CS 2100: Discrete Mathematics for Computer Science

Final Exam

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1. **(5 pts)** a. Prove that $(p \to q) \vee (q \to p)$. (1 point)

   b. Consider the following argument assuming the domain of discourse is the set of real numbers.

   (a) $R(x)$ is the predicate "x is rational."
   (b) $P(x)$ is the predicate "x is positive."
   (c) $\forall x, ((R(x) \to P(x)) \vee (P(x) \to R(x)))$
   (d) Therefore, all rational numbers are positive or all positive numbers are rational.

   At what step is the argument going wrong? (2 points)

   c. Explain why it is wrong in one sentence. (2 points)

2. **(10 pts)** Suppose that $n > 1$ people are positioned (on a plane) so that each has a unique nearest neighbour. Suppose further that each person has a water balloon that he or she throws at the nearest neighbour. A survivor is a person that is not hit by a water balloon. (The points distribution among the following parts is not given on purpose.)

   a. Give an example to show that there may not be a survivor if n is even.

   b. Give an example to show that there may be more than one survivor.

   c. Prove that if n is odd, there is always at least one survivor.

   d. Prove or disprove: If n is odd, one of the two persons farthest apart is a survivor.

   e. Prove or disprove: If n is odd, a person who throws the water balloon the greatest distance is a survivor.

3. **(5 pts)** Given an equal number of 0's and 1's distributed around a circle, show, using induction on the number of 0's, that it is possible to start at some number and proceed around the circle to the original starting position in such a way that, at any point during the cycle, one has seen at least as many 0's as 1's.

4. **(5 pts)** Prove that $2^n \geq n^2$, where $n = 4, 5, \ldots$.

5. **(5 pts)** Formally prove the correctness of the following loop using loop invariant. If your proof is not in accordance with the method described in class, you will lose marks.

   [Precondition: a is a nonnegative integer and d is a positive integer, $r = a$, and $q = 0$.]

   while $(r \geq d)$

   1. $r := r - d$
   2. $q := q + 1$

   end while

   [Post-condition: q and r are nonnegative integers with the property that $a = qd + r$ and $0 \leq r < d$.]

6. **(10 pts)** You want to count the number of ways in which you can climb up a flight of n stairs, where $n \geq 1$. With each step you take, you can move up either one stair or two stairs. As a result, you can climb the entire flight of n stairs (i) taking one staircase every step, (ii) two staircases every step, or (iii) a combination of one and two staircases per step.

   a. Let c_n be the number of different ways to climb a flight of n stairs. Find a recurrence relation for c_n. (5 points)

   b. Write an algorithm to compute c_n efficiently (i.e., as fast as possible) assuming n is given as input. (5 points)

7. **(10 pts)** A circular disk is cut into n distinct sectors, each shaped like a slice of pizza and all meeting at the center point of the disk. Each sector is to be painted red, green, yellow, or blue in such a way that no two adjacent sectors are painted the same colour. Let S_n be the number of ways to paint the disk.

   a. Find a recurrence relation for S_k in terms of S_{k-1} and S_{k-2} for each integer $k \geq 4$. (5 points)

   b. Find an explicit formula for S_n for $n \geq 2$ by solving the recurrence relation obtained in part a. (5 points)

8. **(5 pts)** Suppose $A_1, A_2, A_3, \ldots$ is an infinite sequence of countable sets. Let

$$\bigcup_{i=1}^{\infty} A_i = \{x \mid x \in A_i \text{ for some } i\}.$$

   Prove that $\bigcup_{i=1}^{\infty} A_i$ is countable.

9. **(5 pts)** Let A be a set of six distinct positive integers each of which is less than 13. Show that there must be two distinct subsets $S_1 \subset A$ and $S_2 \subset A$ such that $S_1 \cap S_2 = \phi$ and

$$\sum_{s_1 \in S_1} s_1 = \sum_{s_2 \in S_2} s_2.$$

   For example, if $A = \{10, 3, 5, 6, 1, 12\}$, then the two subsets $S_1 = \{10, 3\}$ and $S_2 = \{1, 12\}$ satisfy the requirements.

10. **(5 pts)**

   a. Can a simple undirected graph have a single odd degree vertex? (2 points)

   b. Prove your answer to part a. (3 points)

11. **(5 pts)** Consider a group $(G, *)$ and a subgroup $(S, *)$. Prove that exactly one of the cosets of S is a subgroup of G.

12. **(5 pts)** Let $(A, *)$ be a group and let B be a subset of A such that $2|B| > |A|$. Show that for any $a \in A$, there exists $b_1 \in B$ and $b_2 \in B$ such that $a = b_1 * b_2$. (Hint: consider the set $C = \{a * b^{-1} \mid b \in B\}$.)

13. **(5 pts)** Fermat's Little Theorem can be stated as follows:

   For a prime number p and a positive integer a such that p does not divide a, p divides $a^{p-1} - 1$.

   Prove Fermat's Little Theorem using Burnside's Theorem. (Hint: think of a general version of the bracelet problem in Tutorial 3 where each bracelet contains p beads and the number of colours is a.)

A relation R is irreflexive if no element is related to itself.
The complementary relation $\overline{R}$ is the set of ordered pairs $\{(a,b) \mid (a,b) \in R\}$.

7.1/52) S.T. the relation R on a set A is reflexive iff ~~the~~ $\overline{R}$ is irreflexive.

7.1/57) Suppose R is irreflexive. Is R^2 necessarily irreflexive? Give reason.

7.5/63) Do we necessarily get an equivalence relation when we form the transitive closure of the symmetric closure of the reflexive closure of a relation?

64) Same as above for the symmetric closure of the reflexive closure of the transitive closure of a relation?

67) Devise an algorithm to find the smallest equivalence relation containing a given relation.

2.3/77) S.T. the polynomial function $f : Z^+ \times Z^+ \to Z^+$ with $f(m,n) = \frac{(m+n-2)(m+n-1)}{2} + m$ is one-to-one and onto.

2.3/30) If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.

31) If f and $f \circ g$ are onto, does it follow that g is onto? Justify.