Relations

Section 8.1, 8.3—8.5 of Rosen

CSCE 235 Introduction to Discrete Structures

Course web-page: cse.unl.edu/~cse235

Relations, Digraphs (07)

3.1 Relations, Paths and Digraphs

3.2 Properties and types of binary relations

3.3 Manipulation of relations, Closures, Warshall‟s algorithm

3.4 Equivalence relations

Introduction

- A relation between elements of two sets is a subset of their Cartesian products (set of all ordered pairs)
- **Definition**: A binary relation from a set A to a set B is a subset $R \subseteq A \times B = \{ (a,b) \mid a \in A, b \in B\}$
- When $(a,b) \in R$, we say that a is related to b.
- Notation: aRb, $a\not{R}b$

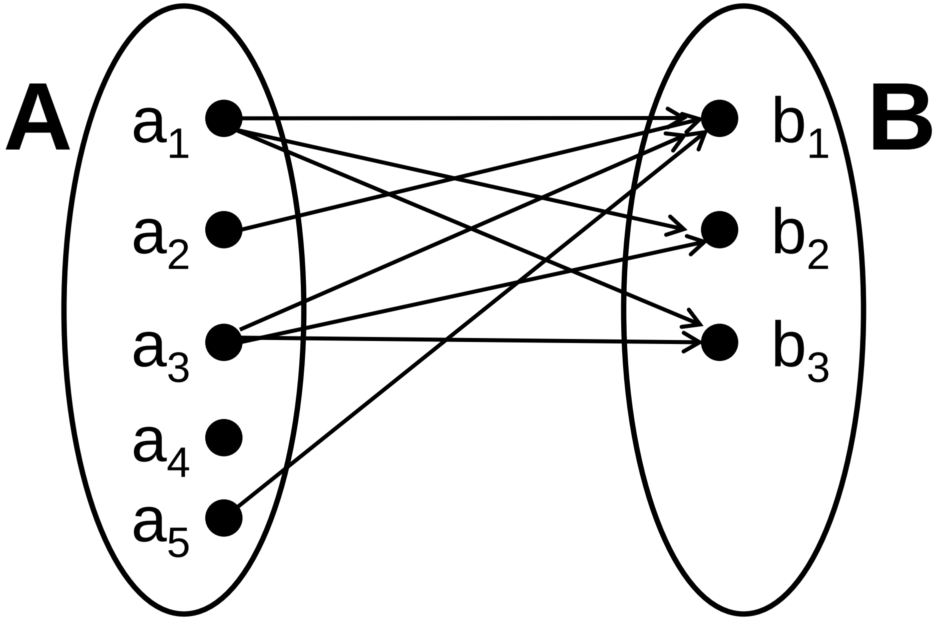
Relations: Representation

- To represent a relation, we can enumerate every element of R
- Example
  - Let A={a_1,a_2,a_3,a_4,a_5} and B={b_1,b_2,b_3}
  - Let *R* be a relation from A to B defined as follows

$$R=\{(a_1,b_1),(a_1,b_2),(a_1,b_3),(a_2,b_1),(a_3,b_1),(a_3,b_2),(a_3,b_3),(a_5,b_1)\}$$

Graphically,

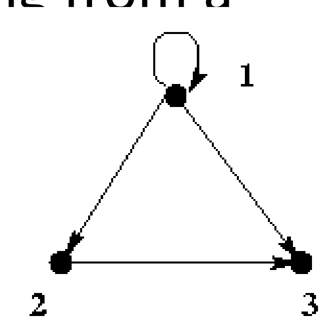
DOMAIN



RANGE

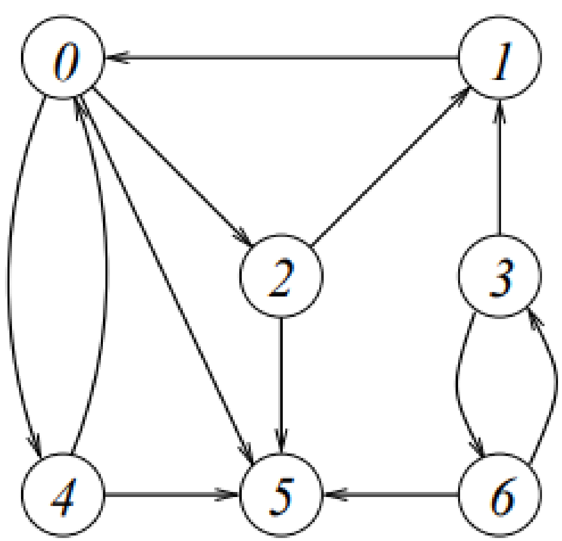
DIGRAPHS-Directed Graphs

A **digraph** (directed graph) is a diagram composed of points called **vertices** (nodes) and arrows called **edges** going from a vertex to a vertex.



Example :-A digraph with 3 vertices and 4 edges

Example: -V = {0, 1, 2, 3, 4, 5, 6 } ,E = {(0, 2),(0, 4),(0, 5),(1, 0),
(2, 1),(2, 5),(3, 1),(3, 6), (4, 0),(4, 5),(6, 3),
(6, 5) }



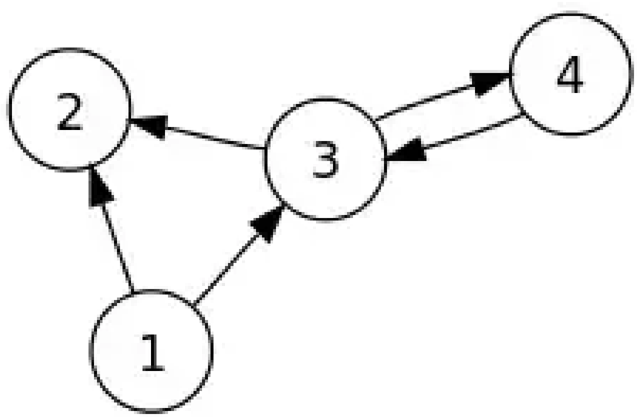
Matrix Representation ?

Degree of Vertex in a Directed Graph

A directed graph, each vertex has an **in-degree** and an **out-degree**.

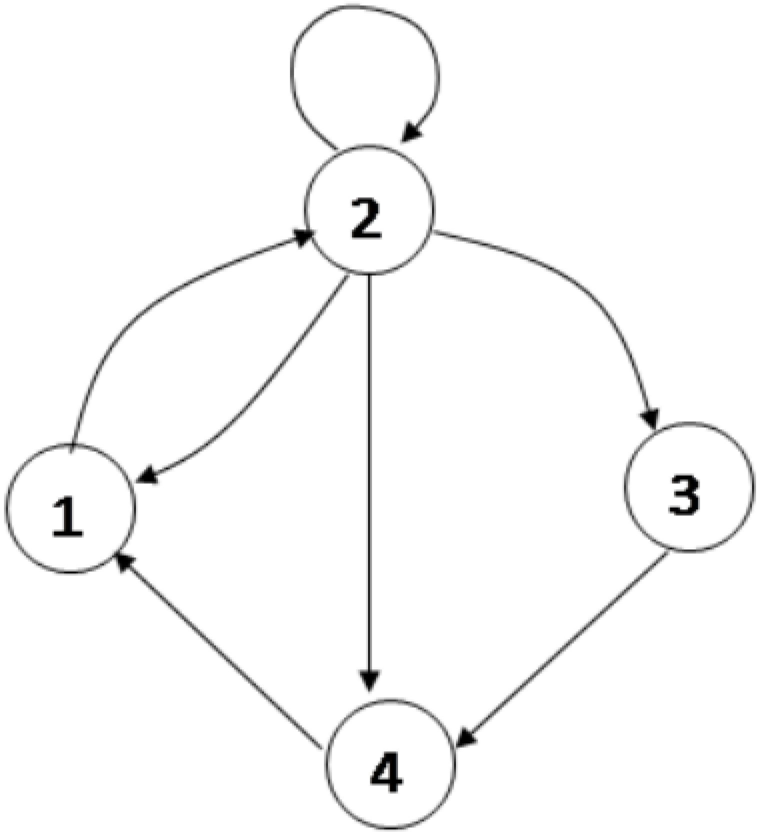
In-degree of a Graph-Number of edges which are coming into the vertex V.

Out-degree of a Graph-Number of edges which are going out from the vertex V



VERTEX	1	2	3	4
In Degree	0	2	2	1
Out-degree	2	0	2	1

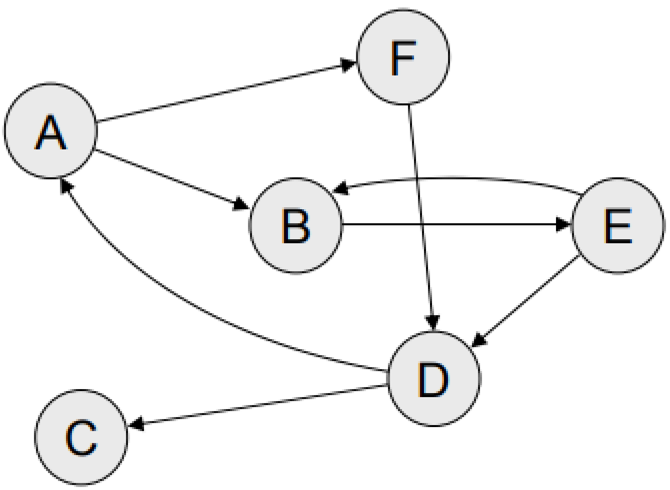
Find out in degree and out degree



VERTEX	1	2	3	4
In Degree	2	2	1	2
Out-degree	1	4	1	1

Problems

For the digraph shown let R be given by digraph shown. Find M_R and R

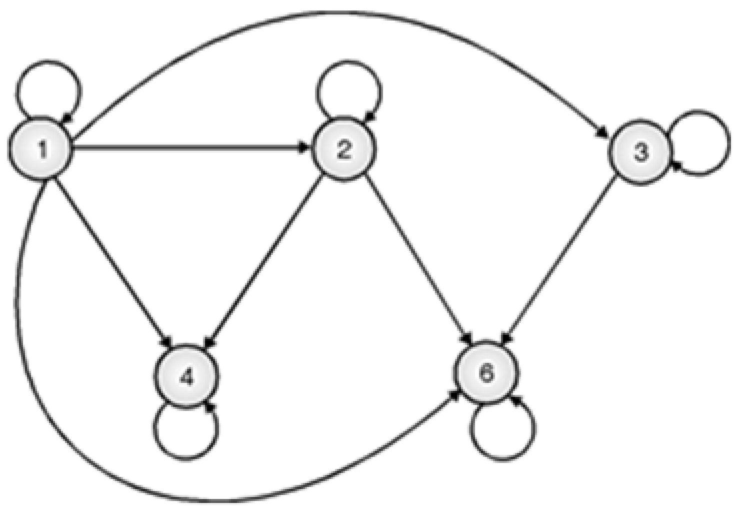


$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}
\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\
\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Example

Let A = {1, 2, 3, 4, 6} and let R be the relation on A defined by 'x divides y'. Find R and draw the digraph of R. Find Matrix of R.

Soln.: $R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (2,2),$ (4,4)
$(2,4), (2,6), (3,3), (3,6), (6,6)\}$



Assume the rows and columns of M are each labelled 1, 2, 3, 4, 6, since R is relation from A to A, the matrix M_R is square, i.e. M_R has the same number of row as column

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Example

Let A = {1, 2, 3, 4, 6} = B, a R b if and only if a is a multiple of b. Find R and draw the digraph of R. Find Matrix of R.

Solution:
R={(1, 1), (2, 1), (2, 2), (3, 1), (3, 3), (4,1), (4, 2), (4, 4), (6, 1), (6, 2), (6, 3), (6, 6)}

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Problems

1. Draw the graphical representation of relation ' less than ' on { 1 , 2 , 3 , 4 }

R= { (1 , 2) , (1 , 3) , (1 , 4) , (2 , 3), (2 , 4), (3 , 4) }

2. A = { 2 , 3 , 4 , 5 } ,

R = { (2 , 3) , (3 , 2) , (3 , 4) , (3 , 5) , (4 , 3) , (4 , 4) , (4 , 5) } Draw Digraph

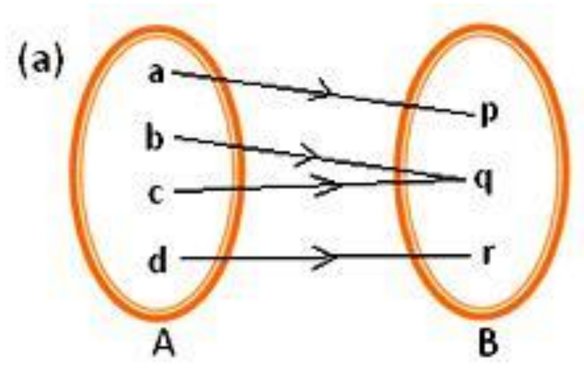
→Domain, Range of Relation R

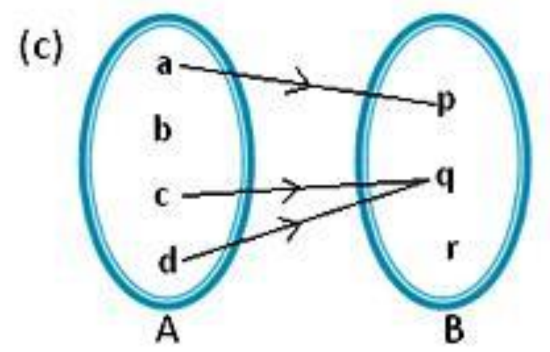
Ex : A={ a , b , c , d } , B = { 1, 2 ,3 }

R= { (a ,1), (a , 2), (b , 1), (c ,2), (d, 1)}

Dom(R)={a , b , c , d }

Ran (R)={ 1 , 2 }





Problems

1. Let A = { 1 , 2 , 3 , 4 , 8 } = B only **if a=b.**

Find the relation R ,draw digraph and also write M_R

2. Let A = { 1 ,2 , 3 , 4 , 8 } = B

    a R b iff a is **a multiple of b**

    a R b iff **a + b < = 9**

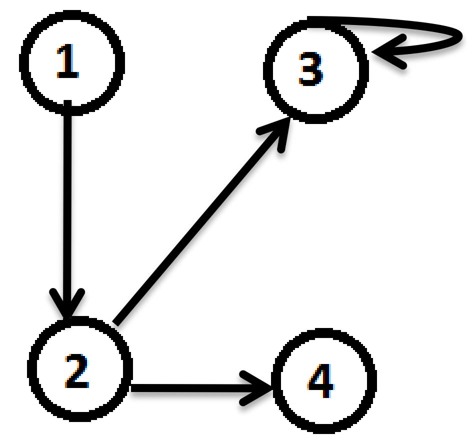
Find the relation R ,draw digraph and also write M_R

3. Let A = { 1,3 ,5 ,7 ,9 }, B = { 2, 4, 6, 8 } ; aRb iff **b < a**

PATHS

R = { (1 , 2) ,(2 , 3) , (2 , 4), (3 , 3) } is a relation on A = {1,2,3,4}

$R^1 = R = \{(1,2),(2,3),(2,4), (3,3)\}$



$R^2 = \{(1,3), (1,4), (2,3), (3,3)\}$

1 R^2 3 Since 1 R 2 and 2 R 3

1 R^2 4 Since 1 R 2 and 2 R 4 ...

$R^3 = \{ (1 , 3) , (2 , 3), (3 , 3) \}$

$R^4 = \{ (1 , 3) , (2 , 3), (3 , 3)\}$

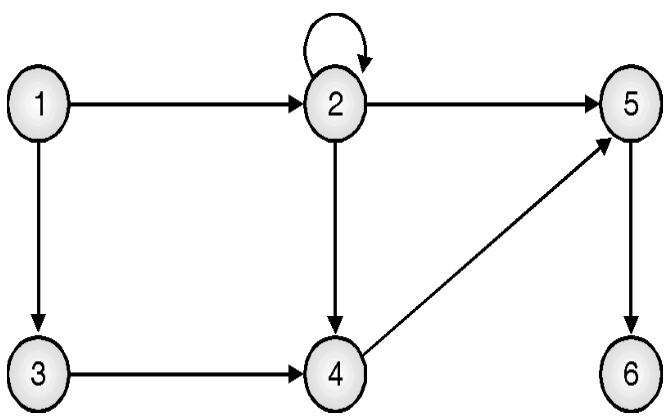
R infinity is all orderd pairs where there is a path of any length

$R^\infty = \{ (1,2) ,(1,3),(1,4),(2,3),(2,4),(3,3)\}$

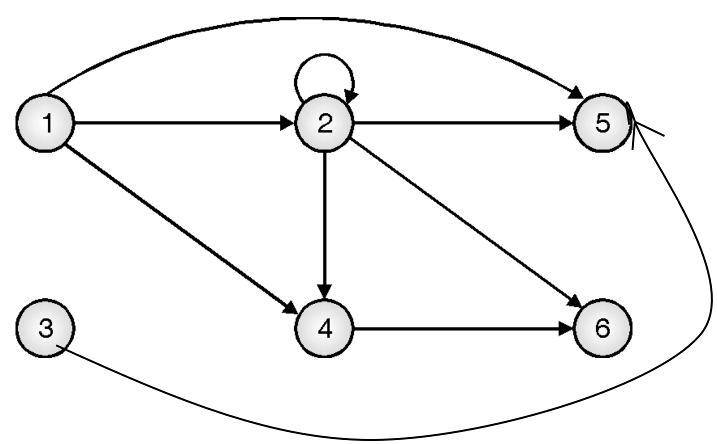
Paths in Relations and Digraphs

Let A = {1, 2, 3, 4, 5, 6}. Let R be the relation whose digraph is shown in Fig.

Find R^2 and draw digraph of the relation R^2.



$1\ R^2\ 2$	Since	1 R 2	and	2 R 2
$1\ R^2\ 4$	Since	1 R 2	and	2 R 4
$1\ R^2\ 5$	Since	1 R 2	and	2 R 5
$2\ R^2\ 2$	Since	2 R 2	and	2 R 2
$2\ R^2\ 4$	Since	2 R 2	and	2 R 4
$2\ R^2\ 5$	Since	2 R 2	and	2 R 5
$2\ R^2\ 6$	Since	2 R 5	and	5 R 6
$3\ R^2\ 5$	Since	3 R 4	and	5 R 5
$4\ R^2\ 6$	Since	4 R 5	and	5 R 6

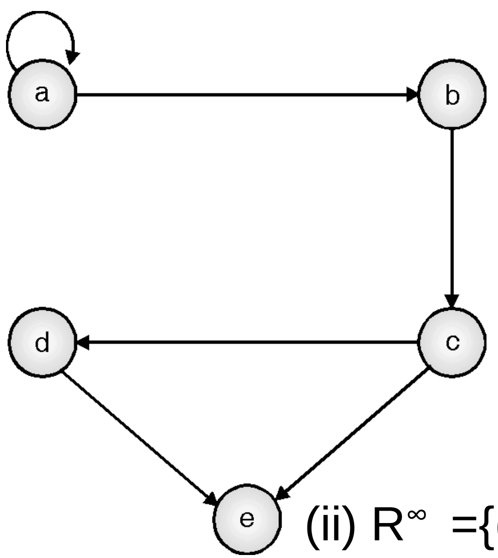


Paths in Relations and Digraphs

Let $A = \{a, b, c, d, e\}$

and $R = \{(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)\}$

Compute (i) R^2 (ii) $R\infty$



$a\,R^2\,a$	Since	$a\,R\,a$	and	$a\,R\,a$
$a\,R^2\,b$	Since	$a\,R\,a$	and	$a\,R\,b$
$a\,R^2\,c$	Since	$a\,R\,b$	and	$b\,R\,c$
$b\,R^2 e$	Since	$b\,R\,c$	and	$c\,R\,e$
$b\,R^2\,d$	Since	$b\,R\,c$	and	$c\,R\,d$
$c\,R^2\,e$	Since	$c\,R\,d$	and	$d\,R\,e$

(ii) R^{∞} ={(a,a),(a,b),(a,c),(a,d),(a,e),(b,c),(b.d),(b,e),(c,d),(c,e),(d,e)}

PROBLEMS

1. Let A = { 1 , 2 , 3 , 4 , 5 } and R be relation defined by a R b iff a < b compute R , R^{2}, R^{3} Draw digraph of R , R^2 and R^3

R = (1 , 2) , (1 , 3) , (1 , 4) , (1 , 5) , (2 , 3) , (2 , 4) , (2 , 5) , (3 , 4) , (3 ,5) , (4 , 5) }

R^2 = { (1 , 3) , (1 , 4) , (1 , 5) , (2 , 4) , (2 , 5) , (3 , 5) }

R^3 ={ (1 ,4) , (1, 5), (2 ,5)}

2. Consider *R* = { (1 , 1) , (2 , 1) , (3 , 2) , (4 , 3) }

Compute R^{2} R^{3} R^{4}

3. Let A = { a , b , c , d , e } , R ={ (a , a) , (a , b) , (b , c) , (c , e) , (c , d) , (d , e) }

Draw digraph of R , M_R , Compute R^{∞}

Properties/Types of Relations

– Reflexive
– Symmetric
– Transitive
– Antisymmetric
– Asymmetric

Properties: Reflexivity

- In a relation on a set, if all ordered pairs (a,a) for every $a \in A$ appears in the relation, R is called reflexive
- **Definition**: A relation *R* <u>on a set A</u> is called <u>reflexive</u> iff

$$\forall a \in A \ (a, a) \in R$$

  - Eg: $A = \{1, 2, 3\}$,

    $R = \{(1, 1), (2, 2), (3, 3)\}$

  - Irreflexive ?

  Assume the relation R on $A = \{1, 2, 3, 4\}$ Is R1/R2 irreflexive?

  $R1 = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$

  $R2 = \{(1, 2), (2, 2), (3, 3)\}$

Properties: Symmetry

- **Definitions**:
  – A relation R on a set A is called <u>symmetric</u> if whenever **a R b and b R a** i.e

$$\forall\, a , b \in A \;\; ((b , a) \in R \Leftrightarrow (a , b) \in R \;)$$

Eg 1 : A = { 1 , 2, 3} , Is R symmetric ?

R = { (1 , 2) , (2 , 1) , (2 , 3), (3 , 2) , (1, 1)) }

Eg 2 : A = { 1 , 2 , 3 , 4 } , Is R symmetric ?

R= { (1 , 2) , (1 , 3) , (1 , 4) , (2 , 1) , (2 , 3) , (2 , 4) , (3 , 1) , (3 , 2), (3 ,4) , (4 , 1) , (4 , 2) , (4 , 3) }

Asymmetric relation: Asymmetric relation is opposite of symmetric relation.

A relation R on a set A is called asymmetric if no (b,a) € R when (a,b) € R

AntiSymmetric Relation: A relation R on a set A is called antisymmetric if (a,b)€ R and (b,a) € R if a = b is called antisymmetric.i.e.

UNLESS there exists (a , b)∈ R and (b , a) ∈ R, AND a ≠ b

Eg : A = { 1 , 2 , 3 , 4 } and R = { (1 , 2) , (2 , 2) , (3 , 3) }

Is R anti-symmetric?

Answer: Yes. It is anti-symmetric as 2,1 is not there

Symmetry versus Antisymmetry

- In a symmetric relation a*R*b $\Leftrightarrow$ b*R*a
- In an antisymmetric relation, if we have a*R*b and b*R*a hold only when a=b
- An antisymmetric relation is not necessarily a reflexive relation
- A relation that is not symmetric is not necessarily asymmetric
- **An anti-symmetric relation is a binary relation where the following two conditions are met:**
- **1) If A is related to B, then B cannot be related to A.**
  2) If A is not related to B, then B cannot be related to A.
- In Maths, we can conclude that a binary relation on a set is called as antisymmetric if there is no pair of distinct elements.

Properties: Transitivity

- **Definition**: A relation R on a set A is called <u>transitive</u> if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ for all $a,b,c \in A$

$$\forall a, b, c \in A\ ((a\ R\ b) \wedge (b\ R\ c)) \Rightarrow a\ R\ c$$

Example

R={ (1 , 2) , (2 , 3) , (1 , 3) } on set

A = { 1 , 2 , 3 } is transitive.

Special cases

1) Let $A = \{ 1 , 2 , 3 , 4 \}$

$R = \{ (1 , 2) , (1 , 3) , (4 , 2) \}$

Is R transitive?

YES

2) $R = \{ \ \}$

3)A relation that is symmetric and anti-symmetric

$R = \{(1,1),(2,2)\}$ on the set $A = \{1,2,3\}$

Properties of Relations

State whether R satisfies property of reflexive , irreflexive , symmetry, asymmetry , antisymmetry , transitivity for A={1,2,3,4}

1. R={(1,1),(1,2),(2,1),(2,2),(3,3),(4,3),(3,4),(4,4)}**R,S,T,**
2. R= {(1,3),(1,1),(3,1),(1,2),(3,3),(4,4)}
3. R={(1,1),(1,2),(2,1),(2,2),(3,3),(3,4)}
4. R={(1,3),(1,4),(2,3),(2,4),(3,1),(3,4)}
5. R={(1,1),(2,2),(3,3),(4,4)}

EQUIVALENCE RELATION

A relation is an **Equivalence Relation** if it is **REFLEXIVE, SYMMETRIC, AND TRANSITIVE**.

Let A = { a , b , c } and

R= { (a , a), (b , b), (b , c), (c , b),(c , c) }

is an equivalence relation since it is

REFLEXIVE, SYMMETRIC, & TRANSITIVE.

Determine whether R is an Equivalence relation

1) $R = \{ (1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2), (1,3), (3,1) \}$

   on set $A = \{1, 2, 3\}$

2) $A = \{1, 2, 3, 4\}$, $R = \{ (1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4) \}$

3) Let $A = \{a, b, c, d\}$

   $R = \{ (a,a), (b,a), (b,b), (c,c), (d,d), (d,c) \}$

Determine whether R is an Equivalence relation

Let A = {a, b, c} and let , $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Determine whether R is an equivalence relation.

Soln.: R = {(a, a), (b, b), (b, c), (c, b), (c, c)}

R is reflexive since (a, a), (b, b), (c, c) $\in$ R

R is symmetric since (b, c) $\in$ R $\rightarrow$ (c, b) $\in$ R

R is transitive since,

(b,b)	and	(b, c) $\in$ R	implies	(b, c) $\in$ R,
(b, c)	and	(c, b) $\in$ R	implies	(b, b) $\in$ R,
(c, c)	and	(c, b) $\in$ R	implies	(c, b) $\in$ R,
(c, b)	and	(b, b) $\in$ R	implies	(c, b) $\in$ R,
(c, b)	and	(b, c) $\in$ R	implies	(c, c) $\in$ R,
(b, c)	and	(c, c) $\in$ R	implies	(b, c) $\in$ R,

Hence R is an equivalence relation.

Equivalence Class and Partitions

- Let $A = \{ 1 , 2 , 3 , 4 \}$ and consider the partition

  $P = \{ \{ 1 , 2 , 3 \} , \{ 4\} \}$ of A.

Find the equivalence relation R on A determined by P

" Each element in a block is related to every other element in the same block and only to those elements "

$R= \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(4,4)\}$

Problems

Find the equivalence relation on A by P and construct its **digraph**

1) Let A ={ a , b , c , d } and P = {{a , b } , { c }, { d } }

2) Let A={1,2,3,4,5} and P={{ 1 ,2 },{ 3 },{ 4, 5}}

R={(1,1),(1,2),(2,1),(2,2),(3,3),(4,4),(4,5),(5,4),(5,5)}

3) If {{1,3,5},{2,4}} is a partition on the set A={1,2,3,4,5},determine the corresponding equivalence relation

R={(1,1),(3,3),(5,5),(1,3),(1,5),(3,5),(3,1),(5,1),(5,3),(2,2),(4,4),(2,4),(4,2)}

EQUIVALENCE CLASS

Let $A = \{1,2,3,4,5,6\}$ and let R be the equivalence relation on A defined by

$R=\{(1,1),(1,5),(2,2),(2,3),(2,6),(3,2),(3,3),(3,6),(4,4),(5,1),(5,5),(6,2),(6,3),(6,6)\}$

Find the equivalence classes of R and find the partition of A induced by R

$R=\{(1,1),(1,5),(2,2),(2,3),(2,6),(3,2),(3,3),(3,6),(4,4),(5,1),(5,5),(6,2),(6,3),(6,6)\}$

Equivalence Classes:

$R(1)=\{1,5\}$ $R(2)=\{2,3,6\}$

$R(3)=\{2,3,6\}$ $R(4)=\{4\}$

$R(5)=\{1,5\}$ $R(6)=\{2,3,6\}$

Therefore, the partition of A induced by R i.e

$A|R=\{\{1,5\},\{2,3,6\},\{4\}\}$

Rank R (Number of distinct equivalence classes)

$= 3$

Problems

1. Let A={1,2,3} and let R={(1,1),(2,2),(1,3),(3,1),(3,3)}.

   Find A|R.

2. Let A ={1,2,3,4},and let

   R={(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)}

   Determine A|R.

3. Let A ={1,2,3,4},and let

   R={(1,1),(1,2),(1,3),(2,1),(2,2),(3,1),(2,3),(3,2),(3,3),(4,4)}

   Show that R is an equivalence relation and determine the equivalence classes and hence find A|R and rank of R

Combining Relations

- Relations are simply… sets (of ordered pairs); subsets of the Cartesian product of two sets
- Therefore, in order to <u>combine</u> relations to create new relations, it makes sense to use the usual set operations
  - **Compliment R**
  - **Intersection ($R_1 \cap R_2$)**
  - **Union ($R_1 \cup R_2$)**
  - **Set difference ($R_1 \setminus R_2$)**
  - **Inverse R^{-1}**

Example: Let $A = \{ 1 , 2 , 3 \}$ and $B = \{ u , v \}$ and

$$R1 = \{ (1 , u) , \; (2 , u) , \; (2 , v) , \; (3 , u) \}$$

and

$$R2 = \{ (1 , v) , (3 , u) , (3 , v) \}$$

$R1 \cup R2 =$

$\{ (1 , u) , (1 , v) , (2 , u) , (2 , v) , (3 , u) , (3 , v) \}$

$R1 \cap R2 =$

$\{ (3 , u) \}$

$R1 - R2 =$

$\{ (1 , u) , (2 , u) , (2 , v) \}$

$R2 - R1 =$

$\{ (1 , v) , (3 , v) \}$

$A = \{a, b, c, d\}$ and

$R=\{(a, b), (b, c), (a, c), (c, d)\}$ then

$\mathbf{R^{-1}}=\{(b, a), (c, b), (c, a), (d, c)\}$

Let $A=\{ 1 , 2 , 3 , 4 \}$ and $B=\{ a , b , c \}$ and let

$R =\{(1,a),(1,b),(2,b),(2,c),(3,b),(4,a)\}$ and

$S=\{(1,b),(2,c),(3,b),(4,b)\}$

Compute $R \cap S$, $R \cup S$, , S^{-1} and R^{-1}

Combining Relations: Example

- Let
  - $A=\{1,2,3,4\}$
  - $B=\{1,2,3,4\}$
  - $R_1=\{(1,2),(1,3),(1,4),(2,2),(3,4),(4,1),(4,2)\}$
  - $R_2=\{(1,1),(1,2),(1,3),(2,3)\}$
- Let
  - $R_1 \cup R_2 =$
  - $R_1 \cap R_2 =$
  - $R_1 - R_2 =$
  - $R_2 - R_1 =$

Composite of Relations

- **Definition**: Let R_1 be a relation from the set A to B and R_2 be a relation from B to C, i.e.

  $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$

  the <u>composite of</u> R_1 and R_2 is the relation consisting of ordered pairs (a,c) where $a \in A$, $c \in C$ and for which there exists an element $b \in B$ such that $(a,b) \in R_1$ and $(b,c) \in R_2$. We denote the composite of R_1 and R_2 by

$$R_2 \circ R_1$$

Ex: Let $A = \{ 1 , 2 , 3 \}$, $B = \{ 0 , 1 , 2 \}$ and $C = \{ a , b \}$

$R = \{ (1 , 0) , (1 , 2) , (3 , 1) , (3 , 2)\}$

$S = \{ (0 , b) , (1 , a) , (2 , b) \}$

$S \text{ o } R = ?$

$\{ (1 , b) , (3 , a) , (3 , b) \}$

Since $(1,0) \in R$ and $(0,b) \in S$, $\therefore$ $(1,b) \in S \text{ o } R$

Since $(1,2) \in R$ and $(2,b) \in S$, $\therefore$ $(1,b) \in S \text{ o } R$

Since $(3,1) \in R$ and $(1,a) \in S$, $\therefore$ $(3,a) \in S \text{ o } R$

Since $(3,2) \in R$ and $(2,b) \in S$, $\therefore$ $(3,b) \in S \text{ o } R$

Problems

1. Let A={1,2,3} and let
   R={(1,1),(1,3),(2,1),(2,2),(2,3),(3,2)} and
   S={(1,1),(2,2),(2,3),(3,1),(3,3)}.
   Find M_{SoR}

SoR={(1,1),(1,3),(2,1),(2,2),(2,3),(3,2),(3,3)}

2. Let A={1,2,3,4}
   R={(1,1),(1,2),(2,3),(2,4),(3,4),(4,1),(4,2)}
   S={(3,1),(4,4),(2,3),(2,4),(1,1),(1,4)}

Compute SoR,RoS,RoR,SoS

SoR={(1,1),(1,3),(2,1),(2,4),(3,4),(4,1),(4,4),(1,4),(4,3)}

RoS={(3,1),(3,2),(4,1),(4,2),(2,4),(2,1),(2,2),(1,1),(1,2)}

RoR=13 elements

SoS=7 elements

Reflexive closure

Let R be a relation on a set A, and R is not reflexive (i.e. some pairs of the diagonal relation Δ are not in R).

A relation $R_1 = R \cup \Delta$ is the reflexive closure of the relation R if $R \cup \Delta$ is the smallest relation containing R which is reflexive.

R_1=R $\cup$ Δ where Δ is the set of elements of the type (a, a) where a $\in$ A.

Example -Reflexive closure

A =$\{1, 2, 3\}$ and the relation R is given by

R =$\{(1, 1), (1, 2), (2, 3)\}$ then

R_1 = $R \cup \Delta$ where

Δ =$\{(1, 1), (2, 2), (3, 3)\}$

$R \cup \Delta=\{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}$

Reflexive closure is,

$$R_1 =\{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}$$

Symmetric closure

Suppose that R is a relation on A that is not symmetric.

Then there must exist pairs (x, y) in R such that (y, x) is not in R. Of course, $(y, x) \in R^{-1}$, so if R is to be symmetric we must add all pairs from R^{-1};

that is we must enlarge R to $\mathbf{R \cup R^{-1}}$**.**

so $R \cup R^{-1}$ is the smallest symmetric relation containing R;

that is $R \cup R^{-1}$ is the 'symmetric closure' of R.

example

A = {a, b, c, d} and

R={(a, b), (b, c), (a, c), (c, d)} then

R^{-1}={(b, a), (c, b), (c, a), (d, c)}

so the symmetric closure of R is

$R \cup R^{-1}$={(a, b), (b, a), (b, c), (c, b), (a, c), (c, a), (c, d), (d, c)}

Transitive closure

Let R be a relation on a set A. Then the 'transitive closure' of a relation R is the smallest transitive relation containing R. The transitive closure of R is just the connectivity relation R^{∞}.

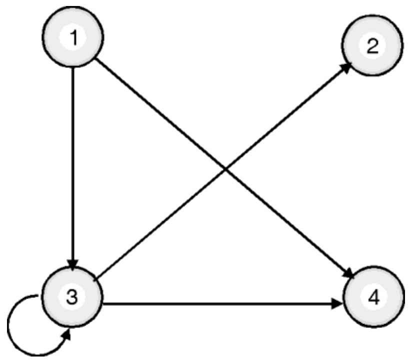
R^*=Transitive closure of R

$=R \cup \{(a, c)$, if and only if $(a, b), (b, c) \in R\}$

example

Find the transitive closure R* of the relation R on A = {1, 2, 3, 4} defined by the directed graph shown

Soln.:



R = {(1, 3), (1, 4), (3, 2), (3, 3), (3, 4)}

Here transitive closure of R is

$=R \cup \{(a, c) \mid \text{if } (a, b), (b, c) \in R\}$

To find transitive closure

(1, 3) $\in$ R and (3, 2) $\in$ R, hence add (1, 2) in R

Transitive closure of R = {(1, 2), (1, 3), (1, 4),(3, 2), (3, 3), (3, 4)}

Warshall's algorithm

Ex. 1 : Let A = {1, 2, 3, 4} and let R = {(1, 2), (2, 3), (3, 4), (2, 1)}.

Find transitive closure of R using Warshall's algorithm.

Solution:

$$W_0 = M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

First we find W_1, so that k = 1. W_0 has 1's in location 2 of column 1 i.e. (2, 1) and location 2 of row 1 i.e. (1, 2)

i j

p_1:(2, 1)

i j

q_1:(1, 2)

add (p_i, q_j) i.e. (2, 2) in W_k

Thus W_1 is just W_0 with a new 1 in position (2, 2)

$$W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & \mathbf{1} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$W_1 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Matrix W_1 has 1's at row 1 and 2 of column 2 and columns 1, 2, and 3 of row 2. i.e.

$$p_1 : \underset{i\ \ j}{(1, 2)} \qquad p_2 : \underset{i\ \ j}{(2, 2)}$$

$$q_1 : \underset{i\ \ j}{(2, 1)} \qquad q_2 : \underset{i\ \ j}{(2, 2)} \qquad q_3 : \underset{i\ \ 1}{(2, 3)}$$

We must put 1's in positions (p_i, q_j) i.e. (1, 1), (1, 2), (1, 3), (2, 1), (2,2) and (2, 3) of matrix W_1 (if 1's are not already there).

$$W_2 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$p_1 : \underset{i\ \ j}{(1, 3)} \qquad p_2 : \underset{i\ \ j}{(2, 3)}$$

$$q_1 : \underset{i\ \ j}{(3, 4)}$$

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally, W_3 has 1's in locations 1, 2, 3 of column 4 and no 1's in row 4, so no new 1's are added and $MR_\infty = W_4 = W_3$.

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{So } M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

We now compute $M_{(R \cup S)\infty}$ by Warshall's algorithm. First, $W_o = M_{R \cup S}$. We next compute W_1 so k = 1. Since W_o has 1's in locations 1 and 2 of column 1 and in locations 1 and 2 of row 1, we find that no new 1's must be adjoined to W_1 . Thus

$$W_0 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

K=1

$$p_1 : \begin{array}{c} i \;\; j \\ (1, 1) \end{array} \qquad p_2 : \begin{array}{c} i \;\; j \\ (2, 1) \end{array}$$

$$q_1 : \begin{array}{c} i \;\; j \\ (1, 1) \end{array} \qquad q_2 : \begin{array}{c} i \;\; j \\ (1, 2) \end{array}$$

To obtain W_1, we must put is in positions (1, 1), (1, 2), (2, 1) and (2, 2). We see that

$$W_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus $W_1 = W_0$

We now compute W2, so k = 2. Since W_1 has 1's in locations 1 and 2 : of column 2 and in locations 1 and 2 of row 2, we find that no new 1's must be added to W_1. That is,

$$\begin{array}{llll} & \text{i j} & & \text{i j} \\ p_1 : & (1, 2) & p_2 : & (2, 2) \\ & \text{i j} & & \text{i j} \\ q_1 : & (2, 1) & q_2 : & (2, 2) \end{array}$$

To obtain W_2, we must put is in positions (1, 1), (1, 2), (2, 1), (2, 2). We see that

$$W_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus $W_2 = W_1$

We next compute W_3, so k = 3. Since W_2 has 1's in locations 3 and 4 of column 3 and in locations 3 and 4 of row 3, we find that no new 1's must be added to W_2. That is

$$\begin{array}{llll} & \text{i j} & & \text{i j} \\ p_1 : & (3, 3) & p_2 : & (4, 3) \\ & \text{i j} & & \text{i j} \\ q_1 : & (3, 3) & q_2 : & (3, 4) \end{array}$$

To obtain W_3, we must put 1's in position (3, 3), (3, 4), (4, 3), (4, 4). We see that

$$W_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus $W_3 = W_2$

Things change when we now compute W4. Since W3 has I's in locations 3, 4, and 5 of column 4 and in locations 3, 4 and 5 of column 4, and in locations 3, 4 and 5 of row 4 we must add new 1's to W3 in positions 3, 5, and 5, 3, i.e.

$$\begin{array}{llllll} & \quad i \;\; j & & \quad i \;\; j & & \quad i \;\; j \\ p_1 : & (3, 4) & p_2 : & (4, 4) & p_3 : & (5, 4) \\ & \quad i \;\; j & & \quad i \;\; j & & \quad i \;\; j \\ q_1 : & (4, 3) & q_2 : & (4, 4) & q_3 : & (4, 5) \end{array}$$

To obtain W4, we must put 1's in positions (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5). We see that,

$$W_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

You may verify that $W_5 = W_4$ and thus

$(R \cup S)^{\infty}$ = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)}

Transitive Closure and Warshall's Algorithm

Compute the Warshall's Algorithm transitive closure of

- R={(a,b),(b,c),(c,d),(b,a)} on set A={a,b,c,d}

- R = {(1,1),(1,2),(1,4),(2,2),(2,3),(3,1),(3,4),(4,1),(4,4)} on the set A={1,2,3,4}

Computer transitive closure using Warshall's algorithm where $A=\{a_1,a_2,a_3,a_4,a_5\}$ and R be a relation on A whose matrix is

$$M_R=W_0=\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$