

University of Oslo

AST3310 - Astrophysical plasma and stellar interiors  
Project 1

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# Modeling the solar core

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## Abstract

I here discuss the properties of the interior of the Sun based on models and simplifications addressed in [1] and [2]. Numerical calculations are used to solve the governing equations of the interior of the Sun. One should expect the luminosity, position and mass to end at zero value simultaneously, while the temperature lies around  $\sim 15$  MK. It is found that the results heavily depend on initial physical properties, and in some cases on the mass step  $\partial m$ . Unexpectedly, it is likely the mass step has little to say given the right initial conditions.

## 1 Introduction

There are four differential equations that govern the internal structure of the Sun in our model. We apply here numerical methods such as the Euler integration scheme to solve these equations. This involves solving equations for nuclear fusion as well as ordinary differential equations. We look at how the inner Sun changes based on different initial parameters, and try to seek out a set of parameters that leads to the case where mass, position and luminosity reach zero value simultaneously.

## 2 Theory

The equations that regulate the physical properties of the interior of the Sun are expressed as

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \quad (1)$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} \quad (2)$$

$$\frac{\partial L}{\partial m} = \varepsilon \quad (3)$$

$$\frac{\partial T}{\partial m} = -\frac{3\kappa L}{256\pi^2 \sigma r^4 T^3}. \quad (4)$$

The variable  $\varepsilon$  in (3) is the total energy generation per unit mass. It is found by looking at the energy generation from nuclear reactions. It depends on the abundancy of different elements, temperature and density. The variable  $\kappa$  is the opacity, which is an average of frequency of photons. The pressure in eq. (2) is a sum of the gas pressure  $P_G$ , and the radiative pressure  $P_R$ .

$$\begin{aligned} P &= P_G + P_R \\ P &= \frac{\rho}{\mu m_u} kT + \frac{a}{3} T^4 \\ \Rightarrow \rho &= \frac{\mu m_u}{kT} \left( P - \frac{a}{3} T^4 \right), \end{aligned} \quad (5)$$

which yields the density derived from the equation of state. Here,  $m_u$  is the atomic mass unit and  $k$  is the Boltzmann constant. The constant  $a$  is defined as  $a = 4\sigma/c$ , where  $\sigma$  is the

Stefan-Boltzmann constant, and  $c$  is the speed of light. The average molecular weight  $\mu$  is found by

$$\mu = \frac{1}{\rho} \frac{\rho}{n_{\text{tot}}}. \quad (6)$$

We can find the total particle density  $n_{\text{tot}}$  with

$$\begin{aligned} n_{\text{tot}} &= n_X + n_Y + n_Z + n_e \\ &= \frac{X\rho}{m_u} + \frac{Y\rho}{4m_u} + \frac{Z\rho}{Am_u} + \left( \frac{X\rho}{m_u} + \frac{2Y\rho}{4m_u} + \frac{Z\rho}{2m_u} \right), \end{aligned}$$

where  $A$  is the average atomic weight of the heavier elements present, which we assume to be  $A = 7$ . This assumption is based on that we only know of two heavier elements present, which are  ${}^7\text{Be}$  and  ${}^7\text{Li}$ . We then get

$$\begin{aligned} \mu &= \frac{\rho m_u}{\rho m_u} \left( \frac{1}{2X + \frac{3}{4}Y + \frac{9}{14}Z} \right) \\ &= \frac{1}{2X + \frac{3}{4}Y + \frac{9}{14}Z}, \end{aligned}$$

assuming all elements are fully ionised.

The total energy generation per unit mass  $\varepsilon$ , is found by

$$\varepsilon = \sum Q'_{ik} r_{ik}, \quad (7)$$

where  $i, k$  represents two elements,  $Q'_{ik}$  is the energy released from the fusion of two elements,  $r_{ik}$  is the reaction rates per unit mass for two elements. The energies  $Q'_{ik}$  from the pp chains are listed in [1, p. 39, Table 2.1]. The reaction rates per unit mass is defined by

$$r_{ik} = \frac{n_i n_k}{\rho(1 + \delta_{ik})} \lambda_{ik}, \quad (8)$$

where  $n_i, n_k$  is the number density for an element,  $\delta_{ik}$  is the Kronecker delta and  $\lambda_{ik}$  is the reaction rate of a fusion. The number density of an element is easily defined as

$$n = \frac{\rho \chi_a}{a m_u}, \quad (9)$$

where  $\chi$  is the number fraction of an element, and  $a$  is the atomic number of the element. We denote  $X, Y, Z$  to be the number fractions of hydrogen, helium and heavier metals, respectively. Finally, the reaction rates  $\lambda_{ik}$  for two elements  $i, k$  can be found in [1, p. 46, Table 2.3].

## 3 Algorithm

### 3.1 Simplifications

In order to make a simple model I needed some simplifications. A list of assumptions and simplifications follows.

Physical property	Unit system			
	CGS		SI	
	Unit name	Unit abbr.	Unit name	Unit abbr.
Length	centimetre	cm	metre	m
Weight	gram	g	kilogram	kg
Time	second	s	second	s
Temperature	kelvin	K	kelvin	K
Energy	erg	erg	joule	J
Pressure	barye	Ba	pascal	Pa

**Table 1:** Overview of the difference between CGS units and SI units.

- There is no change in the composition of elements as a function of radius.
- I assume there is no change in the density of deuterium, so that the rate of destruction of deuterium is the same as the production, and that any reaction involving deuterium happens instantaneous.
- All nuclear energy comes from the three PP-chains. I have not included the CNO cycle.
- I assume all elements to be fully ionised.

### 3.2 Units

Given that  $\kappa$  is in units of  $[\text{cm}^2 \text{ g}^{-1}]$ ,  $\lambda$  in units of  $[\text{cm}^3 \text{ s}^{-1}]$  and so on, I chose to adapt the CGS unit system (centimetre-gram-second). Table 1 shows an overview of the differences.

### 3.3 Structure

We see from eq. (4) that  $\partial T / \partial m$  depends on the opacity,  $\kappa$ . A table of opacities that correspond to different values of temperature and density has been provided. I wrote a function that reads the file and stores values of  $T$ ,  $R$  and  $\kappa$  in separate arrays. Here,  $R = R(T, \rho) = \rho / T_6$ . The function compares the table temperature with the actual temperature, and the same with the variable  $R$ . It returns the  $\kappa$  with table values of  $T$  and  $R$  that most closely resembles the present values.

Next, I wrote a function that calculates the energy generation per mass unit from nuclear reactions. This means entering the energy releases  $Q'_{ik}$  from [1, p. 39, Table 2.1], the reaction rates  $\lambda_{ik}$  from [1, p. 46, Table 2.3] and finding the number densities  $n$  for all particles that are involved in a nuclear reaction. Given the number fractions  $X, Y, Z$  and sub-fractions of these for different elements, I was able to calculate the energy generation per mass by using eq. (7), (8) and (9).

I have a function that calculates the density at present time, given the temperature  $T$  and total pressure  $P$ . This is found from the equation of state, as given in eq. (5). This required me to calculate the average molecular weight  $\mu$ , shown in eq. (6).

Physical properties		Element abundancies	
Parameter	Initial value	Element	Initial value
$L_0$	$L_\odot$	$X$	0.7
$R_0$	$0.5R_\odot$	$Y_3$	$10^{-10}$
$M_0$	$0.7M_\odot$	$Y$	0.29
$\rho_0$	$1 \text{ g cm}^{-3}$	$Z$	0.01
$T_0$	$10^5 \text{ K}$	$Z_{7\text{Li}}$	$10^{-5}$
$P_0$	$10^{12} \text{ Ba}$	$Z_{7\text{Be}}$	$10^{-5}$

**Table 2:** Table that shows the initial physical parameters needed in order to initiate the calculations.

Further, I have four functions that return the right-hand side in eq. (1), (2), (3) and (4). These are called upon in another function that integrates all the equations. I implemented a crude adaptive step system, where I changed the mass step gradually while integrating further into the Sun.

### 3.4 The Euler integration scheme

For simplicity, I have chosen the Euler integration scheme to integrate the differential equations. Listing (1) shows how it is carried out.

Listing 1: Euler integration loop

```

for i in range(n-1):
    r[i+1] = r[i] - drdm * dm
    P[i+1] = P[i] - dPdm * dm
    L[i+1] = L[i] - dLdm * dm
    T[i+1] = T[i] - dTdm * dm
    m[i+1] = m[i] - dm

```

The Euler method advances one step by adding the previous step value to the current value of the right-hand side of the differential equations, multiplied by the mass step. Unfortunately, the scheme carries a local truncation error proportional to the step size squared, and a global truncation error that is proportional to the step size.

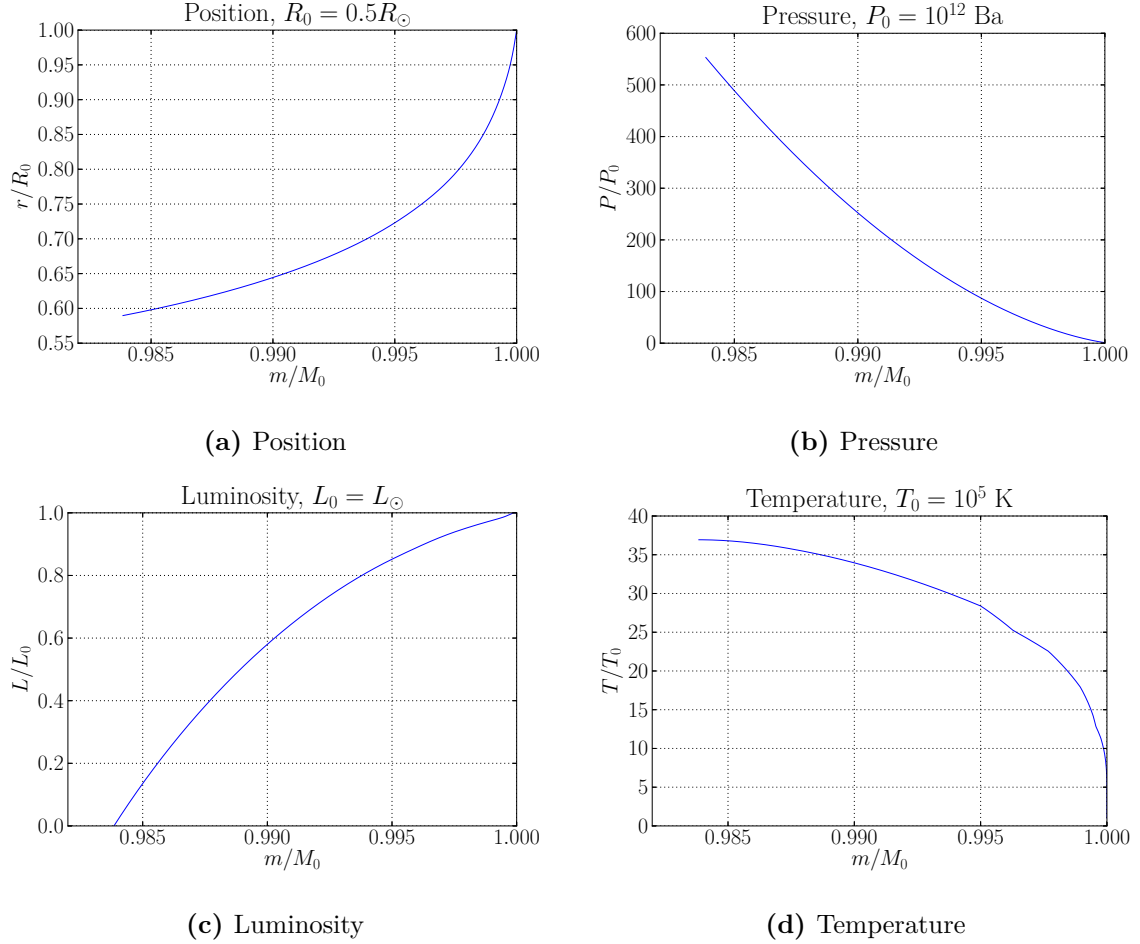
### 3.5 Initial conditions

Finally, we need a set of initial conditions to initiate the calculations. These are shown in Table 2 expressed in the CGS unit system (see Table 1).

## 4 Results

### 4.1 Using original initial values

Carrying out the integrations using the initial conditions shown in Table 2, we get the results shown in Figure 1. As we see from these figures, the luminosity reaches zero before the position does, while we have integrated through just under 0.2 % of the initial mass. Figure 1d shows a sudden leap in temperature to start with. This indicates that the initial temperature is too low relative to the initial pressure and density. The equation of state tries to compensate for this, which leads to a violent increase in temperature. The physical parameter that changes abruptly is *probably* determined by the order the differential equations are solved. Had we solved them in another order, we would probably see a change.



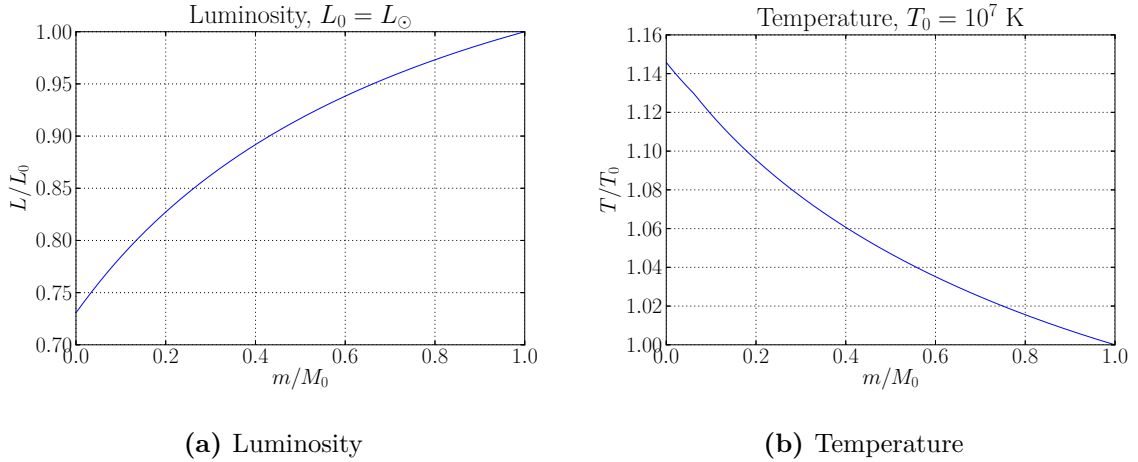
**Figure 1:** Showing results of solving the differential equations using the initial parameters as described in Table 2.

Step number	$M/M_0$	$r/R_0$	$L/L_0$	$\rho/\rho_0$	$P/P_0$	$T/T_0$
1	0.9999	0.9999	1.0000	1.0000	1.0007	1429.6

**Table 3:** The initial results of  $P_0 = 10^{12}$  Ba,  $T_0 = 10^5$  K and  $\partial m$  starting at  $1.39 \times 10^{27}$  g. The program stops after this, as the temperature has increased by a factor of  $10^3$  already in the first step. Compare with Figure 1 that has the same initial values, but smaller  $\partial m$ .

## 4.2 On the effect of changing $\partial m$

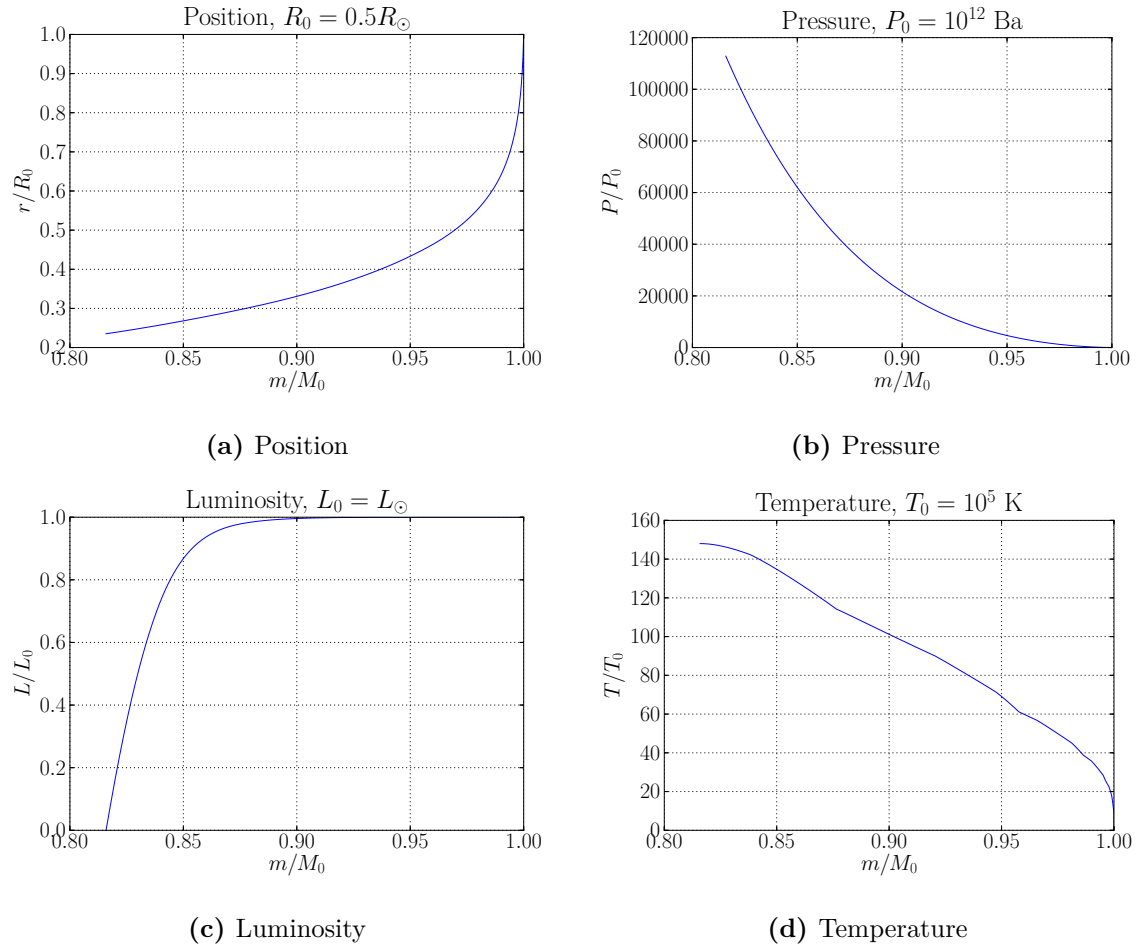
For some sets of initial conditions, it is important to choose a low  $\partial m$  to prevent the luminosity or temperature from increasing too quickly. However, this is highly time consuming. I chose to adapt a very small (of order  $10^{-10}$  of the initial mass) step in the beginning, while gradually increasing it as we work our way into the core. I noticed however that given some initial conditions, I could choose  $\partial m$  to be as large as  $10^{-4}$  of the original mass, and still get identical results as I did in e.g. Figure 4. I can only draw the conclusion that  $\partial m$  needs to be very small if we start with some physical parameters that are too low (or high) compared to others, as in the case with Figure 1d. In those cases it changes abruptly because of the equation of state, and hence we need a very small step so that it does not change too fast. Figure 2 shows that choosing a much higher time step for some initial parameters yield the same results as Figure 4. The output in Table 3 shows the sudden increase in temperature for the original initial conditions, that ultimately fails.



**Figure 2:** Solution for  $P_0 = 10^{16}$  Ba,  $\partial m$  starts at  $1.39 \times 10^{27}$  g. The amount of  ${}^7\text{Li}$  and  ${}^7\text{Be}$  are different than in Figure 1. See Section 4.3. See Figure 4 which yield the same result for much smaller  $\partial m$ .

## 4.3 Decreasing the amount of ${}^7\text{Li}$ and ${}^7\text{Be}$

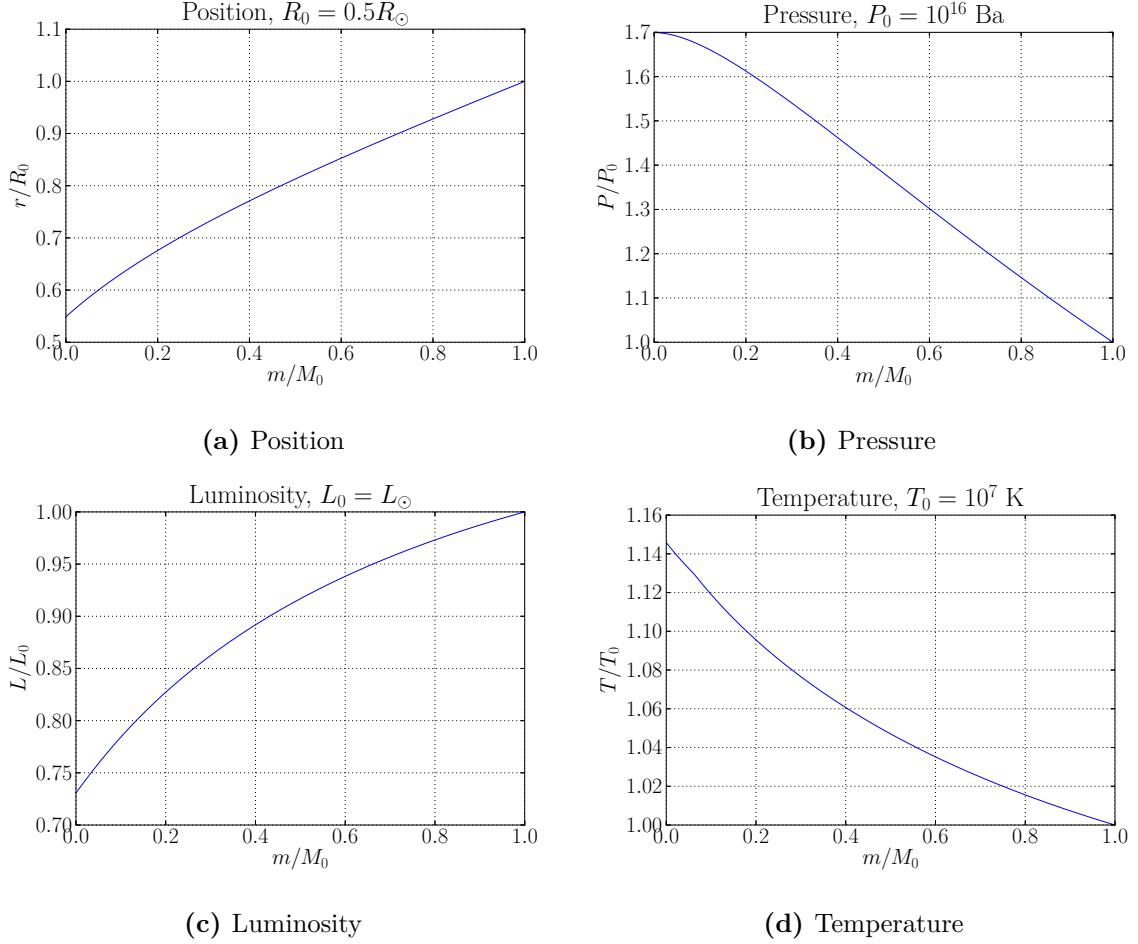
As a solution to the temperature problem, we try to lower the number fraction of  ${}^7\text{Li}$  and  ${}^7\text{Be}$ . We set  $Z_{{}^7\text{Li}} = Z_{{}^7\text{Be}} = 10^{-13}$ . This should lower the energy generation, thus the temperature through the luminosity. The results are shown in Figure 3. We already see a change, especially in the luminosity and temperature. Figure 3c now shows how the luminosity is close to



**Figure 3:** Integration results by using  $Z_{7\text{Li}} = Z_{7\text{Be}} = 10^{-13}$ . Mass step  $\partial m$  starts at  $1.39 \times 10^{23}$  g.



constant to about 10 % of the initial mass. Then it suddenly dips exponentially, reaching zero value at about 82 % of  $M_0$ . Again, the luminosity decreases much faster than the position and mass. Figure 3a shows the position to be near 80 % into the core. The temperature in Figure 3d shows a steadier increase to about  $15 \times 10^6$  K, which sounds reasonable as we approach the core. However, we see an abrupt increase in the pressure. It rises higher than a factor of  $10^5$  of the initial pressure. We increase the initial pressure to see if it stabilises our results further.



**Figure 4:** Integration results by increasing both the initial pressure and temperature. Mass step  $\partial m$  starts at  $1.39 \times 10^{23}$  g.

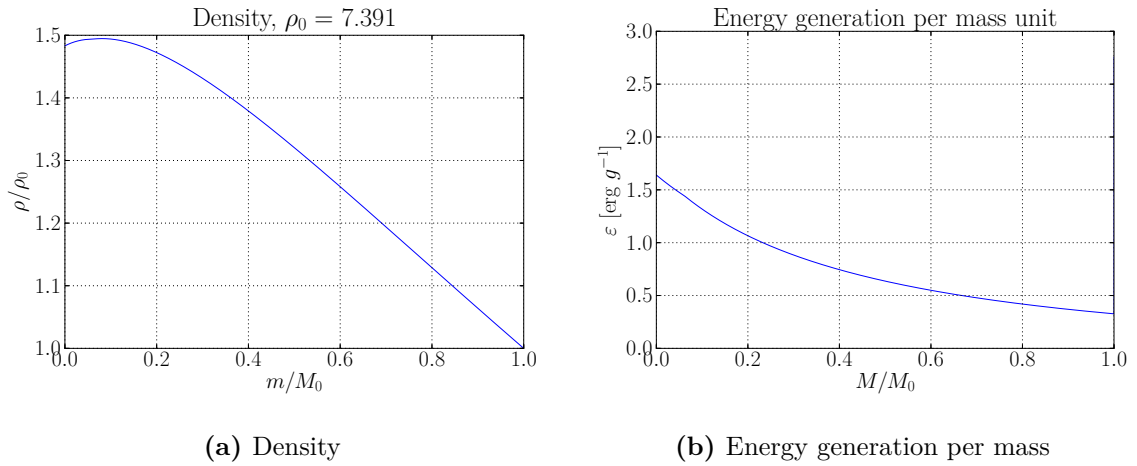
#### 4.4 Increasing initial temperature and pressure

Increasing the pressure by a factor  $10^4$  of the original value, and the temperature by a factor of  $10^2$ , yields Figure 4. We immediately notice that this time, it is the mass  $m$  that reaches zero. However, the luminosity and position stop at 73 % and 55 %, respectively. We also notice from Figure 4d that the temperature now increases faster towards the end, instead of in the beginning as we have had until now. This is probably more likely than the latter. This is the closest I have gotten to having the position, luminosity and mass reach zero values

simultaneously. There might be a combination of  $\rho_0, P_0, T_0$  that makes this possible, but I have not found it.

## 4.5 Density and energy generation

It could be interesting to see how the density and energy generation behave for a given set of initial values. We continue with the most recent initial values and have Figure 5. We see



**Figure 5:** Figures that show how the density and energy generation behave as function of mass, given the same initial conditions as in Figure 4. Mass step  $\partial m$  starts at  $1.39 \times 10^{23}$  g.

that the density increases by a factor of 1.5. What is noticeable however, is it how it decreases again towards the end, making it have the same value for different masses. This does not represent the physical situation, as the density should increase with increasing pressure and temperature without dropping again. We see the energy generation starts at a low value, but gradually increases as the temperature increases towards the core. This makes sense as it is strongly dependent on the temperature and density, where both increase towards the center.

## 5 The code

I have provided the link to my GitHub repository below. The code can be found there.

<https://github.com/vedad/SolarCoreModel/blob/dev/solarCore.py>

## References

- [1] Michael Stix, *The Sun*. Springer, New York, 2nd Edition, 2002.
- [2] Boris Gudiksen *AST3310: Astrophysical plasma and stellar interiors*. 2014.