1 A Proof in Propositional Logic

Theorem 1. *If $p \implies q$ and $q \implies r$, then $p \implies r$.*

Proof. We will prove this theorem using the following steps:

1. $p \implies q$ (Given)
2. $q \implies r$ (Given)
3. p (Assumption)
4. q (Modus Ponens, 1 and 3)
5. r (Modus Ponens, 2 and 4)

From steps 3-5, we have shown that assuming p leads to r. Therefore, by the definition of implication:

$$p \implies r$$

This completes the proof. □

Lemma 2. $\neg(p \land q) \iff (\neg p \lor \neg q)$

Proof. We will prove this equivalence using a truth table:

p $\neg p \lor \neg q$	q	$p \land q$	$\neg(p \land q)$	$\neg p$	$\neg q$
T F	T	T	F	F	F
T T	F	F	T	F	T
F T	T	F	T	T	F
F T	F	F	T	T	T

As we can see, the truth values for $\neg(p \land q)$ and $(\neg p \lor \neg q)$ are identical for all possible combinations of p and q. Therefore, these expressions are logically equivalent. □

Theorem 3. *If $p \implies (q \implies r)$, then $(p \land q) \implies r$.*

Proof. We will prove this using natural deduction:

1. $p \implies (q \implies r)$ (Given)
2. $p \land q$ (Assumption)
3. p (Conjunction Elimination, 2)

4. $q \implies r$ (Modus Ponens, 1 and 3)
5. q (Conjunction Elimination, 2)
6. r (Modus Ponens, 4 and 5)

From steps 2-6, we have shown that assuming $p \wedge q$ leads to r. Therefore, by the definition of implication:

$$(p \wedge q) \implies r$$

This completes the proof. $\square$