SOMAIYA VIDYAVIHAR UNIVERSITY

Semester: March 2022 – July 2022 Examination: End-Semester Examination		
Programme code: 06 **Programme: B.TECH**	**Class:** FY	**Semester:** II **(SVU 2020)**
Name of the Constituent College: **K. J. Somaiya College of Engineering**	**Name of the department:** COMP/ETRX/EXTC/IT/MECH	
Course Code: 116U06C108	**Name of the Course:** Applied Mathematics-II	
Max Marks: 100	**Time :** 3 Hours	

Question No.		Marks
Q.1 (A)	Solve $(x-2e^y)dy+(y+x\sin x)dx=0$ **Soln:** $\frac{\partial M}{\partial y}=1=\frac{\partial N}{\partial x}$, Hence the DE is exact Solution is $\int_{y\ const} M\,dx+\int_{free\ of\ x} N\,dy=C$ $\int_{y\ const} M\,dx=\int(y+x\sin x)dx=xy-x\cos x+\sin x$ $\int_{free\ of\ x} N\,dy=\int -2e^y dy=-2e^y$ Hence the solution is $xy-x\cos x+\sin x-2e^y=c$ **OR**	02 04
	Solve $\frac{d^6y}{dx^6}-64y=e^{2x}$ **Soln:** The A.E. is $(D^6-64)=0\ \ \therefore (D^3-8)(D^3+8)=0$ Roots are $D=2,-1\pm\sqrt{3}i,\ -2,1\pm\sqrt{3}i$ C.F. = $c_1e^{2x}+c_2e^{-2x}+e^{-x}\left(c_3\cos\sqrt{3}x+c_4\sin\sqrt{3}x\right)+e^{x}\left(c_5\cos\sqrt{3}x+c_6\sin\sqrt{3}x\right)$ P.I. $=\frac{1}{D^6-64}e^{2x}$ $(Replace\ D\ by\ 2,\ D'r\ is\ zero)$ $=\frac{x}{6D^5}e^{2x}\ =\frac{x}{192}e^{2x}$ Hence gen solution is C.F. + P.I. $$y=c_1e^{2x}+c_2e^{-2x}+e^{-x}\left(c_3\cos\sqrt{3}x+c_4\sin\sqrt{3}x\right)+e^{x}\left(c_5\cos\sqrt{3}x+c_6\sin\sqrt{3}x\right)+\frac{x}{192}e^{2x}$$	02 04
Q.1 (B)	Attempt any **THREE** of the following	**21**
(i)	Solve $(D^4-16)y=e^{2x}+\cos x\cos 2x$	

Soln: The A.E. is $(D^4 - 16) = 0 \quad \therefore (D^2 - 4)(D^2 + 4) = 0$

Roots are $D = 2, -2, 2i, -2i$

C.F. = $c_1 e^{2x} + c_2 e^{-2x} + (c_3 \cos 2x + c_4 \sin 2x)$ **02**

P.I. $= \frac{1}{D^4 - 16}(e^{2x} + \cos x \cos 2x) = \frac{1}{D^4-16}(e^{2x}) + \frac{1}{D^4-16}\cos x \cos 2x$

$PI_1 = \frac{1}{D^4-16}(e^{2x}) = \frac{x}{4D^3} e^{2x} = \frac{x}{32} e^{2x}$ **04**

$$PI_2 = \frac{1}{D^4 - 16}\cos x \cos 2x = \frac{1}{2}\frac{1}{D^4 - 16}(\cos 4x + \cos 2x)$$

$$= \frac{1}{2}\left(\frac{1}{4^4 - 16}\right)\cos 4x + \frac{1}{2}\frac{x}{4D^3}\cos 2x = \frac{1}{480}\cos 4x - \frac{x}{32}\frac{1}{D}\cos 2x$$

$$= \frac{1}{480}\cos 4x - \frac{x}{64}\sin 2x$$

Now gen solution is C.F. + $PI_1 + PI_2$

y = $c_1 e^{2x} + c_2 e^{-2x} + (c_3 \cos 2x + c_4 \sin 2x) + \frac{x}{32} e^{2x} + \frac{1}{480}\cos 4x - \frac{x}{64}\sin 2x$ **07**

(ii) Solve $\tan y \frac{dy}{dx} + \tan x = \cos y . \cos^3 x$ (1-37)

Sol. : Dividing by cos y, we get $\sec y \tan y \dfrac{dy}{dx} + \tan x \sec y = \cos^3 x$

Put sec $y = v$ and differentiate w.r.t. x, $\therefore$ $\sec y \tan y \dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore$ The equation reduce to $\dfrac{dv}{dx} + \tan x \cdot v = \cos^3 x$ **03**

Now, $\int P\,dx = \int \tan x\,dx = \log \sec x$

$\therefore$ I.F. $= e^{\int P dx} = e^{\log \sec x} = \sec x$

$\therefore$ The solution is

$$v \sec x = \int \sec x \cos^3 x\,dx + c = \int \cos^2 x\,dx + c$$

$$= \frac{1}{2}\int (1 + \cos 2x)\,dx + c$$

$$\therefore \quad \sec y \cdot \sec x = \frac{x}{2} + \frac{\sin 2x}{4} + c.$$

07

(iii) Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = x \log x$ (4-5)

Soln: Putting $z = \log x$ *and* $x = e^z$, $\frac{d}{dz} = D$ we get

	$x^2\frac{d^2y}{dx^2} = D(D-1)y,\ \ x\frac{dy}{dx} = Dy$, Putting in equation $[D(D-1) - D + 2]y = ze^z \quad \therefore (D^2 - 2D + 2)y = ze^z$ $\therefore$ The A.E. is $D^2 - 2D + 2 = 0$, $\therefore D = 1 \pm i$ $\therefore$ The C.F. is $y = e^z(c_1 \cos z + c_2 \sin z)$. $\text{P.I.} = \frac{1}{D^2 - 2D + 2} e^z \cdot z$ $= e^z \cdot \frac{1}{(D+1)^2 - 2(D+1) + 2} \cdot z$ $= e^z \cdot \frac{1}{D^2 + 1} z = e^z \cdot [1 + D^2]^{-1} z$ $= e^z[1 - D^2 + \ldots\ldots]z = e^z \cdot z$ $\therefore$ The complete solution is $y = e^z(c_1 \cos z + c_2 \sin z) + e^z \cdot z$ $\therefore \ y = x(c_1 \cos \log x + c_2 \sin \log x) + x \log x.$	**03** **07**
(iv)	Differential equation of a body of mass m falling from rest subjected to gravity and air resistance is given by $mv\frac{dv}{dx} = ka^2 - kv^2$. If it falls through a distance x and possesses velocity v at that instance then Prove that $\frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right)$ **Soln:** $mv\frac{dv}{dx} = k(a^2 - v^2)$ then $\frac{v}{(a^2 - v^2)}dv = \frac{k}{m}dx$ Then $\frac{\partial M}{\partial v} = 0 = \frac{\partial N}{\partial x}$, Hence the DE is exact Solution will be $\int_{y\ const} \frac{k}{m}\, dx + \int_{free\ of\ x} -\frac{v}{(a^2 - v^2)}\, dv = C$ $\frac{k}{m}x + \frac{1}{2}\log(a^2 - v^2) = \log\ c1$, by data at t = 0, x = 0 and v = 0 $\therefore \log c1 = \frac{1}{2}\log a^2$ then $\frac{k}{m}x + \frac{1}{2}\log(a^2 - v^2) = \frac{1}{2}\log a^2 \therefore \frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right)$	**03** **07**
Q.2 (A)	Find nth derivative y_n for $y = \frac{8x}{x^3 - 2x^2 - 4x + 8}$	

Example 8 (a) : If $y = \dfrac{8x}{x^3 - 2x^2 - 4x + 8}$, find y_n.

Sol. : We have $y = \dfrac{8x}{x^3 - 2x^2 - 4x + 8} = \dfrac{8x}{(x-2)(x^2-4)}$

02

$$\therefore \quad y = \frac{8x}{(x-2)^2(x+2)} = \frac{1}{x-2} + \frac{4}{(x-2)^2} - \frac{1}{x+2}$$

We know that

$$\text{if } y = \frac{1}{ax+b}, \quad y_n = \frac{(-1)^n\, n!\, a^n}{(ax+b)^{n+1}}$$

$$\therefore \quad \text{If } u = \frac{1}{x-2}, \quad u_n = \frac{(-1)^n\, n!(1)^n}{(x-2)^{n+1}}$$

$$\text{If } v = \frac{1}{(x-2)^2}, \quad v_n = \frac{(-1)^n\,(n+1)!(1)^n}{(x-2)^{n+2}}$$

$$\text{If } w = \frac{1}{x+2}, \quad w_n = \frac{(-1)^n\, n!(1)^n}{(x+2)^n}$$

04

$$\therefore \quad y_n = \frac{(-1)^n\, n!}{(x-2)^{n+1}} + 4 \cdot \frac{(-1)^n\,(n+1)!}{(x-2)^{n+2}} - \frac{(-1)^n}{(x+2)^n}.$$

OR

Expand $\tan^{-1} x$ in powers of $\left(x - \frac{\pi}{4}\right)$ (up to 2nd derivative)

Sol. : Let $f(x) = \tan^{-1} x$ and $a = \dfrac{\pi}{4}$.

$$\therefore \quad f(x) = \tan^{-1} x, \quad f'(x) = \frac{1}{1+x^2}, \quad f''(x) = -\frac{2x}{(1+x^2)^2}$$

$$\therefore \quad f\left(\frac{\pi}{4}\right) = \tan^{-1}\left(\frac{\pi}{4}\right), \quad f'\left(\frac{\pi}{4}\right) = \frac{1}{1+(\pi/4)^2}, \quad f''\left(\frac{\pi}{4}\right) = -\frac{\pi/2}{[1+(\pi^2/16)]^2}, \text{ etc.}$$

02

$$\text{Now,} \quad f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots.$$

$$\therefore \quad \tan^{-1} x = \tan^{-1}\frac{\pi}{4} + \left(x - \frac{\pi}{4}\right) \cdot \frac{1}{[1+(\pi^2/16)]} - \left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)^2 \cdot \frac{1}{[1+(\pi^2/16)]^2} + \ldots..$$

04

Q.2 (B)

If $y = \sin(m \sin^{-1} x)$ then prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$$

Sol. : We have $y_1 = -\sin(m\sin^{-1}x)\cdot m\cdot\dfrac{1}{\sqrt{1-x^2}}$

$\therefore\quad \sqrt{1-x^2}\cdot y_1 = -m\sin(m\sin^{-1}x).$ **03**

Differentiating again, we get

$$\sqrt{1-x^2}\cdot y_2 - \frac{xy_1}{\sqrt{1-x^2}} = -m\cos(m\sin^{-1}x)\cdot m\cdot\frac{1}{\sqrt{1-x^2}}$$

$\therefore\quad (1-x^2)\,y_2 - xy_1 + m^2y = 0$ .... (1)

Applying Leibnitz's Theorem to each term, we get

$$(1-x^2)\,y_{n+2} + n(-2x)\,y_{n+1} + \frac{n(n-1)}{2!}(-2)\,y_n - [\,xy_{n+1} + n\,y_n\,] + m^2y_n = 0$$

$$(1-x^2)\,y_{n+2} - 2nxy_{n+1} - n(n-1)\,y_n - xy_{n+1} - n\,y_n + m^2y_n = 0$$ **07**

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$$

OR

If $y = \tan^{-1}x$ then prove that $y_n = (-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta,$ where $\theta = \tan^{-1}\left(\frac{1}{x}\right)$

Sol. : Differentiating $y = \tan^{-1}x$, w.r.t. x, we get,

$$y_1 = \frac{1}{x^2+1} = \frac{1}{(x+i)(x-i)} = \frac{1}{2i}\left[\frac{1}{(x-i)} - \frac{1}{(x+i)}\right]$$

Differentiating $(n-1)$ times, *i.e.* replacing n by $(n+1)$, by result (4-A), page 5-3

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-i)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+i)^n}\right]$$

$$= \frac{(-1)^{n-1}\cdot(n-1)!}{2i}\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right]$$ .... (1) **03**

Putting $x = r\cos\theta,\ 1 = r\sin\theta$

i.e. $\quad r = \sqrt{1+x^2}$ and $\theta = \tan^{-1}(1/x)$, we get

$$\frac{1}{(x-i)^n} = \frac{1}{r^n(\cos\theta - i\sin\theta)^n} = \frac{1}{r^n}(\cos n\theta + i\sin n\theta)$$

Similarly, $\quad \dfrac{1}{(x+i)^n} = \dfrac{1}{r^n(\cos\theta + i\sin\theta)^n} = \dfrac{1}{r^n}(\cos n\theta - i\sin n\theta)$

$$\therefore\quad \frac{1}{(x-i)^n} - \frac{1}{(x+i)^n} = \frac{2i}{r^n}\sin n\theta$$

Hence, from (1), we get,

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\cdot\frac{2i}{r^n}\sin n\theta = (-1)^{n-1}(n-1)!\cdot\frac{1}{r^n}\sin n\theta$$ .... (2)

Since $\frac{1}{r} = \sin\theta$

$$y_n = (-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta, \text{ where } \theta = \tan^{-1}\left(\frac{1}{x}\right)$$ **07**

Q.3 (A)	Evaluate $\int_0^1 (x\log x)^3\,dx$, using gamma function Put $\log x = -t \;\therefore\; x = e^{-t} \quad \therefore\; dx = -e^{-t}\,dt$ $\int_0^1 x^3 (\log x)^3\,dx = \int_\infty^0 e^{-3t}(-t)^3(-e^{-t})\,dt = -\int_0^\infty e^{-4t}\cdot t^3\,dt$ Put $4t = u \therefore\; dt = \frac{1}{4}du$ $\therefore\quad I = -\int_0^\infty e^{-u}\left(\frac{1}{4}u\right)^3 \frac{1}{4}du = -\frac{1}{256}\int_0^\infty e^{-u}u^3\,du$ $= -\frac{1}{256}\left\lvert\overline{4}\right. = -\frac{1}{256}3! = -\frac{3}{128}.$ **OR**	**02** **04**
	Evaluate $\int_0^8 x^4(8-x)^{-1/3}\,dx$, using beta function Sol. : Put $x = 8t \quad \therefore\; dx = 8\,dt$ $\therefore\; I = \int_0^1 (8t)^4\cdot 8^{-1/3}\cdot(1-t)^{-1/3}\cdot 8\,dt$ $= \int_0^1 8^4 t^4\cdot\frac{1}{2}(1-t)^{-1/3}\cdot 8\,dt$ $= \frac{8^5}{2}\int_0^1 t^4(1-t)^{-1/3}\,dt = \frac{8^5}{3}\cdot B\left(5,\frac{2}{3}\right).$	**02** **04**
Q.3 (B)	Attempt any **TWO** of the following	**14**
(i)	Find the value of $\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}dx.\int_0^\infty y^4 e^{-y^6}dy$ Sol. : In I_1, put $x^3 = t, \quad \therefore\; x = t^{1/3} \quad \therefore\; dx = \frac{1}{3}t^{-2/3}dt$ $\therefore\quad I_1 = \int_0^\infty e^{-t}\cdot t^{-1/6}\cdot\frac{1}{3}\cdot t^{-2/3}\,dt$ $= \frac{1}{3}\int_0^\infty e^{-t}\cdot t^{-5/6}\,dt = \frac{1}{3}\left\lvert\overline{\tfrac{1}{6}}\right.$ In I_2, put $y^6 = t, \quad \therefore\; y = t^{1/6} \quad \therefore\; dy = \frac{1}{6}t^{-5/6}\,dt$ $\therefore\quad I_2 = \int_0^\infty t^{4/6}\cdot e^{-t}\cdot\frac{1}{6}t^{-5/6}\,dt = \frac{1}{6}\int_0^\infty e^{-t}\cdot t^{-1/6}\,dt = \frac{1}{6}\cdot\left\lvert\overline{\tfrac{5}{6}}\right.$	**03**

	$I = I_1 \times I_2$ $I = \frac{1}{3}\Gamma\left(\frac{1}{6}\right) \cdot \frac{1}{6}\Gamma\left(\frac{5}{6}\right) = \frac{1}{18}\Gamma\left(\frac{1}{6}\right) \cdot \Gamma\left(\frac{5}{6}\right) = \frac{1}{18} \cdot 2\pi = \frac{\pi}{9}$	**07**
(ii)	Prove that $\int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{4}\beta\left(\frac{n}{2}, \frac{n}{2}\right)$ and hence evaluate $\int_0^\infty \operatorname{sech}^8 x\, dx$ **Sol. :** We have $I = \int_0^\infty \frac{dx}{(e^x + e^{-x})^n} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{(e^x + e^{-x})^n}$ Put $e^x = \tan\theta$ $\therefore$ $e^x dx = \sec^2\theta\, d\theta$ $\therefore$ $dx = \frac{\sec^2\theta\, d\theta}{\tan\theta}$ When $x = \infty$, $e^x = \infty$, $\tan\theta = \infty$ $\therefore$ $\theta = \pi/2$ When $x = -\infty$, $e^x = 0$, $\tan\theta = 0$ $\therefore$ $\theta = 0$	

$$\therefore \quad I = \frac{1}{2}\int_0^{\pi/2} \frac{1}{(\tan\theta + \cot\theta)^n} \cdot \frac{\sec^2\theta}{\tan\theta} \cdot d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1}{\left(\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}\right)^n} \cdot \frac{1}{\cos^2\theta} \cdot \frac{\cos\theta}{\sin\theta} \cdot d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sin^n\theta\cos^n\theta}{\sin\theta\,\cos\theta} \cdot d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{n-1}\theta\cos^{n-1}\theta \cdot d\theta$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot B\left(\frac{n-1+1}{2}, \frac{n-1+1}{2}\right) = \frac{1}{4}B\left(\frac{n}{2}, \frac{n}{2}\right)$$

Since, $\dfrac{e^x + e^{-x}}{2} = \cos h\,x, \quad e^x + e^{-x} = 2\cos\ hx$

Putting $n = 8$ in the integral,

$$\therefore \quad \int_0^\infty \frac{dx}{(e^x + e^{-x})^8} = \int_0^\infty \frac{dx}{2^8 \cos h^8 x} = \frac{1}{4}B(4, 4)$$

$$\therefore \quad \int_0^\infty \sec h^8\, x\, dx = \frac{2^8}{4} \cdot \frac{\overline{|4}\ \overline{|4}}{\overline{|8}} = 2^6 \cdot \frac{3! \cdot 3!}{7!} = \frac{16}{35}.$$

(iii) Using DUIS technique, Show that $\int_0^\infty \frac{\log(1+ax^2)}{x^2}dx = \pi\sqrt{a},\ a > 0$

Sol. : Let I (a) be the given integral. Then by the rule of differentiation under the integral sign

$$\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a}dx = \int_a^\infty \frac{1}{x^2} \cdot \frac{1}{1+ax^2} \cdot x^2\,dx = \int_0^\infty \frac{dx}{1+ax^2}$$

$$= \frac{1}{a}\int_0^\infty \frac{dx}{(1/a)+x^2} = \frac{1}{a} \cdot (\sqrt{a})\left[\tan^{-1} x\sqrt{a}\right]_0^\infty = \frac{1}{\sqrt{a}} \cdot \frac{\pi}{2}$$

$$\therefore \quad \frac{dI}{da} = \frac{\pi}{2\sqrt{a}}$$ **04**

Integrating both sides, $\quad I(a) = \dfrac{\pi}{2}\displaystyle\int \frac{da}{\sqrt{a}} = \pi\sqrt{a} + c$

To find c, put $a = 0$, $\quad \therefore \quad I(0) = c.$

But $\quad I(0) = \int_0^\infty 0\,dx = 0 \quad \therefore\ c = 0 \quad \therefore\ I = \pi\sqrt{a}$

$$\therefore \quad \int_0^\infty \frac{\log(1+ax^2)}{x^2}dx = \pi\sqrt{a}.$$ **07**

<table>
<tr><td>Q.4</td><td>Attempt any TWO of the following</td><td>14</td></tr>
<tr><td>(i)</td><td>
Find the perimeter of cardioid $r = a(1-\cos\theta)$ and find the ratio in which line $\theta = \frac{2\pi}{3}$ divides the upper half of the curve.

Sol.: The shape of the curve is shown in the figure.

We have, $O(0, 0)$ and $B(2a, \pi)$

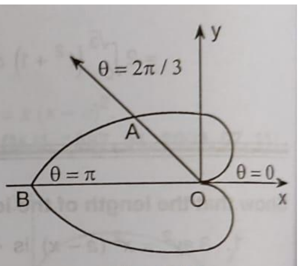

Fig. 8.17

$$\text{Arc } OB = s = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$$

$$\therefore \quad s = \int_0^{\pi} \sqrt{r^2 + a^2 \sin^2\theta} \cdot d\theta$$

$$= \int_0^{\pi} \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta} \cdot d\theta$$

$$= a\int_0^{\pi} \sqrt{2(1-\cos\theta)} \cdot d\theta = \int_0^{\pi} 2a \sin\left(\frac{\theta}{2}\right) d\theta$$

$$= 2a\left[-2\cos\left(\frac{\theta}{2}\right)\right]_0^{\pi} = 4a$$

$\therefore$ Perimeter of the cardioid $= 2s = 8a$.

Now, the arc where the line $\theta = 2\pi/3$, divides the cardioid is given by

$$\text{Arc } OA = \int_0^{2\pi/3} 2a \sin\frac{\theta}{2} \cdot d\theta = 2a\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi/3}$$

$$= -4a\left[\frac{1}{2} - 1\right] = 2a.$$

Hence, the line $\theta = 2\pi/3$ bisects the upper half of the cardioid.
</td><td>05

07</td></tr>
<tr><td>(ii)</td><td>Find the perimeter of the loop of the curve $9ay^2 = (x-2a)(x-5a)^2$</td><td></td></tr>
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Sol.: Differentiating the given equation w.r.t. x,

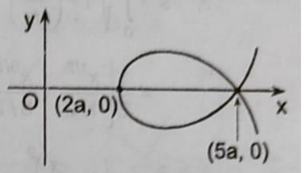


Fig. 8.12

$$18ay\frac{dy}{dx} = (x-2a)\cdot 2(x-5a) + (x-5a)^2$$
$$= (x-5a)(2x-4a+x-5a)$$
$$= 3(x-5a)(x-3a)$$
$$= (x-5a)(3x-9a)$$
$$\therefore \quad \frac{dy}{dx} = \frac{(x-5a)(x-3a)}{6ay}$$
$$\therefore \quad 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(x-5a)^2(x-3a)^2}{36a^2y^2} = 1+\frac{(x-5a)^2(x-3a)^2}{4a(x-2a)(x-5a)^2}$$
$$= 1+\frac{(x-3a)^2}{4a(x-2a)} = \frac{(x-a)^2}{4a(x-2a)}$$

$\therefore$ The perimeter of the loop of the curve

$$= 2\int_{2a}^{5a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx = 2\int_{2a}^{5a}\frac{x-a}{2\sqrt{a}\cdot\sqrt{x-2a}}\,dx$$
$$= \frac{1}{\sqrt{a}}\int_{2a}^{5a}\frac{(x-2a)+a}{\sqrt{x-2a}}\,dx = \frac{1}{\sqrt{a}}\int_{2a}^{5a}\left[\sqrt{x-2a}+a(x-2a)^{-1/2}\right]dx$$
$$= \frac{1}{\sqrt{a}}\left[\frac{2}{3}(x-2a)^{3/2}+a\cdot 2\cdot(x-2a)^{1/2}\right]_{2a}^{5a}$$
$$= \frac{1}{\sqrt{a}}\left[\frac{2}{3}(3a)^{3/2}+2a(3a)^{1/2}\right]$$
$$= \frac{1}{\sqrt{a}}\left[\frac{2}{3}\cdot 3\sqrt{3}\cdot a\sqrt{a}+2a\sqrt{3}\cdot\sqrt{a}\right]$$
$$= 2\sqrt{3}\,a+2a\cdot\sqrt{3} = 4\sqrt{3}\cdot a.$$

(iii) Find the length of curve from one cusp to the next cusp for $x = a(\theta+\sin\theta)$, $y = a(1-\cos\theta)$ **07**

Sol.: The curve is shown on the next page. Let the arc be measured from the origin O. For A, $\theta = -\pi$ and for B, $\theta = \pi$, for O, $\theta = 0$.

Hence, the length of the arc $AB = 2$ arc $OB = 2\int_0^{\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\cdot d\theta$.

But, $\frac{dx}{d\theta} = a(1+\cos\theta)$, $\frac{dy}{d\theta} = a\sin\theta$.

$$\therefore \quad s = 2\int_0^{\pi}\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\cdot d\theta$$
$$= 2a\int_0^{\pi}\sqrt{2(1+\cos\theta)}\cdot d\theta$$
$$= 4a\int_0^{\pi}\cos\left(\frac{\theta}{2}\right)\cdot d\theta = 4a\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8a.$$

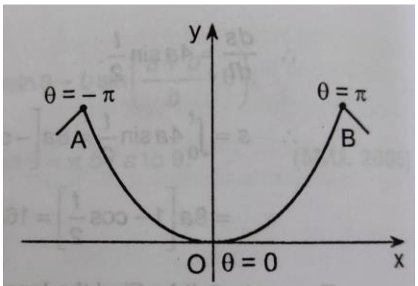
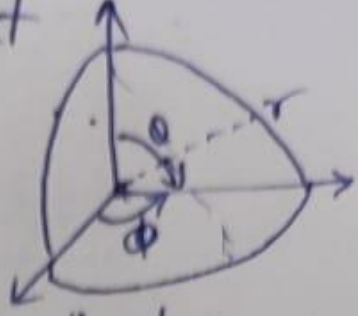


Fig. 8.10

Q.5 (A)	Evaluate $\int_1^2 \int_0^x \frac{1}{x^2+y^2} dy\, dx$ $I = \int_{x=1}^{2} \int_{y=0}^{x} \frac{1}{x^2+y^2} dy\, dx$ $I = \int_1^2 \left[\frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$ $= \int_1^2 \frac{1}{x}\left(\frac{\pi}{4} - 0\right) dx$ $= \frac{\pi}{4}\left[\log x\right]_1^2 = \frac{\pi}{4}(\log 2)$ **OR**	**04**
	Evaluate $\iiint_v \, dxdydz$ over the positive octant of a standard sphere of radius a	

For positive octant of standard sphere of radias a by considering spherical polar coordinates



$dx\,dy\,dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$

r varies from 0 to a,

θ varies from 0 to $\frac{\pi}{2}$ &

ϕ varies from 0 to $\pi/2$

$$\therefore I = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$\therefore I = \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\, d\theta \int_0^a r^2\, dr$$

(Constant limi

$$\therefore I = \left(\frac{\pi}{2} - 0\right)\left[-\cos\theta\right]_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^a$$

$$I = \frac{\pi}{2}\left[1\right]\left[\frac{a^3}{3}\right]$$

$$\boxed{I = \frac{\pi a^3}{6}}$$

Q.5 (B)	Attempt any **FOUR** of the following	28

(i) Evaluate $\iint_R e^{2x-3y}\,dx\,dy$ over the region bounded by lines $x+y=1, x=1$ and $y=1$

Sol. : The region of integration is the triangle ABC as shown in the adjoining figure. $x+y=1$ is the line AB, $x=1$ is the line AC and $y=1$ is the line BC.

Consider a strip parallel to the x-axis in the triangle ABC. On this strip x varies from $x=1-y$ to $x=1$. Then y varies from $y=0$ to $y=1$.

$$\therefore\ I=\int_0^1\int_{1-y}^1 e^{2x-3y}\,dx\,dy=\int_0^1\int_{1-y}^1 e^{2x}\cdot e^{-3y}\,dx\,dy$$

$$=\int_0^1 e^{-3y}\left[\frac{e^{2x}}{2}\right]_{1-y}^1 dy=\frac{1}{2}\int_0^1 e^{-3y}\left[e^2-e^{2(1-y)}\right]dy$$

$$=\frac{1}{2}\int_0^1 e^2\cdot e^{-3y}-\left[e^2\cdot e^{-5y}\right]dy=\frac{e^2}{2}\int_0^1\left[e^{-3y}-e^{-5y}\right]dy$$

$$=\frac{e^2}{2}\left[\frac{e^{-3y}}{-3}-\frac{e^{-5y}}{-5}\right]_0^1=\frac{e^2}{2}\left[-\frac{1}{3}(e^{-3}-1)+\frac{1}{5}(e^{-5}-1)\right]$$

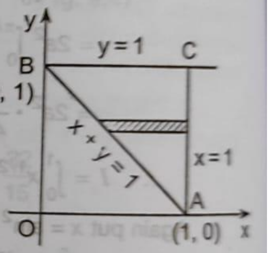


Fig. 9.50 (a)

$$=\frac{e^2}{2}\left[-\frac{e^{-3}}{3}+\frac{e^{-5}}{5}+\frac{2}{15}\right]=-\frac{e^{-1}}{6}+\frac{e^{-3}}{10}+\frac{2e^2}{15}$$

$$=\frac{e^{-3}}{10}-\frac{e^{-1}}{6}+\frac{2}{15}e^2$$

Marks: 04, 07

(ii) Change to polar and evaluate $\iint_R \sqrt{a^2-x^2-y^2}\,dxdy$ where R is area of the upper half of the circle $x^2+y^2=ax$

Sol. : 1. Region in Cartesian Coordinates : The circle

$$x^2+y^2-ax=0 \text{ i.e., } \left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$$ is a circle with centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$.

The region of integration is the upper-half of this circle.

2. Region in Polar Coordinates : Putting $x=r\cos\theta$ and $y=r\sin\theta$, the above circle becomes $r^2=ar\cos\theta$ i.e., $r=a\cos\theta$. The x-axis is $\theta=0$ and the y-axis is $\theta=\pi/2$.

3. Limits of r, θ : Considering a radial strip in the region of integration, we see that r varies from $r=0$ to $r=a\cos\theta$ and θ varies from $\theta=0$ to $\theta=\pi/2$.

4. Integrand : Putting $x=r\cos\theta$ and $y=r\sin\theta$ in the integrand $\sqrt{a^2-x^2-y^2}$, we get $\sqrt{a^2-r^2}$ and we replace $dx\,dy$ by $r\,dr\,d\theta$.

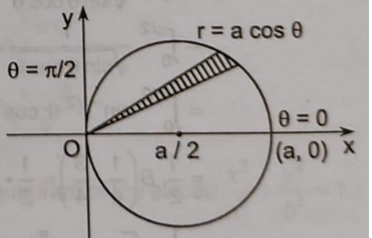


Fig. 9.73

Marks: 04

$$I = \int_0^{\pi/2} \int_0^{a\cos\theta} \sqrt{a^2 - r^2} \cdot r\,dr\,d\theta$$

Now, put $a^2 - r^2 = t \quad \therefore \; -2r\,dr = dt$

When $r = a\cos\theta$, $t = a^2 \sin^2\theta$. When $r = 0$, $t = a^2$.

$$\therefore \; I = \int_0^{\pi/2} \int_{a^2}^{a^2\sin^2\theta} t^{1/2}\left(-\frac{1}{2}\right) dt\,d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} \left[\frac{t^{3/2}}{3/2}\right]_{a^2}^{a^2\sin^2\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2} (a^3 \sin^3\theta - a^3)\,d\theta = \frac{a^3}{3}\int_0^{\pi/2} (1 - \sin^3\theta)\,d\theta$$

$$= \frac{a^3}{3}\left[\int_0^{\pi/2} d\theta - \int_0^{\pi/2} \sin^3\theta\,d\theta\right] = \frac{a^3}{3}\left[\frac{\pi}{2} - \frac{2}{3}\cdot 1\right] = \frac{a^3}{18}(3\pi - 4)$$

(iii) Find by double integration the area between the circles $r = 2a\sin\theta$ and $r = 2b\sin\theta, b > a$

Sol. : We have $r = 2a\sin\theta$

i.e. $\sqrt{x^2 + y^2} = 2a \cdot \dfrac{y}{\sqrt{x^2 + y^2}}$

i.e. $x^2 + y^2 = 2ay$ i.e. $x^2 + (y - a)^2 = a^2$.

Similarly, $r = 2b\sin\theta$ gives $x^2 + (y - b)^2 = b^2$.

These are the circles with centres $(0, a)$, $(0, b)$ and radii a, b.

Now, consider a radial strip. On this strip r varies from $r = 2a\sin\theta$ to $r = 2b\sin\theta$. Then θ varies $\theta = 0$ to $\theta = \pi/2$ in the first quadrant.

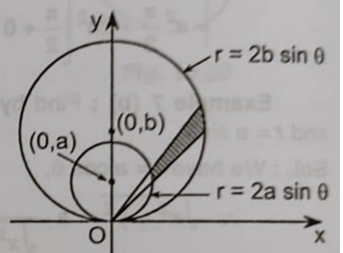


Fig. 10.33

04

$$\therefore \; A = 2\int_0^{\pi/2} \int_{a\sin\theta}^{b\sin\theta} r\,dr\,d\theta = 2\int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{2a\sin\theta}^{2b\sin\theta} d\theta$$

$$= 4\int_0^{\pi/2} (b^2\sin^2\theta - a^2\sin^2\theta)\,d\theta = 4(b^2 - a^2)\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= 4(b^2 - a^2)\cdot\frac{1}{2}\cdot\frac{\pi}{2} = (b^2 - a^2)\pi.$$

07

(iv) Evaluate $\int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2\,dz\,dy\,dx$

Sol. :

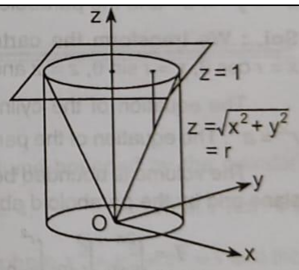
$$I = \int_0^a \int_0^{a-x} \left[x^2 z\right]_0^{a-x-y} dy\,dx = \int_0^a \int_0^{a-x} x^2(a - x - y)\,dy\,dx$$

$$= \int_0^a \left[x^2(a - x)\,y - x^2\frac{y^2}{2}\right]_0^{a-x} dx = \frac{1}{2}\int_0^a x^2(a - x)^2\,dx$$

$$= \frac{1}{2}\int_0^a (a^2x^2 - 2ax^3 + x^4)\,dx = \frac{1}{2}\left[a^2\frac{x^3}{3} - 2a\frac{x^4}{4} + \frac{x^5}{5}\right]_0^a = \frac{a^5}{60}.$$

07

(v) Evaluate $\iiint \sqrt{x^2 + y^2}\,dxdydz$ over the volume bounded by the right circular cone $x^2 + y^2 = z^2,\; z > 0$ and the planes $z = 0$ and $z = 1$

	Sol. : We transform the given integral to cylindrical polar coordinates by putting $x = r\cos\theta,\ y = r\sin\theta,\ z = z$ and $dx\,dy\,dz = r\,dr\,d\theta\,dz.$ Now limits for r are 0 to 1 for θ are 0 to 2π, for z are r to 1. $\therefore\ I = \int_0^1 r^2\,dr \int_0^{2\pi} d\theta \int_r^1 dz = \int_0^1 r^2\,dr\,[\theta]_0^{2\pi}\,[z]_r^1$ $= \int_0^1 r^2 \cdot 2\pi \cdot (1-r)\,dr = 2\pi\left[\frac{r^3}{3} - \frac{r^4}{4}\right]_0^1 = \frac{2\pi}{12} = \frac{\pi}{6}.$  Fig. 12.11	