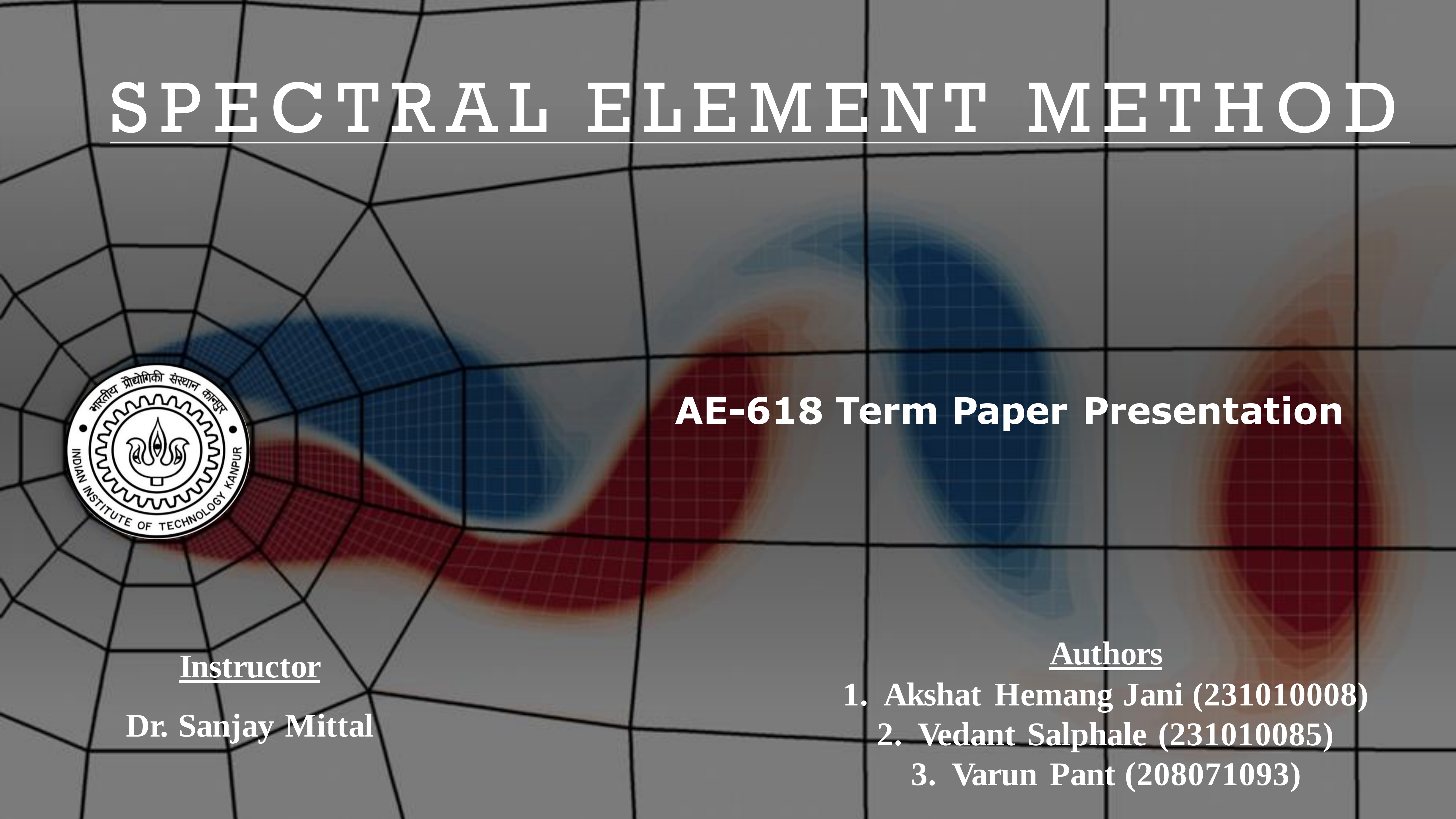

SPECTRAL ELEMENT METHOD

AE-618 Term Paper Presentation

Instructor

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OUTLINE

- Introduction to SEM
- Advantages and Limitations
- Spectral Methods for Differential Equations
- How's it different from conventional h-p FEM
- Applications of SEM

INTRODUCTION TO SEM

- Introduced in 1984 by A.T. Patera
- Legendre, Chebyshev, or Jacobi polynomials
- Interpolation is performed using basis functions and the values at GLL points within each element
- GLL points optimize interpolation accuracy
- Like FEM, weak form of PDE is satisfied using weight function
- Integration is performed using Gauss Legendre Quadrature

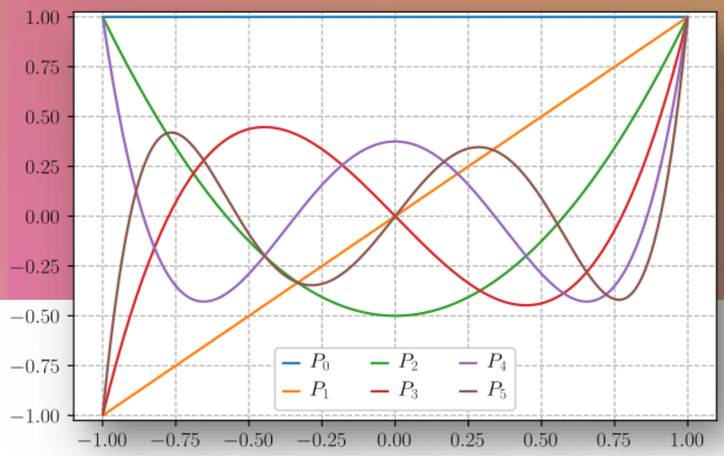


Fig. 1: Legendre polynomial

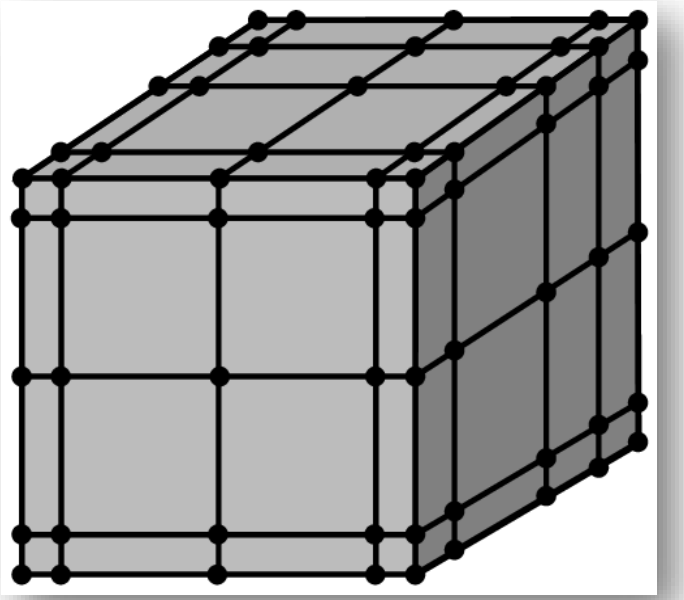
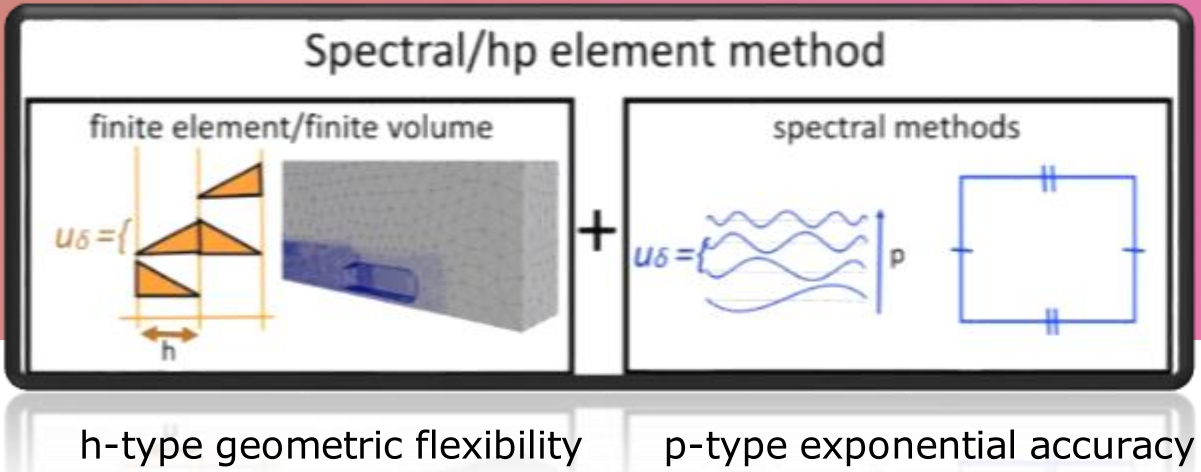


Fig. 2: A cube with 125 GLL points

ADVANTAGES AND LIMITATIONS



- Faster convergence rate
- Captures sharp gradients and complex phenomena better
- Reduced dissipation errors
- Eliminates Runge's phenomenon
- Less number of elements compared to FEM
- Combines h-type geometric flexibility of FEM and p-type exponential accuracy of spectral methods
- Higher computational costs
- Higher storage requirement
- Limitations in multi-dimensional problems
- Stability issues in non-linear problems, discontinuities and strong gradient regions

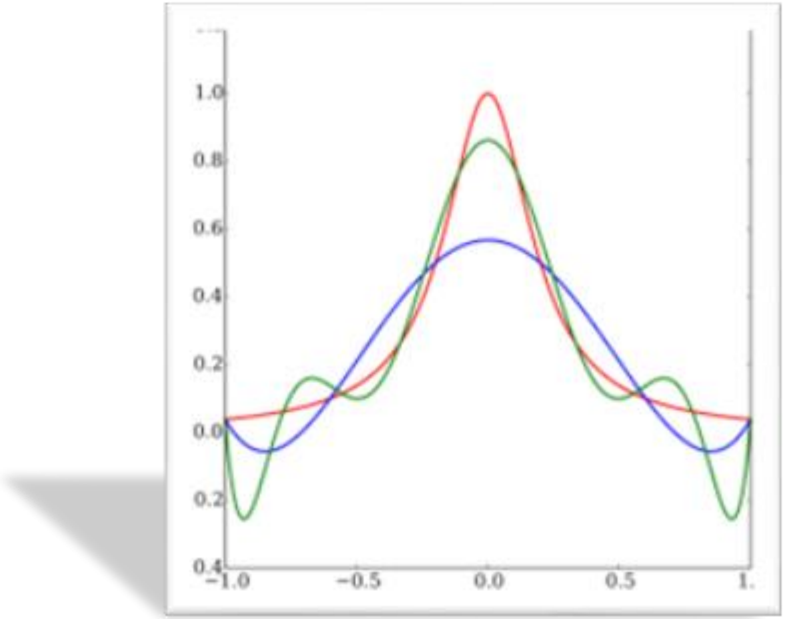


Fig. 3: Runge's phenomenon at endpoints of interval

SOLVING DIFFERENTIAL EQUATIONS NUMERICALLY

- Along with FEM, Spectral Methods also make the use of weighted residuals

$$\mathcal{L}u = f$$

Forcing Function

Approximate Solution

$$\mathcal{L}\hat{u} = f$$

Exact Solution

$$r = f - \mathcal{L}u$$

Residual

- Note that as N →∞, u→ û, and r → 0

$$u(x,y,z,t) = \phi_0(x,y,z) + \sum_{n=1}^{N} a_n(t)\phi_n(x,y,z)$$

Approximate Solution

φ0(x,y,z) is choosen to Satisfy the boundary conditions such that φn(x,y,z) is 0 at Boundary for N = 1, 2, 3…

Time Dependent Coefficients

Spatial Basis/Trial Functions

$$f(x,y,z) = \sum_{n=1}^{N} \hat{a}_n\phi_n(x,y,z)$$

$$\hat{a}_n = \langle f(x,y,z), \phi_n(x,y,z)\rangle$$

- Basis Functions φn(x,y,z) are orthogonal with each other and orthonormal with itself.

CHOICE OF TRIAL FUNCTIONS

- The trial functions are usually smooth functions which are supported in the complete domain D.
- There are many choices possible, in particular trigonometric (Fourier) functions, Chebyshev and Legendre polynomials, but also lower-order Lagrange polynomials with local support (finite element method) or b-splines

1. Fourier Series

- Suitable for periodic functions u(x+L) = u(x), where L is the periodicity and α = 2*π/L is the fundamental wave number.

$$\boxed{u_N(x) = \sum_{|k|\leq K} c_k e^{ik\alpha x} = \sum_{|k|\leq K} c_k \Phi_k \ , \quad \text{with } c_k \in \mathbb{C}}$$

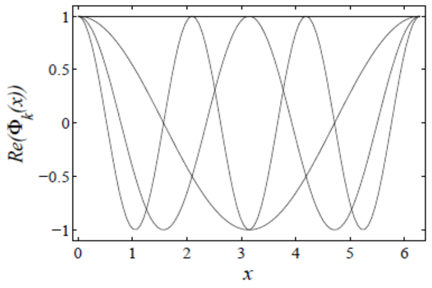


Figure 1: Fourier functions $\Phi_k(x) = \exp(\mathrm{i}\alpha k x)$ for $k = 0, \ldots, 3$ with $\alpha = 2\pi/L = 1$.

- $\boxed{c_k}$ are complex Fourier coefficients for the Fourier mode $\boxed{\Phi_k(x) = \exp(\mathrm{i}\alpha k x)}$
- A Fourier series of a smooth function (also in the derivatives, i.e. part of C∞) shows spectral convergence. Fourier Basis also exhibits mutual orthogonality
- However, if the original function u(x) is non-continuous in at least one of the derivatives, the rate of convergence is severely decreased to order p. $\boxed{||u_N - u|| = \mathcal{O}(N^{-p}) \ , \quad N \rightarrow \infty}$

2. Chebyshev Polynomials

- Popular choice for non-periodic boundary conditions, Proposed by Pafnuty Lvovich Chebyshev (1821–1894)

Chebyshev polynomial of degree $n >= 0$ is defined as

$$T_n(x) = \cos(n \arccos x) \ , \qquad x \in [-1, 1], \text{ or, in a more instructive form,}$$

$$T_n(x) = \cos n\theta \ , \qquad x = \cos\theta \ , \qquad \theta \in [0, \pi] \ .$$

- The first polynomials are thus,

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \end{aligned}$$

- Can be written in recursive form as

$$T_{k+1}(x) + T_{k-1}(x) = 2xT_k(x) \ , \quad k \geq 1$$

- A function u(x) is approximated via a finite series of Chebyshev polynomials as:

$$u_N(x) = \sum_{k=0}^{N} a_k T_k(x)$$

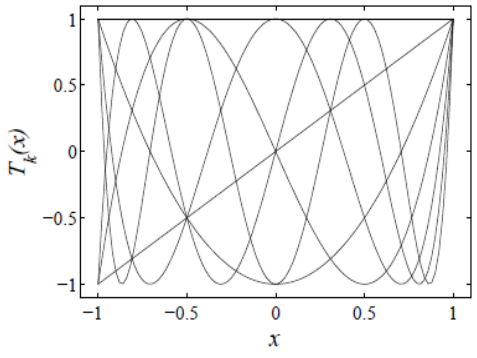


Figure 3: Chebyshev polynomials $T_k(x)$ for $k = 0, \ldots, 6$.

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Fu

- A common distribution of points in particular for

polynomials are the Ga

$$x_j = \cos\frac{\pi j}{N} \ , \quad j = 0, \ldots, N$$

- **Imp Properties:** Orthogonality, Alternate Even & Odd

Boundary Conditions, etc.

WEIGHTED RESIDUALS/WEIGHT FUNCTIONS

$$\langle r(x,t), w_i(x)\rangle = \int_{x_0}^{x_1} r(x,t) w_i(x)\,dx = 0, \quad i = 1,\dots,N.$$

$$\int_{x_0}^{x_1} \left\{ \sum_{n=1}^{N} \hat{a}_n \phi_n(x) - \mathcal{L}\left[\phi_0(x) + \sum_{n=1}^{N} a_n(t)\phi_n(x)\right]\right\} w_i(x)\,dx = 0, \quad i = 1,\dots,N.$$

Weight/Test Functions

- In order to determine the coefficients an(t) for the given set of basis functions, we require that the integral of the weighted residual be zero over the spatial domain.
- Different weighted residual methods correspond to different choices for the weight (test) functions.

Galerkin:

wi (x) = φi(x), where weight (test) functions are the same as the basis (trial) functions.

• Least squares:

wi (x) = ∂r/∂ai , which results in the square of the norm of the residual ∫ r^2 dx being a minimum.

• Collocation:

wi (x) = δ(x − xi), where δ is the Dirac delta function centered at the collocation points xi .

$$\mathcal{L}u = \frac{d^2u}{dx^2} + u = 0, \quad 0 \le x \le 1,$$

B.C. u(0) = 0, u(1) = 1

Spectral Method

Basis Functions : φn(x) = sin(nπx), φ0(x) = x,

N = 1,2,3....

$$u(x) = x + \sum_{n=1}^{N} a_n \sin(n\pi x)$$

$$u(x) = x + \sum_{n=1}^{N} \frac{2(-1)^n}{n\pi\left[1-(n\pi)^2\right]}\sin(n\pi x)$$

Approx. soln

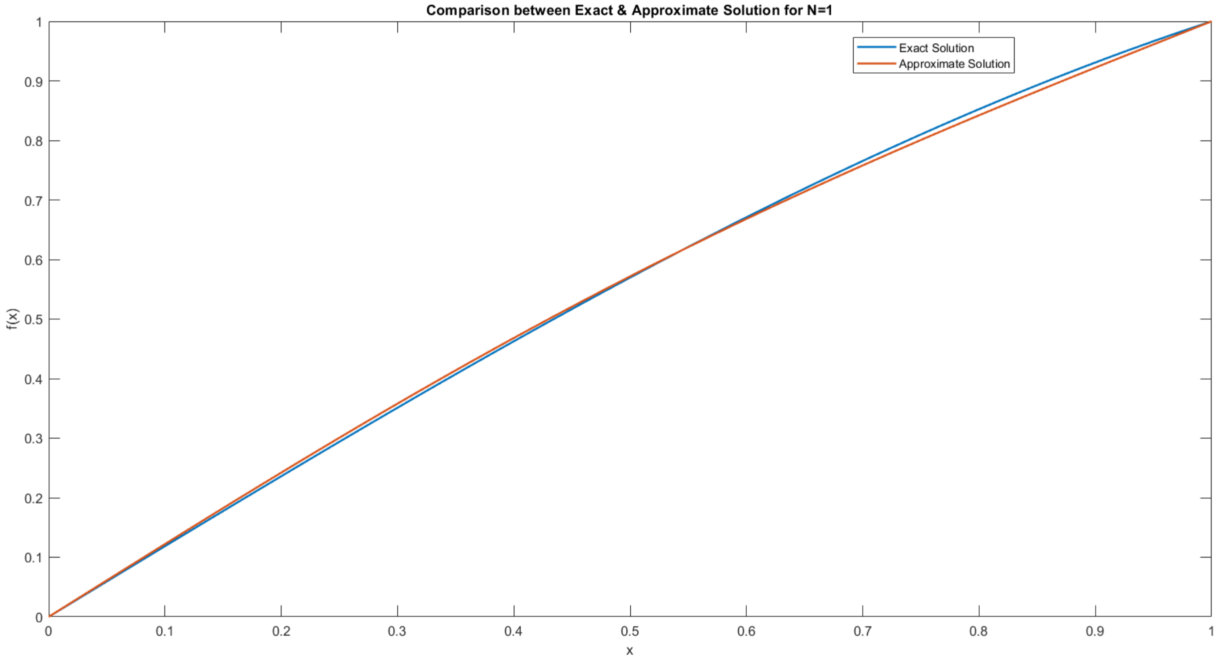
$$\hat{u}(x) = \frac{\sin x}{\sin 1}$$

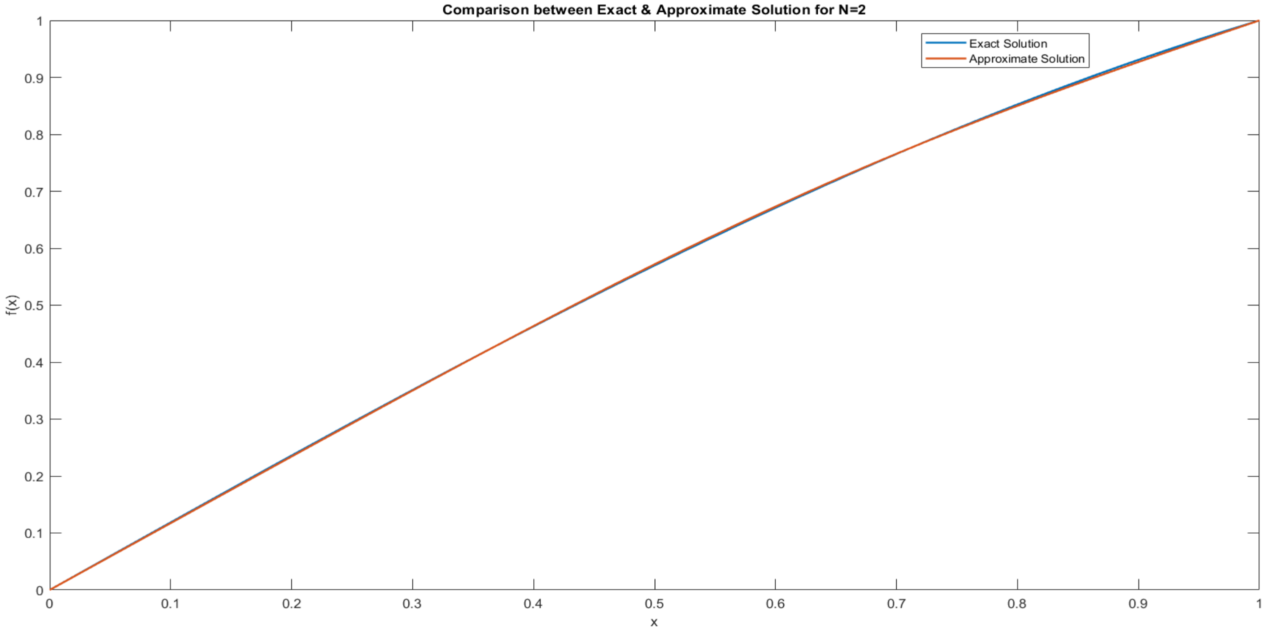
Exact soln

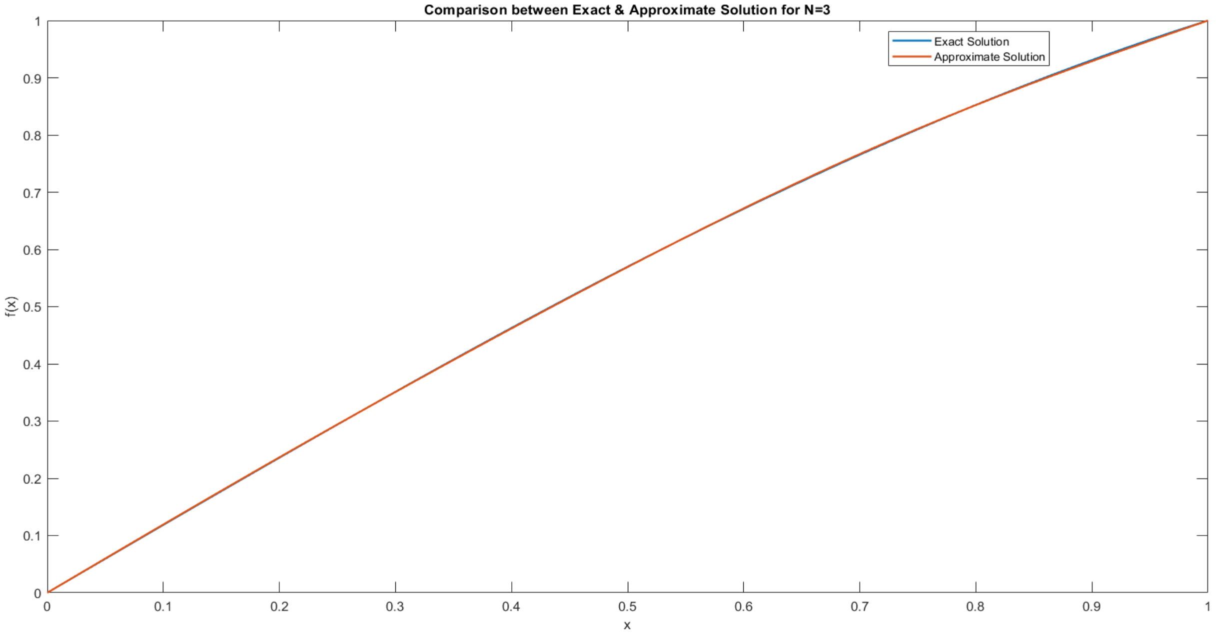
Comparison of the approximate and exact solutions :

$$\boxed{u(x) = x + \sum_{n=1}^{N} \frac{2(-1)^n}{n\pi\left[1-(n\pi)^2\right]} \sin(n\pi x)}$$ Approx. soln

$$\boxed{\hat{u}(x) = \frac{\sin x}{\sin 1}}$$ Exact soln







How is SEM/SFEM different from conventional h-p FEM?

- Usually in conventional h-p FEM, basis functions used for approximating the solution are typically polynomial functions defined over each element of the mesh.(Local Support).
- Mesh refinement (h-refinement) and polynomial degree increase (p-refinement) are used to improve accuracy. [Recall the relative error from Assignment 2!]
- Traditional FEM uses numerical quadrature (Gauss quadrature) to integrate over each element. Higher-order elements or higher-order polynomials may require more integration points.

- In SEM, the spectral basis functions have global support over the entire element (this means that each basis function has non-zero values across the entire spatial domain, not just within a local element. As a result, SEM basis functions can capture global behavior more efficiently, allowing for high-order accuracy with relatively few elements.)
- The accuracy is primarily controlled by the order of the spectral basis functions rather than the mesh density.
- SEM often exploits the properties of orthogonal spectral basis functions, allowing for the use of analytical integration (spectral integration) in many cases. This can significantly reduce the computational cost.

APPLICATIONS OF SEM:

It combines the **accuracy of spectrum approaches** and the **geometric adaptability of FEM**.

- In CFD It has been successfully used to model and simulate technical difficulties in the automotive, oil & gas, and aerospace/aeronautics domains.
- Structural Dynamic Analysis
  -**Soil-Structure Interaction (SSI)**: SFEM is employed to compare planar frame designs considering SSI, offering more accurate findings at high frequencies with lower computational costs.
  -Dynamic Behavior of **Plate Structures and beams**: SFEM is used to evaluate the dynamic behavior of periodic plate structures, particularly those with two parallel supported sides, by constructing spectral equations. It has been used for studies from single-span beams to multi-span beams subjected to dynamic point forces.
  -**Vibrational Behaviors of Structures**: A modified Fourier spectral element technique (SEM) has been created to analyze the vibrational behaviors of structures.
  -**Analysis of Low-Velocity/Low-Energy Impacts on Composite Structures**: SFEM, along with cubic spline layer-wise theory, is used to analyze the effects of low-velocity impacts on composite sandwich laminated plates.

APPLICATIONS OF SEM:

- Wave Propagation: The Buzz area for SEM
  -Starting from **One-Dimensional Elastic Wave Propagation**: SFEM provides more accurate numerical solutions for waves; moving through isotropic rods and Timoshenko beams.
  -**Seismic Body Wave Propagation**: SFEM is utilized for the simulation and validation of seismic wave transmission, even globally, demonstrating its effectiveness in analyzing wave propagation phenomena. SFEM offers a novel numerical method for synthetic computing seismograms in 3-D earth models. Relevant in various scenarios such as active seismic acquisition operations in the oil sector or earthquakes on a continental scale.
  -**Piezo-Induced Ultrasonic Wave Propagation**: SFEM is employed to develop a three-dimensional Piezo-Enabled Spectral Element Analysis tool for simulating piezo-induced ultrasonic wave propagation in composite structures. This tool solves electromechanical governing equations and outputs voltage responses of piezoelectric sensors.

- **Fractional Calculus Problems** - Compared to other numerical approaches, using SFEM to solve fractional differential equations sets effective stability requirements and offers greater flexibility when addressing inhomogeneity and complicated geometries.

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