Assume that C is parametrized by $\gamma(t) = (2 + \cos t, \sin t)$ and show that the induced metric is $dt^2 + (2 + \cos t)^2 d\theta^2$.

Alternatively, the Riemannian metric induced on the torus $\mathbb{T}^n = S^1 \times \ldots \times S^1$ can also be computed via its product structure. We immediately treat a generalization of Riemannian products.

DEFINITION 2.38. Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds, and $f \colon M_1 \to (0, \infty)$ be a smooth function. The *warped product* $M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ endowed with the Riemannian metric $g = g_1 \oplus f^2 g_2$, defined by

$$g_{(p_1, p_2)}((v_1, v_2), (w_1, w_2)) = g_1|_{p_1}(v_1, w_1) + f(p_1)^2 g_2|_{p_2}(v_2, w_2),$$

where $(v_1, v_2), (w_1, w_2) \in T_{p_1} M_1 \oplus T_{p_2} M_2$.

Many examples can be constructed using warped products.

PROBLEM 2.39. Show that metrics induced on surfaces of revolution are isometric to warped products metrics (see [**36**, Ex. 2.24(b) and Problem 2-3]).

EXERCISE 2.40. Let g_0 and g_1 be two Riemannian metrics on M. Then, for any number $\lambda \in [0, 1]$, the convex combination

$$g := \lambda g_0 + (1 - \lambda) g_1$$

is also a Riemannian metric on M.

While it is straightforward to induce Riemannian metrics on submanifolds and products this is, for instance, not the case for quotients. Submersions yield Riemannian structures only in very special cases.

DEFINITION 2.41. Suppose that $(\widetilde{M}, \widetilde{g})$ and (M, g) are Riemannian manifolds and $\pi \colon \widetilde{M} \to M$ is a smooth submersion[7]. Then π is said to be a *Riemannian submersion* if for every $p \in \widetilde{M}$ the differential $d\pi_p \colon H_x \to T_{\pi(p)} M$ is a linear isometry, where $H_x = (\operatorname{Ker} d\pi_p)^{\perp}$ denotes the horizontal tangent space at p. In other words, $\widetilde{g} = \pi^* g$ on H_x.

EXAMPLE 2.42. If $M \times_f N$ is a warped product, then the projection $\pi_M \colon M \times N \to M$ is a Riemannian submersion but π_N generally not.

More about submersions in [**36**, p. 21–24].

2.2. Basic constructions

Riemannian manifolds come with canonical notions such as an associated volume form, various differential operators, and an induced metric space structure. First we need to understand how tensors transform in a way that respects the Riemannian structure (see also [**45**, Sec. 1.5]).

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[7]Recall that a smooth map $F \colon \widetilde{M} \to M$ is called a *smooth submersion* if its differential dF is surjective at every point (equivalently, rank $F = \dim M$).

2.2.1. Raising and lowering indices. An (s,t)-tensor field T is a section on the bundle

$$\underbrace{TM \otimes \ldots \otimes TM}_{s \text{ times}} \otimes \underbrace{T^*M \otimes \ldots \otimes T^*M}_{t \text{ times}} \to M$$

Given a Riemannian metric g on M, we can turn T into an $(s-k, t+k)$-tensor in a canonical way because TM is naturally isomorphic to T^*M. In what follows we make this connection precise.

Let us investigate this idea for g first. For smooth vector fields $X, Y \in \mathfrak{X}(M)$ the map

$$\hat{g}(X)(Y) := g(X,Y)$$

is $C^\infty(M)$-linear in Y and thus $\hat{g}(X)$ a smooth 1-form[8] by the Tensor Characterization Lemma A.17. Moreover, $\hat{g}(X)$ is also $C^\infty(M)$-linear in X and thus $\hat{g}$ a smooth bundle homomorphism (often just denoted by g)

$$\hat{g} \colon TM \to T^*M.$$

Suppose (E_i) is smooth local frame on M and (ε^i) its dual coframe. Then $X = X^i E_i \in \mathfrak{X}(M)$ and

$$\hat{g}(X) = (g_{ij}X^i)\varepsilon^j = X_j \varepsilon^j \in \Omega^1(M),$$

where we have obtained the components $X_j = g_{ij}X^i$ of $\hat{g}(X)$ from X by *lowering an index* (sometimes $\hat{g}(X)$ is called X *flat* and denoted by $X^\flat$).

Since $\hat{g} = (g_{ij})$ is invertible at every p the matrix of $\hat{g}^{-1}$ is the inverse matrix of (g_{ij}). We write $\hat{g}^{-1} = (g^{ij})$ so that $g^{ij}g_{jk} = \delta_k^i$. Thus

$$\hat{g}^{-1}(\omega) = \omega^i E_i \in \mathfrak{X}(M)$$

for $\omega^i = g^{ij}\omega_j$. Thus the vector field $\hat{g}^{-1}(\omega)$ is obtained from the 1-form $\omega \in \Omega^1(M)$ by *raising an index* (sometimes $\hat{g}^{-1}(\omega)$ is called ω *sharp* and denoted by $\omega^\sharp$).

The mutually inverse isomorphisms $\sharp = \hat{g}^{-1} \colon T^*M \to TM$ and $\flat = \hat{g} \colon TM \to T^*M$ are also called the *musical isomorphisms*, and we call the (co)vectors obtained in this way *metrically equivalent*. An important application is the following definition of the gradient[9].

DEFINITION 2.43. Let (M, g) be a Riemannian manifold and $f \colon M \to \mathbb{R}$ smooth. The *gradient of f* is the vector field

$$\operatorname{grad} f := (df)^\sharp.$$

EXERCISE 2.44. Show that $\operatorname{grad} f$ is characterized by

$$df_p(w) = \langle \operatorname{grad} f|_p, w \rangle, \qquad p \in M, w \in T_p M.$$

If (E_i) is a local smooth frame then $\operatorname{grad} f = (g^{ij} E_i f) E_j$ (in particular, if (E_i) is an orthonormal frame then the components of $\operatorname{grad} f$ and df agree).

PROBLEM 2.45. A level set $f^{-1}(c)$ of a smooth function $f \colon M \to \mathbb{R}$ is called regular if every point $p \in f^{-1}(c)$ satisfies $df_p \neq 0$. Prove that the gradient is orthogonal to regular level sets (see [**36**, Prop. 2.37]).

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[8]Lee [**35**, **36**] prefers to call 1-forms covector fields.

[9]Clearly, the gradient therefore depends on the choice of a Riemannian metric, but without the use of a Riemannian metric (and the corresponding musical isomorphism) there really is no way to define gradient vector field in an invariant way. See the discussion in [**45**, Sec. 2.1.1] for why this is.

PROBLEM 2.46. Suppose (M, g) is a Riemannian manifold, $f \in C^\infty(M)$, and $X \in \mathfrak{X}(M)$ is a nowhere-vanishing vector field. Prove that $X = \operatorname{grad} f$ if and only if $Xf \equiv |X|_g^2$ and X is orthogonal to the level sets of f at all regular points.

We initially set out to apply the flat and sharp operators to tensors T which are locally of the form

$$T = T^{i_1 \ldots i_s}_{j_1 \ldots j_t} E_{i_1} \otimes \ldots \otimes E_{i_s} \otimes \varepsilon^{j_1} \otimes \ldots \otimes \varepsilon^{j_t}$$

It is now clear how to do this using the musical isomorphisms, but it is important to keep track of the index that is lowered or raised (see [**36**, p. 27]).

EXAMPLE 2.47. A 3-tensor A of type $(1, 2)$ given in a local frame by

$$A = A_i{}^j{}_k \varepsilon^i \otimes E_j \otimes \varepsilon^k$$

can be turned into the covariant 3-tensor $A^\flat$ with components

$$A_{ijk} = g_{jl} A_i{}^l{}_k.$$

One important application of raising and lowering indices are *contractions*, which lowers the rank of a tensor by 2. Contractions are traces of tensors. Given a covariant 2-tensor field $h = h_{ij} \varepsilon^i \otimes \varepsilon^j$, we first obtain the $(1, 1)$-tensor field $h^\sharp$ and define *the trace of h with respect to g* as

$$\operatorname{tr}_g h := \operatorname{tr}(h^\sharp) = \operatorname{tr}(g^{jk} h_{ij} \varepsilon^i \otimes E_k) = g^{ij} h_{ij} = h_i{}^i.$$

On a Riemannian manifold (M, g) the musical isomorphisms can be further used to carry the *inner product* on $T_p M$ over to covectors in $T_p^* M$ via

$$\langle \omega, \eta \rangle_g := \langle \omega^\sharp, \eta^\sharp \rangle_g, \qquad \omega, \eta \in \Omega^1(M).$$

One can extend this construction to tensor bundles of any rank in the obvious way.

EXERCISE 2.48. Show that in coordinates $\langle \omega, \eta \rangle = g^{ij} \omega_i \eta_j = \omega^j \eta_j$.

EXERCISE 2.49. Let (M, g) be a Riemannian manifold with or without boundary, let (E_i) be a local frame for M, and let (ε^i) be its dual coframe. Show that the following are equivalent:

    (i) (E_i) are orthonormal.
    (ii) (ε^i) are orthonormal.
    (iii) $(\varepsilon^i)^\sharp = E_i$ for each i.

2.2.2. Volume form and integration. A Riemannian metric g induces a canonical volume form on an oriented manifold M. This follows from the usual *signed volume* of the volume of a parallelepiped in $\mathbb{R}^n$ spanned by vectors $(v_1, \ldots, v_n)$. If $(e_1, \ldots, e_n)$ is the canonical basis then the volume of the parallelepiped is

$$\operatorname{Vol}(v_1, \ldots, v_n) = \det[v_i \cdot e_j] = \det([v_1, \ldots, v_n][e_1, \ldots, e_n]^T) = \det[v_1, \ldots, v_n].$$

The same formula also holds for any orthonormal basis, and via the tangent space $T_p M \cong \mathbb{R}^n$ it directly translates to M.

PROPOSITION 2.50 (Riemannian volume form). *Let (M, g) be an oriented Riemannian manifold. There is a unique n-form dV_g on M, called the* Riemannian volume form, *given with respect to an local oriented orthonormal frame $(E_1, \ldots, E_n)$ for TM by*

$$dV_g(E_1, \ldots, E_n) = 1. \tag{2.4}$$

PROOF. Suppose first that the volume form dV_g exists. For an local oriented orthonormal frame $(E_1, \ldots, E_n)$ and $(\varepsilon^1, \ldots, \varepsilon^n)$ its dual coframe the volume form[10] is

$$dV_g = f\varepsilon^1 \wedge \ldots \wedge \varepsilon^n.$$

Because of (2.4) we must have $f = 1$ (the volume of the parallelepiped is 1), and hence uniqueness.

To prove existence, we simply *define* dV_g in a neighborhood of each point by

$$dV_g := \varepsilon^1 \wedge \ldots \wedge \varepsilon^n.$$

It remains to check that this definition is independent of the choice of oriented coframe. Let $(\widetilde{E}_1, \ldots, \widetilde{E}_n)$ is another oriented orthonormal frame and and $(\tilde{\varepsilon}^1, \ldots, \tilde{\varepsilon}^n)$ its dual coframe. We can write

$$\widetilde{E}_i = A_i^j E_j$$

for some transition matrix $A = (A_i^j)$ of smooth functions, and hence

$$\varepsilon^1 \wedge \ldots \wedge \varepsilon^n = \det A \, \tilde{\varepsilon}^1 \wedge \ldots \wedge \tilde{\varepsilon}^n$$

Since both frames are orthonormal we have that $A(p) \in O(n)$ for each p, hence $\det A = \pm 1$. A consistent orientation forces a positive sign, hence $\det A = 1$ and

$$dV_g(\widetilde{E}_1, \ldots, \widetilde{E}_n) = \det(\varepsilon^j(\widetilde{E}_i)) = \det(A_i^j) = 1. \qquad \square$$

EXERCISE 2.51. Prove that, with respect to any local oriented coordinates $(x^1, \ldots, x^n)$ we have

$$dV_g = \sqrt{\det(g_{ij})} dx^1 \wedge \ldots \wedge dx^n.$$

The Riemannian volume form allows to integrate *functions* over oriented Riemannian manifolds.

DEFINITION 2.52. Let (M, g) be an oriented Riemannian manifold and let $f \colon M \to \mathbb{R}$ be continuous and compactly supported function, then

$$\int_M f \, dV_g$$

is called the *integral of f over M*.

DEFINITION 2.53. If (M, g) is a compact oriented Riemannian manifold, then the *volume of M* is defined by

$$\mathrm{Vol}(M) := \int_M dV_g = \int_M 1 \, dV_g.$$

If $D \subseteq M$ is a *regular domain* (a closed, embedded codimension-0 submanifold with boundary), the same definitions apply with respect to the induced Riemannian metric on D.

EXERCISE 2.54. Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are oriented Riemannian manifolds, and $\varphi \colon M \to \widetilde{M}$ is an orientation-preserving isometry. Prove that $\varphi^* dV_{\widetilde{g}} = dV_g$.

---

[10] The notation dV_g with the d is commonly used but slightly misleading, because dV_g is not necessarily an exact form.

PROBLEM 2.55. Suppose M is a hypersurface in an oriented Riemannian manifold $(\widetilde{M}, \widetilde{g})$ and g is the induced metric on M. Then M is orientable if and only if there exists a global unit normal vector field N for M, and in that case the volume form of (M, g) is given by

$$dV_g = (N \lrcorner dV_{\widetilde{g}})|_M$$

(see page 18 for the definition of the interior multiplication $\lrcorner$).

PROBLEM 2.56. Suppose (M_1, g_1) and (M_2, g_2) are oriented Riemannian manifolds of dimensions k_1 and k_2, respectively. Let $f \colon M_1 \to (0, \infty)$ be a smooth function, and let $g = g_1 \oplus f^2 g_2$ be the corresponding warped product metric on $M_1 \times_f M_2$. Prove that the Riemannian volume form of g is given by

$$dV_g = f^{k_2} dV_{g_1} \wedge dV_{g_2},$$

where f, dV_{g_1} and dV_{g_2} are understood to be pulled back to $M_1 \times M_2$ by the projection maps.

Recall that on nonorientable manifolds we can compute integrals of functions using densities instead of differential forms (for more background about densities see [**35**, p. 427–434]). We obtain a canonical Riemannian density as well.

PROPOSITION 2.57 (Riemannian density). *If (M, g) is a Riemannian manifold, then there exists a unique smooth positive density μ_g on M, called the* Riemannian density, *with the property that*

$$\mu_g(E_1, \dots, E_n) = 1$$

for every local orthonormal frame $(E_1, \dots, E_n)$.

EXERCISE 2.58. Prove Proposition 2.57 by showing that μ_g can be defined in terms of any local orthonormal frame by

$$\mu_g = |\varepsilon^1 \wedge \dots \wedge \varepsilon^n|,$$

where $|\omega|(v_1, \dots, v_n) := |\omega(v_1, \dots, v_n)|$ for an n-form $\omega \in \Omega^n(T_p^* M)$.

EXERCISE 2.59. Show that if (M, g) is an oriented Riemannian manifold then $\mu_g = |dV_g|$. For any compactly suppported continuous function $f \colon M \to \mathbb{R}$ we have

$$\int_M f \mu_g = \int_M f dV_g$$

(this is why the Riemannnian density is often also denoted by dV_g).

2.2.3. Differential operators. The most important differential operators on Riemannian manifolds are the gradient, divergence and Laplacian. We have already introduced the gradient in Definition 2.43 using the $\sharp$ isomorphism which raises an index. The divergence and Laplacian can be introduced using the Levi–Civita connection (which we will introduce later, but see also [**43**, p. 85–87]) or via the volume form on (oriented) Riemannian manifolds, as we will see now. On Euclidean space they reduce to the usual definitions.

DEFINITION 2.60. Let (M, g) be an (oriented) Riemannian manifold with volume form dV_g. Let X be a smooth vector field on M. The exterior derivative of the $(n-1)$-form[11] $X \lrcorner dV_g$ can be expressed by a smooth function, called the *divergence of X*, multiplied by dV_g,

$$d(X \lrcorner dV_g) = (\operatorname{div} X) dV_g.$$

---

[11]The symbol $\lrcorner$ denotes the interior multiplication of ω by X, that is, $X \lrcorner \omega(Y_2, \dots, Y_n) := \omega(X, Y_2, \dots, Y_n)$.

REMARK 2.61. Note that div X is well-defined also on nonorientable manifolds because we can still choose an orientation locally around every point to define the divergence as above, and because reversing the orientation changes the sign of dV_g on *both* sides of the equation leaving div X independent of the orientation.

PROBLEM 2.62. Let (M, g) be a Riemannian manifold and let (x^i) be smooth coordinates on an open set $U \subseteq M$. Show that

$$\operatorname{div}(X^i \partial_i) = \frac{1}{\sqrt{\det g}} \partial_i (X^i \sqrt{\det g}),$$

where $\partial_i = \frac{\partial}{\partial x^i}$ and $\det g = \det(g_{kl})$ is the determinant of the component matrix of g in these coordinates.

In particular, on $\mathbb{R}^n$ with the Euclidean metric and the standard coordinates we recover the usual formula $\operatorname{div}(X) = \sum_{i=1}^n \frac{\partial X^i}{\partial x^i}$.

Using the divergence operator one can prove the following important result (which holds also in the nonorientable case by using the Riemannian density [**35**, Thm. 16.48]).

PROBLEM 2.63 (Divergence Theorem). Suppose (M, g) is a compact orientable Riemannian manifold with boundary.

(i) Prove the following *divergence theorem* for $X \in \mathfrak{X}(M)$:

$$\int_M (\operatorname{div} X) dV_g = \int_{\partial M} \langle X, N \rangle_g \, dV_{\tilde{g}},$$

where N is the outward unit normal to ∂M and $\tilde{g}$ is the induced metric on ∂M.

(ii) Show that the divergence operator satisfies the following product rule for $u \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$:

$$\operatorname{div}(uX) = u \operatorname{div} X + \langle \operatorname{grad} u, X \rangle_g,$$

and deduce the following "integration by parts" formula:

$$\int_M \langle \operatorname{grad} u, X \rangle_g \, dV_g = \int_{\partial M} u \langle X, N \rangle_g \, dV_{\tilde{g}} - \int_M u \operatorname{div} X \, dV_g.$$

What does this formula say when M is a compact interval in $\mathbb{R}$?

Using the divergence we can define the Laplace–Beltrami operator (with the convention that the eigenvalues are positive).

DEFINITION 2.64. Let (M, g) be an (orientable) Riemannian manifold. The *Laplace–Beltrami operator* (or Laplacian) is the linear operator $\Delta \colon C^\infty(M) \to C^\infty(M)$ defined by

$$\Delta u := \operatorname{div}(\operatorname{grad} u).$$

PROBLEM 2.65. Let (M, g) be a Riemannian manifold and let (x^i) be smooth coordinates on an open set $U \subseteq M$. Show that

$$\Delta u = \frac{1}{\sqrt{\det g}} \partial_i (g^{ij} \sqrt{\det g} \, \partial_j u),$$

where $\partial_i = \frac{\partial}{\partial x^i}$ and $\det g = \det(g_{kl})$ is the determinant of the component matrix of g in these coordinates.

In particular, on $\mathbb{R}^n$ with the Euclidean metric and the standard coordinates we recover the usual formula $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{(\partial x^i)^2}$.

REMARK 2.66. While on a Riemannian manifold the Laplace–Beltrami operator is elliptic, it is a hyperbolic operator on a Lorentzian manifold (usually called wave operator or d'Alembertian then, and denoted by $\Box$).

Another important second order operator is the Hessian (see later or [**36**]).

2.3. Lengths and Riemannian distance

According to Marcel Berger [**7**, Sec. 14.6] "Gromov's *mm* spaces are [...] the geometry of the future, redefining what we mean by a geometric space, to unify the subjects of probability and metric geometry". Here, *mm* stands for metric measure, the latter connecting the geometric theory to probability. Gromov's theory [**11**, **27**] extends the geometry of Riemannian manifolds to length metric spaces equipped with measures on which one can define upper and/or lower curvature *bounds*. We have already seen in Section 2.2.2 that there is a canonical notion of volumes on Riemannian manifolds. In this section we will establish the connection to metric spaces[12].

2.3.1. Lengths of curves. The basic notions of curves are the following.

DEFINITION 2.67. A (parametrized) *curve* on a smooth manifold M is a continuous map $\gamma\colon I \to M$, where $I \subseteq \mathbb{R}$ is an interval.

A *curve segment* is curve defined on a compact interval I.

We say that γ is a *smooth curve* if it is smooth as a map from the manifold (with boundary) I to M.

A *regular curve* is a smooth curve satisfying $\gamma'(t) \neq 0$ for all $t \in I$.

In what follows with work with piecewise smooth curves in order to allow for "corners" or "kinks". To define these pieces we use a *partition of* $[a,b]$, that is, a finite sequence $(a_0, \ldots, a_k)$ of real numbers such that $a = a_0 < a_1 < \ldots < a_k = b$.

DEFINITION 2.68. A (continuous) curve segment $\gamma\colon [a,b] \to M$ on a smooth manifold M is called *piecewise regular* (or *admissible*) if there exists a partition $(a_0, \ldots, a_k)$ of $[a,b]$ such that $\gamma|_{[a_{i-1},a_i]}$ is a regular curve segment for all $i = 1, \ldots, k$.

Figure 2.3. A piecewise regular curve.
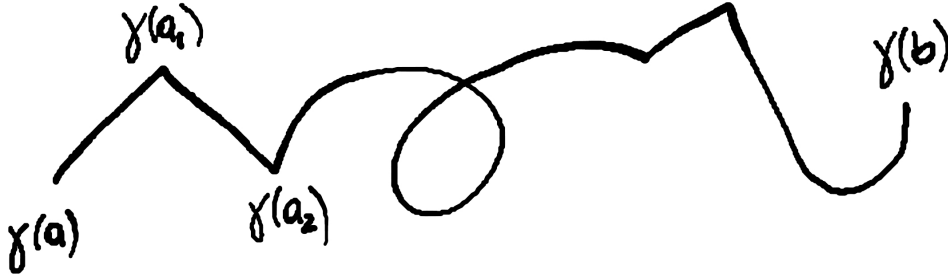
At the partition points $a_1, \ldots, a_{k-1}$ there are two one-sided velocity vectors
$$\gamma'(a_i^-) := \lim_{t \nearrow a_i} \gamma'(t), \qquad \gamma'(a_i^+) := \lim_{t \searrow a_i} \gamma'(t),$$

---

[12]In what follows we reserve the term *metric* for the Riemannian metric (a tensor) and will instead use *distance* for the induced metric (in the usual topological sense).

that are nonzero, but not necessarily equal.

Generally, the parametrization of a curve is not relevant, but in order to remain admissible, only a certain class of reparametrizations is allowed.

DEFINITION 2.69. Let $\gamma\colon [a,b] \to M$ be an admissible curve. A *reparametrization of γ* is a curve $\tilde{\gamma} = \gamma \circ \varphi$ where $\varphi\colon [c,d] \to [a,b]$ is a homeomorphism and for a partition $(c_0,\ldots,c_k)$ of $[c,d]$ the restrictions $\varphi|_{[c_{i-1},c_i]}$ are diffeomorphisms on their image.

A reparametrization is called *forward* if it is increasing, and *backward* if it is decreasing.

DEFINITION 2.70. The *length of an admissible curve* $\gamma\colon [a,b] \to M$ is defined as

$$L_g(\gamma) := \int_a^b |\gamma'(t)|_g dt.$$

Note that due to the regularity of γ this integral is well-defined.

The following result shows that a length structure is obtained from these definitions.

PROPOSITION 2.71 (Properties of lengths). *Suppose (M,g) is a Riemmanian manifold and $\gamma\colon [a,b] \to M$ an admissible curve. The following hold:*
  (i) *Additivity of length: If $a < c < b$, then $L_g(\gamma) = L_g(\gamma|_{[a,c]}) + L_g(\gamma|_{[c,b]})$.*
  (ii) *Parameter independence: If $\widetilde{\gamma}$ is a reparametrization of γ, then $L_g(\gamma) = L_g(\widetilde{\gamma})$.*
  (iii) *Isometry invariance: If (M,g) and $(\widetilde{M},\widetilde{g})$ are Riemannian manifolds and $\varphi\colon M \to \widetilde{M}$ is a local isometry, then $L_g(\gamma) = L_{\widetilde{g}}(\varphi \circ \gamma)$.*
  (iv) *Regularity of the arc-length function: The function $s\colon [a,b] \to \mathbb{R}$, defined by*

$$s(t) := L_g(\gamma|_{[a,t]}) = \int_a^t |\gamma'(u)|_g \, du,$$

  is continuous everyhwere and smooth wherever γ is, with derivative $s'(t) = |\gamma'(t)|$ called the speed of γ.

EXERCISE 2.72. Prove Proposition 2.71.

If $\gamma\colon [a,b] \to M$ is a *unit-speed admissible curve*, i.e., $|\gamma'(t)| = 1$ wherever γ is smooth, then the arc-length function simply reads $s(t) = t - a$. The following results shows that such a *parametrization by arc length* is always possible.

PROPOSITION 2.73. *Suppose (M,g) is a Riemannian manifold.*
  (i) *Every regular curve in M has a unit-speed parametrization.*
  (ii) *Every admissible curve in M has a unique forward reparametrization by arc length.*

SKETCH OF PROOF. For a regular curve show that s is a strictly increasing local diffeomorphism, and thus $\varphi = s^{-1}$ the desired reparametrization (use the chain rule). For an admissible curve it follows from induction on its pieces. (For further details see [**36**, Prop. 2.49].) $\qquad\square$

2.3.2. Riemannian distance function. We extend the most important concept from classical geometry to the Riemannian setting.

DEFINITION 2.74. Suppose (M,g) is a connected Riemannian manifold. The *Riemannian distance* between each pair of points $p,q \in M$ is defined as

$$d_g(p,q) := \inf\{L_g(\gamma); \gamma \text{ admissible curve between } p \text{ and } q\}.$$

The following result guarantees that this definition yields a well-defined map $d\colon M \times M \to [0,\infty)$.

PROPOSITION 2.75. *If M is a connected smooth manifold, then any two points can be joined by an admissible curve.*
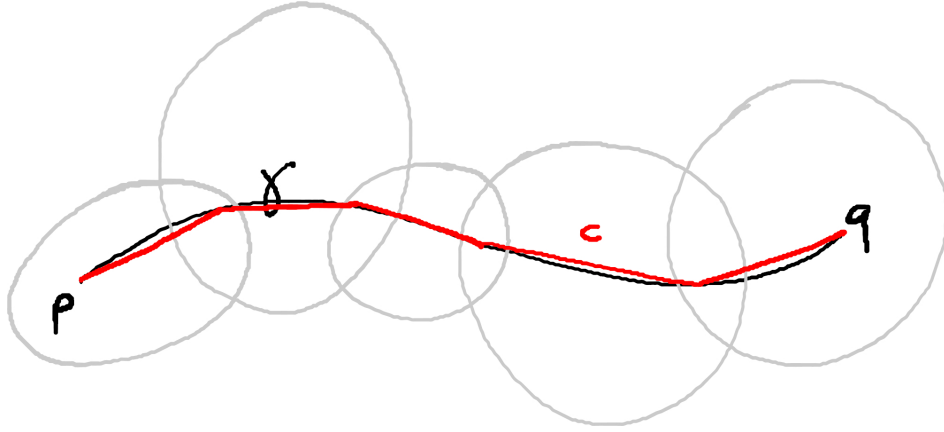
Figure 2.4. Proof of Prop. 2.75, namely that there exists a piecewise regular curve c between any two points p and q on a connected manifold.

PROOF. Let $p, q \in M$ be arbitrary points. Since M is connected it is also path-connected. Hence there exists a continuous path $\gamma\colon [a,b] \to M$. By compactness of $\gamma([a,b])$, there are finitely many curve segments $\gamma([a_{i-1}, a_i])$ each of which are contained in a single coordinate ball. Therefore, they can be replaced by a straight line in coordinates, which in M yields a piecewise regular curve between the same points. $\square$

EXERCISE 2.76. Show that the infimum of curve lengths $d_{\bar{g}}(p, q)$ fails to be realized on the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$.

PROPOSITION 2.77 (Isometry invariance of the Riemannian distance function). *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are connected Riemannian manifolds and $\varphi\colon M \to \widetilde{M}$ is an isometry. Then*

$$d_{\widetilde{g}}(\varphi(p), \varphi(q)) = d_g(p, q), \qquad p, q \in M.$$

EXERCISE 2.78. Prove Proposition 2.77.

Note that unlike lengths of curves, distances are *not* preserved by *local* isometries.

PROBLEM 2.79. Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are connected Riemannian manifolds, and $\varphi\colon M \to \widetilde{M}$ is a local isometry. Show that

$$d_{\widetilde{g}}(\varphi(p), \varphi(q)) \leq d_g(p, q), \qquad p, q \in M.$$

Give an example to show that equality need not hold.

The following important results holds.

THEOREM 2.80 (Riemannian manifolds as metric spaces). *Let (M, g) be a connected Riemannian manifold. The distance function d_g is a metric on M whose metric topology induces the manifold topology.*

EXERCISE 2.81. Show d_g satisfies the triangle inequality and is symmetric.

We will only prove the rest of the statement later, because the proof of positive definiteness is much easier to do with normal neighborhoods. Nevertheless, it is also possible (but much more involved) to show this result in an elementary way (see [**36**, p. 37–39] or [**13**, Sec. 4.1]). In fact, all metric space structures induced by (continuous) Riemannian metrics are equivalent on compact sets [**13**, Thm. 4.5].

Using this metric space structure on (M, g), all notions and results from the analysis and geometry of metric spaces carry over. For instance, we say that (M, d_g) is *metrically complete* if every Cauchy sequences in M converges. A subset $A \subseteq M$ is *bounded* if there exists a constant $C > 0$ such that the *diameter* satisfies

$$\text{diam}(A) = \sup\{d_g(p, q); p, q \in A\} \leq C.$$

Since every compact metric space is bounded, every compact connnected Riemannian manifold has finite diameter.

Later we will prove more important properties related to d_g, such as when the Heine–Borel property holds and the Hopf–Rinow Theorem relating metric and geodesic completeness.

REMARK 2.82. On Lorentzian manifolds one can consider the lengths of timelike curves in much the same fashion and even introduce a Lorentzian distance (based on the *maximization* of curve lengths). While these constructions are still very important and provide some information about the underlying Lorentzian manifold, they are of much more limited use compared to Riemannian manifolds, because they do *not* induce a canonical metric space structure (see [**6**, Ch. 4] for more details). Recent alternative approaches are the null distance [**2**, **15**, **50**, **53**] and Lorentzian length spaces [**34**], both rooted in Lorentzian causality theory (which studies the order structures induced by timelike and causal curves, a unique feature of time-oriented Lorentzian manifolds).

2.4. Symmetries of Riemannian manifolds

Before introducing the technical notions of connections and curvature on Riemannian manifolds, we pause for some considerations related to symmetries of M. Our most important model spaces[13] are (again) Euclidean space, sphere, and hyperbolic space. We will briefly also discuss some other important examples such as Lie groups, homogeneous and isotropic spaces. All of these spaces are highly symmetric and thoroughly discussed in [**36**, Ch. 3] (and more).

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[13]Euclidean space, sphere, and hyperbolic space are called model spaces, because they have high degree of symmetry and–as we will later see—are spaces of constant curvature 0, 1 and −1, see Figure 1.2 on page 3. The Lorentzian analogues Minkowski space, de Sitter space and anti-de Sitter space (not discussed here) appear as important and simplest solutions to the Einstein equations in General Relativity.

By the way, Willem de Sitter (1872–1934) was a Dutch mathematician, physicist and astronomer who made major contributions to physical cosmology.

2.4.1. Isometries. We have already considered isometries between two Riemannian manifolds in Section 2.1.2. In this section we fix the Riemannian manifold to see what kind of information isometries encode about it.

DEFINITION 2.83. An *isometry of M* is an isometry of (M, g) to itself. The set of all isometries of (M, g) forms a group under composition, called the *isometry group of* (M, g) and denoted by $\mathrm{Iso}(M, g)$.

The Myers–Steenrod Theorem [**38**] from 1939 states that if (M, g) is a connected Riemannian manifold then $\mathrm{Iso}(M, g)$ is a Lie group acting smoothly on M. Let us observe this in our most important examples where the isometry groups are (compact) Lie groups.

EXAMPLE 2.84 (Euclidean group). The *Euclidean group* is the semidirect product[14]

$$\begin{aligned} \mathrm{E}(n) :=& \mathbb{R}^n \rtimes \mathrm{O}(n) \\ =& \{F \colon \mathbb{R}^n \to \mathbb{R}^n; F(x) = b + Ax \text{ for } b \in \mathbb{R}^n \text{ and } A \in \mathrm{O}(n)\} \end{aligned}$$

The translational part and rotational part are uniquely determined. It is clear that the maps F preserve the Euclidean metric $\bar{g}$ and hence are isometries, that is, $\mathrm{Iso}(\mathbb{E}^n) \supseteq \mathbb{R}^n \rtimes \mathrm{O}(n)$. The converse is based on a uniqueness result for Riemannian isometries which states that any two (local) isometries that agree at one point p (here, the origin) and whose differentials agrees at p are, in fact, equal (we will prove this statement later). Thus

$$\mathrm{Iso}(\mathbb{E}^n) = \mathrm{E}(n) = \mathbb{R}^n \rtimes \mathrm{O}(n).$$

For the sphere and hyperbolic space we make use of their description as codimension-1 submanifolds of $\mathbb{R}^{n+1}$ and $\mathbb{R}^{1,n}$, respectively (see Section 2.1.5).

EXAMPLE 2.85 (Rotation group). The linear action of the orthogonal group $\mathrm{O}(n+1)$ of $\mathbb{R}^{n+1}$ preserves $\mathbb{S}^n(R)$ (the origin remains fixed) and the Euclidean metric $\bar{g}$, and thus acts isometrically on $\mathbb{S}(R)$ and preserves $\mathring{g}_R$. One can also show the converse (again later), thus spheres (of any Radius $R > 0$) have isometry group

$$\mathrm{Iso}(\mathbb{S}^n(R)) = \mathrm{O}(n+1).$$

EXAMPLE 2.86 (Lorentz group). The $(n+1)$-dimensional *Lorentz group* $\mathrm{O}(1, n)$ denotes the group of linear maps from $\mathbb{R}^{1,n}$ to itself that preserve the Minkowski metric[15] η, i.e.,

$$\mathrm{O}(1, n) := \{L \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}; \bar{\eta}(Lv, Lv) = \bar{\eta}(v, v)\}.$$

One can show (see Ex. 2.87 below) that $\mathrm{O}(1, n)$, just like in the case of the sphere, preserves the two-sheeted hyperboloid. Since hyperbolic space, however, only consists of the upper hyperboloid, we have to restrict to a subgroup "preserving the forward time direction". This subgroup

$$\mathrm{O}^+(1, n) := \{L \in \mathrm{O}(1, n); L^0{}_0 > 0\}$$

---

[14]If G is a group, $H < G$ a subgroup and $N \triangleleft G$ a normal subgroup (a subgroup invariant under conjugation, i.e., $gng^{-1} \in N$ for $g \in G, n \in N$) such that $G = NH$ and $N \cap H = \{1\}$, then we say $G = N \rtimes H$ is a semidirect product of H extended by N. Equivalently, it means that we can write every $g \in G$ as a product nh with a unique $n \in N$ and $h \in H$ (or the other way round). The group operation is given by $(n_1, h_1)(n_2, h_2) := (n_1 h_1 n_2, h_1 h_2)$ for $n_1, n_2 \in N$, $h_1, h_2 \in H$. Note that in case of the Euclidean group the subgroup $\mathrm{T}(n) \cong \mathbb{R}^n$ of translations is the normal subgroup (Why?).

[15]Just like in the Euclidean case, the isometry group of Minkowski space itself is actually larger, namely the *Poincaré group* $\mathbb{R}^{1,n} \rtimes \mathrm{O}(1, n)$.

is called the *orthochronous Lorentz group*. Since $O^+(1,n)$ preserves $\mathbb{H}^n(R)$, and because it preserves the Minkowski metric it acts isometrically on $\mathbb{H}^n(R)$, i.e., preserves $\breve{g}_R$. One can show (later) that it is, in fact, the whole isometry group, that is,

$$\operatorname{Iso}(\mathbb{H}^n(R)) = O^+(1,n).$$

EXERCISE 2.87. Show that each element in $O(1,n)$ preserves the two-sheeted hyperboloid $\{\tau^2 - |\xi|^2 = R^2\}$.

We have seen that Lie groups appear as isometry groups. Conversely, we can ask when a Lie group can be turned into a group of isometries.

EXAMPLE 2.88 (Lie groups). For a Lie group[16] G the tangent bundle can be trivialized, i.e.,

$$TG \cong G \times T_eG$$

by using left (or right) translations on G. Thus fixing an inner product on $T_eG \cong \mathfrak{g}$ induces a left-invariant[17] Riemannian metric g on G (implying, at the same time, that left translations are Riemannian isometries), i.e.,

$$L_\varphi^* g = g, \qquad \varphi \in G,$$

where $L_\varphi(\varphi') = \varphi\varphi'$ denotes a left translation [**36**, Lem. 3.10].

PROBLEM 2.89. Let $\mathfrak{o}(n)$ denote the Lie algebra of $O(n)$, identified with the algebra of skew-symmetric[18] $n \times n$ matrices, and define a bilinear form on $\mathfrak{o}(n)$ by

$$\langle A, B \rangle := \operatorname{tr}(A^T B).$$

Show that this determines a bi-invariant Riemannian metric on $O(n)$ (show that it is Ad-invariant, using [**36**, Prop. 3.12]).

2.4.2. Homogeneity and isotropy. Depending on particular properties of the isometry group various "highly symmetric" Riemannian metrics can be identified. We distinguish between homogeneous Riemannian manifolds that "look the same at every point" and isotropic ones that "look the same in every direction".

DEFINITION 2.90. Let (M,g) be a Riemannian manifold. We say that (M,g) is a *homogeneous Riemannian manifold* if $\operatorname{Iso}(M,g)$ acts transitively on M, that is, for all $p,q \in M$ exists $\varphi \in \operatorname{Iso}(M,g)$ such that $\varphi(p) = q$.

EXAMPLE 2.91. Since a Lie group acts transitively on itself by left translations, every left-invariant metric is homogeneous.

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[16]A *Lie group* is a smooth manifold and a group such that the group operations (multiplication and inversion) are smooth. For more background see [**36**, App. C].

[17]From a Riemannian point of view bi-invariant metrics are much more interesting because their curvatures are intimately tied to the Lie algebra structure. Bi-invariant metrics are much harder to find though, but they always exist for compact Lie groups [**36**, Cor. 3.15]. More generally, John Milnor [**39**] showed that a Lie group admits a bi-invariant Riemannian metric if and only if it is isomorphic to the product $G \times H$ with G compact and H abelian.

[18]A matrix A is *skew-symmetric* if $A^T = -A$.

REMARK 2.92 (Poincaré conjecture). Locally homogeneous Riemannian metrics (where homogeneity holds in a neighborhood) play a key role in the classification of compact 3-manifolds. In 1904 Poincaré conjectured that every simply connected, closed topological 3-manifold is homeomorphic to the 3-sphere. This was called the *Poincaré conjecture* and made it as one of the ten biggest unsolved mathematics problems on the Clay Millenium List[19] compiled in 2000. Later in 1982, William Thurston made a more general conjecture about the classification of 3-manifolds, called the *geometrization conjecture*, stating that every compact orientable 3-manifold can be expressed as the connected sum of compact manifolds, each of which admits a Riemannian covering by a homogenous Riemannian manifold or can be cut along embedded tori so that each piece admits a finite-volume locally homogeneous Riemannian metric. The classification of all simply connected homogeneous Riemannian 3-manifolds that admit finite-volume Riemannian quotients (of which there are 8) played a key role. Eventually, Thurston's geometrization conjecture was shown to be true by Grigori Perelman in 2003, building upon earlier work by Richard Hamilton, using Ricci flow. Perelman published his work in preprints only. He was offered (and declined) the Fields Medal in 2006. He was also awarded (and refused; you may want to read the novel [**59**] for potential further insight) the Clay Millenium Prize worth 1 million US dollars in 2010 for the solution of the Poincaré conjecture, so far the only solved Millenium Prize Problem. In any case, a true win and milestone for Riemannian Geometry!

DEFINITION 2.93. The *isotropy subgroup* $\mathrm{Iso}_p(M, g)$ of $p \in M$ (also called *stabilizer* of p) consists of all isometries that fix p.

EXAMPLE 2.94. The isometries of Euclidean space and Minkowski space generally contain translations, however, isotropies cannot. Therefore,

$$\mathrm{Iso}_0(\mathbb{E}^n) = \mathrm{O}(n), \qquad \mathrm{Iso}_0(\mathbb{R}^{1,n}) = \mathrm{O}(1, n).$$

EXAMPLE 2.95. The isotropy groups of $\mathbb{S}^n(R)$ are isomorphic to $\mathrm{O}(n)$, i.e., the elements of $\mathrm{O}(n + 1)$ fixing a 1-dimensional subspace of $\mathbb{R}^{n+1}$ (the rotation axis).

EXAMPLE 2.96. The isotropy group of hyperbolic space $\mathbb{H}^n(R)$ that preserves the origin (in the hyperboloidal model it can be identified with the timelike vector Re_0) is $\mathrm{O}(n)$.

Note that $\mathbb{R}^n \cong \mathrm{E}(n)/\mathrm{O}(n)$ and $\mathbb{S}^n(R) \cong \mathrm{O}(n+1)/\mathrm{O}(n)$. In fact, one can show that any homogeneous space (M, g) can be written as a quotient $\mathrm{Iso}(M, g)/\mathrm{Iso}_p(M, g)$.

Generally, the differential $d\varphi$ maps TM to itself, and every $\varphi \in \mathrm{Iso}(M, g)$ restricts to linear isometries $d\varphi_p \colon T_pM \to T_{\varphi(p)}M$ for all $p \in M$. For isotropies, $d\varphi_p \colon T_pM \to T_pM$ is even more special, and gives rise to the following definition.

DEFINITION 2.97. A Riemannian manifold (M, g) is called *isotropic at p* if the isotropy representation

$$I_p \colon \mathrm{Iso}_p(M, g) \to \mathrm{GL}(T_pM), \qquad I_p(\varphi) = d\varphi_p,$$

acts transitively on the set of unit vectors in T_pM. In other words, for any two unit vectors $v, w \in T_pM$ there exists an isotropy $\varphi \in \mathrm{Iso}_p(M, g)$ such that[20] $\varphi_*(v) = w$.

If M is isotropic at every p we say that M is *isotropic*.

---

[19]See `https://en.wikipedia.org/wiki/Millennium_Prize_Problems` for the complete list.

[20]Recall that the *pushforward* of $X \in \mathfrak{X}(M)$ is the unique vector field $\varphi_* X \in \mathfrak{X}(\widetilde{M})$ that satisfies $d\varphi_p(X_p) = (\varphi_* X)_{\varphi(p)}$ for all $p \in M$.

The following result shows that for homogeneous spaces we only need to have isotropy at one point and can otherwise push it around.

PROPOSITION 2.98. *Let (M, g) be a Riemannian manifold. If M is homogeneous and isotropic at one point, then it is isotropic everywhere.*

PROOF. Suppose M is isotropic at the point q. Let p be any other point in M. Suppose v and w are (different) unit vectors in $T_p M$. By homogeneity there exists an isometry $\psi \in$ Iso(M, g) such that $\psi(p) = q$. Thus $d\psi_p(v), d\psi_p(w)$ are two unit vectors in $T_q M$, and because M is isotropic at q there exists an isotropy $\varphi \in \text{Iso}_q(M, g)$ that transforms $d\psi_p(v)$ into $d\psi_p(w)$, i.e.,

$$d(\varphi \circ \psi)_p(v) = d\varphi_{\psi(p)}(d\psi_p(v)) = d\varphi_q(d\psi_p(v)) = d\psi_p(w).$$

Hence $\widetilde{\varphi} := \psi^{-1} \circ \varphi \circ \psi \in \text{Iso}_p(M, g)$ satisfies

$$d\widetilde{\varphi}_p(v) = d\psi_q^{-1} \circ d\varphi_q \circ d\psi_p(v) = d\psi_{\psi(p)}^{-1}(d\psi_p(w)) = w.$$

Hence (M, g) is also isotropic at p, and since p was arbitrary, M is isotropic everywhere. □

REMARK 2.99. One can also show that (everywhere!) isotropic Riemannian manifolds are automatically homogeneous, but there are counterexamples for the converse statement. The Berger metrics on $\mathbb{S}^3$ are homogeneous without being isotropic anywhere, see [**36**, Problem 3-10].

There is an even more special class of symmetries, based on the transformation not only of the unit sphere in $T_p M$ as in isotropy but based on the transformation of orthonormal bases, therefore called *frame-homogeneous* (sometimes, isotropy is defined like that, see [**23**]). Frame-homogeneous Riemannian manifolds are both homogeneous and isotropic [**36**, p. 56].

EXAMPLE 2.100. Assuming that we have shown $\text{Iso}(\mathbb{E}^n) = \text{E}(n)$ (see Ex. 2.84) immediately implies that $\mathbb{E}^n$ is frame-homogeneous.

By explicitly constructing the rotation moving the orthonormal basis of the north pole (or origin) to one at another point one obtains the same result for $\mathbb{S}^n(R)$ and $\mathbb{H}^n(R)$ [**36**, Props. 3.2 and 3.9].

We have by now observed several properties of the isometry group on model spaces and Lie groups with regards to homogeneity and isotropy. These results (and more) can be summarized in the following table (symmetric spaces are Riemannian manifolds with point reflections at every point; more details and more spaces can be found in [**36**, Ch. 3]).

	$\mathbb{E}^n$	$\mathbb{S}^n$	$\mathbb{H}^n$	Lie groups	symmetric spaces
homogeneous	x	x	x	x	x
isotropic	x	x	x		
frame-homogeneous	x	x	x		

REMARK 2.101 (Cosmology). Statistically our universe looks "on the large scale" very boring. In Cosmology one therefore studies homogeneous and isotropic solutions of the Einstein equations with cosmological constant (following the *cosmological principle*). The prototypical example are the Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes, which are nothing else but Lorentzian metrics that are warped products $I \times_a \Sigma$, where the first part I is 1-dimensional and negative definite, the second space Σ is a 3-dimensional Riemannian

model space (that is, $\mathbb{E}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$), and the warping function a the scale factor. The Einstein equations with a perfect fluid source reduce to a system of two second-order ordinary differential equations for a, called the *Friedmann equations* (derived by him in the 1920s).

2.4.3. Conformal transformations. An important generalization of an isometry is that of a conformal transformation. It does not preserve the lengths of vectors but the angle between them. In two dimensions (or one complex dimension), conformal geometry is precisely the study of Riemann surfaces. More generally one refers to a manifold equipped with an equivalence class of metrics up to a conformal factor as *conformal manifolds*. Conformal transformations are particularly important in Lorentzian Geometry and General Relativity because they leave the light cone structure, and hence the basic geometric concept of causality, intact.

DEFINITION 2.102. Let M be a manifold. We say that two metrics g_1 and g_2 are *conformal* (or conformally related) to each other if there exists a smooth function $f\colon M \to (0, \infty)$ such that $g_2 = fg_1$.

DEFINITION 2.103. Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifolds. A diffeomorphism $\varphi\colon M \to \widetilde{M}$ is called a *conformal transformation* if the pullback of $\widetilde{g}$ is conformal to g, i.e.,

$$\varphi^*\widetilde{g} = fg \qquad \text{for some positive } f \in C^\infty(M).$$

Two Riemannian manifolds are said to be *conformally equivalent* if there exists a conformal transformation between them.

EXERCISE 2.104.   (i) Show that for every smooth manifold M, conformality is an equivalence relation on the set of Riemannian metrics on M.
  (ii) Show that conformal equivalence is an equivalence relation on the class of all Riemannian manifolds.

EXERCISE 2.105. Suppose g_1 and $g_2 = fg_1$ are conformally related metrics on an oriented n-manifold. Show that their volume forms are related by $dV_{g_2} = f^{\frac{n}{2}} dV_{g_1}$.

Besides global conformal transformation one can also study important local version. In this context, a Riemannian manifold (M, g) is said to be *locally conformally flat* if around ever point in M there is a neighborhood that is conformally equivalent to an open set in $(\mathbb{R}^n, \bar{g})$. Both sphere and hyperbolic space are locally conformally flat (but they are *not* locally flat!).

PROPOSITION 2.106. *Each sphere with a round metric is locally conformally flat.*

SKETCH OF PROOF. A local conformal equivalence between $\mathbb{E}^n$ and $\mathbb{S}^n(R)$ (local, because the north pole N is excluded) is given by the *stereographic projection*

$$\sigma\colon \mathbb{S}^n(R) \setminus \{N\} \to \mathbb{R}^n,$$

$$(\xi^1, \ldots, \xi^n, \tau) \mapsto u := \frac{R\xi}{R - \tau},$$

which can be shown to be a conformal tranformation between $\mathbb{S}^n(R) \setminus \{N\}$ and $\mathbb{R}^n$. In particular, it is a diffeomorphism and satisfies

$$(\sigma^{-1})^* \mathring{g}_R = \frac{4R^4}{(|u|^2 + R^2)^2} \bar{g}. \qquad \qquad \square$$

(For the full proof and pictures see [**36**, p. 59–61].)

Using the (isometric) Poincaré ball or half space models one can also prove that hyperbolic space $\mathbb{H}^n(R)$ is locally conformally flat [**36**, p. 62–66].

REMARK 2.107 (Rigidity of conformal flatness). Conformal flatness is directly related to different curvature components vanishing or being constant. In dimension 2 all spaces with constant sectional curvature are locally conformally flat. Liouville's Theorem from 1850 states that metrics conformal to the flat metric $\bar{g}$ on $\mathbb{E}^n$, for $n \geq 3$, can only be obtained through translation, similarity, orthogonal transformation and inversion. Hence the class of conformal mappings in dimension $n \geq 3$ is much poorer than in dimension 2, adding additional rigidity to metrics that are locally conformally flat. The *Weyl–Schouten Theorem* states that a Riemannian manifold of dimension $n \geq 3$ is locally conformally flat if and only if the Schouten[21] tensor is a Codazzi tensor (if $n = 3$) or if the Weyl[22] tensor vanishes (if $n \geq 4$). More about this in Section 5.4.

REMARK 2.108 (Conformal transformations in General Relativity). In Lorentzian Geometry, conformal transformations are of great importance because they preserve the causal structure. Already in 1921 Kasner [**33**] essentially proved that the a vacuum spacetime that is locally conformally flat is, in fact, flat. Around the same time Brinkmann [**10**] investigated conformal transformations between two Einstein spaces as well as conditions for a space to be conformal to an Einstein space. As a byproduct he found plane waves, which reappear in an important paper of Penrose[23] that also makes use of conformal transformations: In 1976 Penrose [**44**] introduced adapted coordinates around a null geodesic in order to associate with any spacetime metric g a family of conformally equivalent metrics $g_\lambda = \lambda^{-2}g$. The *Penrose limit* is then the metric $g_0 = \lim_{\lambda \to 0} g_\lambda$, and it can be shown that this is a plane wave metric. It follows, for instance, that the Penrose limit of an Einstein manifold is Ricci flat.

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[21]Jan Schouten (1883–1971) was a Dutch mathematician contributing to tensor calculus and Ricci calculus.

[22]Hermann Weyl (1885–1955) was one of the most influential mathematicians of the 20th century, worked in many different areas of mathematics and made major contributions everywhere, also related to theoretical physics.

[23]Sir Roger Penrose (1931–) is a British mathematican and mathematical physicist. In 2020 he won the Nobel Prize for physics for his singularity theorem in General Relativity (from 1965) describing, vaguely put, black hole formulation.

CHAPTER 3

Connections and covariant differentation

In the "Manifolds" course you may have already heard about Lie derivatives and exterior derivative on manifolds [**35**, Ch. 9 and 14]. The main goal of this chapter is to introduce connections and their use for covariant differentiation, which immediately relates to the directional derivative (on functions), the Lie derivative (on vector fields) and the exterior derivative (on forms) by extending them to tensors.

A nice introduction and motivation for connections via parallelism can be found in the book of do Carmo [**23**, Ch. 2]. Also in the book of Lee [**36**, p. 85–88] there is a good recollection of the concept of directional differentiation of vector fields for submanifolds M of $\mathbb{R}^n$. In this setting the projection onto the part that is tangential to M is used. This is clearly *not* an *intrinsic* geometric concept and hence highlights that there is something to resolve in the setting of abstract manifolds. In this section we will therefore introduce covariant differentiation using the basic idea of parallelism, which indeed only depends on the intrinsic geometry of a manifold (since it is invariant by isometry).

Note that connections can be defined on any manifold, however, a Riemannian manifold (and also every semi-Riemannian manifold) can be equipped with a canonical connection, called the *Levi-Civita connection*, which is fundamental for the modern definition of curvature via parallel transport along vector fields and curves. Riemann himself did not know what a connection was. He computed the Riemann curvature tensor as a second-order correction term in the Taylor expansion of the Riemannian metric around a point (in coordinates). The notion of a connection postdates Riemann and was developed by the Italian school around Christoffel and Levi-Civita, Ricci, Bianchi etc. in the context of tensor analysis.

3.1. Affine connections

We define a connection ∇ (pronounced "del" or "nabla") as a way of differentiating vector fields and later apply it to vector fields along curves.

DEFINITION 3.1. Let M be a smooth manifold. An *affine connection on M* is a map

$$\nabla \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),$$
$$(X, Y) \mapsto \nabla_X Y,$$

satisfying the following properties:

(i) $\nabla_X Y$ is $C^\infty(M)$-linear in X: for $f_1, f_2 \in C^\infty(M), X_1, X_2 \in \mathfrak{X}(M)$,

$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y.$$

(ii) $\nabla_X Y$ is $\mathbb{R}$-linear in Y: for $a_1, a_2 \in \mathbb{R}, Y_1, Y_2 \in \mathfrak{X}(M)$,

$$\nabla_X (a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2.$$

(iii) ∇ satiesfies the product rule: for $f \in C^\infty(M)$,

$$\nabla_X(fY) = f\nabla_X Y + X(f)Y.$$

The vector field $\nabla_X Y$ is called the *covariant derivative of Y in direction X*.

REMARK 3.2. Beware, the map $(X, Y) \mapsto \nabla_X Y$ does *not* define a tensor field on M. This is because in place of being $C^\infty(M)$-linear in Y it is only $\mathbb{R}$-linear in Y and satisfies the Leibniz product rule (this follows form the Tensor Characterization Lemma A.17). Nevertheless, ∇ is a well-defined coordinate-independent map. This explains also the use of the word "covariant" in the covariant derivative which just means that it transforms covariantly (that is, correctly).

REMARK 3.3. While $(X, Y) \mapsto \nabla_X Y$ is not a tensor field on M, we will later see that the following modification

$$\tau(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y],$$

is a $(1, 2)$-tensor field, called the *torsion tensor* of ∇. Torsion (or actually, torsion-freeness) will again become relevant when we consider special connections on Riemannian manifolds.

While the connection appears to be a global concept we can, in fact, see that it is a *local operator* because $\nabla_X Y|_p$ only depends on the values of X at p and the values of Y in a neighborhood of p.

LEMMA 3.4 (Locality). *Suppose ∇ is an affine connection on a smooth manifold M. If $X, \widetilde{X}, Y, \widetilde{Y} \in \mathfrak{X}(M)$ such that for a point $p \in M$ we have that $X|_p = \widetilde{X}|_p$ and $Y = \widetilde{Y}$ on a neighborhood of p, then $\nabla_X Y|_p = \nabla_{\widetilde{X}} \widetilde{Y}|_p$.*

PROOF. First consider X. It follows from the definition of ∇ that $X \mapsto \nabla_X Y$ is a tensor, hence we even have that

$$X|_p = \widetilde{X}|_p \implies (\nabla_X Y)|_p = \nabla_{X|_p} Y = (\nabla_{\widetilde{X}} Y)|_p. \tag{3.1}$$

Second consider Y. It suffices to show that on an open set U we have $Y|_U = 0$ implies that $(\nabla_X Y)|_U = 0$. Let $q \in U$ and choose a bump function $\chi \in C^\infty(M)$ with support in U and such that $\chi \equiv 1$ in a neighborhood of q. Hence $\chi Y \equiv 0$ on M, and therefore by (iii) of Definition 3.1 we have for every $X \in \mathfrak{X}(M)$ that

$$0 = (\nabla_X(\underbrace{\chi Y}_{\equiv 0}))|_q \overset{\text{(iii)}}{=} \underbrace{X(\chi)|_q}_{=0} Y|_q + \underbrace{\chi(q)}_{=1}(\nabla_X Y)|_q,$$

hence $\nabla_X Y|_U = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

This result implies that the restriction ∇^U of an affine connection ∇ on an open set U of M defines a unique connection on $TM|_U$ (see [**36**, Prop. 4.2]).

For computations we want to express a connection in terms of a local frame (or local coordinates). Let (E_i) be a smooth local frame on TM on an open set $U \subseteq M$. For every i and j, $\nabla_{E_i} E_j$ is a vector field that we can expand in the same frame, so that

$$\nabla_{E_i} E_j =: \Gamma_{ij}^k E_k. \tag{3.2}$$

The n^3 smooth functions $\Gamma_{ij}^k : U \to \mathbb{R}$ are called the *connection coefficients of ∇* with respect to the given frame.

The connection on U is then completely determined by the connection coefficients $\{\Gamma_{ij}^k\}$: We can write two $X, Y \in \mathfrak{X}(U)$ in the given frame, i.e., $X = X^i E_i$, $Y = Y^j E_j$, and using properties (i)–(iii) of ∇ compute

$$
\begin{aligned}
\nabla_X Y &= \nabla_X (Y^j E_j) \\
&= X(Y^j) E_j + Y^j \nabla_{X^i E_i} E_j \\
&= X(Y^j) E_j + X^i Y^j \nabla_{E_i} E_j \\
&= X(Y^j) E_j + X^i Y^j \Gamma_{ij}^k E_k.
\end{aligned}
$$

Hence locally on U we have we obtain the following result (by relabeling).

PROPOSITION 3.5. *Let M be a smooth manifold, and let ∇ be an affice connection on M. Suppose (E_i) is a smooth local frame over an open subset $U \subseteq M$, and let $\{\Gamma_{ij}^k\}$ be the connection coefficients of ∇ with respect to this frame. For smooth vector fields $X, Y \in \mathfrak{X}(M)$, written in terms of the frame as $X = X^i E_i$, $Y = Y^j E_j$ one has*

$$
\nabla_X Y = (X(Y^k) + X^i Y^j \Gamma_{ij}^k) E_k. \tag{3.3}
$$

$\square$

EXAMPLE 3.6 (Euclidean connection). In $T\mathbb{R}^n$ we define the *Euclidean connection* $\overline{\nabla}$ pointwise by

$$
\overline{\nabla}_X Y := X(Y^1)\partial_1 + \ldots + X(Y^n)\partial_n, \qquad X, Y \in \mathfrak{X}(\mathbb{R}^n), \tag{3.4}
$$

where $\partial_i := \frac{\partial}{\partial x^i}$ with respect to the standard coordinates $(x^1, \ldots, x^n)$ in $\mathbb{R}^n$.

EXERCISE 3.7. Check that the Euclidean connection, defined in (3.4), is indeed a connection (satisfies the required properties) and that its connection coefficients in the standard coordinate frame are all zero.

There is a natural way to define an affine connection on a submanifold M of $\mathbb{R}^n$ by orthogonally projecting onto TM (the same idea can be used if $\mathbb{R}^n$ is replaced by another ambient semi-Riemannian manifold $\widetilde{M}$). Here enter some ideas from considering connections of vector bundles, which we briefly describe in Section 3.2. For now we introduce some notation to make this idea in the context of submanifolds precise.

DEFINITION 3.8. Let M be a submanifold embedded in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. At every point $p \in M$ we can defined the *normal space at p* as

$$
N_p M := (T_p M)^\perp := \{v \in T_p \widetilde{M}; \widetilde{g}_p(v, w) = 0 \text{ for all } w \in T_p M\},
$$

and the *normal bundle of M* as

$$
NM := \bigsqcup_{p \in M} N_p M.
$$

One can show that NM is a smooth rank-$(m - n)$ vector subbundle (see Section 3.2) of the ambient tanget bundle $T\widetilde{M}|_M$, and that there are smooth bundle homomorphisms

$$
\pi^\top \colon T\widetilde{M}|_M \to TM, \qquad \pi^\perp \colon T\widetilde{M}|_M \to NM,
$$

called the *tangential* and *normal projection*, that for each p restrict to orthogonal projections from $T_p \widetilde{M}$ to $T_p M$ and $N_p M$, respectively (the proof uses an adapted orthonormal frame, see [**36**, Prop. 2.16]).

EXAMPLE 3.9 (Tangential connection on a submanifold of $\mathbb{R}^n$). Let M be an embedded submanifold in $\mathbb{R}^n$ and $X, Y \in \mathfrak{X}(M)$. By using smooth extensions $\widetilde{X}$ and $\widetilde{Y}$ of X and Y to an open set of $\mathbb{R}^n$ (one can show that such extensions exist, but they are not unique, see [**36**, Ex. A.23]) and the orthogonal projection $\pi^\top$ onto TM we can define the *tangential connection* $\nabla^\top$ on TM by

$$\nabla^\top_X Y := \pi^\top \left(\overline{\nabla}_{\widetilde{X}} \widetilde{Y}|_M \right). \tag{3.5}$$

PROBLEM 3.10. Show that the tangential connection $\nabla^\top$ defined in (3.5) for submanifolds M of $\mathbb{R}^n$ is indeed a connection, that is, verify that

    (i) $\nabla^\top$ is well-defined (independent of the extensions $\widetilde{X}$ and $\widetilde{Y}$), and

    (ii) $\nabla^\top$ satisfies the axioms of an affine connection on M.

(*Hint*: (i) Note that the value of $\overline{\nabla}_{\widetilde{X}} \widetilde{Y}$ at a point $p \in M$ only depends on $\widetilde{X}_p = X_p$ and is hence independent of the extension. Show then that the tangential directional derivative $\nabla^\top_{X_p} Y := \pi^\top(\overline{\nabla}_{X_p} \widetilde{Y})$ is also independent of the choice of extension $\widetilde{Y}$, so $\nabla^\top$ is well-defined. Smoothness of $\overline{\nabla}_{\widetilde{X}} \widetilde{Y}$ holds by using an adapted orthonormal frame [**36**, Prop. 2.14]. (ii) It is straightforward to check that (i)–(iii) of Definition 3.1 hold. For verifying the product rule extend $f \in C^\infty(M)$ to a smooth function $\widetilde{f}$ on a neighborhood of M.)

Now that we have seen some examples of important connections, the question remains whether there always exists a connection in the tangent bundle of a manifold. Let us start with a simplified assumption, and work our way up.

On any manifold admitting a *global frame* (E_i) a connection can easily be obtained by picking n^3 smooth real-valued functions $\{\Gamma^k_{ij}\}$ and then defining ∇ via (3.3), i.e.,

$$\nabla_X Y := (X(Y^k) + X^i Y^j \Gamma^k_{ij}) E_k$$

(see Exercise 3.12 below). Using a partition of unity $\{\varphi_\alpha\}$ subordinate to an atlas $\{U_\alpha\}$ then allows one to obtain a connection on *any* manifold M. More precisely, on each neighborhood U_α a connection ∇^α exists by the above, and those connections can be put together to a connection

$$\nabla_X Y := \sum_\alpha \varphi_\alpha \nabla^\alpha_X Y, \qquad X, Y \in \mathfrak{X}(M),$$

on M (for the details see the proof of [**36**, Prop. 4.12]). Thus we argued that connections always exist.

PROPOSITION 3.11. *The tangent bundle of every smooth manifold admits a connection.*
$\square$

EXERCISE 3.12. Suppose M is a smooth n-manifold admitting a global frame (E_i). Then (3.3) gives a one-to-once correspondence between connections on TM and sets of n^3 functions $\Gamma^k_{ij} \in C^\infty(M)$.

(*Hint*: We have already seen that every connection determines functions $\{\Gamma^k_{ij}\}$ by (3.3) in a unique way. On the other hand, one can easily check that $\nabla_X Y$ defined by (3.3) using given smooth real-valued functions $\{\Gamma^k_{ij}\}$ yield a expression that is smooth in X and Y, $\mathbb{R}$-linear in Y, and $C^\infty(M)$-linear in X. Checking the product rule is a straightforward computation.)

3.2. Connections in vector bundles

Recall that a vector field is just a section of the tangent bundle TM, i.e., $\mathfrak{X}(M) = \Gamma(TM) = C^\infty(M, TM)$. In this spirit one can extend Definition 3.1 to sections $\Gamma(E)$ of vector bundles. For a short intro to vector bundles see Section A.1 of Appendix A as well as [**36**, p. 382–384], and [**35**, Ch. 10] for a more complete treatment.

DEFINITION 3.13. Let $\pi \colon E \to M$ be a smooth vector bundle over a smooth manifold M. A *connection in E* is a map

$$\nabla \colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),$$
$$(X, Y) \mapsto \nabla_X Y,$$

for which (i)–(iii) of Definition 3.1 hold for sections $\Gamma(E)$ in place of $\mathfrak{X}(M)$.

The expression $\nabla_X Y$ is called the *covariant derivative of Y in direction X*.

EXERCISE 3.14 (Trivial connection). Show that any connection in the trivial line bundle $\mathrm{pr}_M \colon M \times \mathbb{R} \to M$ is given by the exterior derivative $d \colon C^\infty(M) \to \Omega^1(M)$ (recall that by Example A.6 the sections are $\Gamma(M \times \mathbb{R}) = C^\infty(M)$; for the differential of a function see [**35**, p. 280–284] and, more generall, for the exterior derivative see [**35**, p. 362–373]) with covariant derivative along X being of the form $\nabla_X f = df(X) + f\omega(X)$ for any 1-form ω (the special case $\nabla_X f = df(X) = X(f)$ is the flat connection, for which the curvature tensor vanishes).

How can we extend this definition to a connection in the trivial bundle $M \times \mathbb{R}^k$ of rank k?

EXERCISE 3.15. Show that there is a connection in any vector bundle $\pi \colon E \to M$ over a smooth manifold M.

Considering connections on vector bundles is a far-reaching concept that can also be used in connection with, for instance, Lie group actions on manifolds. These ideas are relevant, in particular, in Gauge Theory and Algebraic Geometry. We will later use connections in tensor bundles and the normal bundle, and one can either view those as extensions of affine connections or in this wider framework of vector bundles. Vector bundles themselves are widely used in Differential Geometry, and if you follow a Master course in Differential Geometry, you will learn a lot more about them. Since the main focus of this course is to provide a short introduction to Riemannian Geometry, we keep the notions brief and focus on the implications and use of connections in this context. See [**36**, p. 88–91] and [**29**, Ch. 2] for more details.

3.3. Covariant differentiation of tensor fields

An affine connection on M is a connection in the tangent bundle TM and gives us a way to compute the covariant derivatives of vector fields. We will see that every such connection induces a canonical connection in all tensor bundles over M, which can also be interpreted in the sense of Section 3.2 because every tensor bundle $T^{(k,l)}TM$ is a vector bundle with sections being the tensor fields $\mathcal{T}^{(k,l)}(M) = \Gamma(T^{(k,l)}TM)$ (recall Section A.3 for the definition of tensor fields).

To indeed obtain a unique tensor derivation on $\mathcal{T}^{(k,l)}(M)$ for a given connection in TM, we extend it via the exterior derivative d on $C^\infty(M)$ to a tensor derivation (essentially using the idea of Exercise 3.14 in (ii) of Proposition 3.16 below). In Section 3.4 we will furthermore see that on a Riemannian manifold we can already pick a canonical connection in TM.

PROPOSITION 3.16. *Let M be a smooth manifold and ∇ be an affine connection on TM. Then there exists a unique connection on each tensor bundle $T^{(k,l)}TM$ (denoted also by ∇),*

$$\nabla \colon \mathfrak{X}(M) \times \mathcal{T}^{(k,l)}(M) \to \mathcal{T}^{(k,l)}(M),$$

such that the following four properties are satisfied:

(i) *In $T^{(1,0)}TM = TM$, ∇ agrees with the given connection.*

(ii) *In $T^{(0,0)}TM = M \times \mathbb{R}$, ∇ is given by the ordinary differentiation of functions, i.e.,*

$$\nabla_X f = X(f), \qquad f \in C^\infty(M).$$

(iii) *∇ obeys the product rule with respect to tensor products, i.e.,*

$$\nabla_X (F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G).$$

(iv) *∇ commutes with all contractions, i.e., if trace acts on any pair of covariant and contravariant indices, then*

$$\nabla_X (\operatorname{tr} F) = \operatorname{tr}(\nabla_X F).$$

In addition, ∇ satisfies:

(a) *∇ obeys the product rule with respect to the natural pairing of a 1-form and a vector field, i.e.,*

$$\nabla_X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y).$$

(b) *For all $F \in \mathcal{T}^{(k,l)}(M)$ with $\omega^1, \ldots, \omega^k \in \Omega^1(M)$, $Y_1, \ldots, Y_l \in \mathfrak{X}(M)$ we have*

$$
\begin{aligned}
(\nabla_X F)(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l) = {} & X(F(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l)) \\
& - \sum_{i=1}^{k} F(\omega^1, \ldots, \nabla_X \omega^i, \ldots, \omega^k, Y_1, \ldots, Y_l) \qquad (3.6) \\
& - \sum_{j=1}^{l} F(\omega^1, \ldots, \omega^k, Y_1, \ldots, \nabla_X Y_j, \ldots, Y_l).
\end{aligned}
$$

PROOF. *Step 1.* (i)–(iv) $\Rightarrow$ (a)–(b). This is an easy computation, since for (a) we have $\omega(Y) = \operatorname{tr}(\omega \otimes Y)$ (prove this in coordinates, where both are $\omega_i Y^i$) and by (i)–(iv)

$$
\begin{aligned}
\nabla_X(\omega(Y)) = \nabla_X(\operatorname{tr}(\omega \otimes Y)) & \overset{(iv)}{=} \operatorname{tr}(\nabla_X(\omega \otimes Y)) \\
& \overset{(iii)}{=} \operatorname{tr}(\nabla_X \omega \otimes Y + \omega \otimes \nabla_X Y) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y),
\end{aligned}
$$

and (b) is obtained by using induction applied to

$$F(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l) = \underbrace{\operatorname{tr} \circ \ldots \circ \operatorname{tr}}_{k+l}(F \otimes \omega^1 \otimes \ldots \otimes \omega^k \otimes Y_1 \otimes \ldots \otimes Y_l).$$

Step 2. Uniqueness. Assume ∇ is a connection satisfying (i)–(iv), then by the above also (a)–(b) holds. For every 1-form ω we obtain

$$(\nabla_X \omega)(Y) \overset{(a)}{=} \nabla_X(\omega(Y)) - \omega(\nabla_X Y) \overset{(ii)}{=} X(\omega(Y)) - \omega(\nabla_X Y), \qquad (3.7)$$

hence the connection on 1-forms is uniquely determined by the original affine connection on TM. Similarly, (b) gives a formula that determines the covariant derivative of an arbitrary tensor field F in terms of the covariant derivatives of vector fields (using (i)) and 1-forms (using (3.7)), thus the connection in every tangent bundle is also unique.

Step 3. Existence. The covariant derivatives on 1-forms is defined by (3.7), then define ∇ on all tensor bundles via (3.6). We first note that $\nabla_X F$ is a tensor field since it is multilinear over $C^\infty(M)$, i.e., for $f \in C^\infty(M)$ and all i and j a computation (cancellation of two terms involving f) yields, for instance,

$$(\nabla_X F)(\omega^1, \ldots, f\omega^i + \widetilde{\omega}^i, \ldots, \omega^k, Y_1, \ldots, Y_l) = \ldots$$
$$= f(\nabla_X F)(\omega^1, \ldots, \omega^i, \ldots, \omega^k, Y_1, \ldots, Y_l) + (\nabla_X F)(\omega^1, \ldots, \widetilde{\omega}^i, \ldots, \omega^k, Y_1, \ldots, Y_l).$$

Thus indeed we obtain a map $\nabla \colon \mathfrak{X}(M) \times \mathcal{T}^{(k,l)}(M) \to \mathcal{T}^{(k,l)}(M)$, and it remains to check that ∇ satisfies the properties (i)–(iii) of a connection from Definition 3.1. Here, $C^\infty(M)$-linearity in X and $\mathbb{R}$-linearity in F follow from (3.6) and (3.7), while the product rule follows from the usual differention of functions by X since

$$X(fg) = X(f)g + fX(g)$$

applied to $F \in \mathcal{T}^{(k_1, l_1)}(M)$ and $G \in \mathcal{T}^{(k_2, l_2)}(M)$ "pointwise" via the smooth functions

$$F \otimes G(\omega^1, \ldots, \omega^{k_1+k_2}, Y_1, \ldots, Y_{l_1+l_2})$$
$$= F(\omega^1, \ldots, \omega^{k_1}, Y_1, \ldots, Y_{l_1}) G(\omega^{k_1+1}, \ldots, \omega^{k_2}, Y_{l_1+1}, \ldots, Y_{l_1+l_2}). \qquad \square$$

In coordinates we obtain the following expressions.

PROBLEM 3.17. Let M be a smooth manifold and ∇ an affine connection on TM. Suppose (E_i) is a local frame for M, (ε^j) its dual coframe, and $\{\Gamma_{ij}^k\}$ the corresponding connection coefficients of ∇. Let $X = X^i E_i$ be a smooth vector field. Show that

(i) the covariant deriviative of a 1-form $\omega = \omega_i \varepsilon^i$ is locally given by

$$\nabla_X \omega = (X(\omega_k) - X^j \omega_i \Gamma_{jk}^i)\varepsilon^k,$$

(ii) if $F \in \mathcal{T}^{(k,l)}(M)$, locally given by $F = F_{j_1 \ldots j_l}^{i_1 \ldots i_k} E_{i_1} \otimes \ldots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \varepsilon^{j_l}$, then the covariant derivative of F is locally given by

$$\nabla_X F = \left(X(F_{j_1 \ldots j_l}^{i_1 \ldots i_k}) + \sum_{s=1}^{k} X^m F_{j_1 \ldots j_l}^{i_1 \ldots p \ldots i_k} \Gamma_{mp}^{i_s} - \sum_{s=1}^{l} X^m F_{j_1 \ldots p \ldots j_l}^{i_1 \ldots i_k} \Gamma_{m j_s}^{p} \right) E_{i_1} \otimes \ldots \otimes \varepsilon^{j_l}. \quad (3.8)$$

The covariant derivative $\nabla_X F$ of a (k,l)-tensor field F is $C^\infty(M)$-linear in X and (as can be shown) $C^\infty(M)$-linear in all its $k + l$ arguments, hence by the Tensor Characterization Lemma $\nabla_X F$ can be seen as a $(k, l+1)$-tensor field on M. This gives rise to the following definition.

DEFINITION 3.18. Let M be a smooth manifold and let ∇ be an affine connection in TM. Let $F \in \mathcal{T}^{(k,l)}M$. Then the *total covariant derivative of F* is $\nabla F \in \mathcal{T}^{(k,l+1)}(M)$ given by

$$(\nabla F)(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l, X) := (\nabla_X F)(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l). \qquad (3.9)$$

In a local frame (E_i) we denote the total covariant derivative with ; (semicolon): For instance, for a vector field $Y = Y^i E_i$ the $(1,1)$-tensor field ∇Y reads

$$\nabla Y = Y^i{}_{;j} E_i \otimes \varepsilon^j, \qquad \text{with } Y^i{}_{;j} = E_j Y^i + Y^k \Gamma_{jk}^i,$$

following (3.8). Similarly, for 1-form ω we obtain

$$\nabla \omega = \omega_{i;j} \varepsilon^i \otimes \varepsilon^j, \qquad \text{with } \omega_{i;j} = E_j \omega_i - \omega_k \Gamma_{ji}^k.$$

EXERCISE 3.19. Let M be a smooth manifold and let ∇ be an affine connection on TM. Suppose (E_i) is a smooth local frame for TM and $\{\Gamma^k_{ij}\}$ are the corresponding connection coefficients. Show that the components of the total covariant deriviative of a (k, l)-tensor field F with respect to this frame are given by

$$F^{i_1...i_k}_{j_1...j_l;m} = E_m \left(F^{i_1...i_k}_{j_1...j_l} \right) + \sum_{i=1}^{k} F^{i_1...p...i_k}_{j_1...j_l} \Gamma^{i_s}_{mp} - \sum_{i=1}^{l} F^{i_1...i_k}_{j_1...p...i_l} \Gamma^{p}_{mj_s}.$$

3.4. Levi-Civita connection

Everything that has been said so far (and much more [**36**, Ch. 4]) holds for any connection in any vector bundle on any manifold. Since we want to use the covariant derivative (and parallel transport) to study Riemannian manifolds, it is important to pick a connection that goes well with its special structure. In this section we single out connections by requiring two additional properties, namely *compatibility with the metric* and *symmetry* (also called *torsion-freeness*). Both of these properties are inspired by properties of the tangential connection $\nabla^\top$ for submanifolds of $\mathbb{R}^n$ (see Example 3.9). In fact, it turns out that these two additional properties already determine a unique affine connection on a Riemannian manifold (and, similarly, on a semi-Riemannian manifold), called the *Levi-Civita connection*, named after the Italian geometer Tullio Levi-Civita who defined it in 1917 (though several other mathematics such as Christoffel, Brouwer, Schouten and Weyl were also involved and obtained similar results independently).

3.4.1. Metric connections. The Euclidean connection $\overline{\nabla}$ on $\mathbb{E}^n$ satisfies

$$\overline{\nabla}_X \langle Y, Z \rangle = \langle \overline{\nabla}_X Y, Z \rangle + \langle Y, \overline{\nabla}_X Z \rangle, \tag{3.10}$$

as well as the tangential connection $\nabla^\top$ on an embedded submanifold of $\mathbb{E}^n$ (same in the semi-Riemannian setting).

EXERCISE 3.20. Prove (3.10) by computing in terms of the standard basis (the left hand side is simply $X \langle Y, Z \rangle$ by Prop. 3.16 (ii)).

EXERCISE 3.21. Show that the tangential connection on an embedded submanifold of $\mathbb{E}^n$ satisfies the product rule (3.10) (with $\overline{\nabla}$ replaced by $\nabla^\top$, and with respect to the induced metric and tangential vector fields).

This concept can be extended to abstract manifolds in the same way.

DEFINITION 3.22. Let (M, g) be a Riemannian manifold. An affine connection ∇ on TM is said to be *compatible with g* (or a *metric connection*) if for all $X, Y, Z \in \mathfrak{X}(M)$ we have the product rule[1]

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle. \tag{3.11}$$

This condition can be characterized as follows.

PROPOSITION 3.23 (Characterization of metric connections). *Let (M, g) be a Riemannian manifold and ∇ an affine connection on M. Then ∇ is compatible with g (as in Def. 3.22) if and only if g is parallel with respect to ∇, i.e., $\nabla g \equiv 0$.*

---

[1]Recall that, if g is fixed, we often simply write $\langle ., . \rangle$ in place of $g(., .)$.

PROOF. By (3.9) and (3.6) the total covariant derivative of the symmetric 2-tensor g is given by

$$(\nabla g)(Y, Z, X) \stackrel{(3.9)}{=} (\nabla_X g)(Y, Z) \stackrel{(3.6)}{=} X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z).$$

This expression vanishes for all X, Y, Z if and only if (3.11) holds. $\square$

One can show that the condition of metric compatibility is *not* sufficient to fix a unique connection on a Riemannian manifold, since any other affine connection whose difference tensor is in some variables antisymmetric is also compatible to g [**36**, Prob. 5-1]. In fact, the space of metric connections on a Riemannian manifold (of dimension at least 2) is an infinite-dimensional affine space. This is why we have to include more properties of the tangential connection.

3.4.2. Symmetric connections. On Euclidean space $\mathbb{E}^n$ the Lie bracket[2] $[X, Y]$ of two smooth vector fields X, Y can be written in the standard coordinates as

$$[X, Y] = X(Y^i)\partial_i - Y(X^i)\partial_i \stackrel{(3.4)}{=} \overline{\nabla}_X Y - \overline{\nabla}_Y X.$$

Due to the coordinate-independence of the expression in terms of the Euclidean connection, this property can be defined and studied for general affine connections. Importantly, we do *not* need a Riemannian metric for this concept (in contrast to metric connections).

DEFINITION 3.24. Let M be a smooth manifold and let ∇ be a connection on M. The map $\tau \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, defined by

$$\tau(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y], \tag{3.12}$$

is called the *torsion tensor of* ∇.

We say that ∇ is *symmetric* (or torsion-free) if $\tau \equiv 0$.

PROBLEM 3.25. Let M be a smooth manifold and let ∇ be a connection on M, and let τ be given by (3.12).
   (i) Show that τ is a $(1, 2)$-tensor field (thus justifying the name torsion tensor).
   (ii) Show that ∇ is symmetric if and only if $\Gamma_{ij}^k = \Gamma_{ji}^k$ with respect to every *coordinate* frame (not necessarily with respect to other frames!).
   (iii) Show that ∇ is symmetric if and only if the *covariant Hessian* $\nabla^2 u := \nabla(du)$ (this definition is inspired by the fact that $\nabla u(X) = \nabla_X u = X(u) = du(X)$) of every $u \in C^\infty(M)$ is a symmetric 2-tensor field. (See [**36**, Ex. 4.22] for a formula of the covariant Hessian.)

Not only is the Euclidean connection $\overline{\nabla}$ symmetric, also the tangential connections $\nabla^\top$ of submanifolds embedded in $\mathbb{R}^n$ is.

EXERCISE 3.26. Let M be an embedded submanifold of Euclidean space. Then the tangential connection $\nabla^\top$ is symmetric. (*Hint*: Extend the vector fields X, Y to the ambient space as $\widetilde{X}, \widetilde{Y}$, and use that the Lie bracket is natural [**36**, Prop. A.39]; in particular $[\widetilde{X}, \widetilde{Y}]$ is tangent to M and restricted to M equals $[X, Y]$.)

---

[2] Given two vector fields $X, Y \in \mathfrak{X}(M)$ the map $[X, Y] \colon C^\infty(M) \to C^\infty(M)$, defined by $[X, Y](f) := X(Y(f)) - Y(X(f))$, is a derivation an thus $[X, Y] \in \mathfrak{X}(M)$, called the *Lie bracket of X and Y*. Conceptually, the Lie bracket $[X, Y]$ is a sort commutator of vector fields, and also the derivative of Y along the flow generated by X, and the *Lie derivative $\mathcal{L}_X Y$ of Y along X*.

3.4.3. Fundamental Theorem of Riemannian Geometry. We have started out this section by proposing that on Riemannian manifolds we want to mimick the properties of the tangential connection $\nabla^\top$ of embedded submanifolds. We have found that $\nabla^\top$ is both compatible with the induced Riemannian metric as well as symmetric. The following result states that both of these properties combined already uniquely determine a connection on a Riemannian manifold.

THEOREM 3.27 (Fundamental Theorem of Riemannian Geometry). *Let (M, g) be a Riemannian manifold. There exists a unique affine connection ∇, called the* Levi-Civita *connection, that in addition to (i)–(iii) of Definition 3.1 satisfies*

(iv) ∇ *is symmetric: for $X, Y \in \mathfrak{X}(M)$,*

$$[X, Y] = \nabla_X Y - \nabla_Y X.$$

(v) ∇ *is compatible with g: for $X, Y, Z \in \mathfrak{X}(M)$;*

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle.$$

Moreover, the Levi-Civita connection ∇ satisfies the Koszul formula, *that is,*

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \big(X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle$$
$$- \langle Y, [X, Z] \rangle - \langle Z, [Y, X] \rangle + \langle X, [Z, Y] \rangle \big), \tag{3.13}$$

for smooth vector fields $X, Y, Z \in \mathfrak{X}(M)$.

PROOF. *Step 1. Uniqueness and Koszul formula.* We prove uniqueness by deriving an explicit formula for ∇. For $X, Y, Z \in \mathfrak{X}(M)$ we have

$$X \langle Y, Z \rangle \overset{\text{(v)}}{=} \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \overset{\text{(iv)}}{=} \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_Z X \rangle + \langle Y, [X, Z] \rangle.$$

Cyclic permuation and adding up/substracting implies

$$X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_Z X \rangle + \langle Y, [X, Z] \rangle$$
$$+ \langle \nabla_Y Z, X \rangle + \langle Z, \nabla_X Y \rangle + \langle Z, [Y, X] \rangle$$
$$- \langle \nabla_Z X, Y \rangle - \langle X, \nabla_Y Z \rangle - \langle X, [Z, Y] \rangle$$
$$= 2 \langle \nabla_X Y, Z \rangle + \langle Y, [X, Z] \rangle + \langle Z, [Y, X] \rangle - \langle X, [Z, Y] \rangle.$$

Hence we can solve for $\langle \nabla_X Y, Z \rangle$, and obtain the Koszul formula

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \left(X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle - \langle Y, [X, Z] \rangle - \langle Z, [Y, X] \rangle + \langle X, [Z, Y] \rangle \right),$$

with a right hand side that does not depend on the connection ∇. Hence for any two connections ∇^1, ∇^2 satisfying (i)–(v), and all vector fields $X, Y, Z \in \mathfrak{X}(M)$, we have

$$\langle \nabla^1_X Y - \nabla^2_X Y, Z \rangle = 0,$$

thus (by injectivity of the musical isomorphism $\flat$) $\nabla^1_X Y = \nabla^2_X Y$ and hence $\nabla^1 = \nabla^2$ is uniquely determined.

Step 2. Existence. Let $F(X, Y, Z)$ be the right hand side of (3.13), i.e.,

$$F(X, Y, Z) := \frac{1}{2} \big(X \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle - \langle Y, [X, Z] \rangle - \langle Z, [Y, X] \rangle + \langle X, [Z, Y] \rangle \big).$$

For fixed $X, Y \in \mathfrak{X}(M)$ the map $Z \mapsto F(X, Y, Z)$ is $C^\infty(M)$-linear (check!), hence an element $\omega_{(X,Y)} \in \Omega^1(M)$. By the musical isomorphism $\sharp$ there exists a unique metrically equivalent

vector field $\omega^{\sharp}_{(X,Y)} \in \mathfrak{X}(M)$, i.e., $\langle \omega^{\sharp}_{(X,Y)}, Z \rangle = F(X, Y, Z)$ for all $Z \in \mathfrak{X}(M)$. We denote $\nabla_X Y := \omega^{\sharp}_{(X,Y)}$ and check that it satisfies all desired properties (i)–(v) of a symmetric metric connection. $\square$

PROBLEM 3.28. Finish the proof of Theorem 3.27, that is, prove that $Z \mapsto F(X, Y, Z)$ is a 1-form and that the expression $\nabla_X Y$ defined in Step 2 of the proof indeed satisfies all properties (i)–(v) from Definition 3.1 and Theorem 3.27.

Whenever we are in the setting of Riemannian Geometry (thus, almost always in this course), we assume that ∇ is the Levi-Civita connection of the Riemannian metric g. As desired, we have managed to single out the natural connections in Euclidean space.

EXAMPLE 3.29 (Euclidean connection). The Levi-Civita connection on the Euclidean space $\mathbb{E}^n$ is the Euclidean connection $\overline{\nabla}$.

EXAMPLE 3.30 (Tangential connection). Let M be an embedded submanifold of $\mathbb{E}^n$. By Exercise 3.26 the tangential connection $\nabla^{\top}$ (as defined in Example 3.9) is symmetric, and by Exercise 3.21 it is compatible with the induced metric. Hence, due to uniqueness, $\nabla^{\top}$ is the Levi-Civita connection on M.

Similar to the local computations necessary for the tangential connection in Exercise 3.9 we can use Lemma 3.4 (and [36, Prop. 4.2]) to restrict ∇ to a (coordinate) neighborhood U (of p) and obtain a well-defined element $\nabla_X Y \in \mathfrak{X}(U)$. The connection coefficients of the Levi-Civita connection (which, by default, is the connection we use on a Riemannian manifold) in coordinates have a particular name.

DEFINITION 3.31. Let $(U, (x^i))$ be a coordinate chart of a Riemannian manifold (M, g). The *Christoffel symbols of g* with respect to this chart are the smooth functions $\Gamma^k_{ij} \colon U \to \mathbb{R}$ such that (with coordinate vector fields $\partial_i = \frac{\partial}{\partial x^i}$)

$$\nabla_{\partial_i} \partial_j =: \Gamma^k_{ij} \partial_k. \tag{3.14}$$

Since, the Lie bracket vanishes for coordinate vector fields, i.e.,

$$[\partial_i, \partial_j] = 0,$$

it immediately follows from the symmetric condition (iv) (see also Problem 3.25) that

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

Recall, however, that ∇ is not a tensor field and so the Christoffel symbols do not transform in the usual way. Nonetheless, they are easy compute via the metric coefficients

$$g_{ij} = \langle \partial_i, \partial_j \rangle$$

and their inverses.

COROLLARY 3.32. *Let $(U, (x^i))$ be a coordinate chart on a Riemannian manifold (M, g). The Christoffel symbols are given by*

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \tag{3.15}$$

PROOF. By the Koszul formula (3.13), taking into account that $[\partial_i, \partial_j] = 0$, we obtain

$$\langle \nabla_{\partial_i} \partial_j, \partial_l \rangle = \frac{1}{2}(\partial_i \langle \partial_j, \partial_l \rangle + \partial_j \langle \partial_l, \partial_i \rangle - \partial_l \langle \partial_i, \partial_j \rangle).$$

In terms of the coefficients this reads

$$\Gamma_{ij}^m g_{ml} = \frac{1}{2}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}),$$

and multiplying both formulas with the inverse matrix g^{kl}, that is, $g_{ml}g^{kl} = \delta_l^k$, yields (3.15). $\square$

Of course, all general formulas that we have derived about affine connections and their coefficients still hold. For instance, we can can compute the Levi-Civita covariant derivative $\nabla_X Y$ of Y in direction X via the local formula (3.3). Similarly, the Levi-Civita connection in TM extends to a unique *tensor covariant derivative* ∇_X in $T^{(k,l)}TM$ via $\nabla_X f = X(f)$ etc. (as shown in Proposition 3.16) and allows us to define a *total covariant differential* ∇ *of a tensor field* (via Definition 3.18).

An important consequence of the coordinate-independent definition of the Levi-Civita connection is that it respects isometries (via pullbacks).

PROPOSITION 3.33 (Naturality of the Levi-Civita connection). *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian (or semi-Riemannian) manifolds, and let ∇ and $\widetilde{\nabla}$ denote the Levi-Civita connection of g and $\widetilde{g}$, respectively. If $\varphi \colon M \to \widetilde{M}$ is an isometry, then $\varphi^* \widetilde{\nabla} = \nabla$, where $\varphi^* \widetilde{\nabla}$ is the pullback of $\widetilde{\nabla}$ by φ defined by*

$$(\varphi^* \widetilde{\nabla})_X Y := (\varphi^{-1})_* (\widetilde{\nabla}_{\varphi_* X}(\varphi_* Y)).$$

The proof is straight-forward and can be found in [**36**, Prop. 5.13] based on the earlier result [**36**, Lem. 4.37] that the pullback connection is indeed a connection.

3.5. Parallel transport

The basic idea behind using connections is that it provides us with a tool to "parallel transport" vectors, meaning that we want to effortlessly move a tangent vector from one point to another along curves on a manifold. This idea will be crucial for the definition of curvature, and we will explore it in detail now. Note that one can study parallel transport with respect to any connection (even in vector bundles) but towards the end of this section we will already see that on Riemannian manifolds there is more to gain.

3.5.1. Parallel vector and tensor fields. Before we discuss how to transport something parallel along a curve, we first want to understand what it means for a vector field (or tensor field) to be parallel on all of M. Afterwards we study this notion on curves.

DEFINITION 3.34. Let M be a smooth manifold with affine connection ∇. A vector field X on M is called *parallel* if $\nabla_Y X = 0$ for all $Y \in \mathfrak{X}(M)$.

EXAMPLE 3.35. The coordinate vector fields on $\mathbb{R}^n$ are parallel (with respect to the Euclidean connection) because for any $Y = Y^j \partial_j$ we obtain $\overline{\nabla}_Y \partial_i = Y^j \overline{\nabla}_{\partial_j} \partial_i = 0$. In fact, precisely all constant vector fields are parallel on $\mathbb{R}^n$. (The same result holds for semi-Euclidean space $\mathbb{R}^{\nu, n-\nu}$.)

PROBLEM 3.36. Suppose G is a Lie group.

(i) Show that there is a unique connection[3] ∇ in TG with the property that every left-invariant vector field is parallel.

(ii) Show that the torsion tensor of ∇ is zero if and only if G is abelian.

The following result shows that one can use the total covariant derivative ∇ of Definition 3.18 to characterize parallel vector and tensor fields. In fact, we have already mentioned this when characterizing metric connections in Proposition 3.23.

PROPOSITION 3.37. *Let M be a smooth manifold with affine connection ∇. Then a smooth vector (or tensor) field A is parallel on M if and only if $\nabla A \equiv 0$.*

PROBLEM 3.38. Prove Propostion 3.37.

3.5.2. Vector and tensor fields along curves. Our initial motivation for introducing connections was that we want to make sense of the derivative of a vector field along a curve, and we want to study vector fields that are parallel along curves. Thus we first have to say what we even mean by a vector field along a curve, and how to use the covariant derivative in this context. We first pin down what we mean by such a vector field.

DEFINITION 3.39. Let M be a smooth manifold and let $\gamma\colon I \to M$ be a smooth curve. A *smooth[4] vector field along γ* is a smooth map $V\colon I \to TM$ such that $V(t) \in T_{\gamma(t)}M$.
By $\mathfrak{X}(\gamma)$ we denote the set of all smooth vector fields along γ.

The set $\mathfrak{X}(\gamma)$ is real vector space with respect to pointwise addition scalar multiplication. In fact, it is a module over $C^\infty(M)$ with respect to pointwise multiplication, i.e., $(fX)(t) := f(t)X(t)$.

REMARK 3.40 (Vector fields along functions). Apart from curves, on can also consider vector fields along functions between arbitrary smooth manifolds M and N. A *vector field along $f \in C^\infty(N, M)$* is a smooth map $Z\colon N \to TM$ such that $\pi \circ Z = f$, where $\pi\colon TM \to M$ is the natural projection. Similarly, we write $\mathfrak{X}(f)$ for vector fields along f, which also is a $C^\infty(N)$-module. Vector fields along curves appear as the special case where N is an interval in $\mathbb{R}$.

EXAMPLE 3.41 (Velocity field of a curve). If γ is a smooth curve on M, then $\gamma'(t) \in T_{\gamma(t)}M$ and smooth because locally the *velocity* $\gamma'(t)$ is of the form

$$\gamma'(t) = \dot{\gamma}^1(t)\partial_1|_{\gamma(t)} + \ldots + \dot{\gamma}^n(t)\partial_n|_{\gamma(t)}, \tag{3.16}$$

hence $\gamma' \in \mathfrak{X}(\gamma)$.

EXAMPLE 3.42 (Normal field of a curve). If γ is a curve in $\mathbb{R}^2$, consider $N(t) := R\gamma'(t)$, where R is the counterclockwise rotation by $\frac{\pi}{2}$, i.e., $N(t) = (-\dot{\gamma}^2(t), \dot{\gamma}^1(t))$. Thus N is smooth and hence $N \in \mathfrak{X}(\gamma)$.

EXAMPLE 3.43 (Restriction of vector field to curve). If $\gamma\colon I \to M$ is a smooth curve on M and $\widetilde{V}$ a smooth vector field defined on an open subset containing $\gamma(I)$. Then the restriction $V := \widetilde{V} \circ \gamma \in \mathfrak{X}(\gamma)$.

---

[3]This is usually *not* the Levi-Civita connection.
[4]Often continuity is sufficient, but we will restrict ourselves to the smooth setting.

While every vector field can be restricted to a smooth curve, not every vector field V along a curve can be extended to a vector field $\widetilde{V}$ in a neighborhood of the curve (as seen in Figure 3.1, this is clearly impossible if there exists $\gamma(t_1) = \gamma(t_2)$ but $\gamma'(t_1) \neq \gamma'(t_2)$). This prompts the following definition.

DEFINITION 3.44. A smooth vector field V along a curve γ on M is *extendible* if there exists a smooth vector field $\widetilde{V}$ on a neighoorhood of $\gamma(I)$ such that $V = \widetilde{V} \circ \gamma$.

Tensor fields along curves, and their extendibility, are defined in the same way.
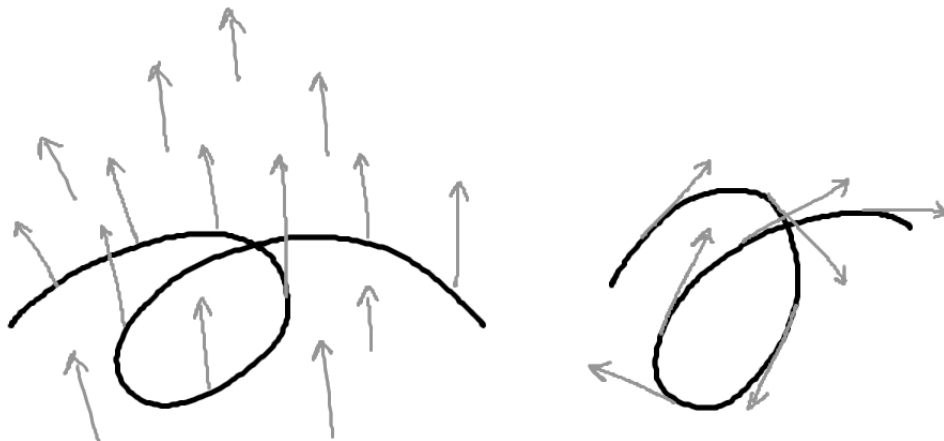

Figure 3.1. A restricted/extendible and nonextendible vector field along a curve.

PROBLEM 3.45. Consider the *figure eight curve* $\gamma\colon (-\pi, \pi) \to \mathbb{R}^2$, defined by

$$\gamma(t) := (\sin 2t, \sin t).$$

Show that γ is an injective smooth immersion, but that the velocity vector field γ' is not extendible.

3.5.3. Covariant derivative along curves. The following result gives a bit more insight into why and how an affine connection gives rise to a derivative of vector fields along curves, thereby also giving meaning to the *acceleration* of a curve in M (relevant in Chapter 4). The same result holds also for tensor fields along curves.

THEOREM 3.46. *Let M be a smooth manifold with an affine connection ∇. Then each smooth curve $\gamma\colon I \to M$ gives rise to a unique operator*

$$D_t\colon \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma),$$

called the covariant derivative along γ, *satisfying the following properties for $V, W \in \mathfrak{X}(\gamma)$, $a, b \in \mathbb{R}$, and $f \in C^\infty(I)$:*

  (i) linearity over $\mathbb{R}$*: $D_t(aV + bW) = aD_t(V) + bD_t(W)$,*
  (ii) product rule*: $D_t(fV) = f'V + fD_tV$,*
  (iii) *If $V \in \mathfrak{X}(\gamma)$ is extendible, then for every extension $\widetilde{V}$ of V,*

$$D_tV = \nabla_{\gamma'(t)}\widetilde{V}.$$

REMARK 3.47. Note that property (iii) makes sense because $\nabla_X Y(p)$ only depends on the value of X_p and the value Y along a curve tangent to X at p.

PROOF. *Step 1. Uniqueness.* Suppose D_t is such an operator satisfying (i)–(iii). Let $t_0 \in I$ arbitrary. Similar to the proof of Lemma 3.4 one can see that the value $D_t V|_{t_0}$ only depends on the value of V in a small interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ (if t_0 is an endpoint extend the coordinate representation of γ beyond t_0, show the result, and restrict back to I). For smooth coordinates (x^i) on M in a neighborhood of $\gamma(t_0)$ we can write

$$V(t) = V^j(t)\partial_j|_{\gamma(t)}$$

for t near t_0. By the properties of D_t and ∇ we have

$$
\begin{aligned}
D_t V(t) &\overset{\text{(ii),(iii)}}{=} \dot{V}^j(t)\partial_j|_{\gamma(t)} + V^j(t)\nabla_{\gamma'(t)}\partial_j|_{\gamma(t)} \\
&\overset{(3.2),(3.16)}{=} \left(\dot{V}^k(t) + \dot{\gamma}^i(t)V^j(t)\Gamma_{ij}^k(\gamma(t)) \right)\partial_k|_{\gamma(t)},
\end{aligned}
\tag{3.17}
$$

and hence uniqueness (if it exists).

Step 2. Existence. If $\gamma(I)$ is contained in a single coordinate chart, simply *define* D_t by (3.17) and check that (i)–(iii) hold. In the general case we can cover $\gamma(I)$ by coordinate charts and check that the definitions of $D_t V$ agree when the charts overlap. $\qquad\square$

EXERCISE 3.48. Complete the proof of Theorem 3.46 by showing that D_t as defined in (3.17) indeed satisfies properties (i)–(iii).

COROLLARY 3.49. *Suppose γ is a smooth curve on a Riemannian manifold (M, g) and D_t the induced covariant derivative of Theorem 3.46. Then, in addition to (i)–(iii), D_t also satisfies*

(iv) $\frac{d}{dt}\langle V, W \rangle = \langle D_t V, W \rangle + \langle V, D_t W \rangle$.

PROBLEM 3.50. Prove Corollary 3.49. (*Hint*: Use (3.17) and the compatibility of the Levi-Civita connection with g.)

3.5.4. Parallel transport along curves. The idea of parallel transport is that of moving around tangent vectors along a curve in an effortless way. The notion is made precise by restricting the initial Definition 3.34 of parallel vector fields to a curve with help of the induced covariant derivative D_t.

DEFINITION 3.51. Let M be a smooth manifold with affine connection ∇. A smooth vector field V along a smooth curve γ is said to be *parallel along γ* (with respect to ∇) if $D_t V \equiv 0$.

EXERCISE 3.52. Let $\gamma \colon I \to \mathbb{R}^n$ be a smooth curve, and let V be a smooth vector field along γ. Show that V is parallel along γ (with respect to the Euclidean connection $\overline{\nabla}$) if and only if its component functions (with respect to the standard basis) are constants.

In (3.17) we have derived a local formula for $D_t V$ in terms of the connection coefficients Γ_{ij}^k and the coefficients V^j of V. Based on this local differential identity $D_t V(t) = 0$ we can prove that we can indeed parallel transport vectors along curves.
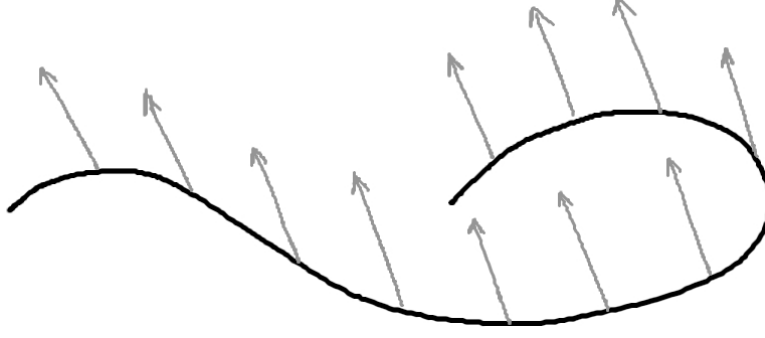
Figure 3.2. A vector field that is parallel along a curve in $\mathbb{R}^2$.

THEOREM 3.53 (Existence and uniqueness of parallel fields). *Suppose $\gamma\colon I \to M$ is a smooth curve on a smooth manifold M with affine connection ∇ (and induced covariant derivative D_t along γ). If $t_0 \in I$ and $v \in T_{\gamma(t_0)}M$, then there is a unique parallel vector field $V \in \mathfrak{X}(\gamma)$, called the* parallel transport *of v along γ, defined on all of I with $V(t_0) = v$.*

PROOF. In local coordinates V has to satisfy (3.17)= 0, i.e.,

$$D_t V(t) = \left(\dot{V}^k(t) + \dot{\gamma}^i(t)V^j(t)\Gamma_{ij}^k(\gamma(t)) \right) \partial_k|_{\gamma(t)} = 0.$$

This is a system of first-order linear ordinary differential equations for the coefficients $V^1, \ldots, V^n$, which possesses a unique solution on the entire interval for given initial data $V^1(t_0), \ldots, V^n(t_0)$ (see [**36**, Thm. 4.31] to recall the required result from the theory of ordinary differential equations). The claim thus follows by covering $\gamma(I)$ with chart domains. $\square$

Theorem 3.53 gives rise to the following definition.

DEFINITION 3.54. Let M be a smooth manifold with affine connection ∇. Let $\gamma\colon I \to M$ be a smooth curve. For each $t_0, t_1 \in I$ the map

$$P^\gamma = P_{t_0 t_1}^\gamma \colon T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M,$$
$$v \mapsto V(t_1),$$

where V is the unique vector field obtained in Theorem 3.53, is called the *parallel transport map*.

It is straightforward to extend this result and definition to admissible curves $\gamma\colon [a, b] \to M$ and thus piecewise smooth vector fields $V \in \mathfrak{X}(\gamma)$: In [**36**, Cor. 4.33] it is shown that Theorem 3.53 immediately yields a parallel transport along piecewise smooth curves, the only difference being that the vector field obtained is continuous everywhere but smooth only where γ is.

In fact, one can obtain a *parallel frame along γ* in the following way: By transporting any basis $(b_1, \ldots, b_n)$ for $T_{\gamma(t_0)}M$ along γ, one obtains an n-tuple of parallel vector fields $(E_1, \ldots, E_n)$ along γ. By Proposition 3.55 below these vector fields again form a basis $(E_i(t))$ for $T_{\gamma(t)}M$ at each point $\gamma(t)$. Then each vector field $V \in \mathfrak{X}(\gamma)$ in such a frame reads $V(t) = V^i(t)E_i(t)$, and since the E_i's are parallel it is easy to compute the covariant derivative along γ, i.e.,

$$D_t V(t) = \dot{V}^i(t)E_i(t), \tag{3.18}$$

where γ and V are smooth.

PROPOSITION 3.55. *Let M be a smooth manifold with affine connection ∇. Suppose $\gamma\colon [a,b] \to M$ is a smooth curve on M. Then the parallel transport $P^\gamma\colon T_{\gamma(a)}M \to T_{\gamma(b)}M$ is an invertible linear map.*

COROLLARY 3.56. *If (M,g) is a Riemannian manifold, then P^γ is a linear isometry.*

Corollary 3.56 implies that any orthonormal basis at a point on γ can be extended to a *parallel orthonormal frame along γ* (see Figure 3.3 and [**36**, Prop. 5.5]).
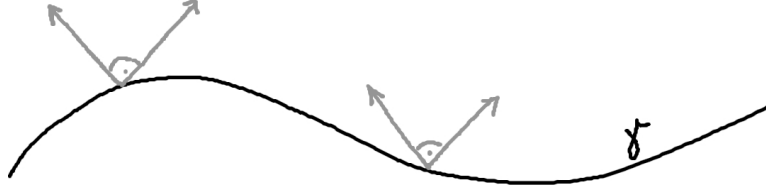

Figure 3.3. A parallel orthonormal frame along a curve in $\mathbb{R}^2$.

PROBLEM 3.57.   (i) Prove Proposition 3.55. (*Hint*: Consider $v_1, v_2 \in T_pM$ and corresponding parallel vector fields $V_1, V_2 \in \mathfrak{X}(\gamma)$. Then $V_1 + V_2$ is also parallel and $P^\gamma(v_1 + v_2) = P^\gamma(v_1) + P^\gamma(v_2)$ by unique solvability of the ordinary differential equation. Analogously for av_1, $a \in \mathbb{R}$. Show that P^γ is injective and use that $\dim T_pM = \dim T_qM$ to argue that it is bijective.)
   (ii) Prove Corollary 3.56. (*Hint*: Show, in addition, that $\langle P^\gamma(v_1), P^\gamma(v_2)\rangle = \langle v_1, v_2\rangle$.)

We have now seen that an affine connection really determines a way of differentiation and a notion for parallel transporting vectors along curves, which really "connects" nearby tangent spaces. It is natural to ask whether parallel transport also uniquely determines the covariant derivative along a curve and ultimately the connection. This is indeed the case.

THEOREM 3.58 (Parallel transport determines covariant differentiation). *Let M be a smooth manifold with affine connection ∇. Suppose $\gamma\colon I \to M$ is a smooth curve and $V \in \mathfrak{X}(\gamma)$. Then*

$$D_t V(t_0) := \lim_{t_1 \to t_0} \frac{P^\gamma_{t_1 t_0} V(t_1) - V(t_0)}{t_1 - t_0}, \qquad t_0 \in I. \tag{3.19}$$

PROOF. If (E_i) is a parallel frame along γ, then $V = V^j E_j$ and by (3.18) thus, on the one hand, $D_t V(t_0) = \dot{V}^i(t_0) E_i(t_0)$.

On the other hand, the parallel vector field of $V(t_1)$ along γ is the constant coefficient field $V^i(t_1)E_i(t)$, and thus $P^\gamma_{t_0 t_1} V(t_1) = V^i(t_1)E_i(t_0)$. Taking the limit yields also $\lim_{t_1 \to t_0} \frac{P^\gamma_{t_1 t_0} V(t_1) - V(t_0)}{t_1 - t_0} = \dot{V}^i(t_0)E_i(t_0)$. Hence both expressions on the left and right of (3.19) are equal. $\qquad\square$

By restricting a vector field Y to γ, Theorem 3.46(iii) implies that if $\gamma(0) = p$ and $\gamma'(0) = X_p$ then $\nabla_X Y|_p = D_t Y(\gamma(0))$. Hence Theorem 3.53 immediately implies a result for the connection everywhere.

COROLLARY 3.59 (Parallel transport detMermines the connection). *Let M be a smooth manifold with affine connection ∇. Suppose $X, Y \in \mathfrak{X}(M)$. Then, for every $p \in M$,*

$$\nabla_X Y|_p = \lim_{h \to 0} \frac{P_{h0}^\gamma Y_{\gamma(h)} - Y_p}{h}, \tag{3.20}$$

where $\gamma \colon I \to M$ is a smooth curve through $\gamma(0) = p$ with $\gamma'(0) = X_p$. $\qquad\square$

EXERCISE 3.60. In Definition 3.34 we said that a vector field X is parallel if $\nabla_Y X = 0$ for all $Y \in \mathfrak{X}(M)$. Show that this is equivalent to X being parallel along every smooth curve in M. (*Remark*: This is how Lee [**36**, p. 110] defines parallel vector fields.)

We mostly deal with parallel vector fields, however, also parallel differential forms naturally occur as the following exercise shows.

EXERCISE 3.61. Let (M, g) be an oriented Riemannian manifold. The Riemannian volume form dV_g is parallel with respect to the Levi-Civita connection. (Use Exercise 3.60 and choose a parallel oriented orthonormal frame along each curve.)

It is crucial to note that while it is *always* possible to extend a vector at a point on a curve to a parallel vector field along it by Theorem 3.53, it is generally *not* possible to extend it to a parallel vector field on an open subset of a point. In fact, the impossibility of this is intimately tied to curvature, as we will soon see.

One immediate consequence of the failure of a connection "to commute", even without defining curvature explictely, can be seen by studying *admissible loops*, i.e., piecewise smooth curves $\gamma \colon [a, b] \to M$ with $\gamma(a) = \gamma(b) = p$. This is based on the fact that the corresponding parallel transport defines an invertible linear map on $T_p M$ via Proposition 3.55, i.e., $P^\gamma \in \mathrm{GL}(T_p M)$.

DEFINITION 3.62. Let M be a smooth manifold. The *holonomy group* of an affine connection ∇ at a point $p \in M$ is defined by

$$\mathrm{Hol}(p) := \{P^\gamma \in \mathrm{GL}(T_p M); \gamma \text{ is an admissible loop at } p\}.$$

The *restricted holonomy group* $\mathrm{Hol}^0(p)$ based at p is the subgroup of $\mathrm{Hol}(p)$ coming from contractible loops γ, i.e., loops that are path-homotopic to the constant loop.

PROBLEM 3.63. Let (M, g) be a connected Riemannian manifold and $p \in M$.
  (i) Show that $\mathrm{Hol}(p)$ is a subgroup of $\mathrm{O}(T_p M)$ (the set of all linear isometries on $T_p M$).
  (ii) Show that $\mathrm{Hol}^0(p)$ is a normal subgroup of $\mathrm{Hol}(p)$.
  (iii) Show that the holonomy group depends on the base point only up to conjugation with an element in $\mathrm{GL}(\mathbb{R}^n)$. (*Hint:* Use that M is connected and parallel transport along an admissible path γ from p to q, i.e., explicitly $\mathrm{Hol}(q) = P^\gamma \mathrm{Hol}(p)(P^\gamma)^{-1}$. With this understanding one often drops the base point and just writes $\mathrm{Hol}(M)$.)
  (iv) Show that M is orientable if and only if $\mathrm{Hol}(p) \subseteq \mathrm{SO}(T_p M)$ (the set of all isometries with determinant $+1$) for some $p \in M$.
  (v) Show that g is flat (locally isometric to Euclidean space) if and only if $\mathrm{Hol}^0(p)$ is the trivial group for some $p \in M$.

REMARK 3.64 (Berger's list). In 1955, Marcel Berger classified all possible holonomy groups for simply connected n-dimensional Riemannian manifolds which are irreducible (not locally a product space) and nonsymmetric (not locally a Riemannian symmetric space). There are only 7 of those: $SO(n)$, $U(\frac{n}{2})$, $SU(\frac{n}{2})$, $Sp(\frac{n}{4})$, $Sp(\frac{n}{4})Sp(1)$, G_2, and $Spin(7)$. Several of those are Ricci flat. See Berger's own summary in [**7**, Sec. 13.4] for more background. Riemannian manifolds with special holonomy play an important role in string theory compactifications, for instance, Calabi–Yau manifolds.

Holonomy groups are important in all kinds of settings, also in connection with vector bundles. The Ambrose–Singer Theorem from 1953, for instance, relates the holonomy of a connection in a principal bundle (equipped with a group action) with the curvature form of the connection.

CHAPTER 4

Geodesics

By equipping a smooth manifold with an affine connection we obtained a tool to parallel transport vectors along curves, from one tangent space to another. We have seen that on Riemannian manifolds equipped with the canonical Levi-Civita connection parallel transport is actually an isometry.

Given that the velocity vector of a curve $\gamma\colon I \to M$ is a special vector field along this curve (recall Example 3.41) it is natural to ask when this velocity vector field γ' is itself parallel along γ. Such curves with "zero acceleration" are called *geodesics*. In principle, one can study them on any manifold with affine connection, but on Riemannian manifolds (M, g) they provide us with a fundamental tool to understand the underlying geometry. For instance, geodesics are also "locally length-minimizing" curves with respect to the induced Riemannian distance d_g (recall Definition 2.74). Strictly speaking, we will only now prove that d_g indeed gives rise to a metric space structure on M.

Essentially all of the local behavior of geodesics is encoded in the *exponential map* (with corresponding special coordinates and neighborhoods), which provides us with a powerful theoretical as well as computational tool. It is the aim of this section to rigorously introduce and analyze geodesics and all their related standard concepts.

4.1. Geodesics

4.1.1. Basic definitions and existence. We have already defined the velocity of a curve γ in Example 3.41. By covariant differentiation along γ we can also define the acceleration.

DEFINITION 4.1. Let M be a smooth manifold with affine connection ∇ and let $\gamma\colon I \to M$ be a smooth curve. The *acceleration of γ* is the vector field $D_t\gamma'$ along γ.

Curves where the velocity γ' is parallel along γ have a special name.

DEFINITION 4.2. Let M be a smooth manifold with affine connection ∇. A smooth curve $\gamma\colon I \to M$ is called *geodesic* (with respect to ∇) if its acceleration is zero, i.e., $D_t\gamma' \equiv 0$.

EXERCISE 4.3. Let (M, g) be a Riemannian manifold. Show that for a geodesic γ the length of the tangent vector $|\gamma'(t)|$ is constant.

In smooth coordinates $(x^1, \ldots, x^n)$, if we write the component functions of γ (by abuse of notation[1]) as $\gamma(t) = x(t) = (x^1(t), \ldots, x^n(t))$, it follows immediately from (3.17) that γ is a geodesic if and only if the component functions satisfy the *geodesic equation*

$$\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma_{ij}^k(x(t)) = 0, \qquad 1 \le k \le n. \tag{4.1}$$

---

[1]More carefully one should actually write $(x \circ \gamma)(t) = ((x^1 \circ \gamma)(t), \ldots, (x^n \circ \gamma)(t))$ or $= (\gamma^1(t), \ldots, \gamma^n(t))$.

This is a system of n second-order ordinary differential equations and the standard Picard–Lindelöf Theorem from ODE theory implies existence and uniqueness of solutions to the following initial value problem (this result is more involved for manifolds with boundary).

THEOREM 4.4 (Existence and uniqueness of geodesics). *Let M be a smooth manifold with affine connection ∇. For every $p \in M$, $w \in T_pM$, and $t_0 \in \mathbb{R}$, there exists an open interval $I \subseteq \mathbb{R}$ containing t_0 and a geodesic $\gamma \colon I \to M$ satisfying $\gamma(t_0) = p$ and $\gamma'(t_0) = w$. Any two such geodesics agree on their common domain.*

The proof follows a standard argument, where the only additional step needed is the rewriting of (4.1) as a system of $2n$ first-order equations in local charts.

PROOF. *Step 1. Local existence and uniqueness.* Let (x^i) be smooth coordinates on some neighborhood U of p. A smooth curve $\gamma(t) = (x^1(t), \ldots, x^n(t))$ in U is a geodesics if and only if it satisfies the geodesic equation (4.1). By introducing auxiliary variables $v^i = \dot{x}^i$ the n second-order equations (4.1) can be written as a system of $2n$ first-order equations,

$$\dot{x}^k(t) = v^k(t), \tag{4.2}$$

$$\dot{v}^k(t) = -v^i(t)v^j(t)\Gamma^k_{ij}(x(t)), \tag{4.3}$$

for the variables $(x^1, \ldots, x^n, v^1, \ldots, v^n)$ on $U \times \mathbb{R}^n$. In other words, the equations (4.2)–(4.3) are the equations of flow for the vector field $G \in \mathfrak{X}(U \times \mathbb{R}^n)$ given by

$$G_{(x,v)} = v^k \frac{\partial}{\partial x^k}\bigg|_{(x,v)} - v^i(t)v^j(t)\Gamma^k_{ij}(x(t))\frac{\partial}{\partial v^k}\bigg|_{(x,v)}. \tag{4.4}$$

By the fundamental theorem of flows [**35**, Thm. 9.12], for each $(p, w) \in U \times \mathbb{R}^n$ and $t_0 \in \mathbb{R}$, there exists an open interval I_0 containing t_0 and a unique smooth solution $\zeta \colon I_0 \to U \times \mathbb{R}^n$, $\zeta(t) = (x^i(t), v^i(t))$, to this system satisfying the initial conditions $\zeta(t_0) = (p, w)$. The curve $\gamma(t) := (x^1(t), \ldots, x^n(t))$ in U is thus the desired solution.

Step 2. Global uniqueness. If there are two geodesics $\gamma, \tilde{\gamma} \colon I \to M$ with the same initial conditions $\gamma(t_0) = \tilde{\gamma}(t_0)$ and $\gamma'(t_0) = \tilde{\gamma}'(t_0)$, then due to the uniqueness of ODE solutions, they agree on an interval $(t_0 - \varepsilon, t_0 + \varepsilon)$. Suppose

$$\beta := \inf\{b \in I \cap [t_0, \infty); \gamma(b) \neq \tilde{\gamma}(b)\}.$$

Clearly, $\beta > t_0$, and by continuity, $\gamma(\beta) = \tilde{\gamma}(\beta)$ as well as $\gamma'(\beta) = \tilde{\gamma}'(\beta)$. Thus by local uniqueness we must have equality also in a neighborhood of β, a contradiction to it being the infimum. $\square$

DEFINITION 4.5. Let M be a smooth manifold with affine connection ∇. A geodesic $\gamma \colon I \to M$ is called *maximal* if there exists no geodesic $\tilde{\gamma} \colon \tilde{I} \to M$ with $I \subsetneq \tilde{I}$ and $\tilde{\gamma}|_I = \gamma$.

A *geodesic segment* is a geodesic with compact domain.

DEFINITION 4.6. A smooth manifold with affine connection is called *geodesically complete* if each maximal geodesic is defined on all of $\mathbb{R}$.

Theorem 4.4 immediately implies the existence of maximal geodesics.

COROLLARY 4.7. *Let M be a smooth manifold with affine connection ∇. For each $p \in M$ and $v \in T_pM$, there exists a unique maximal geodesic $\gamma \colon I \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$, defined on some open interval I containing 0.* $\square$

The unique maximal geodesic of Corollary 4.7 is often called the *geodesic with initial point p and initial velocity v*, and denoted by γ_v. The point p is omitted because it can be recovered from the natural projection $\pi\colon TM \to M$ via $p = \pi(v)$.

EXERCISE 4.8 (Geodesics in $\mathbb{R}^n$). Show that the maximal geodesics on $\mathbb{R}^n$ with the Euclidean connection $\overline{\nabla}$ (see Example 3.6) are exactly the constant curves and the straight lines with constant-speed parametrizations. See Figure 4.1 (Same in $\mathbb{R}^{\nu,n-\nu}$.)
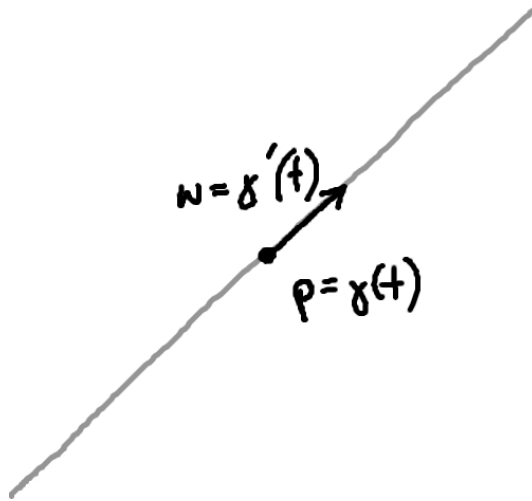
Figure 4.1. The maximal geodesics in $\mathbb{R}^n$ are straight lines with constant speed w.

EXERCISE 4.9 (Geodesics on a cylinder). Let M be cylinder in $\mathbb{R}^3$ with radius 1 and consider the chart

$$(\cos\theta, \sin\theta, z) \mapsto (\theta, z), \qquad \theta \in (0, 2\pi), z \in \mathbb{R}.$$

Show that the geodesics on the cylinder are generally helixes, with special cases circles of latitute and generating straight lines. (*Hint*: Either compute and solve the geodesic equation directly in these coordinates, or "unwinde" M in $\mathbb{R}^2$ via an isometry then use the solutions of Exercise 4.8).

PROBLEM 4.10. See [**36**, Problem 5-4] about geodesics on surfaces of revolution.

PROBLEM 4.11. Suppose (M_1, g_1) and (M_2, g_2) are Riemannian manifolds
   (i) Prove that if $M_1 \times M_2$ is endowed with the product metric, then a curve $\gamma\colon I \to M_1 \times M_2$ of the form $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is a geodesic if and only if γ_i is a geodesic in (M_i, g_i) for $i = 1, 2$.
   (ii) Suppose $f\colon M_1 \to \mathbb{R}^*$ is a strictly positive smooth function, and $M_1 \times_f M_2$ is the resulting warped product. Let $\gamma_1\colon I \to M_1$ be a smooth curve and q_0 a point in M_2, and define $\gamma\colon I \to M_1 \times_f M_2$ by $\gamma(t) := (\gamma_1(t), q_0)$. Prove that γ is a geodesic with respect to the warped product metric if and only if γ_1 is a geodesic with respect to g_1.

4.1.2. Geodesic flow. By using the projection $\pi \colon TM \to M$ we can reformulate the approach via the vector field G used in the proof of Theorem 4.4 in a more invariant way. The importance of G is that it actually defines a *global* vector field on TM.

THEOREM 4.12. *Let M be a smooth manifold with affine connection ∇. Then there is a unique vector field $G \in \mathfrak{X}(TM)$, called the* geodesic vector field, *with the property that the projection $\pi \colon TM \to M$ provides a one-to-one correspondence between (maximal) integral curves of G and (maximal) geodesics on M.*

PROOF. *Step 1. Existence and properties of G in a chart.* Let (x^i) be any smooth local coordinates on an open set $U \subseteq M$, and let (x^i, v^i) be the associated natural coordinates on $\pi^{-1}(U)$, that is, by following the local trivialization $\Phi \colon U \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ with $(p, (v^1, \ldots, v^n)) \mapsto (x^i(p), v^i(p))$. Thus G as defined in (4.4) is a smooth vector field on $\pi^{-1}(U) \subseteq TM$, and the integral curves $\eta(t) = (x^i(t), v^i(t))$ of G satisfy (4.2)–(4.3), which was equivalent to the geodesic equation under the substitution $v^k = \dot{x}^k$ (as seen in Theorem 4.4). In other words: Every integral curve of G on $\pi^{-1}(U)$ projects to a geodesic under $\pi \colon TM \to M$ (just $\pi(x, v) = x$); conversely, every geodesic $\gamma(t) = (x^1(t), \ldots, x^n(t))$ in U lifts to an integral curve of G in $\pi^{-1}(U)$ by setting $v^i(t) = \dot{x}^i(t)$.

Step 2. Global existence, properties, and uniqueness of G. To obtain a global vector field G we can simply patch together the chart versions of Step 1. That this is possible rests on the fact that G can be shown to be coordinate independent. One (tedious) way to prove this is to show that G behaves well where the charts overlap. Instead, we show that G acts on a function $f \in C^\infty(TM)$ by

$$Gf(p, v) = \frac{d}{dt}\bigg|_{t=0} f(\gamma_v(t), \gamma_v'(t)). \qquad (4.5)$$

Due to the coordinate independence of this formula, the various coordinate dependend definitions of G given by (4.4) must then agree (and G is unique).

To prove (4.5) we use the usual notation for the components $x^i(t)$ of a geodesic γ_v and its velocity as $v^i(t) = \dot{x}^i(t)$. The chain rule and geodesic equation (4.2)–(4.3) allow us to rewrite the right hand side of (4.5) as

$$\frac{d}{dt}\bigg|_{t=0} f(\gamma_v(t), \gamma_v'(t)) = \left(\frac{\partial f}{\partial x^k}(x(t), v(t))\dot{x}^k(t) + \frac{\partial f}{\partial v^k}(x(t), v(t))\dot{v}^k(t) \right)\bigg|_{t=0}$$

$$= \frac{\partial f}{\partial x^k}(p, v)v^k - \frac{\partial f}{\partial v^k}(p, v)v^i v^j \Gamma_{ij}^k(p) = Gf(p, v).$$

By the fundamental theorem of flows [**35**, Thm. 9.12] there exists an open neighborhood $\mathcal{D} \subseteq \mathbb{R} \times TM$ containing $\{0\} \times TM$ and a smooth map $\theta \colon \mathcal{D} \to TM$ such that each curve

$$\theta^{(p,v)}(t) = \theta(t, (p, v))$$

is the unique maximal integral curve of G starting at (p, v), defined on the open interval containing 0. By the local description (4.4) of G and Theorem 4.4 these are precisely the (maximal) geodesics. □

Based on the properties of the pullback connection in Proposition 3.33, together with the fact that being a geodesic is a local property one can directly show the following.

PROPOSITION 4.13 (Naturality of geodesics). *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds, and $\varphi \colon M \to \widetilde{M}$ is a local isometry. If γ is a geodesic in M, then $\varphi \circ \gamma$ is a geodesics in $\widetilde{M}$.* □

4.1.3. Geodesics on submanifolds. To compute geodesics of a submanifold M embedded in $\mathbb{R}^n$ one uses the covariant derivative along a curve $\gamma \colon I \to M$ with respect to the tangential connection $\nabla^\top$, which by a standard calculation in [**36**, Prop. 5.1] is shown to be given by

$$D_t^\top V(t) = \pi^\top(\overline{D}_t V(t)). \tag{4.6}$$

The geodesics on M then satisfy the following constraints.

PROPOSITION 4.14. *Suppose $M \subseteq \mathbb{R}^n$ is an embedded submanifold. A smooth curve $\gamma \colon I \to \mathbb{R}^n$ is a geodesic with respect to the tangential connection $\nabla^\top$ on M if and only if its ordinary acceleration $\gamma''(t)$ is orthogonal to $T_{\gamma(t)} M$ for all $t \in I$.*

PROOF. The Christoffel symbols of the Euclidean connection are all zero. Thus by (3.17)

$$\overline{D}_t \gamma'(t) = \gamma''(t).$$

The orthogonality of $\gamma''(t)$ to $T_{\gamma(t)} M$ follows from (4.6) together with the condition $D_t^\top \gamma'(t) \equiv 0$ for a geodesic. □

Recall that the model spaces $\mathbb{E}^n$, $\mathbb{S}^n(R)$, and $\mathbb{H}^n(R)$ are highly symmetric. While it is possible to compute their geodesics via the geodesic equation, it is more insightful to use their symmetries and Proposition 4.14 for embedded submanifolds. We will give a short overview here. More details can be found in [**36**, p. 136–145].

EXAMPLE 4.15 (Euclidean space). We have seen that the Euclidean connection $\overline{\nabla}$ is the Levi–Civita connection of Euclidean space (Example 3.29). Therefore, constant-coefficient vector fields are parallel (Example 3.35), and the Euclidean geodesics are straight lines with constant-speed parametrizations (Exercise 4.8). Every Euclidean space $\mathbb{E}^n$ is geodesically complete.

EXAMPLE 4.16 (Spheres). Using Proposition 4.14 the geodesics on a sphere can be obtained from its normal space. One can show that a nonconstant curve on $\mathbb{S}^n(R)$ is a maximal geodesics if and only if it is a periodic constant-speed curve whose image is a *great circle*, that is, a subset of the form $\mathbb{S}^n(R) \cap \Pi$, where $\Pi \subseteq \mathbb{R}^{n+1}$ is a 2-dimensional linear subspace:

Firstly, one shows that for any $p \in \mathbb{S}^n(R)$ the set of vectors orthogonal to p is exactly the tangent space $T_p \mathbb{S}^n(R)$ (by checking for the defining function $f(x) = |x|^2 = R^2$ of the sphere that $df_p(v) = 2\langle v, p \rangle = 0$).

Secondly, for any nonzero $v \in T_p \mathbb{S}^n(R)$ one considers the smooth curve $\gamma \colon \mathbb{R} \to \mathbb{R}^{n+1}$, with $a = |v|/R$, defined by

$$\gamma(t) := (\cos at)p + (\sin at)\frac{v}{a},$$

and checks that $\gamma(t) \in \mathbb{S}^n(R)$ and that $\gamma''(t)$ is proportional to $\gamma(t)$. Hence it is $\bar{g}$-orthogonal to $T_{\gamma(t)} \mathbb{S}^n(R)$ and thus by Proposition 4.14 a geodesic in $\mathbb{S}^n(R)$, namely γ_v. The image of γ_v is the great circle formed by $\Pi = \operatorname{span}\{p, \frac{v}{a}\}$ with period $\frac{2\pi}{a}$ and constant speed (check!).

Conversely, if Π is given by an orthonormal basis $\{v, w\}$, then a great circle C is the image of a geodesic with initial point $p = Rw$ and initial velocity v.

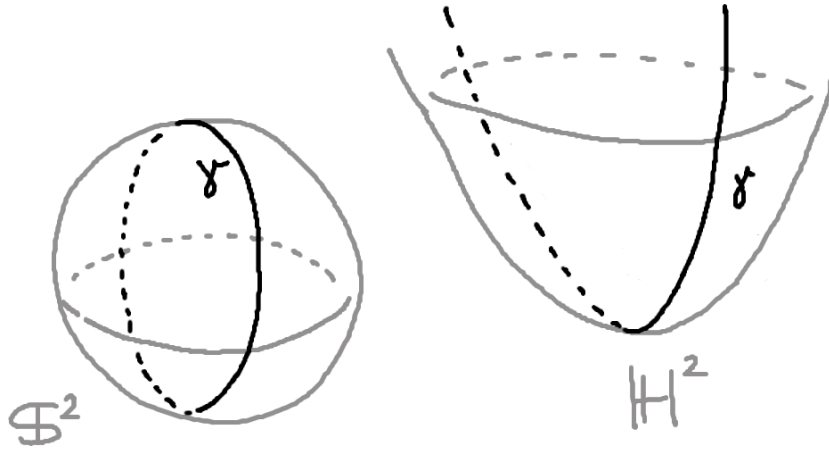It follows, in particular, that every sphere is geodesically complete.

Figure 4.2. The images of maximal geodesics of $\mathbb{S}^n$ and $\mathbb{H}^n$ are great circles and great hyperbolas, respectively, and can be obtained by intersecting 2-dimensional linear subspaces with $\mathbb{S}^n$ and $\mathbb{H}^n$ (see Examples 4.16 and 4.17).

EXAMPLE 4.17 (Hyperbolic spaces). The geodesics on $\mathbb{H}^n(R)$ can be obtained in the same way using the hyperboloidal model. The intersection of $\mathbb{H}^n(R)$ with a 2-dimensional linear subspace of $\mathbb{R}^{1,n}$ is called a *great hyperbola*. One can show that the geodesics are of the form

$$\gamma(t) = (\cosh at)p + (\sinh at)\frac{v}{a}.$$

For this computation and to see the behavior of geodesics also in the other models of hyperbolic space, see [**36**, p. 138–142].

EXERCISE 4.18. Prove the claim of Example 4.17. More precisely, show that the maximal geodesics of $\mathbb{H}^n(R)$ are the constant-speed embeddings of $\mathbb{R}$ whose image is a great hyperbola. (*Hint*: Use Proposition 4.14, following essentially the proof used in Example 4.16 but with defining function $f(x) = \bar{\eta}(x,x) = -R^2$ given by the Minkowski metric $\bar{\eta}$ and grad $f = 2p \in \mathbb{R}^{1,n}$.)

4.2. Exponential map

Since we want to understand the behavior of a Riemannian manifold locally around a point it is not enough to know about the behavior geodesics in certain directions, but it is important to study the collective behavior of all geodesics emanating from this point in *all* possible directions. This information is conveniently encoded in the exponential map[2]. Note that everything in this section works verbatim for semi-Riemannian manifolds and, in fact, even for any other affine connection.

Since γ' is parallely propagated along a geodesic γ, and parallel transport P^γ is an isometry by Corollary 3.56, geodesics always have constant velocity $|\gamma'(t)| = |\gamma'(t_0)|$. Therefore, only linear reparametrizations are allowed without loosing the geodesic property.

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[2]Note that, a priori, this exponential map (denoted by exp) has nothing to with the exponential map of a Lie group G (denoted by $\exp^G$), however, they do agree for Lie groups with bi-invariant metrics.

LEMMA 4.19 (Rescaling Lemma). *Let M be a smooth manifold with affine connection ∇. For every $p \in M$, $v \in T_pM$ and $c, t \in \mathbb{R}$ we have that*

$$\gamma_{cv}(t) = \gamma_v(ct), \tag{4.7}$$

whenever both sides are defined.

PROOF. If $c = 0$ both sides define the constant curve p, so assume that $c \neq 0$. It suffices to show that $\gamma_{cv}(t)$ exists and (4.7) holds whenever the right hand side is defined. (The other way round follows in the same way.)

Suppose $\gamma = \gamma_v$ with maximal domain $I \subseteq \mathbb{R}$. Define a new curve

$$\tilde{\gamma} \colon c^{-1}I \to M,$$
$$t \mapsto \tilde{\gamma}(t) := \gamma(ct).$$

Clearly, $\tilde{\gamma}(0) = \gamma(0) = p$ and by the chain rule $\tilde{\gamma}'(t) = c\gamma'(ct)$. In particular, $\tilde{\gamma}'(0) = c\gamma'(t) = cv$. Hence initial point and velocity of both sides of (4.7) are equal. It remains to show that $\tilde{\gamma}$ is a geodesic, then the statement follows by uniqueness and maximality of geodesics (Corollary 4.7). It follows by the chain and product rule of Theorem 3.46 (and the fact that $(ct)'' = 0$) that

$$\widetilde{D}_t\tilde{\gamma}'(t) = \left(\frac{d}{dt}\dot{\tilde{\gamma}}^k(t) + \Gamma_{ij}^k(\tilde{\gamma}(t))\dot{\tilde{\gamma}}^i(t)\dot{\tilde{\gamma}}^j(t) \right) \partial_k$$
$$= \left(c^2\ddot{\gamma}^k(ct) + c^2\Gamma_{ij}^k(\gamma(ct))\dot{\gamma}^i(ct)\dot{\gamma}^j(ct) \right) \partial_k = c^2 D_t\gamma'(ct) = 0,$$

hence $\tilde{\gamma}$ is a geodesic, and so $\tilde{\gamma} = \gamma_{cv}$. $\qquad\square$

The Rescaling Lemma allows us define a map $v \mapsto \gamma_v$ in a sensible way.

DEFINITION 4.20. Let M be a smooth manifold with affine connection. The *domain of the exponential map* is defined by

$$\mathcal{E} := \{v \in TM; \gamma_v \text{ is defined on } [0, 1]\},$$

and the *exponential map* on M by

$$\exp \colon \mathcal{E} \to M,$$
$$v \mapsto \exp(v) := \gamma_v(1).$$

For each $p \in M$ the *exponential map of M at p* is the restriction

$$\exp_p \colon \mathcal{E}_p := \mathcal{E} \cap T_pM \to M.$$

Clearly, by definition, the domain $\mathcal{E}$ is maximal. In general, exp is only *locally defined* because it is based on the local result about the existence and uniqueness of solutions to ordinary differential equations. If M is geodesically complete, however, then $\mathcal{E} = TM$ and exp well-defined globally.

By the Rescaling Lemma it follows immediately that

$$\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t).$$

In other words, the straight lines $t \mapsto tv$ through the origin in T_pM are mapped by $\exp_p$ to geodesics through p on M (see Figure 4.3).
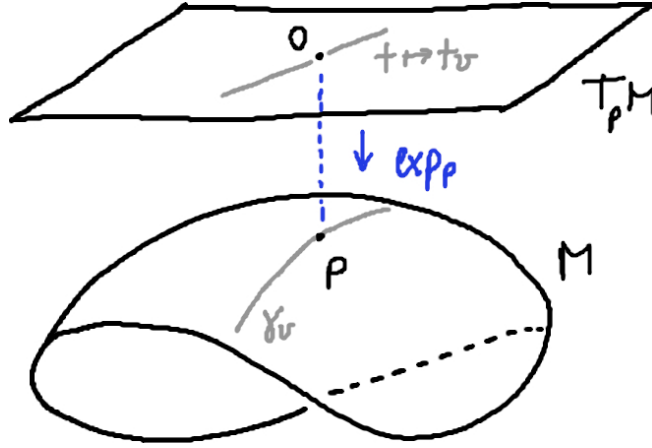
Many more nice properties hold.

Figure 4.3. The exponential map at a point p maps a straight line $t \mapsto tv$ in T_pM to the geodesic γ_v in M.

PROPOSITION 4.21 (Properties of the exponential map). *Let* (M, g) *be a Riemannian manifold and* $\exp \colon \mathcal{E} \to M$ *the exponential map. The following hold:*

(i) *$\mathcal{E}$ is an open subset of TM containing the image of the zero section[3] TM_0, and the set $\mathcal{E}_p \subseteq T_pM$ is star-shaped[4] with respect to 0.*

(ii) *For each $v \in TM$, the geodesic γ_v is given by*

$$\gamma_v(t) = \exp(tv)$$

for all t such that either side is defined.

(iii) *The exponential map is smooth.*

(iv) *For each point $p \in M$, the differential*

$$d(\exp_p)_0 \colon T_0(T_pM) \cong T_pM \to T_pM$$

is the identity map of T_pM.

PROOF. Suppose $n = \dim M$.

(ii) As discussed above, this follows immediately from the Rescaling Lemma.

(i) We first prove that $\mathcal{E}_p$ is star-shaped. If $v \in \mathcal{E}_p$, then by definition of γ_v it is defined on at least $[0, 1]$, and thus by the rescaling lemma

$$\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$$

is defined for $t \in [0, 1]$. Hence $\mathcal{E}_p$ is star-shaped.

To prove that $\mathcal{E}$ is open we use the geodesic vector field $G \in \mathfrak{X}(TM)$ from Theorem 4.12. By the fundamental theorem of flows [**35**, Thm. 9.12] there exists an open set $\mathcal{D} \subseteq \mathbb{R} \times TM$ containing $\{0\} \times TM$ and a unique smooth maximal flow $\theta \colon \mathcal{D} \to TM$ whose infinitesimal generator is G, i.e., each curve $t \mapsto \theta(t, (p, v))$ is the unique maximal integral curve of G starting at (p, v), defined on an open interval containing 0.

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[3]The *zero section* of a vector bundle is the submanifold of the bundle that consists of all the zero vectors, i.e., $TM_0 := \{0_p; p \in M\}$.

[4]A subset S of a vector space is called *star-shaped around* 0 if $v \in S$ and $t \in [0, 1]$ imply that $tv \in S$. It follows that every convex set is star-shaped.

Suppose $(p, v) \in \mathcal{E}$. Thus the geodesic γ_v is defined at least on $[0, 1]$, and so is the integral curve of G starting at $(p, v) \in TM$ by Theorem 4.12. Hence $(1, (p, v)) \in \mathcal{D}$, and therefore there exists a neighborhood of $(1, (p, v))$ in $\mathbb{R} \times TM$ on which the flow of G is defined. In particular, there is a neighborhood of (p, v) on which the flow exists for $t \in [0, 1]$, on which therefore also the exponential map is defined. Hence $\mathcal{E}$ is open.

(iii) By Theorem 4.12 the geodesics are projections of the integral curves of G. Hence the exponential map can be expressed as

$$\exp_p(v) = \gamma_v(1) = \pi \circ \theta(1, (p, v)),$$

wherever it is defined. Since π and θ are smooth, $\exp_p(v)$ depends smoothly on (p, v).

(iv) For any $v \in T_0(T_pM) \cong T_pM$ chose a curve τ in T_pM starting at 0 with initial velocity $v = \tau'(0) = d_0\tau$. For convenience we use the curve $\tau(t) = tv$. Then

$$d(\exp_p)_0(v) = \frac{d}{dt}\Big|_{t=0} (\exp_p \circ \tau)(t) = \frac{d}{dt}\Big|_{t=0} \exp_p(tv) = \frac{d}{dt}\Big|_{t=0} \gamma_v(t) = v.$$

Thus $d(\exp_p)_0 = \mathrm{Id}_{T_pM}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Properties (iii) and (iv) together with the inverse function theorem (see, for instance, [**35**, Thm. 4.5]) thus immediately imply the following.

THEOREM 4.22. *The exponential map $\exp_p$ at every point $p \in M$ is a local diffeomorphism, that is, there exist a neighborhood V of 0 in T_pM and a neighborhood U of p in M such that $\exp_p\colon V \to U$ is a diffeomorphism.* $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

EXAMPLE 4.23 (Exponential map for $\mathbb{E}^n$). If $v \in T_p\mathbb{R}^n \cong \mathbb{R}^n$, then the geodesic through p with initial velocity v is $t \mapsto p + tv$. Hence

$$\exp_p(v) = p + v,$$

which is a global diffeomorphism (even an isometry). The same holds for $\mathbb{R}^{\nu, n-\nu}$.
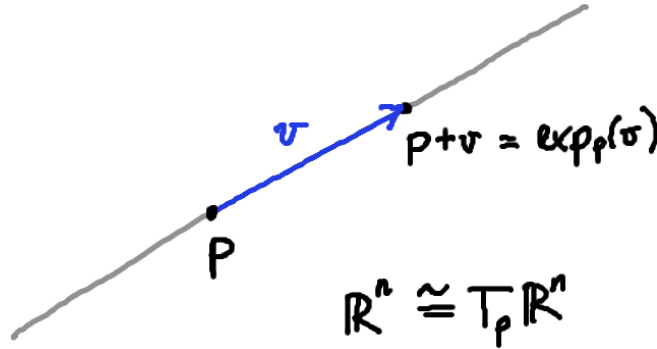

Figure 4.4. The exponential map at a point p in $\mathbb{R}^n$ in direction $v \in T_p\mathbb{R}^n \cong \mathbb{R}^n$. It is a global diffeomorphism, i.e., $\mathcal{E}_p = T_p\mathbb{R}^n$.

In this spirit we will later prove the Gauss Lemma 6.3 which essentially states that $\exp_p$ is always a "partial isometry".

EXAMPLE 4.24 (Exponential map for $\mathbb{S}^n$). In Example 4.16 we have seen that the geodesics of $\mathbb{S}^n$ are great circles parametrized proportionally to arc length. Given $p \in \mathbb{S}^n$ and $v \in T_p\mathbb{S}^n$

the point $\exp_p v \in \mathbb{S}^n$ is obtained by running along the geodesic $\gamma_{v/|v|}$ a length equal to $|v|$, starting from p.

It is clear that $\exp_p$ is defined over the entire tangent space and transforms $B_\pi(0)$ injectively into $\mathbb{S}^n \setminus \{q\}$, where q is the antipodal point to p. The boundary $\partial B_\pi(0)$ is transformed to q and the open annulus $B_{2\pi}(0) \setminus \overline{B_\pi(0)}$ is transformed injectively onto $\mathbb{S}^n \setminus \{p, q\}$, and $\partial B_{2\pi}(0)$ collapes to p, etc.
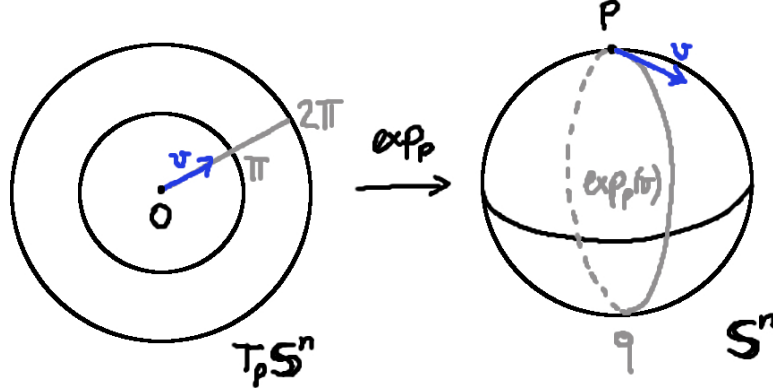

Figure 4.5. The exponential map at a point p in direction $v \in T_p\mathbb{S}^n$ of the sphere. It is a diffeomorphism on $\mathcal{E}_p = B_\pi(0) \subseteq T_p\mathbb{S}^n \cong \mathbb{R}^n$ but no longer even injective once it hits the antipodal point q of p.

If, instead, we consider the Riemannian manifold $\mathbb{S}^n \setminus \{q\}$, $\exp_p$ is defined only on $B_\pi(0) \subsetneqq T_p(\mathbb{S}^n \setminus \{q\})$.

REMARK 4.25. Theorem 4.22 can easily be extended to the full exponential map. One can think of points in TM as given by $p \in M$ and $v \in T_pM$ and thus

$$\exp(p, v) = (p, \exp_p(v)).$$

Then one can show (see [**45**, Prop. 5.5.1] or [**43**] for the full details) that for each $p \in M$ and $0 \in T_pM$ the map

$$d\exp \colon T_{(p,0)}(TM) \to T_{(p,p)}(M \times M)$$

is of the form

$$\begin{pmatrix} \mathrm{Id} & 0 \\ * & \mathrm{Id} \end{pmatrix},$$

hence also nonsingular, and by the inverse function theorem the map $\exp$ is therefore a (local) diffeomorphism of an open neighborhood of the zero section TM_0 in TM to an open neighborhood of the diagonal Δ_M in $M \times M$.

Proposition 4.13 on the naturality of geodesics translates to the same property of the exponential map.

PROPOSITION 4.26 (Naturality of the exponential map). *Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds and $\varphi \colon M \to \widetilde{M}$ is a local isometry. Then for every $p \in M$ the*

following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{E}_p & \xrightarrow{\ d\varphi_p\ } & \widetilde{\mathcal{E}}_{\varphi(p)} \\
\downarrow{\scriptstyle \exp_p} & & \downarrow{\scriptstyle \exp_{\varphi(p)}} \\
M & \xrightarrow{\ \varphi\ } & \widetilde{M}.
\end{array}
$$

EXERCISE 4.27. Prove Proposition 4.26.

An important consequence of this result is that local isometries (on connected manifolds) are completely determined by their values and differentials at a single point.

PROPOSITION 4.28. *Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be Riemannian manifolds, and M connected. Suppose $\psi, \varphi \colon M \to \widetilde{M}$ are local isometries such that for some $p \in M$ we have $\varphi(p) = \psi(p)$ and $d\varphi_p = d\psi_p$. Then $\varphi \equiv \psi$.*

PROBLEM 4.29. Prove Proposition 4.28.

We can use Proposition 4.28 to finish up an old problem related to the isometry groups of the modal spaces.

PROBLEM 4.30. Recall the groups $\mathrm{E}(n)$, $\mathrm{O}(n+1)$, and $\mathrm{O}^+(1, n)$ defined in Section 2.4.1 which act isometrically on the model Riemannian manifolds $\mathbb{E}^n$, $\mathbb{S}^n(R)$, and $\mathbb{H}^n(R)$, respectively. Show that, in fact,

$$
\mathrm{Iso}(\mathbb{E}^n) = \mathrm{E}(n),
$$
$$
\mathrm{Iso}(\mathbb{S}^n(R)) = \mathrm{O}(n+1),
$$
$$
\mathrm{Iso}(\mathbb{H}^n(R)) = \mathrm{O}^+(1, n).
$$

4.2.1. Normal neighborhoods and normal coordinates. In Proposition 4.21 we have shown that $\mathcal{E}_p$ is a star-shaped region around 0 in T_pM. Moreover, by Theorem 4.22 $\exp_p$ is a local diffeomorphism. This gives rise to the following definition.

DEFINITION 4.31. Let (M, g) be a Riemannian manifold. A neighborhood U of $p \in M$ is that is the diffeomorphic image under $\exp_p$ of a star-shaped neighborhood of $0 \in T_pM$ is called a *normal neighborhood of p*.

On any normal neighborhood there exist distinguished coordinate systems, which are very helpful for tensor calculations.

DEFINITION 4.32. Let (M, g) be a Riemannian manifold, $p \in M$ and $U = \exp_p(V)$ be a normal neighborhood of p. Suppose (b_i) is an orthonormal basis for T_pM, which determines a basis isomorphism $B \colon \mathbb{R}^n \to T_pM$ via $B(x^1, \ldots, x^n) = x^i b_i$. The *(Riemannian) normal coordinates $(U, \varphi = (x^i))$ centered at p* induced by (b_i) are obtained by combining B with $\exp_p$ to get $\varphi = B^{-1} \circ (\exp_p |_V)^{-1} \colon U \to \mathbb{R}^n$ as

$$
\begin{array}{ccc}
T_pM & \xrightarrow{\ B^{-1}\ } & \mathbb{R}^n \\
{\scriptstyle (\exp_p |_V)^{-1}}\Big\uparrow & \nearrow{\scriptstyle \varphi} & \\
U & &
\end{array}
$$

which assigns to each $q \in U$ the coordinates of $\exp_p^{-1}(q) \in T_p M$ with respect to (b_i), i.e.,

$$(\exp_p |_V)^{-1}(q) = x^i(q)b_i, \qquad q \in U.$$

One can show that the normal coordinate chart associated with a given orthonormal basis is unique and satisfies

$$\partial_i|_p = d\varphi_p^{-1}(\partial_i|_0) = \underbrace{d(\exp_p)_0}_{=\mathrm{Id}_{T_p M}} \circ \underbrace{dB_0}_{=B}(\partial_i|_0) = B(\partial_i|_0) = b_i, \tag{4.8}$$

since $d(\exp_p)_0$ is the identity [**36**, Prop. 5.23]. Moreover, any two normal coordinate charts $(\tilde{x}^j)$ and (x^i) are related by some (constant) orthogonal matrix $(A_i^j) \in O(n)$, i.e.,

$$\tilde{x}^j = A_i^j x^i.$$

PROPOSITION 4.33 (Properties of normal coordinates). *Let (M, g) be a Riemannian manifold, and let $(U, (x^i))$ be any normal coordinate chart centered at $p \in M$.*
   (i) *The coordinates of p are $(0, \ldots, 0)$.*
   (ii) *The components of the metric are $g_{ij}(p) = \delta_{ij}$.*
   (iii) *For every $v = v^i \partial_i|_p \in T_p M$, the geodesic γ_v is represented in normal coordinates by the line*

$$\gamma_v(t) = (tv^1, \ldots, tv^n), \tag{4.9}$$

   as long as t is in some interval I containing 0 with $\gamma_v(I) \subseteq U$.
   (iv) *The Christoffel symbols vanish at p, i.e., $\Gamma_{ij}^k(p) = 0$.*

PROOF. (i) follows from the definition of normal coordinates.
(ii) follows immediately from (4.8) since

$$g_{ij}(p) = \langle \partial_i|_p, \partial_j|_p \rangle = \langle b_i, b_j \rangle = \delta_{ij}.$$

(iii) follows from Proposition 4.21(ii).
(iv) Let $v = v^i \partial_i|_p \in T_p M$. The geodesic equation (4.1) for $\gamma_v(t) = (tv^1, \ldots, tv^n)$ is

$$\Gamma_{ij}^k(\gamma_v(t))v^i v^j = 0.$$

At $t = 0$ we see that for all k the quadratic form $(\Gamma_{ij}^k(p))_{ij}$ is zero. Hence by the polarisation identity $\Gamma_{ij}^k(p)v^i w^j = 0$ for all v, w, and therefore identitically zero. $\square$

PROBLEM 4.34. Suppose (M, g) is a Riemannian manifold and (U, φ) is a smooth coordinate chart on a neighborhood of $p \in M$ such that $\varphi(p) = 0$ and $\varphi(U)$ is star-shaped with respect to 0. Prove that this chart is a normal coordinate chart for g if and only if $g_{ij}(p) = \delta_{ij}$ and $x^i x^j \Gamma_{ij}^k(x) \equiv 0$ is satisfied on U.

PROBLEM 4.35. Suppose (M, g) is a Riemannian manifold, and let div and Δ be the divergence and Laplace operators definde in Section 2.2.3.
   (i) Show that for every vector field $X \in \mathfrak{X}(M)$, $\mathrm{div} \, X$ can be written in terms of the total covariant derivative as[5]

$$\mathrm{div} \, X = \mathrm{tr}_g(\nabla X),$$

---

[5]Based on this result, one can then define the *divergence* of any smooth k-tensor field F by $\mathrm{div} \, F := \mathrm{tr}_g(\nabla F)$, where the trace is taken on the last two indices of the $(k+1)$-tensor field ∇F.

and that if $X = X^i E_i$ in terms of some local frame, then div $X = X^i{}_{;i}$. (*Hint*: Show that it suffices to prove the formulas at the origin in normal coordinates.)

(ii) Show that the Laplace operator acting on a smooth function u can be expressed as

$$\Delta u = \operatorname{tr}_g(\nabla^2 u),$$

and in terms of any local frame,

$$\Delta u = g^{ij} u_{;ij} = u_{;i}{}^i.$$

PROBLEM 4.36. Let (M, g) be a Riemannian or pseudo-Riemannian manifold and $p \in M$. Show that for every orthonormal basis $(b_1, \ldots, b_n)$ for $T_p M$, there is a smooth orthonormal frame (E_i) on a neighborhood of p such that $E_i|_p = b_i$ and $(\nabla E_i)_p = 0$ for each i.

The geodesics starting at p and lying in a normal neighborhood of p have the very simple form (4.9), and a special name.

PROPOSITION 4.37. *Let U be a normal neighborhood of $p \in M$. Then for each $q \in U$ there exists a unique geodesic $\sigma \colon [0, 1] \to U$ connecting p and q, called* radial geodesic. *It satisfies $\sigma'(0) = \exp_p^{-1}(q) \in V$.*

PROBLEM 4.38. Prove Proposition 4.37. (*Hint*: Existence is clear. To prove uniqueness assume that τ is any such geodesic with initial velcity w and use Theorem 4.4 and the properties of the exponential map to obtain $w = v$ and thus $\tau = \sigma$.)

While normal neighborhoods are crucial, we will learn about the even more special *convex neighborhoods* C in Section 4.3.2. These are neighborhoods that are normal neighborhoods for *all* of their points. By Proposition 4.37 we then know that there exists a unique geodesic σ in C between any two points p and q in C. But beware, there may still be other geodesics between p and q that leave C (see Figure 4.6)! This ambiguity will be resolved in Section 4.3 when we bring the Riemannian distance d_g back into play to further reduce the size of these convex neighborhoods.
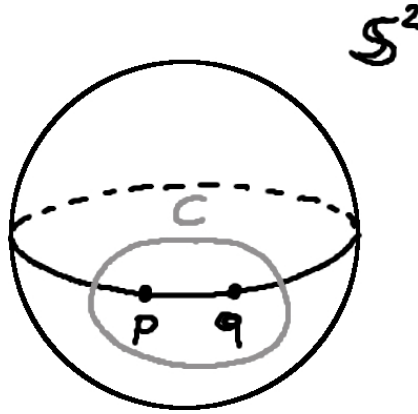

Figure 4.6. Within the convex neighborhood C there is a unique geodesic between p and q along the great circle connecting it. Globally, however, there are two geodesics between p and q (also the long arc).

PROBLEM 4.39. Show that (M, g) is connected if and only if any two points in M can be connected by a broken geodesic, which is a piecewise smooth curve whose smooth parts are geodesics. (*Hint*: One direction is clearly sufficient. Conversely, show that the set $\{p \in M; q$ can be connected to p by a broken geodesic$\}$ is nonempty, open and closed – and hence equal to M.)

REMARK 4.40 (Tubular neighborhoods). One can generalize the construction of normal neighborhoods to embedded submanifolds P of a Riemannian manifold (M, g). If $\mathcal{E}$ is the domain of the exponential map of M, then one calls the restriction of the exponential map to the normal bundle NP, i.e., $E \colon \mathcal{E}_P \to M$ for $\mathcal{E}_P = \mathcal{E} \cap NP$, the *normal exponential map of P in M*. Normal neighborhoods are diffeomorphic images of open subsets $V \subseteq \mathcal{E}_P$ whose intersection with each fiber $N_x P$ is star-shaped with respect to 0. A particularly important type is that of a *tubular neighborhood*, when $V \subseteq \mathcal{E}_P$ has the special form

$$V := \{(x, v) \in NP; |v|_g < \delta(x)\},$$

for some continuous function $\delta \colon P \to (0, \infty)$. If $\delta(x) \equiv \varepsilon$ it is called an *ε-tubular neighborhood*. It is not difficult to prove that every embedded submanifold of M has a tubular neighborhood in M, and that every every compact submanifold has a uniform ε-tubular neighborhood [**36**, Thm. 5.25]. In 1939 Hermann Weyl [**56**] showed (based on earlier work of Harold Hotelling [**31**] for curves) that the volume of such a tubular ε-neighborhood of an embedded compact submanifold P in Euclidean space is an *intrinsic* quantity, more precisely, that it is a polynomial with coefficients that only depend on certain integrals of the intrinsic curvature of P. This result can be generalized to a much larger class of normal neighborhoods whose sections satisfy certain symmetry conditions based on the dimension and codimension of P in M [**16**]. For instance, in the case P is an embedded curve it is sufficient that V in each fiber has the center of mass in the origin, that is, on the curve. For many more insights about tubes and their relation to curvature see the book of Gray [**25**].

4.3. Minimizing curves

There is an important link between geodesics and length-minimizing curves. In this Section we will show that every length-minimizing curve is a geodesic and that, almost conversely, every geodesic is locally length-minizmizing.

Recall that the definition of an admissible curve $\gamma \colon [a, b] \to M$ as a continuous and piecewise smooth curve, and its length on a connected Riemannian manifold (M, g) given by

$$L_g(\gamma) := \int_a^b |\gamma'(t)|_g dt$$

as introduced in Section 2.3.

DEFINITION 4.41. Let (M, g) be a Riemannian manifold. An admissible curve γ in M is said to be a *minimizing curve* if $L_g(\gamma) \leq L_g(\widetilde{\gamma})$ for every admissible curve $\widetilde{\gamma}$ with the same endpoints.

From the definition of the Riemannian distance d_g it follows that a curve γ between two points $p, q \in M$ is minimizing if and only if $d_g(p, q) = L_g(\gamma)$.

4.3.1. Minimizing curves are geodesics. Our first goal is to prove that every minimizing curve is a geodesic. The best approach to this problem is via the *calculus of variations* although for our purposes we will not (and cannot, in this short time) introduce this theory in full generality[6]. The idea is to consider an admissible class of functions (in our case the admissible curves between given end points) together with a functional that should be minimized or maximized (in this case the length functional). Analogous to multivariable calculus on then uses differentiation to compute extremals and show convexity/concavity properties. In this context, the admissible class of functions is infinite-dimensional, and so instead of derivatives one has to use the *first variation* to compute potential extrema (this translates then to a set of ordinary or partial differential equations, called the Euler–Lagrange equation) and the *second variation* or other arguments to conclude that the critical point obtained in the first step is not a saddle point but indeed a minimum or maximum. In our situation the Euler–Lagrange equation turns out to be the geodesic equation. Many important equations in geometry and almost all equations in physics arise from the same variational approach (think of the Einstein equations, the Yamabe equation, or the minimal surface equation).

Since the first variation requires a local computation, it is sufficient to restrict to the following one-parameter family of curves. Thanks to the exponential map we can restrict to a smooth family.

DEFINITION 4.42. Let M be a smooth manifold and let $I, J \subseteq \mathbb{R}$ be intervals. A (continuous) one-parameter family $\Gamma \colon J \times I \to M$ is called an *admissible family of curves* if

(i) the domain is of the form $J \times [a, b]$ for J an open interval,
(ii) there is a partition $(a_0, \ldots, a_k)$ of $[a, b]$ such that Γ is smooth on every $J \times [a_{i-1}, a_i]$ (called *admissible partition*), and
(iii) $\Gamma_s(t) := \Gamma(s, t)$ is an admissible curve for every $s \in J$.

Such a family defines two collections of curves in M (see Figure 4.7): the *main curves* $\Gamma_s(t) := \Gamma(s, t)$, and the *transverse curves* $\Gamma^{(t)}(s) := \Gamma(s, t)$.
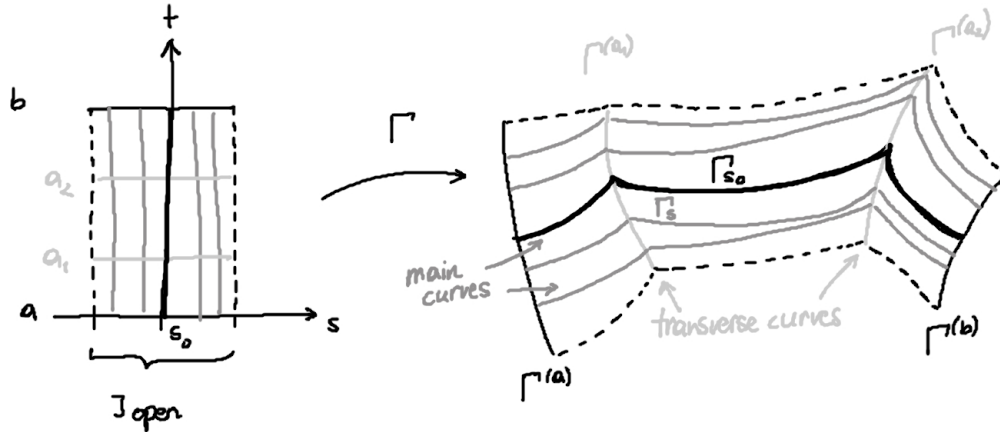

Figure 4.7. An admissible family of curves $\Gamma \colon J \times [a, b] \to M$ with main curves Γ_s and transverse curves $\Gamma^{(t)}$.

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[6]See my Bachelor course "Optimization in Geometry and Physics" [**14**] for the classical one-dimensional theory and Master courses for the full modern theory via direct methods.

The velocity vectors of the main and transverse curves (they exist where Γ is sufficiently smooth) are denoted by

$$\partial_t \Gamma(s,t) = (\Gamma_s)'(t) \in T_{\Gamma(s,t)}M, \quad \text{and} \quad \partial_s \Gamma(s,t) = \Gamma^{(t)'}(s) \in T_{\Gamma(s,t)}M,$$

respectively. Note that for an admissible family the transverse curves are smooth on J but the main curves are generally only piecewise regular. Thus the vector fields $\partial_s \Gamma$ and $\partial_t \Gamma$ are smooth on each rectangle $J \times [a_{i-1}, a_i]$, but not necessarily everywhere.

EXERCISE 4.43. Suppose Γ is an admissible family of curves.
   (i) Show that $\partial_s \Gamma$ is a piecewise smooth vector field along Γ, that is, that it is a continuous vector field along Γ whose restrictions to each $J \times [a_{i-1}, a_i]$ are smooth for an admissible partition.
   (ii) Argue that $\partial_t \Gamma$ is generally not continuous at $t = a_i$.

Based on such the notion of such an admissible family of curves we define the variation of an admissible curve.

DEFINITION 4.44. Let M be a smooth manifold and let $\gamma\colon [a,b] \to M$ be an admissible curve. A *variation of* γ is an admissible family of curves $\Gamma\colon J \times [a,b] \to M$ such that J is an open interval containing 0 and $\Gamma_0 = \gamma$.

The variation Γ is called a *proper variation* if all the main curves have the same starting and end point, i.e., $\Gamma_s(a) = \gamma(a)$ and $\Gamma_s(b) = \gamma(b)$ for all $s \in J$.
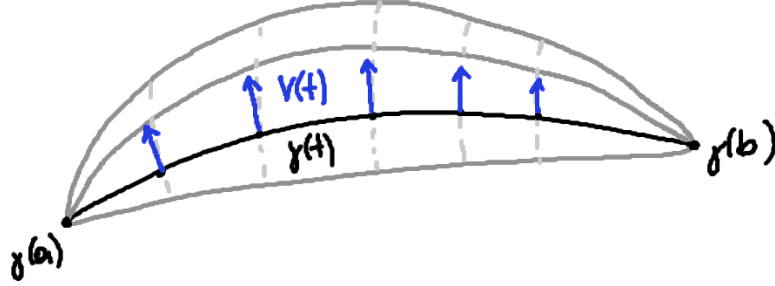

Figure 4.8. A proper variation Γ of γ with variation field $V(t) = \partial_s \Gamma(0,t)$ of Γ. Every vector field V along γ is a variation field of some variation of γ (see Exercise 4.45).

If Γ is a variation of γ, then the piecewise smooth vector field $V(t) = \partial_s \Gamma(0,t)$ along γ (see Exercise 4.43(i)) is called the *variation field of* Γ. It is called *proper* if $V(a) = V(b) = 0$.

Clearly, the variation field of a proper variation is proper. One can also prove the converse in the following sense.

EXERCISE 4.45. Suppose γ is an admissible curve and V is a piecewise smooth vector field along γ. Show that V is the variation field of some variation of γ. Moreover, if V is proper also the variation of γ can be chosen proper. (*Hint*: Define $\Gamma(s,t) := \exp_{\gamma(t)}(sV(t))$.)

In order to relate the minimization property of curves to the geodesic equation, we need to compute the covariant derivative along the main curves and use the symmetry of the Levi-Civita connection[7].

_____

[7]The following proof works, in fact, for any symmetric connection.

LEMMA 4.46 (Symmetric Lemma). *Let* $\Gamma\colon J\times[a,b]\to M$ *be an admissible family of curves in a Riemannian manifold. On every rectangle* $J\times[a_{i-1},a_i]$ *where* Γ *is smooth we have*

$$D_s\partial_t\Gamma = D_t\partial_s\Gamma.$$

PROOF. Since this is a local property we can fix local coordinates (x^i) around a point $\Gamma(s_0,t_0)$. Writing the components of Γ as $\Gamma(s,t) = (x^1(s,t),\dots,x^n(s,t))$ we have

$$\partial_t\Gamma = \frac{\partial x^k}{\partial t}\partial_k, \qquad \partial_s\Gamma = \frac{\partial x^k}{\partial s}\partial_k,$$

and by the coordinate formula (3.17) for D_t we obtain

$$D_s\partial_t\Gamma = \left(\underbrace{\frac{\partial^2 x^k}{\partial s\partial t}}_{=\frac{\partial^2 x^k}{\partial t\partial s}} + \frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial s}\underbrace{\Gamma_{ij}^k}_{=\Gamma_{ji}^k}\right)\partial_k$$

$$= \left(\frac{\partial^2 x^k}{\partial t\partial s} + \frac{\partial x^i}{\partial s}\frac{\partial x^j}{\partial t}\Gamma_{ij}^k\right)\partial_k = D_t\partial_s\Gamma,$$

where we used the symmetry condition $\Gamma_{ij}^k = \Gamma_{ji}^k$ and the fact that Γ is smooth on each rectangle (and hence the derivatives commute). $\square$

In the next step we compute a differential equation that every length-minimizing curve has to satisfy that the *first variation* vanishes. The first variation is essentially the first derivative of a functional on a function space. In our case it is somewhat simpler because it is the length functional on a one-parameter family (alternatively, one can and often does use the energy functional which then also determines the curve with a unit speed; we assume this conditions holds).

THEOREM 4.47 (First variation). *Let* (M,g) *be a Riemannian manifold. Suppose* $\gamma\colon [a,b]\to M$ *is a unit-speed admissible curve,* $\Gamma\colon J\times[a,b]\to M$ *is a proper variation of* γ *and* $V = \partial_s\Gamma(0,.)$ *is its variation field. Then* $L_g(\Gamma_s)$ *is a smooth function of* s, *and*

$$\frac{d}{ds}\bigg|_{s=0} L_g(\Gamma_s) = -\int_a^b \langle V, D_t\gamma'\rangle dt - \sum_{i=1}^{k-1}\langle V(a_i), \Delta_i\gamma'\rangle, \qquad (4.10)$$

where $(a_0,\dots,a_k)$ *is an admissible partition of* V, *and for each* $i = 1,\dots,k-1$, $\Delta_i\gamma' = \gamma'(a_i^+) - \gamma'(a_i^-)$ *is the "jump" in the velocity field* γ' *at* a_i.

PROOF. On each compact $[a_{i-1},a_i]$ the integrand of $L_g(\Gamma_s)$ is smooth, and so we can exchange differentiation with integration and obtain by the chain rule and the Symmetric Lemma 4.46 that

$$\frac{d}{ds}L_g(\Gamma_s|_{[a_{i-1},a_i]}) = \int_{a_{i-1}}^{a_i}\frac{\partial}{\partial s}\langle\partial_t\Gamma,\partial_t\Gamma\rangle^{1/2}dt$$

$$= \int_{a_{i-1}}^{a_i}\frac{1}{2}\langle\partial_t\Gamma,\partial_t\Gamma\rangle^{-1/2}2\langle D_s\partial_t\Gamma,\partial_t\Gamma\rangle dt = \int_{a_{i-1}}^{a_i}\frac{1}{|\partial_t\Gamma|}\langle D_t\partial_s\Gamma,\partial_t\Gamma\rangle dt.$$

At $s = 0$ we have $\partial_s \Gamma(0, t) = V(t)$ and $\partial_t \Gamma(0, t) = \gamma'(t)$ (which has unit length), and thus

$$\frac{d}{ds}\bigg|_{s=0} L_g(\Gamma_s|_{[a_{i-1}, a_i]}) = \int_{a_{i-1}}^{a_i} \langle D_t V, \gamma' \rangle dt \overset{\text{(iv)}}{=} \int_{a_{i-1}}^{a_i} \left(\frac{d}{dt} \langle V, \gamma' \rangle - \langle V, D_t \gamma' \rangle \right) dt$$

$$= \langle V(a_i), \gamma'(a_i^-) \rangle - \langle V(a_{i-1}), \gamma'(a_{i-1}^+) \rangle - \int_{a_{i-1}}^{a_i} \langle V, D_t \gamma' \rangle dt,$$

where we used property (iv) of the Levi-Civita connection (compatibility with g) as derived for D_t in Corollary 3.49. Summing over i yields

$$\frac{d}{ds}\bigg|_{s=0} L_g(\Gamma_s) = -\int_a^b \langle V, D_t \gamma' \rangle dt - \sum_{i=1}^{k-1} \langle V(a_i), \Delta_i \gamma' \rangle + \langle V(b), \gamma'(b) \rangle - \langle V(a), \gamma'(a) \rangle. \quad (4.11)$$

Since the variation is proper, we have $V(a) = V(b) = 0$ and thus (4.10). $\qquad\square$

Setting the first variation to zero yields critical points of the length functional via the *Euler–Lagrange equation*, which turn out to be the geodesic equation.

THEOREM 4.48. *In a Riemannian manifold, every minimizing curve which is parametrized by unit speed is a geodesic.*

PROOF. Suppose $\gamma \colon [a, b] \to M$ is a minimizing unit-speed curve with admissible partition $(a_0, \ldots, a_k)$. Thus for every proper variation Γ of γ, $L_g(\Gamma_s)$ is smooth and by elementary calculus the minimization property of γ in this one-parameter family implies

$$\frac{d}{ds} L_g(\Gamma_s)\bigg|_{s=0} = 0.$$

Thus for the corresponding proper variation field V, by Theorem 4.47 we obtain that

$$-\int_a^b \langle V, D_t \gamma' \rangle dt - \sum_{i=1}^{k-1} \langle V(a_i), \Delta_i \gamma' \rangle = 0. \quad (4.12)$$

We show that, in fact, all of these summands vanish. First we show that $D_t \gamma' = 0$ on each subinterval $[a_{i-1}, a_i]$, which means that γ is a "broken geodesic": Suppose $\varphi \in C^\infty(\mathbb{R})$ is a bump function with $\varphi > 0$ on (a_{i-1}, a_i) and else zero. For the vector field $V = \varphi D_t \gamma'$ we obtain from (4.12) that

$$0 = -\int_{a_{i-1}}^{a_i} \varphi |D_t \gamma'|^2 dt,$$

and since $\varphi > 0$ on this interval we must have $D_t \gamma' = 0$ on (a_{i-1}, a_i).

Second we show that $\Delta_i \gamma' = 0$ for each $i = 1, \ldots, k-1$, which means that γ has no corners: For each i there is a piecewise smooth vector field V along γ such that $V(a_i) = \Delta_i \gamma'$ and 0 at all other points a_j, $j \neq i$. Thus (4.12) implies that

$$-|\Delta_i \gamma'|^2 = 0,$$

and so $\Delta_i \gamma' = 0$ for each i.

Since the two one-sided velocity vectors agree at each a_i, the uniqueness of geodesics implies that $\gamma|_{[a_{i-1}, a_i]}$ can be continued by $\gamma|_{[a_i, a_{i+1}]}$ as geodesic. $\qquad\square$
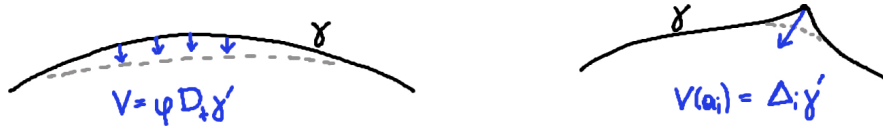
Figure 4.9. Minimizing the length of a curve γ can be achieved by deforming it in the direction of the accelartion (left) and by rounding the corners (right).

REMARK 4.49. Note that the geometric interpretion of what is being optimized in a length-minimizing curve that appears for the choice of the vector fields V used in the proof of Theorem 4.48: When we use $V = \varphi D_t \gamma'$ we deform in the direction of the acceleration, and for V with $V(a_i) = \Delta_i \gamma'$ we round the corners.

Actually we did not use the minimization feature of a curve but rather that it is a critical point of the length functional, i.e., $\frac{d}{ds}|_{s=0} L_g(\Gamma_s) = 0$.

COROLLARY 4.50. *A unit-speed admissible curve γ is a critical point for L_g if and only if it is a geodesic.*

PROOF. If γ is a critical point, then the proof of Theorem 4.48 goes through verbatim. Conversely, if γ is a geodesic, we know that is has constant-speed parametrization, is smooth and satisfies $D_t \gamma' = 0$, hence the right hand side of (4.10) vanishes. $\square$

PROBLEM 4.51. Instead of minimizing the length functional it is often more convenient to consider the *energy functional* for an admissible curve $\gamma\colon [a,b] \to M$, defined by

$$E(\gamma) := \frac{1}{2} \int_a^b |\gamma'(t)|^2 dt.$$

(i) Prove that an admissible curve is a critical point of E with respect to proper variations) if and only if it is a geodesic (which means, in particular, that it has constant speed).
(ii) Prove that if γ is an admissible curve that minimizes the energy among admissible curves with the same endpoints, then it also minimizes the length.
(iii) Prove that if γ is an admissible curve that minimizes the length among admissible curves with the same endpoints, then it minimizes the energy if and only if it has constant speed.

(*Remark*: Since the integrand of the energy functional is smooth (no square root), its critical points have automatically constant-speed parametrizations. Hence it is sometimes more useful to prove the existence of geodesics with certain properties via the energy.)

REMARK 4.52. Using the Gauss Lemma (see Section 4.3.2 below) and the more special convex neighborhoods one can prove that minimizing curves are geodesics also in a different way (see [**36**, p. 165 f]) without the use of variations.

REMARK 4.53. In Lorentzian geometry it does not make sense to consider length-minimizing curves because those are simply the null curves. Instead, the class of admissible curves consists only of timelike curves and one seeks to *maximize* the length functional between two points that are chronologically related. The result is quite similar to the Riemannian situation though. In flat Minkowski space, for instance, the curves connecting such points are the straight lines, and those with constant-speed parametrization are again geodesics.

4.3.2. Geodesics are locally minimizing. It is clear that the full converse of Theorem 4.48 does not hold. We can see this by picking two (not antipodal) points on a great circle of the unit sphere $\mathbb{S}^n$. Then only one of the two geodesic segments is length-minimizing (see Figure 4.6). We will see, however, that if the two points on a geodesic are not too far apart (in the case of the sphere less than π, that is, half the length of a great circle) then the length-minimizing property still holds. Formally, we say these curves are *locally* minimizing. How small local really needs to be could be made precise with the notion of the injectivity radius (for the sphere, $\mathrm{inj}(\mathbb{S}^n) = \pi$).

DEFINITION 4.54. Let (M, g) be a Riemannian manifold. An admissible curve $\gamma\colon I \to M$ is said to be *locally minimizing* if for every $t_0 \in I$ there exists a neighborhood $I_0 \subseteq I$ containing t_0 such that $\gamma|_{[a,b]}$ is minimizing for every $[a, b] \subseteq I_0$.

LEMMA 4.55. *Every minimizing admissible curve on a Riemannian manifold is locally minimizing.*

EXERCISE 4.56. Prove Lemma 4.55.

The key ingredient in the proof that geodesics are locally minimizing is the fundamental Gauss Lemma. It states that any sufficiently small sphere centered at a point in a Riemannian manifold is perpendicular to every geodesic through the point (see, e.g., [**36**, Thm. 6.9] and [**45**, Lem. 5.5.5]). One can also understand it in the way that the exponential map is a local radial isometry (see [**23**, Lem. 3.5] or [**43**, p. 127]). We start with this more general approach, because it holds verbatim also in the semi-Riemannian setting (and is the formulation of the Gauss Lemma used on Wikipedia!), and subsequently prove that the radial vector field is orthogonal to small geodesic spheres. In order to formulate the first result, we identify for $p \in M$ and $x \in T_pM$ the spaces $T_x(T_pM) \cong T_pM$ and say that $v_x \in T_x(T_pM)$ is *radial* if it is a scalar multiple of x.

THEOREM 4.57 (Gauss Lemma). *Let (M, g) be a Riemannian manifold, $p \in M$ and $x \in \mathcal{E}_p \subseteq T_pM$ (so that $\exp_p x$ is defined, see Definition 4.20). Let $v_x, w_x \in T_x(T_pM) \cong T_pM$ with v_x radial. Then*

$$\langle d(\exp_p)_x(v_x), d(\exp_p)_x(w_x)\rangle = \langle v_x, w_x\rangle. \tag{4.13}$$

PROOF. Since v_x is radial, we may assume without loss of generality that $v_x = x$ (since (4.13) is linear in v_x), and write v instead of x. We also write w in place of w_x.

First observe that since $v \in \mathcal{E}_p$ the expression $\exp_p v$ is defined. Consider the two-parameter map

$$u(s, t) := t(v + sw)$$

in T_pM. Since $v \in \mathcal{E}_p$, $\mathcal{E}_p$ is open and the curve $v(s) := v + sw$ is continuous (with[8] $v(0) = v$, $v'(0) = w$, and constant $|v'(s)|$), we also have that $v + sw \in \mathcal{E}_p$ for $s \in (-\varepsilon, \varepsilon)$. Therefore, since $[0, 1]$ is compact,

$$u(s, t) \in \mathcal{E}_p, \qquad (s, t) \in (-\varepsilon, \varepsilon) \times [0, 1].$$

We can thuse define a parametrized surface in M by

$$\Gamma\colon (-\varepsilon, \varepsilon) \times [0, 1] \to M,$$
$$\Gamma(s, t) = \exp_p(tv(s)) = \exp_p(t(v + sw)).$$

---

[8]Note that any other curve v with these properties would also work for the proof. See also Figure 4.10.

Note that $\partial_t \Gamma(0,1) = d(\exp_p)_x(v)$ and $\partial_s \Gamma(0,1) = d(\exp_p)_x(w)$, hence the left hand side of (4.13) is now expressed in terms of partial derivatives $\partial_t \Gamma$ and $\partial_s \Gamma$, i.e.,

$$\langle \partial_s \Gamma, \partial_t \Gamma \rangle(0,1) = \langle d(\exp_p)_x(w), d(\exp_p)_x(v) \rangle. \tag{4.14}$$

For fixed s, let us consider the main curves $\Gamma_s = \Gamma(s,.) \colon t \mapsto \Gamma(s,t)$. They are geodesics with initial velocity $\partial_t \Gamma(s,0) = v + sw$ and constant speed $|\partial_t \Gamma(s,.)|$. In particular, $D_t \partial_t \Gamma = 0$. (see Figure 4.10).
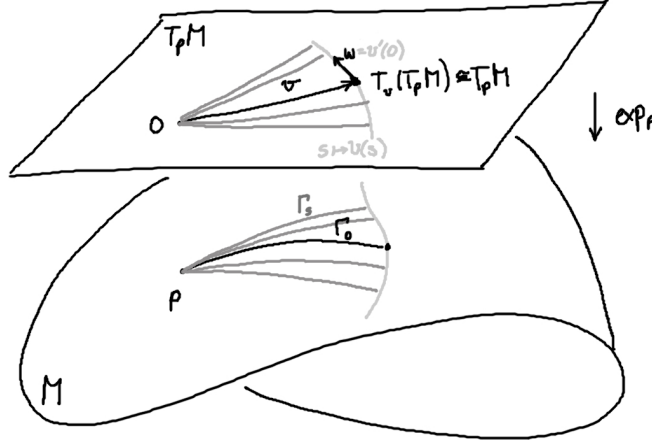

Figure 4.10. The map Γ is a variation of the curve $t \mapsto \exp_p(tv)$ which is proper on one side since $\Gamma_s(0) = \Gamma(s,0) = p$. The main curves $\Gamma_s(t) = \exp_p(tv(s))$ are all geodesics, thus $D_t \partial_t \Gamma = 0$. (Note that in this picture the curve $s \mapsto v(s)$ in $T_p M$ is not a straight line as chosen for convenience in the proof of Theorem 6.3. This is to emphasize that the particular shape of the curve is largely irrelevant. Verify that all we need for the proof is that $v(0) = v$, $v'(0) = w$, and that the curve has constant speed.)

By the Symmetric Lemma 4.46 we furthermore have that $D_s \partial_t \Gamma = D_t \partial_s \Gamma$. Thus from the metric compatibility (iv) of D_t from Corollary 3.49 it follows that for all (s,t)

$$\partial_t \langle \partial_s \Gamma, \partial_t \Gamma \rangle \stackrel{(iv)}{=} \langle \underbrace{D_t \partial_s \Gamma}_{=D_s \partial_t \Gamma}, \partial_t \Gamma \rangle + \langle \partial_s \Gamma, \underbrace{D_t \partial_t \Gamma}_{=0} \rangle$$

$$= \langle D_s \partial_t \Gamma, \partial_t \Gamma \rangle \stackrel{(iv)}{=} \frac{1}{2} \partial_s \underbrace{\langle \partial_t \Gamma, \partial_t \Gamma \rangle}_{=|v+sw|^2} = \langle v, w \rangle + s \langle w, w \rangle.$$

For $s = 0$ we already know that $\partial_t \Gamma(0,0) = v$, and furthermore obtain

$$\partial_s \Gamma(0,0) = \lim_{t \to 0} \partial_s \Gamma(0,t) = \lim_{t \to 0} d(\exp_p)_{tv}(tw) = \mathrm{Id}_{T_p M}(0) = 0.$$

Thus by the above and the fundamental theorem of calculus

$$\langle d(\exp_p)_x(w), d(\exp_p)_x(v) \rangle \stackrel{(4.14)}{=} \langle \partial_s \Gamma, \partial_t \Gamma \rangle(0,1)$$

$$= \underbrace{\langle \partial_s \Gamma, \partial_t \Gamma \rangle(0,0)}_{=\langle 0,v \rangle = 0} + \int_0^1 \langle v, w \rangle dt = \langle v, w \rangle,$$

which is the desired equality. $\qquad\square$

We can reinterpret the Gauss Lemma in terms of geodesic balls and the radial vector field, which we now make precise.

DEFINITION 4.58. Let (M, g) be a Riemannian manifold and $p \in M$. Suppose $\varepsilon > 0$ is such that $\exp_p$ is a diffeomorphism from the ball $B_\varepsilon(0) \subseteq T_pM$ (with respect to g_p), then the image $\exp_p(B_\varepsilon(0))$ is a normal neighborhood of p and called a *geodesic ball in M*.

If $\overline{B_\varepsilon(0)}$ is contained in an open neighborhood $V \subseteq T_pM$ where $\exp_p$ is a diffeomorphism onto its image, then $\exp_p(\overline{B_\varepsilon(0)})$ is called a *closed geodesic ball*, and $\exp_p(\partial B_\varepsilon(0))$ a *geodesic sphere*.

In Riemannian normal coordinates centered at p the open and closed geodesic balls, and the geodesic sphere are just the coordinate balls and spheres.

DEFINITION 4.59. Let (M, g) be a Riemanian manifold and U be a normal neighborhood of $p \in M$. The *radial distance function* $r \colon U \to [0, \infty)$ is defined by

$$r(q) := |\exp_p^{-1}(q)|, \qquad q \in U,$$

and ∂_r on $U \setminus \{p\}$ is the *radial vector field*.

Clearly, both r and ∂_r are smooth on $U \setminus \{p\}$ and r^2 is smooth on U, because $\exp_p$ is a diffeomorphism on U and g is smooth (but $\sqrt{\cdot}$ is only smooth away from 0).

In other words, the radial function is simply the Euclidean distance function from the origin in T_pM in exponential coordinates. To be more precise, one can show[9] that in any normal coordinates (x^i) on U centered at p we have

$$r(x) = \sqrt{\sum_{i=1}^{n}(x^i)^2}, \qquad \text{and} \qquad \partial_r(x) = \sum_{i=1}^{n} \frac{x^i}{r(x)} \frac{\partial}{\partial x^i}. \tag{4.15}$$

EXERCISE 4.60. Prove (4.15).

EXAMPLE 4.61. In $\mathbb{E}^n$, $r(x)$ is the distance of x to the origin, and ∂_r is the unit vector field point radially outward from the origin.

EXERCISE 4.62. Suppose σ is the radial geodesic from p to some q in the (maximal) normal neighborhood of p. Then $L_g(\sigma) = r(q)$.

Another way to formulate the Gauss Lemma is the following.

THEOREM 4.63 (Gauss Lemma). *Let (M, g) be a Riemannian manifold and let U be a geodesic ball centered at $p \in M$. Then the radial vector field ∂_r is a unit vector field orthogonal to the geodesic spheres in $U \setminus \{p\}$.*

PROBLEM 4.64. Use the first version of the Gauss Lemma (Theorem 6.3) to prove the second version (Theorem 4.63). (*Hint*: In normal coordinates the geodesic spheres $\Sigma_\delta := \exp_p(\partial B_\delta(0))$ are given by $\sum_{i=1}^{n}(x^i)^2 = \delta^2$. For any $q \in U \setminus \{p\}$ with $r(q) = \delta$ show that $\langle \partial_r|_q, w \rangle_g = 0$, for all $w \in T_qM$ that are tangent to Σ_δ at q, using the Gauss Lemma.)

---

[9]This is actually how Lee [**36**, p. 158] define the radial distance function r and corresponding radial vector field ∂_r. This definition is more intuitive, but the downside is that one then still has to show that r and ∂_r are well-defined, i.e., that they are independent of the choice of normal coordinates.

Petersen [**45**, Lemma 5.5.5] actually calls the following formulation the Gauss Lemma.

COROLLARY 4.65. *Let (M, g) be a Riemannian manifold and let U be a geodesic ball centered at $p \in M$. Then $\operatorname{grad} r = \partial_r$ on $U \setminus \{p\}$.*

PROOF. By Problem 2.46 this is equivalent to ∂_r being orthogonal to the level sets of r (follows from the Gauss Lemma) and $\partial_r(r) \equiv |\partial_r|_g^2$ (a direct computation in normal coordinates yields $\partial_r(r) \equiv 1$ and this is equal to $|\partial_r|_g^2$ by the Gauss Lemma). □

REMARK 4.66. In semi-Riemannian Geometry it is also possible to equip $T_x(T_pM)$ with a scalar product and show that the exponential map is a radial isometry in this sense (see, e.g., [**6**, Thm. 10.18] and [**43**, p. 126 ff]). Instead of geodesics balls and radial distance functions, however, one has to use the quadratic form $\tilde{q}(x) := \langle x, x \rangle$ on T_pM and $q := \tilde{q} \circ \exp_p^{-1}$ on normal neighborhoods. Instead of geodesic spheres, the level sets of q are *local hyperquadrics* on semi-Riemannian manifolds.

We set out to prove that geodesics are locally length-minimizing. In a first step, we will prove this result for radial geodesics from Proposition 4.37.

PROPOSITION 4.67. *Let (M, g) be a Riemannian manifold. Suppose $p \in M$ and q contained in a geodesic ball around p. Then (up to reparametrization) the radial geodesic σ from p to q is the* unique *minimizing curve in M from p to q.*

Moreover, the radial distance function is equal to the Riemannian distance function within every geodesic ball, that is,

$$r(q) = L_g(\sigma) = d_g(p, q).$$

PROOF. Choose $\varepsilon > 0$ such that $U = \exp_p(B_\varepsilon(0))$ is a geodesic ball containing q. Let $\sigma: [0, c] \to M$ be the radial geodesic from p to q (see Proposition 4.37), and assume that it is parametrized by unit speed, i.e., $\sigma(t) = \exp_p(tv)$ for $v \in T_pM$, $|v| = 1$. Then the velocity is equal to ∂_r and $L_g(\sigma) = c = r(q)$ (see also Exercise 4.62).

To show that σ is minimizing, we need to consider an arbitrary admissible curve $\gamma: [0, b] \to M$ from p to q (without loss of generality we assume it is parametrized by arc length, and that $\gamma(t') \neq p$ for $t' \in (0, b]$) and show that $L_g(\gamma) \geq c$ with equality if and only if $\gamma \equiv \sigma$.

Step 1. Assume that $\gamma([0, b]) \subseteq U$. Then the composition $r \circ \gamma$ with the radial function r is continuous on $[0, b]$ and piecewise smooth on $(0, b)$. By the fundamental theorem of calculus

$$c = r(\gamma(b)) - \underbrace{r(\gamma(0))}_{=0} = \int_0^b \frac{d}{dt} r(\gamma(t)) dt = \int_0^b dr(\gamma'(t)) dt = \int_0^b \langle \operatorname{grad} r|_{\gamma(t)}, \gamma'(t) \rangle dt,$$

and by the Cauchy–Schwarz inequality,

$$L_g(\sigma) = c \leq \int_0^b \underbrace{|\operatorname{grad} r|_{\gamma(t)}|}_{=|\partial_r|=1} |\gamma'(t)| dt = L_g(\gamma),$$

so σ is minimizing.

Next, assume that $L_g(\gamma) = c$. Then $b = c$, and the Cauchy–Schwarz inequality must be an equality almost everywhere, which is the case if and only if $\gamma'(t)$ and $\operatorname{grad} r|_{\gamma(t)}$ are collinear for all t. Since we assumed that γ has unit speed, in fact,

$$\gamma'(t) = \operatorname{grad} r|_{\gamma(t)} = \partial_r|_{\gamma(t)}.$$

Therefore, both γ and σ are integral curves of ∂_r, passing through q at time $t = c$, and so by uniqueness $\gamma = \sigma$.

Step 2. Assume that γ leaves U. If $\gamma([0, b])$ is not contained in $U = \exp_p(B_\varepsilon(0))$, then let $b_0 \in (0, b]$ be the first parameter point for which $\gamma(b_0)$ belongs to ∂U. Then

$$L_g(\gamma) \geq L_g(\gamma|_{[0,b_0]}) \geq \varepsilon > c = L_g(\sigma),$$

and equality cannot be achieved. $\hspace{2cm} \square$

REMARK 4.68 (Normal neighborhoods are not good enough). Note that geodesic balls B at a point p are in many ways better than arbitrary normal neighborhoods U of p. Although there exists a unique radial geodesic between p and $q \in U$, this geodesic need not be the shortest curve in M. If, however, $q \in B$, then the radial geodesic between p and q is the unique shortest curve in M.

The following example illustrates the problem clearly: Suppose $M = \mathbb{S}^1 \times \mathbb{R}$ is a cylinder in $\mathbb{R}^3$ and L a vertical line. Then $U = M \setminus L$ is a normal neighborhood of any point $p \in U$ but in general radial geodesics from p to some point $q \in U$ need not be minimizing *in M*. On the other hand, within geodesic balls B centered at p they are (see Figure 4.11 and Exercise 4.9 for a description of the geodesics in M).
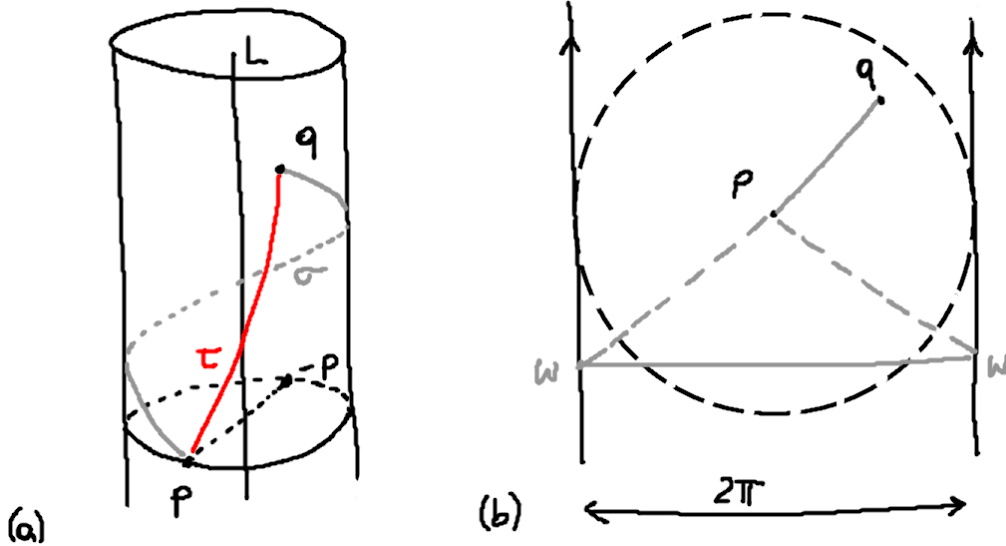

Figure 4.11. (a) The set $U = M \setminus L$ is a normal neighborhood of p and the radial geodesic σ connecting p and $q \in M \setminus L$ is the unique shortest curve *in U*, however, there is a shorter curve τ connecting p and q in M. (b) On the other hand, the largest possible geodesic ball B centered at p has radius π. If q is in this geodesic ball, then the radial geodesic is indeed the unique shortest cuve *in M* connecting p and q. If w is a point in $M \setminus B$ then there are still shortest curves from p to w but uniqueness is no longer guaranteed (it fails when w is a point on the vertical line through $-p$).

It is important to note that Proposition 4.67 relates the *local* radial distance function r defined in (4.15) to the *global* Riemannian distance function d_g from Definition 2.74. We have already seen, however, that geodesics can cease to be minimizing after a while. In this sense, Proposition 4.67 is *not* a global result. It may be even worse, namely that there doesn't even

exist a shortest curve between two (even arbitrarily close) points on a Riemannian manifold: think about the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$ with the induced Euclidean metric and consider the points $p = (\varepsilon, 0)$ and $q = (-\varepsilon, 0)$.

Since the radial distance function and Riemannian distance function are equal on geodesic balls by Proposition 4.67, one can show the following corollary relating the geodesic and metric balls globally.

COROLLARY 4.69. *In a connected Riemannian manifold the geodesic balls (and spheres) are also metric balls (and spheres) of the same radius.*

SKETCH OF PROOF. For $V = \exp_p(B_\varepsilon(0))$, it follows from Proposition 4.67 that if $q \in V$ then $d_g(p,q) < \varepsilon$. If $q \notin V$, then one can argue by contradiction (similar to the proof of 4.67) that $d_g(p,q) > \varepsilon$. For the detailled proof see [**36**, Cor. 6.13]. $\square$

We can now also prove the remaining property of the Riemannian distance function, namely that d_g indeed is a metric on M (see Theorem 2.80), and show some other important relations between g and d_g which are crucial from the metric geometric point of view (see also Section 4.4).

PROBLEM 4.70. Let (M, g) be a connected Riemannian manifold.
  (i) Show that $d_g(p,q) = 0$ if and only if $p = q$.
  (ii) Argue that d_g induces the manifold topology.

PROBLEM 4.71. Let (M, g) be a connected Riemannian manifold. Show that the distance function on a smooth manifold determines the Riemannian metric:
  (i) Show that if $\gamma \colon (-\varepsilon, \varepsilon) \to M$ is any smooth curve, then
  $$|\gamma'(0)|_g = \lim_{t \searrow 0} \frac{d_g(\gamma(0), \gamma(t))}{t}.$$
  (ii) Show that if g and $\bar{g}$ are two Riemannian metrics on M such that $d_g(p,q) = d_{\bar{g}}(p,q)$ for all $p, q \in M$, then $g = \bar{g}$.

Finally, we sketch how to extend Proposition 4.67 to show that every geodesic is locally minimizing. For this we use that we can find not just normal but actually convex neighborhoods around each point.

DEFINITION 4.72. An open set in a Riemannian manifold is called (geodesically) *convex* provided it is a normal neighborhood of each of its points.

REMARK 4.73. While geodesic balls are clearly related to Euclidean balls in the tangent space and normal neighborhoods are related to star-shaped domains, the same is not true for convex neighborhoods: In the cylinder example used in Remark 4.68 the set $U = M \setminus L$ is clearly a convex set, but *not* a convex neighborhood.

LEMMA 4.74. *Every point in a Riemannian manifold possesses a base of convex neighborhoods.*

We will not prove this Lemma here but see, for instance, [**43**, p. 130] and [**36**, Problem 6-5]. The proof is based on the fact that the full exponential map on M is a local diffeomorphism (which was discussed in Remark 4.25), so that we can remove the dependence on the center point p of normal neighborhoods. The convex neighborhoods constructed are, in fact, sufficiently small geodesic balls.

THEOREM 4.75. *Every geodesic in a Riemannian manifold is locally minimizing.*

PROOF. Suppose $\gamma\colon I \to M$ is a geodesic with I an open interval. Let $t_0 \in I$ and let C be a convex neighborhood of $\gamma(t_0)$. Consider the connected component I_0 of $\gamma^{-1}(C)$ containing t_0. If $a, b \in I_0$, $a < b$, then $\gamma(b)$ is in C and hence, in particular, in a geodesic ball of $\gamma(a)$. Thus by Proposition 4.67, the radial geodesic σ from $\gamma(a)$ to $\gamma(b)$ is the unique minimizing curve between these points. On the other hand, by construction, the restriction $\gamma|_{[a,b]}$ is also a geodesic that is contained entirely in C. Thus by the uniqueness of radial geodesics obtained in Proposition 4.37, the geodesics $\gamma|_{[a,b]}$ and σ must coincide. $\square$

REMARK 4.76. The proof shows that one does not need the full generality of a convex neighborhood. It is sufficient to use *uniformly normal neighborhoods*, that is, neighborhoods W such that for some $\delta > 0$ the set W is contained in a geodesic ball of radius δ around each of its points (see [**36**, p. 163], and [**23**, p. 72] who calls them totally normal neighborhoods).

4.4. Completeness (not covered in the course)

In the previous Section we have seen how geodesic and metric balls are related. In what follows we sketch that also geodesic and metric completeness are intimately tied to each other. This is the Hopf–Rinow Theorem.

DEFINITION 4.77. A Riemannian manifold is said to be *geodesically complete* (at p) if every maximal geodesic (through p) is defined on all of $\mathbb{R}$.

DEFINITION 4.78. A Riemannian manifold (M, g) is said to be *metrically complete* if it is complete as metric space with respect to the Riemannian distance function d_g in the sense that every Cauchy sequence converges.

The following result [**30**] of Heinz Hopf and Willi Rinow from 1931 is a crucial feature of Riemannian manifolds (it is false for Lorentzian manifolds, see Clifton–Pohl torus in [**43**, p. 193]). One can prove it with the methods we have developed so far, but since we do not want to dwelve further in the metric direction, we only state the result. For a proof see, for instance, [**45**, Theorem 5.7.1], [**36**, p. 166 ff], or [**43**, p. 138 ff].

THEOREM 4.79 (Hopf–Rinow). *Let (M, g) be a (connected) Riemannian manifold. The following statements are equivalent:*

  (i) *M is geodesically complete.*
  (ii) *M is geodesically complete at p.*
  (iii) *M satisfies the Heine–Borel property, i.e., every closed bounded set is compact.*
  (iv) *M is metrically complete.*

An essential step of the proof is the following observation.

LEMMA 4.80. *Suppose (M, g) is a connected Riemannian manifold and that $\exp_p$ is defined on all of T_pM. Then for every $q \in M$ there exists a minimizing geodesic from p to q.*

SKETCH OF PROOF OF LEMMA 4.80. Choose $\varepsilon > 0$ such that the closed geodesic ball $\overline{B_\varepsilon(p)}$ around p does not contain q. By continuity of d_g and compactness of the geodesic sphere $S_\varepsilon(p) = \partial B_\varepsilon(p)$, the function $y \mapsto d_g(y, q)$ attains a minimum at $x \in S_\varepsilon(p)$. Suppose γ is the maximal unit speed geodesic whose initial segment $\gamma|_{[0,\varepsilon]}$ is the radial geodesic from p to x. By assumption, γ is defined on $\mathbb{R}$.

One then shows that
$$d_g(p,q) = d_g(p,x) + d_g(x,q).$$
Then let $T := d_g(p,q)$ and consider the set
$$\mathcal{A} := \{b \in [0,T]; \gamma|_{[0,b]} \text{ is minimizing and satisfies } b + d_g(\gamma(b),q) = T\}.$$
Since $\varepsilon \in \mathcal{A}$, $A := \sup \mathcal{A} \geq \varepsilon > 0$. By continuity of the distance function, $\mathcal{A}$ is closed and hence $A \in \mathcal{A}$. If $T = A$ we are done. One assumes $A < T$, then extends it a bit a minimizing curve (without additional corner) that still satisfies the condition of $\mathcal{A}$ and so arrives at a contradiction. $\square$

SKETCH OF PROOF OF THEOREM 4.79. (i)$\Rightarrow$(ii) is trival.

For (ii)$\Rightarrow$(iv) suppose M is geodesically complete at p. Then $\exp_p$ is defined on all of T_pM. Let (q_i) be a Cauchy sequence in M. For each i, let $\gamma_i(t) = \exp_p(tv_i)$ be a unit-speed minimizing geodesic from p to q_i. Setting $d_i = d_g(p,q_i)$ yields $q_i = \exp_p(d_iv_i)$. Since Cauchy sequences are bounded, so is (d_i) and since $|v_i| = 1$ also the sequence (d_iv_i) of vectors in T_pM is bounded. Thus a subsequence $(d_{i_k}v_{i_k})$ converges to some $v \in T_pM$, and by continuity of the exponential map $\lim_{k \to \infty} \exp_p(d_{i_k}v_{i_k}) = \exp_p v$. Since the original sequence (q_i) was already Cauchy, it converges as a whole to the limit $q = \exp_p v$ in M.

To show (iv)$\Rightarrow$(i) assume that M is metrically complete but not geodesically complete. Then there exists a unit-speed geodesic $\gamma \colon [0,b) \to M$ with no extension to b. Then for a sequence (t_i) approaching b one can show that $q_i = \gamma(t_i)$ satisfies
$$d_g(q_i,q_j) \leq |t_i - t_j|,$$
hence (q_i) is a Cauchy sequence and hence converges to some point q in M. By considering a convex neighborhood $C(q)$ around q one can construct geodesic extensions $\widetilde{\gamma}$ of γ for all $q_j \in C(q)$ with j sufficiently large such that $\widetilde{\gamma}'(t_j) = \gamma'(t_j)$ but $\widetilde{\gamma}$ is defined for $t_j + \delta > b$, a contradiction.

Hence (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iv). In addition, one can prove (ii)$\Rightarrow$(iii) using Lemma 4.80 by picking a point q in a closed and bounded set A and considering the minimizing geodesic $\sigma \colon [0,1] \to M$ from p to q. From Theorem 4.67 we know that $|\sigma'(0)| = L(\sigma) = d_g(p,q)$. The triangle inequality together with the boundedness of A implies that each such $\sigma'(0)$ for all $q \in A$ is in a compact ball $B_r(0) \subseteq T_pM$. Since $A \subseteq \exp_p(B_r(0))$ and $\exp_p(B_r(0))$ is compact, also A is.

Finally, (iii)$\Rightarrow$(iv) follows because every Cauchy sequence is bounded, and hence the closure is compact. Thus the sequence contains a convergent subsequence, and since it is actually Cauchy, it converges itself. $\square$

In addition, some more useful and important implications follow.

COROLLARY 4.81. *On a connected complete Riemannian manifold any two points can be joined by a minimizing geodesic.* $\square$

In some situations the Heine–Borel property holds (almost) trivially. For instance, we get the following results.

COROLLARY 4.82. *Every compact Riemannian manifold is complete.* $\square$

COROLLARY 4.83. *If a Riemannian manifold (M,g) admits a proper Lipschitz function $f \colon M \to \mathbb{R}$, then M is complete.*

EXERCISE 4.84. Prove Corollary 4.83. How can this result be used for warped products?

PROBLEM 4.85. A curve $\gamma\colon [0,b) \to M$ (with $0 < b \leq \infty$) is said to *diverge to infinity* if for every compact set $K \subseteq M$, there is a time $T \in [0,b)$ such that $\gamma(t) \notin K$ for all $t > T$. Prove that a connected Riemannian manifold is complete if any only if every regular curve that diverges to infinity has infinite length. (The length of a curve whose domain is not compact is just the supremum of the length of its restrictions to compact submanifolds.)

REMARK 4.86. The Hopf–Rinow Theorem does not make sense on semi-Riemannian manifolds (the Clifton–Pohl torus [**43**, p. 193] is a counterexample to Corollary 4.82), and it also false in infinite dimensions [**5**]. It does, however, generalize to locally compact length metric spaces. This result is due to Stefan Cohn–Vossen, see [**11**, Theorem 2.5.28], and an important cornerstone of Metric Geometry. In Lorentzian geometry one can still characterize globally hyperbolic spacetimes (which in many ways resemble complete Riemannian manifolds) by metric completeness of a null distance [**15**, Theorem 1.4].

Curvature

The notion of curvature is due to Bernhard Riemann and was presented in his habilitation lecture "Über die Hypothesen, welche der Geometrie zu Grunde liegen" (German for "On the hypotheses which lie at the bases of geometry") in 1854, but was published[1] only in 1868 after his death. For some historical background and the original spirit from a modern point of view see, for instance, the recent book by Jürgen Jost [**47**] (or the much older re-edition [**48**] in German and with comments by Hermann Weyl). It is important to know that Riemann's mentor Gauss had at the time already extensively studied surfaces in the Euclidean 3-space. In his *Theorema Egregium*[2] (Latin for "Remarkable Theorem") Gauss showed that the curvature of a surface can be expressed solely in terms of the first fundamental form and that the way the surface is embedded does not matter. In order words, Gaussian curvature is an *intrinsic local invariant* of a surface.

In Gauss's spirit, Riemann introduced curvature at a point p on an abstract Riemannian manifold[3] M by considering 2-dimensional subspaces $\Pi \subseteq T_pM$ and using the Gaussian curvature $K(p, \Pi)$ of the corresponding 2-dimensional submanifold S generated by geodesics starting at p and tangent to Π (essentially the image of Π under $\exp_p$). Nowadays we call $K(p, \Pi)$ the sectional curvature of M at p with respect to Π. To be more precise, Riemann computes an expansion of the metric in normal coordinates to second order and recovers so the Gaussian curvature. This approach thus allows for the desired geometric interpretation.

Moreover, Riemann considers constant curvature spaces. Clearly, if $M = \mathbb{E}^n$ we have $K \equiv 0$. In addition, Riemann computed the sectional curvature of a sphere of radius R and showed that it is everywhere $K \equiv \frac{1}{R^2}$. But he did not stop there, he also considered the case when K is negative. At the time, many mathematicians tried to find geometries that satisfied the first four of Euclid's axioms but not the fifth "parallel postulate"[4]. This is how hyperbolic space was discovered independently by Nikolai Lobachevsky, János Bolyai and Carl Friedrich Gauss. Riemann essentially introduced it as constant curvature space with $K \equiv -\frac{1}{R^2}$ (see, for instance, [**7**, Section 4.3.2] for some more details). In retrospect, the reason why hyperbolic space was not found earlier is due to Hilbert's Theorem: the hyperbolic plane $\mathbb{H}^2$ is not isometrically embeddable in $\mathbb{E}^3$ (we already know from Example 2.32 that hyperbolic space

---

[1]See `https://www.emis.de/classics/Riemann` for Riemann's original (transcripted) texts.

[2]This theorem is covered in the 2nd Year Bachelor course "Curves and Surfaces".

[3]Note that even the notion of an abstract manifold wasn't yet available at this point and had to be introduced by Riemann, which shows just how pioneering and influential his work was (see [**37**, p. 295 ff] for a wider discussion).

[4]Euclid's fifth postulate states: *If a straight line falling on two straight lines makes two interior angles on the same side with sum less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.*

is a codimension-one submanifold of Minkowski space though, and from Nash's Embedding Theorem we know we can still isometrically embed it in some $\mathbb{E}^n$ for $n \geq 4$).

Note that Riemann did not make use of parallel transport and connections (which we have already heard in Chapter 3 were introduced by the Italian school only several decades later), in fact, he didn't even use tensors. We are going to go a very different, modern, and rather abstract route but try to relay some of Riemann's original observations as well.

The notion of curvature is indispensible in mathematics and physics today. Note that everything we learn in this chapter works verbatim for semi-Riemannian manifolds (see [**43**], modulo different sign conventions that different authors use). As an important application, we will discuss briefly how curvature is used to formulate the theory of General Relativity.

EXERCISE 5.1. Look at Riemann's original paper and try to find evidence that he uses normal coordinates and discusses constant curvature spaces. (*Hint*: Mannigfaltigkeit = manifold, Krümmung = curvature)

5.1. Riemann curvature tensor

So far we have studied Riemannian manifolds individually, occassionally also by looking at (local) isometries between two Riemannian manifolds. Our goal now is to fully capture via *local invariants* when two Riemannian manifolds are locally equivalent (or not). We start by analyzing, once more, the specific properties of Euclidean space and describe its *flatness* in an abstract way. As an extension of this observation we then introduce the Riemann curvature tensor as a failure of the covariant derivatives to commute.

5.1.1. Motivation. Suppose M is a 2-dimensional Riemannian manifold and $z \in T_pM$. If $M = \mathbb{E}^2$ we can extend z trivially to a smooth parallel (constant) vector field on all of M. For a general manifold M (such as the 2-sphere) assume we are given local coordinates (x^1, x^2) at p. Then we can extend z at least locally to a vector field Z, first by parallel transport along the x^1-axis and then by parallel transport along x^2-axis (see Figure 5.1).

By construction, $\nabla_{\partial_2} Z \equiv 0$, but it is unclear whether Z is still parallel with respect to any other x^1-line, i.e., whether $\nabla_{\partial_1} Z \equiv 0$ (by contruction it is true at $x^2 = 0$). Because parallel transport is unique and because the parallel transport of zero is zero, we can ask, equivalently, whether

$$\nabla_{\partial_2} \nabla_{\partial_1} Z = 0. \tag{5.1}$$

Suppose we know that this expression commutes, i.e., that

$$\nabla_{\partial_2} \nabla_{\partial_1} Z = \nabla_{\partial_1} \nabla_{\partial_2} Z, \tag{5.2}$$

then we obtain immediately from $\nabla_{\partial_2} Z \equiv 0$ that also $\nabla_{\partial_1} Z \equiv 0$. But can we assume (5.2)? The formula (5.2) holds for $\mathbb{E}^n$ with the Euclidean connection because

$$\overline{\nabla}_{\partial_i} \overline{\nabla}_{\partial_j} Z = (\partial_i \partial_j Z^k) \partial_k = (\partial_j \partial_i Z^k) \partial_k = \overline{\nabla}_{\partial_j} \overline{\nabla}_{\partial_i} Z,$$

but may not hold for general Riemannian manifolds. Moreover, in order to check whether the commutativity condition (5.2) is well-defined (and satisfied), it should be formulated in an coordinate-independent manner. Since on $\mathbb{E}^n$ we know that $\overline{\nabla}_Y \overline{\nabla}_X Z = YX(Z^k)\partial_k$ we can use the Lie bracket $[X, Y] = XY - YX$ to write

$$\overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z = \overline{\nabla}_{[X,Y]} Z.$$
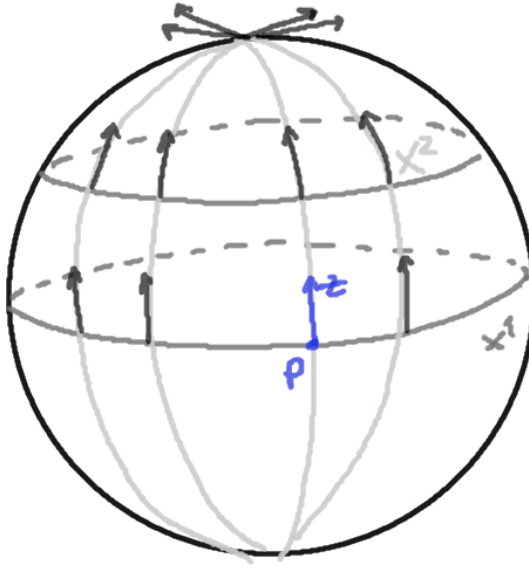
Figure 5.1. The vector $z \in T_p M$ is parallel transported along x^1 and x^2. The different tangent vectors on the north pole hint that commutativity is not satisfied on $\mathbb{S}^2$.

The above observations immediately translate into a necessary condition for Riemannian manifolds that are *flat*, that is, locally isometric to Euclidean space.

PROPOSITION 5.2. *If (M, g) is a flat Riemannian manifold, then its Levi-Civita connection satisfies the* flatness criterion, *that is, for any smooth vector fields X, Y, Z defined on an open subset of M we have*

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z. \tag{5.3}$$

EXERCISE 5.3. Fill in the details in the above exposition and prove Proposition 5.2.

PROBLEM 5.4. Show that the round 2-sphere is not locally isometric to Euclidean space via Proposition 5.2. (*Hint*: Let $X(\varphi, \theta) = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi)$ be the spherical coordinate parametrization of an open set U of $\mathbb{S}^2$, and let $X_\theta = X_*(\partial_\theta)$ and $X_\varphi = X_*(\partial_\varphi)$ denote the corresponding coordinate vector fields. Compute $\nabla_{X_\theta}(X_\varphi)$ and $\nabla_{X_\varphi}(X_\varphi)$ and note that X_φ is parallel along the equator $\varphi = \frac{\pi}{2}$ and along each meridian $\theta = \theta_0$.)

5.1.2. Definition. In general, as Figure 5.1 shows for the sphere, parallel transport along closed curves in an arbitrary Riemannian manifold does not commute. Curvature measures precisely this noncommutativity.

DEFINITION 5.5. Let (M, g) be a Riemannian manifold. The map $R \colon \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$, defined by[5]

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{5.4}$$

---

[5]Warning! Different authors use different conventions, e.g., doCarmo [**23**] and O'Neill [**43**] define the Riemann curvature tensor with the opposite sign, but their sectional curvatures have the same sign. We use Lee's book [**36**] and thus his convention. Note that also the index notation discussed later varies considerably, and it is important to be aware of the convention in use when reading a book or article.

is called the *Riemann curvature tensor*.

The following calculation verifies that R is indeed a $(1,3)$-tensor field via the tensor characterization lemma, hence justifying the name a posteriori.

PROPOSITION 5.6. *The map R defined in (5.4) is multilinear over $C^\infty(M)$, and thus a $(1,3)$-tensor field.*

PROOF. For $f \in C^\infty(M)$ it follows from the properties (i) and (iii) of affine connections and the Lie bracket that

$$
\begin{aligned}
R(X, fY)Z &= \nabla_X \nabla_{fY} Z - \nabla_{fY} \nabla_X Z - \nabla_{[X,fY]} Z \\
&= \nabla_X (f \nabla_Y Z) - f \nabla_Y \nabla_X Z - \nabla_{f[X,Y]+(Xf)Y} Z \\
&= (Xf)\nabla_Y Z + f \nabla_X \nabla_Y Z - f \nabla_Y \nabla_X Z - f\nabla_{[X,Y]} Z - (Xf)\nabla_Y Z \\
&= f R(X,Y)Z.
\end{aligned}
$$

Since $R(X,Y)Z = -R(Y,X)Z$ the same is true in X. The linearity in Z is an exercise. $\qquad\square$

PROBLEM 5.7. Complete the proof of Proposition 5.6 by showing that $R(X,Y)(fZ) = f R(X,Y)Z$ for all $X, Y, Z \in \mathfrak{X}(M)$, $f \in C^\infty(M)$.

Due to $R \in \mathcal{T}^{(1,3)}(M)$ we can write it in local coordinates (x^i) as

$$
R = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l,
$$

where the coefficients $R_{ijk}{}^l$ are given implicitly by

$$
R(\partial_i, \partial_j)\partial_k = R_{ijk}{}^l \partial_l.
$$

One can easily compute the coefficients via the Christoffel symbols which in turn, namely by (3.15), follows directly from the metric coefficients g_{ij}.

PROPOSITION 5.8. *Let (M,g) be a Riemannian manifold. In local coordinates, the components of the Riemann curvature tensor are given by*

$$
R_{ijk}{}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{jk}^m \Gamma_{im}^l - \Gamma_{ik}^m \Gamma_{jm}^l. \tag{5.5}
$$

PROBLEM 5.9. Prove Proposition 5.8.

The endomorphism $Z \mapsto R(X,Y)Z$ is also called the (directional) *curvature operator*.

It is, however, often more covenient to describe the Riemann curvature tensor in its equivalent *covariant* form, namely as a $(0,4)$-tensor field $Rm = R^\flat$ (recall that $\flat$ denotes a musical isomorphism introduced in Section 2.2.1 to lower an index). In terms of vector fields we thus have

$$
Rm(X,Y,Z,W) = \langle R(X,Y)Z, W \rangle, \tag{5.6}
$$

and in local coordinates

$$
R_{ijkl} = g_{lm} R_{ijk}{}^m.
$$

PROBLEM 5.10. Let G be a Lie group with a bi-invariant metric g. Let $X, Y, Z \in \mathfrak{X}(G)$ be left invariant vector fields on G. Show that

$$
R(X,Y)Z = \frac{1}{4}[Z, [X,Y]].
$$

(*Hint*: First show that $\nabla_X Y = \frac{1}{2}[X, Y]$ by using the symmetry of the connection and the fact that $\nabla_X X = 0$. This is [**36**, Prob. 5-8(b)]. See also Ex 3 (on p. 80 f) and Exercise 1 (on p. 103) in [**23**].)

REMARK 5.11 (Curvature and curves). So far we have not said how curvature is related to (variations of) curves. This connection is made precise in [**36**, Prop. 7.5], where it is shown that for a smooth one-parameter family Γ of curves in (M, g) and a smooth vector field V along Γ one obtains

$$D_s D_t V - D_t D_s V = R(\partial_s \Gamma, \partial_t \Gamma) V. \tag{5.7}$$

Before introducing other curvature notions, we first establish some basic properties of the Riemann curvature tensor.

5.1.3. Flatness. By defining the curvature tensor we have indeed achieved what we hoped for, namely to obtain a local invariant of a Riemannian manifold.

PROPOSITION 5.12. *The curvature tensor is a local isometry invariant: if (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds and $\varphi \colon M \to \widetilde{M}$ is a local isometry, then $\varphi^* \widetilde{Rm} = Rm$.*

EXERCISE 5.13. Prove Proposition 5.12.

In what follows we show the converse in the flat case, which yields a more qualitative description of curvature as a geometric obstruction of a Riemannian manifold to being locally isometric to Euclidean space. The proof rests one the construction of a local parallel vector field, which should not come as a surprise after the motivating discussion in Section 5.1.1 and our earlier introduced notion of a holonomy group (see Definition 3.62), both essentially related to the noncommutativity of parallel transport along closed curves.

LEMMA 5.14. *Suppose M is a smooth manifold with affine connection ∇ satisfying the flatness condition (5.3). Then for any $p \in M$ and $v \in T_p M$ there exists a parallel vector field V in a neighborhood of p such that $V_p = v$.*

PROOF. Let $p \in M$, $v \in T_p M$ and (x^i) smooth coordinates on M centered at p. Without loss of generality we assume that the image of the coordinate domain is an open cube C_ε with side length 2ε (same notation used for the domain).

We parallel transport v along the x^1-axis, then from each point of the x^1-axis along the x^2-axis etc. through to the x^n-axis (see Figure 5.2). The result is a vector field V on C_ε. It is smooth because we essentially solve linear ODEs when parallel transporting and thus have smooth dependence on the initial data. By construction, $\nabla_{\partial_1} V = 0$ on the x^1-axis, and generally $\nabla_{\partial_k} V = 0$ on the set $M_k := \{x^{k+1} = \ldots = x^n = 0\} \subseteq C_\varepsilon$.

We will show by induction on k that

$$\nabla_{\partial_1} V = \ldots = \nabla_{\partial_k} V = 0 \qquad \text{on } M_k. \tag{5.8}$$

For $k = 1$ it is true by construction. Assume that (5.8) is true for some $k < n$. By construction, $\nabla_{\partial_{k+1}} V = 0$ on $M_{k+1} \supseteq M_k$. Consider $i \leq k$. Since $[\partial_{k+1}, \partial_i] = 0$ the flatness condition (5.3) translates to

$$\nabla_{\partial_{k+1}}(\nabla_{\partial_i} V) = \nabla_{\partial_i}(\underbrace{\nabla_{\partial_{k+1}} V}_{=0}) = 0 \qquad \text{on } M_{k+1}.$$
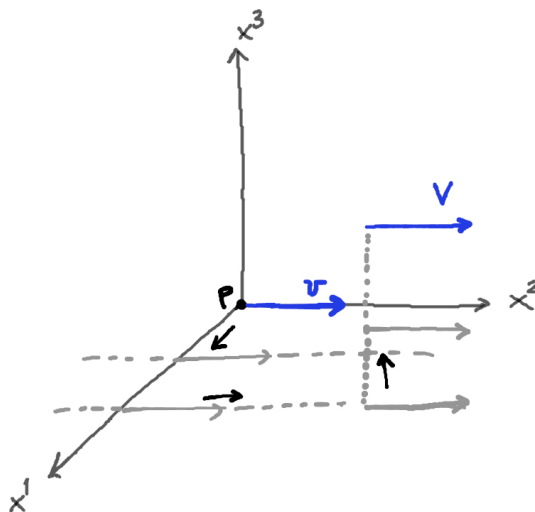
Figure 5.2. Construction of the vector field V in the proof of Lemma 5.14 via parallel transport of v along the coordinate axes.

Thus $\nabla_{\partial_i} V$ is parallel along the x^{k+1}-curves starting on M_k. Moreover, $\nabla_{\partial_i} V$ vanishes on M_k by construction, hence by uniqueness of the parallel transport, $\nabla_{\partial_i} V = 0$ on each x^{k+1}-curve, and hence on all of M_{k+1}. This proves (5.8) for $k + 1$.   □

We can now characterize when a Riemannian manifold is locally isometric to Euclidean space. For the proof we will use another distinctive feature of Euclidean space, namely that it admits a parallel orthonormal frame globally (see Example 3.35). Using Lemma 5.14 we will show that flat Riemannian manifolds share this feature locally.

THEOREM 5.15. *A Riemannian manifold is flat if and only if its curvature tensor vanishes identically.*

PROOF. We have already seen in Proposition 5.2 that flatness of the Levi-Civita connection implies (5.3), which means that the curvature tensor (5.4) vanishes identically.

Conversely, suppose that the curvature tensor of (M, g) vanishes. We first show that g then admids a parallel orthonormal frame in a neighborhood of each point: For any $p \in M$ and an orthonormal basis $(b_1, \ldots, b_n)$ for $T_p M$ there exist by Lemma 5.14 parallel vector fields $E_1, \ldots, E_n$ on a neighborhood U of p such that

$$E_i|_p = b_i.$$

Since parallel transport is an isometry by Corollary 3.56, the vector fields (E_j) are also orthonormal on U.

By the symmetry of the Levi-Civita connection (property (iv) in Theorem 3.27),

$$[E_i, E_j] = \nabla_{E_i} E_j - \nabla_{E_j} E_i = 0,$$

and so (E_j) is a commuting orthonormal frame on U.

Thus by the canonical form theorem for commuting vector fields (see [**35**, Theorem 9.46]) there exist coordinates (y^i) on a (possibly smaller) neighborhood V of p such that

$$E_i = \partial_i \qquad \text{for } i = 1, \ldots, n.$$

In these coordinates, we therefore have that on the whole neighborhood[6]

$$g_{ij} = g(\partial_i, \partial_j) = g(E_i, E_j) = \delta_{ij},$$

and hence the chart $y = (y^1, \ldots, y^n)$ is an isometry from the neighborhood V of p to an open subset of the n-dimensional Euclidean space. $\qquad\square$

5.1.4. Symmetries. We have already used that $R(X,Y)Z = -R(Y,X)Z$ in the proof of Proposition 5.6, but besides this trivial symmetry of R there are several others that follow from properties of the definition or the Levi-Civita connection, and the Bianchi symmetries obtained by cyclic permutation.

PROPOSITION 5.16 (Symmetries of the curvature tensor). *Let (M,g) be a Riemannian manifold. The $(0,4)$-curvature tensor has the following symmetries for all $X, Y, Z, W \in \mathfrak{X}(M)$:*

  (i) $Rm(W, X, Y, Z) = -Rm(X, W, Y, Z)$.
  (ii) $Rm(W, X, Y, Z) = -Rm(W, X, Z, Y)$.
  (iii) $Rm(W, X, Y, Z) = Rm(Y, Z, W, X)$.
  (iv) $Rm(W, X, Y, Z) + Rm(X, Y, W, Z) + Rm(Y, W, X, Z) = 0$.

SKETCH OF PROOF. (i) is immediate from the definition of the curvature tensor as endomorphism because $R(W, X)Y = -R(X, W)Y$.

(ii) follows from the compatibility of the Levi-Civita connection with the metric g. More precisely, one needs to show $Rm(W, X, Y, Y) = 0$ for all Y (see [**36**, p. 203] for details) and then expand $Rm(W, X, Y + Z, Y + Z) = 0$.

(iv) follows from the symmetry of the connection. In particular, we notice that Z remains at the same position, so it is sufficient to prove

$$R(W, X)Y + R(X, Y)W + R(Y, W)X = 0.$$

By writing everything out one notices that this is just the Jacobi identity

$$[W, [X, Y]] + [X, [Y, W]] + [Y, [W, X]] = 0.$$

(iii) follows form (i), (ii) and (iv) by adding up four cyclically permuted Bianchi identities. $\qquad\square$

The curvature identity (iv) is called the *first Bianchi identity* (or algebraic Bianchi identity). We next prove the *second Bianchi identity* (or differential Bianchi identity). It has vast implications for General Relativity, which we discuss later.

PROPOSITION 5.17 (Differential Bianchi Identity). *The total covariant derivative[7] of the curvature tensor of a Riemannian manifold satisfies*

$$\nabla Rm(X, Y, Z, V, W) + \nabla Rm(X, Y, V, W, Z) + \nabla Rm(X, Y, W, Z, V) = 0. \qquad (5.9)$$

---

[6]Recall that by using normal coordinates we can always have this property $g_{ij}(p) = \delta_{ij}$ (see Proposition 4.33(ii)), but generally not in a whole neighborhood. This proof thus also explains why curvature has to become "visible" in higher order terms when the metric components are expressed in normal coordinates.

[7]Recall the unique construction of a covariant derivative ∇F of a tensor field $F \in \mathcal{T}^{(k,l)}(M)$ from Section 3.3.

PROOF. Note that (5.9) is a pointwise condition, and so it suffices to assume that X, Y, Z, V, W are coordinate vector fields. This way the commutators vanish, i.e., $[\partial_i, \partial_j] \equiv 0$. If we use normal coordinates at p we know, in addition, that the covariant derivatives vanish at p, i.e., $\Gamma_{ij}^k(p) = 0$ (see Proposition 4.33(iv)).

By Proposition 5.16 the symmetries of Rm imply that (5.9) is equivalent to

$$\nabla Rm(Z, V, X, Y, W) + \nabla Rm(V, W, X, Y, Z) + \nabla Rm(W, Z, X, Y, V) = 0. \qquad (5.10)$$

Then by definition of the total derivative ∇ and the compatibility condition (iv) of ∇ with g we have

$$\begin{aligned}
\nabla Rm(Z, V, X, Y, W) &= (\nabla_W Rm)(Z, V, X, Y) = \nabla_W \langle R(Z, V)X, Y \rangle \\
&= \nabla_W \langle \nabla_Z \nabla_V X - \nabla_V \nabla_Z X - \nabla_{[Z,V]}X, Y \rangle \\
&\overset{\text{(iv)}}{=} \langle \nabla_W \nabla_Z \nabla_V X - \nabla_W \nabla_V \nabla_Z X, Y \rangle.
\end{aligned}$$

Cyclic permutation and adding up the three expressions in (5.10) yields

$$\begin{aligned}
\nabla Rm(Z, V, X, Y, W) &+ \nabla Rm(V, W, X, Y, Z) + \nabla Rm(W, Z, X, Y, V) \\
&= \langle R(W, Z)(\nabla_V X) + R(Z, V)(\nabla_W X) + R(V, W)(\nabla_Z X), Y \rangle.
\end{aligned}$$

Due to the choice of normal coordinates at p we have $\nabla_V X = \nabla_W X = \nabla_Z X = 0$ at p, and so we are done. $\square$

In local coordinates, the following expressions hold.

COROLLARY 5.18. *In terms of the components of the Riemann curvature tensor we have*

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij},$$

as well as the algebraic Bianchi identity

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0,$$

and the differential Bianchi identity

$$R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} = 0. \qquad (5.11)$$

EXERCISE 5.19. Prove Corollary 5.18.

REMARK 5.20 (Ricci identities). One can also prove that the Riemann curvature appears directly as the obstruction to commutation of the total *second* covariant derivatives (which we did not define but are introduced in [**36**, p. 99 f]). It can be shown that

$$\nabla_{X,Y}^2 Z - \nabla_{Y,X}^2 Z = R(X, Y)Z, \qquad (5.12)$$

for smooth vector fields $X, Y, Z \in \mathfrak{X}(M)$. This statement can also be translated to 1-forms and (k, l)-tensor fields (see [**36**, Thm. 7.14] for more details).

5.1.5. Expansion in normal coordinates. We conclude this section about the local curvature invariants of a Riemannian manifold by making a connection to Riemann's original approach to curvature via an expansion in normal coordinates. Surprisingly, these results are hard to find and not really covered in standard text books on Riemannian Geometry. One can find them without proofs in Riemann's habilitation lecture and Berger's overview book [**7**, p. 202]. Almost complete proofs in the analytic setting can be found in Heckman's lecture notes [**29**, Sec. 3.5], Gray's book [**25**, Ch. 9] and Weyl's comments [**48**, p. 25 ff] to Riemann's original article. Apparently a derivation is also contained in Spivak's volumes [**54**]. We will sketch the approach used in [**25**, Ch. 9] (unfortunately Gray also uses the opposite sign convention but below it is converted).

Our aim is to observe how curvature describes the deviation of the Riemannian metric to flat space in a very geometric way. We can also understand Ricci and scalar curvature in this way. In my opinion, this geometric understanding of curvature is crucial and naturally resurfaces in the context of smooth curvature comparison theorems (see, for instance, [**18**]). Subsequently, such geometric interpretations of curvature reappeared as definitions in what is now called Synthetic Geometry, that is, as sectional curvature bounds in length metric spaces (see, for instance, [**1**, **11**, **27**]) and as Ricci curvature bounds in metric measure spaces (used in geometric measure theory mixed with optimal transport theory, see [**58**]). These ideas are the forefront of research in Riemannian Geometry today.

Assume that (M, g) is an *analytic* Riemannian manifold, that is, a manifold with an atlas whose transition maps and the Riemannian metric (and thus the exponential map) are real analytic and thus have locally converging power series expansions.

Let $p \in M$ and $(x^1, \dots, x^n)$ be normal coordinates at p. The normal coordinate vector fields ∂_i are orthonormal at p, and the components of a covariant tensor field $W \in \mathcal{T}^{(0,l)}(M)$ satisfy

$$W(\partial_{a_1}, \dots, \partial_{a_l}) = W_{a_1 \dots a_l} = \sum_{k=0}^{\infty} \frac{1}{k!} (\partial_{i_1} \dots \partial_{i_k} W_{a_1 \dots a_l})(p) x^{i_1} \dots x^{i_k}.$$

One can express the right hand side in terms of covariant derivatives of W and R [**27**, Thm. 9.6], which in the case that W is parallel is of the form

$$W_{a_1 \dots a_l} = W_{a_1 \dots a_l}(p) + \frac{1}{6} \sum_{k=1}^{l} R_{i a_k j}{}^s W_{a_1 \dots a_{k-1} s a_{k+1} \dots a_l}(p) x^i x^j + \text{higher order terms}.$$

The expansion of g in normal coordinates about p can then be obtain as a consequence (we will not prove it, but see [**25**, Cor. 9-8]).

THEOREM 5.21. *Let (M, g) be an analytic Riemannian manifold and $(x^1, \dots, x^n)$ be normal coordinates about a point $p \in M$. Then the metric tensor is given by*

$$g_{ij} = \delta_{ij} + \frac{1}{3} R_{kilj}(p) x^k x^l + \frac{1}{6} \nabla_k R_{limj}(p) x^k x^l x^m + \text{higher order terms}.$$

5.2. Sectional curvature

At this point we can readily define the sectional curvature of a point p in the plane $\Pi \subseteq T_p M$ and show that it contains the *same* information as the Riemann curvature tensor. This is indeed how many authors of text books on Riemannian Geometry proceed (see, for instance, [**23**, **29**, **43**, **45**]). The relation to Riemann's original approach via the Gaussian curvature of a

surface, however, is then lost and needs to be established a posteriori. Indeed, this connection can only be made precise once we understand the instrinsic (and extrinsic) curvature of submanifolds. We will therefore return to this geometric interpretation of sectional curvature in Section 6.2.2.

5.2.1. Definition. As already mentioned, the sectional curvature at a point p depends on two-dimensional subspaces $\Pi \subseteq T_pM$. We think of Π as being spanned by a pair of (linearly independent) vectors v, w in an inner product space V. By

$$|v \wedge w| := \sqrt{|v|^2 |w|^2 - \langle v, w \rangle^2}$$

we denote the area of the parallelogram spanned by v and w (compare this to the definition of the canonical Riemannian volumes in Section 2.2.2!).

Recall that, by the Cauchy–Schwarz inequality, $|v \wedge w| \geq 0$ with equality if and only if v and w are linearly dependent. If v and w are orthonormal then $|v \wedge w| = 1$. Hence the following definition makes sense.

DEFINITION 5.22. Let (M, g) be a Riemannian manifold of dimension $n \geq 2$, and $p \in M$. If v, w are two linearly independent vectors in T_pM, then the *sectional curvature* of the plane Π spanned by v and w is defined by

$$\sec(v, w) := \frac{Rm_p(v, w, w, v)}{|v \wedge w|^2}. \tag{5.13}$$

Note that the sectional curvature can be defined equally well for vector fields $X, Y \in \mathfrak{X}(M)$ (at least on a maximal open subset of M where X_p and Y_p are linearly independent, in order to guarantee that the denominator is nonzero). For $\sec(v, w)$ to be a sensible definition it remains to observe that $\sec(v, w)$ only depends on $\Pi = \mathrm{span}\{v, w\}$ but not the particular choice of v and w.

LEMMA 5.23. *Let (M, g) be a Riemannian manifold. Suppose $\Pi \subseteq T_pM$ is a nondegenerate tangent plane at p. Then*

$$\sec(\Pi) = \sec(v, w) \tag{5.14}$$

is independent of the choice of basis v, w for Π.

EXERCISE 5.24. Prove Lemma 5.23. (*Hint*: Avoid long calculations by passing from the basis (v, w) to another basis $(\tilde{v}, \tilde{w})$ and iterating the elementary transformations (a) $(v, w) \to (w, v)$, (b) $(v, w) \to (\lambda v, w)$, and (c) $(v, w) \to (v + \lambda w, w)$. Observe that $\sec(v, w)$ is invariant under these transformations, which completes the proof.)

Note that the sectional curvature is often denoted by K in place of sec (see [**23, 29, 43**]). This is because the sectional curvature of the plane $\Pi \subseteq T_pM$ is directly related to the Gaussian curvature K at p of the embedded surface $S_\Pi = \exp_p(\Pi \cap V)$ in $U \subseteq M$ (where V is a star-shaped region around 0 in T_pM, which via $\exp_p$ is diffeomorphic to an open neighborhood U of p in M), called the *plane section determined by* Π. We will make this connection of the sectional and Gaussian curvature precise in Section 6.1, but the following example already hints that we are on the right track.

EXAMPLE 5.25 (Curvature of surfaces in $\mathbb{R}^3$). Suppose M is a smooth surface in $\mathbb{E}^3$ with local parametrization $X \colon \mathbb{R}^2 \supseteq U \to \mathbb{R}^3$ and first fundamental form g with coefficients $E = |\frac{\partial X}{\partial u}|^2$, $F = \frac{\partial X}{\partial u} \cdot \frac{\partial X}{\partial v}$, and $G = |\frac{\partial X}{\partial v}|^2$ (see Example 2.33), i.e.,

$$g = E \, du^2 + 2F \, du \, dv + G \, dv^2.$$

Let K denote the Gaussian curvature and recall that it is given by (see [**45**, Sec. 8.1] or the "Curves and Surfaces" course [**28**])

$$K = \frac{LN - M^2}{EG - F^2},$$

where $L = \frac{\partial^2 X}{\partial^2 u} \cdot n$, $M = \frac{\partial^2 X}{\partial u \partial v} \cdot n$ and $N = \frac{\partial^2 X}{\partial^2 v} \cdot n$ with respect to the unit normal vector $n = \frac{X_u \times X_v}{|X_u \times X_v|}$ are the coefficients of the second fundamental form. In terms of the Riemann curvature tensor (via the Christoffel symbols, see (5.5)) the Gauss equations read

$$EK = R_{211}{}^2, \quad FK = R_{212}{}^2 = R_{121}{}^1, \quad GK = R_{122}{}^1,$$

which in turn implies that

$$R_{1221} = R_{122}{}^k g_{k1} = GKE - FKF = K(EG - F^2)$$

and hence

$$\sec(\partial_u, \partial_v) = \frac{R_{1221}}{EG - F^2} = K.$$

EXERCISE 5.26. Fill in the details in Example 5.25 (in particular, the computation of EK, FK, and GK, using the Gauss equations).

5.2.2. Relation to the Riemann curvature tensor. Besides the (upcoming) simpler geometric interpretation of the sectional curvature, we can still show that $\sec(\Pi)$, for all nondegenerate planes Π, completely determines the Riemann curvature tensor R. While this result is of great importance, with the above definition of sectional curvature the proof is purely algebraic (using the algebraic symmetries of R obtained in Proposition 5.16) and no deep insights are needed.

THEOREM 5.27. *The Riemann curvature of a Riemannian manifold is completely determined by its sectional curvature, and vice versa.*

PROOF. It follows immediately from Definition 5.22 (together with Lemma 5.23¸) that the sectional curvature is fully determined by the Riemann curvature.

To prove the converse observe that, since both expressions are given pointwise, it suffices to solve the associated linear problem on a real vector space V of dimension $n \geq 2$: Suppose Rm is a quadrilinear map $V \times V \times V \times V \to \mathbb{R}$ that satisfies properties (i)–(iv) of Proposition 5.16 (sometimes called form of curvature type on V or *algebraic curvature tensor*). Then one can show that Rm is completely determined by the knowledge of the values $Rm(v, w, v, w)$ for all $v, w \in V$ as follows. By assumption we know $Rm(v, w, v, w)$ for all $v, w \in V$. By replacing v with $v + x$ in $Rm(v, w, v, w)$ and using the multilinearity and symmetry (iii) we have

$$Rm(v + x, w, v + x, w) = Rm(v, w, v, w) + 2Rm(v, w, x, w) + Rm(x, w, x, w),$$

and thus also know $Rm(v, w, x, w)$ for all $v, w, x \in V$. In the next step we replace w with $w + y$, and hence obtain

$$Rm(v, w + y, x, w + y) = Rm(v, w, x, w) + Rm(v, w, x, y) + Rm(v, y, x, w) + Rm(v, y, x, y),$$

and therefore $A(v, w, x, y) := Rm(v, w, x, y) + Rm(v, y, x, w)$ is known for all $v, w, x, y \in V$. By (iii) and (i) furthermore

$$
\begin{aligned}
A(v, w, x, y) &:= Rm(v, w, x, y) + Rm(v, y, x, w) \\
&= Rm(v, w, x, y) + Rm(x, w, v, y) \\
&= Rm(v, w, x, y) - Rm(w, x, v, y).
\end{aligned}
$$

Thus if $\widetilde{Rm}$ is another such quadrilateral form of curvature type with $\sec = \widetilde{\sec}$, then for all $v, w, x, y \in V$

$$
A(v, w, x, y) = \widetilde{A}(v, w, x, y),
$$

and thus

$$
Rm(v, w, x, y) - \widetilde{Rm}(v, w, x, y) = Rm(w, x, v, y) - \widetilde{Rm}(w, x, v, y).
$$

In other words, the expression $Rm(v, w, x, y) - \widetilde{Rm}(v, w, x, y)$ is invariant by cyclic permutations in the first three slots. Applying the algebraic Bianchi identitiy (iv) of Proposition 5.16 thus yields

$$
3[Rm(v, w, x, y) - \widetilde{Rm}(v, w, x, y)] = 0,
$$

hence Rm is uniquely determined by the sectional curvature. $\qquad \square$

5.2.3. Constant sectional curvature. We consider some important special cases related to sectional curvature.

DEFINITION 5.28. A Riemannian manifold/metric is said to have *constant sectional curvature* if the sectional curvatures are the same for all planes at all points.

Let us first understand when the sectional curvature is independent of the section in terms of the Riemann curvature tensor. We already know that this is possible, in principle, by Theorem 5.27.

PROPOSITION 5.29. *Let (M, g) be a Riemannian manifold. Then $\sec(\Pi) = c$ for all planes $\Pi \subseteq T_p M$ if and only if*

$$
R(v, w)x = c(\langle w, x \rangle v - \langle v, x \rangle w) \tag{5.15}
$$

for all $v, w, x \in T_p M$.

For the proof (and more characterizations) we refer to [**45**, Prop. 3.1.3]. A neat charaterization of constant curvature spaces can be given by using the Kulkarni–Nomizu product $\owedge$ which is defined in (5.29) below (see [**36**, Prop. 8.36] for more details).

PROPOSITION 5.30. *Let M be an n-dimensional manifold. A Riemannian metric g on M has constant curvature c if and only if*

$$
Rm = \frac{1}{2} cg \owedge g.
$$

In this case, the Ricci tensor and scalar curvature of g are given by

$$
Rc = (n-1)cg, \qquad S = n(n-1)c.
$$

Once we have understand sectional curvature via the curvature of submanifolds S_Π by the planes Π we will see in Section 6.2.2 that our standard examples $\mathbb{E}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$ have constant sectional curvatures 0, 1 and -1, respectively. This is not surprising, because we have seen that they are highly symmetric and we know that (local) isometries preserve the curvature tensor by Proposition 5.12. By using this very argument it is easy to see that all frame-homogeneous Riemannian manifolds have constant sectional curvature.

LEMMA 5.31. *If a Riemannian manifold (M, g) is frame-homogenous, then it has constant sectional curvature.*

PROOF. Recall that a frame-homogeneous means that we can move one orthonormal basis to another via an isometry. Thus given any two 2-planes at the same or different points, there is an isometry taking one to the other. Thus the result follows from the isometry invariance of the curvature tensor (Proposition 5.12). $\square$

5.3. Ricci and scalar curvature

While we have tried to motivate why the Riemann curvature tensor is defined as it is, it remains difficult to interpret it geometrically or even grasp what a general Riemannian (sub)manifold with a particular curvature tensor looks like. As often, it is most insightful to restrict ourselves to special (but not too special) classes of examples that are easier to grasp. In the context of sectional curvature we have already mentioned space forms. But there are also other ways to obtain geometric insight. In what follows we construct well-known simpler curvature tensors that contain a lot (but in dimensions ≥ 4 not all) information of the Riemann curvature tensor. The Ricci and scalar curvature are naturally obtained via metric contractions (briefly already mentioned in Section 2.2.1) but can geometrically be better interpreted by using the sectional curvature. Both are of immense importance in both Physics and Geometry.

5.3.1. Definition and symmetries. Note that the definitions of the Ricci and scalar curvature are different for the opposite sign convention of the Riemann curvature tensor. Thus, independent of the convention used, the sign of Rc and S are eventually the same. In our case they read as follows.

DEFINITION 5.32. Let (M, g) be a Riemannian manifold. The *Ricci curvature tensor* $Rc \colon \mathfrak{X}(M)^2 \to \mathbb{R}$ (or Ric) is a covariant 2-tensor field defined as the trace of the curvature operator, i.e.,

$$Rc(X, Y) := \operatorname{tr}(Z \mapsto R(Z, X)Y), \qquad X, Y \in \mathfrak{X}(M).$$

The components of Rc are denoted by

$$R_{ij} := R_{kij}{}^k = g^{km} R_{kijm}.$$

The symmetries of the Riemann curvature tensor obtained in Section 5.1.4 naturally descend to its contractions.

LEMMA 5.33. *The Ricci curvature is a symmetric 2-tensor field. Componentwise it can be expressed in any of the following ways:*

$$R_{ij} = R_{kij}{}^k = R_{ik}{}^k{}_j = -R_{ki}{}^k{}_j = -R_{ikj}{}^k.$$

EXERCISE 5.34. Prove Lemma 5.33 by applying the symmetries of the Riemann curvature tensor.

DEFINITION 5.35. Let (M, g) be a Riemannian manifold. The scalar curvature $S \colon M \to \mathbb{R}$ is the function defined as the trace of the Ricci tensor, i.e.,

$$S = \operatorname{tr}_g Rc = R_i{}^i = g^{ij} R_{ij}.$$

Also the differential Bianchi identity can be contracted. To formulate it in a clean way we use the *exterior*[8] *covariant derivative* $DT \in \mathcal{T}^{(0,3)}(M)$ of a 2-tensor field T, defined by

$$(DT)(X, Y, Z) := -(\nabla T)(X, Y, Z) + (\nabla T)(X, Z, Y).$$

In components we have

$$(DT)_{ijk} = -T_{ij;k} + T_{ik;j}.$$

PROPOSITION 5.36 (Contracted Bianchi identites). *Let (M, g) be a Riemannian manifold. The covariant derivatives of Riemann, Ricci, and scalar curvature satisfy (the trace being on the first and last indices)*

$$\operatorname{tr}_g(\nabla Rm) = -D(Rc), \tag{5.16}$$

$$\operatorname{tr}_g(\nabla Rc) = \frac{1}{2} dS, \tag{5.17}$$

which in components reads

$$R_{ijkl;}{}^i = R_{jk;l} - R_{jl;k}, \tag{5.18}$$

$$R_{il;}{}^i = \frac{1}{2} S_{;l}. \tag{5.19}$$

PROOF. The differential Bianchi identity for R was obtained in (5.11) and reads

$$R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} = 0.$$

Raising the index m and contracting on i and m yields, by Corollary 5.18,

$$0 = R_{ijkl;}{}^i + R_{ijl}{}^i{}_{;k} + \underbrace{R_{ij}{}^i{}_{k;l}}_{=-R_{ijk}{}^i{}_{;l}} = R_{ijkl;}{}^i + R_{jl;k} - R_{jk;l},$$

because ∇ commutes with both trace (Proposition 3.16(iv)) and the musical isomorphism[9]. This establishes the first equation (5.18).

Contraction on j and k in (5.18) furthermore yield

$$R_{il;}{}^i = R_{kil}{}^k{}_{;}{}^i = -R_{ikl}{}^k{}_{;}{}^i \overset{(5.18)}{=} -R_{kl;}{}^k + \underbrace{R_k{}^k{}_{;l}}_{=S_{;l}},$$

hence (5.19). These are the components of (5.16)–(5.17), and so we are done. $\qquad\square$

---

[8]It is called *exterior* covariant derivative because it is a generalization of the ordinary exterior derivative of a 1-form, since $(d\eta)(Y, Z) = -(\nabla\eta)(Y, Z) + (\nabla\eta)(Z, Y)$ (see [**36**, Problem 5-13]).

[9]We did not prove this earlier, but it follows immediately from the properties of the covariant derivative for tensor fields because $F^\flat = \operatorname{tr}(F \otimes g)$, g is parallel and therefore $\nabla_X(F \otimes g) = (\nabla_X F) \otimes g$ by Proposition 3.16(iii), and again because ∇ commutes with tr by Proposition 3.16(iv) therefore

$$\nabla_X(F^\flat) = \nabla_X(\operatorname{tr}(F \otimes g)) = \operatorname{tr}((\nabla_X F) \otimes g) = (\nabla_X F)^\flat$$

(substituting $F = G^\sharp$ also yields $(\nabla_X G)^\sharp = \nabla G^\sharp$). See also [**36**, Prop. 5.17].

PROBLEM 5.37. Suppose (M, g) is a Riemannian manifold and $u \in C^\infty(M)$. Prove *Bochner's formula*:

$$\Delta(\frac{1}{2}| \operatorname{grad} u|^2) = |\nabla^2 u|^2 + \langle \operatorname{grad}(\Delta u), \operatorname{grad} u \rangle + Rc(\operatorname{grad} u, \operatorname{grad} u).$$

(*Hint*: Use that the Laplace operator in any local frame reads $\Delta u = g^{ij} u_{;ij} = u_{;i}{}^i$ (see Problem 4.35(ii)) and the Ricci identity $\beta_{j;pq} - \beta_{j;qp} = R_{pqj}{}^m \beta_m$ for $\beta \in \Omega^1(M)$ (you can assume this without proof, the details are in [**36**, Thm. 7.14]).)

5.3.2. Geometric interpretation of Ricci and scalar curvature. In Section 5.2 we have already shown that the full curvature information of the Riemann curvature is also contained in the sectional curvature. So it is only natural to ask how the translates to the Ricci and scalar curvature. Since the Ricci tensor is symmetric and bilinear it is sufficient to know the values of $Rc(v, v)$ for unit vectors v, analogous to the polarization identity (2.2).

PROPOSITION 5.38. *Let (M, g) be an n-dimensional Riemannian manifold and $p \in M$. For every unit vector $v \in T_pM$ and any orthonormal basis $(b_1 = v, \ldots, b_n)$ for T_pM we have*

$$Rc_p(v, v) = \sum_{k=2}^{n} \sec(v, b_k), \tag{5.20}$$

$$S(p) = \sum_{j \neq k} \sec(b_j, b_k). \tag{5.21}$$

PROOF. Given $v \in T_pM$ and $(b_1, \ldots, b_n)$, as in the hypothesis we have $|b_j \wedge b_k| = \delta_{jk}$, and thus it follows immediately from the definitions that

$$Rc_p(v, v) = R_{11}(p) = R_{k11}{}^k(p) = \sum_{k=1}^{n} Rm_p(b_k, b_1, b_1, b_k) = \sum_{k=2}^{n} \sec(b_1, b_k)$$

and

$$S(p) = R_j{}^j(p) = \sum_{j=1}^{n} Rc_p(b_j, b_j) = \sum_{j,k=1}^{n} Rm_p(b_k, b_j, b_j, b_k) = \sum_{j \neq k} \sec(b_j, b_k). \qquad \square$$

REMARK 5.39 (More averaging). One can also geometrically describe the Ricci curvature in terms of an integral over the sectional curvature that does not refer to a basis (see [**36**, Problem 8-22]). For each $v \in T_pM$

$$Rc_p(v, v) = \frac{n-1}{\operatorname{Vol}(\mathbb{S}^{n-2})} \int_{w \in S_v^\perp} \sec(v, w) dV_{\hat{g}},$$

where $S_v^\perp$ denotes the set of unit vectors in T_pM that are orthogonal to v and $\hat{g}$ denotes the Riemannian metric on $S_v^\perp$ induced from the flat metric g_p on T_pM. Similarly, one can obtain the scalar curvature as an integral over the Ricci curvature [**23**, Ex. 9 on p. 107].

Furthermore, in Proposition 4.33 and Theorem 5.21 we have seen that in normal coordinates $g_{ij}(p) = \delta_{ij} + O(r^2)$ and that curvature appears in the second order terms of this expansion. This expansion immediately also yields geometric interpretations of the Ricci and scalar curvature in terms of volumes.

COROLLARY 5.40. *The expansion of the volume form dV_g of an analytic Riemannian manifold (M, g) in normal coordinates about p is given by*

$$dV_g(\partial_1, \ldots, \partial_n) = (dV_g)_{1\ldots n}$$
$$= 1 - \frac{1}{6}Rc_{ij}(p)x^i x^j - \frac{1}{12}\nabla_i Rc_{jk}(p)x^i x^j x^k + \text{higher order terms.}$$

EXERCISE 5.41. Compute the second order coefficient in the expansion of Corollary 5.40 by assuming Theorem 5.21. (*Hint*: Recall that in local coordinates $dV_g = \sqrt{\det(g_{ij})}dx^1 \wedge \ldots \wedge dx^n$.)

We already know that on surfaces only scalar curvature if relevant. Hence the following result from 1848 should not come as a surprise.

COROLLARY 5.42 (Bertrand–Diguet–Puiseux Theorem). *Let (M, g) be a surface. Then the area of a geodesic ball at a point p with small radius $r > 0$ is given by*

$$\text{Vol}(p, r) = \pi r^2 \left(1 - \frac{1}{24}Sr^2 + O(r^4)\right).$$

EXERCISE 5.43. Prove Corollary 5.42

Originally the above theorem was used as another argument to show that the Gaussian curvature K is intrinsic and does not depend on the embedding (simply because $\text{Vol}(p, r)$ does not). Closely related to is the following conjecture from 1979.

CONJECTURE 5.44 (Gray–Vanhecke [**26**]). *Let (M, g) be a Riemannian manifold. Suppose the volume of a geodesic balls in M coincides with the volume of the corresponding Euclidean ball, that is,*

$$\text{Vol}(p, r) = \frac{(\pi r^2)^{\frac{n}{2}}}{(\frac{n}{2})!},$$

for all p and all sufficiently small $r > 0$. Then (M, g) is flat.

The converse is obviously true. By Corollary 5.42 the conjecture holds for $n = 2$ because the Gaussian curvature is the only relevant curvature. The conjecture also holds for $n = 3$. One can prove it by including higher order terms and using that the Ricci curvature is sufficient. In dimensions $n \geq 4$ the conjecture is still open. It has only been resolved in some special cases (such as locally symmetric spaces).

PROBLEM 5.45. Let (M, g) be a connected Riemannian manifold. Show that if (M, g) is homogeneous, then it has constant scalar curvature.

5.3.3. Traceless Ricci curvature and Einstein metrics. The scalar curvature is the trace of the Ricci tensor, but the Ricci tensor generally contains information that is *not* already encoded in its trace. In this section we investigate what information this traceless part of the Ricci tensor contains.

DEFINITION 5.46. Let (M, g) be an n-dimensional Riemannian manifold. The *traceless Ricci tensor of g* is the symmetric 2-tensor

$$\mathring{Rc} := Rc - \frac{1}{n}Sg.$$

PROPOSITION 5.47. *Let (M, g) be a Riemannian manifold. Then* $\operatorname{tr}_g \mathring{R}c \equiv 0$, *and the Ricci tensor decomposes orthogonally as* $Rc = \mathring{R}c + \frac{1}{n}Sg$. *Therefore,*

$$|Rc|_g^2 = |\mathring{R}c|_g^2 + \frac{1}{n}S^2. \tag{5.22}$$

PROOF. Since in every local frame $\operatorname{tr}_g g = g_{ij}g^{ij} = \delta_i^i = n$, it follows immediately from the definition that $\operatorname{tr}_g \mathring{R}c \equiv 0$.

Orthogonality is evident because for any symmetric 2-tensor h (and thus also $\mathring{R}c$) we have

$$\langle h, g \rangle = g^{ik}g^{jl}h_{ij}g_{kl} = g^{ij}h_{ij} = \operatorname{tr}_g h,$$

and thus, in particular, $\langle \mathring{R}c, g \rangle = 0$.

Finally, (5.22) holds because $\langle g, g \rangle = \operatorname{tr}_g g = n$. □

We know establish a connection of the traceless Ricci curvature to the (Riemannian) Einstein equations. More about the Lorentzian context and application in General Relativity in Section 5.3.4 below.

DEFINITION 5.48. A Riemannian metric g is called an *Einstein metric* if

$$Rc = \lambda g \qquad \text{for some constant } \lambda.$$

The following result is a consequence of the twice-contracted second Bianchi identity and shows that we do not actually need to assume that λ is a constant.

PROPOSITION 5.49 (Schur's Lemma). *Suppose (M, g) is a connected Riemannian manifold of dimension $n \geq 3$ whose Ricci tensor satisfies*

$$Rc = fg$$

for some $f \in C^\infty(M)$. Then f is constant and g is an Einstein metric.

PROOF. Assume that $Rc = fg$, then taking the trace on both sides yields

$$nf = \operatorname{tr}_g(fg) = \operatorname{tr}_g(Rc) = S.$$

Hence $\mathring{R}c = Rc - \frac{1}{n}Sg \equiv 0$, and therefore $\nabla \mathring{R}c \equiv 0$. Since also $\nabla g \equiv 0$ by Proposition 3.23 we have

$$\nabla Rc = \frac{1}{n}(\nabla S)g = \frac{1}{n}(dS)g, \tag{5.23}$$

which in coordinates reads $R_{ij;k} = \frac{1}{n}S_{;k}g_{ij}$. Taking the trace in the first and last components i and k (the last one corresponds to the covariant derivative, *not* both components of g) yields by the contracted second Bianchi identity from Proposition 5.36

$$\frac{1}{2}dS \overset{(5.17)}{=} \operatorname{tr}_g(\nabla Rc) \overset{(5.23)}{=} \frac{1}{n}dS.$$

Since $n \geq 3$ we have that $dS \equiv 0$, and thus S (and therefore f) is constant on the connected components of M. □

COROLLARY 5.50. *If (M, g) is a connected Riemannian manifold of dimension $n \geq 3$, then g is Einstein if and only if $\mathring{R}c \equiv 0$.*

PROOF. If g is an Einstein metric with $Rc = \lambda g$, then taking the trace on both sides yields $\lambda = \frac{1}{n}S$, and therefore

$$\mathring{Rc} = Rc - \frac{1}{n}Sg = Rc - \lambda g.$$

Conversely, if $\mathring{Rc} = 0$, then $Rc = \frac{1}{n}Sg$, and by Schur's Lemma g is Einstein. $\qquad\square$

Schur's Lemma is often employed to prove roundness of geometric objects, for instance, to characterize the limits of convergent geometric flows (such as Ricci flow and mean curvature flow). So-called *almost rigidity* results are also true: Camillo De Lellis and Peter Topping [**22**] have recently shown that if the traceless Ricci tensor is approximately zero then the scalar curvature is approximately constant. See Wikipedia for more information (but note that there is also an unrelated Schur Lemma in Representation Theory).

PROBLEM 5.51. Show that a Riemannian 3-manifold is Einstein if and only if it has constant sectional curvature.

5.3.4. Curvature in General Relativity (not covered in the course). The name "Einstein metric" in Section 5.3.3 originates from the general theory of Relativity[10] obtained by Albert Einstein in 1915. In this theory graviation is no longer described as a force but by the geometry of a 4-dimensional (time-oriented) Lorentzian manifold. More precisely, General Relativity is the study of solutions (M, g) of the *Einstein equation*[11]

$$Rc - \frac{1}{2}Sg = T, \tag{5.24}$$

where T is the *stress-energy tensor* describing the matter content of the universe. The reason for formulating the Einstein equation not just with the Ricci tensor is due to the fact that[12] $\mathrm{div}\, Rc \neq 0$ but from a physical perspective some kind of local energy conservation in the form of

$$\mathrm{div}\, T = 0,$$

in coordinates $T^{\alpha\beta}{}_{;\beta} = 0$, is desired.

EXERCISE 5.52. Check that the *Einstein tensor* $G := Rc - \frac{1}{2}Sg$ is indeed divergence-free.

The special case $T \equiv 0$ describes the *vacuum Einstein equation*. Note that taking the trace on both sides of (5.24) then yields $S = 2S$ (since $\mathrm{tr}_g\, g = \dim M = 4$), which implies $S = 0$. Hence the vacuum Einstein equation is equivalent to

$$Rc = 0, \tag{5.25}$$

---

[10]Note that Riemann has not made any direct connections to Physics in his habilitation lecture. In fact, after mentioning the work of Newton he concludes with "Es führt dies hinüber in das Gebiet einer andern Wissenschaft, in das Gebiet der Physik, welches wohl die Natur der heutigen Veranlassung nicht zu betreten erlaubt." (German for "This leads us into the domain of another science, of physics, into which the object of this work does not allow us to go to-day."). It is nice to see that today there is actually a very elegant, deep, and fruitful connection of Riemann's Geometry to Physics.

[11]Here (5.24) is normalized so that the discussion of the physical constants can be neglected.

[12]We have defined the divergence of a vector field in Definition 2.60, but by using the result of Problem 4.35 one generalizes it by $\mathrm{div}\, F = \mathrm{tr}_g(\nabla F)$ to any tensor field F.

which means that the metric g is indeed Einstein in the sense of Definition 5.48. Minkowski space (described in Example 2.24) is the simplest solution to equation 5.25, the Schwarzschild–Droste and Kerr metrics describing static and rotating black holes, respectively, are other prominent solutions.

In 1917 Einstein added a cosmological constant Λ to (5.24) in the hope to obtain static solutions to (5.24) (a then prevalent point of view) yielding

$$Rc - \frac{1}{2}Sg + \Lambda g = T, \qquad (5.26)$$

then removed it again in 1931 after Hubble's observation of the expanding universe. In the 1990s a positive cosmological constant was added again by the physics community after the discovery of the accelerating expansion of the universe, which also offers the currently simplest explanation to dark energy. In any case, in the vacuum setting this would mean that

$$Rc = \Lambda g, \qquad (5.27)$$

which coincides precisely with the Riemannian definition of Einstein metrics.

EXERCISE 5.53. Verify that (5.26) with $T \equiv 0$ indeed implies (5.27)

In the quest for finding a variational formulation of the Einstein equations (which is highly relevant from a physics perspective), David Hilbert was searching for a suitable action. He showed that the Einstein equations appear as critical points of the *Einstein–Hilbert action*

$$\mathcal{S}(g) := \int_M S dV_g. \qquad (5.28)$$

In the language of the calculus of variations, the Einstein equation is thus simply the Euler–Lagrange equation $\delta S(g) = 0$ of the functional 5.28.

However, unlike in other physical theories there is no sensible total energy[13]. In 1915 Hilbert suggested to Emmy Noether to investigate this problem of energy and conservation of energy. This is how she obtained not only an answer to the problem in General Relativity but also her first theorem (now called *Noether's Theorem*) [42] (see also [17, 46]) determining the conserved quantities for *every* system of physical laws that possesses some continuous symmetry (think of conservation of energy, momentum, center of mass etc.). Unfortunately, for General Relativity with its much larger infinite symmetry group (the Einstein equations are invariant under coordinate change, thus diffeomorphism invariant) her first result cannot be used. Even Noether's second theorem which is applicable in this setting yields nothing new in General Relativity, only the contracted second Bianchi identity that we have already seen holds on *any* semi-Riemannian manifold anyway.

Nonetheless, conservation laws for energies/masses are still very relevant in General Relativity. The ADM mass, for instance, is well-defined and nonnegative for asymptotically flat Riemannian initial data slices with nonnegative scalar curvature. This is the celebrated Positive Mass Theorem of Richard Schoen and Shing-Tung Yau [51, 52] (for dimension $n < 8$) and Edward Witten [57] (valid for all dimensions on spin manifolds), both derived around 1980, and for which both Yau and Witten in part obtained their Fields medals.

If you would like to learn more about the interplay between Geometry and General Relativity I recommend the Master course "Singularities and Black Holes" as a follow-up to this

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[13]See also the discussion of Michael Weiss and John Baez trying to answer Is energy conserved in General Relativity?

course, and if you like it very much the Gravity$^+$ synergy track in the Mathematical Physics specialization. But even if physics is not for you, let me assure you that the Einstein manifolds that show up as critical points of the total scalar curvature functional (5.28) are relevant also in pure Riemannian Geometry today (but note that there are even smooth compact manifolds that do not admit any Einstein metrics at all, see [**8**, Ch. 6]).

5.4. Weyl curvature (not covered in the course)

The Ricci and scalar curvature introduced in Section 5.3 contain a lot of important information of the Riemann curvature tensor but clearly not all. The Weyl curvature tensor encodes all the rest. We will see that it is closely related to the conformal structure of manifold (recall Section 2.4.3 about conformal transformations) and as such, for instance, also of great importance in General Relativity. We will only be sketchy in this section, further details can be found in Lee's book [**36**, p. 212–222] and in Petersen's book [**45**, Ex. 3.4.23–26].

5.4.1. Definition. The first question to ask is: *How much information exactly is contained in the Ricci curvature?* A rough answer can be obtained by counting dimensions. The approach is a bit similar to what we have done in Theorem 5.27 for sectional curvature. Assuming we are in a real-vector space V, then the space $\mathcal{R}(V^*) \subseteq T^4(V^*)$ of covariant 4-tensors that have the symmetries (i)–(iv) of Proposition 5.16 of the Riemann curvature tensor turns out to be

$$\dim \mathcal{R}(V^*) = \frac{n^2(n^2 - 1)}{12}$$

(see [**36**, Prop. 7.21]). The elements of $\mathcal{R}(V^*)$ are called *algebraic curvature tensors*. The Ricci tensor, on the other hand, due to being symmetric contains the information of order $n(n-1)$, so clearly some information is lost in the process of taking the trace as soon as $n \geq 4$. Our approach is to recover as much as possible of the Riemann curvature tensor, and express the remaining part by suitable new tensor that is trace-free.

If we endow V with a scalar product $g \in \Sigma^2(V^*)$ (denoting the space of symmetric tensors) and consider the trace operation $\mathrm{tr}_g \colon \mathcal{R}(V^*) \to \Sigma^2(V^*)$ with respect to the first and last indices (as in $Rc = \mathrm{tr}_g(Rm)$), then we would like to know how far from being bijective the tr_g-operator is. Assuming surjectivity, injectivity can be studied by constructing a right inverse to the trace, i.e., a map

$$G \colon \Sigma^2(V^*) \to \mathcal{R}(V^*)$$

such that

$$\mathrm{tr}_g(G(h)) = h, \qquad h \in \Sigma^2(V^*).$$

It turns out that this is possible with the help of the *Kulkarni–Nomizu product*, which for $h, k \in \Sigma^2(V^*)$ is defined by

$$h \owedge k(w, x, y, z) := h(w, z)k(x, y) + h(x, y)k(w, z) - h(w, y)k(x, z) - h(x, z)k(w, y), \quad (5.29)$$

and in components reads

$$(h \owedge k)_{ijlm} = h_{im}k_{jl} + h_{jl}k_{im} - h_{il}k_{jm} - h_{jm}k_{il}.$$

The right inverse for tr_g is given by

$$G(h) := \frac{1}{n-2}\left(h - \frac{\mathrm{tr}_g h}{2(n-1)}g\right) \owedge g,$$

and furthermore

$$\mathrm{Im}\, G = \mathrm{Ker}(\mathrm{tr}_g)^\perp$$

(see [**36**, Prop. 7.23]).

DEFINITION 5.54. Let (M, g) be a Riemannian manifold. The *Schouten tensor of g* is the symmetric 2-tensor field

$$P := \frac{1}{n-2}\left(Rc - \frac{S}{2(n-1)}g\right),$$

and the *Weyl tensor of g* is the algebraic curvature tensor field

$$W := Rm - P \varotimes g = Rm - \frac{1}{n-2}Rc \varotimes g + \frac{S}{2(n-1)(n-2)}g \varotimes g.$$

The Weyl tensor is trace-free.

PROPOSITION 5.55. *For every Riemannian manifold (M, g) of dimension $n \geq 3$,*

$$\operatorname{tr}_g W = 0,$$

and

$$Rm = W + P \varotimes g$$

is the orthogonal decomposition of Rm corresponding to $\mathcal{R}(V^*) = \operatorname{Ker}(\operatorname{tr}_g) \oplus \operatorname{Ker}(\operatorname{tr}_g)^\perp$.

EXERCISE 5.56. Prove Proposition 5.55 by using the formula for G above and $P \varotimes g = G(Rc) = G(\operatorname{tr}_g Rm)$ (show this).

This result simplifies $\mathcal{R}(V^*)$ is lower dimensions, and one can show that for dimension $n = 3$ the map G is an isomorphism [**36**, Cor. 7.25]. Therefore, in dimension $n = 3$ the composition $\operatorname{tr}_g \circ G$ is the identity and thus also tr_g an isomorphism. Since $\operatorname{tr}_g W = 0$ by Proposition 5.55 it follows that W itself is always zero. Hence in dimension 3 the Ricci curvature determines the *entire* curvature tensor. More generally, in dimension $n \geq 3$ one can prove the following formula.

PROPOSITION 5.57 (Ricci decomposition of the curvature tensor). *Let (M, g) be a Riemannian manifold of dimension $n \geq 3$. Then the $(0, 4)$-curvature tensor of g has the following orthogonal decomposition:*

$$Rm = W + \frac{1}{n-2}\mathring{Rc} \varotimes g + \frac{1}{2n(n-1)}Sg \varotimes g. \tag{5.30}$$

EXERCISE 5.58. Prove the formula (5.30) in Proposition 5.57. (*Hint*: Substitute the formula for the traceless Ricci curvature, that is, $Rc = \mathring{Rc} + \frac{1}{n}Sg$ into (5.30) and simplify. The orthogonality follows from [**36**, Lemma 7.22].)

PROBLEM 5.59. Derive the formulas of the Riemann curvature tensor Rm in terms of Rc and S in dimension $n = 2$ and $n = 3$. (*Hint*: The case $n = 3$ follows from above. See [**36**, p. 215–216] for more info on $n = 2$.)

5.4.2. Conformal flatness. The strength of the Weyl and Schouten tensors lie in their remarkable behavior with respect to conformal transformations. Recall that, by Definition 2.102, conformal metrics g and $\widetilde{g}$ on a manifold M are always related by

$$\widetilde{g} = e^{2f}g \qquad \text{for some } f \in C^\infty(M).$$

One can show that the Levi–Civita connections of g and $\widetilde{g}$ are related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + (Xf)Y + (Yf)X - \langle X, Y \rangle_g \operatorname{grad} f. \qquad (5.31)$$

EXERCISE 5.60. Show that $\widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + f_{;i}\delta_j^k + f_{;j}\delta_i^k - g^{kl}f_{;l}g_{ij}$ by using (3.15). Then derive (5.31) by expanding it in coordinates.

By using the properties of the Kulkarni–Nomizu product one can prove the following result (for the proof see [**36**, Thm. 7.30]).

THEOREM 5.61 (Conformal transformation of curvature). *Let (M, g) be a Riemannian manifold of dimension n, $f \in C^\infty(M)$, and $\widetilde{g} = e^{2f}g$. Then the curvature tensors of $\widetilde{g}$ are related to those g by*

$$\widetilde{Rm} = e^{2f}\left(Rm - (\nabla^2 f) \otimes\!\!\!\!\wedge\, g + (df \otimes df) \otimes\!\!\!\!\wedge\, g - \frac{1}{2}|df|_g^2(g \otimes\!\!\!\!\wedge\, g)\right), \qquad (5.32)$$

$$\widetilde{Rc} = Rc - (n-2)(\nabla^2 f) + (n-2)(df \otimes df) - (\Delta f + (n-2)|df|_g^2)g, \qquad (5.33)$$

$$\widetilde{S} = e^{-2f}(S - 2(n-1)\Delta f - (n-1)(n-2)|df|_g^2), \qquad (5.34)$$

where $\Delta f = \operatorname{div}(\operatorname{grad} f)$ is the Laplace–Beltrami operator with respect to g as defined in Section 2.2.3.

If, in addition, dimension $n \geq 3$, then

$$\widetilde{P} = P - \nabla^2 f + (df \otimes df) - \frac{1}{2}|df|_g^2 g, \qquad (5.35)$$

$$\widetilde{W} = e^{2f}W. \qquad (5.36)$$

EXERCISE 5.62. Suppose (M, g) is a Riemannian manifold and $\widetilde{g} = \lambda g$ for some $\lambda > 0$. Use Theorem 5.61 to prove that for every $p \in M$ and plane $\Pi \subseteq T_p M$, the sectional curvatures of Π with respect to $\widetilde{g}$ and g are related by $\widetilde{\sec}(\Pi) = \lambda^{-1}\sec(\Pi)$.

We see from (5.36) that the conformal curvature structure of a metric tensor is encoded in the Weyl tensor. In fact, there is an important characterization of local conformal flatness that was already mentioned earlier (for the proof see [**36**, p. 216–222]).

DEFINITION 5.63. A Riemannian manifold is said to be *locally conformally flat* if every point has an open neighborhood that is conformally equivalent to an open subset in Euclidean space.

THEOREM 5.64 (Weyl–Schouten Theorem). *Suppose (M, g) is a Riemannian manifold of dimension ≥ 3.*

(i) *If $n \geq 4$, then (M, g) is locally conformally flat if and only if the Weyl tensor W is identically zero.*

(ii) *If $n = 3$, then (M, g) is locally conformally flat if and only if the* Cotton tensor[14] *$C = -DP$ is identically zero.*

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[14]As before, D denotes the exterior covariant derivative.

PROBLEM 5.65. Show that, as long as $n \geq 3$, the Weyl tensor vanishes for locally conformally flat g. (*Hint*: Use that for an embedding $\varphi \colon U \to \mathbb{R}^n$ we have $\varphi^* \bar{g} = e^{2f} g =: \widetilde{g}$.)

The only other nontrivial case $n = 2$ is handled using isothermal coordinates. In Section 2.4.3 we have sketched that the round sphere $\mathbb{S}^2$ and hyperbolic plane $\mathbb{H}^2$ are locally conformally flat. It actually turns out that *every* Riemannian 2-manifold is locally conformally flat (see, for instance, [**19**]).

All results above translate verbatim to semi-Riemannian manifolds. The 2-dimensional Lorentzian result is obtained as follows.

PROBLEM 5.66. Suppose (M, g) is a 2-dimensional Lorentzian manifold, and $p \in M$.
   (i) Show that there is a smooth local frame (E_1, E_2) in a neighborhood of p such that $g(E_1, E_1) = g(E_2, E_2) = 0$.
  (ii) Show that there are smooth coordinates (x, y) in a neighborhood of p such that $(dx)^2 - (dy)^2 = fg$ for positive function $f \in C^\infty(M)$. (*Hint*: Use [**36**, Prop. A.45] to show that there exist coordinates (t, u) in which $E_1 = \partial_t$, and coordinates (v, w) in which $E_2 = \partial_v$, and set $x = u + w$, $y = u - w$.)
 (iii) Show that (M, g) is locally conformally flat.

PROBLEM 5.67. Let (M, g) be a Riemannian manifold of dimension $n \geq 3$. We define the *conformal Laplacian* $L \colon C^\infty(M) \to C^\infty(M)$ by

$$Lu := -\frac{4(n-1)}{n-2}\Delta u + Su,$$

where Δ is the Laplace–Beltrami operator of g and S is its scalar curvature.
   (i) Suppose $\widetilde{g} = e^{2f} g$ for some $f \in C^\infty(M)$, and $\widetilde{L}$ denotes the conformal Laplacian with respect to $\widetilde{g}$. Show that $e^{\frac{n+2}{2}f}\widetilde{L}u = L\left(e^{\frac{n-2}{2}fu}\right)$.
  (ii) Conclude that $\widetilde{g}$ conformal to g has constant scalar curvature λ if and only if $\widetilde{g} = \varphi^{\frac{4}{n-2}}g$, where φ is a smooth positive solution to the *Yamabe equation* $L\varphi = \lambda\varphi^{\frac{n+2}{n-2}}$.

CHAPTER 6

Extrinsic curvature of submanifolds

Until now we have looked at the intrinsic curvature of a Riemannian manifold, and we have seen that it is a local invariant and such independent of the embedding. In order to understand the Riemann curvature tensor better as a generalization of Gaussian curvature of surfaces in $\mathbb{R}^3$ to higher dimensions (which means, ultimately, to understand sectional curvature geometrically) we also need to understand the interplay of curvature and its extrinsic counterparts that do depend on the embedding.

Many constructions from the "Curves and Surfaces" course will thus reappair here in a broader context, since we consider embeddings in arbitrary Riemannian manifolds and not just in Euclidean space.

Some results we present here, in particular, the Gauss and Codazzi equations also hold in semi-Riemannian geometry and are prominently used in the context of the initial value formulation of the Einstein equations in General Relativity. Moreover, we will will define minimal (hyper)surfaces and show some basic properties relating to their curvature. Besides geodesics, minimal surfaces are another prominent example of variational problems studied in pure Riemannian Geometry and are highly relevant to this day.

6.1. Second fundamental form

6.1.1. Definition. Let (M, g) be a Riemannian submanifold of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$, that is, with respect to an inclusion $\iota_M \colon M \hookrightarrow \widetilde{M}$ we are concerned with the pullback metric $g = \iota_M^* \widetilde{g}$. In what follows we want to better understand the relationship between the geometric of M and that of $\widetilde{M}$. We call $\widetilde{M}$ the *ambient manifold*.

Recall that (see page 33) the orthogonal direct sum decomposition $T\widetilde{M}|_M = TM \oplus NM$ yields the orthogonal *tangential* and *normal projections*, denoted by

$$\pi^\top \colon T\widetilde{M}|_M \to TM,$$
$$\pi^\perp \colon T\widetilde{M}|_M \to NM.$$

We often write $X^\top := \pi^\top X$ and $X^\perp := \pi^\perp X$ for a section X of $T\widetilde{M}|_M$.

Via the tangential projection we defined the tangential connection $\overline{\nabla}^\top$ which we have seen is the Levi–Civita connection of the submanifold in the Euclidean case. Now we are more concerned with the normal component, but basically proceed in the same way: We extend vector fields $X, Y \in \mathfrak{X}(M)$ to vector fields on an open subset of $\widetilde{M}$ (still denoted by X and Y), apply the ambient covariant derivative operator $\widetilde{\nabla}$ and then decompose on M into

$$\widetilde{\nabla}_X Y = (\widetilde{\nabla}_X Y)^\top + (\widetilde{\nabla}_X Y)^\perp. \tag{6.1}$$

DEFINITION 6.1. Let (M, g) be a Riemannian manifold embedded in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. The *second fundamental form* is the map $\mathrm{II} \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(NM)$ given by

$$\mathrm{II}(X, Y) := (\widetilde{\nabla}_X Y)^\perp,$$

where X and Y are arbitrarily extended to an open subset of $\widetilde{M}$.

Note that II is indeed well-defined since $\pi^\perp$ maps smooth sections to smooth sections. Moreover, the following properties hold.

PROPOSITION 6.2 (Properties of the second fundamental form). *Suppose (M, g) is an embedded Riemannnian submanifold of the Riemannian (or semi-Riemannian) manifold $(\widetilde{M}, \widetilde{g})$, and let $X, Y \in \mathfrak{X}(M)$.*

  (i) *$\mathrm{II}(X, Y)$ is independent of the extensions of X and Y.*
  (ii) *$\mathrm{II}(X, Y)$ is $C^\infty(M)$-bilinear in X and Y.*
  (iii) *$\mathrm{II}(X, Y)$ is symmetric in X and Y.*
  (iv) *The value $\mathrm{II}(X, Y)|_p$ depends only on X_p and Y_p.*

PROOF. (iii) By symmetry of the connection $\widetilde{\nabla}$ we have

$$\mathrm{II}(X, Y) - \mathrm{II}(Y, X) = (\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X)^\perp = [X, Y]^\perp.$$

Since X and Y are tangent to M, so is $[X, Y]$ [**36**, Cor. A.40]. Thus $[X, Y]^\perp = 0$, and II is symmetric.

(i and iv) Recall that $\widetilde{\nabla}_X Y|_p$ only depends on X_p by (3.1), and is thus independent of the extension.

(ii) Since around every point p a function $f \in C^\infty(M)$ can be extended to a smooth function in a neighborhood of M in $\widetilde{M}$ linearity of $\mathrm{II}(X, Y)$ in X and Y follows. $\square$

6.1.2. Gauss formula. Let us now turn to the tangential component. In the case that $(\widetilde{M}, \widetilde{g})$ is Euclidean space we have discussed in Example 3.30 that $\widetilde{\nabla}^\top$ coincides with the induced Levi–Civita connection of (M, g) which follows from the Fundamental Theorem 3.27 of Riemannian Geometry. The same procedure can be applied in this general setting. Again it suffices to prove that the map $\nabla^\top \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$\nabla_X^\top Y := (\widetilde{\nabla}_X Y)^\top$$

is a symmetric connection on M that is compatible with g.

THEOREM 6.3 (Gauss Formula). *Suppose (M, g) is an embedded Riemannian manifold of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. For $X, Y \in \mathfrak{X}(M)$ we have*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X, Y). \tag{6.2}$$

PROOF. Due to the orthogonal decomposition $\widetilde{\nabla}_X Y = (\widetilde{\nabla}_X Y)^\top + (\widetilde{\nabla}_X Y)^\perp$ it is sufficient to prove that $\nabla_X^\top Y := (\widetilde{\nabla}_X Y)^\top = \nabla_X Y$ for all points in M. It is easy to check that $\nabla^\top \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is an affine connection on M.

Compatibility with respect to g follows directly from the compatibility of $\widetilde{\nabla}$ with respect to $\widetilde{g}$: For $X, Y, Z \in \mathfrak{X}(M)$ and smooth extensions $\widetilde{X}, Y, Z$ to an open subset of M in $\widetilde{M}$ we

obtain

$$\nabla_X^\top \langle Y, Z \rangle = X \langle Y, Z \rangle = \widetilde{X} \langle \widetilde{Y}, \widetilde{Z} \rangle = \widetilde{\nabla}_{\widetilde{X}} \langle \widetilde{Y}, \widetilde{Z} \rangle$$
$$= \langle \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}, \widetilde{Z} \rangle + \langle \widetilde{Y}, \widetilde{\nabla}_{\widetilde{X}} \widetilde{Z} \rangle = \langle \pi^\top (\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}), \widetilde{Z} \rangle + \langle \widetilde{Y}, \pi^\top (\widetilde{\nabla}_{\widetilde{X}} \widetilde{Z}) \rangle$$
$$= \langle \nabla_X^\top Y, Z \rangle + \langle Y, \nabla_X^\top Z \rangle,$$

since $\widetilde{Y}$ and $\widetilde{Z}$ are tangent to M.

We prove symmetry: The vector fields $X, Y \in \mathfrak{X}(M)$ are ι-related to the smooth extensions $\widetilde{X}$ and $\widetilde{Y}$. Thus by the naturality of the Lie bracket [**36**, Prop. A.39] also $[X, Y]$ is ι-related to $[\widetilde{X}, \widetilde{Y}]$. In particular, $[\widetilde{X}, \widetilde{Y}]$ is tangent to M and equal to $[X, Y]$. Thus

$$\nabla_X^\top Y - \nabla_Y^\top X = \pi^\top (\underbrace{\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}|_M - \widetilde{\nabla}_{\widetilde{Y}} \widetilde{X}|_M}_{=[\widetilde{X}, \widetilde{Y}]|_M}) = [\widetilde{X}, \widetilde{Y}]|_M = [X, Y].$$

By the Fundamental Theorem of Riemannian Geometry, it follows that $\nabla^\top$ is the unique Levi–Civita connection of (M, g) and therefore equal to ∇. $\square$

6.1.3. Geometric interpretation of the second fundamental form. One can use the Gauss formula to compare intrinsic and extrinsic covariant derivatives along curves. Suppose (M, g) is an embedded Riemannian manifold of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$, and $\gamma \colon I \to M$ is a smooth curve. For $X \in \mathfrak{X}(\gamma)$ tangent to M we have (see [**36**, Cor. 8.3])

$$\widetilde{D}_t X = D_t X + \mathrm{II}(\gamma', X). \tag{6.3}$$

If γ has unit speed, we define the intrinsic *(geodesic) curvature of* γ by the map $\kappa \colon I \to \mathbb{R}$,

$$\kappa(t) := |D_t \gamma'(t)|.$$

Thus γ has vanishing geodesic curvature (in M) if and only if it is a g-geodesic. However, γ also has an extrinsic curvature $\widetilde{\kappa}$ as curve in $\widetilde{M}$. By (6.3),

$$\widetilde{D}_t \gamma' = \mathrm{II}(\gamma', \gamma').$$

One can thus interpret the second fundamental form $\mathrm{II}(v, v)$ for $v \in T_p M$ as the $\widetilde{g}$-acceleration of the g-geodesic γ_v at p. This observation about curves (and geodesics) can be applied also to submanifolds.

DEFINITION 6.4. An embedded Riemannian submanifold (M, g) in $(\widetilde{M}, \widetilde{g})$ is called *totally geodesic* if every geodesic in M is also a geodesic in $\widetilde{M}$.

Thus II can be viewed as measuring the failure of M being totally geodsic in $\widetilde{M}$ (see also Figure 6.1).

PROPOSITION 6.5. *Suppose (M, g) is an embedded Riemannian submanifold in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then M is totally geodesic in $\widetilde{M}$ if and only if the second fundamental form* II *vanishes identically.*

PROBLEM 6.6. Prove Proposition 6.5.

PROBLEM 6.7. Let G be a Lie group with bi-invariant metric.
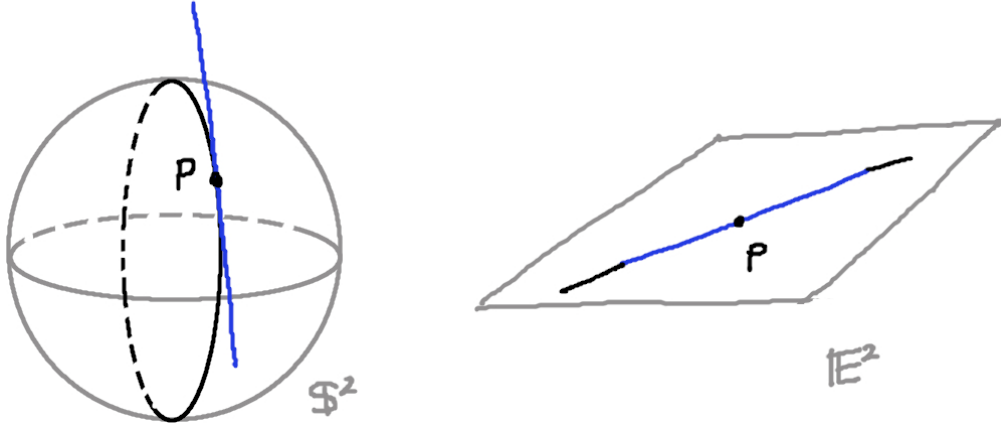
Figure 6.1. In the Euclidean space $\mathbb{E}^3$ the sphere $\mathbb{S}^2$ is not totally geodesic but $\mathbb{E}^2$ is (because $\bar{g}$-geodesics are simply straight lines, indicated in blue).

(i) Suppose X and Y are orthonormal elements of $\mathrm{Lie}(G)$, the Lie algebra of left-invariant vector fiels on G. Show that $\sec(X_p, Y_p) = \frac{1}{4}|[X,Y]|^2$ for each $p \in G$, and conclude that the sectional curvature of (G, g) are all nonnegative.
(ii) Show that every Lie subgroup of G is totally geodesic in G.
(iii) Now suppose G is connected. Show that G is flat if and only if it is abelian.

6.1.4. Gauss and Codazzi equations. The second fundamental form is also crucial in describing the difference in the curvatures of M and $\widetilde{M}$.

THEOREM 6.8 (Gauss Equation). *Suppose (M, g) is an embedded Riemannian submanifold in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. For $W, X, Y, Z \in \mathfrak{X}(M)$ the following equation holds:*

$$\widetilde{Rm}(W, X, Y, Z) = Rm(W, X, Y, Z) - \langle \mathrm{II}(W,Z), \mathrm{II}(X,Y)\rangle + \langle \mathrm{II}(W,Y), \mathrm{II}(X,Z)\rangle. \qquad (6.4)$$

PROOF. Extend W, X, Y, Z arbitrarily to an open neighborhood of $\widetilde{M}$. Then by definition of $\widetilde{Rm}$

$$\widetilde{Rm}(W, X, Y, Z) = \langle \widetilde{\nabla}_W \widetilde{\nabla}_X Y, Z\rangle - \langle \widetilde{\nabla}_X \widetilde{\nabla}_W Y, Z\rangle - \langle \widetilde{\nabla}_{[W,X]} Y, Z\rangle.$$

By the Gauss formula (6.2)

$$\langle \widetilde{\nabla}_W \widetilde{\nabla}_X Y, Z\rangle = \langle \widetilde{\nabla}_W \nabla_X Y, Z\rangle + \langle \widetilde{\nabla}_W \mathrm{II}(X,Y), Z\rangle.$$

Since $\langle \mathrm{II}(X,Y), Z\rangle = 0$ on M, the metric compatibility of $\widetilde{\nabla}$ (and again the Gauss formula) implies

$$\langle \widetilde{\nabla}_W \mathrm{II}(X,Y), Z\rangle = -\langle \mathrm{II}(X,Y), \widetilde{\nabla}_W Z\rangle = -\langle \mathrm{II}(X,Y), \nabla_W Z + \mathrm{II}(W,Z)\rangle$$
$$= -\langle \mathrm{II}(X,Y), \mathrm{II}(W,Z)\rangle.$$

The same decomposition can be obtained for the second coefficient in $\widetilde{Rm}(W, X, Y, Z)$. Thus

$$\widetilde{Rm}(W, X, Y, Z) = \langle \widetilde{\nabla}_W \nabla_X Y, Z\rangle - \langle \mathrm{II}(X,Y), \mathrm{II}(W,Z)\rangle$$
$$- \langle \widetilde{\nabla}_X \nabla_W Y, Z\rangle + \langle \mathrm{II}(W,Y), \mathrm{II}(X,Z)\rangle - \langle \widetilde{\nabla}_{[W,X]} Y, Z\rangle.$$

In each term involving $\widetilde{\nabla}$ only the tangential component survives (because Z is tangential to M), thus we can use g in terms of $\widetilde{g}$ and replace the terms by Rm, and so the Gauss equation (6.4) remains. $\square$

Note that the Gauss equation is a scalar equation. In a similar fashion, one can obtain another vectorial equation in the normal direction. In order to formulate it, one uses the *normal connection* $\nabla^{\perp} \colon \mathfrak{X}(M) \times \Gamma(NM) \to \Gamma(NM)$, defined by

$$\nabla^{\perp}_X N := (\widetilde{\nabla}_X N)^{\perp},$$

where N is extended to a smooth vector field on a neighborhood M of $\widetilde{M}$.

EXERCISE 6.9. Show that $\nabla^{\perp}$ is a well-defined connection on NM which is compatible with $\widetilde{g}$, that is,

$$X \langle N_1, N_2 \rangle = \langle \nabla^{\perp}_X N_1, N_2 \rangle + \langle N_1, \nabla^{\perp}_X N_2 \rangle.$$

Analogous to the Gauss formula we obtain the Weingarten equation for the Weingarten map.

DEFINITION 6.10. Let (M, g) an embedded Riemannian submanifold of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. For each normal vector field $N \in \Gamma(NM)$, the self-adjoint linear map $W_N \colon \mathfrak{X}(M) \to \mathfrak{X}(M)$, characterized by

$$\langle W_N(X), Y \rangle := \langle N, \mathrm{II}(X, Y) \rangle,$$

is called the *Weingarten map in direction of N*.

Since II is bilinear over $C^{\infty}(M)$, also W_N is $C^{\infty}(M)$-linear and thus defines a smooth bundle homomorphism from TM onto itself. It can be used to measure the change in the normal direction. For the proof see [**36**, Prop. 8.4] (basically analogous to the Gauss formula).

PROPOSITION 6.11 (Weingarten equation). *Suppose (M, g) is an embedded Riemannian submanifold in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. For every $X \in \mathfrak{X}(M)$ and $N \in \Gamma(NM)$ we have the orthogonal decomposition*

$$\widetilde{\nabla}_X N = \nabla^{\perp}_X N - W_N(X).$$

Just like the Gauss formula implies Gauss equation, does the Weingarten equation imply the following vector-valued constraint equation, independently discovered by Karl M. Petersen (1853), Gaspare Mainardi (1856), and Delfio Codazzi (1868–1869). This equation is formulated using a combined connection of the tangential connection $\nabla^{\top}$ and normal connection $\nabla^{\perp}$ to form a new connection ∇^F on any tensor product of copies of TM and NM. In particular, for any smooth section $B \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(NM)$, we have

$$(\nabla^F_X B)(X, Y) := \nabla^{\perp}_X (B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z).$$

EXERCISE 6.12. Prove that ∇^F is a connection on F, where $F \to M$ denotes the bundle whose fiber at each point $p \in M$ is a set of bilinear maps $T_p M \times T_p M \to N_p M$ (this is a smooth vector bundle).

THEOREM 6.13 (Petersen–Codazzi–Mainardi Equation). *Suppose (M, g) is an embedded Riemannian submanifold in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. For all $W, X, Y \in \mathfrak{X}(M)$ the following equation holds:*

$$(\widetilde{R}(W, X)Y)^{\perp} = (\nabla^F_W \mathrm{II})(X, Y) - (\nabla^F_X \mathrm{II})(W, Y). \tag{6.5}$$

The proof is similar to that of the Gauss equation, by starting to compute $\widetilde{Rm}(W, X, Y, N)$ for N a normal vector field along M using the Gauss equation (see [**36**, Thm. 8.9] for details).

6.2. Sectional curvature revisited

6.2.1. Geometric interpretation of sectional curvature. By using the Gauss equation we can finally make a connection to Riemann's original definition of curvature. Let us recall some notation first. In Section 5.2 we have defined the sectional curvature of a plane $\Pi \subseteq T_pM$ spanned by linearly independent vectors v and w (where the choice does not matter) via the Riemann curvature tensor as

$$\sec(\Pi) := \sec(v, w) = \frac{Rm_p(v, w, w, v)}{|v \wedge w|^2}.$$

Suppose $\exp_p \colon T_pM \supseteq V \to U \subseteq M$ is a local diffeomorphism. Then we call

$$S_\Pi := \exp_p(\Pi \cap V)$$

the *plane section determined by* Π. It is a 2-dimensional embedded submanifold of M, and swept out by geodesics with initial velocities in $\Pi = T_pS_\Pi$.

From the "Curves and Surfaces" course we know what the Gaussian curvature K of an embedded Riemannian submanifold of $\mathbb{R}^3$ is (see also Example 5.25), and we know that it is an intrinsic quantity by the *Theorema Egregium*. How to define it for an abstract Riemannian 2-manifold (M, g), not necessarily embedded it $\mathbb{R}^3$? It turns out that for surfaces in $\mathbb{R}^3$ we have that the scalar curvature satisfies $S = 2K$ (see Exercise 6.14 below and also Proposition 5.30), so we simply define

$$K := \frac{1}{2}S \tag{6.6}$$

also in the general case.

EXERCISE 6.14. Suppose (M, g) is an embedded Riemannian surface in $\mathbb{E}^3$. Show that $2S = K$. (*Hint:* Use the Gauss equation and express the scalar second fundamental form with respect to an orthonormal basis.)

THEOREM 6.15. *Let (M, g) be a Riemannian manifold, $p \in M$ and Π a plane in T_pM. Then*

$$\sec(\Pi) = K(S_\Pi)(p), \tag{6.7}$$

where $K(S_\Pi)(p)$ denotes the Gaussian curvature of S_Π at p.

PROOF. Let $\hat{g}$ denote the induced Riemannian metric on S_Π, $\widehat{Rm}$ the associated Riemann curvature tensor, and $\widehat{K}(p)$ its Gaussian curvature at p. In the first step we show how $\widehat{K}$ can be computed directly from $\widehat{Rm}_p$. In the second step we show that the second fundamental form of S_Π vanishes at p. Thus by the Gauss equation the curvature tensor of M and S_Π agree at p. Since Rm is used for the definition of the sectional curvature, we obtain equation 6.7.

Step 1. Expression of $\widehat{K}(p)$. We already know by Lemma 5.23 that the $\sec(\Pi)$ is independent of the choice of basis vectors of Π. Assume that (b_1, b_2) is an orthonormal basis of

Π. Then, by Proposition 5.38 we have that

$$\widehat{K}(p) = \frac{1}{2}\widehat{S}(p) = \frac{1}{2}\sum_{i,j=1}^{2}\widehat{Rm}_p(b_i, b_j, b_j, b_i)$$

$$= \frac{1}{2}(\widehat{Rm}_p(b_1, b_2, b_2, b_1) + \widehat{Rm}_p(b_2, b_1, b_1, b_2)) = \widehat{Rm}_p(b_1, b_2, b_2, b_1)$$

by the symmetries of the Riemann curvature tensor.

Step 2. $\Pi = 0$ *at p.* Let $z \in \Pi = T_p(S_\Pi)$ be arbitrary, and let $\gamma = \gamma_z$ be the g-geodesic with initial velocity z. Then $\gamma_z(t) \in S_\Pi$ for t sufficiently close to 0, and by the Gauss formula (6.3) for vector fields along curves therefore

$$0 = D_t\gamma' = \widehat{D}_t\gamma' + \mathrm{II}(\gamma', \gamma').$$

Due to the right hand side being an orthogonal direct sum both terms must vanish identically. Thus at $t = 0$ we have $\mathrm{II}(z, z) = 0$. Since z was arbitrary and Π is symmetric, by the polarization identity $\Pi = 0$ at p.

Step 3. Relation of Rm and $\widehat{Rm}$. By the Gauss equation, since $\mathrm{II} = 0$ at p by Step 2, we have

$$Rm_p(b_1, b_2, b_2, b_1) = \widehat{Rm}_p(b_1, b_2, b_2, b_2)$$

for an orthonormal basis (b_1, b_2) of $\Pi \subseteq T_pM$. Thus by Step 1 it follows immediately from the definition of sectional curvature in Section 5.2 that $K(p) = \sec(b_1, b_2) = \sec(\Pi)$. $\qquad\square$

6.2.2. Constant sectional curvature and model spaces. We now return to complete manifolds with constant sectional curvature, called *space forms*. Let us understand the special cases that we are well acquainted with.

THEOREM 6.16. *The following Riemannian manifolds have the indicated constant sectional curvature:*

  (i) $(\mathbb{R}^n, \bar{g})$ *has constant sectional curvature* 0.
  (ii) $(\mathbb{S}^n(R), \mathring{g})$ *has consant sectional curvature* $1/R^2$.
  (iii) $(\mathbb{H}^n(R), \breve{g})$ *has constant sectional curvature* $-1/R^2$.

For the proof we use Lemma 5.31 for highly symmetric Riemannian manifold that already implies that we are dealing with constant curvature spaces.

PROOF. By Lemma 5.31 is suffices to compute the sectional curvature for one plane at one point.

(i) Since $Rm \equiv 0$ on $\mathbb{R}^n$ it follows from the definition[1] that $\sec \equiv 0$.

(ii) Consider the plane Π spannes by (∂_1, ∂_2) at the point $(0, \ldots, 0, R)$. The geodesics are great circles with initial velocities in Π in the (x^1, x^2, x^{n+1}) subspace. Thus S_Π is isometric to the 2-sphere of radius R embedded in $\mathbb{R}^3$. Since $\mathbb{S}^2(R)$ has Gaussian curvature $1/R^2$ it follows that $\mathbb{S}^n(R)$ has constant sectional curvature $1/R^2$.

(iii) This is Problem 6.17 below. $\qquad\square$

For every real number c we thus have a model space with constant sectional curvature c. These spaces thus play a crucial role in comparison results and local-to-global theorems.

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[1]If we were to use the more geometric definition of sectional curvature via the Gaussian curvature of generated submanifolds, it also follows immediately, because the planes are actually planes and thus have Gaussian curvature zero.

PROBLEM 6.17. Complete the proof of Theorem 6.16 by showing in that the hyperbolic space of radius R has constant sectional curvature equal to $-1/R^2$ in the hyperboloidal model: Compute the sectional curvature form of $\mathbb{H}^n(\mathbb{R}) \subseteq \mathbb{R}^{1,n}$ at the point $(0, \ldots, 0, R)$ and use the Gauss equation.

EXERCISE 6.18. Show that the metric on real projective space $\mathbb{RP}^n$ has constant sectional curvature.

Riemannian manifolds of constant sectional curvature have been widely studied and can be seen to be special also in other ways: In Lemma 5.31 we have mentioned that a frame-homogeneous Riemannian manifold has constant sectional curvature, and in Problem 5.51 Riemannian Einstein 3-manifolds are characterized by having constant sectional curvature. A related result is the following (also taken from [**36**, Ch. 8]).

PROBLEM 6.19. Suppose (M, g) is a 3-dimensional Riemannian manifold is homogeneous and isotropic. Show that g has constant sectional curvature. (*Remark*: An analogous result in dimension 4 is not true, see [**36**, Prob. 8-13].)

6.3. Hypersurfaces

We consider now the special situation that M is a *hypersurface*, that is, a codimension-1 Riemannian submanifold, of $(\widetilde{M}, \widetilde{g})$. Thus at each point there are exactly two unit normal vectors, and due to local orientability we can always choose a smooth unit normal vector field along M in a small neighborhood of each point. If both M and $\widetilde{M}$ are orientable, we can do so globally. In what follows we only do local computations to ensure existence of a unit normal field, but it is still important to realize that several concepts and formulas depend on the chosen orientation.

6.3.1. Scalar second fundamental form and shape operator. Assume we have chosen a unit normal vector field N on a hypersurface $M \subseteq \widetilde{M}$ and $X, Y \in \mathfrak{X}(M)$. We define the *scalar second fundamental form of M* as the symmetric covariant 2-tensor field $h \in \Gamma(\Sigma^2 T^*M)$ given by

$$h(X, Y) := \langle N, \mathrm{II}(X, Y) \rangle \overset{(6.2)}{=} \langle N, \widetilde{\nabla}_X Y \rangle,$$

where the second equality follows from the Gauss formula (6.2) together with the fact that $\nabla_X Y \perp N$. Furthermore, since N is unital and spans NM,

$$\mathrm{II}(X, Y) = h(X, Y)N$$

(note that the sign of h depends on the choice of N, see Figure 6.2 for curves).

The choice of unit normal field N also determines the Weingarten map $s := W_N \colon \mathfrak{X}(M) \to \mathfrak{X}(M)$, in this situation called the *shape operator of M*. By the above it is given by

$$\langle sX, Y \rangle := \langle N, \mathrm{II}(X, Y) \rangle = h(X, Y).$$

Since h is symmetric, s is a self-adjoint endomorphism of TM, that is,

$$\langle sX, Y \rangle = \langle X, sY \rangle, \qquad X, Y \in \mathfrak{X}(M).$$

The fundamental equations of submanifolds derived earlier, thus simplify for hypersurfaces.
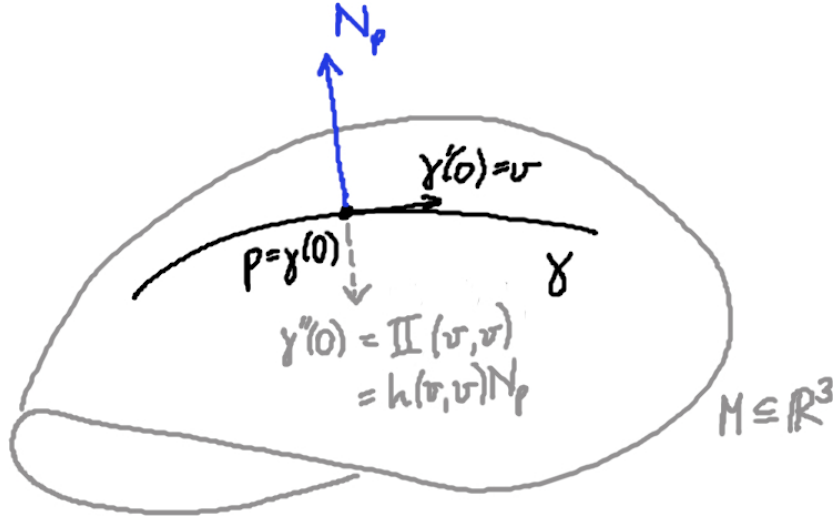
Figure 6.2. The scalar second fundamental form h satisfies $\text{II}(X,Y) = h(X,Y)N$, here indicated along a curve γ (with "natural" acceleration γ'').

PROBLEM 6.20. Prove the following equations for a Riemannian hypersurface (M, g) in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$, and N a smooth unit normal vector field along M:

   (i) Gauss formula: $\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)N$,
   (ii) Weingarten equation: $\widetilde{\nabla}_X N = -sX$,
   (iii) Gauss equation: $\widetilde{Rm}(W,X,Y,Z) = Rm(W,X,Y,Z) - \frac{1}{2}(h \otimes h)(W,X,Y,Z)$,
   (iv) Codazzi equation: $\widetilde{Rm}(W,X,Y,N) = (Dh)(Y,W,X)$.

PROBLEM 6.21. Suppose (M, g) is a Riemannian hypersurface in an $(n + 1)$-dimensional Lorentzian manifold $(\widetilde{M}, \widetilde{g})$, and N is a smooth unit normal vector field along M. Define the scalar second fundamental form h and the shape operator s by requiring that $\text{II}(X,Y) = h(X,Y)N$ and $\langle sX, Y \rangle = h(X,Y)$ for all $X, Y \in \mathfrak{X}(M)$. Prove the following Lorentzian analogues of the formulas of Problem 6.20:

   (i) Gauss formula: $\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)N$,
   (ii) Weingarten equation: $\widetilde{\nabla}_X N = sX$,
   (iii) Gauss equation: $\widetilde{Rm}(W,X,Y,Z) = Rm(W,X,Y,Z) + \frac{1}{2}(h \otimes h)(W,X,Y,Z)$,
   (iv) Codazzi equation: $\widetilde{Rm}(W,X,Y,N) = -(Dh)(Y,W,X)$.

REMARK 6.22 (Constraint equations in general relativity). Suppose $(\widetilde{M}, \widetilde{g})$ is an $(n + 1)$-dimensional Lorentzian manifold, and assume that $\widetilde{g}$ satisfies the Einstein equation (5.26) with a cosmological constant Λ,

$$Rc - \frac{1}{2}Sg + \Lambda g = T.$$

Suppose (M, g) is a Riemannian hypersurface in $\widetilde{M}$ with scalar second fundamental form h as defined in Problem 6.21. One can use the results of Problem 6.21 to show that g and h

have to satisfy the *Einstein constraint equations* on M, i.e.,

$$S - 2\Lambda - |h|_g^2 + (\operatorname{tr}_g h)^2 = 2\rho,$$
$$\operatorname{div} h - d(\operatorname{tr}_g h) = J,$$

where $\rho := T(N, N)$, and $J := T(N, X)$. They are the Lorentzian versions of the scalar Gauss equation and vector-valued Codazzi equation, and after some rewriting can be seen to form a system on nonlinear elliptic partial differential equations. Solving these constraint equations is highly nontrivial and several approaches are in use. It is a crucial step in understanding the initial value formulation of the Einstein equations in General Relativity. In 1952 Yvonne Choquet-Bruhat [24] showed (in the vacuum case) that these conditions are sufficient to obtain a well-posed initial value problem for the vacuum Einstein equations by suitable transforming the Einstein equations to a second-order quasilinear wave equation. The step from local existence to the existence of a maximal globally hyperbolic development is again non-trivial. The uniqueness of such a maximal development was only settled in 1969 in joint work with Robert Geroch [20] (for more details on the initial value problem of the Einstein equations see the excellent and self-contained book by Hans Ringström [49], and his list of errata).

PROBLEM 6.23. Suppose (M, g) is a Riemannian hypersurface in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$, and N is a unit normal vector field along M. We say that M is *convex* (with respect to N) if its scalar second fundamental form satisfies $h(v, v) \leq 0$ for all $v \in TM$. Show that if M is convex and $\widetilde{M}$ has sectional curvatures bounded below by a constant c, then all sectional curvatures of M are bounded below by c.

6.3.2. Principal curvatures. Recall that the shape operator s is a self-adjoint linear endomorphism of the tangent space T_pM (or generally, any such operator on a finite-dimensional inner product space V). Consider the smooth map

$$f(v) := \langle sv, v \rangle.$$

On the compact set $C = \{v; q(v) := \langle v, v \rangle = 1\}$ the maximum is achieved for some $v_0 \in C$. By the Lagrange multiplier rule there is a real number λ_0 such that $df_v = \lambda_0 dq_v$, hence by linearity and selfadjointness of s it follows that for all $w \in V$,

$$\langle sv_0, w \rangle = \lambda_0 \langle v_0, w \rangle,$$

and thus all such maximal v_0 are s-eigenvectors with real eigenvalues λ_0 due to the nondegeneracy of the inner product. Consider $b_1 = v_0$ to be the first unit basis vector, and set $B := \operatorname{span}\{b_1\}$. Clearly, $s(B) \subseteq B$, and thus by self-adjointness $s(B^\perp) \subseteq B^\perp$. We can apply induction to obtain the following result.

PROPOSITION 6.24 (Finite-Dimensional Spectral Theorem). *Suppose V is a finite-dimensional inner product space and $s \colon V \to V$ is a self-adjoint linear endomorphism. Then V has an orthonormal basis of s-eigenvectors, and all eigenvalues are real.*

Suppose (M, g) is a Riemannian hypersurface of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ and $p \in M$. Then the shape operator $s \colon T_pM \to T_pM$ has real eigenvalues $\kappa_1, \ldots, \kappa_n$ and there is an orthonormal basis $(b_1, \ldots, b_n)$ for T_pM consisting of s-eigenvectors with

$$sb_i = \kappa_i b_i, \qquad i = 1, \ldots, n.$$

Hence s (and h) are in this basis represented by diagonal matrices, and thus

$$h(v, w) = \langle sv, w \rangle = \sum_{i=1}^{n} \kappa_i v^i w^i.$$

DEFINITION 6.25. Let (M, g) be a Riemannian hypersurface of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. The eigenvalues of the shape operator at a point $p \in M$ are called the *principal curvatures of M at p*, and the corresponding eigenspaces are called the *principal directions*

The following combinations are particularly important.

DEFINITION 6.26. Let (M, g) be a Riemannian hypersurface of an $(n + 1)$-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$. The *Gaussian curvature* is defined as

$$K := \det(s),$$

and the *mean curvature* as

$$H := \frac{1}{n} \operatorname{tr}(s) = \frac{1}{n} \operatorname{tr}_g(h).$$

Since the determinant and trace of a linear endomorphism are independent of the basis, they are well-defined for a chosen unit normal. In terms of the principal curvatures

$$K = \kappa_1 \kappa_2 \cdots \kappa_n, \qquad H = \frac{1}{n}(\kappa_1 + \ldots + \kappa_n). \tag{6.8}$$

EXERCISE 6.27. Prove the formulas (6.8) for the mean and Gaussian curvature. Do these quantities depend on the choice of normal vector N? If yes, how?

EXERCISE 6.28. At this point we have two definitions of the Gaussian curvature for hypersurfaces embedded in a Riemannian manifold (via the scalar curvature in (6.6) and via the shape operator in Definition 6.26). Do they coincide?

PROBLEM 6.29. Suppose U is an open set in $\mathbb{R}^n$ and $f \in C^\infty(U)$. Let $\Gamma(f) = \{(x, f(x)); x \in U\} \subseteq \mathbb{R}^{n+1}$ be the graph of f, endowed with the Riemannian metric and upward unit normal.
   (i) Compute the components of the shape operator in graph coordinates, in terms of f and its partial derivatives.
   (ii) Let $\Gamma(f) \subseteq \mathbb{R}^{n+1}$ be the n-dimensional paraboloid defined as the graph of $f(x) = |x|^2$. Compute the principal curvatures of $\Gamma(f)$.

PROBLEM 6.30. Let (M, g) be an embedded Riemannian hypersurface in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$, let F be the local defining function for M and let $N = \operatorname{grad} F / |\operatorname{grad} F|$.
   (i) Show that the scalar second fundamental form of M with respect to the unit normal N is given by

$$h(X, Y) = -\frac{-\widetilde{\nabla}^2 F(X, Y)}{|\operatorname{grad} F|}, \qquad X, Y \in \mathfrak{X}(M).$$

   (ii) Show that the mean curvature of M is given by

$$H = -\frac{1}{n} \operatorname{div}_{\widetilde{g}} \left(\frac{\operatorname{grad} F}{|\operatorname{grad} F|} \right),$$

where $n = \dim M$ and $\operatorname{div}_{\widetilde{g}}$ is the divergence operator of $\widetilde{g}$ as introduced in Problem 4.35. (*Hint*: First prove the following linear algebra lemma: If V is a finite-dimensional inner product space, $w \in V$ is a unit vector, and $A \colon V \to V$ is a linear map that takes $w^\perp$ to $w^\perp$, then $\operatorname{tr}(A|_{w^\perp}) = \operatorname{tr} B$, where $B \colon V \to V$ is defined by $Bx = Ax - \langle x, w \rangle Aw$.)

PROBLEM 6.31. Let $M \subseteq \mathbb{R}^{n+1}$ be a Riemannian hypersurface, and let N be a smooth unit normal vector field along M. At each point $p \in M$, $N_p \in T_p\mathbb{R}^{n+1}$ ca ben thought of as unit vector in $\mathbb{R}^{n+1}$ and therefore as a point in $\mathbb{S}^n$. Thus each choice of unit normal vector field defines a smooth map $\nu \colon M \to \mathbb{S}^n$, called the *Gauss map of M*. Show that $\nu^* dV_{\hat{g}} = (-1)^n K dV_g$, where K is the Gaussian curvature of M.

6.3.3. Minimal hypersurfaces.
Let us present some important application in the field of Geometric Analysis of the above concepts, namely that of minimal (hyper)surfaces, which is a paramount research problem that has stimulated new mathematics since ancient times.

In $\mathbb{R}^3$ the question can be phrased as follows (see Figure 6.3): *Suppose C is a simple closed curve in $\mathbb{R}^3$. Is there an embedded (or immersed) surface M that hast least area among all surfaces with boundary $\partial M = C$? If so, what is it?*
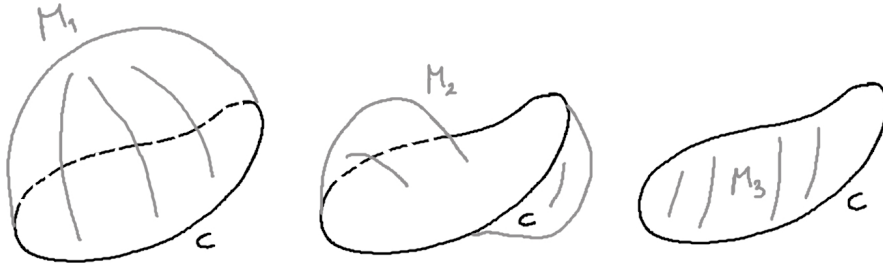
Figure 6.3. A simple closed curve C in $\mathbb{R}^3$, and different surfaces M_i with the same boundary $\partial M_i = C$. Which one has least area?

Note that there is a close analogy to the study of length-minimizing curves of Section 4.3, which is a similar (but simpler) variational problem in one less dimension.

We will treat the above question in the general case where $\mathbb{R}^n$ is replaced by an abstract Riemannian manifold $(\widetilde{M}, \widetilde{g})$ in which M is embedded as hypersurface. Let us make the problem precise by giving some definitions.

DEFINITION 6.32. Suppose M is a compact codimension-1 submanifold with nonempty boundary in an $(n + 1)$-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$. We say that M is *area-minimizing* if it has the smallest area[2] among all compact embedded hypersurfaces in $\widetilde{M}$ with the same boundary.

Before fixing the boundary, we derive a necessary condition of area-minimizing compact hypersurfaces. For the proof, note that most of results we have shown earlier are true for manifolds with boundary, although we did not mention it explicitly. Moreover, some details

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[2]The term area is only used due to it's analogy with the 2-dimensional question above. Strictly speaking it is the n-dimensional volume of M with respect to its induced Riemannian metric.

are only sketched, since we did not cover Fermi coordinates (a generalization of normal coordinates) and will simply take their existence for granted and refer to literature for proofs (see, e.g., [**25**, **36**]).

THEOREM 6.33. *Let M be a compact codimension-1 submanifold with nonempty boundary in an $(n+1)$-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$. If M is area-minimizing, then its mean curvature is identically zero.*

PROOF. Let g be the induced metric on M. By assumption M minimizes the area among hypersurfaces with the same boundary. Thus, in particular, M minimizes the area among small perturbations of M in a neighborhood of an interior point $p \in \mathrm{Int}(M)$. We use this idea to set up a one-dimensional variational problem and show that M must have zero mean curvature H.

Step 1. Write $\widetilde{g}$ in Fermi coordinates adapted to M. For each $p \in \mathrm{Int}(M)$ one can construct so-called Fermi coordinates[3] $(x^1, \ldots, x^n, v)$ on a neighborhood $\widetilde{U} \subseteq \widetilde{M}$ of $p \in M$ [**36**, p. 135 f], that is, coordinates such that $(x^1, \ldots, x^n)$ are coordinates on M and $\widetilde{g}$ is of the form

$$\widetilde{g} = dv^2 + g_{\alpha\beta}(x, v) dx^\alpha dx^\beta \tag{6.9}$$

[**36**, Prop. 6.37, Ex. 6.43, p. 238 f]. Set $U := \widetilde{U} \cap M$ and assume without loss of generality that it is a regular coordinate ball in M and $\widetilde{U} \cap \partial M = \emptyset$.

Step 2. Construct perturbations M_t of M. Take an arbitrary funciton $\varphi \in C^\infty(M)$ with compact support in U. For sufficiently small t we can define a set (see Figure 6.4)

$$M_t := (M \setminus U) \cup \{z \in \widetilde{U}; v(z) = t\varphi(x^1(z), \ldots, x^n(z))\} \subseteq \widetilde{M}.$$
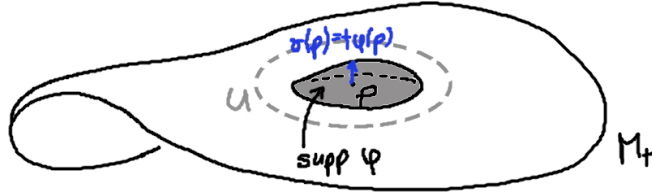
Figure 6.4. The submanifold M_t is a perturbation of M and only differs in a neighborhood U of p.

Then M_t is an embedded hypersurface in $\widetilde{M}$ which largely agrees with M but is a graph $v = t\varphi$ in $\widetilde{U}$. One can then construct two new Riemannian metrics using the graph parametrization $f_t(x) = (x, t\varphi(x))$ and resulting diffeomorphism $F_t \colon M \to M_t$, given by,

$$F_t(z) = \begin{cases} z, & z \in M \setminus \mathrm{supp}\,\varphi, \\ f_t(z), & z \in U, \end{cases}$$

with the following properties:

---

[3]Fermi coordinates, named after the Italian mathematician Enrique Fermi (1922), are a generalization of normal coordinates when one moves from (a geodesic ball around) a point p to (a neighborhood of) a submanifold M by making use of properties of the exponential map $\exp_{NM} \colon NM \to M$ defined on the normal bundle of M in $\widetilde{M}$ (namely that it is a local diffeomorphism of the zero section of NM). Many results about normal coordinates can be generalized to this setting as well. See [**25**, Ch. 2] for full details.

(i) the induced Riemannian metric $\hat{g}_t := \iota^*_{M_t}\widetilde{g}$ on M_t, and

(ii) the pullback metric $g_t := F^*_t\hat{g}_t = F^*_t\widetilde{g}$ on M.

Note that, for $t = 0$, $M_0 = M$ and $g_0 = \hat{g}_0 = g$. It follows from (6.9) that

$$
g_t = \begin{cases} g, & \text{on } M \setminus U, \\ \left[t^2\partial_\alpha\varphi(x)\partial_\beta\varphi(x) + g_{\alpha\beta}(x, t\varphi(x))\right] dx^\alpha dx^\beta, & \text{in } U, \end{cases}
$$

By assumption and construction we have thus obtained a smooth hypersurface M_t with same boundary $\partial M_t = \partial M$ such that $\text{Area}(M_t, \hat{g}_t)$ is minimal for $t = 0$.

Step 3. Compute the area of $(M_t, \hat{g}_t)$. By definition of the parametrized metrics, $F_t \colon (M, g_t) \to (M_t, \hat{g}_t)$ is an isometry. Thus

$$
\text{Area}(M_t, \hat{g}_t) = \text{Area}(M, g_t) = \text{Area}(M \setminus U, g) + \text{Area}(U, g_t),
$$

and it remains to compute the second term on U. In the chosen coordinates,

$$
\text{Area}(U, g_t) = \int_U \sqrt{\det g_t}\, dx^1 \cdots dx^n,
$$

where $\det g_t = \det((g_t)_{\alpha\beta})$. Note that the integrand depends smoothly on t and $(x^1, \ldots, x^n)$.

Step 4. Vary the area. By Step 3, $t \mapsto \text{Area}(M_t, \hat{g}_t)$ is a smooth function and we can compute the variation by

$$
\frac{d}{dt}\bigg|_{t=0} \text{Area}(M_t, \hat{g}_t) = \int_U \frac{\partial}{\partial t}\bigg|_{t=0} \sqrt{\det g_t}\, dx^1 \cdots dx^n
$$

$$
= \int_U \frac{1}{2}(\det g)^{-1/2} \frac{\partial}{\partial t}\bigg|_{t=0} (\det g_t)\, dx^1 \cdots dx^n,
$$

since the integrand has compact support on U. One can show that

$$
\frac{\partial}{\partial t}\bigg|_{t=0} (\det g_t) = (\det g)g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial v}\varphi = (\det g)(-2nH)\varphi,
$$

where H is the mean curvature of (M, g) (see [**36**, p. 241] and [**36**, Prop. 8.17] for the computation of the mean curvature in these coordinates, see also Exercise 6.34 below). Thus

$$
\frac{d}{dt}\bigg|_{t=0} \text{Area}(M_t, \hat{g}_t) = -n \int_U H\varphi dV_g. \tag{6.10}
$$

Step 5. Conclude that mean curvature $H \equiv 0$. By assumption, $\text{Area}(M_t, \hat{g}_t)$ attains a minimum at $t = 0$, thus (6.10) implies that for *every* such φ,

$$
\int_U H\varphi dV_g = 0. \tag{6.11}
$$

Suppose, for contradiction, that $H(p) \neq 0$. Suppose, without loss of generality, that $H(p) > 0$. Then we can pick a nonnegative bump function φ with $\varphi(p) > 0$ and $\text{supp}\,\varphi \subseteq \{H > 0\}$ Then $\int_U H\varphi dV_g > 0$, which contradicts (6.11). Thus $H \equiv 0$ on U, and since p was arbitrary, $H \equiv 0$ on $\text{Int}(M)$. Thus by continuity, $H \equiv 0$ on all of M. $\qquad\square$

EXERCISE 6.34. Verify that $H = -\frac{1}{2n}g^{\alpha\beta}\partial_v g_{\alpha\beta}|_{v=0}$ used in Step 4 indeed holds. (*Hint*: Use (6.9) and look into [**36**, Prop. 8.17].)

Because of Theorem 6.33 the following terminology has been established.

DEFINITION 6.35. Let M be a compact codimension-1 submanifold with $\partial M \neq \emptyset$ in Riemannian manifold $(\widetilde{M}, \widetilde{g})$. If the mean curvature of M vanished identically, i.e., $H \equiv 0$, then we call M a *minimal (hyper)surface*.

Note that, strictly speaking, the condition $H \equiv 0$ only implies that M is a critical point of the area. But one can, just as in the case of geodesics, show that minimal hypersurfaces are *locally* area-minimizing.

REMARK 6.36 (Minimal surfaces in General Relativity). Certain extensions and modifications of the notion of minimal surface are significant in General Relativity. They are known as *apparent horizons*, and provide a curvature-based approach to understanding the boundary of black holes. More precisely, they mark the boundary between outward-moving light that moves outward or inward. Unlike the event horizon (which is a global boundary), however, apparent horizons are local and depend on the slicing. An apparent horizon is often also called *marginally outer trapped surface* (MOTS) because it is the outermost of all trapped surfaces.

In the language of Calculus of Variations, Theorem 6.33 is a variational problem without contraints. There, we fixed the boundary, but now we want to require that the interior has a certain fixed volume instead. We can turn our initial question into a problem with contraints analogous to Dido's problem in $\mathbb{R}^2$: Queen Dido considered a regular domain D in $\mathbb{R}^2$ which is enclosed by a curve C. Her aim was to find the domain D with maximal area for fixed length $L(\partial D) = L(C)$ (but arbitrary shape). Clearly, the solution is a ball D of a certain radius, and the boundary C is the corresponding circle (see [**14**] for several proofs and a presentation to the general public with an elementary geometric proof). Instead of fixing the length of C and maximizing the area of D one can, equivalently, fix the area of D and minimize the length of C. This is what we do in more generality now. The role of the curve C is played by a hypersurface M, and $(\widetilde{M}, \widetilde{g})$ replaces $\mathbb{E}^2$.

THEOREM 6.37. *Suppose $(\widetilde{M}, \widetilde{g})$ is an $(n+1)$-dimensional Riemannian manifold, $D \subseteq \widetilde{M}$ is a compact regular domain[4], and $M = \partial D$. If M is surface area-minizing among all boundaries of compact regular domains with the same volume as D, then M has constant mean curvature (computed with respect to the outward unit normal).*

The proof is somewhat analogous (but more involved, because 2-paramater perturbations of D are needed) than that of Theorem 6.33.

PROOF. Let g be the induced metric on M. Assume, for contradiction, that the mean curvature of M is not constant, that is, that there are points $p, q \in M$ with $H(p) < H(q)$.

Step 1. Write $\widetilde{g}$ in Fermi coordinates adapted to M. Since M is compact it has an ε-tubular neighborhood for some $\varepsilon > 0$ (see Remark 4.40). As in the proof of Theorem 6.33 there are Fermi coordinates $(x^1, \ldots, x^n, v)$ for M on an open set $\widetilde{U} \subseteq \widetilde{M}$ containing p. Let $U := \widetilde{U} \cap M$ and assume, without loss of generality, that U is a regular coordinate ball in M and that the image of the chart is a cylindrical set of the form $\widehat{U} \times (-\varepsilon, \varepsilon)$ for some open set $\widehat{U} \subseteq \mathbb{R}^n$. Similarly, let $(y^1, \ldots, y^n)$ be Fermi coordinates for M on $\widetilde{W} \subseteq \widetilde{M}$ containing q, and let $W := \widetilde{W} \cap M$ (with same properties as U).

---

[4]Recall that a regular domain is a closed, embedded codimension-0 submanifold with boundary.

By potentially changing the signs of v and/or w, we have that

$$D \cap \widetilde{U} = \{v \le 0\}, \qquad D \cap \widetilde{W} = \{w \le 0\}.$$

By further shrinking the domains U and W we can assume that we have barriers, i.e., constants H_1 and H_2 with $H(p) < H_1 < H_2 < H(q)$, so that in the whole domains still

$$H|_U \le H_1 < H_2 \le H|_W. \tag{6.12}$$

Step 2. Construct perturbations $D_{s,t}$ of D. Consider $\varphi, \psi \in C^\infty(M)$ with compact supports in U and W, respectively, and normalization

$$\int_U \varphi \, dV_g = \int_W \psi \, dV_g = 1. \tag{6.13}$$

For $s, t \in \mathbb{R}$ sufficiently small, define the sets $D_{s,t} \subseteq \widetilde{M}$ by (see Figure 6.5)

$$D_{s,t} := \{z \in \widetilde{U}; v(z) \le s\varphi(x(z))\} \cup \{z \in \widetilde{W}; w(z) \le t\psi(y(z))\} \cup (D \setminus (\widetilde{U} \cup \widetilde{W}),$$

and set

$$M_{s,t} := \partial D_{s,t}$$

Clearly, for $s = t = 0$, we obtain $D_{0,0} = D$ and $M_{0,0} = M$. For sufficiently small s and t, $D_{s,t}$ is a regular domain and $M_{s,t}$ a compact smooth hypersurface, and the functions

$$(s,t) \mapsto V(s,t) := \mathrm{Vol}(D_{s,t}), \qquad (s,t) \mapsto A(s,t) := \mathrm{Area}(M_{s,t})$$

are smooth.
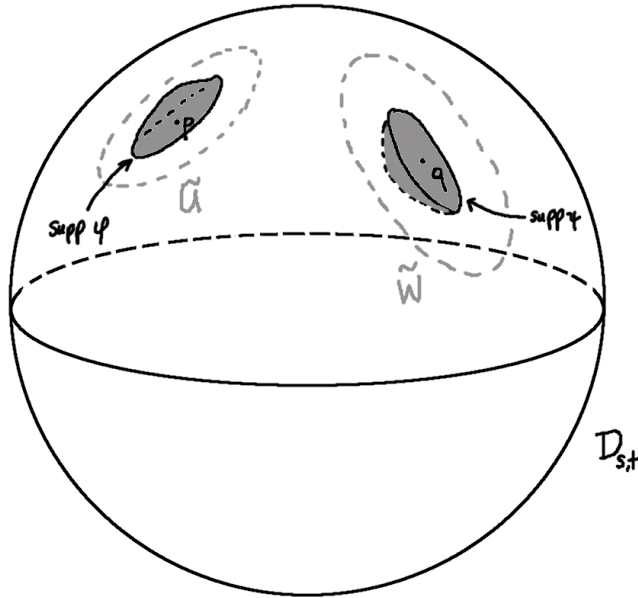
Figure 6.5. The regular domain $D_{s,t}$ is a perturbation of D and only differs in the disjoint neighborhoods U of p and W of q.

Step 3. Vary the volumes of $D_{s,t}$ and areas of $M_{s,t} = \partial D_{s,t}$. As in (6.10) one can show that

$$\frac{\partial A}{\partial s}(0,0) = -n \int_U H\varphi dV_g, \qquad \frac{\partial A}{\partial t}(0,0) = -n \int_W H\psi dV_g. \qquad (6.14)$$

It remains to compute the variation of the volume at $(s,t) = (0,0)$. If we hold $t = 0$ fixed and vary s, the only volume change of $D_{s,t}$ takes place in $\widetilde{U}$. Thus by the fundamental theorem of calculus,

$$\begin{aligned}
\frac{\partial V}{\partial s}(0,0) &= \frac{d}{ds}\bigg|_{s=0} \mathrm{Vol}(D_{s,0} \cap \widetilde{U}) \\
&= \frac{d}{ds}\bigg|_{s=0} \int_U \left(\int_{-\varepsilon}^{s\varphi(x)} \sqrt{\det \widetilde{g}(x,v)} dv \right) dx^1 \cdots dx^n \\
&= \int_U \left(\frac{d}{ds}\bigg|_{s=0} \int_{-\varepsilon}^{s\varphi(x)} \sqrt{\det \widetilde{g}(x,v)} dv \right) dx^1 \cdots dx^n \\
&= \int_U \varphi(x) \sqrt{\det \widetilde{g}(x,0)} dx^1 \cdots dx^n
\end{aligned}$$

Since in the chosen coordinates $g_{\alpha\beta}(x) = \widetilde{g}_{\alpha\beta}(x,0)$,

$$\frac{\partial V}{\partial s}(0,0) = \int_U \varphi \, dV_g = 1. \qquad (6.15)$$

Similarly, $\frac{\partial V}{\partial t}(0,0) = 1$.

Step 4. Conclude that H is constant. Step 3 allows us to combine s and t as follows. Since $\frac{\partial V}{\partial t}(0,0) \neq 0$, the implicit function theorem guarantees that there is a smooth function $\lambda: (-\delta, \delta) \to \mathbb{R}$ for some small $\delta > 0$ so that

$$V(s, \lambda(s)) \equiv V(0,0) = \mathrm{Vol}(D).$$

Differentiation with respect to s yields by the chain rule and Step 3

$$0 = \frac{d}{ds}\bigg|_{s=0} V(s, \lambda(s)) = \frac{\partial V}{\partial s}(0,0) + \lambda'(0)\frac{\partial V}{\partial t}(0,0) \overset{(6.15)}{=} 1 + \lambda'(0),$$

i.e., $\lambda'(0) = -1$.

By assumption, M minimizes the area, hence also

$$0 = \frac{d}{ds}\bigg|_{s=0} A(s, \lambda(s)),$$

and, similarly, by the chain rule, Step 3, and $\lambda'(0) = -1$,

$$0 = \frac{\partial A}{\partial s}(0,0) + \lambda'(0)\frac{\partial A}{\partial t}(0,0) \overset{(6.14)}{=} -n \int_U H\varphi dV_g + n \int_W H\psi dV_g.$$

Thus

$$\int_U H\varphi dV_g = \int_W H\psi dV_g.$$

On the other hand, due to our restrictions of U and W so that (6.12) holds, and the normalization (6.13) of φ and ψ, imply that

$$\int_U H\varphi dV_g \leq H_1 < H_2 \leq \int_W H\psi dV_g,$$

a contradiction.   □

Theorem 6.37 suggest the following definition.

DEFINITION 6.38. An immersed hypersurface M in a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ is said to be a *constant-mean-curvature (CMC) surface* if its mean curvature is constant.

Figure 6.6 summarizes the results of Theorems 6.33 and 6.37 visually for $(\widetilde{M}, \widetilde{g}) = \mathbb{E}^3$. In the first setting, we showed that a area-minimizing surface M with fixed boundary must have zero mean curvature (here, a flat soap film). In the second setting, we only fixed the volume of the interior domain D and searched for the surface M enclosing it with least surface area. We concluded that M must have constant mean curvature (here, a round soap bubble).
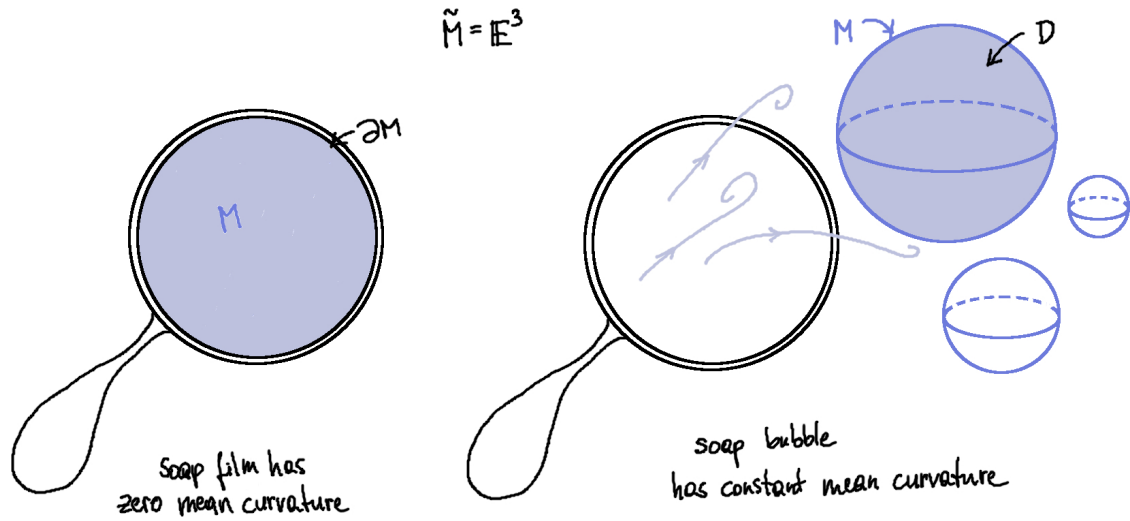
Figure 6.6. The results of Theorems 6.33 (left) and 6.37 (right) visualized in Euclidean 3-space. On the left hand side the boundary ∂M is fixed and the surface M with minimal area must have zero mean curvature, and is a *minimal (hyper)surface*. On the right hand side, instead, only the volume of a compact regular domain D is given, and the surface M which encloses this domain with minimal area must have constant mean curvature, and is a *CMC surface*.

In the spirit of our original question we can further ask ourselves: *What are the compact CMC surfaces in $\mathbb{R}^3$?* Certainly, spheres $\mathbb{S}^2(R)$ have constant mean curvature and are compact, but are they the only ones?

The answer is far from trivial. Already in 1841 Charles-Eugène Delaunay showed that the only surfaces of revolution with constant mean curvature are surfaces obtained by rotating the roulettes of the conics. These are the plane, cylinder, sphere, the catenoid, the unduloid and nodoid. All of those, except the sphere, however, are not compact. On the other hand, there may be other compact CMC surfaces that cannot be obtained by rotation. In 1853 J. H. Jellet showed that also in the class of compact star-shaped surfaces only the sphere remains. Finally, A. D. Alexandrov proved in a series of six papers in 1956–1960 that a compact embedded surface in $\mathbb{R}^3$ with constant mean curvature $H \neq 0$ must indeed be a sphere. At this point one would expect, and Heinz Hopf indeed conjectured it in 1956, that this result can be extended to higher dimensions and to immersions as well, namely that

any immersed compact orientable CMC hypersurface in $\mathbb{R}^{n+1}$ must be a standard embedded n-sphere. But this is false! In 1982 Wu-Yi Hsiang constructed a counterexample in $\mathbb{R}^4$. It is also false in $\mathbb{R}^3$ if one replaces embedded by immersed, an example being the Wente torus constructed in 1984.

Generally, even in the non-compact setting, up until then only a few examples of general CMC surfaces in $\mathbb{R}^3$ were known. This has changed with more modern glueing constructions in the 1990s. See [**9**] for an up-to-date (May 2022) and thourough survey of current research on CMC (hyper)surfaces using conservation laws and glueing.

REMARK 6.39 (CMC slices in General Relativity). When studying the initial value problem of the Einstein equation it is necessary to pick a Cauchy surface, a suitable Riemannian 3-manifold representing an instance of "time". One way is to pick/assume a CMC slicing of the spacetime. In this case, the time is exactly the constant mean curvature of the hypersurface Σ_t, i.e., $H(\Sigma_t) = t$. Global existence for the Einstein vacuum evolution equations in CMC time is a long standing conjecture in relativity. Not all spacetimes have CMC slices, and so a crucial question concerning CMC slices is also their generality. See, for instance, [**3**, **4**, **21**] for some discussion of these issues.

Tensors and tensor fields

The most important constructions on smooth manifolds for Riemannian Geometry are tensor fields. They are sections of tensor bundles, which is a special kind of vector bundle. We briefly review all these constructions here. Most are already covered in the "Manifolds" course. See [**36**, Appendix] and [**35**] for more details and exercises. Here we just repeat the basic definitions. We assume a background in basic Linear Algebra.

Note that there is a shorter equivalent way to introduce the set of tensor fields as a $C^\infty(M)$-module (see [**43**, Chapter 2]) but then the relation to vector bundles is still open, and this vector bundle viewpoint is sometimes beneficial (in this course, for example, in the context of connections). For us this interpretation as a $C^\infty(M)$-module is a consequence and stated in Lemma A.17.

A.1. Vector bundles

DEFINITION A.1. A (real) *smooth[1] vector bundle of rank k* is a pair of smooth manifolds E and M (with or without boundary), together with a smooth surjective map $\pi \colon E \to M$ satisfying

  (i) For each $p \in M$, the set $E_p = \pi^{-1}(p)$ is a k-dimensional vector real vector space.
  (ii) For each $p \in M$, there exists a neighborhood U of p and a diffeomorphism $\Phi_U \colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that
    • the diagram

$$E_U = \pi^{-1}(U) \xrightarrow{\ \Phi_U\ } U \times \mathbb{R}^k$$
$$\searrow{\scriptstyle \pi} \qquad \swarrow{\scriptstyle \mathrm{pr}_U}$$
$$U$$

commutes, where $\mathrm{pr}_U \colon U \times \mathbb{R}^k \to U$ is the projection onto the first factor; and
    • for each $q \in U$, Φ_U restricts to a linear isomorphisms $E_q \to \{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

The space M is called the *base*, E is called the *total space*, and $\pi \colon E \to M$ its *projection*. Each set $E_p = \pi^{-1}(p)$ is called the *fiber of E over p*, and each diffeomorphism $\Phi_U \colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ as above a *smooth local trivialization* of E over U.

There are certain special cases of vector bundles. For instance, a rank-1 vector bundle is called a *line bundle*. If there exists a *global trivialization of E* over all of M, i.e., E is diffeomorphic to $M \times \mathbb{R}^k$, then E is called a *trivial bundle* (or *smoothly trivial*).

The simplest example of a rank-k bundle is therefore the following.

---

[1]Generally, smoothness is not required, only continuity for π (and Φ_U being a homeomorphisms), which is why we call it a *smooth* vector bundle. We only consider smooth vector bundles.

EXAMPLE A.2 (Product bundle). The product space $E = M \times \mathbb{R}^k$ with $\pi = \mathrm{pr}_1 \colon M \times \mathbb{R}^k \to M$ as projection is a *product bundle*. The identity map is a global trivialization, hence it is trivial.

Most vector bundles are not trivial, hence they require more than one local trivialization.

EXAMPLE A.3 (Möbius strip). On $\mathbb{S}^1 \times \mathbb{R}$ define the equivalence relation

$$(\theta, t) \sim (\theta', t') :\Longleftrightarrow (\theta', t') = (\theta + \pi, -t),$$

and let $E = (\mathbb{S}^1 \times \mathbb{R})/\mathbb{Z}_2$ be the quotient space with quotient map $q \colon \mathbb{S}^1 \times \mathbb{R} \to E$ (see Figure A.1). One can show that the projection $\mathrm{pr}_1 \colon \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1$ onto the first factor descends to a (smooth) surjective map $\pi \colon E \to \mathbb{S}^1$ with the desired properties, turning $\pi \colon E \to \mathbb{S}^1$ into a smooth real line bundle over $\mathbb{S}^1$, called the *Möbius strip*. (For more details see [**35**, Ex. 10.3].)
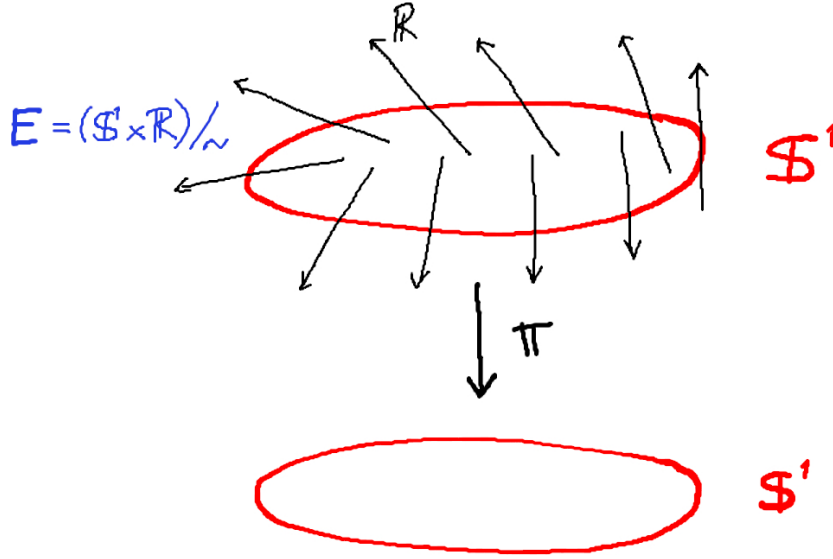
Figure A.1. Rough sketch of the Möbius strip (you can find nice vidoes online, craft one your self, or read a bit more about its math and relevance for the world).

The most important examples of vector bundles are tangent bundle TM, cotangent bundle T^*M, and tensor bundles $T^{(k,l)}TM$ (for the latter see Section A.3).

EXERCISE A.4 (Tangent bundle). Let M be a smooth n-dimensional manifold and let TM be its tangent bundle. Show that with its standard projection map, its natural vector space structure on each fiber, and the natural smooth structure of a $2n$-dimensional manifold (see, e.g., [**35**, Prop. 3.18]), TM is a smooth vector bundle of rank n over M. (*Hint*: Locally on any smooth chart $(U, \varphi = (x^1, \ldots, x^n))$ define $\Phi_U(v^i \frac{\partial}{\partial x^i}|_p) = (p, d\varphi_p(v)) = (p, (v^1, \ldots, v^n))$, and verify that it is a smooth local trivialization.)

We now turn to define sections on vector bundles $E \to M$ in an attempt to generalize vector fields $\mathfrak{X}(M)$ on the tangent bundle TM. We will use it to define tensor fields in Section A.3.

DEFINITION A.5. A *smooth[2] section of a vector bundle* $\pi\colon E \to M$ is a smooth map $\sigma\colon M \to E$ such that $\pi \circ \sigma = \mathrm{Id}_M$ (equivalently, $\sigma(p) = E_p$ for all $p \in M$).

The set of all sections of E is denoted by $\Gamma(E)$.

EXAMPLE A.6 (Smooth functions). The sections of the trivial line bundle $M \times \mathbb{R}$ are simply the smooth real-valued functions, i.e., $\Gamma(M \times \mathbb{R}) = C^\infty(M)$.

EXAMPLE A.7 (Vector fields and 1-forms). The sections of the tangent bundle TM are precisely the vector fields, i.e., $\Gamma(TM) = \mathfrak{X}(M)$. The sections of the cotangent bundle T^*M are the 1-forms, i.e., $\Gamma(T^*M) = \Omega^1(M)$. For smooth k-forms we have $\Omega^k(M) = \Gamma(\Lambda^k T^*M)$.

A.2. Tensors on a vector space

Let V be an n-dimensional real vector space and V^* the *dual space of V*, i.e., the set of linear maps $V \to \mathbb{R}$. While the elements of V are called *vectors*, the elements of V^* are called *covectors*.

DEFINITION A.8. Let $k, l \in \mathbb{N} \cup \{0\}$. A multilinear map

$$F\colon \underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \times \underbrace{V \times \cdots \times V}_{l \text{ copies}} \to \mathbb{R}$$

is called a (mixed) (k, l)-*tensor*, or k-*contravariant, l-covariant tensor*.

The spaces of tensors on V are denoted by

$$T^l(V^*) := \{\text{covariant } l\text{-tensors on } V\},$$
$$T^k(V) := \{\text{contravariant } k\text{-tensors on } V\},$$
$$T^{(k,l)}(V) := \{\text{mixed } (k,l)\text{-tensors on } V\}.$$

The *rank* of a tensor is the number or arguments it takes, i.e., $k + l$.

By convention, $T^{(0,0)}(V) = \mathbb{R}$. Note that $T^1(V^*) = V^*$ and $T^1(V) = V$. One can show (see, for instance, [**36**, Prop. B.1]) that

$$T^{(1,1)}(V) \cong \mathrm{End}(V).$$

Natural and important operations on tensors are tensor products and trace (or contraction).

DEFINITION A.9. Let $F \in T^{(k,l)}(V)$, $G \in T^{(p,q)}(V)$ be tensors. The *tensor product* $F \otimes G \in T^{(k+p,l+q)}(V)$ is defined by

$$F \otimes G(\omega^1, \ldots, \omega^{k+p}, v_1, \ldots, v_{l+q})$$
$$= F(\omega^1, \ldots, \omega^k, v_1, \ldots, v_l)G(\omega^{k+1}, \ldots, \omega^{k+p}, v_{l+1}, \ldots, v_{l+q}).$$

The tensor product is associate because multiplication in $\mathbb{R}$ is.

EXERCISE A.10. Suppose (b_i) is a basis for the n-dimensional vector space V, and (β^j) is the corresponding dual basis for V^*. Show that the set of all tensors

$$b_{i_1} \otimes \ldots \otimes b_{i_k} \otimes \beta^{j_1} \otimes \ldots \otimes \beta^{j_l}$$

---

[2]Again, we only consider smooth sections, while generally one can also study continuous sections.

is a basis for $T^{(k,l)}(V)$, i.e., every tensor $F \in T^{(k,l)}(V)$ is of the form (using the Einstein summation convention)

$$F = F^{i_1 \dots i_k}_{j_1 \dots j_l} b_{i_1} \otimes \dots \otimes b_{i_k} \otimes \beta^{j_1} \otimes \dots \otimes \beta^{j_l},$$

which components $F^{i_1 \dots i_k}_{j_1 \dots j_l} = F(\beta^{i_1}, \dots, \beta^{i_k}, b_{j_1}, \dots, b_{j_l})$. Compute $\dim T^{(k,l)}(V)$.

DEFINITION A.11. The *trace* (or *contraction*) is the operator $\mathrm{tr} \colon T^{(k+1,l+1)}(V) \to T^{(k,l)}(V)$, which for $F \in T^{k+1,l+1}(V)$ is defined by

$$(\mathrm{tr}\, F)(\omega^1, \dots, \omega^k, v_1, \dots, v_l) := \mathrm{tr}\big(\underbrace{F(\omega^1, \dots, \omega^k, \cdot, v_1, \dots, v_l, \cdot)}_{\in T^{(1,1)}(V)}\big).$$

In terms of a basis, the componentes of $\mathrm{tr}\, F$ are $(\mathrm{tr}\, F)^{i_1 \dots i_k}_{j_1 \dots j_l} = F^{i_1 \dots i_k m}_{j_1 \dots j_l m}$.

EXERCISE A.12. Show that the trace on any pair of indices (one upper and one lower) is a well-defined linear map $T^{(k+1,l+1)}(V) \to T^{(k,l)}(V)$.

To define differential forms one needs alternating tensors, for Riemannian geometry symmetric tensors are more important.

DEFINITION A.13. A covariant tensor $F \in T^l(V^*)$ is said to be *symmetric* if for any pair of arguments $1 \le i < j \le l$

$$F(v_1, \dots, v_i, \dots, v_j, \dots, v_l) = F(v_1, \dots, v_j, \dots, v_i, \dots, v_l).$$

The set of symmetric l-tensors on V is a linear subspace, denoted by $\Sigma^l(V^*)$.

A.3. Tensor fields on a manifold

On a smooth manifold M, for each $p \in M$, the tangent space $T_p M$ is a vector space and thus one can consider tensors on $T_p M$. The disjoint union of tensors of the same type defines a tensor bundle of this type.

DEFINITION A.14. The *bundle of (k,l)-tensors* on M is given by

$$T^{(k,l)} TM := \bigsqcup_{p \in M} T^{(k,l)}(T_p M).$$

The *tangent bundle* $TM = \bigsqcup_{p \in M} T_p M$ and the *contangent bundle* $T^*M = \bigsqcup_{p \in M} T_p^* M$ are special cases.

EXERCISE A.15. Show that each tensor bundle is a smooth vector bundle (see Definition A.1) over M, with a local trivialization over every open subset that admits a smooth local frame for TM.

Recall that we have seen in Example A.7 that vector fields and 1-forms are sections of the tangent and cotangent bundle, respectively. They are also special cases of tensor fields (with (k,l) being $(1,0)$ and $(0,1)$). Given Exercise A.4, we now define a tensor field on M as a section (see Definition A.5) of a smooth tensor bundle over M.

DEFINITION A.16. The sections of the tensor bundle $T^{(k,l)} TM$ are the *(k,l)-tensor fields* [3]

$$\mathcal{T}^{(k,l)}(M) := \Gamma(T^{(k,l)} TM).$$

---

[3]There are also other ways to denote (k,l)-tensor fields, for example, as $\mathcal{T}_l^k(M)$ [**43**, p. 35].

Note that $\mathcal{T}^{(1,0)}(M) = \mathfrak{X}(M)$ and $\mathcal{T}^{(0,1)}(M) = \Omega^1(M)$. For smooth covariant l-tensor fields we write $\mathcal{T}^l(M) := \Gamma(T^l T^* M) := \Gamma(T^{(0,l)} TM)$.

Multilinearity over the space of smooth functions characterizes tensor fields [**35**, Lemma 12.24] as a module. O'Neill [**43**, p. 55 ff] actually defines tensor fields in this way.

LEMMA A.17 (Tensor Characterization Lemma). *A map*

$$\mathcal{F} \colon \underbrace{\Omega^1(M) \times \ldots \times \Omega^1(M)}_{k \ copies} \times \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{l \ copies} \to C^\infty(M)$$

is induced by a smooth (k,l)-tensor field if and only if is multilinear over $C^\infty(M)$.

The idea for why this result is true can be seen by considering for $F \in \mathcal{T}^{(k,l)}(M)$, $\omega^i \in \Omega^1(M)$, $X_j \in \mathfrak{X}(M)$ the real-valued (smooth) function $\mathcal{F}$, pointwise defined by

$$\mathcal{F}(\omega^1, \ldots, \omega^k, X_1, \ldots, X_l)(p) := F_p(\omega^1|_p, \ldots, \omega^k|_p, X_1|_p, \ldots, X_l|_p)$$

(how to establish smoothness is explained in [**36**, Ex. B.5]).

Analogous to the tensor case (see Exercise A.10), given a smooth local frame (E_i) with dual coframe (ε^i), it follows that the tensor fields $E_{i_1} \otimes \ldots E_{i_k} \otimes \varepsilon^{j_1} \otimes \ldots \varepsilon^{j_l}$ form a smooth local frame for $\mathcal{T}^{(k,l)}(M)$. In particular, for local coordinates (x^i) on U, a (k,l)-tensor field F has the local expression

$$F = F^{i_1 \ldots i_k}_{j_1 \ldots j_l} \partial_{i_1} \otimes \ldots \otimes \partial_{i_k} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_l},$$

with coefficients $F^{i_1 \ldots i_k}_{j_1 \ldots j_l} \in C^\infty(U)$.

Just like vectors one also pull back tensor fields in the natural way.

DEFINITION A.18. Suppose M and N are manifolds, $F \colon M \to N$ is a smooth map and $A \in \mathcal{T}^{(0,l)}(N)$. For every $p \in M$ the pointwise pullback of A by F is the tensor $dF_p^*(A) \in T^l(T_p^* M)$, defined by

$$dF_p^*(A)(v_1, \ldots, v_l) := A(dF_p(v_1), \ldots, dF_p(v_l)),$$

where $v_i \in T_p M$. The *pullback of A by F* is the tensor field $F^* A \in \mathcal{T}^{(0,l)}(M)$ defined by

$$(F^* A)_p := dF_p^*(A_{F(p)}).$$

Exterior derivatives

For more details see Lee [**35**, p. 280–284, 362–372].

B.1. Differential of a function

Let M be a smooth manifold. For every smooth real-valued function $f \in C^\infty(M)$ we define the *differential of f*, piontwise by

$$df_p(v) := vf, \qquad v \in T_pM.$$

It is clear that df is a smooth 1-form, because df_p it depends linearly on $v \in T_pM$ for each $p \in M$ and because $df(X) = Xf$ is smooth for every smooth $X \in \mathfrak{X}(M)$.

EXAMPLE B.1. The coordinate 1-forms dx^j associated with the coordinate functions $x^j \colon U \to \mathbb{R}$ on an open set $U \subseteq M$.

In coordinates, we can therefore write

$$df = \frac{\partial f}{\partial x^i} dx^i.$$

EXERCISE B.2. For $f(x, y) = x^2 y \cos x$ on $\mathbb{R}^2$ compute df in terms of dx and dy.

The differential satisfies the usual rules of differentiation (linearity, product rule, chain rule etc.), see [**35**, Prop. 11.20]. In particular, $df = 0$ if and only if f is constant.

B.2. Exterior derivative of a form

On forms one can define a natural generalization of the differential on functions, called the *exterior derivative*.

Every $f \in C^\infty(M)$ defines a 1-form df, but in order for a 1-form ω to satisfy $\omega = df$ it needs to be *closed* [**35**, Prop. 11.44], i.e., in every coordinate chart (but independent of the choice coordinates)

$$\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} = 0. \tag{B.1}$$

Note that (B.1) is antisymmetric in i and j, and thus can be interpreted as components of an alternating tensor field, i.e., a 2-form, locally given by

$$d\omega = \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j.$$

In other words, ω is closed if and only if $d\omega = 0$ in all charts.

In fact, $d\omega$ can be defined globally, and for differential forms of any degree. One can show the following result [**35**, Thm. 14.24].

THEOREM B.3 (Existence and uniqueness of exterior differentation). *Let M be a smooth manifold. For all k, there are unique operators*

$$d\colon \Omega^k(M) \to \Omega^{k+1}(M),$$

called exterior differentation, *satisfying*

(i) *d is linear over $\mathbb{R}$.*
(ii) *If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(iii) *$d \circ d \equiv 0$.*
(iv) *For $f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f, given by $df(X) = Xf$.*

In any smooth coordinate chart (x^i), d is given as follows: For a smooth k-form locally given by $\omega = \sum_J' \omega_J dx^J = \sum_{\{J: j_1 < \ldots < j_k\}} \omega_J dx^J$ (the primed summation indicates that the multi-indices are increasing) we have

$$d\left(\sum_J {}'\omega_J dx^J\right) = \sum_J {}'d\omega_J \wedge dx^J,$$

where $d\omega_J$ is the differential of the function ω_J (see [**35**, p. 363] for more details).

PROBLEM B.4. Show that for any $\omega \in \Omega^1(M)$ and $X, Y \in \mathfrak{X}(M)$ we have

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]),$$

where $[,]$ is the Lie bracket.

One can furthermore show that the pullback commutes with d [**35**, Prop. 14.26], i.e., for $F\colon M \to N$, $F^*\colon \Omega^k(N) \to \Omega^k(M)$ for any k, and all $\omega \in \Omega^k(M)$

$$F^*(d\omega) = d(F^*\omega).$$

Also the *Lie derivative* commutes with d [**35**, Cor. 14.36], i.e., for $V \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$,

$$\mathcal{L}_V(d\omega) = d(\mathcal{L}_V\omega).$$

The latter follows from Cartan's Magic Formula [**35**, Thm. 14.35], which states that for any $V \in \mathfrak{X}(M), \omega \in \Omega^1(M)$

$$\mathcal{L}_V\omega = V \lrcorner (d\omega) + d(V \lrcorner \omega).$$

Using differential forms and the exterior derivative one can generalize the fundamental theorem of calculus to manifolds in a very elegant way.

THEOREM B.5 (Stokes' Theorem). *Let M be an oriented smooth n-manifold with boundary, and let ω be a compactly supported $(n-1)$-form on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

For the proof see [**35**, p. 411–415].

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