Midterm II Preparation Notes

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Notes on Convergence Tests

Divergence Test

The divergence test (also known as the nth-term test for divergence) can be used to check if a series is diverging. With the divergence test, you can only conclude that the series is divergent if the test fails.

If the limit of the nth term of a series $\sum a_n$ does not approach zero, i.e.,

$$\lim_{n\to\infty} a_n \neq 0,$$

then the series $\sum a_n$ diverges.

Example: Consider the series $\sum \frac{1}{n}$. Here, $a_n = \frac{1}{n}$. We have:

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Since the limit is zero, the divergence test is inconclusive. However, for the series $\sum 1$, where $a_n = 1$, we have:

$$\lim_{n\to\infty} 1 = 1 \neq 0,$$

so the series diverges by the divergence test.

Comparison Test

The Comparison Test can be used to determine if a series is converging or diverging by comparing it to another series which we are able to evaluate

If $0 \leq a_n \leq b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ also converges.

Conversely, if $\sum b_n$ diverges and $a_n \geq b_n \geq 0$, then $\sum a_n$ also diverges.

Example: Consider the series $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n^3}$. Since $\frac{1}{n^3} \leq \frac{1}{n^2}$ for all $n \geq 1$ and $\sum \frac{1}{n^2}$ converges (p-series with $p > 1$), by the Comparison Test, $\sum \frac{1}{n^3}$ also converges.

Ratio Test

The Ratio Test involves the limit of the absolute value of the ratio of consecutive terms in a series. For a series $\sum a_n$, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$:

- If $L < 1$, the series converges absolutely.
- If $L > 1$ or $L = \infty$, the series diverges.
- If $L = 1$, the test is inconclusive.

Example: Consider the series $\sum \frac{n!}{n^n}$. Let $a_n = \frac{n!}{n^n}$. Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right| = \left|\frac{(n+1)\cdot n!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right| = \left|\frac{n^n}{(n+1)^n} \cdot \frac{1}{n+1}\right| \approx \frac{1}{e} < 1$$

Thus, the series converges by the Ratio Test.

Root Test

The Root Test uses the n-th root of the absolute value of the terms in a series. For a series $\sum a_n$, if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$:

- If $L < 1$, the series converges absolutely.
- If $L > 1$ or $L = \infty$, the series diverges.
- If $L = 1$, the test is inconclusive.

Example: Consider the series $\sum \left(\frac{1}{2}\right)^n$. Let $a_n = \left(\frac{1}{2}\right)^n$. Then,

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{1}{2}\right)^n} = \frac{1}{2}$$

Since $\frac{1}{2} < 1$, the series converges by the Root Test.

Alternating Series Test

The Alternating Series Test applies to series of the form $\sum(-1)^n a_n$ or $\sum(-1)^{n+1} a_n$ where $a_n \geq 0$. The series converges if:

- a_n is monotonically decreasing, and
- $\lim_{n\to\infty} a_n = 0$.

Example: Consider the series $\sum(-1)^n \frac{1}{n}$. Here, $a_n = \frac{1}{n}$, which is monotonically decreasing and $\lim_{n\to\infty} \frac{1}{n} = 0$. Thus, the series converges by the Alternating Series Test.

Absolute Convergence

A series $\sum a_n$ converges absolutely if the series of absolute values $\sum |a_n|$ converges. Absolute convergence implies convergence, but not vice versa.

Example: Consider the series $\sum(-1)^n \frac{1}{n^2}$. The series of absolute values is $\sum \frac{1}{n^2}$, which converges (p-series with $p > 1$). Therefore, $\sum(-1)^n \frac{1}{n^2}$ converges absolutely.

Notes on Maclaurin and Taylor Polynomials

Maclaurin Polynomials

A Maclaurin polynomial is a special case of the Taylor polynomial centered at $x = 0$. The n-th degree Maclaurin polynomial for a function $f(x)$ is:

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

Example: For $f(x) = e^x$, the Maclaurin polynomial of degree 3 is:

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Taylor Polynomials

A Taylor polynomial approximates a function $f(x)$ near a point a. The n-th degree Taylor polynomial for $f(x)$ centered at a is:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example: For $f(x) = \sin(x)$ centered at $a = \pi/4$, the Taylor polynomial of degree 2 is:

$$P_2(x) = \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)(x - \frac{\pi}{4}) - \frac{\sin\left(\frac{\pi}{4}\right)}{2!}(x - \frac{\pi}{4})^2$$

Estimating Function Values

To estimate the value of a function $f(x)$ to within 3 decimal places using either a Taylor or Maclaurin polynomial, follow these steps:

1. Choose the appropriate polynomial (Taylor or Maclaurin) based on the function and the point of interest.
2. Determine the degree n of the polynomial such that the remainder term $R_n(x)$ is less than 0.001.
3. Calculate the polynomial $P_n(x)$ and use it to approximate $f(x)$.

Example: Estimate $e^{0.1}$ using the Maclaurin polynomial for e^x.
The Maclaurin series for e^x is:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

To estimate $e^{0.1}$ to within 3 decimal places, we need to find the degree n such that the remainder term $R_n(0.1)$ is less than 0.001. For e^x, the remainder term is:

$$R_n(x) = \frac{e^c x^{n+1}}{(n+1)!}$$

where c is some value between 0 and 0.1.
Using the first four terms of the series:

$$P_3(0.1) = 1 + 0.1 + \frac{0.1^2}{2!} + \frac{0.1^3}{3!} = 1 + 0.1 + 0.005 + 0.0001667 = 1.1051667$$

The actual value of $e^{0.1}$ is approximately 1.1051709, so the error is less than 0.001, and the estimate is accurate to within 3 decimal places.

Notes on Maclaurin and Taylor Series

Maclaurin Series

The Maclaurin series is the infinite series representation of a function $f(x)$ centered at $x = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Example: For $f(x) = \cos(x)$, the Maclaurin series is:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Taylor Series

The Taylor series is the infinite series representation of a function $f(x)$ centered at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Example: For $f(x) = e^x$ centered at $a = 1$, the Taylor series is:

$$e^x = \sum_{n=0}^{\infty} \frac{e}{n!} (x-1)^n$$

Notes on Power Series

A power series is an infinite series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where c_n are coefficients and a is the center of the series. The series converges within a certain radius R around a, known as the radius of convergence.

Example: Consider the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. This series converges for all x (radius of convergence $R = \infty$).

Properties of Power Series

1. *Term-by-Term Differentiation and Integration*: If a power series converges within its radius of convergence, it can be differentiated and integrated term-by-term within that interval.

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$\int \left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) dx = C + \sum_{n=0}^{\infty} \frac{c_n (x-a)^{n+1}}{n+1}$$

2. **Uniform Convergence**: A power series converges uniformly on any closed interval within its radius of convergence.

Examples of Power Series

1. **Geometric Series**:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for} \quad |x| < 1$$

2. **Exponential Function**:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

3. **Sine and Cosine Functions**:

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Notes on Convergence of Taylor Series

The convergence of a Taylor series depends on the function being represented and the point around which the series is centered. For a function $f(x)$ with a Taylor series centered at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

The series converges within a certain interval around a, known as the interval of convergence. The radius of convergence R is the distance from a to the boundary of this interval. The series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$.

To determine the radius of convergence, one can use the Ratio Test or the Root Test. For the Ratio Test, consider the limit:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

where $a_n = \frac{f^{(n)}(a)}{n!}(x-a)^n$. The radius of convergence R is given by:

$$R = \frac{1}{L}$$

For the Root Test, consider the limit:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

where $a_n = \frac{f^{(n)}(a)}{n!}(x-a)^n$. The radius of convergence R is given by:

$$R = \frac{1}{L}$$

Within the interval of convergence, the Taylor series converges to the function $f(x)$. At the endpoints of the interval, convergence must be checked separately.

Example: For the Taylor series of $f(x) = \ln(1+x)$ centered at $a = 0$:

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

The radius of convergence is $R = 1$, so the series converges for $|x| < 1$.