2025 Wolverine Math Tournament Middle School Division: Individual Round

Westview Math Club

May 17th, 2025

1 MS Individual Round Questions:

The questions are in increasing order of difficulty. Please write your answers on the answer sheets given. Simplify all fractions (write 1/2 instead of 2/4). Do not express your answer as a decimal (write 1/2, not 0.5). Write all expressions containing square roots in simplest radical form. Good luck!

Time limit: 60 minutes

1. Ryan the hammerhead shark and Shriya the sea otter are swimming in the sea. Ryan has 26 fish, and Shriya has 6 fish. Ryan feels bad for Shriya, so he wants to share some of his fish. How many fish does Ryan need to give to Shriya so that they both have an equal number of fish?
2. Let $a \heartsuit b = (a+b)^2 - b$. What is the value of $(1 \heartsuit 2) \heartsuit 3$?
3. Julian the Wolverine is covering his floor with Wolverine bucks. If a Wolverine buck measures 4 inches by 3 inches, how many Wolverine bucks would it take him to cover a square floor with side length 5 feet?
4. A chord of a circle has length 12, and it's midpoint lies 5 units from the center of the circle. Find the square of the radius of the circle.
5. A triangle has side lengths of 14, 25, and 25. What is the positive difference between the area and perimeter of the triangle?
6. David's phone's battery life (in hours) is inversely proportional to its screen brightness (expressed as a percentage). When the brightness is set at 80%, the battery lasts for 4.5 hours. David is planning to have a 6 hour gaming session on his phone. At what percent brightness should he use for his battery to last exactly enough for his session?
7. The Westview Math Club is selecting a 4-person team for an upcoming math competition. There are 9 students eligible to join, including Raunav and Jack. How many ways can the team be chosen if the team must include either Raunav or Jack but not both of them?
8. In a convex pentagon, the measures of the interior angles form an arithmetic progression. If the smallest angle measures 80 degrees, what is the measure of the largest angle (in degrees)? (An arithmetic progression is a sequence of numbers where the differences between consecutive numbers is the same. 1, 2, 3, 4, 5... and 2, 6, 10, 14, 18... are arithmetic progressions, but 1, 2, 4 is not)
9. At Westview High School, the ratio of students who play tennis to those who play badminton is 3:2. If 60 students play at least one of these sports and 10 students play both sports, how many students play tennis but not badminton?
10. Ryan and Aiden start a race. Aiden runs at 12 feet per second but stops for 5 seconds after every 20 feet. Ryan moves at a steady pace of 2 feet per second without stopping. If Ryan and Aiden take A and B seconds to finish a 100-foot race, respectively, find $|A - B|$.

11. There exists a unique positive integer n less than 100 such that n has 6 factors, $n+2$ has 8 factors, and $n+4$ has 10 factors. Find n.

12. Mihir is running from Station A to Station B, which are 90 miles apart. If he increases his speed by 5 miles per hour, his travel time decreases by 15 minutes. What is Mihir's original running speed in miles per hour?

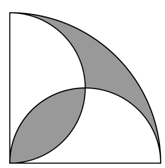
13. While rolling a pair of regular 6-sided dice, Sarah notices that the sum of the numbers on the top faces is an even number. Find the probability that Sarah rolled two primes.

14. An empty water tank is filled over a span of 5 hours by adding a fixed amount of water at the start of each hour. The water is added instantaneously. Every hour, 50% of the water evaporates. After 5 hours, just before adding water for the 6th hour, the tank holds exactly 93 liters. Find the amount of water (in liters) added at the beginning of each hour.

15. When Aarav computes the median and mean of the set $\{3, 6, 9, 10, n\}$ (n is not necessarily the largest number in the set), he discovers that they are the same value! Find the sum of all possible values of n.

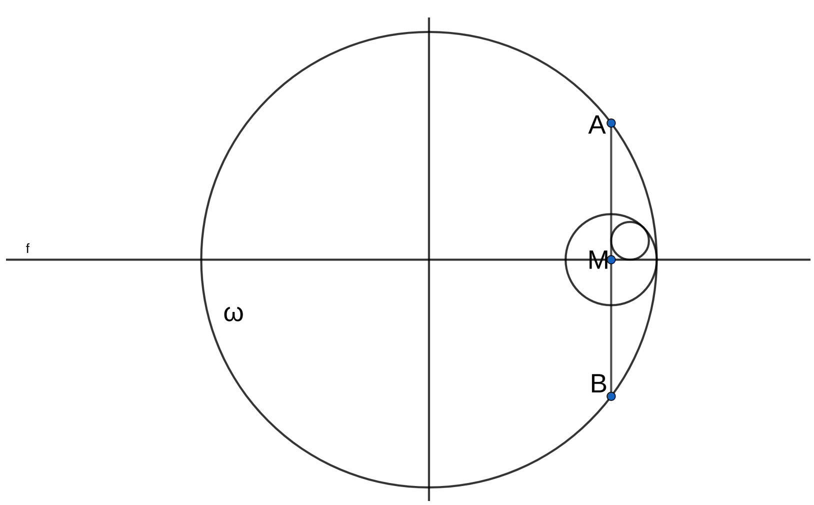
16. At the Wolverine bus station, a bus is scheduled to depart every 15 minutes from 11:00 AM to 1:00 PM (inclusive). However, due to traffic, the delay (in minutes) for the bus originally scheduled at exactly t minutes after 11:00 AM is given by delay $= kt^2$. If the sum of the delays for all buses that day is exactly 2025 minutes, find k.

17. In a quarter circle with radius 8, two semicircles are drawn with diameters on the radius of the quarter circle, as shown. If the area of the shaded region can be written as $a\pi - b$, find $a + b$.



18. Aneesh rides from town A to town B at a constant speed of v mph. On the return trip, the headwind reduces his speed by 20%. It turns out that the total round-trip time in minutes is exactly equal to v^2 in mph. If the distance between towns A and B is d, and v is the smallest positive integer that makes d an integer, find the total round-trip time (in hours).

19. A circle ω with radius 10 has its center on the origin of the cartesian plane. A chord AB of ω has length 12, and has its midpoint, M, on the x-axis. A smaller circle with center M is drawn such that it is internally tangent to ω. Find the radius of the largest circle that can be placed between the x-axis, the chord AB, and the circle with center M.



20. Jack has 8 wooden letters of different colors that spell "WESTVIEW". How many ways can he rearrange these letters so that the resulting word is not in alphabetical order or reverse-alphabetical order? For example, the word "MOOR" is in alphabetical order, since every letter is either the same or further in the alphabet than the previous letter.

21. Triangle ABC has side $\overline{AB} = 14$. If the inradius is 4, and the tangency point of the incircle to AB splits the side into line segments of lengths 6 and 8, what is the area of the triangle?
The inradius of a triangle is the radius of the circle inscribed inside the triangle.

22. Find the sum of all possible values of x that satisfy the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{30}$, where x and y are positive integers.

23. Let z be a complex number $a + bi$ (where $i^2 = -1$) with $|z| = 1$ and suppose that the imaginary part of $\frac{1}{1-z}$ is 1. Determine the real part of z.
Note: The magnitude of a complex number $z = a + bi$, $|z|$, is $\sqrt{a^2 + b^2}$.

24. Westview Math Club offers its members a week-long challenge. Each day, a student chooses one of 5 different puzzles, labeled S, I, G, M, and A, but they cannot choose the same puzzle on two consecutive days. Moreover, over the 7-day week, the puzzle labeled S must be chosen at least twice. Compute the total number of valid 7-day schedules.

25. In triangle ABC, the medians from A and B are perpendicular, with $\overline{AC}$=22 and $\overline{BC} = 19$. If the circle with diameter AB intersects AC at a point D not equal to A, what is $\overline{AD}$?

A special thanks to our sponsor Jane Street for helping make this contest possible!

2 MS Individual Round Answer Key

1. 10
2. 97
3. 300
4. 61
5. 104
6. 60
7. 70
8. 136
9. 32
10. $\frac{65}{3}$
11. 76
12. 40
13. $\frac{5}{18}$
14. 96
15. 26
16. $\frac{3}{68}$
17. 48
18. $\frac{15}{4}$
19. $2\sqrt{2}-2$
20. 40312
21. 84
22. 3631
23. $\frac{3}{5}$
24. 9140
25. $\frac{73}{11}$

3 MS Individual Round Solutions

1. Ryan the hammerhead shark and Shriya the sea otter are swimming in the sea. Ryan has 26 fish, and Shriya has 6 fish. Ryan feels bad for Shriya, so he wants to share some of his fish. How many fish does Ryan need to give to Shriya so that they both have an equal number of fish?

   We know that the total amount of fish won't change, since the only action is moving fish from one person to another. We add the number of fish that Ryan has to the number of fish that Shriya has. This gives 32 fish in total. For them to have an equal amount of fish, both need to have 16 fish. Ryan currently has 26 fish, but needs to get to 16 fish, which is just $26 - 16 = \boxed{\mathbf{10}}$ fish.

2. Let $a \heartsuit b = (a+b)^2 - b$. What is the value of $(1 \heartsuit 2) \heartsuit 3$?

   $(1 \heartsuit 2) \heartsuit 3$ is the original expression. $1 \heartsuit 2 = (1+2)^2 - 2 = 7$. $7 \heartsuit 3 = (7+3)^2 - 3 = \boxed{\mathbf{97}}$

3. Julian the Wolverine is covering his floor with Wolverine bucks. If a wolverine buck measures 4 inches by 3 inches, how many Wolverine bucks would it take him to cover a square floor with side length 5 feet?

   There are 12 inches in a foot. Therefore, the number of Wolverine bucks in 1 square foot is $\frac{12}{3} * \frac{12}{4} = 4 * 3 = 12$. There are $5^2 = 25$ square feet in a square with side length 5 feet, so it would take $25 * 12 = \boxed{\mathbf{300}}$ Wolverine bucks to cover the floor.

4. A chord of a circle has length 12, and it's midpoint lies 5 units from the center of the circle. Find the radius of the circle.

   Draw the radius of the circle from the center of the circle to one of the endpoints of the chord. Then, draw the line segment connecting the center of the circle with the midpoint of the chord. By the Pythagorean theorem, $r^2 = 5^2 + (\frac{12}{2})^2 = 25 + 36 = \boxed{\mathbf{61}}$.

5. A triangle has side lengths of 14, 25, and 25. What is the positive difference between the area and perimeter of the triangle?

   Notice that this triangle is isosceles. Draw the altitude of the triangle from the vertex. The altitude splits the opposite side into 2 line segments of length 7. The height is then $\sqrt{25^2 - 7^2} = 24$. Thus, the area is $\frac{24*14}{2}$=168. The perimeter of the triangle is $25 + 25 + 14 = 64$. Thus, the positive difference between the area and perimeter is $168 - 64 = \boxed{\mathbf{104}}$.

6. David's phone's battery life (in hours) is inversely proportional to its screen brightness (expressed as a percentage). When the brightness is set at 80%, the battery lasts for 4.5 hours. David is planning to have a 6 hour gaming session on his phone. At what percent brightness should he use for his battery to last exactly enough for his session?

   Since battery life is inversely proportional to screen brightness, we find that battery life * brightness = constant. We find that this constant is equal to $4.5 * 80 = 360$. Therefore, we get that when battery life=6, the brightness should be $\frac{360}{6} = \boxed{\mathbf{60}}$ percent.

7. The Westview Math Club is selecting a 4-person team for an upcoming math competition. There are 9 students eligible to join, including Raunav and Jack. How many ways can the team be chosen if the team must include either Raunav or Jack but not both of them?

Let's consider the case where Jack is on the team and multiply that by 2. When Jack is on the team, Raunav isn't. Thus, there are $9-2$ other people from whom to select the team. The number of ways to do this is $\binom{7}{3}=35$. The total number of ways to select the team is $35*2=\boxed{\mathbf{70}}$.

8. In a convex pentagon, the measures of the interior angles form an arithmetic progression. If the smallest angle measures 80 degrees, what is the measure of the largest angle (in degrees)? (An arithmetic progression is a sequence of numbers where the differences between consecutive numbers is the same. 1, 2, 3, 4, 5... and 2, 6, 10, 14, 18... are arithmetic progressions, but 1, 2, 4 is not)

   The sum of the interior angles of a pentagon is $3*180=540$. Let the common difference be r. We have that the angle measures of the pentagon are $80, 80+r, 80+2r, 80+3r$, and $80+4r$. The sum of these angle measures is $400+10r$. Set this equal to 540 and solve for r to obtain $400+10r=540 \Rightarrow 10r=140 \Rightarrow r=14$. Thus, the measure of the largest angle is $80+4r=80+4*14=80+56=\boxed{\mathbf{136}}$.

9. At Westview High School, the ratio of students who play tennis to those who play badminton is 3:2. If 60 students play at least one of these sports and 10 students play both sports, how many students play tennis but not badminton?

   Let the number of students playing tennis be $3x$. Then, the number of students playing badminton is $2x$. By PIE, the number of students playing either badminton or tennis is $3x+2x-10$. But, we are given that 60 students play either badminton or tennis, so this expression equals 60. Solving for x, we obtain $5x-10=60 \Rightarrow 5x=70 \Rightarrow x=14$. Thus, $3x=3*14=42$ students play tennis. To find the number of students that only play tennis, we subtract the number of students that play both tennis and badminton, to get a final answer of $42-10=\boxed{\mathbf{32}}$.

10. Aiden and Ryan start a race. Aiden runs at 12 feet per second but stops for 5 seconds after every 20 feet. Ryan moves at a steady pace of 2 feet per second without stopping. If Aiden and Ryan take a and b seconds to finish a 100-foot race, respectively, find $|a-b|$.

    Ryan takes $\frac{100}{2}=50$ seconds to complete the race, so $b=50$. Aiden takes $\frac{20}{12}=\frac{5}{3}$ seconds to complete each 20 feet, but he takes a 5 second break every 20 feet. Aiden takes 4 breaks in total: after 20 feet, 40 feet, 60 feet, and 80 feet. Thus, his time is $5*\frac{5}{3}+4*5=\frac{25}{3}+\frac{60}{3}=\frac{85}{3}$, so $a=\frac{85}{3}$. $|a-b|=|\frac{85}{3}-50|=|\frac{85}{3}-\frac{150}{3}|=|\frac{-65}{3}|=\boxed{\mathbf{\frac{65}{3}}}$.

11. There exists a unique positive integer n less than 100 such that n has 6 factors, $n+2$ has 8 factors, and $n+4$ has 10 factors. Find n.

    For a number to have 10 positive factors, its prime factorization must be either $p_1^4p_2$ or p_1^9. Since 2 is the smallest prime, and $2^9>100$, the number must be of the form $p_1^4p_2$. Since $3^4*2>100$, p_1 has to equal 2. Thus, we can further deduce that the number is of the form $2^4*p_2=16p_2$ for some prime p_2 other than 2. Since $16*7=112>100$, the only two options for the value of $n+4$ are $16*3=48$, and $16*5=80$. Lets test $n+4=48$ first.
    When $n+4=48$, $n+2=46=2*23$, which only has $2*2$=4 factors, not 6. Thus, $n+4$ has to be 80, so n=76. Lets test $n+4=80$ to make sure though. When $n+4=80$, $n+2=78=2*3*13$, which indeed has 8 factors. $76=2^2*19$, which indeed has 6 factors. Thus, $n=\boxed{\mathbf{76}}$

12. Mihir is running from Station A to Station B, which are 90 miles apart. If he increases his speed by 5 miles per hour, his travel time decreases by 15 minutes. What is Jack's original running speed in miles per hour?

Let Mihir's speed in mph be x. 15 minutes $= \frac{15}{60} = \frac{1}{4}$ of an hour. We can write the given information as $\frac{90}{x} - \frac{90}{x+5} = \frac{1}{4}$. Multiplying by $4(x)(x+5)$, we obtain $360x + 1800 - (360x) = x^2 + 5x \Rightarrow x^2 + 5x - 1800 = 0 \Rightarrow (x+45)(x-40) = 0$, so the solutions are $x = 40$ and $x = -45$. Since speed is positive, we ignore the negative solution to get x= $\boxed{\mathbf{40}}$.

13. While rolling a pair of regular 6-sided dice, Sarah notices that the sum of the numbers on the top faces is an even number. Find the probability that Sarah rolled two primes.

    We find the number of ways to roll two primes that sum to an even number and divide by the total number of ways to roll an even sum. Since there are an equal number of even and odd sums, the total number of ways to roll an even sum is $\frac{6*6}{2} = 18$. We count the number of ordered pairs of primes that sum to an even number: (2, 2), (3, 3), (5, 5), (3, 5), (5, 3), for a total of 5 ways. Thus, the probability is $\boxed{\mathbf{\frac{5}{18}}}$.

14. An empty water tank is filled over a span of 5 hours by adding a fixed amount of water at the start of each hour. The water is added instantaneously. Every hour, 50% of the water evaporates. After 5 hours, just before adding water for the 6th hour, the tank holds exactly 93 liters. Find the amount of water (in liters) added at the beginning of each hour.

    Let the fixed amount of water (in liters) be x. After the first hour, we have $\frac{x}{2}$ liters. After the second hour, we have $\frac{\frac{x}{2}+x}{2} = \frac{3x}{4}$ liters left. After the third hour, we have $\frac{\frac{3x}{4}+x}{2} = \frac{7x}{8}$ liters. After the fourth hour, we have $\frac{\frac{7x}{8}+x}{2} = \frac{15x}{16}$ liters. After the fifth hour, we have $\frac{\frac{15x}{16}+x}{2} = \frac{31x}{32}$ liters. Since this equals 93, we find that $x = \frac{93*32}{31} = 3*32 = \boxed{\mathbf{96}}$.

15. When Aarav computes the median and mean of the set 3, 6, 9, 10, n (n is not necessarily the largest number in the set), he discovers that they are the same value! Find the sum of all possible values of n.

    There are three possible cases for the median: it can be either $6, n,$ or 9. The mean is $\frac{3+6+9+10+n}{5} = \frac{28+n}{5}$.

    When the median is 6, n has to be less than 6. $28 + n = 5*6 = 30 \Rightarrow n = 2$. 2 is less than 6, so this is one possible value of n.

    When the median is n, n has to be between 6 and 9. $28 + n = 5n \Rightarrow 4n = 28 \Rightarrow n = 7$. 7 is between 6 and 9, so this is another possible value of n.

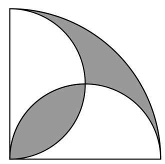
    When the median is 9, n has to be more than 9. $28 + n = 5*9 = 45 \Rightarrow n = 45 - 28 = 17$. 17 is greater than 9, so this is another value of n.

    The sum of all possible values of n is $2 + 7 + 17 = \boxed{\mathbf{26}}$

16. At the Wolverine bus station, a bus is scheduled to depart every 15 minutes from 11:00 AM to 1:00 PM (inclusive). However, due to traffic, the delay (in minutes) for the bus originally scheduled at exactly t minutes after 11:00 AM is given by delay $= kt^2$. If the sum of the delays for all buses that day is exactly 2025 minutes, find k.

    There are 9 buses scheduled to depart. The sum of the delays is $k(0^2+15^2+30^2+45^2+...+120^2) = k(15^2)(0^2+1^2+2^2+...+8^2)$. Using the formula for the sum of the first n positive perfect squares, we find that this expression is equal to $225k(\frac{8*9*17}{6}) = 225k*12*17$. Setting this equal to 2025, we can then solve for k. $225k*12*17 = 2025 \Rightarrow k*12*17 = 9 \Rightarrow k = \frac{9}{12*17} = \boxed{\mathbf{\frac{3}{68}}}$.

17. In a quarter circle with radius 8, two semicircles are drawn with diameters on the radius of the quarter circle, as shown. If the area of the shaded region can be written as $a\pi - b$, find $a + b$.

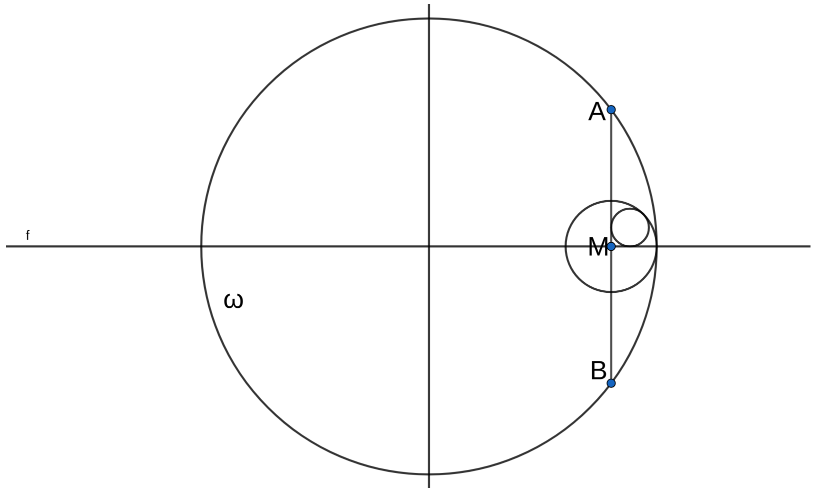


We first find the area of the white region, and then subtract from the total area. The area of one semicircle is $\frac{4^2\pi}{2} = 8\pi$. The sum of the areas of both semicircles is then $2*8\pi = 16\pi$. To find the area of the white region, we have to subtract twice the area of the overlap between the two semicircles (we subtract it twice because initially it is counted twice (one for each semicircle), and we want to count it 0 times). The area of the overlap between the two semicircles is $2(\frac{4^2\pi}{4} - \frac{4*4}{2}) = 2(4\pi - 8) = 8\pi - 16$. Multiplying by 2 and subtracting from the area of both semicircles combined, we find that the white region has area $16\pi - 2(8\pi - 16) = 16\pi - 16\pi + 32 = 32$. The quarter circle has area $\frac{8^2\pi}{4} = 16\pi$, so the area of the shaded region is $16\pi - 32$, for a final answer of $16 + 32 = \boxed{\mathbf{48}}$.

18. Aneesh rides from town A to town B at a constant speed of v mph. On the return trip, the headwind reduces his speed by 20%. It turns out that the total round-trip time (in minutes) is exactly equal to v^2 (in mph). If the distance between towns A and B is d, and v is the smallest positive integer that makes d an integer, find the total round-trip time (in hours).

The time it takes for Aneesh to ride from town A to town B (in minutes) is $\frac{d}{\frac{v}{60}} = \frac{60d}{v}$. The time it takes for Aneesh to ride from town B to town A (in minutes) is $\frac{d}{\frac{0.8v}{60}} = \frac{d}{\frac{v}{75}} = \frac{75d}{v}$. The total round-trip time (in minutes) is then $\frac{60d}{v} + \frac{75d}{v} = \frac{135d}{v}$. Setting this equal to v^2, we get that $\frac{135d}{v^2} = v \Rightarrow v^3 = 135d = d*5*3^3$. The smallest v that makes d an integer is $v = 3*5 = 15$, for which $d = \frac{v^3}{135} = \frac{15^3}{135} = 25$. Thus, the total round-trip time (in hours), is $\frac{1}{60}(\frac{135d}{v}) = \frac{1}{60} = \frac{9*25}{60} = \boxed{\mathbf{\frac{15}{4}}}$.

19. A circle ω with radius 10 has its center on the origin of the cartesian plane. A chord AB of ω has length 12, and has its midpoint, M, on the x-axis. A smaller circle with center M is drawn such that it is internally tangent to ω. Find the radius of the largest circle that can be placed between the x-axis, the chord AB, and the circle with center M.



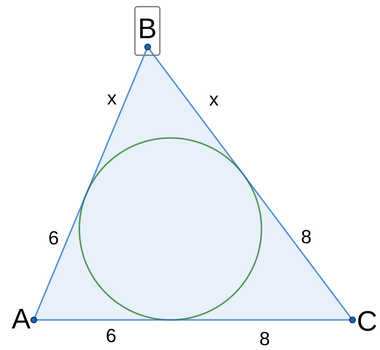
Label the origin as point O. Since M is the midpoint of AB, $AM = \frac{12}{2} = 6$. By the Pythagorean theorem, $OM = \sqrt{10^2 - 6^2} = 8$. Therefore, the radius of the circle with center M is $10 - 8 = 2$.

We want to find the radius of the largest circle that can fit into a quarter circle with radius 2. Let this radius be x. Notice that the circle is tangent to the quarter circle at its curved edge. Let this point of tangency be A. Notice that the distance from the center of the quarter circle to A is $x\sqrt{2} + x$. However, since the quarter circle has radius 2, and since this length is simply a radius, it also equals 2. We then simplify and solve for x. $x(\sqrt{2} + 1) = 2 \Rightarrow x = \frac{2}{\sqrt{2}+1} = 2(\sqrt{2} - 1) = \boxed{\mathbf{2\sqrt{2} - 2}}$

20. Jack has 8 wooden letters of different colors that spell "WESTVIEW". How many ways can he rearrange these letters so that the resulting word is not in alphabetical order or reverse-alphabetical order? For example, the word "MOOR" is in alphabetical order, since every letter is either the same or further in the alphabet than the previous letter.

The total amount of combinations, disregarding the conditions, is $8! = 40320$. The amount of combinations in alphabetical order is the same as the amount in reverse alphabetical order. There are $2 * 2 = 4$ ways to order the blocks in alphabetical order, so the answer is $40320 - (4 * 2) = \boxed{\mathbf{40312}}$

21. Triangle ABC has side length $\overline{AB} = 14$. If the inradius is 4, and the tangency point of the incircle to AB splits the side into line segments of lengths 6 and 8, what is the area of the triangle? Note: The inradius of a triangle is the radius of the circle inscribed inside the triangle.



Since the two tangents from a point to a circle are equal, the other tangents from A and C are equal to 6 and 8, respectively. Label the length of the tangent from B as x. We express the area two different ways, using Heron's formula and area=inradius$*$semiperimeter. The semiperimeter is $\frac{6+6+8+8+x+x}{2} = 14 + x$. The side lengths of the triangle are $x + 6, x + 8$, and 14.

Heron's formula: area=$\sqrt{(x + 14)(8)(6)(x)}$.

$a = rs$: area=$4 * (x + 14)$

Setting these expressions equal to each other, we solve for x. $\sqrt{(x + 14)(x)(6)(8)} = 4(x + 14) \Rightarrow (x+14)(x)(6)(8) = 16(x+14)^2 \Rightarrow 3x = x+14 \Rightarrow x = 7$. The semiperimeter is $x+14 = 7+14 = 21$. The area is then $rs = 4 * 21 = \boxed{\mathbf{84}}$.

22. Find the sum of all possible values of x that satisfy the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{30}$, where x and y are positive integers.

    Multiplying both sides by $30xy$, we obtain the equation $30x + 30y = xy$. Moving all the terms to one side and using SFFT, we find that $xy - 30x - 30y = 0 \Rightarrow xy - 30x - 30y + 900 = 900 \Rightarrow (x-30)(y-30) = 900$. Since x and y are positive integers, both terms on the left are integers. For the terms on the left to multiply to 900, they must be a factor pair of 900. Notice that this factor pair must have both factors be positive, as the maximum product obtained when both factors are negative is $(1 - 30)(1 - 30) = 841 < 900$. $900 = 30^2 = 2^2 * 3^2 * 5^2$ has $3 * 3 * 3 = 27$ factors. We observe that x is always 30 greater than a factor of 900, and factors are never repeated. Therefore, the sum of all values of x is the sum of the divisors + 30$*$(number of divisors). The sum of the divisors is $(1 + 2 + 2^2)(1 + 3 + 3^2)(1 + 5 + 5^2) = 7 * 13 * 31 = 2821$. Thus, the sum of all values of x is $2821 + 30 * 27 = \boxed{\mathbf{3631}}$.

23. Let z be a complex number $a + bi$ (where $i^2 = -1$) with $|z| = 1$ and suppose that the imaginary part of $\frac{1}{1-z}$ is 1. Determine the real part of z.
    Note: Given that $z = a + bi$, the magnitude of a complex number, denoted $|z|$, is $\sqrt{a^2 + b^2}$.

    Let $z = a + bi$. Thus, we can represent $\frac{1}{1-z}$ as $\frac{1}{1-(a+bi)} = \frac{1}{(1-a)-bi}$. We can rationalize the denominator by multiplying the numerator and denominator by the denominator's conjugate, $(1 - a) + bi$. $\frac{1}{1-a-bi} = \frac{1-a+bi}{(1-a-bi)(1-a+bi)} = \frac{1-a+bi}{(1-a)^2+b^2} = \frac{1-a+bi}{a^2-2a+1+b^2}$. We are told that the imaginary part of $\frac{1-a+bi}{a^2-2a+1+b^2} = 1$. The imaginary part of this expression is the coefficient of i, which is $\frac{b}{a^2-2a+1+b^2}$. Thus, we get that $\frac{b}{a^2-2a+1+b^2} = 1$. We're also told that $|z| = 1$, so $\sqrt{a^2 + b^2} = 1$. Squaring both sides, we get $a^2 + b^2 = 1$. Plugging into the first equation, $\frac{b}{a^2-2a+1+b^2} = 1$, we get that $\frac{b}{-2a+2} = 1$. Multiplying by $-2a + 2$ on both sides, we get that $b = -2a + 2$. Plugging back into $a^2 + b^2 = 1$, we get that $a^2 + (-2a + 2)^2 = 1$. Expanding, we get that $5a^2 - 8a + 4 = 1$, which leads to $5a^2 - 8a + 3 = 0$. Factoring, we get $(5a-3)(a-1) = 0$. Plugging in $a = 1$ to $a^2 + b^2 = 1$ yields $b = 0$, but this solution isn't possible because $\frac{1}{1-z} = \frac{1}{0}$, which isn't defined. This leaves the solution $a = \boxed{\frac{3}{5}}$.

24. Westview Math Club offers its members a week-long challenge. Each day, a student chooses one of 5 different puzzles, labeled S, I, G, M, and A, but they cannot choose the same puzzle on two consecutive days. Moreover, over the 7-day week, the puzzle labeled S must be chosen at least twice. Compute the total number of valid 7-day schedules.

We count the total number of schedules without restrictions, then subtract the number of schedules with 0 or 1 S. Without restrictions, there are 5 choices on the 1st day, and 4 choices every other day, for a total of $5 * 4^6$ schedules. The amount of schedules with no S is $4 * 3^6$ (4 choices for the first day, 3 for the rest). To find the amount of schedules with only 1 S, we do casework on the position of the S.

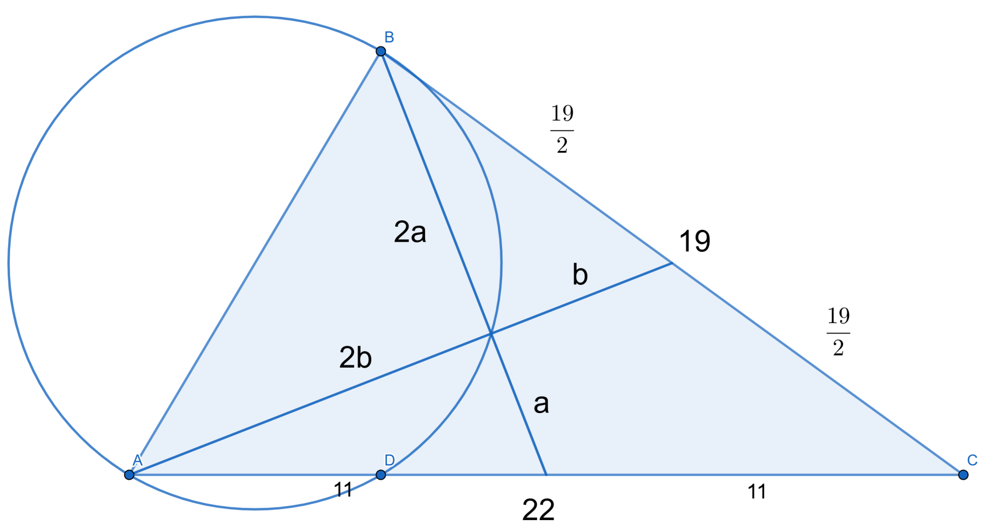
The S is chosen on day 1: There are 4 choices for day 2, and 3 choices for all days after that. $4 * 3^5$ schedules.

The S is chosen on days 2-6: There are 4 choices for day 1, 4 choices for the day after the day with the S, and 3 choices for the other days, for a total of $4^2 * 3^4$ schedules. There are 5 days from 2-6, so we multiply by 5 to get $5 * 4^2 * 3^4$ total schedules for these cases.

The S is chosen on day 7: There are 4 choices for day 1, and 3 choices for days $2 - 6$, for a total of $4 * 3^5$ schedules.

The final answer is then $5*4^6 - 4*3^6 - (2*4*3^5 + 5*4^2*3^4) = 20480 - 2916 - 1944 - 6480 = \boxed{\mathbf{9140}}$

25. In triangle ABC, the medians from A and B are perpendicular, with $\overline{AC}$=22 and $\overline{BC} = 19$. If the circle with diameter AB intersects AC at a point D not equal to A, what is $\overline{AD}$?



We start by finding the length $\overline{AB}$. The medians intersect at the centroid. The centroid divides medians into a 2:1 ratio, with the longer length being closer to the vertex. Let the median to the side length of length 22 have length $3a$, so the shorter part of the median has length a, and the longer has length $2a$. Let the median to the side of length 19 have length $3b$, so the shorter part of the median has length b and the longer has length $2b$. By the Pythagorean theorem, we get the following two equations:

$a^2 + (2b)^2 = (\frac{22}{2})^2 = 11^2$
$(2a)^2 + b^2 = (\frac{19}{2})^2 = \frac{361}{4}$

We wish to find the length of the third side length, $\overline{AB}$, which is $\sqrt{4a^2 + 4b^2}$. Adding the two equations above, we find that $5a^2 + 5b^2 = 121 + \frac{361}{4} = \frac{845}{4}$. Multiplying by $\frac{4}{5}$ and taking the square root, we get that $4a^2 + 4b^2 = \frac{4}{5} * \frac{845}{4} = 169 \Rightarrow \overline{AB} = \sqrt{4a^2 + 4b^2} = \sqrt{169} = 13$.

Lets look at the circle with diameter AB. Since D is on the circle with diameter AB, $\triangle ADB$ is a right triangle, with $\angle ADB$ being a right angle. But, since D is on AC, D must be the base of the altitude from B.

We want to find $\overline{AD}$. We let $\overline{AD}$ be x, and $\overline{BD}$ be y. By the Pythagorean theorem, $x^2 + y^2 = 13^2 = 169$, and $(22 - x)^2 + y^2 = 19^2 = 361$. Subtracting the first equation from the second, we obtain $(x^2 - 44x + 484) - x^2 = 361 - 169 = 192 \Rightarrow 484 - 44x = 192 \Rightarrow 44x = 292 \Rightarrow 11x = 73 \Rightarrow x = \overline{AD} = \boxed{\mathbf{\frac{73}{11}}}$.

Scrapped Problems

1. Find the sum of all K less than 500, such that the sum of all positive integers less than or equal to K is a perfect square.

   The sum of the first K positive integers is $\frac{K(K+1)}{2}$. For this to equal a square, say N^2, we must have $K(K+1) = 2N^2$. Notice that K and $K+1$ are consecutive integers, meaning that one of them is even, and the other is odd. For K and $K+1$ to be twice a perfect square, the even number must be twice a perfect square, and the odd one must be a perfect square. We then check all odd perfect squares less than 500. If the even number that is 1 less or 1 more than it is twice a perfect square, then that value of K that is a valid solution.

   $1^2 = 1$: $K = 0$: $\frac{0}{2} = 0$ is a perfect square, so $K = 0$ is a valid solution. $K = 1$: $\frac{2}{2} = 1$ is a perfect square, so $K = 1$ is a valid solution.

   $3^2 = 9$: $K = 8$: $\frac{8}{2} = 4$ is a perfect square, so $K = 8$ is a solution. $K = 9$: $\frac{10}{2} = 5$ is not a perfect square.

   $5^2 = 25$: $K = 24$ and $K = 25$: $\frac{24}{2} = 12$ and $\frac{26}{2} = 13$ are not perfect squares.

   $7^2 = 49$: $K = 48$: $\frac{48}{2} = 24$ is not a perfect square. $K = 49$: $\frac{50}{2} = 25$ is a perfect square, so $K = 49$ is a solution.

   $9^2 = 81$: $K = 80$ and $K = 81$: $\frac{80}{2} = 40$ and $\frac{82}{2} = 41$ are not perfect squares.

   $11^2 = 121$: $K = 120$ and $K = 122$: $\frac{120}{2} = 60$ and $\frac{122}{2} = 61$ are not perfect squares.

   $13^2 = 169$: $K = 168$ and $K = 169$: $\frac{168}{2} = 84$ and $\frac{170}{2} = 85$ are not perfect squares.

   $15^2 = 225$: $K = 224$ and $K = 225$: $\frac{224}{2} = 112$ and $\frac{226}{2} = 113$ are not perfect squares.

   $17^2 = 289$: $K = 288$: $\frac{288}{2} = 144$, which is a perfect square, so $K = 288$ is a valid solution. $K = 289$: $\frac{290}{2} = 145$, which is not a perfect square.

   $19^2 = 361$: $K = 360$ and $K = 361$: $\frac{360}{2} = 180$ and $\frac{362}{2} = 181$ are not perfect squares.

   $21^2 = 441$: $K = 440$ and $K = 441$: $\frac{440}{2} = 220$ and $\frac{442}{2} = 221$ are not perfect squares.

   $23^2 = 529 > 500$, so the sum of the solutions is $0 + 1 + 8 + 49 + 288 = \boxed{\mathbf{346}}$.

2. Let a, b, and c the roots of $x^3 - 6x^2 + 15x - 9 = 0$. Find $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}$.

   By Vieta's formulas, $a + b + c = 6$. Therefore, $b + c = 6 - a$, $a + c = 6 - b$, and $a + b = 6 - c$. We can then rewrite the desired expression as $\frac{a}{6-a} + \frac{b}{6-b} + \frac{c}{6-c}$. Notice that $\frac{a}{6-a} + 1 = \frac{6-a+a}{6-a} = \frac{6}{6-a} \Rightarrow \frac{a}{6-a} = \frac{6}{6-a} - 1$. We can then further manipulate our expression to $\frac{6}{6-a} + \frac{6}{6-b} + \frac{6}{6-c} - 3 = 6(\frac{1}{6-a} + \frac{1}{6-b} + \frac{1}{6-c}) - 3$. To find the expression inside the parenthesis, we find the polynomial with roots $\frac{1}{6-a}$, $\frac{1}{6-b}$, and $\frac{1}{6-c}$. Notice that the polynomial with roots $6-a, 6-b$, and $6-c$, is $(6-x)^3 - 6(6-x)^2 + 15(6-x) - 9 = -x^3 + 12x^2 - 51x + 81$, so the polynomial with roots $\frac{1}{6-a}$, $\frac{1}{6-b}$, and $\frac{1}{6-c}$ is $81x^3 - 51x^2 + 12x - 1$. By Vieta's, $\frac{1}{6-a} + \frac{1}{6-b} + \frac{1}{6-c} = \frac{51}{81} = \frac{17}{27}$. Plugging this into our previous expression, we find that the desired result is $6(\frac{17}{27}) - 3 = \frac{34}{9} - \frac{27}{9} = \boxed{\mathbf{\frac{7}{9}}}$.

3. Let n be the largest possible value of $x^4 + 8x^3 - 20x^2 + 8x + 1$, given that $x^2 - 3x + 1 = 0$. If n can be written as $\frac{a+b\sqrt{c}}{d}$, find $a - b + c^d$.

   Notice that this polynomial is symmetric, so we can rewrite it in terms of $x + \frac{1}{x}$. $x^4 + 8x^3 - 20x^2 + 8x + 1 = x^2(x^2 + 8x - 20 + \frac{8}{x} + \frac{1}{x^2})$. $x^2 + \frac{1}{x^2} = (x + \frac{1}{x})^2 - 2$, so we can further manipulate the expression to get $x^2((x + \frac{1}{x})^2 + 8(x + \frac{1}{x}) - 22)$. The given expression can be rewritten as $x^2 + 1 = 3x$. Dividing both sides by x, we get that $x + \frac{1}{x} = 3$. Plugging this back into our desired expression, we find that it is equal to $x^2(3^2 + 8 * 3 - 22) = 11x^2$. To find the largest possible value, we simply have to maximize x^2. By using the quadratic formula on $x^2 - 3x + 1 = 0$, we find that the value of x with larger absolute value is $\frac{3+\sqrt{5}}{2}$. Thus, $x^2 = \frac{14+6\sqrt{5}}{4} = \frac{7+3\sqrt{5}}{2}$, and $11x^2 = \frac{77+33\sqrt{5}}{2}$. The final answer is then $77 - 33 + 5^2 = 44 + 25 = \boxed{\mathbf{69}}$.

4. Determine the number of ordered pairs of relatively prime positive integers (a, b) for which $\frac{a}{b} + \frac{14b}{9a}$ is an integer.

   Rewrite the expression as $\frac{9a^2}{9ab} + \frac{14b^2}{9ab} = \frac{9a^2+14b^2}{9ab}$. Since a divides $9a^2$, for the expression to be an integer, a must divide $14b^2$ as well. Since a and b are relatively prime, a cannot divide b, so a must divide 14. Therefore, a is one of $(1, 2, 7, 14)$. Similarily, for the expression to be an integer, b must divide $9a^2$. Since b cannot divide a^2, b must divide 9. Furthermore, since 9 divides $9a^2$, 9 must divide $14b^2$ as well. Since 14 and 9 are relatively prime, 9 must divide b^2. Combining these two observations, we see that b must be either 3 or 9. We do casework on the value of b.

   $b = 3$:
   $a = 1$: $\frac{1}{3} + \frac{14*3}{9} = \frac{45}{9} = 5$, which is a integer.
   $a = 2$: $\frac{2}{3} + \frac{14*3}{18} = \frac{2}{3} + \frac{7}{3} = \frac{9}{3} = 3$, which is a integer.
   $a = 7$: $\frac{7}{3} + \frac{14*3}{63} = \frac{49}{21} + \frac{14}{21} = \frac{63}{21} = 3$, which is a integer.
   $a = 14$: $\frac{14}{3} + \frac{14*3}{126} = \frac{14}{3} + \frac{1}{3} = \frac{15}{3} = 5$, which is a integer.

   $b = 9$:
   $a = 1$: $\frac{1}{9} + \frac{126}{9}$, which is not an integer.
   $a = 2$: $\frac{2}{9} + \frac{126}{18}$, which is not an integer.
   $a = 7$: $\frac{7}{9} + \frac{126}{63}$, which is not an integer.
   $a = 14$: $\frac{14}{9} + \frac{126}{126}$, which is not an integer.

   There are a total of $\boxed{\mathbf{4}}$ relatively prime integer pairs (a, b) that produce an integer value.