University of Oslo

AST3310 - Astrophysical plasma and stellar interiors
Project 2

Numerical modeling of stellar interior

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Abstract

I here discuss the physical properties of stellar interiors based on models and simplifications addressed in [1] and [2]. Numerical calculations are used to solve the governing equations of the interior of stars. Models for convective, as well as radiative, energy transfer are included and discussed. For a given set of initial conditions, one should expect the luminosity, position and mass to end at zero value simultaneously as the core is reached. The best model to achieve this within the limit of 10% of initial value has initial physical properties $L_0 = L_\odot$, $M_0 = 1.5M_\odot$, $R_0 = 9R_\odot$, $T_0 = 5770$ K, $\rho_0 = 4.0 \times 10^{-7}$ g cm^{-3}. To this end, I apply an adaptive step method. It is found that the results heavily depend on initial properties, as well as the constraints on the mass step dm.

1 Introduction

There are four differential equations that govern the internal structure of stellar objects in our model. I apply here numerical methods such as the Euler integration scheme to solve these equations. This involves solving equations for nuclear fusion as well as ordinary differential equations. We include radiation and convection models, and look at how the interior of solar-like stars changes based on different initial parameters. I try to seek out a set of parameters that leads to the case where mass, radius and luminosity reach zero value simultaneously.

2 Theory

2.1 General background

The equations that regulate the physical properties of stellar interiors are expressed as

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \tag{1}$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} \tag{2}$$

$$\frac{\partial L}{\partial m} = \varepsilon \tag{3}$$

$$\frac{\partial T}{\partial m} = -\frac{3\kappa L}{256\pi^2 \sigma r^4 T^3}. \tag{4}$$

The parameter ε in Eq. (3) is the total energy generation per unit mass. It is found by looking at the energy generation from nuclear reactions. It depends on the abundancy of different elements, temperature and density. The variable κ is the opacity, which is an average of frequency of photons. The pressure in Eq. (2) is a sum of the gas pressure P_G, and the radiative pressure P_R.

$$P = P_\mathrm{G} + P_\mathrm{R}$$

$$P = \frac{\rho}{\mu m_\mathrm{u}} kT + \frac{a}{3} T^4$$

$$\Rightarrow \rho = \frac{\mu m_\mathrm{u}}{kT}\left(P - \frac{a}{3}T^4\right), \tag{5}$$

which yields the density derived from the equation of state. Here, m_{u} is the atomic mass unit and k is the Boltzmann constant. The constant a is defined as $a = 4\sigma/c$, where σ is the Stefan-Boltzmann constant, and c is the speed of light. The average molecular weight μ is found by

$$\mu = \frac{1}{m_{\mathrm{u}}}\frac{\rho}{n_{\mathrm{tot}}}. \tag{6}$$

We can find the total particle density n_{tot} with

$$\begin{aligned} n_{\mathrm{tot}} &= n_X + n_Y + n_Z + n_e \\ &= \frac{X\rho}{m_{\mathrm{u}}} + \frac{Y\rho}{4m_{\mathrm{u}}} + \left(\frac{X\rho}{m_{\mathrm{u}}} + \frac{2Y\rho}{4m_{\mathrm{u}}}\right) + \sum_i (n_{Z_i} + n_{e,Z_i}) \end{aligned}$$

The sum is over the heavier metals present in the star. The number of atoms and electrons for an element i is

$$\begin{aligned} n_{Z_i} &= \frac{Z_i\rho}{m_{Z_i}} = \frac{Z_i\rho}{A_i m_{\mathrm{u}}} \\ n_{e,Z_i} &= \frac{1}{2}A_i\frac{Z_i\rho}{A_i m_{\mathrm{u}}}, \end{aligned}$$

where A_i is the mass number; the total number of particles in the element nuclei. For the heavier elements to be fully ionised, we have assumed that these elements have released half as many electrons as their atomic mass, A_i. We therefore assume that the nucleus is half neutrons and half protons. The number density of heavier metals is then

$$\sum_i (n_{Z_i} + n_{e,Z_i}) = \sum_i \left(\frac{Z_i\rho}{A_i m_{\mathrm{u}}} + \frac{1}{2}A_i\frac{Z_i\rho}{A_i m_{\mathrm{u}}}\right) \approx \frac{Z\rho}{2m_{\mathrm{u}}},$$

where we assumed that the first term in the sum is negligible compared to the second term, because A can be rather large. We then have for the total number density

$$n_{\mathrm{tot}} = \frac{2X\rho}{m_{\mathrm{u}}} + \frac{3Y\rho}{4m_{\mathrm{u}}} + \frac{Z\rho}{2m_{\mathrm{u}}},$$

and the average molecular weight is then

$$\begin{aligned} \mu &= \frac{\rho m_{\mathrm{u}}}{\rho m_{\mathrm{u}}}\left(\frac{1}{2X + \frac{3}{4}Y + \frac{1}{2}Z}\right) \\ &= \frac{1}{2X + \frac{3}{4}Y + \frac{1}{2}Z}, \end{aligned}$$

assuming all elements are fully ionised.

The total energy generation per unit mass ε, is found by

$$\varepsilon = \sum Q'_{ik} r_{ik}, \tag{7}$$

where i, k represents two elements, Q'_{ik} is the energy released from the fusion of two elements, r_{ik} is the reaction rates per unit mass for two elements. The energies Q'_{ik} from the pp chains are listed in [1, p. 39, Table 2.1]. The reaction rates per unit mass is defined by

$$r_{ik} = \frac{n_i n_k}{\rho(1+\delta_{ik})}\lambda_{ik}, \tag{8}$$

where n_i, n_k is the number density for an element, δ_{ik} is the Kronecker delta and λ_{ik} is the reaction rate of a fusion. The number density of an element is easily defined as

$$n_i = \frac{\rho \chi_i}{A_i m_{\mathrm{u}}}, \tag{9}$$

where χ is the mass fraction of an element, and A is the mass number of the element. We denote X, Y, Z to be the mass fractions of hydrogen, helium and heavier metals, respectively. Finally, the reaction rates λ_{ik} for two elements i, k can be found in [1, p. 46, Table 2.3].

2.2 Convection

In this model, we consider two means of energy transport in the star: radiation and convection. Convection is the mass movement of plasma where warm plasma ascends and cooled plasma descends through the star. A parcel of gas that is adiabatically lifted from its equilibrium position can either be heavier or lighter than its new environment. If lighter, it will continue to rise, meaning the original equilibrium was unstable. If it is heavier, it will return to its original position, meaning it was in a stable equilibrium. The onset of convection is determined from the Schwarzschild criterion, which describes the properties for stability and instability. The criterion is often written as

$$\nabla > \nabla_{\mathrm{ad}},$$

where

$$\nabla = \frac{\partial \ln T}{\partial \ln P}, \tag{10}$$

is the temperature gradient, and ∇_{ad} is its adabatic value. Eq. (4) only holds for radiative energy transport, which means we need a way of finding ∂T, the change in temperature over an infinitessimal step in a convective layer. From Eq. (10) we find

$$\begin{aligned} \nabla &= \frac{\partial \ln T}{\partial \ln P} \\ &= \frac{P}{T}\frac{\partial T}{\partial P} \\ &= \frac{P}{T}\frac{\partial T}{\partial m}\frac{\partial m}{\partial P} \\ \Rightarrow \partial T &= \frac{T}{P}\partial P \nabla, \end{aligned} \tag{11}$$

which gives us the change in temperature in a convectively unstable layer.

We now need to find an expression for the true temperature gradient ∇. We know that the total energy flux is

$$\begin{aligned} F_{\mathrm{R}} + F_{\mathrm{C}} &= \frac{L}{4\pi r^2} \\ &= \frac{4acGT^4 m}{3\kappa P r^2}\nabla_{\mathrm{rad}}, \end{aligned}$$

where

$$F_{\mathrm{R}} = \frac{4acGT^4m}{3\kappa Pr^2}\nabla$$
$$F_{\mathrm{C}} = \rho c_P T\sqrt{g\delta}H_P^{-3/2}\left(\frac{l_{\mathrm{m}}}{2}\right)^2(\nabla - \nabla^*)^{3/2}.$$

We use these to get an expression between $\nabla, \nabla_{\mathrm{rad}}$ and ∇^*.

$$F = F_{\mathrm{R}} + F_{\mathrm{C}}$$
$$\Rightarrow \frac{4acGT^4m}{3\kappa Pr^2}\nabla_{\mathrm{rad}} = \frac{4acGT^4m}{3\kappa Pr^2}\nabla + \rho c_P T\sqrt{g\delta}H_P^{-3/2}\left(\frac{l_{\mathrm{m}}}{2}\right)^2(\nabla - \nabla^*)^{3/2}$$
$$\Rightarrow (\nabla - \nabla^*)^{3/2} = \frac{\rho H_P G m}{Pr^2 l_{\mathrm{m}}}\frac{16acT^3}{3\kappa\rho^2 c_P}\sqrt{\frac{H_P}{g\delta}}(\nabla_{\mathrm{rad}-\nabla}).$$

We insert $a = 4\sigma/c$ and note that

$$U = \frac{64\sigma T^3}{3\kappa\rho^2 c_P}\sqrt{\frac{H_P}{g\delta}}$$
$$H_P = \frac{Pr^2}{G\rho m},$$

and then have

$$(\nabla - \nabla^*)^{3/2} = \frac{U}{l_{\mathrm{m}}^2}(\nabla_{\mathrm{rad}} - \nabla).$$

Furthermore, we have

$$(\nabla^* - \nabla_{\mathrm{ad}}) = (\nabla - \nabla_{\mathrm{ad}}) - (\nabla - \nabla^*)$$
$$\Rightarrow U\left(\frac{S}{dQl_{\mathrm{m}}}\right)(\nabla - \nabla^*)^{1/2} = (\nabla - \nabla_{\mathrm{ad}}) - (\nabla - \nabla^*),$$

which can be rewritten to a second order equation for $(\nabla - \nabla^*)^{1/2}$. We rename $(\nabla - \nabla^*)^{1/2} = \xi$ and find

$$0 = \xi^2 + K\xi - (\nabla - \nabla_{\mathrm{ad}})$$
$$\Rightarrow \nabla = \xi^2 + K\xi + \nabla_{\mathrm{ad}}, \tag{12}$$

where $K = 4R$, $R = U/l_{\mathrm{m}}^2$. We now have an expression for the true temperature gradient, but we need ξ. It can be found that

$$\xi^3 + R\xi^2 + RK\xi - (\nabla_{\mathrm{rad}} - \nabla_{\mathrm{ad}}) = 0.$$

This equation yields three roots. Two are complex conjugates and one is real. We can easily solve this equation numerically to find the real root, and insert it in Eq. (12) to find the temperature gradient. This can then be used to calculate ∂T in Eq. (11).

	Unit system			
	CGS		SI	
Physical property	Unit name	Unit abbr.	Unit name	Unit abbr.
Length	centimetre	cm	metre	m
Weight	gram	g	kilogram	kg
Time	second	s	second	s
Temperature	kelvin	K	kelvin	K
Energy	erg	erg	joule	J
Pressure	barye	Ba	pascal	Pa

Table 1: Overview of the difference between CGS units and SI units.

3 Algorithm

3.1 Simplifications

In order to make a simple model I needed some simplifications. A list of assumptions and simplifications follows.

- There is no change in the composition of elements as a function of radius.
- I assume there is no change in the density of deuterium, so that the rate of destruction of deuterium is the same as the production, and that any reaction involving deuterium happens instantaneous.
- All nuclear energy comes from the three PP-chains. I have not included the CNO cycle.
- I assume all elements to be fully inonised.

3.2 Units

Given that κ is in units of $[\mathrm{cm}^2\ \mathrm{g}^{-1}]$ and λ in units of $[\mathrm{cm}^3\ \mathrm{s}^{-}1]$, I chose to adapt the CGS unit system (centimetre-gram-second). Table 1 shows an overview of the differences.

3.3 Structure

We see from Eq. (4) that $\partial T/\partial m$ depends on the opacity, κ. A table of opacities that correspond to different values of temperature and density has been provided. I wrote a function that reads the file and stores values of T, R and κ in separate arrays. Here, $R = R(T, \rho) = \rho/T_6$. The function compares the table temperature with the actual temperature, and the same with the variable R. It returns the κ with table values of T and R that most closely resembles the present values.

Next, I wrote a function that calculates the energy generation per mass unit from nuclear reactions. This means entering the energy releases Q'_{ik} from [1, p. 39, Table 2.1], the reaction rates λ_{ik} from [1, p. 46, Table 2.3] and finding the number densities n for all particles that are involved in a nuclear reaction. Given the mass fractions X, Y, Z and sub-fractions of these for different elements, I was able to calculate the energy generation per mass by using Eqs. (7), (8) and (9). The function also returns the distribution of energy generation between the three pp chains.

I have a function that calculates the density at present time, given the temperature T and total pressure P. This is found from the equation of state, as given in Eq. (5). This required me to calculate the average molecular weight μ, shown in Eq. (6). Another function calculates the pressure at any layer, which is also derived from the equation of state.

Three functions calculate the convective, radiative and total flux. These are useful to see the fraction of energy transported by convection and radiation.

Further, I have four functions that return the right-hand side in Eqs. (1), (2), (3) and (4). These are called upon in another function that integrates all the equations.

Lastly, the function that integrates the equations contains a test for convective stability, calculating $\mathrm{d}T$ based on convection if the layer is convective unstable. This is done by finding the one real root of a cubic polynomial numerically, then using the solution in finding the true temperature gradient, including convection, of the star. This function also implements an adaptive step method. See Section (3.5) for details.

3.4 The Euler integration scheme

For simplicity, I have chosen the Euler integration scheme to integrate the differential equations. Listing (1) shows how it is carried out.

Listing 1: Euler integration loop

```
while m > 0:
    r_new = r_old + (dr/dm) * dm
    P_new = P_old + (dP/dm) * dm
    L_new = L_old + (dL/dm) * dm
    T_new = T_old + (dT/dm) * dm
    m_new = m_old + dm
```

The Euler method advances one step by adding the previous step value to the current value of the right-hand side of the differential equations, multiplied by the mass step. Unfortunately, the scheme carries a local truncation error proportional to the step size squared, and a global truncation error that is proportional to the step size.

3.5 Adaptive mass step

To avoid sudden and large changes in r, T, L, P, I implement an adaptive step method that calculates the mass step $\mathrm{d}m$ by applying constraints on how much the physical parameters are allowed to increase or decrease from one step to another. By applying a constraint on the relative change in radius, i.e. $\mathrm{d}r/r \leq 0.2$, we are guaranteed that the relative change in

Physical properties		Element abundancies	
Parameter	Initial value	Element	Initial value
L_0	$L_\odot$	X	0.7
R_0	$R_\odot$	Y_3	10^{-10}
M_0	$M_\odot$	Y	0.29
ρ_0	4.0×10^{-7} g cm^{-3}	Z	0.01
T_0	5770 K	$Z_{^7\mathrm{Li}}$	10^{-12}
P_0	3.1×10^5 Ba	$Z_{^7\mathrm{Be}}$	10^{-12}

Table 2: Table that shows the initial physical parameters needed in order to initiate the calculations.

radius will not exceed 20 %. The corresponding mass step can be easily found from

$$\begin{aligned} \frac{\mathrm{d}r}{r} &\leq 0.2 \\ \frac{\mathrm{d}r}{r}\frac{\mathrm{d}m}{\mathrm{d}m} &\leq 0.2 \\ \mathrm{d}m\frac{\mathrm{d}r}{\mathrm{d}m}\frac{1}{r} &\leq 0.2 \\ \mathrm{d}m &\leq 0.2\frac{r}{\left(\frac{\mathrm{d}r}{\mathrm{d}m}\right)}, \end{aligned}$$

and when integrating inwards, d$m < 0$, so that

$$\mathrm{d}m \leq -0.2\left|\frac{r}{\left(\frac{\mathrm{d}r}{\mathrm{d}m}\right)}\right|.$$

The same calculation can be done for the other physical parameters with the same constraint, or another. By finding the dm that is smallest in absolute value of $r/(\mathrm{d}r/\mathrm{d}m)$, $T/(\mathrm{d}T/\mathrm{d}m)$, $L/(\mathrm{d}L/\mathrm{d}m)$, $P/(\mathrm{d}P/\mathrm{d}m)$, with each their own constraint, we are guaranteed that they will all be fulfilled. I apply another constraint on dm itself, so that it cannot increase or decrease more than a certain percentage of its previous value.

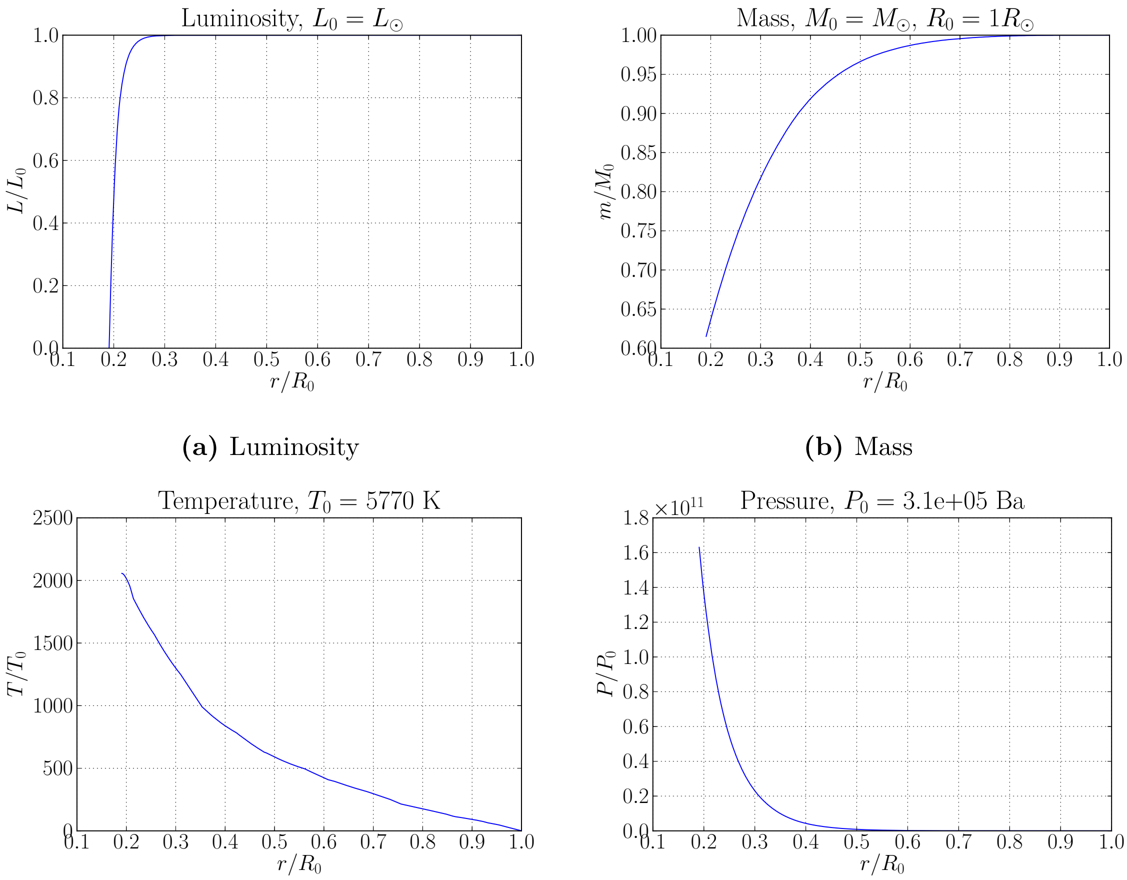
3.6 Initial conditions

Finally, we need a set of initial conditions to initiate the calculations. These are shown in Table 2 expressed in the CGS unit system (see Table 1).

4 Results

4.1 Model with original initial conditions

If we carry out the integration using the initial values as given in Table 2, we get the evolution for the physical parameters r, m, T, P as shown in Fig. 1. Fig. 1a shows that the luminosity reaches zero a little premature in relation to the position. This means that the luminosity is the limiting parameter here, stopping the integration. However, it shows what we expect of a luminosity graph as we integrate towards the center. The mass plot is shown in Fig. 1b. It falls short because of the luminosity, but we see it lags considerably behind the position as well. The temperature graph shows a larger increase towards the end which seems reasonable. It ends at $T \approx 12$ MK, which is within reason at $r < 0.2R_\odot$. The pressure grows incredibly fast towards the end, as the radiative pressure increases from the increase in temperature.



(a) Luminosity

(b) Mass

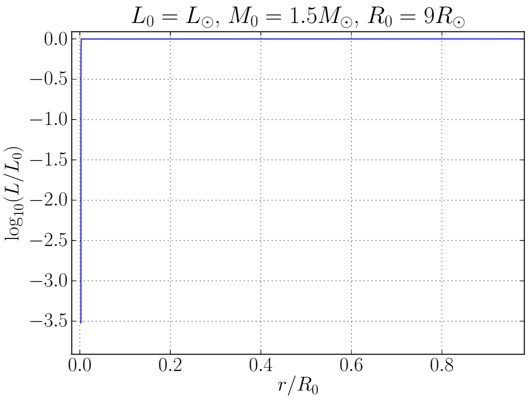
(c) Temperature

(d) Pressure

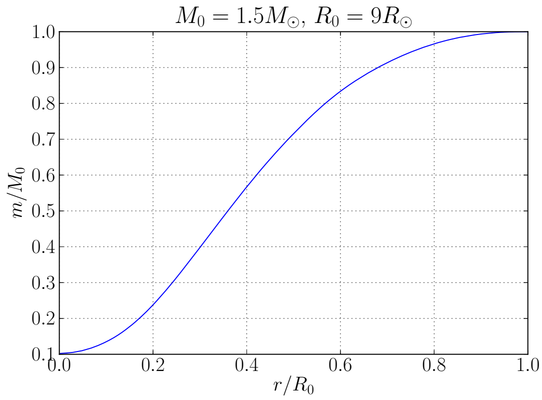
Figure 1: The results from using initial conditions as described in Table 2.

4.2 The best model

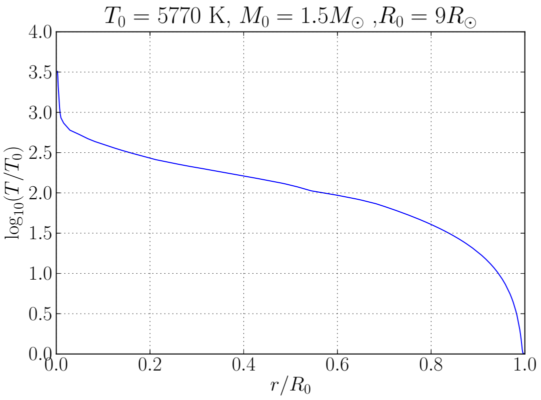
The best parameters were found by trial and error. The result is shown in Fig. 2. We were able to integrate all the way to the core, ending with mass $M_{\text{end}} = 0.1M_\odot$. The temperature in the core is of the order $T \approx 18$ MK. which is slightly higher than the temperature in the core of the Sun. However, we are considering a star with a higher mass and larger radius, making the temperature reasonable. The pressure has increased by a factor 100 billion in the core, relative to the surface pressure. This seems reasonable as the radiative pressure increases by temperature to the power of four, making it within limits.



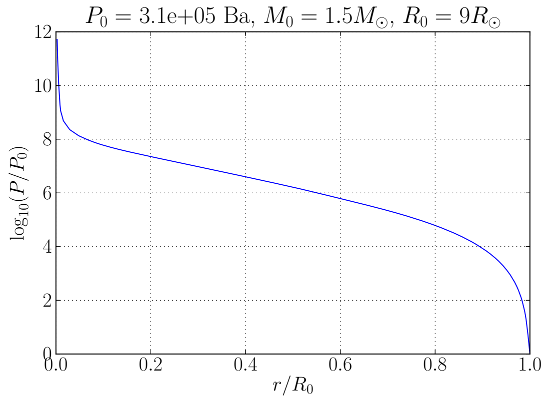
(a) Luminosity



(b) Mass



(c) Temperature



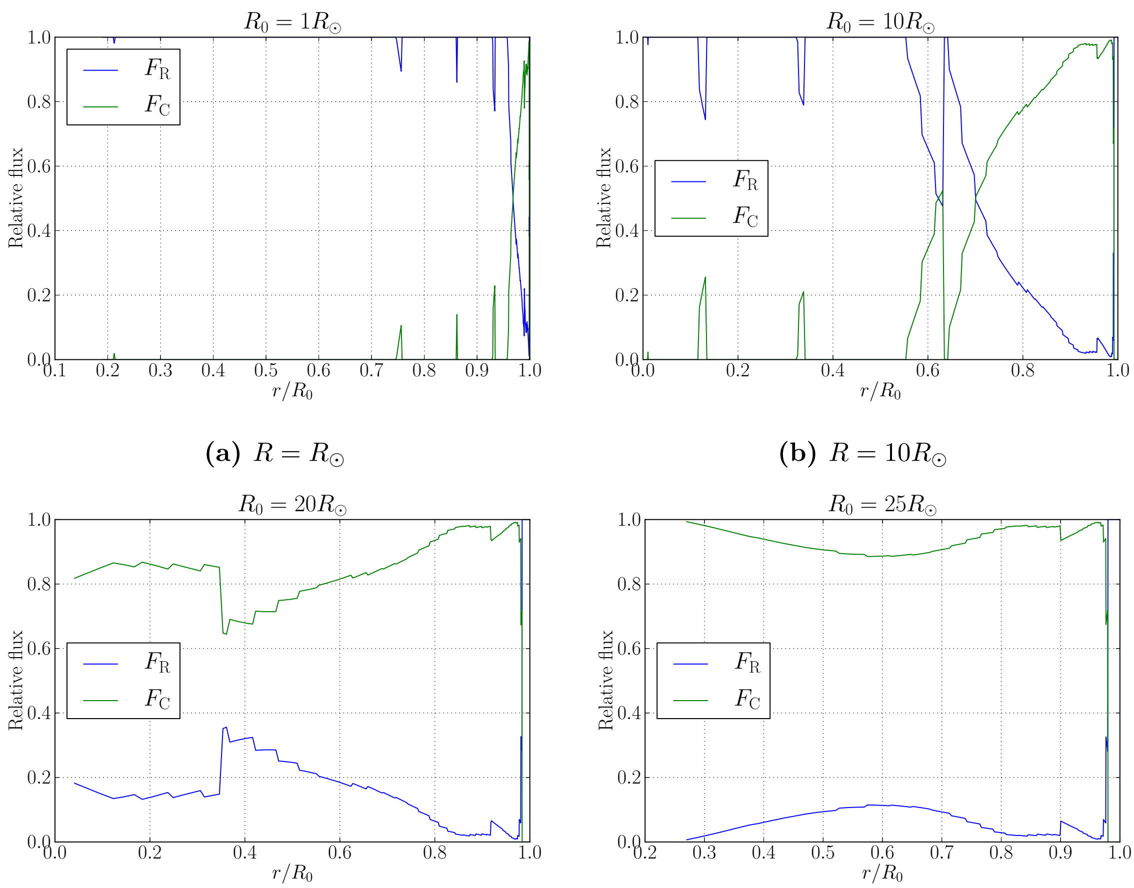
(d) Pressure

Figure 2: The results from using the physical parameters giving the best model of a star. $M_0 = 1.5M_\odot$, $R0 = 9R_\odot$, $T0 = 5770$ K, $P0 = 3.1 \times 10^5$ Ba.

4.3 Changing the initial radius

4.3.1 Energy transport

It is of interest to see how the fraction of energy transport changes as function of initial radius. Fig. 3 shows the change for four different initial stellar radii. Fig. 3a shows convective transport only in the outermost layers. For a star like our sun, the convective zone starts at a depth of $\sim$ 200000 km below the surface, corresponding to $\sim$ 30% into the Sun. For this initial radius, our convection model is not satisfactory. Increasing the initial radius by a factor 10 yields convection dominance from about 30% beneath the surface, which agrees more for a star like our Sun. Fig. 3b also show some interesting fluctuations below the convective zone. It displays some fluctuations which might be due to numerical error. Fig. 3c shows that convection dominates as energy transport in the entire star. This agrees with the case of a giant star, but also a dwarf star. Increasing the radius while keeping the other physical parameters constant might give the effect of a giant star carrying its energy outwards by convection. This might also be the case in Fig. 3d, as the effect is even more prominent there.



(a) $R = R_\odot$

(b) $R = 10R_\odot$

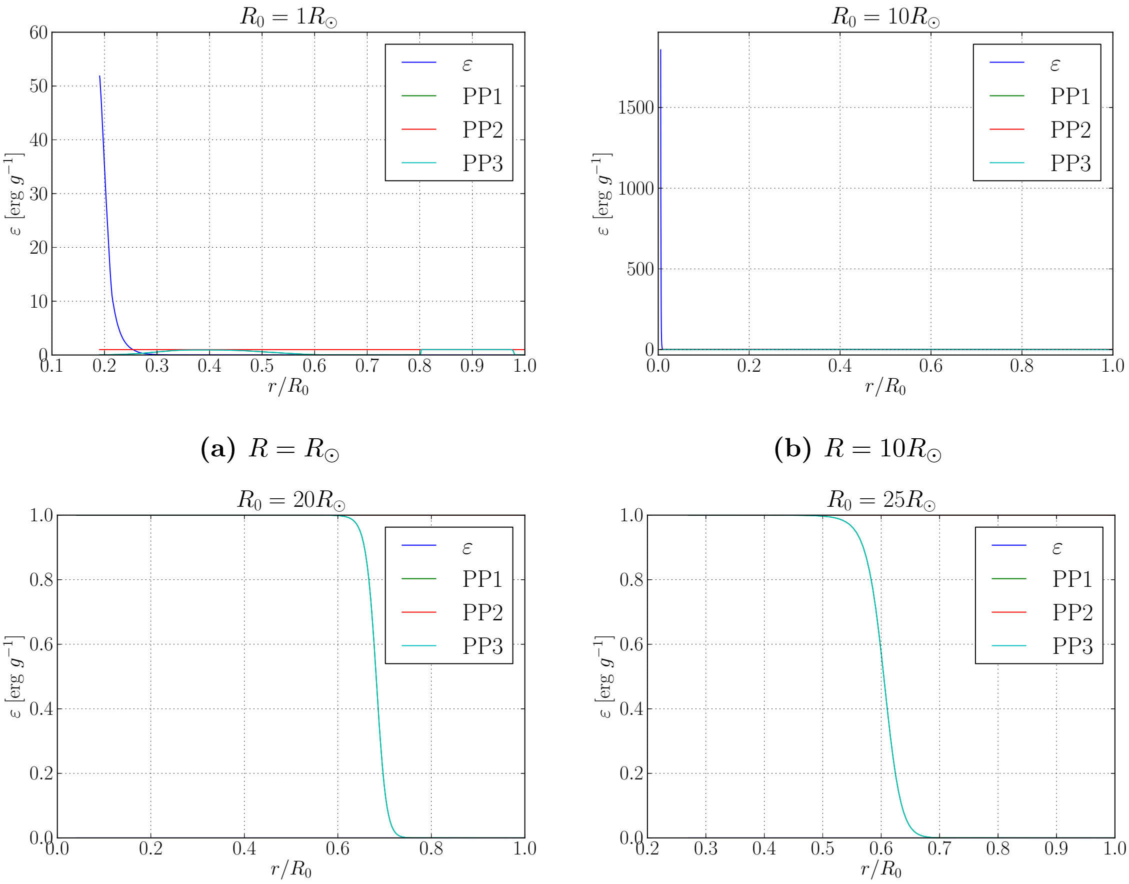
(c) $R = 20R_\odot$

(d) $R = 25R_\odot$

Figure 3: The change in energy transport for increasing initial stellar radius.

4.3.2 Energy generation

We now consider the energy production as function of intitial stellar radius. Fig. 4a shows the energy generation for a star like our Sun. We expect an increase in the energy production towards the core, which we see starts at $R \approx 0.25R_{\odot}$. The PP1-chain is probably what is causing the increase in energy output, as we see PP2 and PP3 are all very small. An increase in initial radius, Fig. 4b, shows a more sudden and violent jump in energy output. The jump is much higher in this model, probably because we reach the center of the star, unlike in the Sun-like case. Increasing the initial radius even more shows a sudden energy production much earlier. Furthermore, the energy production seems to originate from the PP3-chain. The energy production is heavily dependent on temperature, so the temperature must have increased enough for PP3 to be effective.



(a) $R = R_{\odot}$

(b) $R = 10R_{\odot}$

(c) $R = 20R_{\odot}$

(d) $R = 25R_{\odot}$

Figure 4: The change in energy generation for increasing initial stellar radius.

5 Conclusion

It is found that the model we implement in this project works better for initial physical parameters to that of larger and more massive stars than the Sun. We also found the results to be heavily dependent on how fast the physical parameters change. By applying constraints to how fast they can change in one step by using an adaptive step method, one avoids several problems.

6 The code

I have provided the link to my GitHub repository below. The code can be found there.

References

[1] Michael Stix, *The Sun*. Springer, New York, 2nd Edition, 2002.

[2] Boris Gudiksen *AST3310: Astrophysical plasma and stellar interiors*. 2014.