Unit-1
Linear Differential Equations with Constant Coefficients

Second Year Engineering-Mech
Engineering Mathematics-III

SPPU DPU DIT

Linear Differential Equations with Constant Coefficients

Introduction to LDE

General Solution of LDE

Method of Finding Complimentary Function

Content

- Pre-requisites
- Introduction to LDE
- General Solution of LDE
- Method of finding Complimentary Function

Pre-requisites : To Find Roots of the Equation

Method 1: Find the roots of equation **Using Calculator**

<u>Quadratic equation</u>:

$$ax^2 + bx + c = 0,$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

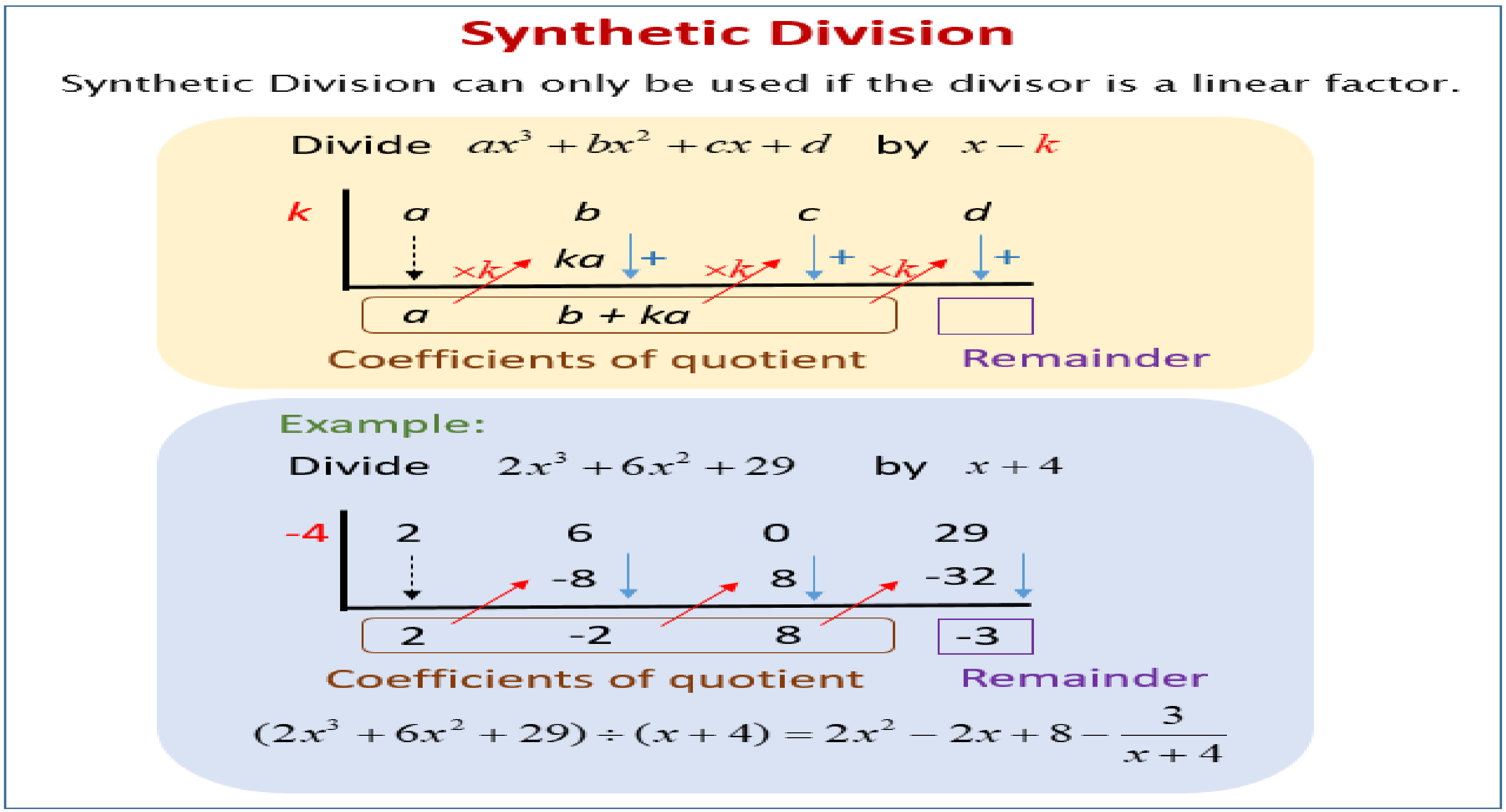
For higher order equations ,Use

-Synthetic division Method

- Factorization Method

Synthetic Division

- Make sure that equation is complete with respect to all powers
- Place coefficients in descending order on the inside.



Prerequisites:

- Find the roots of the following equation
- Quadratic equation (Second degree Polynomials)
- 1. $x^2 - 5x - 6 = 0$ 2. $x^2 - 4x + 4 = 0$ 3. $x^2 + x + 1 = 0$

- Cubic or Third Degree Polynomials :

1. $x^3 - 2x + 4 = 0$ 2. $x^3 + x^2 - 2x + 12 = 0$

3. $x^3 - 3x^2 + 3x - 1 = 0$ 4. $x^3 + 8 = 0$

- Fourth Degree Polynomials and their Factorization :

1 $x^4 + 8x^2 + 16 = 0$ 2. $x^4 + 2x^3 + 3x^2 + 2x + 1 = 0$

3 $x^4 + 6x^2 + 25 = 0$ 4. $x^4 - 13x^2 + 36 = 0$

Prerequisites:

Some Integration Formulas	
$1.\int dx = x + c$	$7.\int \sin x dx = -\cos x + c$
$2.\int a dx = ax + c$	$8.\int \cos x dx = \sin x + c$
$3.\int x^n dx = \frac{x^{n+1}}{n+1} + c, (n \neq 1)$	$9.\int \sec x \tan x dx = \sec x + c$
$4.\int e^x dx = e^x + c$	$10.\int \cos ec x \cot x dx = -\cos ec x + c$
$5.\int a^x dx = \frac{a^x}{\log a} + c$	$11.\int \sec^2 x dx = \tan x + c$
$6.\int \frac{1}{x} dx = \log x + c$	$12.\int \cos ec^2 x dx = -\cot x + c$

Prerequisites:

Integration Formulae for Some Specific Functions

Integration By Parts : $\int uv\,dx = u\int v\,dx - \int u'\left(\int v\,dx\right)dx$

$$\int e^x\left(f(x)+f'(x)\right)dx = e^x \cdot f(x) + c$$

eg: $\int e^x(\tan x + \sec^2 x)dx = e^x \tan x + c$

$$\int e^{f(x)} f'(x)dx = e^{f(x)}$$

Eg: $\int e^{e^x} e^x dx = e^{e^x} + c$

Prerequisites:

Integration Formulae for Some Specific Functions

$$\int \frac{f'(x)}{f(x)}\,dx = \log[f(x)] + c$$

Eg: $$\int 3\frac{\cos 3x}{1+\sin 3x}\,dx = \log(1+\sin 3x) + c$$

$$\int \frac{p(x)}{q(x)}\,dx$$

Case1 :Degree P(x) > Degree of Q(x) : Actual division Method

Case2 :Degree P(x) > Degree of Q(x) : Partial Fraction Method

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx] + c$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx] + c$$

Learning Outcomes

At the end of this topic students will be able to :

- Find out Complementary Function for the given Linear Differential Equation

Linear Differential Equation with constant coefficients

Definition :

A differential equation which contains the dependent variable and its derivatives in the first degree only and does not contain the product of dependent variable with any of its derivative or product of two different order derivatives is called linear differential equation

The general form of n th order linear differential equation of order 'n' is

$$\boxed{a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots. + a_{n-1} \frac{dy}{dx} + a_n y = Q(x)} \quad \text{--------(1)}$$

where a_i's are constants.

General Form

Equation (1) can be written in the operator form as

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \dots.. + a_{n-1}D + a_n)y = Q(x) \qquad \dots..(2)$$

where $\quad D^r = \dfrac{d^r}{dx^r}$

By Putting $\; F(D) = a_0D^n + a_1D^{n-1} + a_2D^{n-2} + a_3D^{n-3} + \dots. + a_{n-1}D + a_n$

Equation (2) reduces to $\; F(D)y = Q(x) \quad \dots\dots(3)$

To find General Solution

Consider the DE $\frac{dy}{dx} + py = Q(x)$

Let p=m(a constant)

Integrating Factor of the diff. eqn $e^{\int p\,dx}$ is e^{mx}

General Solution is $ye^{mx} = \int Qe^{mx}\,dx + c$

$$y = e^{-mx}\int Qe^{mx}\,dx + ce^{-mx}$$

$$y = \{ce^{-mx}\} + \{e^{-mx}\int Qe^{mx}\,dx\}$$

$$y = \{CF\} + \{PI\}$$

General Solution

- The general solution of the equation $F(D)y = Q(x)$ can be written as

General Solution(y) =

Complimentary Function (y_c)+Particular Integral (y_p),

CF + PI = GS

Complimentary function(y_c)

is defined as the solution of the differential equation $F(D)y = 0$

[C.F involves ' n' arbitrary constants (number arbitrary constants is equal to order of differential equation)]

The Particular integral(y_p) is defined as $y_p = \frac{1}{F(D)} Q(x)$

[P.I does not contain any arbitrary constants]

To find CF

Consider the differential equation $a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = Q(x)$

Let y= e ^{mx} be the solution of the given equation

$$F(D)y = 0, \qquad (a_0 D^2 + a_1 D + a_2)y = 0,$$

Substituting the values of y, y',y'', in $F(D)y = 0,$ we get

$(a_0 m^2 + a_1 m + a_2)e^{mx} = 0$, Since e $^{mx} \neq 0$,for all 'x'

$$(a_0 m^2 + a_1 m + a_2) = 0 \qquad F(m) = 0$$

To find CF

The algebraic equation $F(m)=0$ is called as Auxiliary equation or Characteristic equation.
(This can be obtained by replacing 'D by m)

The values of 'm' obtained are called as characteristic roots.
Let the roots be $m_1, m_2, m_3 \ldots.$

Corresponding solutions are $c_1 e^{m_1 x}, c_2 e^{m_2 x}, c_3 e^{m_3 x} etc\ldots$
Since equation is linear , General solution can be written as linear combination of these solutions

$$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + --$$

To find CF

Case-1 If the roots $m_1, m_2, m_3 \ldots$ are real and distinct ,

then C.F is written as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + - - + c_r e^{m_r x}$$

Case -2: In case of repeated roots,

Let 'm' be the root repeated r times ,then C.F is written as

$$y = (c_1 + c_2 x + c_3 x^2 + - - - - + c_r x^{r-1}) e^{mx}$$

To find CF

Case 3: Roots are Complex Conjugates, say m_1, m_2

$$m_1 = \alpha + i\beta \qquad m_2 = \alpha - i\beta$$

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} \qquad y_c = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

$$y_c = e^{\alpha x}(c_1 e^{(i\beta)x} + c_2 e^{(-i\beta)x})$$

$$y_c = e^{\alpha x}(c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x))$$

$$y_c = e^{\alpha x}[(c_1 + ic_2)\cos\beta x + (c_1 - ic_2)\sin\beta x)$$

$$\boxed{y_c = e^{\alpha x}[A\cos\beta x + B\sin\beta x]}$$

To find CF

Case 4:

Roots are repeated Complex Conjugates,

say m_1, m_1 , m_2, m_2, where

$m_1 = \alpha + i\beta$, and $m_2 = \alpha - i\beta$

Then Complimentary Function is written as

$$y_c = e^{\alpha x}[(A_1 x + A_2)\cos \beta x + (B_1 x + B_2)\sin \beta x]$$

Steps to find Complimentary function

Write the differential equation in the operator form

$$F(D)y = Q(x)$$

Form the auxiliary equation/Characteristic Equation F(m) = 0

(replacing D→m in F(D))

Solve the auxiliary equation

Find the roots say m_1, m_2,….,m_n ,Identify nature of roots (real .complex)

Formula for finding C.F.

Nature of roots	Complimentary Function
Roots are real & distinct say $m_1, m_2, \ldots\ldots m_n$	$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots c_n e^{m_n x}$
Roots are real & repeated say $m_1 = m_2$ & $m_3, m_4, \ldots\ldots m_n$	$y_c = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_1 x}$
Roots are complex conjugates $m_1 = \alpha + i\beta \quad m_2 = \alpha - i\beta$	$y_c = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$
Roots are complex and repeated $m_1 = m_2 = \alpha + i\beta$ $m_3 = m_4 = \alpha - i\beta$	$y_c = e^{\alpha x}((c_1 x + c_2)\cos \beta x + (c_3 x + c_4)\sin \beta x)$

Second order linear differential equation

Consider second order linear differential equation

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = Q(x)$$

Step1: Write the differential equation in the operator form

$$(a_0 D^2 + a_1 D + a_2) y = Q(x)$$

Step2 : Form the Auxiliary Equation

F(m)=0 $\Rightarrow$ $a_0 m^2 + a_1 m + a_2 = 0$

Step3: Find the roots $m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Second order linear differential equation

Step 4: Apply appropriate formula as per the nature of roots

Case1: Roots are real and distinct say m_1 and m_2

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Case 2: Roots are real and repeated say m_1,m_1

$$y_c = (c_1 + c_2 x) e^{m_1 x}$$

Case 3: Roots are complex conjugates

$$m_1 = \alpha + i\beta \quad m_2 = \alpha - i\beta$$

$$y_c = e^{\alpha x}[A\cos\beta x + B\ \sin\beta x]$$

Case-1 : Roots are real and Distinct

Q1. Solve the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0$

find C.F

$D^2y - 5Dy - 6y = 0$ put $\frac{d}{dx} = D$

Step1

- **Write the given diff. equation in operator form F(D)y=0**
- Eq ^{n} is $(D^2 - 5D - 6)y = 0$

$\phi(D)y = 0$

Step2

$\phi(D) = D^2 - 5D - 6$

- Form Auxiliary Eq ^{n} F(m)=0 $\quad m^2 - 5m - 6 = 0$

A.E $\phi(m) = m^2 - 5m - 6 = 0 \quad (m-6)(m+1)=0$

Step3

- Solve and find roots
- (m - 6) (m + 1) = 0 ; m = -1, 6 **Roots are real and distinct**

find roots : 6, -1 (real and distinct)

Step4

- C.F is $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ $\quad y_c = c_1 e^{-x} + c_2 e^{6x}$

$y_c = c_1 e^{6x} + c_2 e^{-x}$

Case-1 : Roots are real and Distinct

Q2. Solve the differential equation $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 10y = 0$

Step1
- **Write the given diff. equation in operator form F(D)y=0**
- Eq n is $(2D^2 - D - 10)y = 0$

Step2
- Form Auxiliary Eq n F(m)=0 $2m^2 - m - 10 = 0$

$\frac{5}{2}, \quad -2$ real & distinct

Step3
- Solve and find roots
- (m +2) (m –(5/2)) = 0 m = -2, 5/2 **Roots are real and distinct**

Step4
- C.F is $y_c = c_1e^{m_1x} + c_2e^{m_2x}$ $y_c = c_1e^{-2x} + c_2e^{(5/2)x}$

$y_c = c_1 e^{\frac{5}{2}x} + c_2 e^{-2x}$

Case-2 : Roots are real and repeated

Q3 :Solve the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

Step1

- **Write the given diff. equation in operator form F(D)y=0**
- Eq ^{n} is $(D^2 - 4D + 4)y = 0$

Step2

- Form Auxiliary Eq ^{n} F(m)=0 $m^2 - 4m + 4 = 0$

Step3

- Solve and find roots
- $(m - 2)^2 = 0$ m = 2,2, **Roots are real and repeated**

Step4

- C.F is $y_c = (c_1 + c_2 x)e^{2x}$

Case-2 : Roots are real and repeated

Q4 :Solve the differential equation $y''+6y'+9y=0$

Step1

- **Operator form F(D)y=0** $(D^2+6D+9)y=0$

Step2

- Form Auxiliary Eq ^{n} F(m)=0 $(m+3)^2=0$

Step3

- Solve and find roots
- $(m + 3)^2 = 0$ m = -3,-3, **Roots are real and repeated**

Step4

- C.F is $y_c=(c_1+c_2x)e^{-3x}$

Case-3 : Roots are Complex numbers

Q5 :Solve the differential equation $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+10y=0$

Step1

- **Write the given diff. equation in operator form F(D)y=0**
- Eq ^{n} is $(D^2+6D+10)y=0$

Step2

- Form Auxiliary Eq ^{n} F(m)=0 $m^2+6m+10=0$

Step3

- Solve and find roots ,m= $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $m=-3\pm i4$

Step4

- Roots are Complex conjugates $y_c=e^{-3x}(c_1\cos 4x+c_2\sin 4x)$

Case-3 : Roots are Complex numbers

Q6 :Solve the differential equation $4\frac{d^2y}{dx^2}+9y=0$

Step1

- **Write the given diff. equation in operator form F(D)y=0**
- Eq ^{n} is $(4D^2+9)y=0$

Step2

- Form Auxiliary Eq ^{n} F(m)=0 $4m^2+9=0$

Step3

- Solve and find roots ,m= $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $m=0\pm i(3/2)$

Step4

- Roots are Complex conjugates $y_c=e^{0x}(c_1\cos(3/2)x+c_2\sin(3/2)x)$

Case-4 : Roots are Complex & repeated

Q7 :Solve the differential equation $(D^2+2D+5)^2 y=0$

Step1

- **Write the given diff. equation in operator form F(D)y=0**
- Eq ^{n} is $(D^2+2D+5)^2 y=0$

Step2

- Form Auxiliary Eq ^{n} F(m)=0 $(m^2+2m+5)^2=0$

Step3

- Solve and find roots ,m= $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ $m=-1\pm 2i,-1\pm 2i$

Step4

Roots are repeated complex conjugates $y_c=e^{-x}((A_1x+A_2)\cos 2x+(A_3x+A_4)\sin 2x)$

Third order differential equation (Real roots)

Q1 :Solve the differential equation $(D^3+6D^2+11D+6)y=0$

Step1
- **Write the given diff. equation in operator form F(D)y=0**
- Eq ^{n} is $(D^3+6D^2+11D+6)y=0$

Step2
- Form Auxiliary Eq ^{n} F(m)=0 $(m^3+6m^2+11m+6)=0$

Step3
- Solve and find roots ,m= -1,m= -2, m= -3 (use synthetic division)

Step4
- Roots are real and distinct $y=c_1e^{-x}+c_2e^{-2x}c_3e^{-3x}$

Third order differential equation (Repeated Real roots)

Q.2) Find the solution of $(D^3+3D^2+3D+1)y=0$

Step1

Given $(D^3+3D^2+3D+1)y=0$

Step2

Aux.Eq is $m^3+3m^2+3m+1=0$

Step3

Aux.Eq is $(m+1)^3=0$, m=-1,-1,-1

Step4

$$y=(c_1x^2+c_2x+c_3)e^{-x}$$

Third order differential equation
(Real and Complex Conjugates)

Q.3) Find the solution of $\left(D^3+2D^2+4D+3\right)y=0$

Given $D^3+2D^2+4D+3=0$ Aux.Eq is $m^3+2m^2+4m+3=0$

Since m=-1 is one real roots of the equation. By synthetic division

$$\begin{array}{r|rrrr} -1 & 1 & 2 & 4 & 3 \\ & & -1 & -1 & -3 \\ \hline & 1 & 1 & 3 & 0 \end{array}$$

$$(m+1)\left(m^2+m+3\right)=0$$

$m=-1, \frac{-1}{2} \pm \frac{\sqrt{11}}{2} i$ one root is real and other roots are complex conjugates

$$y=y_c=c_1e^{-x}+e^{\frac{-1}{2}x}\left(c_2\cos\frac{\sqrt{11}}{2}x+c_3\sin\frac{\sqrt{11}}{2}x\right)$$

Fourth order differential equation (Factorization Method)

Q.1) Find the solution of $(D^4 - a^4)y = 0$

Step1.Operator form

Given $(D^4 - a^4)y = 0$

Step2.Aux .Eqn

A.E is $D^4 - a^4 = 0$

Step3.

Solve and Find Ch.Roots

$$(m^2 - a^2)(m^2 + a^2) = 0$$
$$(m - a)(m + a)(m^2 + a^2) = 0$$
$$m = -a, a, \pm ia$$

Two roots are real and two roots are complex

Step4. CF

$$y = c_1 e^{-ax} + c_2 e^{ax} + c_3 \cos ax + c_4 \sin ax$$

Fourth order differential equation (Biquadratic equation)

Q.2) Find the solution of $(D^4 + 8D^2 + 16)y = 0$

Step1

Given $(D^4 + 8D^2 + 16)y = 0$

Step2

A.E is $m^4 + 8m^2 + 16 = 0$

Step3

$$(m^2 + 4)^2 = 0$$

$$m^2 = -4, -4$$

$$m = \pm 2i, \pm 2i$$

Roots are complex and repeated twice,

Step4

$$y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$$

Fourth order differential equation (Factorization Method)

Q.3) Find the solution of $\left(D^4 - 2D^3 + D^2\right)y = 0$

Step1

Given $\left(D^4 - 2D^3 + D^2\right)y = 0$

Step2

A.E is $D^2\left(D^2 - 2D + 1\right) = 0$

Step3

$D^2(D-1)^2 = 0$

$D = 0,0,1,1$

Roots are real and repeated

Step4

$y = c_1 + c_2 x + (c_3 + c_4 x)e^x$

Higher order differential equations

Q.1 Find the solution of $(D+2)(D^4+6D^2+9)y=0$

Step1

Given $(D+2)(D^4+6D^2+9)y=0$

Step2

A.E is $(m+2)(m^4+6m^2+9)=0$

Step3

$$(m+2)(m^4+6m^2+9)=0$$

$$m=-2$$

&

$$(m^2+3)^2=0$$

$$m=\pm\sqrt{3}i, \pm\sqrt{3}i$$

Roots are real and imaginary and repeated twice

Step4

$$y=c_1e^{-2x}+(c_2+c_3x)\cos\sqrt{3}x+(c_4+c_5x)\sin\sqrt{3}x$$

Higher order differential equations

Q.2 Find the solution of $\left(D^6+2D^4+D^2\right)y=0$

Step1

Given $\left(D^6+2D^4+D^2\right)y=0$

Step2

A.E is $m^6+2m^4+m^2=0$

Step3

$$m^2\left(m^4+2m^2+1\right)=0$$

$$m^2\left(m^2+1\right)^2=0$$

$$m=0.0\pm i,\pm i$$

Roots are real and imaginary and also they are repeated,

Step4

$$y=\left(c_1+c_2x\right)+\left(c_3+c_4x\right)\cos x+\left(c_5+c_6x\right)\sin x$$

Particular Solution

Solve the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0, x(0) = 2, x'(0) = 0$$

Solution:

Operator Form $(D^2 + 2D + 5) = 0$

Characteristic Equation $(m^2 + 2m + 5) = 0$

Characteristic roots $m = -1 \pm 2i$

General Solution $x(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$

Particular Solution

Substituting the initial conditions, we get

$$x(0) = 2, x'(0) = 0$$

$$x(0) = 2 \Rightarrow 2 = c_1$$

$$x'(t) = -e^{-t}(c_1 \cos 2t + c_2 \sin 2t) + e^{-t}(c_1(-2\sin 2t) + 2(c_2)\cos 2t)$$

$$x'(0) = 0 \rightarrow -c_1 + 2c_2 = 0 \Rightarrow c_2 = 1$$

$$x(t) \quad = e^{-t}(2\cos 2t + \sin 2t)$$

Particular Solution

Q2:Find Particular solution of the differential equation

$$\frac{d^2s}{dt^2} = -16\frac{ds}{dt} - 64s, s = 0, \frac{ds}{dt} = -4, at, t = 0$$

Solution $\quad \frac{d^2s}{dt^2} + 16\frac{ds}{dt} + 64s = 0 \longrightarrow (D^2 + 16D + 64)s = 0$

Ch.eqn is $\quad m^2 + 16m + 64 = 0 \longrightarrow (m+8)^2 = 0, m = -8, -8$

Ch. Roots are real and repeated $\quad s = (c_1 t + c_2)e^{-8t}$

Particular Solution

$$s = (c_1 t + c_2)e^{-8t}$$

Substituting initial conditions , $s = 0, \dfrac{ds}{dt} = -4, at, t = 0$

putting S=0 at t=0 ,we get , $0 = c_2$

$$\frac{ds}{dt} = c_1 e^{-8t} - 8c_1 t e^{-8t}$$

Putting $\dfrac{ds}{dt} = -4,$ at 't'=0 , we get $-4 = c_1$

$c_1 = -4$ $c_2 = 0$ $s = -4te^{-8t}$

Complementary Functions-MCQ's

1. The solution of the differential equation $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$ is

A) $c_1e^{2x}+c_2e^{-3x}$ B) $c_1e^{-2x}+c_2e^{3x}$ C) $c_1e^{-2x}+c_2e^{-3x}$ D) $c_1e^{2x}+c_2e^{3x}$

2. The solution of the differential equation $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$ is

A) $c_1e^{-2x}+c_2e^{2x}$ B) $c_1e^{2x}+c_2e^{2x}$

C) $(c_1x+c_2)e^{-2x}$ D) $c_1e^{3x}+c_2e^{x}$

3. The solution of the differential equation $\frac{d^2y}{dx^2}+9y=0$ is

A) $c_1\cos 3x+c_2\sin 3x$ B) $(c_1x+c_2)e^{-3x}$

C) $c_1e^{3x}+c_2e^{-3x}$ D) $c_1\cos 2x+c_2\sin 2x$

Complementary Functions-MCQ's

4. The solution of the differential equation $\frac{d^6y}{dx^6}+6\frac{d^4y}{dx^4}+9\frac{d^2y}{dx^2}=0$ is

A) $c_1x+c_2+(c_3x+c_4)\cos\sqrt{3}x+(c_3x+c_4)\sin\sqrt{3}x$

B) $(c_1x+c_2)\cos\sqrt{3}x+(c_3x+c_4)\sin\sqrt{3}x$

C) $c_1x+c_2+(c_3x+c_4)\cos 3x+(c_3x+c_4)\sin 3x$

D) $c_1x+c_2+(c_3x+c_4)e^{\sqrt{3}x}$

5. The solution of the differential equation $\frac{d^4y}{dx^4}-y=0$ is

A) $c_1e^{x}+c_2e^{-x}+c_3\cos x+c_4\sin x$

B) $(c_1+c_2x+c_3x^2+c_4x^3)e^{x}$

C) $(c_1x+c_2)\cos x+(c_3x+c_4)\sin x$

D) $(c_1x+c_2)e^{-x}+c_3\cos x+c_4\sin x$

Practice Problems

Solve the following differential equations

Niral

Textbook

B S Grewal

B V Ramana

Higher Engg. Maths.

References

1> AEM by Kreizig

2> AEM by P O Neil

1. $4y'' - 8y' + 7y = 0$

2. $(D^3 + 6D^2 + 11D + 6)y = 0$

3. $(D^4 + a^4)y = 0$

4. $(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0$

Practice Problems

Q6. An electric circuit consists of an inductance 0.1 henry , a resistance R of 20 ohms and a condenser of capacitance ‘C of 25 microfarads. If the differential equation of electric circuit is

$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{c}=0$ then find the charge ‘q’ and current ‘ i’ at any time ‘t’, given that at t=0,q=0.05 coulombs, i= dq/dt=0,when t=0

Q7.The equation for bending of a strut is $EI\frac{d^2y}{dx^2}+py=0$, If

y=0 when x=0and y=a when x= l/2, find ‘y’

MCQ's

1. If the roots $m_1, m_2, m_3, \ldots, m_n$ of auxiliary equation $\phi(D) = 0$ are real and distinct, then solution of $\phi(D)\,y = 0$ is (1)

   (A) $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$
   (B) $c_1 \cos m_1 x + c_2 \cos m_2 x + \ldots + c_n \cos m_n x$
   (C) $m_1 e^{c_1 x} + m_2 e^{c_2 x} + \ldots + m_n{}^{c_n x}$
   (D) $c_1 \sin m_1 x + c_2 \sin m_2 x + \ldots + c_n \sin m_n x$

2. The roots $m_1, m_2, m_3 \ldots, m_n$ of auxiliary equation $\phi(D) = 0$ are real. If two of these roots are repeated say $m_1 = m_2$ and the remaining roots $m_3, m_4, \ldots, m_n$ are distinct then solution of $\phi(D)\,y = 0$ is (1)

   (A) $c_1 e^{m_1 x} + c_2{}^{m_2 x} + \ldots + c_n{}^{m_n x}$
   (B) $(c_1 x + c_2) \cos m_1 x + c_3 \cos m_3 x + x \ldots + c_n \cos m_n x$
   (C) $(c_1 x + c_2)\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$
   (D) $(c_1 x + c_2) \sin m_1 x + c_3 \sin m_3 x + \ldots + c_n \sin m_n x$

3. The roots $m_1, m_2, m_3 \ldots, m_n$ of auxiliary equation $\phi(D) = 0$ are real. If three of these roots are repeated, say, $m_1 = m_2 = m_3$ and the remaining roots $m_4, m_5, \ldots m_n$ are distinct then solution of $\phi(D)\,y = 0$ is (1)

   (A) $c_1 e^{m_1 x} + c_2{}^{m_2 x} + \ldots + c_n\, e^{m_n x}$
   (B) $(c_1 x^2 + c_2 x + c_3)\, e^{m_1 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$
   (C) $(c_1 x^2 + c_2 x + c_3) \cos m_1 x + c_4 \cos m_4 x + \ldots + c_n \cos m_n x$
   (D) $(c_1 x^2 + c_2 x + c_3) \sin m_1 x + c_4 \sin m_4 x + \ldots + c_n \sin m_n x$

4. If $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ are two complex roots of auxiliary equation of second order DE $\phi(D)\,y = 0$ then it's solution is (1)

   (A) $e^{\beta x} [c_1 \cos \alpha x + c_2 \sin \alpha x]$
   (B) $e^{\alpha x} [(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x]$
   (C) $c_1 e^{\alpha x} + c_2 e^{\beta x}$
   (D) $e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]$

5. If the complex roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ of auxiliary equation of fourth order DE $\phi(D)\,y = 0$ are repeated twice then it's solution is (1)

   (A) $e^{\beta x} [c_1 \cos \alpha x + c_2 \sin \alpha x]$
   (B) $e^{\alpha x} [(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x]$
   (C) $(c_1 x + c_2)\, e^{\alpha x} + (c_3 x + c_4)\, e^{\beta x}$
   (D) $e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]$

6. The solution of differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ is (1)

(A) $c_1e^{2x} + c_2e^{-3x}$
(B) $c_1e^{-2x} + c_2e^{3x}$
(C) $c_1e^{-2x} + c_2e^{-3x}$
(D) $c_1e^{2x} + c_2e^{3x}$

7. The solution of differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0$ is (1)

(A) $c_1e^{-x} + c_2e^{6x}$
(B) $c_1e^{-2x} + c_2e^{-3x}$
(C) $c_1e^{3x} + c_2e^{2x}$
(D) $c_1e^{-3x} + c_2e^{-2x}$

8. The solution of differential equation $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 10y = 0$ is (1)

(A) $c_1e^{2x} + c_2e^{\frac{5}{2}x}$
(B) $c_1e^{-2x} + c_2e^{-\frac{5}{2}x}$
(C) $c_1e^{-2x} + c_2e^{\frac{5}{2}x}$
(D) $c_1e^{-2x} + c_2e^{\frac{3}{2}x}$

9. The solution of differential equation $\frac{d^2y}{dx^2} - 4y = 0$ is (1)

(A) $(c_1x + c_2)\,e^{2x}$
(B) $c_1e^{4x} + c_2e^{-4x}$
(C) $c_1 \cos 2x + c_2 \sin 2x$
(D) $c_1e^{2x} + c_2e^{-2x}$

10. The solution of differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$ is (1)

(A) $c_1e^{2x} + c_2e^{x}$
(B) $c_1e^{2x} + c_2e^{-x}$
(C) $c_1e^{-2x} + c_2e^{x}$
(D) $c_1e^{-2x} + c_2e^{-x}$

11. The solution of differential equation $2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = 0$ is (1)

(A) $c_1e^{x} + c_2e^{\frac{3}{2}x}$

(B) $c_1e^{2x} + c_2e^{-3x}$

(C) $c_1e^{-x} + c_2e^{\frac{3}{2}x}$

(D) $c_1e^{\frac{x}{2}} + c_2e^{\frac{3}{2}x}$

12. The solution of differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ is (1)

(A) $c_1e^{2x} + c_2e^{x}$

(B) $c_1e^{x} + c_2e^{-x}$

(C) $(c_1x + c_2)\, e^{-x}$

(D) $(c_1x + c_2)\, e^{x}$

13. The solution of differential equation $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$ is (1)

(A) $c_1e^{\frac{x}{2}} + c_2e^{-\frac{x}{2}}$

(B) $(c_1 + c_2x)\, e^{-2x}$

(C) $c_1 \cos 2x + c_2 \sin 2x$

(D) $(c_1 + c_2x)\, e^{\frac{x}{2}}$

14. The solution of differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ is (1)

(A) $(c_1x + c_2)\, e^{2x}$

(B) $(c_1x + c_2)\, e^{-2x}$

(C) $c_1e^{4x} + c_2e^{-4x}$

(D) $c_1e^{2x} + c_2e^{-2x}$

15. The solution of differential equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$ is (1)

(A) $c_1e^{-6x} + c_2e^{-9x}$

(B) $(c_1x + c_2)\, e^{-3x}$

(C) $(c_1x + c_2)\, e^{3x}$

(D) $c_1e^{3x} + c_2e^{2x}$

16. The solution of differential equation $\frac{d^2y}{dx^2} + y = 0$ is (1)

(A) $c_1e^x + c_2e^{-x}$

(B) $(c_1x + c_2)\, e^{-x}$

(C) $c_1 \cos x + c_2 \sin x$

(D) $e^x (c_1 \cos x + c_2 \sin x)$

17. The solution of differential equation $\frac{d^2y}{dx^2} + 9y = 0$ is (1)

(A) $c_1 \cos 2x + c_2 \sin 2x$

(B) $(c_1x + c_2)\, e^{-3x}$

(C) $c_1e^{3x} + c_2e^{-3x}$

(D) $c_1 \cos 3x + c_2 \sin 3x$

18. The solution of differential equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 0$ is (1)

(A) $e^{-3x} (c_1 \cos x + c_2 \sin x)$

(B) $e^x (c_1 \cos 3x + c_2 \sin 3x)$

(C) $c_1e^{5x} + c_2e^{2x}$

(D) $e^x (c_1 \cos x + c_2 \sin x)$

19. The solution of differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$ is (1)

(A) $e^x (c_1 \cos x + c_2 \sin x)$

(B) $e^{x/2}\left[c_1 \cos\left(\frac{3}{2}\right)x + c_2 \sin\left(\frac{3}{2}\right)x\right]$

(C) $e^{-\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\right)x + c_2 \sin\left(\frac{\sqrt{3}}{2}\right)x\right]$

(D) $c_1e^x + c_2e^{-x}$

20. The solution of differential equation $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$ is (1)

(A) $e^{-x} (c_1 \cos 2x + c_2 \sin 2x)$

(B) $e^{-x/2} [c_1 \cos x + c_2 \sin x]$

(C) $e^{-2x} (c_1 \cos x + c_2 \sin x)$

(D) $c_1e^{-4x} + c_2e^{-5x}$

21. The solution of differential equation $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$ is

(A) $c_1e^x + c_2e^{2x} + c_3e^{3x}$

(B) $c_1e^{-x} + c_2e^{2x} + c_3e^{-3x}$

(C) $c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$

(D) $c_1e^x + c_2e^{-2x} + c_3e^{3x}$

22. The solution of differential equation $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$ is

(A) $c_1e^x + c_2e^{2x} + c_3e^{3x}$

(B) $c_1e^{-x} + c_2e^{-2x} + c_33^{6x}$

(C) $c_1e^{-x} + c_2e^{2x} + c_3e^x$

(D) $c_1e^{-x} + c_2e^{-2x} + c_3e^{3x}$

23. The solution of differential equation $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ is

(A) $c_1 + e^x(c_2x + c_3)$

(B) $c_1 + e^{-x}(c_2x + c_3)$

(C) $e^{-x}(c_2x + c_3)$

(D) $c_1 + c_2e^x + c_3e^{-x}$

24. The solution of differential equation $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 4y = 0$ is

(A) $c_1e^x + (c_2x + c_3)e^{2x}$

(B) $c_1e^x + c_2e^{2x} + c_3e^{3x}$

(C) $(c_2x + c_3)e^{2x}$

(D) $c_1e^{-x} + (c_2x + c_3)e^{-2x}$

25. The solution of differential equation $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 0$ is

(A) $c_1e^{2x} + c_2e^{-2x}$

(B) $c_1 + c_2\cos 2x + c_3\sin 2x$

(C) $c_1e^x + c_2e^{-2x} + c_3e^{-3x}$

(D) $c_1 + c_2e^{2x} + c_3e^{-2x}$

26. The solution of differential equation $\frac{d^3y}{dx^3} + y = 0$ is

(A) $c_1e^x + e^x\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$

(B) $c_1e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\frac{1}{2}x + c_3 \sin\frac{1}{2}x\right)$

(C) $c_1e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$

(D) $(c_1 + c_2x + c_3x^2)\, e^{-x}$

27. The solution of differential equation $\frac{d^3y}{dx^3} + 3\frac{dy}{dx} = 0$ is

(A) $c_1 + c_2 \cos x + c_3 \sin x$

(B) $c_1 + c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x$

(C) $c_1 + c_2e^{\sqrt{3}x} + c_3e^{-\sqrt{3}x}$

(D) $c_1 \cos x + c_2 \sin x$

28. The solution of differential equation $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 12y = 0$ is

(A) $c_1e^{-3x} + e^x\left(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right)$

(B) $c_1e^{-3x} + (c_2 \cos 3x + c_3 \sin 3x)$

(C) $c_1e^{3x} + e^{-x}\left(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right)$

(D) $c_1e^{-x} + c_2e^{-\sqrt{3}x} + c_3e^{\sqrt{3}x}$

29. The solution of differential equation $(D^3 - D^2 + 3D + 5)\, y = 0$ where $D = \frac{d}{dx}$ is

(A) $c_1e^{-x} + e^x(c_2 \cos 2x + c_3 \sin 2x)$

(B) $c_1e^{-x} + (c_2 \cos 3x + c_3 \sin 3x)$

(C) $c_1e^x + e^{-x}(c_2 \cos 2x + c_3 \sin 2x)$

(D) $c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$

30. The solution of differential equation $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 4y = 0$ is

(A) $(c_1 + c_2x)\, e^{-2x} + c_3e^{-x}$

(B) $c_1e^x + c_2 \cos 4x + c_3 \sin 4x$

(C) $c_1e^x + c_2 \cos 2x + c_3 \sin 2x$

(D) $c_1e^x + c_2e^{2x} + c_3e^{-2x}$

31. The solution of differential equation $\frac{d^4y}{dx^4} - y = 0$ is

(A) $(c_1x + c_2)\, e^{-x} + c_3 \cos x + c_4 \sin x$

(B) $(c_1x + c_2) \cos x + (c_3x + c_4) \sin x$

(C) $(c_1 + c_2x + c_3x^2 + c_4x^3)\, e^x$

(D) $c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x$

32. The solution of differential equation $(D^4 + 2D^2 + 1)\, y = 0$ where $D = \frac{d}{dx}$ is

(A) $(c_1x + c_2)\, e^x + (c_3x + c_4)\, e^{-x}$

(B) $(c_1x + c_2) \cos x + (c_3x + c_4) \sin x$

(C) $c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x$

(D) $(c_1x + c_2) \cos 2x + (c_3x + c_4) \sin 2x$

33. The solution of differential equation $(D^2 + 9)^2\, y = 0$, where $D = \frac{d}{dx}$ is

(A) $(c_1x + c_2)\, e^{3x} + (c_3x + c_4)\, e^{-3x}$

(B) $(c_1x + c_2) \cos 3x + (c_3x + c_4) \sin 3x$

(C) $(c_1x + c_2) \cos 9x + (c_3x + c_4) \sin 9x$

(D) $(c_1x + c_2) \cos x + (c_3x + c_4) \sin x$

34. The solution of differential equation $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$ is

(A) $c_1e^{2x} + c_2e^{-x} + c_3e^x + c_4e^{-2x}$

(B) $(c_1x + c_2)\, e^{2x} + (c_3x + c_4)\, e^{-2x}$

(C) $(c_1x + c_2) \cos 4x + (c_3x + c_4) \sin 4x$

(D) $(c_1x + c_2) \cos 2x + (c_3x + c_4) \sin 2x$

35. The solution of differential equation $\frac{d^6y}{dx^6} + 6\frac{d^4y}{dx^4} + 9\frac{d^2y}{dx^2} = 0$ is

(A) $c_1x + c_2 + (c_3x + c_4) \cos \sqrt{3}x + (c_5x + c_6) \sin \sqrt{3}x$

(B) $c_1x + c_2 + (c_3x + c_4) \cos 3x + (c_5x + c_6) \sin 3x$

(C) $(c_1x + c_2) \cos \sqrt{3}x + (c_3x + c_4) \sin \sqrt{3}x$

(D) $c_1x + c_2 + (c_3x + c_4)\, e^{\sqrt{3}x}$

ANSWERS

1. (A)	2. (C)	3. (B)	4. (D)	5. (B)	6. (D)	7. (A)	8. (C)
9. (D)	10. (B)	11. (A)	12. (C)	13. (D)	14. (A)	15. (B)	16. (C)
17. (D)	18. (A)	19. (C)	20. (B)	21. (C)	22. (D)	23. (B)	24. (A)
25. (D)	26. (C)	27. (B)	28. (A)	29. (A)	30. (C)	31. (D)	32. (B)
33. (B)	34. (D)	35. (A)					

Exercise

Solve the following Differential Equations

1. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} - 6y = 0.$

Ans. $y = c_1 e^{-x} + c_2 e^{6x}$

2. $2\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 10y = 0.$

Ans. $y = c_1 e^{-2x} + c_2 e^{(5/2)\,x}$

3. $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0.$

Ans. $y = c_1 + e^{-x}(c_2 x + c_3)$

4. $(D^4 - 2D^3 + D^2)\,y = 0.$

Ans. $y = c_1 x + c_2 + (c_3 x + c_4)\,e^x$

5. $(D^6 - 6D^5 + 12D^4 - 6D^3 - 9D^2 + 12D - 4)\,y = 0.$

Ans. $y = (c_1x^2 + c_2x + c_3)\,e^x + (c_4x + c_5)\,e^{2x} + c_6e^{-x}$

6. $(D^3 + 6D^2 + 11D + 6)\,y = 0.$

Ans. $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$

7. $4y'' - 8y' + 7y = 0.$

Ans. $y = e^x\left[A\cos\left(\dfrac{\sqrt{3}}{2}x\right) + B\sin\left(\dfrac{\sqrt{3}}{2}x\right)\right]$

8. $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 5x = 0,\ x(0) = 2,\ x'(0) = 0.$

Ans. $x = e^{-t}(2\cos 2t + \sin 2t)$

9. $\dfrac{d^2s}{dt^2} = -16\dfrac{ds}{dt} - 64\,s,\ s = 0,\ \dfrac{ds}{dt} = -4$ when $t = 0.$

Ans. $s = -4e^{8t}\,t$

10. $(D^3 + D^2 - 2D + 12)\,y = 0.$

Ans. $y = c_1 e^{-3x} + e^x\left[A\cos\sqrt{3}\,x + B\sin\sqrt{3}\,x\right]$

11. $(D^2 + 1)^3(D^2 + D + 1)^2\,y = 0.$

Ans. $y = (c_1 + c_2x + c_3x^2)\cos x + (c_4 + c_5x + c_6x^2)\sin x + e^{-x/2}\left[(c_7 + c_8x)\cos\left(\dfrac{\sqrt{3}}{2}x\right) + (c_9 + c_{10}x)\sin\left(\dfrac{\sqrt{3}}{2}x\right)\right]$

Exercise

12. $\dfrac{d^4y}{dx^4} + m^4y = 0.$

Ans. $y = e^{(mx/\sqrt{2})}\left[A\cos\left(\dfrac{mx}{\sqrt{2}}\right) + B\sin\left(\dfrac{mx}{\sqrt{2}}\right)\right] + e^{-(mx/\sqrt{2})}\left[C\cos\left(\dfrac{mx}{\sqrt{2}}\right) + D\sin\left(\dfrac{mx}{\sqrt{2}}\right)\right]$

13. $4\dfrac{d^2s}{dt^2} = -9s.$

Ans. $s = c_1 \sin\dfrac{3t}{2} + c_2 \cos\dfrac{3t}{2}.$

14. The equation for the bending of a strut is $EI\dfrac{d^2y}{dx^2} + Py = 0$. If $y = 0$ when $x = 0$ and $y = a$ when $x = \dfrac{l}{2}$, find y.

Ans. $y = \dfrac{a\sin\sqrt{\dfrac{P}{EI}}\,x}{\sin\sqrt{\dfrac{P}{EI}\cdot\dfrac{l}{2}}}$