CS 2400 : Assignment 1

G.V. Vishwanath CS14B056
S.K. Sai Kritin CS14B062

Introduction/Problem Description:

In this assignment, we generate and sample two sinusoidal signals, and observe their properties in the frequency domain using the DFT (Discrete Fourier Transform). The FFT (Fast Fourier Transform) of the signals were calculated using the function (FFTCompute) provided. The frequencies at which peaks appear are calculated and the frequency of the sampled signal is estimated. Properties of DFT like Addition, Convolution, Multiplication and Shifting were also verified.

Sinusoidal signals are periodic functions, which can also serve as basis signals. Their Fourier transform yields two infinitely tall impulses (at equal distances from origin). Their parameters are amplitude and angular frequency.

Equation for DFT:

$$X_k \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} x_n \cdot e^{-2\pi i k n/N}, \quad k \in \mathbb{Z}$$ [1]

Here $x_n = x(n * T_S)$ where T_s is the sampling rate. N is the FFTSize.

To calculate the FFT of a signal, a divide-and-conquer algorithm called the Cooley - Tukey algorithm is used.It breaks a DFT of size N into smaller DFTs of size N1 and N2 such that N = N1 * N2. The complexity of calculating the DFT is reduced from $O(N^2)$ to O(N*Log(N)).[2]

Problem 1 : Experiment/Output/Discussion

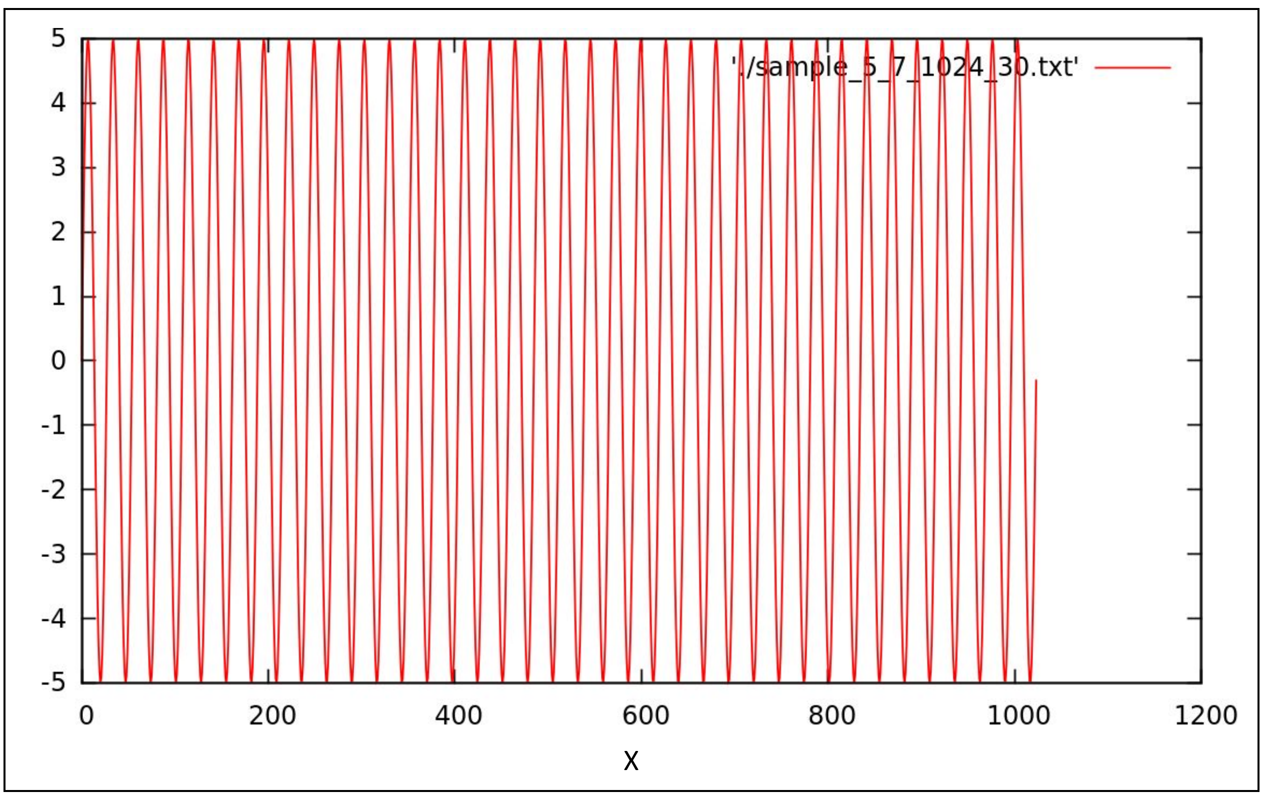
1(a):

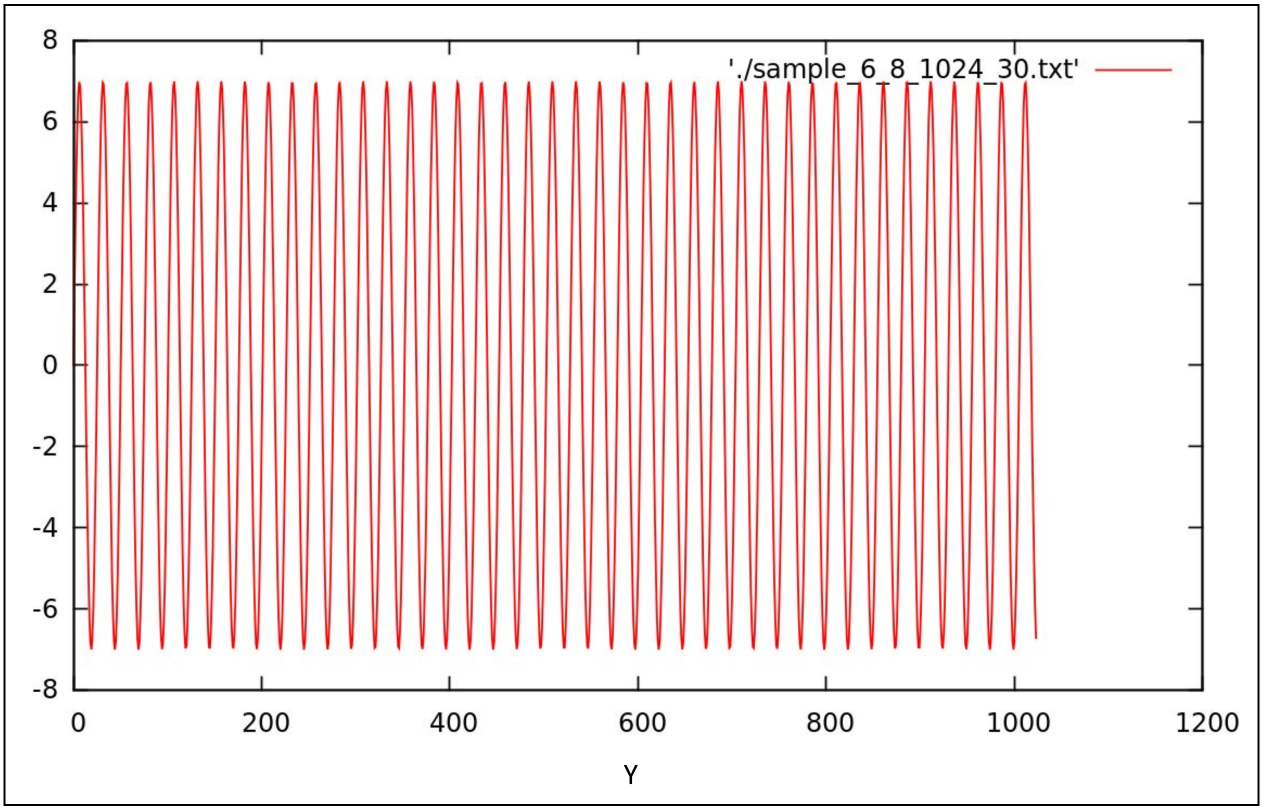
Sampling:

- To sample the sinusoidal signal, values at fixed intervals (i * T_S) were calculated and written into a file.
- The sampling rate is chosen so that the Nyquist sampling criterion for effective sampling (R >= 2 * f) is satisfied.
- Because of a large value of N, many cycles of the signal are sampled.

Results and Plots:

- We consider two sinusoidal waves: one (X) with amplitude A1 = 5 and angular frequency w1 = 7, the other (Y) with amplitude A1 = 6 and angular frequency w1 = 8. Their sampled versions at sampling rate 30 Hz are plotted below.





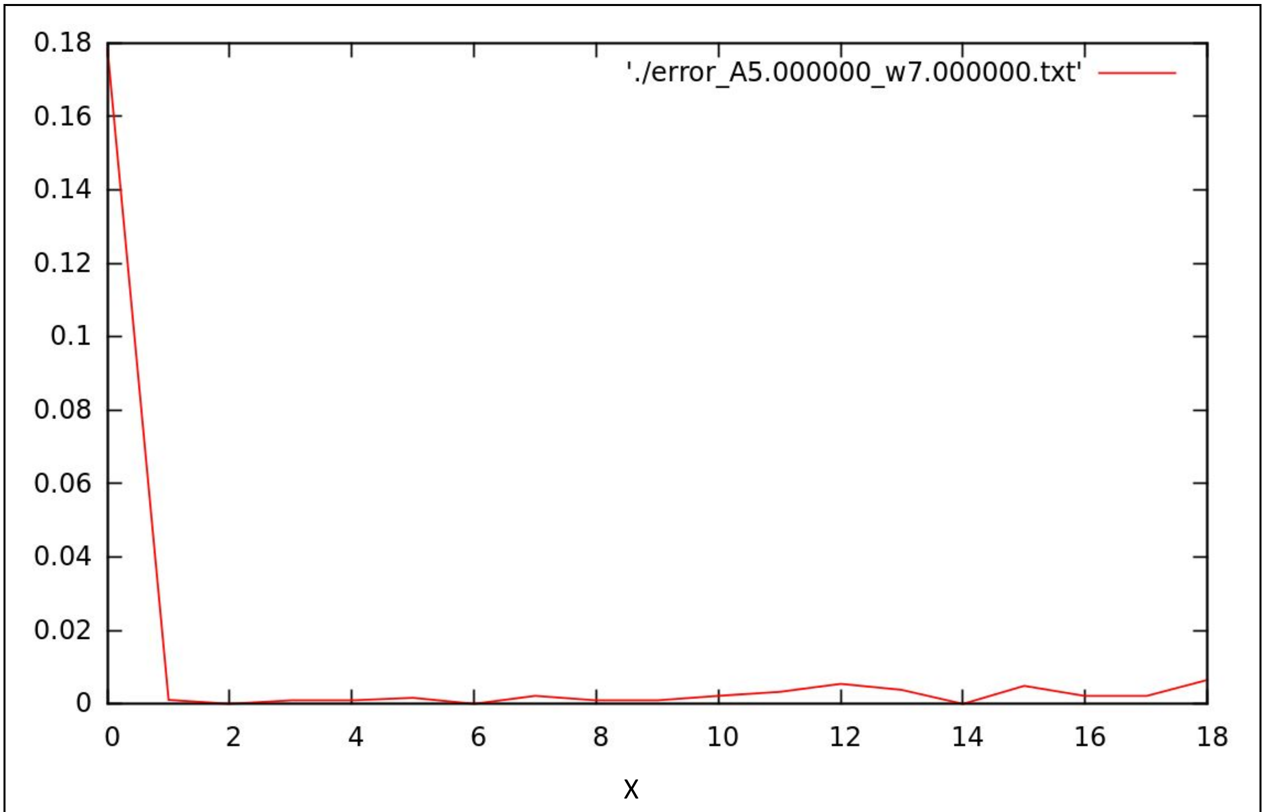
Frequency estimation from zero crossings:

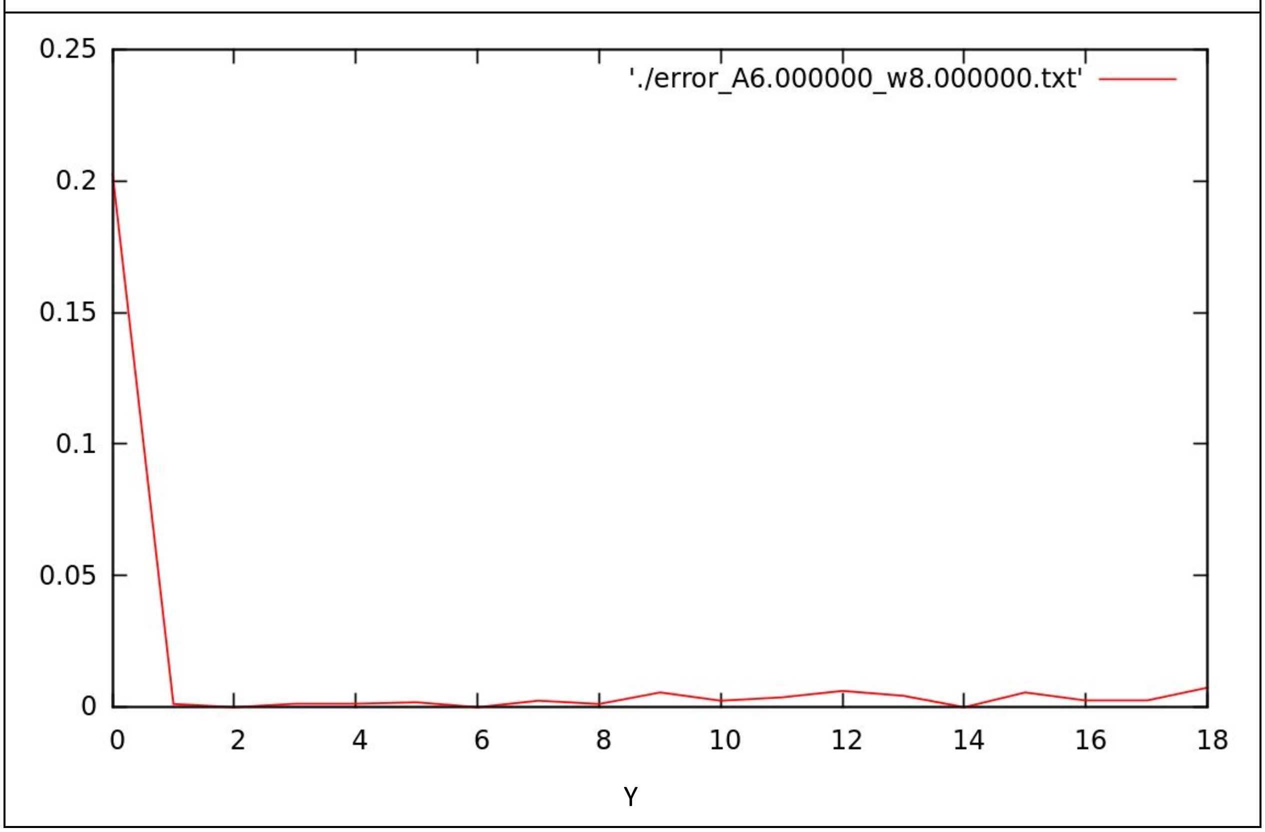
- The total number of zero crossings were found. Each zero crossing is found by comparing consecutive samples, if their product is negative (one is +ve and the other -ve) or the current sample is zero then we add one to the zero crossing count.
- The relation between number of zero crossings (Z) and N, R and frequency f is given by $f = \frac{R * Z}{2 * N}$.

Error in frequency estimation:

- The error is found for different values of the sampling rate (R) = 2 * f_o to 20 * f_o where f_o is the actual frequency of the sinusoidal signal.

- We observe that the error gradually decreases starting from the Nyquist sampling rate. This can be explained by the fact that as we take more samples, we get a more accurate sampled signal.


'./error_A5.000000_w7.000000.txt'
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
2
4
6
8
10
12
14
16
18
X

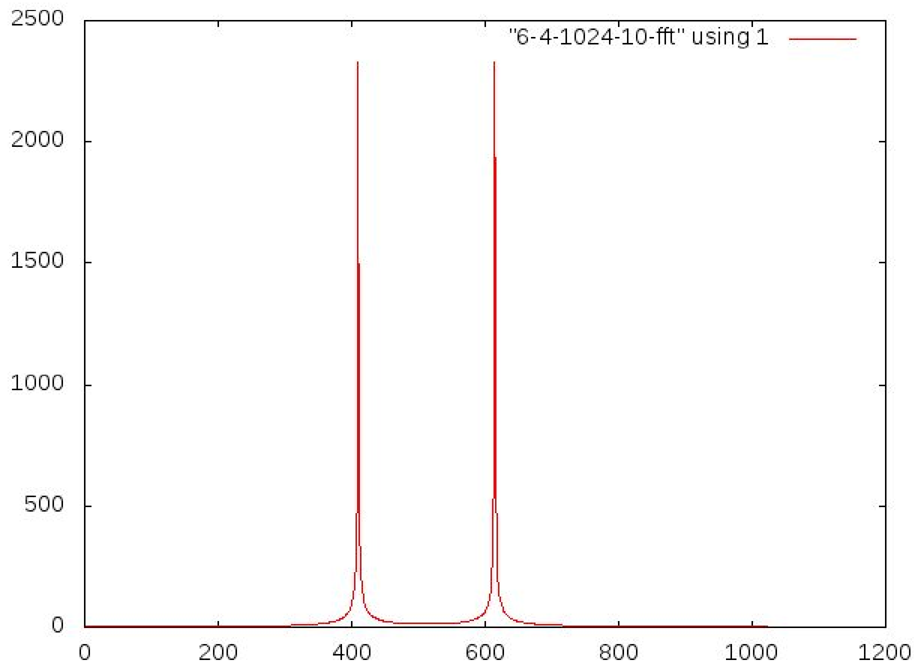

'./error_A6.000000_w8.000000.txt'
0.25
0.2
0.15
0.1
0.05
0
0
2
4
6
8
10
12
14
16
18
Y

1(b):

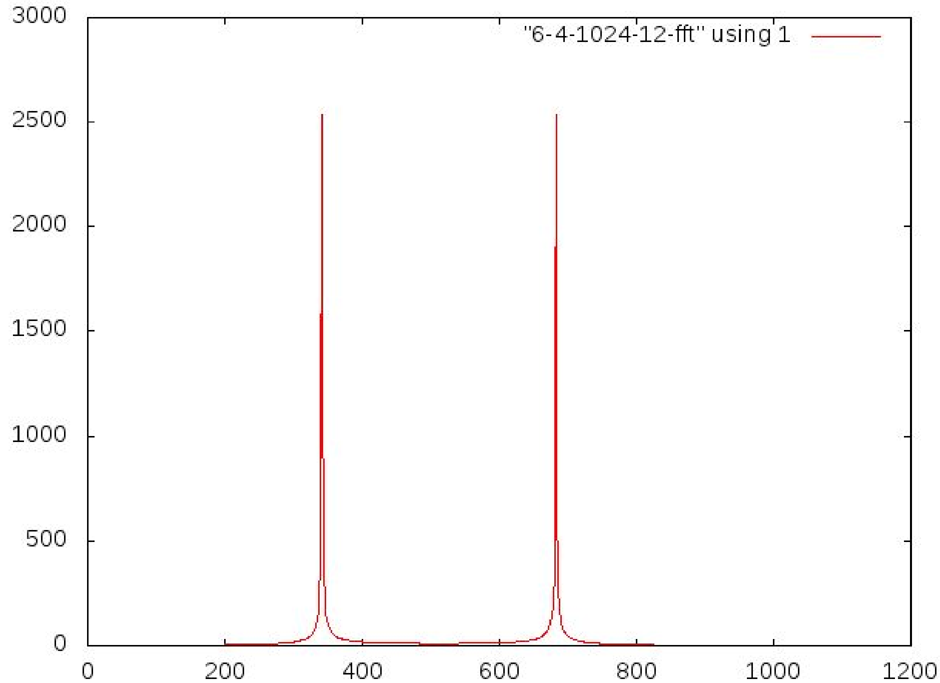
FFT Magnitude spectrum:

- The FFT magnitude spectra of $x = 6 * sin(4 * 2\pi * t)$ and $y = 10 * sin(1 * 2\pi * t)$ sampled at 10, 12 and 16 Hz with FFTSize = 1024 were obtained and plotted.
- The error in the estimation of frequency was calculated for the above two cases.
- Plots for different values of the sampling rate are as follows:

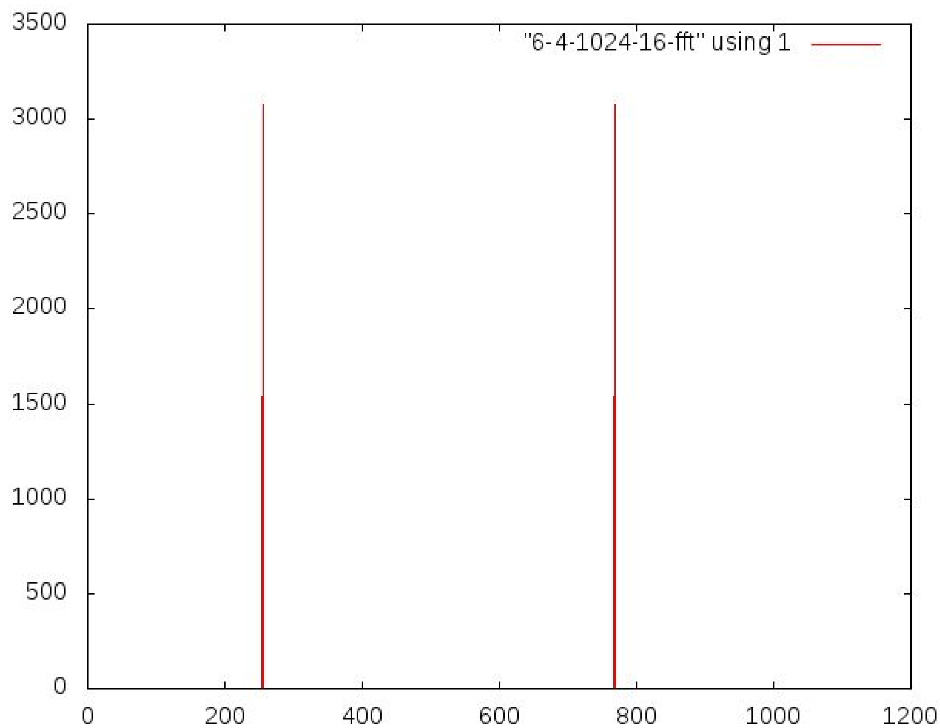
Results and plots:



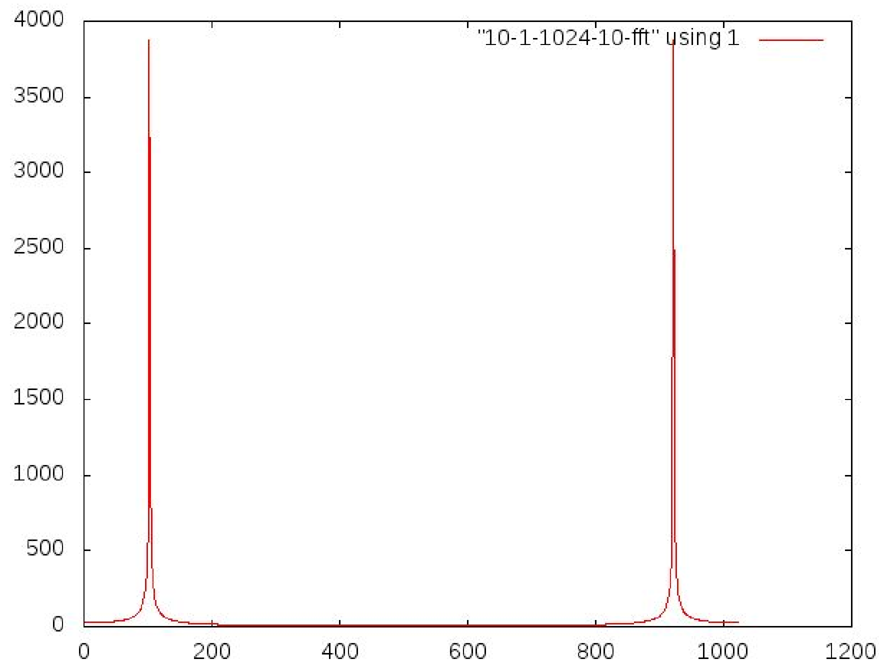
X at 10 Hz



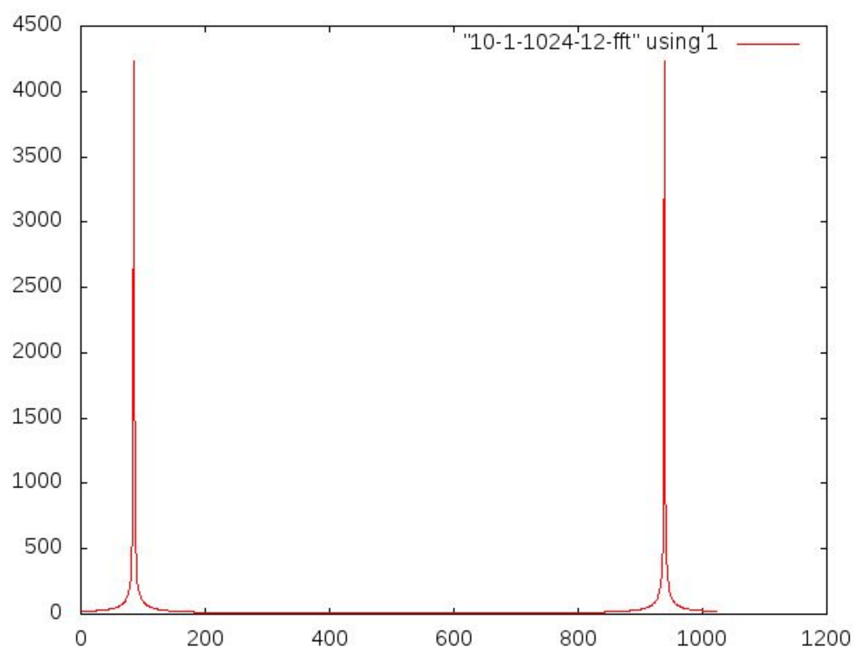
X at 12 Hz



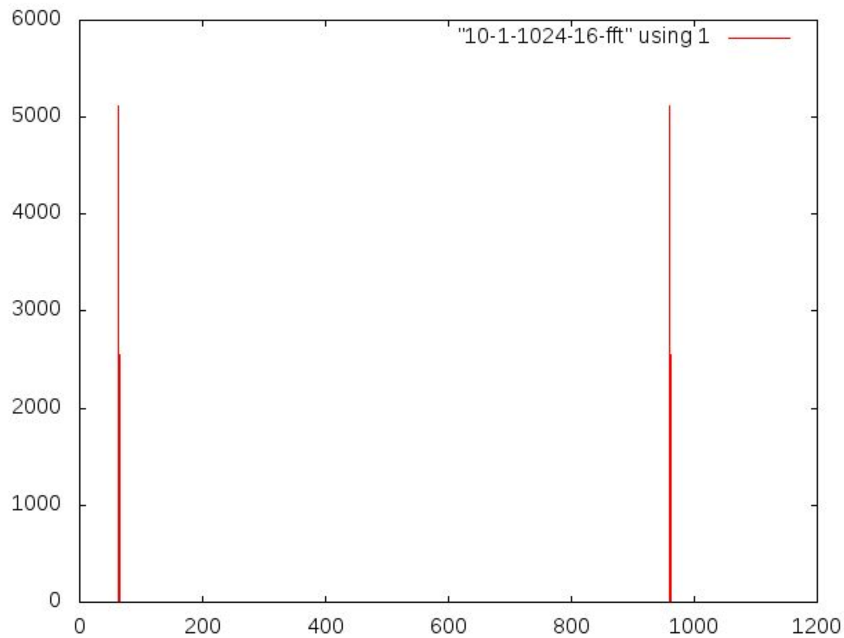
X at 16 Hz



Y at 10 Hz



Y at 12 Hz



Y at 16 Hz

The frequencies calculated from the peaks (in Hz) are respectively:

Sampling rate (Hz)	Estimated frequency (Hz) for x	Error in X (Hz)	Estimated frequency (Hz) for y	Error in Y (Hz)
10	4.003906	0.003906	0.996094	0.003906
12	3.996094	0.003906	0.996094	0.003906
16	4.000000	0	1.000000	0

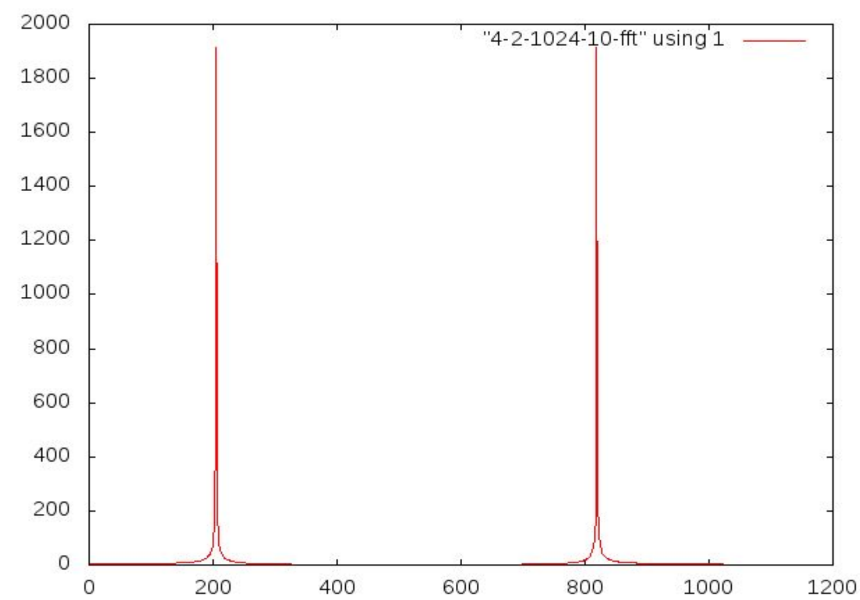
Problem 2 : Experiment/Output/Discussion

2(a) -> Addition:

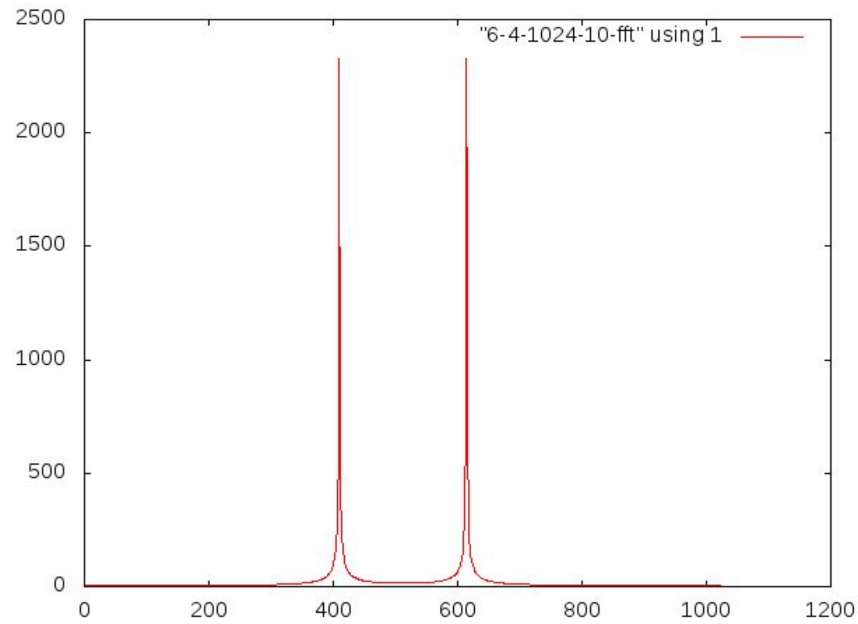
- The verification of the property $F(x + y) = F(x) + F(y)$ was attempted.
- The two input signals were added using an add function and the result was stored in a file.
- The fourier transform of the result was then calculated using ComputeFFT function.
- $F(X) + F(Y)$ was calculated using the same add function.
- As can be seen from the plots below fourier transform of the sum of x and y is almost equal to the sum of the fourier transforms of x and y

Plots and Results:

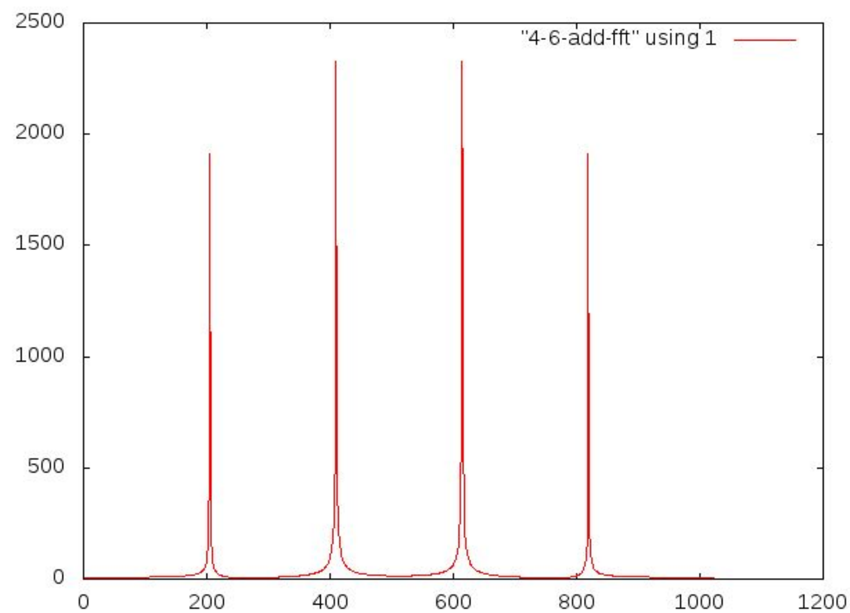
- Here $x = 4 * sin(2 * 2\pi * t)$ and $y = 6 * sin(4 * 2\pi * t)$ and were sampled at 10 hertz with FFT size = 1024.



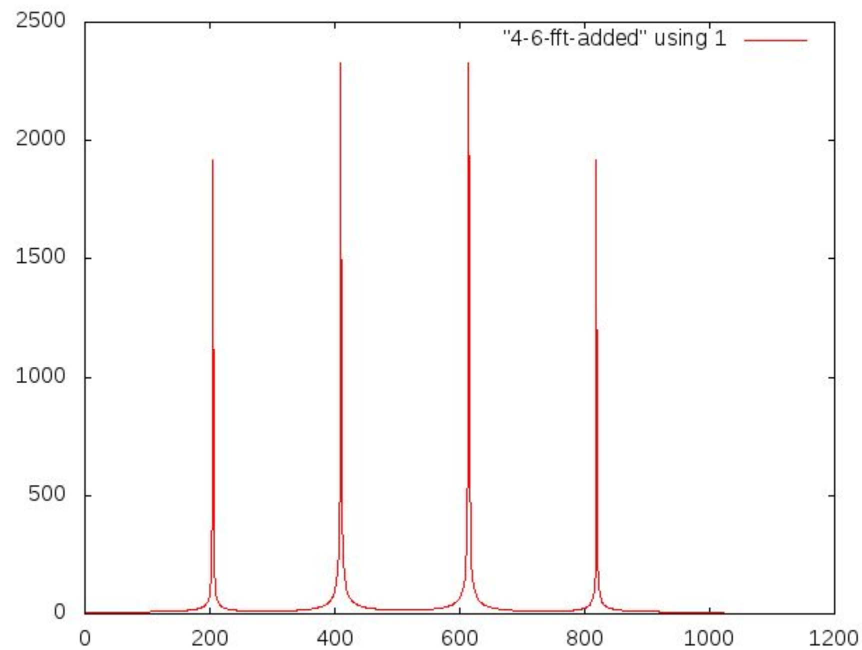
$F(X)$


"6-4-1024-10-fft" using 1
2500
2000
1500
1000
500
0
0
200
400
600
800
1000
1200

$F(Y)$


"4-6-add-fft" using 1
2500
2000
1500
1000
500
0
0
200
400
600
800
1000
1200

$F(X+Y)$



$F(X) + F(Y)$

- $F(X) + F(Y) == F(X+Y)$ is verified from the above two figures

2(b) -> Convolution:

- The fourier transform of the two signals was calculated separately
- The convolution of the two signals was calculated according to the formula:

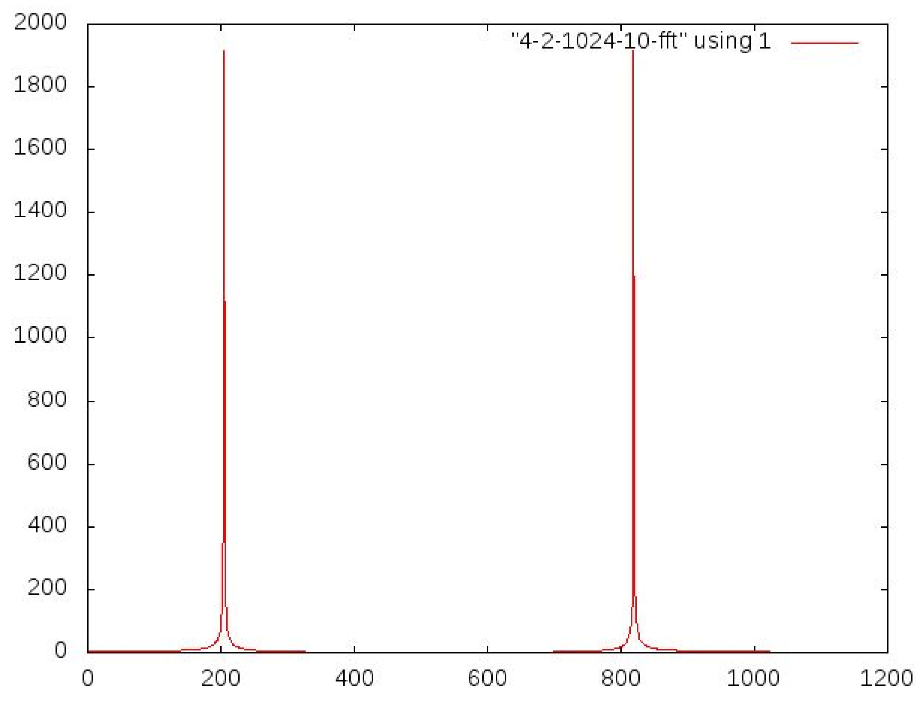
$$z[i] = \sum_{-\infty}^{+\infty} x[k] * y[i - k]$$

- Here z is the convolved signal and x, y are the input signals. As the input signals have the same size, say 1024 with range [0, 1023], the range of z will be [0, 2046], so z at each value in the range is calculated according to the above equation. Two cases arise:
    a. If i $\in$ [0, 1023], then $z[i] = \sum_{k=0}^{i} x[k] * y[i - k]$
    b. If i $\in$ [1024, 2046], then $z[i] = \sum_{k=i-1023}^{1023} x[k] * y[i - k]$

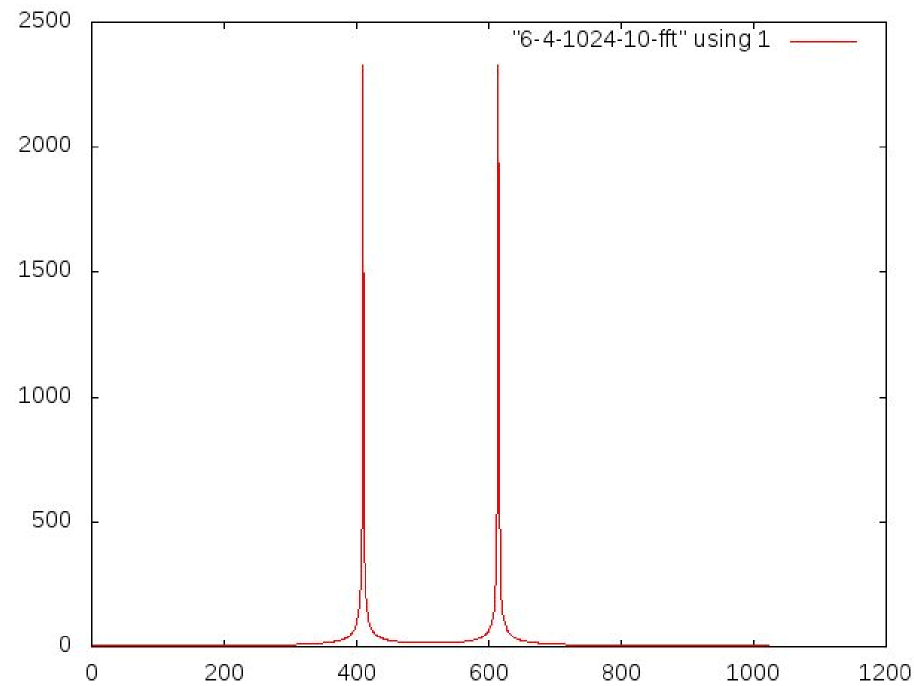
- The sums in the above two cases were calculated according to the value of i.
- The obtained convoluted signal was padded with a zero to make its size 2048, this is then given as the input to ComputeFFT function with order 11 and size 2048.
- The plots of the fourier transforms of the input signals are as follows:
- Preprocessing is done by adding zero samples at the end to make the total number of samples a power of 2.

Plots:

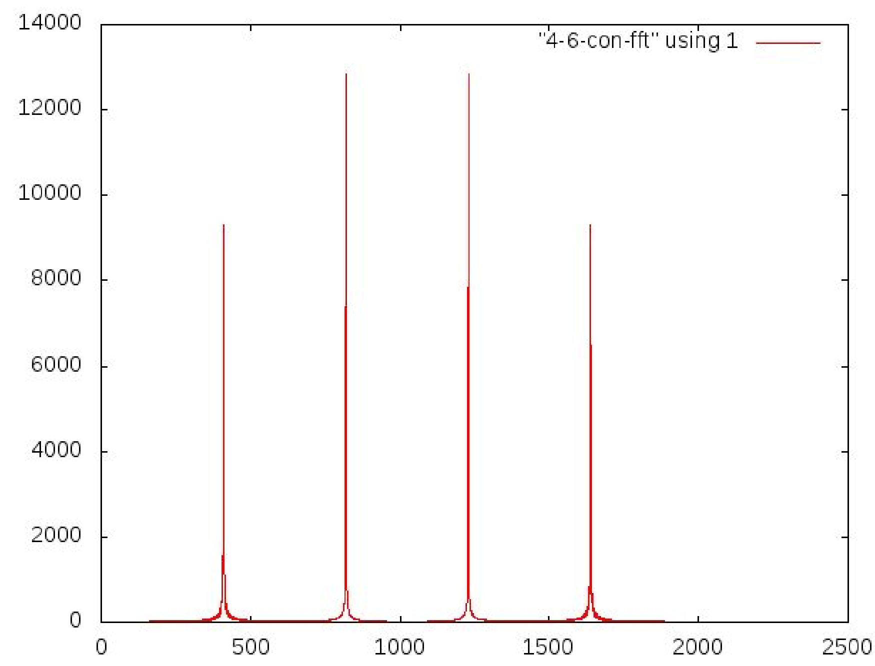
- Here $x = 4 * sin(2 * 2\pi * t)$ and $y = 6 * sin(4 * 2\pi * t)$ which were sampled at 10 hertz with FFT size = 1024.
- As can be seen from the below plots the fourier of the convolution is the product of the fourier transform of the input signals.
- The fourier transform of the convolution might have a small error because of the preprocessing step of adding a single zero.



Fourier transform of X



Fourier transform of Y



Fourier transform of $X \otimes Y$

Choice of FFTSize(N):

- $\frac{N}{R} = Sampling\ Width >= Time\ Period(T)$ $\Rightarrow N >= \frac{R}{f}$ is a constraint which needs to be satisfied as at least one full cycle of the graph should be sampled.
- FFTSize should be a power of 2 for faster FFT calculation.

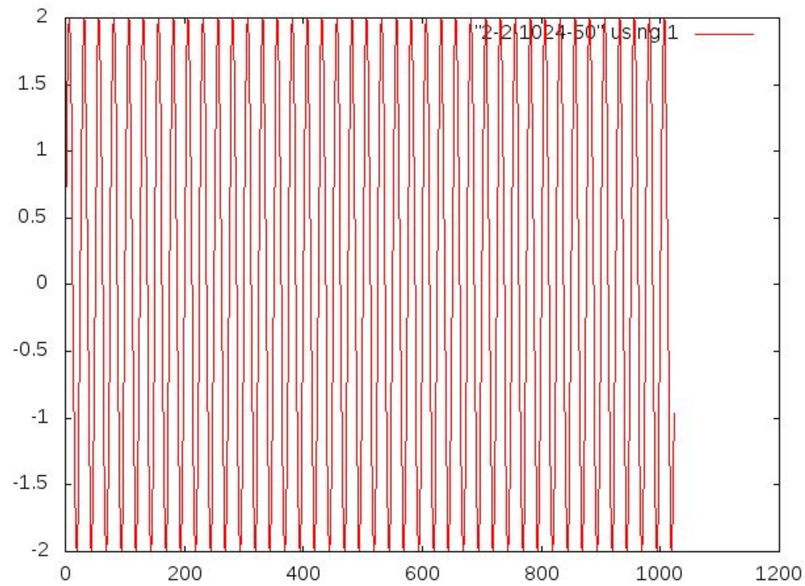
2(c) -> other DFT properties:

Multiplication of two signals:

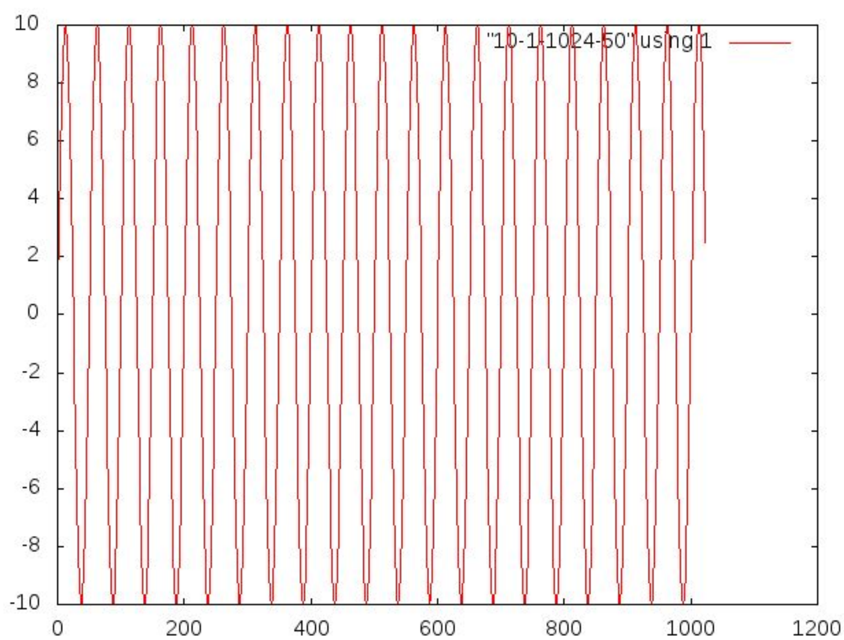
- The verification of the property $F(x * y) = F(x) \otimes F(y)$ is attempted, where $\otimes$ represents convolution.
- The input signals x, y are multiplied and the output is given to the ComputeFFT function to calculate $F(x * y)$.
- The fourier transforms of x and y are calculated and convolved using the same function as in 2(b) to get $F(x) \otimes F(y)$
- The obtained plots are as follows:

Plots:

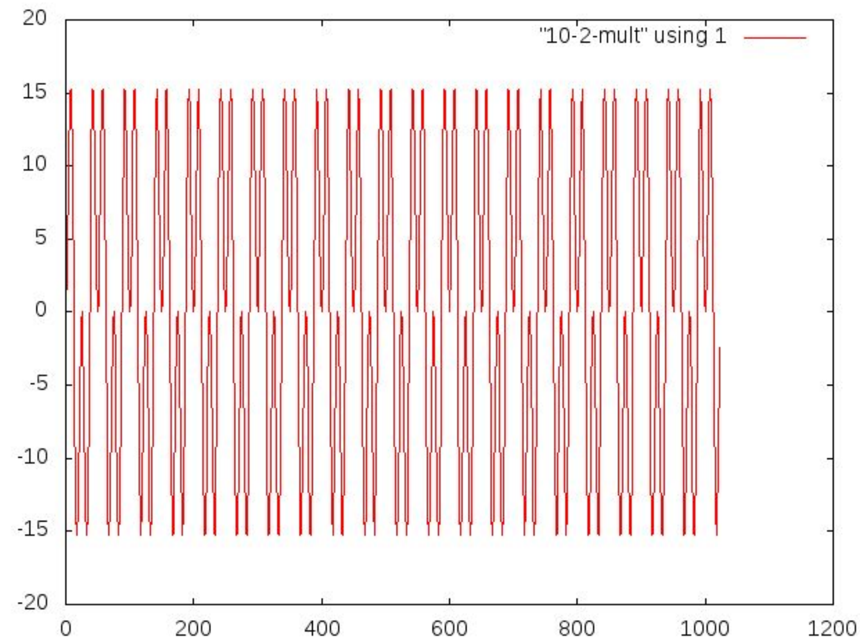
- Here $x = 2 * sin(2 * 2\pi * t)$ and $y = 10 * sin(1 * 2\pi * t)$ which were sampled at 50 hertz with FFT size = 1024.
- Input and output signals in time domain:


"2-2-1024-50" using 1
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
200
400
600
800
1000
1200

X

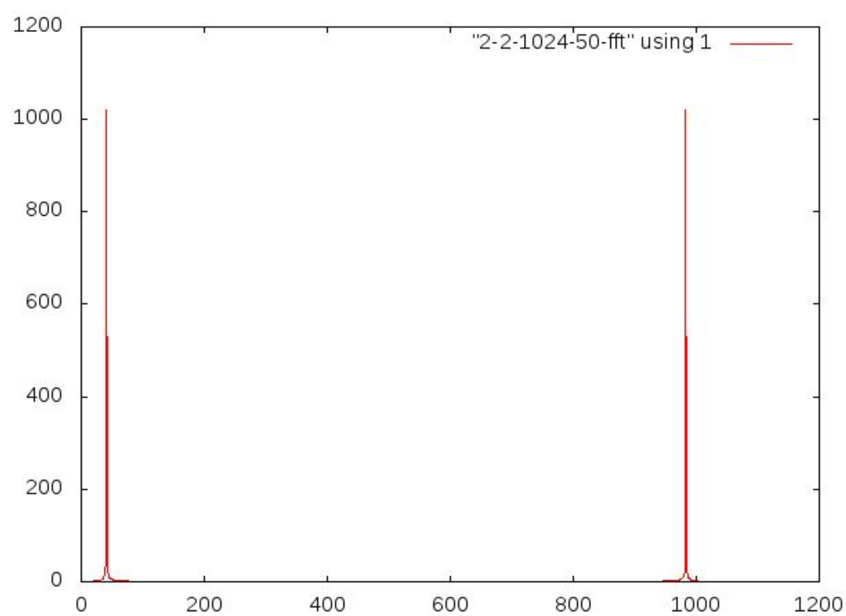

"10-1-1024-50" using 1
10
8
6
4
2
0
-2
-4
-6
-8
-10
0
200
400
600
800
1000
1200

Y

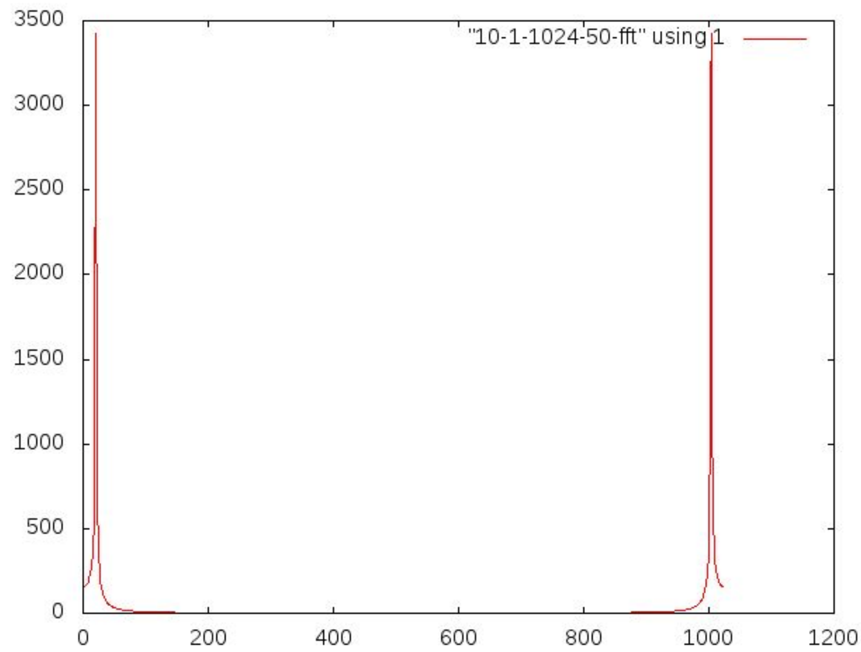


$X * Y$

- Input and Output signals in frequency domain:

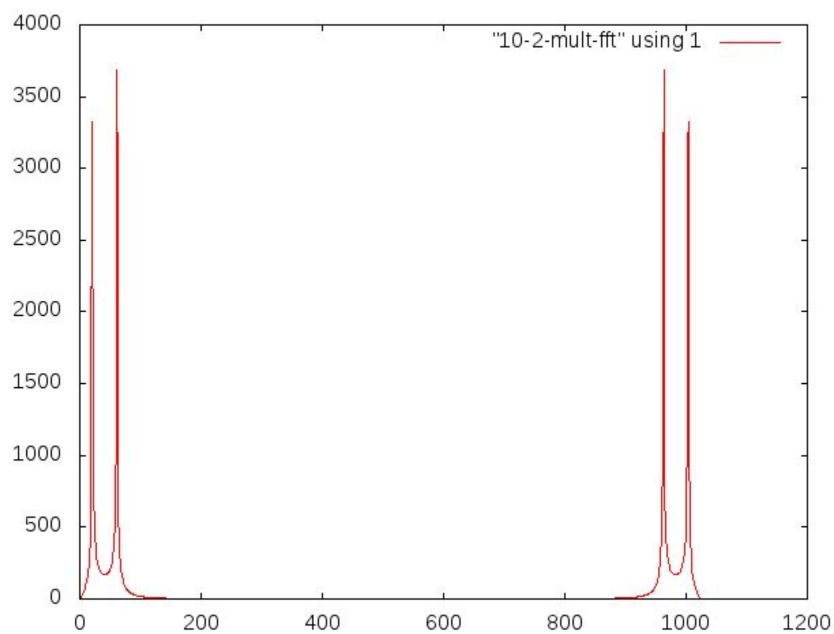


$F(X)$



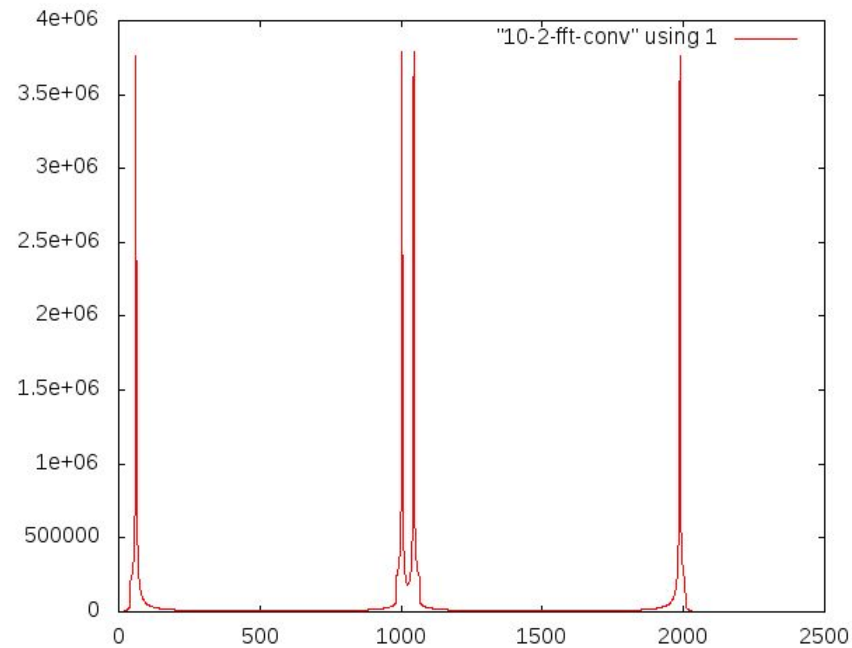
"10-1-1024-50-fft" using 1
3500
3000
2500
2000
1500
1000
500
0
0
200
400
600
800
1000
1200

$F(Y)$



"10-2-mult-fft" using 1
4000
3500
3000
2500
2000
1500
1000
500
0
0
200
400
600
800
1000
1200

$F(X * Y)$



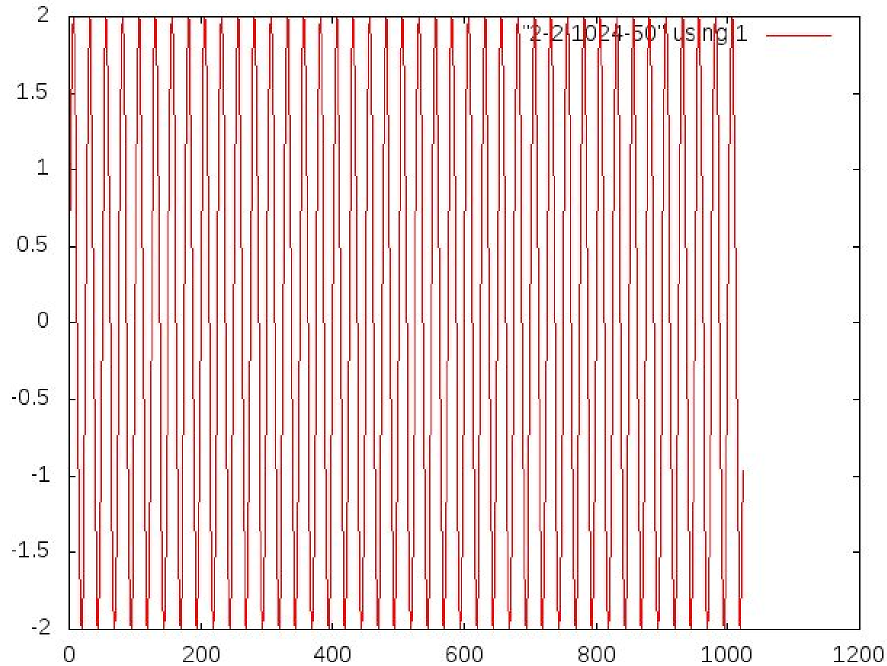
$F(X) \otimes F(Y)$

Shifting Property:

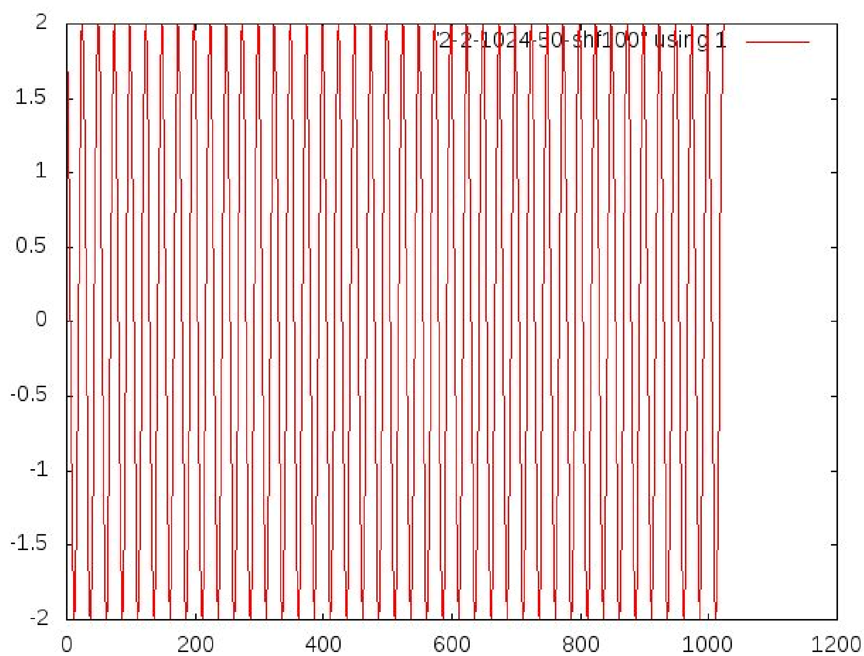
- The verification of the property $F(x(t-n)) = F(x(t)) * e^{-2\pi i * km/N}$ is attempted.
- The input signals x is shifted by an integer n, the shifting is performed modulo N (FFTSize), the corresponding fourier transform is calculated.
- The amplitude spectra of x(t) and x(t-n) are compared.
- The obtained plots are as follows:

Plots:

- Here $x = 2 * sin(2 * 2\pi * t)$ which is sampled at 50 hertz with FFT size = 1024.
- The signals in the time domain:

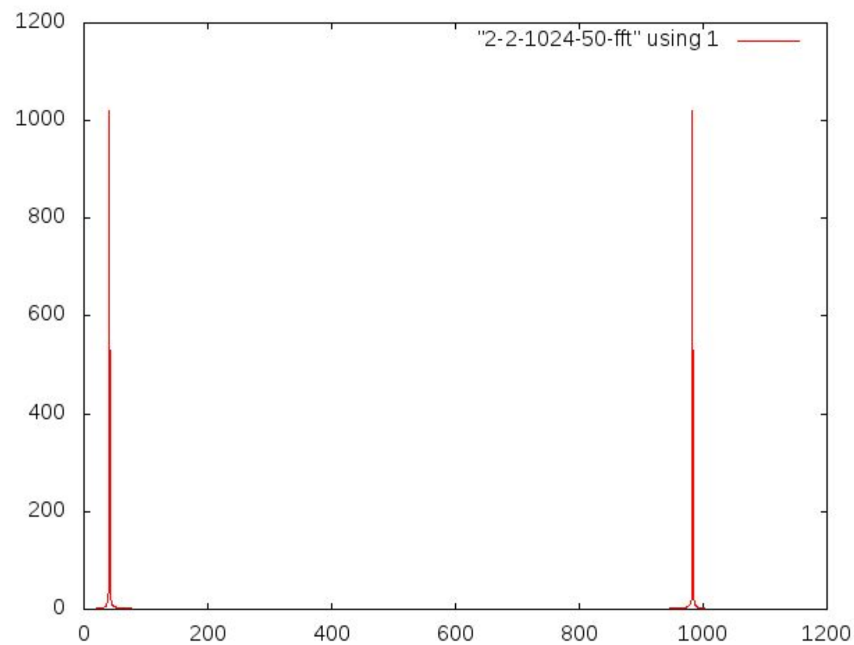

"2-2-1024-50" using 1
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
200
400
600
800
1000
1200

$X(t)$

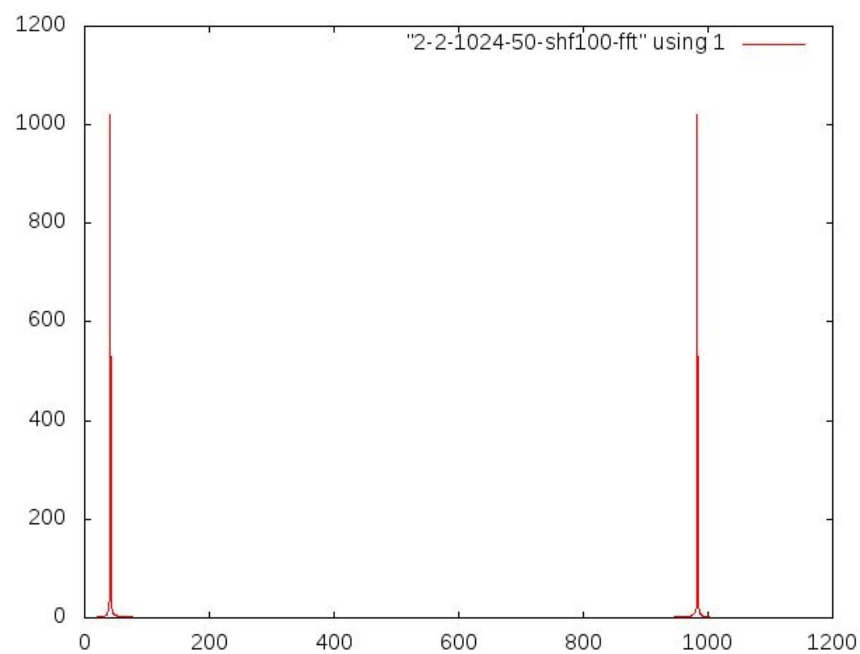

"2-2-1024-50-shf100" using 1
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
200
400
600
800
1000
1200

$X(t-100)$

- The signals in the frequency domain:



$F(X(t))$



$F(X(t-100))$

- Comparing the above two fourier transforms verifies the fact that shifting in the time domain does not affect the amplitude spectra in the frequency domain

General observations/conclusions

- Theory and practice differ in quantity, i.e., values rather than quality, i.e., general shape of the graph.
- The DFT of the sinusoidal signals is periodic, this property causes the DFT of a sinusoidal signal to have two peaks in the positive frequency range.
- Two peaks appear in the DFT of a sinusoidal signal, one of them corresponds to the frequency of the actual signal while the other is = $R - f_o$, R = sampling rate; f_o = frequency of the signal.

References:

[1] https://en.wikipedia.org/wiki/Discrete_Fourier_transform

[2] https://en.wikipedia.org/wiki/Fast_Fourier_transform