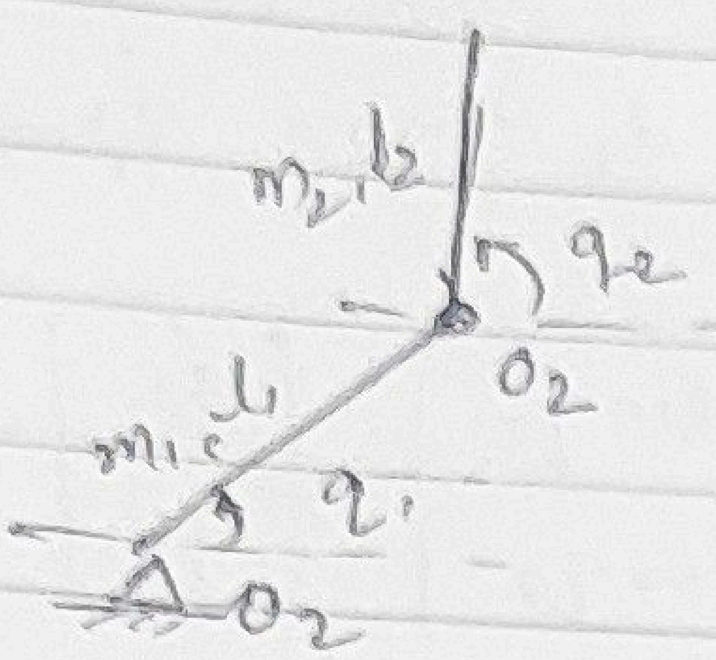
2R Manipulator:



$E \rightarrow$ end effector.

$(x,y) \rightarrow$ end effector position

Note: q_2 convention,

Assume origin at O_1

Let us assume motors are connected to ~~both~~ each links at O_1 and O_2.

$(q_1, q_2) \rightarrow$ angle joints.

Also called planar elbow manipulator

Let us assume we have a way to control either the torque τ_1 and τ_2 applied to the two joints or control angles q_1 and q_2 directly.

~~Also, called~~ We will later study how (hardware, algorithm, software, etc) we can control τ_1, τ_2 or q_1, q_2.

Note: Angles are sometimes θ_1, θ_2 or p_1, p_2 in various textbooks.

Let us consider three tasks

Task 1 (T1) - Given an arbitrary trajectory of end effector (given (x,y) as a function of time), make the robot follow the trajectory.

Task 2 (T2) - Given the location on a wall, make the robot touch the wall ~~at~~ and apply a prespecified constant force at that location.

Task 3 (T3) - Make the robot behave like a virtual spring connected from E to a given point (x_0, y_0).

Now,

$x = l_1 \cos q_1 + l_2 \cos q_2$
$y = l_1 \sin q_1 + l_2 \sin q_2$

or using simplified notation.

$$\left.\begin{aligned} x &= l_1 c q_1 + l_2 c q_2 \\ y &= l_1 s q_1 + l_2 s q_2 \end{aligned}\right\} \to (1).$$

Differentiating (1). we get

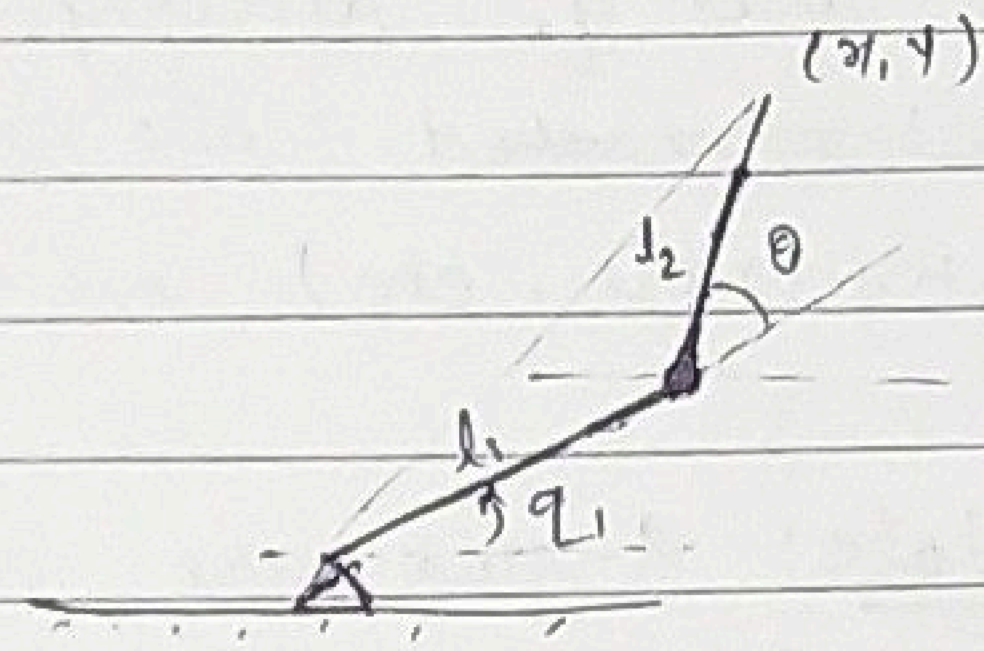
~~$\dot{x} = -l_1 \sin q_1 - l_2 s q_2$~~
~~$\dot{y} =$~~

$$\left.\begin{aligned} \dot{x} &= -l_1 s q_1 \cdot \dot{q}_1 - l_2 s q_2 \cdot \dot{q}_2 \\ \dot{y} &= l_1 \cos q_1 \, \dot{q}_1 + l_2 \cos q_2 \, \dot{q}_2 \end{aligned}\right\} \to (2).$$

$\Rightarrow$ End effector velocity. $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 \sin q_1 & -l_2 \sin q_2 \\ l_1 \cos q_1 & l_2 \cos q_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$

We will need ~~~~ the inverse relationship. Given (x, y) we need to be able to find q_1, q_2.

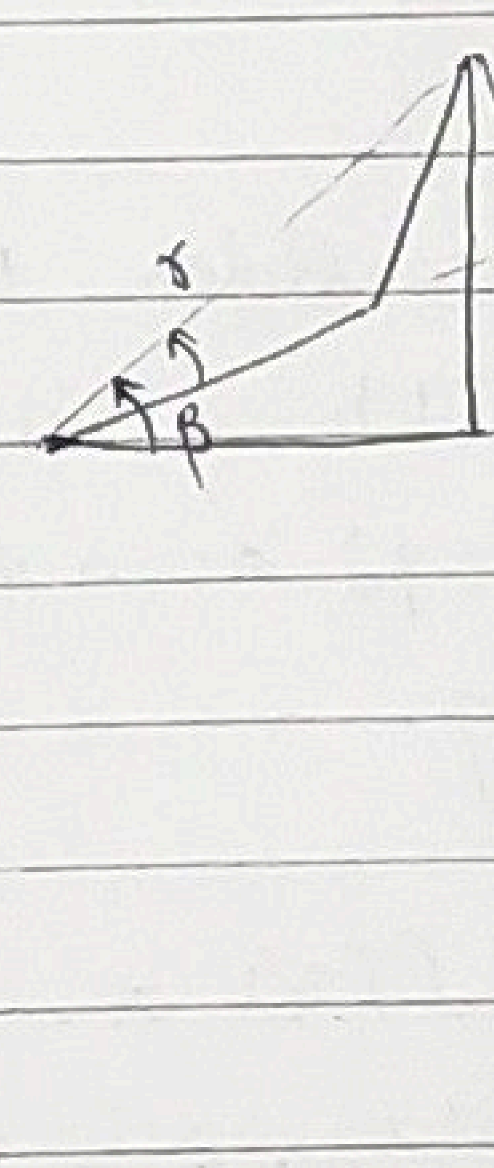
Option 1 : Solve numerically.
Option 2 : Derive a closed-form expression. $\begin{cases} q_1 = \dots \\ q_2 = \dots \end{cases}$
- Hard in general.
- Multiple solution.



Cosine rule + switching to acute angles.

$(x^2 + y^2) = l_1^2 + l_2^2 + 2 l_1 l_2 \cos\theta$

$\Rightarrow \quad \theta = \cos^{-1}\left(\dfrac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right)$



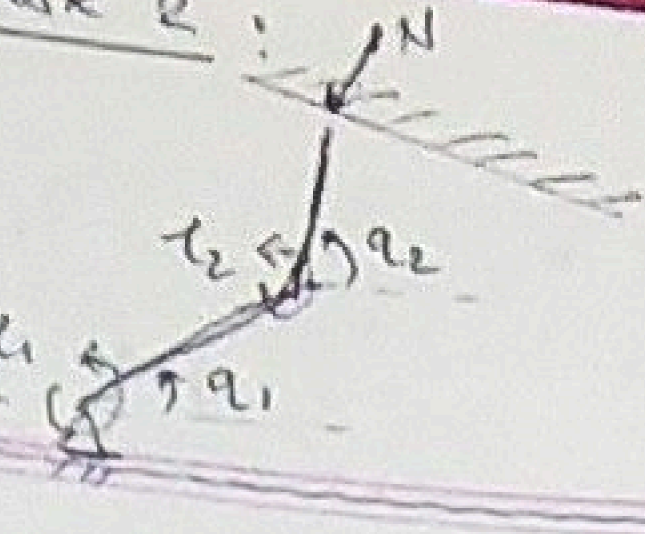
$q_1 = \beta - \gamma$
$\quad = \tan^{-1}\left(\dfrac{y}{x}\right) - \tan^{-1}\left(\dfrac{l_2 \sin\theta}{l_1 + l_2 \cos\theta}\right)$

$q_2 = q_1 + \theta$

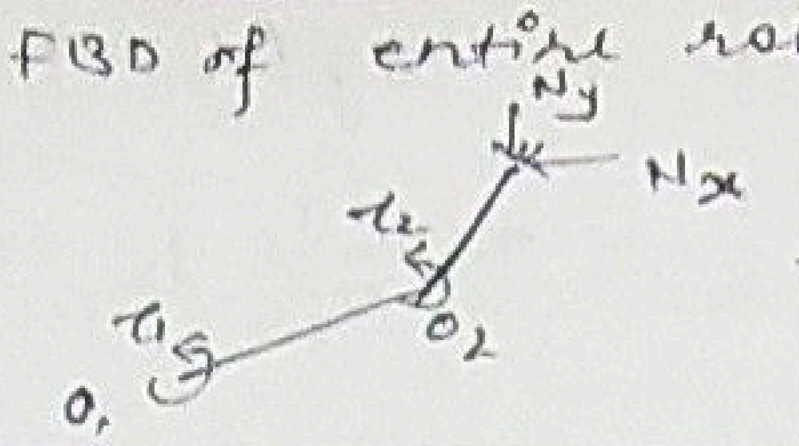
This is 1st level answer to T1.

We will later start using the notation x_d and y_d (and q_{1d} and q_{2d}) here for desired values. (they are not ~~necessarily~~ necessary actual value)

Task 2 :



FBD of entire robot.



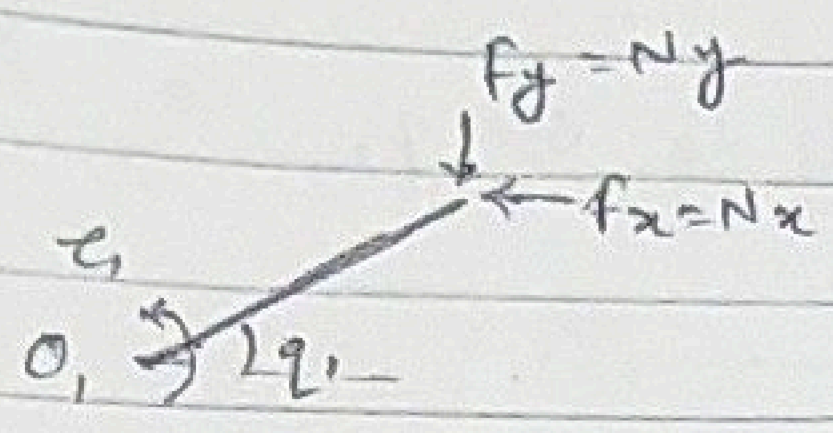
Forces applied by the manipulator

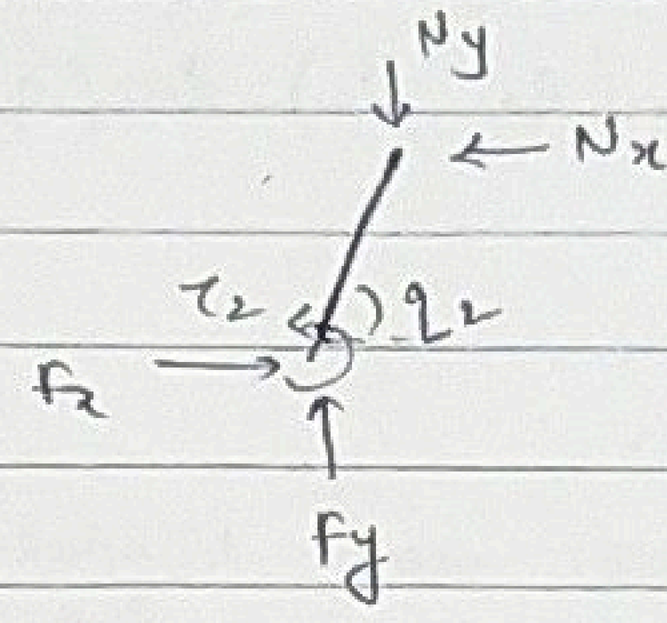
$F_x = -N_x$

$F_y = -N_y$

Neglecting gravity.

FBD of each link ~~sp~~ seperately.





$\Sigma M_{O_2} = 0$

~~$N_y l_2 c q_2$~~

C.C.W +ve

$\Sigma M_{O_1} = 0$

$$\Rightarrow \boxed{N_y l_1 \cos q_1 - N_x l_1 \sin q_1 = \tau_1}$$

$\Sigma M_{O_2} = 0$

$$\Rightarrow \boxed{N_y l_2 \cos q_2 \; \cancel{+} \; N_x l_2 \sin q_2 = +\tau_2}$$

$\Rightarrow$

$$\cancel{\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} l_1 c q_1 & -l_1 s q_1 \\ l_2 c q_2 & -l_2 s q_2 \end{bmatrix} \begin{bmatrix} N_x \\ N_y \end{bmatrix}}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 \sin q_1 & l_1 \cos q_1 \\ -l_2 \sin q_2 & l_2 \cos q_2 \end{bmatrix} \begin{bmatrix} N_x \\ N_y \end{bmatrix} \quad \text{— (4)}$$

For T_3 and next-level answer to T1. Need to understand dynamics. Lagrange's Equation:

$$\mathcal{L} = K - V$$

K - kinetic energy , V - Potential energy

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \left(\frac{\partial \mathcal{L}}{\partial q_i}\right) = Q_i} \quad (5)$$

Q_i are generalized forces derived using principle of virtual work.

$$K = \underbrace{\frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right) \dot{q}_1^2}_{\text{pure rotation of } l_1} + \frac{1}{2}\left(\frac{1}{12} m_2 l_2^2\right) \dot{q}_2^2 + \frac{1}{2} m_2 v_{c_2}^2$$

velocity of COM of l_2

where $v_{c_2}^2 = (l_1 \dot{q}_1)^2 + \left(\frac{l_2}{2} \dot{q}_2\right)^2 + 2 \, l_1 \dot{q}_1 \cdot \frac{l_2}{2} \dot{q}_2 \cos(q_2 - q_1)$

$$V = m_1 g \frac{l_1}{2} \sin q_1 + m_2 g \left(l_1 \sin q_1 + \frac{l_2}{2} \sin q_2 \right)$$

$$\frac{1}{3} m_1 l_1^2 \ddot{q}_1 + m_2 l_1^2 \ddot{q}_1 + m_2 \frac{l_1 l_2}{2} \ddot{q}_2 \cos(q_2 - q_1) - m_2 \frac{l_1 l_2}{2} \dot{q}_2 (\dot{q}_2 - \dot{q}_1) \sin(q_2 - q_1)$$

$$+ m_1 g \frac{l_1}{2} \cos q_1 + m_2 g l_1 \cos q_1 = \tau_1$$

$$\frac{1}{3} m_2 l_2^2 \ddot{q}_2 + m_2 \frac{l_2^2}{4} \ddot{q}_2 + m_2 \frac{l_1 l_2}{2} \ddot{q}_1 \cos(q_1 - q_2)$$

$$- m_2 \frac{l_1 l_2}{2} \dot{q}_1 (\dot{q}_2 - \dot{q}_1) \sin(q_2 - q_1) + m_2 g \frac{l_2}{2} \sin q_2 = \tau_2$$

— (6).

Save this for later.

Next, we ~~had~~ ~~force~~ note that (4) is valid for any forces F_x, F_y.

We want,

$$F_x = kx \qquad F_y = ky$$

[more Generally
$$f_x = k_x (x - x_0)$$
$$F_y = k_y (y - y_0)$$]

From (1),
$$F_x = k(l_1 c q_1 + l_2 c q_2)$$
$$F_y = k(l_1 s q_1 + l_2 s q_2)$$

From (4),

$$\left.\begin{aligned} k(l_1 s q_1 + l_2 s q_2)\, l_2 c q_2 - k(l_1 c q_1 + l_2 c q_2)\, l_2 s q_2 &= \tau_{23} \\ k(l_1 s q_1 + l_2 s q_2)\, l_1 c q_1 - k(l_1 c q_1 + l_2 c q_2)\, l_1 s q_1 &= \tau_{13} \end{aligned}\right\} \text{— (7).}$$

Set motor torques to be $\tau_1 + \tau_{13}$ and $\tau_2 + \tau_{23}$ respectively!

Answer to T3.

Another way to tackle T1 → solve for q_{1d} & q_{2d} from (3)

find derivative & double derivative.

↓

τ_1 & τ_2 from (6) -