جامعة الملك عبدالله
للعلوم والتقنية
King Abdullah University of
Science and Technology

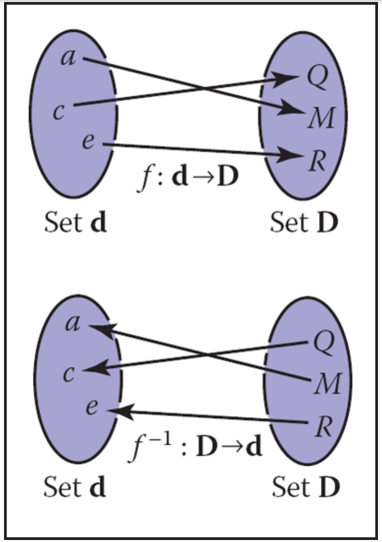
VCC VISUAL COMPUTING CENTER

Computer Graphics CS248
Math for Computer Graphics

Ivan Viola
NANOVISUALIZATION GROUP

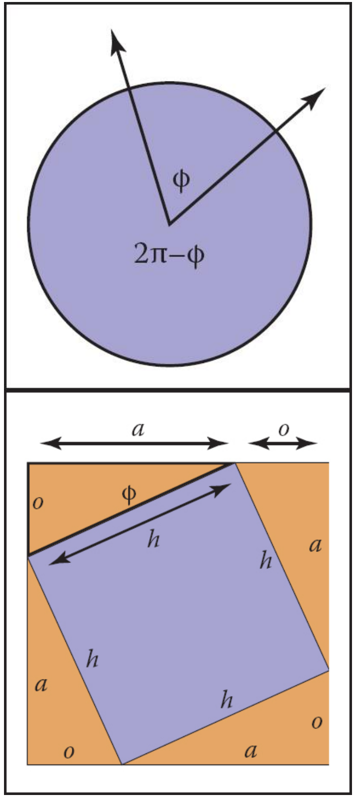
Basic Notations

- Function $f\colon \mathbb{R} \to \mathbb{Z}$
- Domain $\mathbb{R}$, target $\mathbb{Z}$
- $f(a)$ is image of a
- Image of the whole domain is *range* of the function
- $f^{-1}\colon \boldsymbol{B} \to \boldsymbol{A}$ is inverse function to $f\colon \boldsymbol{A} \to \boldsymbol{B}$ (bijective = injective+surjective)
- $f\colon \mathbb{R} \to \mathbb{Z}, f(x) = x^3$ is bijective, $f(x) = x^2$ is not



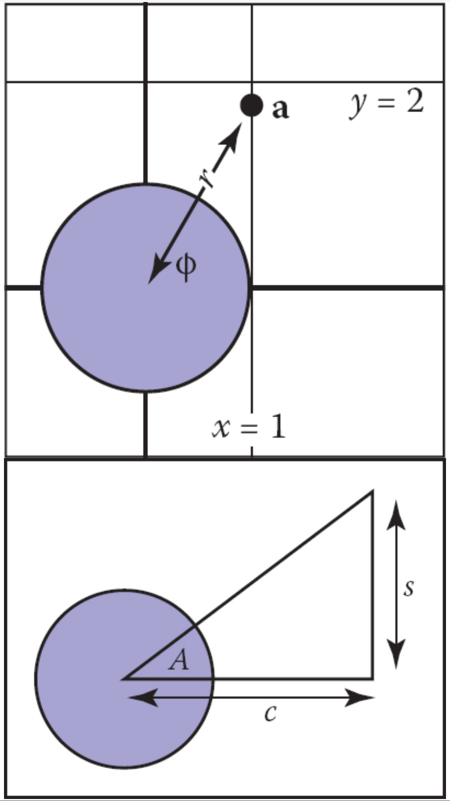
Angles

- Angle: the arclength of unit circle segment defined by two halflines
- Convention: range $[-\pi, \pi]$
- Opposite, adjacent sides, hypotenuse
- $a^2 + o^2 = h^2 = (a+o)^2 - 2ao$
- $\sin\varphi = \frac{o}{h}, \cos\varphi = \frac{a}{h}, \tan\varphi = \frac{o}{a}$



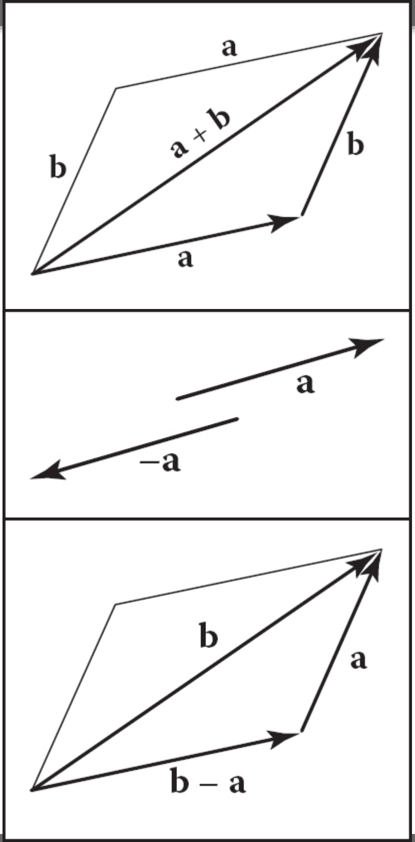
Polar Coordinates

- Cartesian coordinates define positions through distance of projection on coordinate system
- Polar coordinates define position by distance r from origin and angle φ
- Function $\mathrm{atan2}(s, c)$ converts Cartesian coordinates to angle A in polar coordinates



Vectors

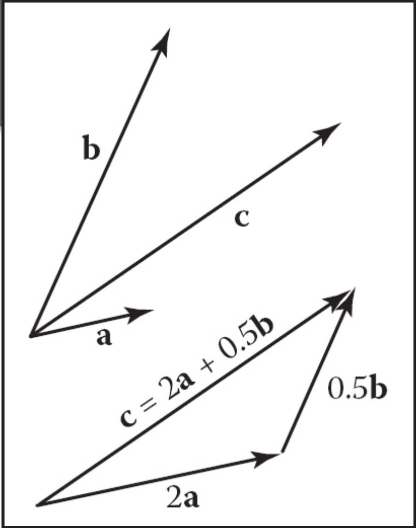
- Vectors geometrically describe length and direction
- Vectors form offset, offset from origin define point locations
- In graphics, unit vectors are used
- Vectors can be added, negated, subtracted, multiplied

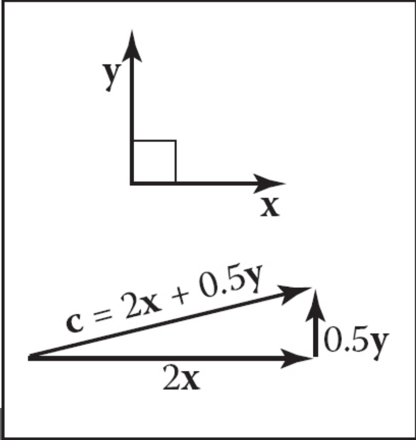


Cartesian Coordinates

- Vector can be described as a sum of other linearly independent basis vectors
- Orthonormal basis vectors form Cartesian coordinate system

$$\boldsymbol{a} = x_a \boldsymbol{x} + y_a \boldsymbol{y}$$

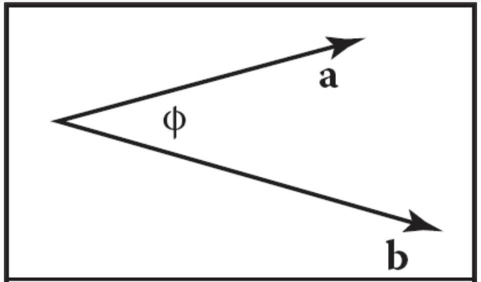


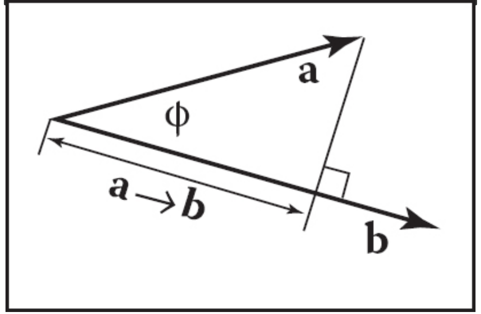


Vector Operations

- Vector length $\|\boldsymbol{a}\| = \sqrt{x_a^2 + y_a^2}$
- Dot Product $\boldsymbol{a}.\boldsymbol{b} = x_a x_b + y_a y_b$
$$= \|\boldsymbol{a}\|\|\boldsymbol{b}\| \cos\varphi$$
- Projection

$$\boldsymbol{a} \rightarrow \boldsymbol{b} = \|\boldsymbol{a}\| \cos\varphi = \frac{\boldsymbol{a}.\boldsymbol{b}}{\|\boldsymbol{b}\|}$$





Vector Operations (cont.)

- Cross Product

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} y_a z_b - z_a y_b \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{bmatrix}$$

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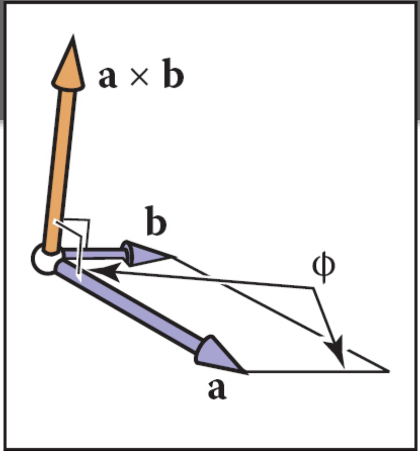
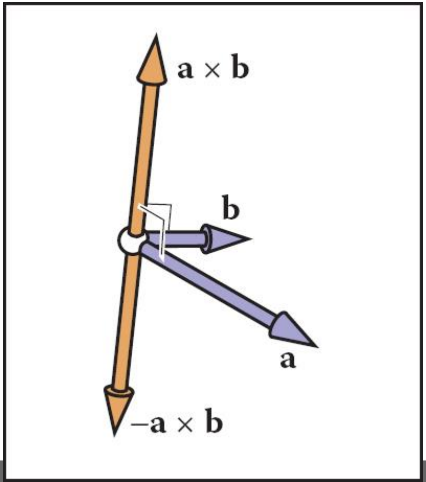
$$\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \sin \varphi$$





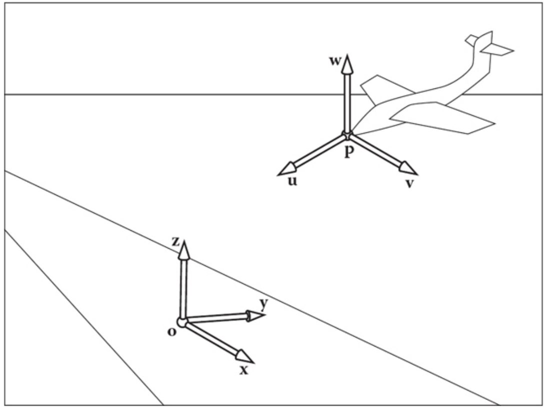
Coordinate Systems

- Canonical (global, world) coordinate system
- Vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, origin $\boldsymbol{o}$ not explicitly stored
- Frame of reference (local coordinate system)

$$\boldsymbol{u} = x_u\boldsymbol{x}+y_u\boldsymbol{y}+z_u\boldsymbol{z}$$

$$\boldsymbol{p} = \boldsymbol{o} + x_p\boldsymbol{x}+y_p\boldsymbol{y}+z_p\boldsymbol{z}$$

$$\boldsymbol{p} = [x_p, y_p, z_p]$$



Coordinate Systems (cont.)

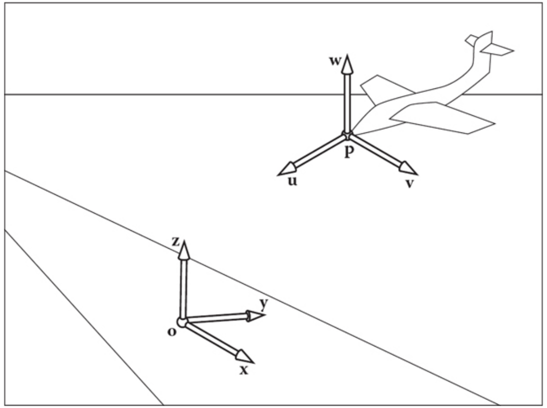
$\boldsymbol{u} = x_u\boldsymbol{x}+y_u\boldsymbol{y}+z_u\boldsymbol{z}$

$\boldsymbol{a} = u_a\boldsymbol{u}+v_a\boldsymbol{v}+w_a\boldsymbol{w}$ transform into global coord.

$\boldsymbol{b} = x_b\boldsymbol{x}+y_b\boldsymbol{y}+z_b\boldsymbol{z}$

Projection of $\boldsymbol{b}$ onto $\boldsymbol{u}$-$\boldsymbol{v}$-$\boldsymbol{w}$ frame

$u_b = \boldsymbol{u}.\boldsymbol{b}, v_b = \boldsymbol{v}.\boldsymbol{b}, w_b = \boldsymbol{w}.\boldsymbol{b}$



Constructing a Basis from two Vectors

Given are vectors transform $\boldsymbol{a}$ and $\boldsymbol{b}$

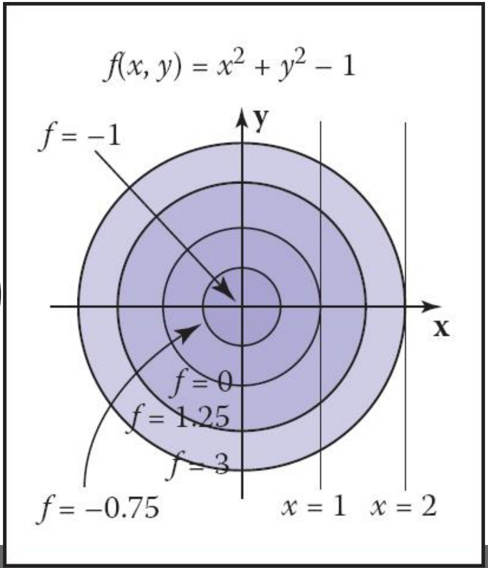
Searching for coordinate system $\boldsymbol{u}$–$\boldsymbol{v}$–$\boldsymbol{w}$ that is as close as possible to $\boldsymbol{a}$ and $\boldsymbol{b}$

- $\boldsymbol{w} = \dfrac{\boldsymbol{a}}{||\boldsymbol{a}||}$
- $\boldsymbol{u} = \dfrac{\boldsymbol{b}\times\boldsymbol{w}}{||\boldsymbol{b}\times\boldsymbol{w}||}$
- $\boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{u}$

Curves and Surfaces

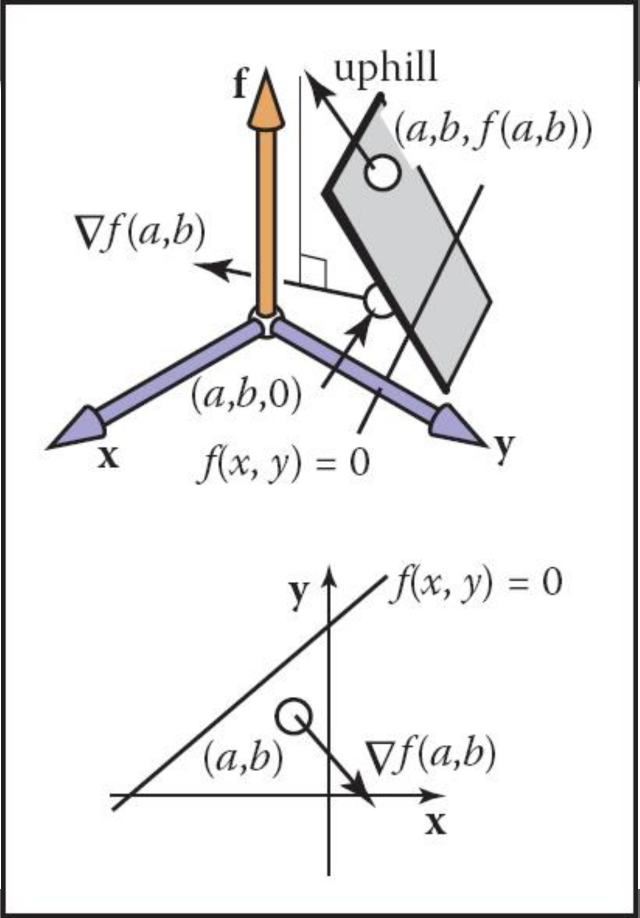
- Curve described in implicit form $f(x, y) = 0$
- Circle $(x - x_c)^2 + (y - y_c)^2 - R^2 = 0$
- Inside the curve $f(x, y) < 0$
- Vector form

$$(\boldsymbol{p} - \boldsymbol{c}).(\boldsymbol{p} - \boldsymbol{c}) - R^2 = 0$$
$$\|\boldsymbol{p} - \boldsymbol{c}\| - R = 0$$



Gradient

- Heightfield $h = f(x, y)$
- Gradient points in the maximum slope
- $f(x, y) = c$
  $\nabla f(x, y)$ is a normal to f



Partial Derivative and Gradient

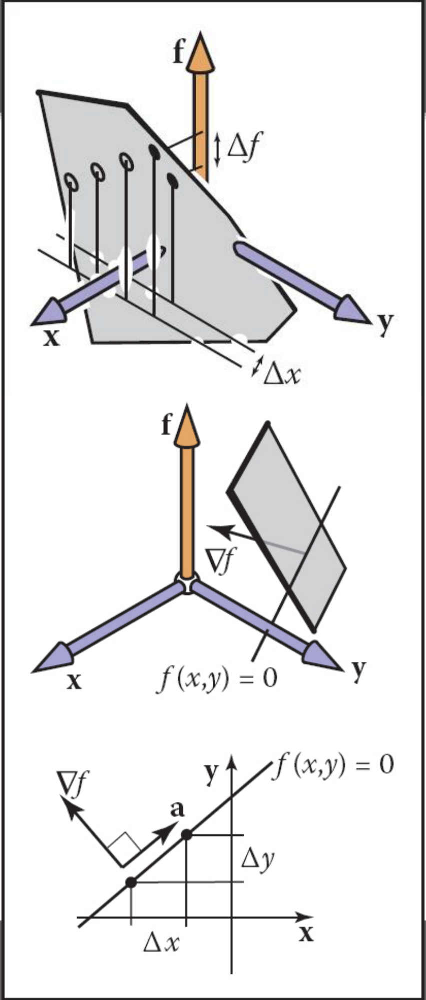
Expresses how much f changes when x is infinitesimally changing while y is constant

$$(\boldsymbol{\nabla f}).\boldsymbol{a} = (x_\nabla, y_\nabla).(x_a, y_a) = x_\nabla \Delta x + y_\nabla \Delta y = 0$$

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y = \frac{\partial f}{\partial x} x_a + \frac{\partial f}{\partial y} y_a = 0$$

$$\rightarrow (x_a, y_a) = k\left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right) \rightarrow (x_\nabla, y_\nabla) = k'\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\nabla f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$



Implicit 2D Lines

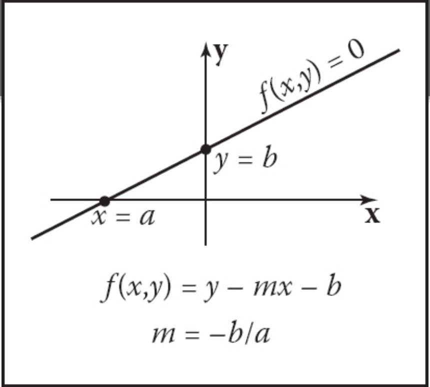
Slope-intercept form $y = mx + b$

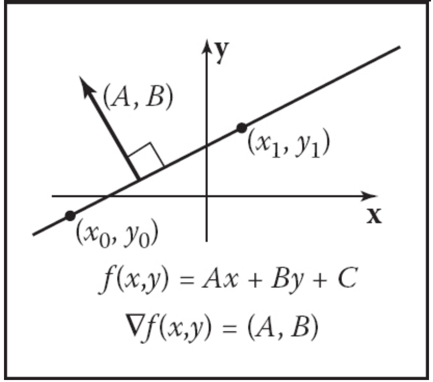
in implicit form $y - mx - b = 0$

For (x_0, y_0) and (x_1, y_1) the corresponding implicit form is

$$Ax + By + C = 0$$

$$(y_0 - y_1)x + (x_1 - x_0)y + x_0 y_1 - x_1 y_0 = 0$$





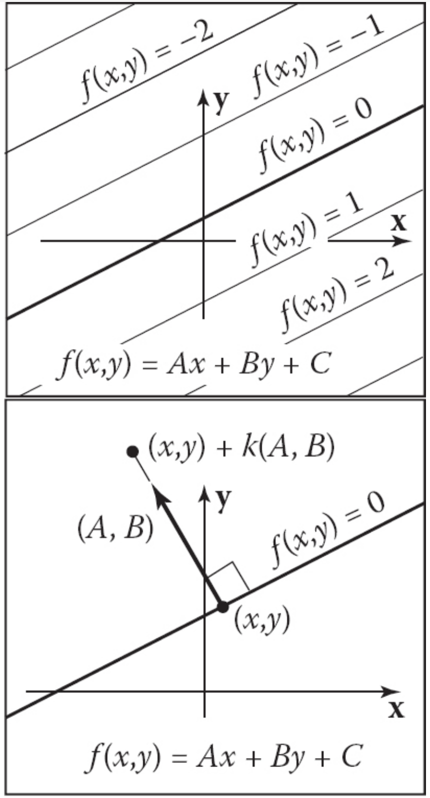
Implicit 2D Lines (cont.)

Implicit eq. $f(x, y) = Ax + By + C$ can be used to find the distance of a point $P = (x_p, y_p)$ to line $f(x, y) = 0$

$$(x_p, y_p) = (x, y) + k(A, B)$$

Distance $d = k\sqrt{A^2 + B^2}$

$$f(x + kA, y + kB) = k(A^2 + B^2)$$

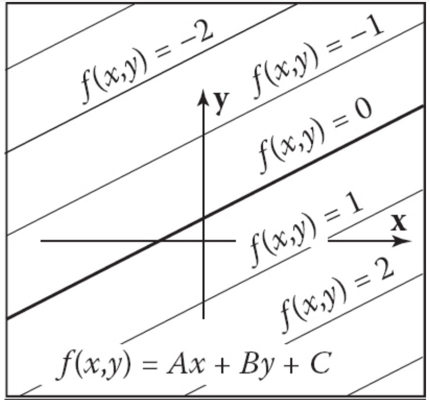


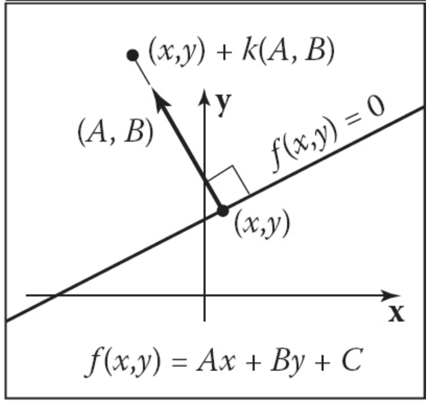
Implicit 2D Lines (cont.)

$$\text{Distance} = k\sqrt{A^2 + B^2}$$

$$f(x + kA, y + kB) = k(A^2 + B^2)$$

$$\text{Distance} = \frac{f(x_p, y_p)}{\sqrt{A^2+B^2}}$$





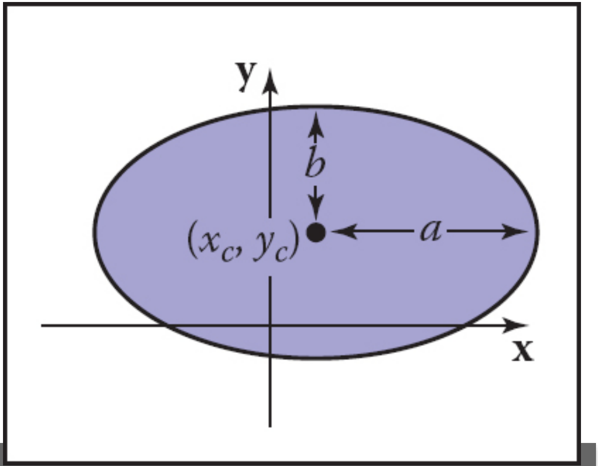
Implicit Quadric Curves

Quadratic function

$$f(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Ellipse:

$$\frac{(x - x_c)^2}{a^2} + \frac{(y - y_c)^2}{b^2} - 1 = 0$$



Implicit Surfaces

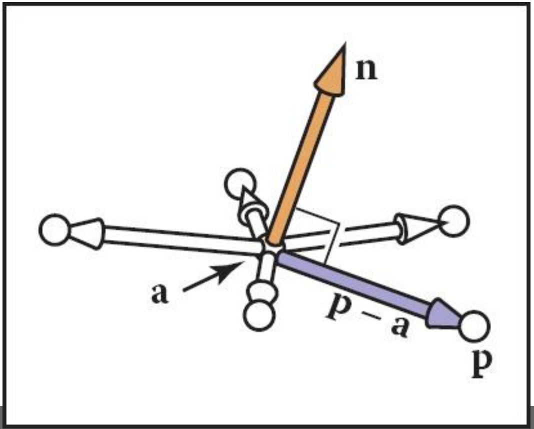
Function $f(x, y, z) = 0$, $\boldsymbol{p} = (x, y, z)$, $f(\boldsymbol{p}) = 0$

Normal $\boldsymbol{n} = \nabla f(\boldsymbol{p}) = \left(\frac{\partial f(\boldsymbol{p})}{\partial x}, \frac{\partial f(\boldsymbol{p})}{\partial y}, \frac{\partial f(\boldsymbol{p})}{\partial z}\right)$

Plane $(\boldsymbol{p} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0$

For plane through points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$

$(\boldsymbol{p} - \boldsymbol{a}) \cdot ((\boldsymbol{b} - \boldsymbol{a}) \times (\boldsymbol{c} - \boldsymbol{a})) = 0$



Quadric Surfaces

Sphere: $f(\boldsymbol{p}) = (\boldsymbol{p} - \boldsymbol{c})^2 - r^2 = 0$

Ellipsoid:

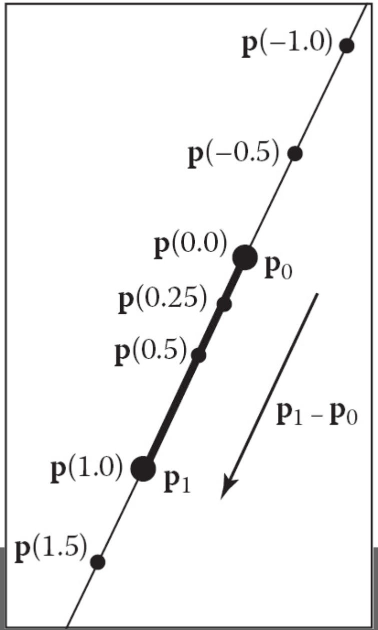
$$f(\boldsymbol{p}) = \frac{(x - x_c)^2}{a^2} + \frac{(y - y_c)^2}{b^2} + \frac{(z - z_c)^2}{c^2} - 1 = 0$$

Parametric Curves

A curve is defined through a single parameter t that continuously moves along the curve.

$$f(t)\colon \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g(t) \\ h(t) \end{bmatrix}$$

Line: $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 + t(x_1 - x_0) \\ y_0 + t(y_1 - y_0) \end{bmatrix}$$



Parametric Curves (cont.)

Circle: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_c + r\cos\varphi \\ y_c + r\sin\varphi \end{bmatrix}$ $\varphi \epsilon [-\pi, \pi]$

Axis-aligned ellipse:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_c + a\cos\varphi \\ y_c + b\sin\varphi \end{bmatrix}$$

Spiral: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}$

Parametric Curves (cont.)

Circle: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_c + r\cos\varphi \\ y_c + r\sin\varphi \end{bmatrix}$ $\varphi \epsilon [-\pi, \pi]$

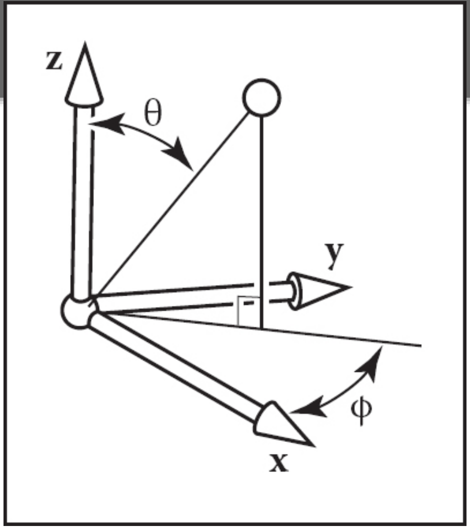
Axis-aligned ellipse:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_c + a\cos\varphi \\ y_c + b\sin\varphi \end{bmatrix}$$

Spiral: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}$

Parametric Surfaces

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boldsymbol{p}(u, v), \boldsymbol{p}\colon \mathbb{R}^2 \rightarrow \mathbb{R}^3$$



Sphere:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\varphi\sin\theta \\ r\sin\varphi\sin\theta \\ r\cos\theta \end{bmatrix}, \quad \theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\varphi = \operatorname{atan2}(y, x)$$

Derivative of Parametric Surfaces

$\boldsymbol{q}(t) = \boldsymbol{p}(t, v_0)$, $\boldsymbol{q}$ is an iso-parametric curve

Derivative $\boldsymbol{q}'$ is a tangent vector to $\boldsymbol{q}$ and partial derivative of $\boldsymbol{p}$ wrt. u ($\boldsymbol{p}_u$). The same holds for varying v.

Normal to the surface can be obtained through

$\boldsymbol{n} = \boldsymbol{p}_u \times \boldsymbol{p}_v$

Linear Interpolation

$\boldsymbol{p} = (1-t)\boldsymbol{a} + t\boldsymbol{b}$ interpolates position $\boldsymbol{a}$ to position $\boldsymbol{b}$ with $t \in [0,1]$.

Linear interpolation between set of positions x_0, x_1, x_2, … x_n

$$f(x) = y_i + \frac{x - x_i}{x_{i+1} - x_i}(y_{i+1} - y_i)$$

Triangles

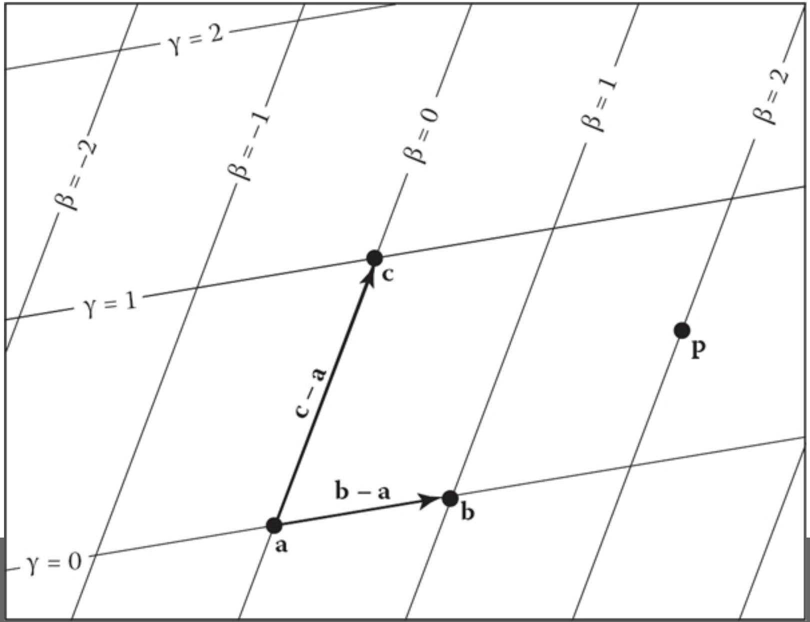
Interpolation: Barycentric coordinates

using non-orthogonal coordinate system: $\boldsymbol{ab}, \boldsymbol{ac}$

Points represented by ordered pair (β, γ)

$\boldsymbol{p} = (2.0, 0.5)$

$$\boldsymbol{p} = \boldsymbol{a} + 2.0(\boldsymbol{b} - \boldsymbol{a}) + 0.5(\boldsymbol{c} - \boldsymbol{a})$$



Triangles (cont.)

$$\boldsymbol{p} = \boldsymbol{a} + \beta(\boldsymbol{b} - \boldsymbol{a}) + \gamma\,(\boldsymbol{c} - \boldsymbol{a})$$
$$\boldsymbol{p} = \boldsymbol{a}(1 - \beta - \gamma) + \beta\boldsymbol{b} + \gamma\boldsymbol{c}$$

$$\alpha = (1 - \beta - \gamma)$$

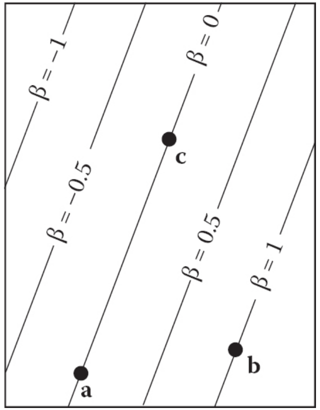
Within a triangle $\alpha, \beta, \gamma \in [0,1]$

$$\boldsymbol{p}\text{:} \begin{bmatrix} x_b - x_a & x_c - x_a \\ y_b - y_a & y_c - y_a \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} x_p - x_a \\ y_p - y_a \end{bmatrix}$$

Triangles (cont.)

Scaled distance from a line $f(x, y) = 0$

$$(y_a - y_c)x + (x_c - x_a)y + x_a y_c - x_c y_a = 0$$



$$\beta = \frac{f_{ac}(x, y)}{f_{ac}(x_b, y_b)}$$

$$\beta = \frac{(y_a - y_c)x + (x_c - x_a)y + x_a y_c - x_c y_a}{(y_a - y_c)x_b + (x_c - x_a)y_b + x_a y_c - x_c y_a}$$

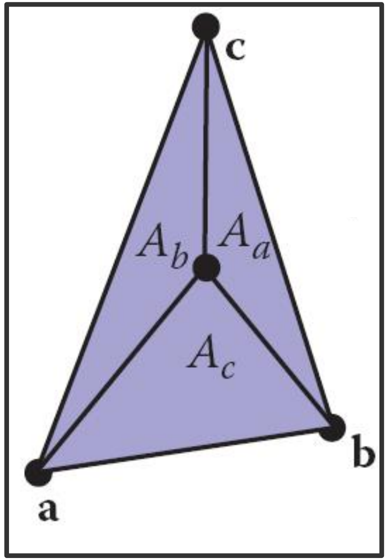
Triangles (cont.)

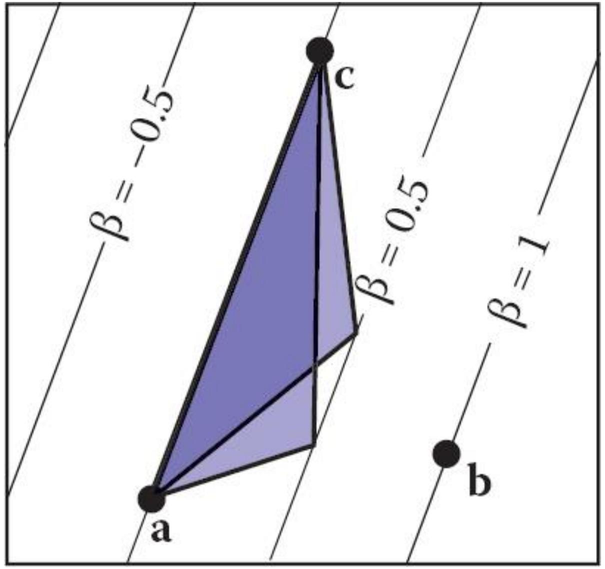
$$\alpha = \frac{A_a}{A}$$

$$\beta = \frac{A_b}{A}$$

$$\gamma = \frac{A_c}{A}$$

$$A = A_a + A_b + A_c$$





Area: $A = \frac{1}{2}\begin{vmatrix} x_b - x_a & x_c - x_a \\ y_b - y_a & y_c - y_a \end{vmatrix}$

Triangles (cont.)

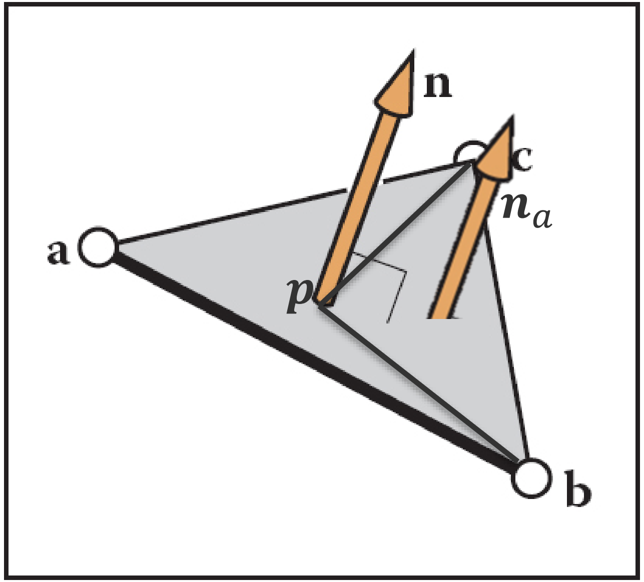
Triangles in 3D extend naturally from 2D case.

Normal computed by the cross

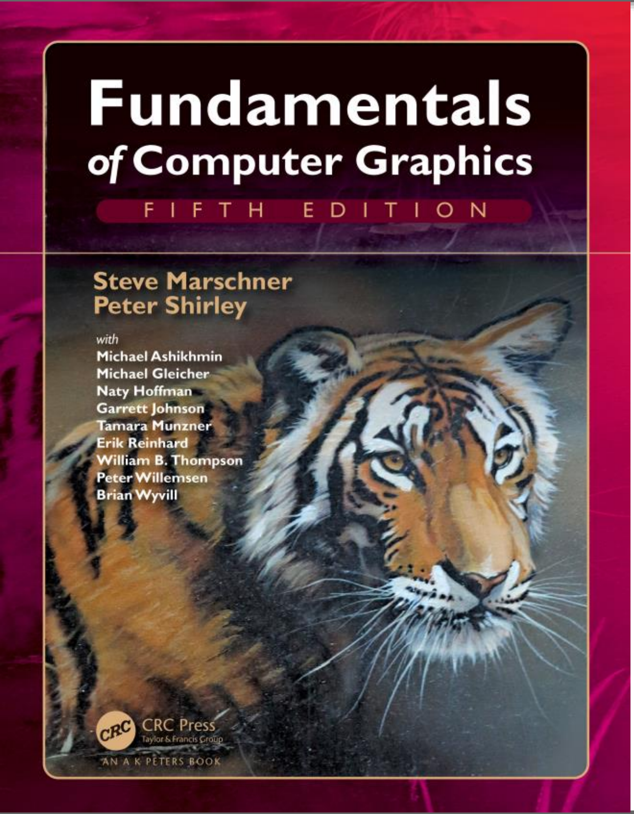
$$\boldsymbol{n} = (\boldsymbol{b} - \boldsymbol{a}) \times (\boldsymbol{c} - \boldsymbol{a})$$

Area: $\frac{1}{2}|(\boldsymbol{b} - \boldsymbol{a}) \times (\boldsymbol{c} - \boldsymbol{a})|$

$$\alpha = \frac{\boldsymbol{n}.\boldsymbol{n}_a}{||\boldsymbol{n}||^2}$$



Credits



Fundamentals of Computer Graphics, 5th Edition

by Peter Shirley, Steve Marschner

Publisher: A K Peters/CRC Press

Release Date: September 30, 2021

ISBN: 9781000426359

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