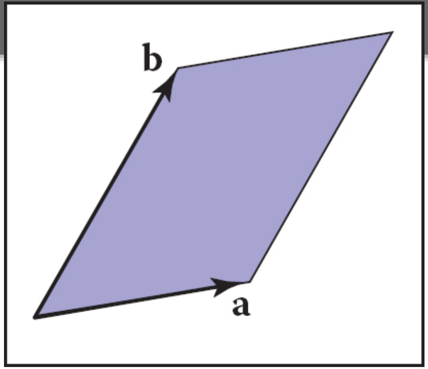
جامعة الملك عبدالله للعلوم والتقنية
King Abdullah University of Science and Technology
VCC VISUAL COMPUTING CENTER

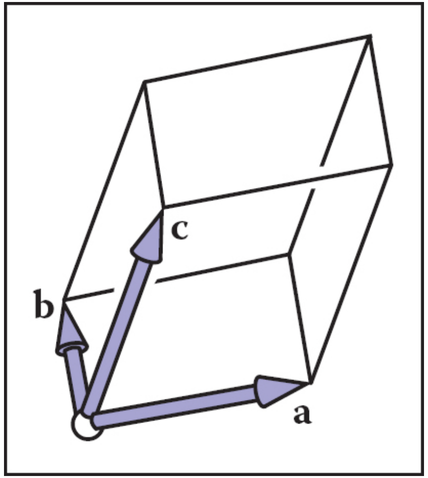
Computer Graphics CS248
Linear Algebra for Computer Graphics

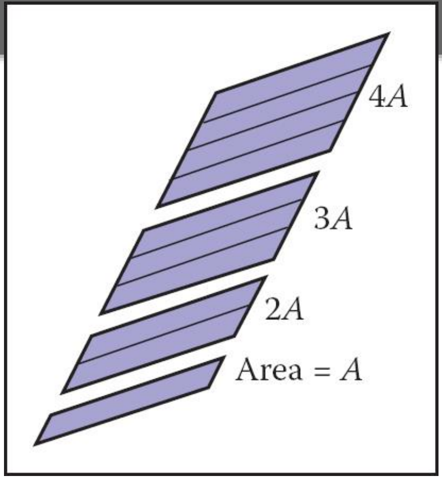
Ivan Viola
NANOVISUALIZATION GROUP

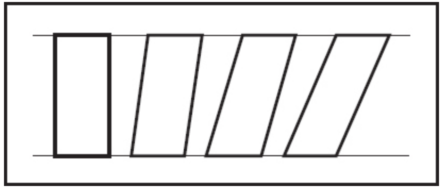
Determinants

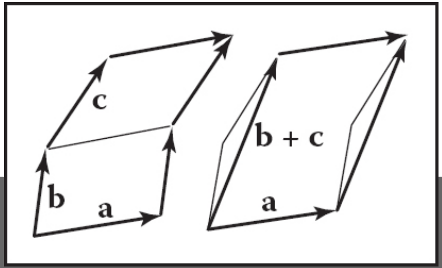
- $|\boldsymbol{ab}| = -|\boldsymbol{ba}|$ is signed area
- $|\boldsymbol{abc}|$ is a signed volume
- Scaling parallelogram: $|(k\boldsymbol{a})\boldsymbol{b}| = |\boldsymbol{a}(k\boldsymbol{b})| = k|\boldsymbol{ab}|$
- Shearing: $|(\boldsymbol{a} + k\boldsymbol{b})\boldsymbol{b}| = |(\boldsymbol{a}(\boldsymbol{b} + k\boldsymbol{a})| = |\boldsymbol{ab}|$
- Slide edge: $|\boldsymbol{a}(\boldsymbol{b} + \boldsymbol{c})\ | = |\boldsymbol{ab}| + |\boldsymbol{ac}|$











Determinants (cont.)

$$|\boldsymbol{ab}| = |(x_a\boldsymbol{x} + y_a\boldsymbol{y})(x_b\boldsymbol{x} + y_b\boldsymbol{y})|$$
$$= x_a x_b|\boldsymbol{xx}| + x_a y_b|\boldsymbol{xy}| + y_a x_b|\boldsymbol{yx}| + y_a y_b|\boldsymbol{yy}|$$
$$= x_a x_b|0| + x_a y_b|+1| + y_a x_b|-1| + y_a y_b|0|$$
$$= x_a y_b - y_a x_b$$

$$|\boldsymbol{abc}|$$
$$= |(x_a\boldsymbol{x} + y_a\boldsymbol{y} + z_a\boldsymbol{z})(x_b\boldsymbol{x} + y_b\boldsymbol{y} + z_b\boldsymbol{z})(x_c\boldsymbol{x} + y_c\boldsymbol{y} + z_c\boldsymbol{z})|$$
$$= x_a y_b z_c - x_a z_b y_c - y_a x_b z_c + y_a z_b x_c + z_a x_b y_c - z_a y_b x_c$$

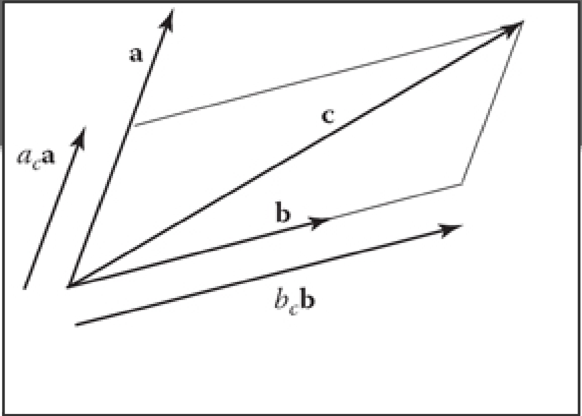
Determinants (cont.)

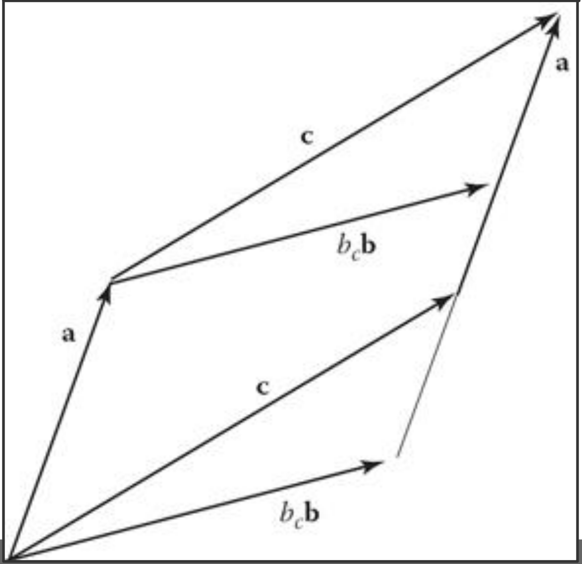
$$\boldsymbol{c} = a_c \boldsymbol{a} + b_c \boldsymbol{b}$$

$$|(b_c \boldsymbol{b})\boldsymbol{a}| = |\boldsymbol{c}\boldsymbol{a}|$$

$$b_c = \frac{|\boldsymbol{c}\boldsymbol{a}|}{|\boldsymbol{b}\boldsymbol{a}|}$$

$$a_c = \frac{|\boldsymbol{b}\boldsymbol{c}|}{|\boldsymbol{b}\boldsymbol{a}|}$$





Matrices

Addition:

$$p_{i,j} = a_{i,j} + b_{i,j}$$

Multiplication:

$$p_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + a_{i,3}b_{3,j} + \cdots + a_{i,m}b_{m,j}$$

$$\boldsymbol{AB} \neq \boldsymbol{BA}$$

Inverse:

$$\boldsymbol{AA}^{-1} = \boldsymbol{I},\ (\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

Matrices (cont.)

Transpose:

$$\boldsymbol{A}^T : a^T_{i,j} = a_{j,i}$$

$$(\boldsymbol{A}\boldsymbol{B})^T = \boldsymbol{B}^T\boldsymbol{A}^T$$

Determinant:

$$|\boldsymbol{A}\boldsymbol{B}| = |\boldsymbol{A}||\boldsymbol{B}|$$

$$|\boldsymbol{A}^{-1}| = \frac{1}{|\boldsymbol{A}|}; \; |\boldsymbol{A}^T| = |\boldsymbol{A}|$$

Vector Operations in Matrix Form

Matrix-column-vector multiplication:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ y_a \end{bmatrix} = \begin{bmatrix} -y_a \\ x_a \end{bmatrix}$$

Row-vector-matrix multiplication:

$$\begin{bmatrix} x_a & y_a \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -y_a & x_a \end{bmatrix}$$

Vector Operations in Matrix Form (cont.)

Inner Product:

$$\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} x_a & y_a & z_a \end{bmatrix} \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} x_a x_b + y_a y_b + z_a z_b \end{bmatrix}$$

Outer Product:

$$\boldsymbol{a}\boldsymbol{b}^{\mathrm{T}} = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} \begin{bmatrix} x_b & y_b & z_b \end{bmatrix} = \begin{bmatrix} x_a x_b & x_a y_b & x_a z_b \\ y_a x_b & y_a y_b & y_a z_b \\ z_a x_b & z_a y_b & z_a z_b \end{bmatrix}$$

Special Types of Matrices

- Identity Matrix
- Diagonal Matrix
- Symmetric Matrix: $\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R}$
- Orthogonal Matrix:

$$\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{I} = \boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} \Rightarrow \boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R}^{-1}$$

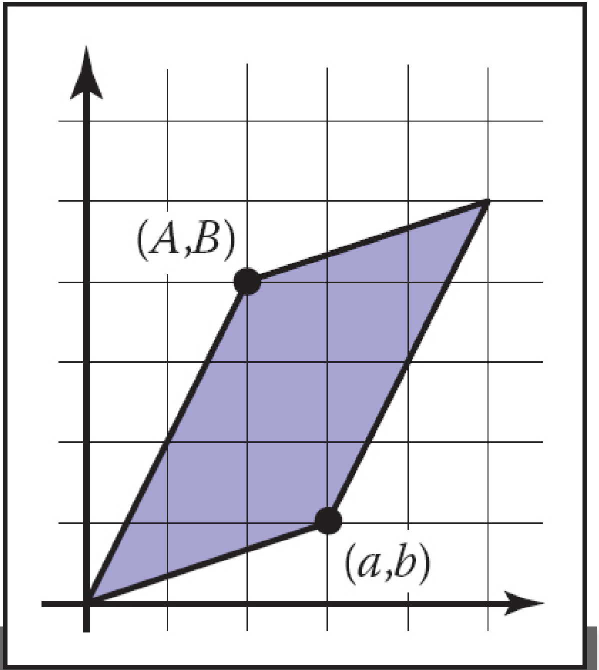
$$|\boldsymbol{R}| = \pm 1$$

Matrices and Determinants

$$\left|\begin{bmatrix} a \\ b \end{bmatrix} \quad \begin{bmatrix} A \\ B \end{bmatrix}\right| \equiv \begin{vmatrix} a & A \\ b & B \end{vmatrix} = \begin{vmatrix} a & b \\ A & B \end{vmatrix} = aB - Ab$$

Plane equation:

$$\begin{vmatrix} x - x_0 & x - x_1 & x - x_2 \\ y - y_0 & y - y_1 & y - y_2 \\ z - z_0 & z - z_1 & z - z_2 \end{vmatrix} = 0$$



Computing Determinants using Co-Factors

$$\boldsymbol{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \qquad a_{11}^{c} = \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$a_{12}^{c} = -\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} \qquad a_{13}^{c} = \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

$$|\boldsymbol{A}| = a_{11}a_{11}^{c} + a_{12}a_{12}^{c} + a_{13}a_{13}^{c} + a_{14}a_{14}^{c}$$

Computing Inverses

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{11}^c & a_{21}^c & a_{31}^c & a_{41}^c \\ a_{12}^c & a_{22}^c & a_{32}^c & a_{42}^c \\ a_{13}^c & a_{23}^c & a_{33}^c & a_{43}^c \\ a_{14}^c & a_{24}^c & a_{34}^c & a_{44}^c \end{bmatrix}$$

Adjoint matrix to $\boldsymbol{A}$

Cramer's rule

$$x_1 = \frac{D_{x_1}}{D}, \qquad \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{1,2} & a_{1,3} \\ b_2 & a_{2,2} & a_{2,3} \\ b_3 & a_{3,2} & a_{3,3} \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}}$$

Eigenvalues and Matrix Diagonalization

$$\boldsymbol{A}\boldsymbol{a} = \lambda \boldsymbol{a}$$
$$\boldsymbol{A}\boldsymbol{a} = \lambda \boldsymbol{I}\boldsymbol{a}$$
$$\boldsymbol{A}\boldsymbol{a} - \lambda \boldsymbol{I}\boldsymbol{a} = 0$$
$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{a} = 0$$
$$\boldsymbol{a} \neq \boldsymbol{0} \Rightarrow |A - \lambda \boldsymbol{I}| = 0$$

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^{-1}$$
$$= \boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^{\mathrm{T}}$$

Ex.: $\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

Singular Value Decomposition

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^{\mathrm{T}}$$

$$\boldsymbol{M} = \boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} = \left(\boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^{\mathrm{T}}\right)\left(\boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^{\mathrm{T}}\right)^{\mathrm{T}} = \boldsymbol{U}\boldsymbol{S}\left(\boldsymbol{V}^{\mathrm{T}}\boldsymbol{V}\right)\boldsymbol{S}\boldsymbol{U}^{\mathrm{T}} = \boldsymbol{U}\boldsymbol{S}^2\boldsymbol{U}^{\mathrm{T}}$$

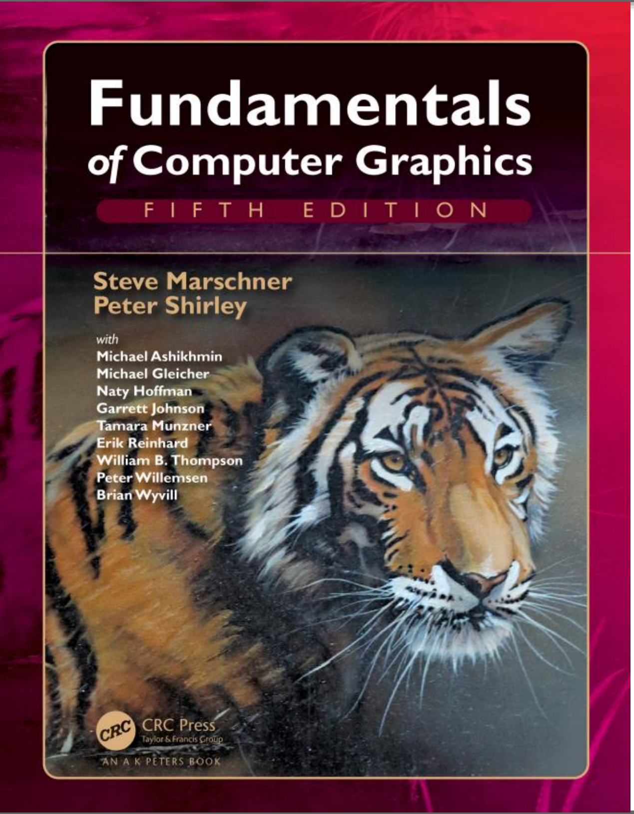
$$\boldsymbol{N} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \left(\boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^{\mathrm{T}}\right)^{\mathrm{T}}\left(\boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^{\mathrm{T}}\right) = \boldsymbol{V}\boldsymbol{S}\left(\boldsymbol{U}^{\mathrm{T}}\boldsymbol{U}\right)\boldsymbol{S}\boldsymbol{V}^{\mathrm{T}} = \boldsymbol{V}\boldsymbol{S}^2\boldsymbol{V}^{\mathrm{T}}$$

or

$$\boldsymbol{V} = \left(\boldsymbol{S}^{-1}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{A}\right)^{\mathrm{T}}$$

Ex.: $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Credits



Fundamentals of Computer Graphics, 5th Edition

by Peter Shirley, Steve Marschner

Publisher: A K Peters/CRC Press

Release Date: September 30, 2021

ISBN: 9781000426359

https://learning.oreilly.com/library/view/fundamentals-of-computer/9781000426359/