Quantum Galton Board — Comprehensive Summary

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1 Introduction

The **Galton Board** is a classical device that demonstrates the binomial distribution through a physical random walk: balls fall through an array of pegs, being deflected left or right with equal or fixed bias probabilities, accumulating into a distribution at the bottom. The **Quantum Galton Board** (QGB) adapts this concept into the quantum domain by replacing classical probabilistic decisions with quantum superposition and entanglement. A single quantum circuit can encode all possible trajectories simultaneously, providing potential exponential sampling advantages.

2 Uses and Relevance

QGBs are useful for:

- Statistical simulation and random sampling.
- Quantum-enhanced modelling of random walks.
- Generating biased or custom probability distributions for finance, physics, and machine learning tasks.
- Exploring resource-efficient universal statistical simulators.

3 Relation to Statistical Simulation

In classical simulations, each trial corresponds to one trajectory through the pegs. In QGB, superposition allows representation of 2^n trajectories for n levels. Measurement collapses the state to a single outcome, and repeating this process reconstructs the probability distribution. This mirrors a classical Monte Carlo process but can offer quantum parallelism.

4 Unbiased Quantum Circuit Design

An unbiased QGB peg is implemented by applying a Hadamard gate to a control qubit, creating equal amplitudes for left/right paths. Controlled-SWAP (Fredkin) gates then route the "ball" qubit accordingly. After each peg, the control qubit is reset for reuse. This pattern repeats for all levels.

4.1 Mathematical Basis

For n unbiased pegs, the probability of k right turns is given by:

$$P(k) = \frac{1}{2^n}\binom{n}{k}$$

This yields a symmetric distribution centred at $n/2$.

4.2 Simulation and Post-Processing

Simulations with Qiskit's AerSimulator often use 40,000 shots, grouping outcomes into bins of 8 to smooth the histogram and approximate a continuous normal curve. This aggregation reduces statistical fluctuations and improves visual comparison to theoretical predictions.

5 Biased Quantum Circuit

Bias is introduced by replacing the Hadamard with $R_x(\theta)$ gates:

$$R_x(\theta)|0\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle - i\sin\left(\frac{\theta}{2}\right)|1\rangle$$

The probability of a left turn is $p = \cos^2(\theta/2)$, giving:

$$\theta = 2\arccos(\sqrt{p})$$

The function `find_theta_values` computes peg-specific angles to produce desired distributions, including exponential and skewed forms.

6 Noise Model

IBM noise models, generated via `NoiseModel.from_backend`, enable realistic noisy simulations. Multiple CSWAPs increase depth, leading to fidelity loss in hardware runs compared to ideal simulations.

7 Distribution Comparison: Wasserstein Distance

The Wasserstein distance (Earth Mover's Distance) quantifies how close QGB output distributions are to classical targets, considering both probability differences and outcome positions. It is valuable for:

- Evaluating hardware vs ideal outputs.
- Comparing optimization and noise mitigation strategies.
- Guiding parameter tuning to minimize distributional divergence.

8 Bootstrap Method for Stochastic Consideration

Bootstrap resampling can estimate uncertainty in QGB output statistics. By repeatedly resampling measurement results with replacement, one can build empirical distributions of summary statistics (e.g., mean bin index, Wasserstein distance). From these, 95%

9 Optimization Strategies

- **Transpilation:** High optimisation levels and custom layouts reduce SWAP overhead.
- **Gate decomposition:** Replace CSWAPs with hardware-native equivalents.
- **Mid-circuit reset:** Reuse control qubits to reduce total qubit count.
- **Hardware acceleration:** GPU-accelerated simulation for large-shot experiments is very useful. **Note:** The code doesn't use GPU, due to Hardware unavailability

10 Conclusion

The Quantum Galton Board showcases how quantum circuits can replicate and extend classical statistical processes. By incorporating unbiased and biased designs, precise parameter tuning, realistic noise models, quantitative comparison metrics like Wasserstein distance, and statistical tools such as bootstrap confidence intervals, QGBs become powerful testbeds for exploring quantum statistical behaviour and potential computational advantages.