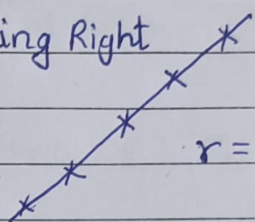
2/1/24 # MODULE - 2 : CORRELATION & REGRESSION

* <u>Correlation</u>

Coeff. of Correlation : $-1 \leq r \leq 1$

* <u>Scattered diagram</u>

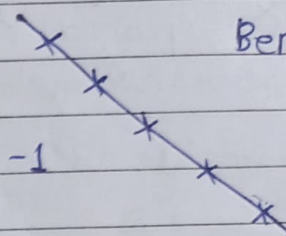
Bending Right



$r = 1.$

Perfect +ve correlation

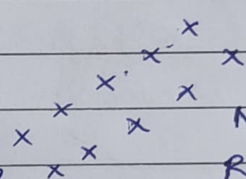
Bending Left.



$r = -1$

Perfect -ve correlation.

(Not a straight line)

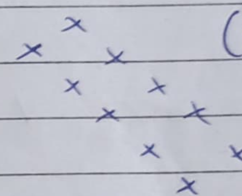


(closely associated, Not scattered, Right of plane)

$r = +ve$

~~(some +ve no. close to 1)~~

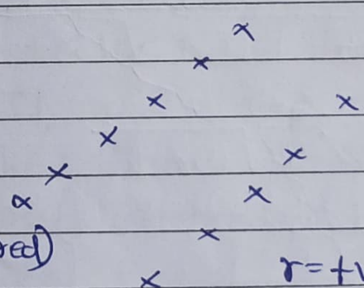
High degree +ve correlation.



(closely associated to the left of plane)

$r = -ve$
(close to -1)

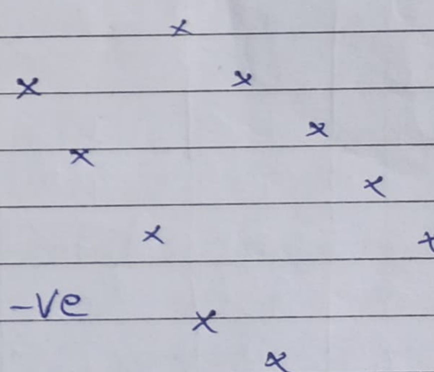
High degree -ve correlation



(Scattered)

$r = +ve$

Low Degree +ve correlation



$r = -ve$

Low Deg. -ve correlation.

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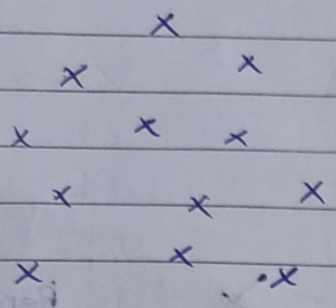

→ No correlation. (Highly scattered)

r = 0

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r = Karl Pearson's coeff. of correlation

$$= \frac{cov(x,y)}{\sigma_x \times \sigma_y}$$

$$= \frac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{N(\sigma_x \times \sigma_y)}$$

where, $\sigma_x = \sqrt{\frac{\Sigma(x_i-\bar{x})^2}{N}}$

$\sigma_y = \sqrt{\frac{\Sigma(y_i-\bar{y})^2}{N}}$

cov = covariance.
σ_x = std. devia^n of x.
σ_y = std. devia^n of y
σ_x^2 = variance of x = var(x)
σ_y^2 = variance of y = var(y)
N = No. of observations or Frequency.

$\bar{x} = \frac{\Sigma x_i}{N}$, $\bar{y} = \frac{\Sigma y_i}{N}$

$$\therefore r = \frac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\Sigma(x_i-\bar{x})^2}\sqrt{\Sigma(y_i-\bar{y})^2}}$$

$$\boxed{r = \frac{\Sigma x_i y_i - N\bar{x}\bar{y}}{\sqrt{\Sigma x_i^2 - N\bar{x}^2}\sqrt{\Sigma y_i^2 - N\bar{y}^2}}}$$

NOTE/Observa^n:
① $-1 \leq r \leq 1$.
② If x & y are independent then there is no correlation b/w them.
③ r is independent of : Change of scale & change of origin.

$\boxed{r_{xy} = r_{uv}}$ if $x, y \to U = x_i - \bar{x}$, $V = y_i - \bar{y}$ → Change of origin.

$U = \frac{x_i - \bar{x}}{h}$, $V = \frac{y_i - \bar{y}}{z}$ → Change of scale.

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Q) Calculate r for following data:

X	Y
23	18
27	22
28	23
29	24
30	25
31	26
33	28
35	29
36	30
39	32

30 Taken as $\bar{x}$; 25 taken as $\bar{y}$

$\rightarrow$ $u = X - 30$, $V = Y - 25$

u	v	uv	u^2	v^2
-7	-7	49	49	49
-3	-3	9	9	9
-2	-2	4	4	4
-1	-1	1	1	1
0	0	0	0	0
1	1	1	1	1
3	3	9	9	9
5	4	20	25	16
6	5	30	36	25
9	7	63	81	49
$\Sigma u = 11$	$\Sigma v = 7$	$\Sigma uv = 186$	$\Sigma u^2 = 215$	$\Sigma v^2 = 163$

$\bar{u} = \frac{\Sigma u}{N} = \frac{11}{10} = 1.1$

$\bar{V} = \frac{\Sigma V}{N} = 0.7$

$r = \frac{186 - 10 \times 1.1 \times 0.7}{\sqrt{215 - 10(1.1)^2}\sqrt{163 - 10(0.7)^2}}$

$= \frac{178.3}{\sqrt{202.9}\sqrt{158.1}}$

$= \frac{178.3}{179.1}$

$r_{uv} = 0.995 \simeq 1$

* Spearman's Rank Correlation Coeff. (R)

$$R = 1 - \left(\frac{6\sum d^2}{n^3 - n}\right)$$, where: d = difference b/w 2 ranks

↳ $-1 \leq R \leq 1$

Q) Calculate R for the following data:

given

Student No.	Rank in English (R_1)	Rank in Maths (R_2)	$d^2 = (R_1 - R_2)^2$
1	1	3	4
2	3	1	4
3	7	4	9
4	5	5	0
5	4	6	4
6	6	9	9
7	2	7	25
8	10	8	4
9	9	10	1
10	8	2	36
			$\sum d^2 = 96$

$$R = 1 - \left(\frac{6 \times 96}{10^3 - 10}\right) = 1 - \frac{576}{990} \doteq 0.4182$$

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2) ~~$\dfrac{\Sigma xy - n\bar{x}\bar{y}}{\sqrt{\Sigma x^2 - n\bar{x}^2}\sqrt{\Sigma y^2 - n\bar{y}^2}}$~~

$\Sigma d = 0$

$$R = 1 - \left\{ \frac{6\left[\Sigma d^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \cdots\right]}{n^3 - n} \right\}$$

where, m_i = No. of times particular item is repeated.

3)

X	R_1	Y	R_2	$d = R_1 - R_2$	d^2
32	3	40	5	-2	4
55	9	30	3.5	5.5	30.25
49	7.5	70	9	-1.5	2.25
60	10	20	1	9	81
43	5.5	30	3.5	2	4
37	4	50	7	-3	9
43	5.5	72	10	-4.5	20.25
49	7.5	60	8	-0.5	0.25
10	1	45	6	-5	25
20	2	25	2	0	0

$\Sigma d^2 = 176$

Equal So, The Rank is given as the avg of Respective Ranks.

$n = 10$, $m_1 = 2$ (for 43), $m_2 = 2$ (for 49), $m_3 = 2$ (for 30).

$$R = 1 - \left\{ \frac{6\left[\Sigma d^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \frac{1}{12}(m_3^3 - m_3)\right]}{n^3 - n} \right\}$$

$$= 1 - \left[\frac{6\left(176 + \frac{3}{12}(8-2)\right)}{990} \right]$$

$$= 1 - 6\left[\frac{176 + 3/2}{990}\right]$$

$$= 1 - \frac{\cancel{6}}{}\left(\frac{177.5}{\cancel{990}\ \cancel{330}\ 165}\right) = 1 - 1.0757 = -0.757.$$

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Q) $r_{xy} = 0.4$
$cov(x,y) = 1.6$
$\sigma_y^2 = 25$, find σ_x.

→ $r_{xy} = \frac{cov(x,y)}{\sigma_x \times \sigma_y}$

$\cancel{0.4} = \frac{\cancel{1.6}\,4}{5 \times \sigma_x}$.

$\boxed{\sigma_x = 0.8}$

Q) $R_{xy} = 0.143$.
$\Sigma d^2 = 48$. find n.

→ $R_{xy} = 1 - \left[\frac{6 \Sigma d^2}{n^3 - n}\right]$

$0.143 = 1 - \left[\frac{6 \times 48}{n^3 - n}\right]$

$0.857 = \frac{288}{n^3 - n}$.

$n^3 - n = \frac{288}{0.857}$.

$n^3 - n = 336.056$.

$\cancel{n(n^2-1) = 336.056}$.

$n^3 - n - 336.05 = 0$.

$\boxed{n = 7}$

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Q) Find r for: $N = 10,\ \Sigma x = 140,\ \Sigma y = 150$
$\Sigma(x-10)^2 = 180,\ \Sigma(y-15)^2 = 215$
$\Sigma(x-10)(y-15) = 60.$

→ Let $u = x - 10$
$v = y - 15$
then $\Sigma u^2 = 180$
$\Sigma v^2 = 215.$
$\Sigma uv = 60.$

$\Sigma u = \Sigma(x-10)$
$= \Sigma x - \Sigma 10$
$= 140 - 10 \times 10$
$= 40$

$\Sigma v = \Sigma(y-15)$
$= \Sigma y - \Sigma 15$
$= 150 - 15 \times 10$
$= 0$

$\bar{u} = \frac{\Sigma u}{N} = \frac{40}{10} = 4$

$\bar{v} = \frac{\Sigma v}{n} = 0$

$$r_{uv} = \frac{\Sigma uv - N\bar{u}\cdot\bar{v}}{\sqrt{\Sigma u^2 - N\bar{u}^2}\sqrt{\Sigma v^2 - N\bar{v}^2}}$$

$$= \frac{60 - 10(4)(0)}{\sqrt{\cancel{1600}\,180 - 10(16)}\ \cancel{\sqrt{0 - 10(0)}}\ \sqrt{215 - 0}}$$

$$= \frac{60}{\sqrt{20}\sqrt{215}}$$

$$= \frac{60}{65.57} = 0.915.$$

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Q) 10 competitors in a musical test were ranked by 3 judges, x, y & z, in the following order.

Rank by x	Rank by y	Rank by z	d_{xy}	d_{yz}	d_{xz}
1	3	6	-2	-3	-5
6	5	4	1	1	2
5	8	9	-3	-1	-4
10	4	8	6	-4	2
3	7	1	-4	6	2
2	10	2	-8	8	0
4	2	3	2	-1	1
9	1	10	8	-9	-1
7	6	5	1	1	2
8	9	7	-1	2	1

d_{xy}^2	d_{yz}^2	d_{xz}^2
4	9	25
1	1	4
9	1	16
36	16	4
16	36	4
64	64	0
4	1	1
64	81	1
1	1	4
1	4	1
$\Sigma d_{xy}^2 = 200$	$\Sigma d_{yz}^2 = 214$	$\Sigma d_{xz}^2 = 50$

$$R_{xy} = 1 - \frac{6 \Sigma d_{xy}^2}{n^3 - n}$$

$$= 1 - \frac{6(200)}{990}$$

$$R_{xy} = -0.212$$

$$R_{yz} = 1 - \frac{6(214)}{990}$$

$$= -0.296$$

$$R_{xz} = 1 - \frac{6(50)}{990}$$

$$= \cancel{0.642} \quad 0.636$$

Judge x & z have ~~same~~ similar liking in music.

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Q) Sample of 25 pairs of x, y values gives the following results:

$\Sigma x = 127$	Later on it was found that,
$\Sigma y = 100$	2 pairs of values when taken as
$\Sigma x^2 = 760$	$(8,14)$ & $(8,6)$ instead of
$\Sigma y^2 = 449$	$(8,12)$ & $(6,8)$.
$\Sigma xy = 500$	Find correct correlation coefficient.

→ Correct $\Sigma x =$ Wrong $\Sigma x - (8+8) + (8+6)$
$= 127 - 2$
Correct $\Sigma x = 125$

Correct $\Sigma y =$ Wrong $\Sigma y - (14+6) + (12+8)$
$= 100 - 20 + 20$
Correct $\Sigma y = 100$

Correct $\Sigma x^2 =$ Wrong $\Sigma x^2 - (8^2+8^2) + (8^2+\cancel{36}\,6^2)$
$= \cancel{127}\,760 - (128) + (100)$
$= \cancel{99}\,732$

Correct $\Sigma y^2 =$ Wrong $\Sigma y^2 - (14^2+6^2) + (12^2+8^2)$
$= \cancel{760}\,449 - (196+36) + (144+64)$
$= \cancel{760}\,449 - 232 + 208$
$= \cancel{736} = 425$

Correct $\Sigma xy = 500 - (8\times14 + 8\times6) + (8\times12 + 6\times8)$
$= 500 - (112+48) + (96+48)$
$= 484.$

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$\therefore \bar{x} = \frac{\Sigma x}{n} = \frac{125}{25} = 5,\ \bar{y} = \frac{\Sigma y}{n} = \frac{100}{25} = 4.$

$$\therefore r = \frac{\Sigma xy - n\bar{x}\bar{y}}{\sqrt{\Sigma x^2 - n\bar{x}^2}\sqrt{\Sigma y^2 - n\bar{y}^2}}$$

$$= \frac{484 - 25(5)(4)}{\sqrt{732 - 25(25)}\sqrt{425 - 25(16)}}$$

$$= \frac{-16}{\sqrt{107}\sqrt{25}}$$

$$= -0.309.$$

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* Regression Lines:

1) **Regression Line of y on x**

$$y-\bar{y} = b_{yx}(x-\bar{x})$$

where, $b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$.

Regression coefficient.

2) **Regression line of x on y.**

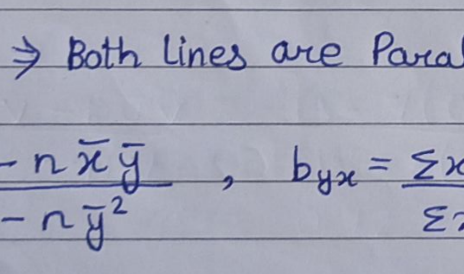
$$x-\bar{x} = b_{xy}(y-\bar{y})$$

where, $b_{xy} = r\dfrac{\sigma_x}{\sigma_y}$.

Regression coefficients are slopes of the given lines

Remarks:

① If $r=0$, then $x=\bar{x}$ & $y=\bar{y}$



② If $r=\pm 1 \Rightarrow$ Both Lines are Parallel to each other.

③ $b_{xy} = \dfrac{\Sigma xy - n\bar{x}\bar{y}}{\Sigma y^2 - n\bar{y}^2}$, $b_{yx} = \dfrac{\Sigma xy - n\bar{x}\bar{y}}{\Sigma x^2 - n\bar{x}^2}$

④ $r = \pm\sqrt{b_{xy}\cdot b_{yx}}$

⑤ $\dfrac{b_{xy}+b_{yx}}{2} \geq r$

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(6) If $b_{xy} > 1$ then $b_{yx} < 1$.

(7) If θ is angle b/w 2 Regression Lines then,

$$\therefore \tan\theta = \left(\frac{1-r^2}{r}\right)\frac{\sigma_x \times \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

If $r = 0$, $\tan\theta = \infty \Rightarrow \theta = \pi/2$

If $r = \pm 1$, $\tan\theta = 0 \Rightarrow \theta = 0$

- Regression coeff. are independent of change of origin but "not" change of scale. unlike correlation coeff.

(8) $u = \frac{x-a}{h}$, $v = \frac{y-b}{k}$ then,

$$\sigma_x = h\sigma_u \;\&\; \sigma_y = k\sigma_v$$

$$\therefore b_{xy} = r\frac{\sigma_x}{\sigma_y} = r\frac{h\sigma_u}{k\sigma_v} = \boxed{\frac{h}{k} b_{uv} = b_{xy}}$$

$$\therefore \boxed{b_{yx} = \frac{k}{h} b_{vu}}$$

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Q) A panel of 2 Judges A & B graded the following performances.

Perf. No.	Marks given by: A(X)	B(Y)	$U=X-\bar{X}$	$V=Y-\bar{Y}$	U^2	UV
1	36	35	3	2	9	6
2	32	33	-1	0	1	0
3	34	31	1	-2	1	-2
4	31	30	-2	-3	4	6
5	32	34	-1	1	1	-1
6	32	32	-1	-1	1	1
7	34	36	1	3	1	3
8	38	-	$\Sigma U=0$	$\Sigma V=0$	$\Sigma U^2=18$	$\Sigma UV=13$

Judge A awarded 38 to 8th performance while Judge B was absent. If B would have been present, how many marks would he award?

→ $\Sigma x = 231$ (Only from 1-7)

$\bar{x} = \frac{231}{7} = 33.$

$\Sigma y = 231$

$\bar{y} = \frac{231}{7} = 33$

$\bar{U} = 0, \bar{V} = 0.$

$b_{VU} = \frac{\Sigma UV - n\bar{U}\bar{V}}{\Sigma U^2 - n\bar{U}^2} = \frac{13 - 7(0)(0)}{18 - 7(0)^2}$

$= 0.722$

$V - \bar{V} = b_{VU}(U - \bar{U})$

$V = (0.722)U$

$Y - \bar{Y} = (0.722)(X - \bar{X}) = Y - 33 = (0.722)(5) = 36.61$

$= 37$

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Q) Following is Data for 60 students.

	Maths (X)	English (Y)
Mean	80 ($\bar{X}$)	50 ($\bar{Y}$)
S.D	15 (σ_x)	10 (σ_y)

$r = 0.4$

Estimate marks of students in maths who scored 60 marks in English & Marks of Student in Eng. who scored 70 in maths.

→ $b_{xy} = r\dfrac{\sigma_x}{\sigma_y} = 0.4 \times \dfrac{15}{10} = 0.6$

$b_{yx} = r\dfrac{\sigma_y}{\sigma_x} = 0.4 \times \dfrac{10}{15} = 0.26$

$Y=60 \Rightarrow X - \bar{X} = b_{xy}(Y - \bar{Y}) \Rightarrow X - 80 = 0.6(60-50)$

$X=70 \Rightarrow Y - \bar{Y} = b_{yx}(X - \bar{X})$

$X - 80 = 6$

$X = 86$

$Y - 50 = 0.26(70-80)$

$= -2.6$

$Y = 47.4 = 47$

Q) Given: $6y = 5x + 90$ — I

$15x = 8y + 130$ — II

$\sigma_x^2 = 16$

Find: ① $\bar{x}, \bar{y}$, ② r, ③ σ_y^2.

→ Solving I & II

$x = \bar{x} = 30$
$y = \bar{y} = 40$ } ①

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$r = \pm\sqrt{b_{yx} b_{xy}}$

(I) → $y = \frac{5}{6}x + \frac{90}{6} \Rightarrow b_{yx} = 5/6.$

(II) → $x = \frac{8}{15}y + \frac{130}{15} \Rightarrow b_{xy} = 8/15$

$r = \sqrt{b_{yx} \cdot b_{xy}} = 0.66. \longrightarrow$ (2).

Now, $b_{xy} = \frac{r\sigma_x}{\sigma_y}.$

$= \frac{8}{15} = 0.66 \times \frac{4}{\sigma_y} \Rightarrow \cancel{\sigma_y = 0.05}.$

$\sigma_y = 4.95. \Rightarrow \sigma_y^2 = 25.$

Q) If $\sigma_x = \sigma_y = \sigma$ & $\theta = \tan^{-1} 3$
find r.

→ $\tan\theta = \frac{1-r^2}{r} \cdot \frac{\sigma_x \times \sigma_y}{\sigma_x^2 + \sigma_y^2}$

$3 = \frac{1-r^2}{r} \frac{\sigma^2}{2\sigma^2}$

$6r = 1 - r^2.$

$r^2 + 6r - 1 = 0.$

$r = -3 + \sqrt{10}.$

$= 0.162$

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Q) If $\bar{x}=5, \bar{y}=10$ & Line of Regression of y on x is $\parallel^{r}$ to the Line $20y = 9x+40$. — (1)
Find y for $x=30$.

→ $y-\bar{y} = b_{yx}(x-\bar{x})$ — (2)
From (1), slope ⇒ $m_1 = 9/20$.
From (2), $m_2 = b_{yx}$.
$\therefore b_{yx} = 9/20$.

$y-10 = \frac{9}{20}(30-5)$.

$y-10 = \frac{9(\cancel{25})^{5}}{\cancel{20}_{4}}$

$y = \frac{45}{4} + 10$

$= 11.25 + 10$

$\boxed{y = 21.25}$

Q) If the tangent of the angle made by line of Regression of y on x is 0.6 & $\sigma_y = 2\sigma_x$
Find r.

→ $\tan\theta = 0.6$.

$0.6 = \frac{1-r^2}{r} \cdot \frac{\sigma_x \times \sigma_y}{\sigma_x^2+\sigma_y^2}$.

$= \frac{1-r^2}{r} \times \frac{2\sigma_x^2}{4\sigma_x^2+\sigma_x^2}$

$\Rightarrow \frac{1-r^2}{r} \times \frac{2}{5}$

This is for Angle b/w 2 lines. Here we have a line & x-axis.

$3r = 2 - 2r^2$.

$2r^2 + 3r - 2 = 0$.

$r = 0$.

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$b_{yx} = 0.6 = r\frac{\sigma_y}{\sigma_x}$

$0.6 = r \times \frac{2\sigma_x}{\sigma_x}$

$\boxed{r = 0.3}$

Q) Given:

	x	y
Arithmetic mean:	36	85
Std. deviation:	11	8

Correlaⁿ coeff. b/w. x & y = 0.66 = r

① Find 2 reg. eqⁿ.

② Estimate value of x when $y = 75$.

→ $\therefore \bar{x} = 36, \bar{y} = 85$
$\sigma_x = 11, \sigma_y = 8$ } Given.

y on x,

$y - \bar{y} = b_{yx}(x - \bar{x})$

$y - 85 = r\frac{\sigma_y}{\sigma_x}(x - 36)$

$y - 85 = 0.66\left(\frac{8}{11}\right)(x - 36)$

$y - 85 = 0.48(x - 36)$

$\boxed{y = 0.48x + 67.72}$

x on y,

$x - \bar{x} = b_{xy}(y - \bar{y})$

$x - 36 = 0.66\left(\frac{11}{8}\right)(y - 85)$

$x - 36 = 0.9(y - 85)$

$\boxed{x = 0.9y - 40.5}$

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(2) $y = 75 \rightarrow$ given.

$$\therefore x = 0.9y - 40.5$$
$$= 0.9(75) - 40.5$$
$$= 67.5 - 40.5$$
$$\boxed{x = 27}$$

* **Formulas**

- r = Karl Pearson's coeff. of correlation.

$$r = \frac{cov(x,y)}{\sigma_x \cdot \sigma_y} = \frac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{N(\sigma_x \cdot \sigma_y)} \longrightarrow \sigma_x = \sqrt{\frac{\Sigma(x_i-\bar{x})^2}{N}}$$

$\sigma_y \rightarrow$

$$\bar{x} = \frac{\Sigma x_i}{N}$$

$$= \frac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\Sigma(x_i-\bar{x})^2 \Sigma(y_i-\bar{y})^2}}$$

$$\boxed{r = \frac{\Sigma x_i y_i - N\bar{x}\bar{y}}{\sqrt{\Sigma x_i^2 - N\bar{x}^2}\sqrt{\Sigma y_i^2 - N\bar{y}^2}}}$$

- Spearman's Rank correlation coeff. (R):

$$R = 1 - \left(\frac{6\Sigma d^2}{n^3-n}\right) \longrightarrow d = \text{diff. b/w 2 ranks.}$$

$$F = \cancel{m(m^2-1)} \frac{m^3-m}{12} = \frac{m(m^2-1)}{12} \rightarrow \text{for repeatation.}$$

- **Regression Line**:

y on x $\Rightarrow y - \bar{y} = b_{yx}(x - \bar{x})$

x on y $\Rightarrow x - \bar{x} = b_{xy}(y - \bar{y})$

$b_{yx} \rightarrow r\frac{\sigma_y}{\sigma_x}$, $b_{xy} \rightarrow r\frac{\sigma_x}{\sigma_y}$ } regression coefficients.

$$b_{xy} = \frac{\Sigma x_i y_i - n\bar{x}\bar{y}}{\Sigma y^2 - n\bar{y}^2}, \quad b_{yx} = \frac{\Sigma x_i y_i - n\bar{x}\bar{y}}{\Sigma x^2 - n\bar{x}^2}$$

$$r = \pm\sqrt{b_{xy} \cdot b_{yx}}, \quad r \leq \frac{b_{xy} + b_{yx}}{2}$$

$$\tan\theta = \left(\frac{1-r^2}{r}\right)\frac{\sigma_x \cdot \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

If $b_{xy} > 1$, $b_{yx} < 1$.

b_{xy} & $b_{yx} \geq 0 \Rightarrow r = +ve$, b_{xy} & $b_{yx} \leq 0 \Rightarrow r = -ve$.

MODULE-1 - PROBABILITY & PROBABILITY DISTRIBUTION.

Conditional Probability:

The probability of occurence of any event A when another event B in relaⁿ to A has already occured is known as conditional probability $\cdot (P(A|B))$.

S = Sample space.

A event of S $(A \subseteq S)$

$$P(A/B) = \frac{n(A\cap B)}{n(B)} = \frac{n(A\cap B)/n(S)}{n(B)/n(S)} = \frac{P(A\cap B)}{P(B)}$$

Eg: A: Card is ~~Red~~ King

B: Card is Red (is already chosen)

$$P(A/B) = \frac{n(A\cap B)}{n(B)} = \frac{2}{26}$$

Remarks:

① Multiplication Theorem

$P(A\cap B) = P(A)\, P(B/A)$

$= P(B)\, P(A/B)$

② $P(A/B) \leq P(A)$

③ $P(B/B) = 1$

④ Independent Events:

A is independent if $P(A/B) = P(A)$

B is independent if $P(B/A) = P(B)$

Also, $P(A\cap B) = P(A)\, P(B)$

⑤ For any event A in S,

→ A & ϕ are independent; $\because P(A\cap\phi) \Rightarrow P(\phi) = 0 = P(\phi)\cdot P(A)$

→ A & S are independent. $\because P(A\cap S) \Rightarrow P(A) = P(A)\cdot 1 = P(A)\cdot P(S)$

Addⁿ Thm. ⑥ $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$A \cup B \Rightarrow A$ or B. $B/A \Rightarrow B$ given A.

$A/B \Rightarrow A$ given B.

$A \cap B \Rightarrow A$ and B.

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Q) From a City population, Probability of selecting ~~a male or smoker is 7/10~~,

① Male or smoker is 7/10

② Male Smoker is 2/5

③ A Male if smoker is selected is 2/3.

Find the probability of selecting: ⓐ Non Smoker.
ⓑ a male
ⓒ smoker if a male is selected first.

→ Define A: male is selected.
B: smoker is selected.

Given: $P(A \cup B) = 7/10$
~~$P(A) = 2/5$~~ $P(A \cap B) = 2/5$
$P(A/B) = 2/3$.

ⓐ $P(\bar{B}) = 1 - P(B) = 1 - \frac{P(A \cap B)}{P(A/B)}$

$= 1 - \frac{2/5}{2/3} = 1 - \frac{6}{10} = \frac{2}{5} = P(\bar{B})$ — ⓐ ↳ non smoker

ⓑ $P(A) = P(A \cup B) - P(B) + P(A \cap B)$

$= \frac{7}{10} - \frac{3}{5} + \frac{2}{5} \Rightarrow \cancel{\frac{7}{10} = P(A)}$ — ⓑ

$= \frac{5}{10} = \frac{1}{2}$ — ⓑ → a male.

ⓒ $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{2/5}{1/2} = \frac{4}{5}$ → Smoker if male is selected.

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* ~~Baye's Theorem~~
* Baye's Theorem:

Let $E_1, E_2, \ldots E_n$ be mutually disjoint ~~ie~~ Events.
(i.e. $E_i \cap E_j = \phi, \forall i \neq j$) $P(E_i) \neq 0$

Then for every event A such that $A \subset \left(\bigcup_{i=1}^{n} E_i\right)$ and $P(A) > 0$, we have ~~$P(E_i/A) = P(E_i)P(A/E_i)$~~

$$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)}$$

$$= \frac{P(E_i)\, P(A/E_i)}{P(A)}$$

Eg: A Product is produced in 3 factories x, y & z.
~~For~~ 'x' produces thrice as many times as 'y'.
'y' & z produce same no. of items.
It is known that 3% of items produced by each of the factories, x & z, are defective while 5% produced by y are defective. An item is selected at random:
① Probablity of item being defective.
② If item selected is found to be defective, what is prob. that, it was produced by x, y & z respectively

→ ~~Let~~ Assume, items produced by $y = n$
∴ Items produced by $x = 3n$.
Items produced by $z = n$.

Let E_i = Producⁿ of items produced by x, y, z respecti
A = Item selected is defective.

$P(E_1) = \frac{3n}{5n} = 3/5$ → Prob. of x producing an ~~defective~~ item.

$P(E_2) = 1/5 = P(E_3)$

$P(A/E_1) = 0.03$ → Prob. of item being defective & being produced in factory 'x'

$P(A/E_2) = 0.05$

$P(A/E_3) = 0.03$

① $P(A) = \sum_{i=1}^{3} P(E_i)\, P(A/E_i)$

$$= \left(\frac{3}{5} \times \frac{3}{100}\right) + \left(\frac{1}{5} \times \frac{5}{100}\right) + \left(\frac{1}{5} \times \frac{3}{100}\right)$$

$$= \cancel{0.63 + 0.01 + 0.23} \quad \frac{9}{500} + \frac{5}{500} + \frac{3}{500}$$

$$= \cancel{0.87}$$

$$= 17/500$$

② $P(E_1/A) = \dfrac{P(E_1)\,P(A/E_1)}{P(A)} = \dfrac{3/5 \times 3/100}{17/500} = \dfrac{9}{17}$

$P(E_2/A) = \dfrac{P(E_2)\,P(E_2/A)}{P(A)} = \dfrac{1/5 \times 5/100}{17/500} = \dfrac{5}{17}$

$P(E_3/A) = \dfrac{P(E_3)\,P(E_3/A)}{P(A)} = \dfrac{1/5 \times 3/100}{17/500} = \dfrac{3}{17}$

Q. 3 Candidates for the posi^n of principals x, y, z whose chances of getting selected are in a proportion $4:2:3$ respectively. Prob. that Mr. x gets selected, who will introduce the co-educa^n, in college is 0.3. Prob. of y & z doing the same are 0.5 & 0.8 resp.
① What is the prob. that there will be co-edu. in college?
② If there is already co-edu. in college, what is the prob. that Mr. x or y or z get selected?

→ Let E_i be the event, Mr x, y, z get selected resp.
Let A be the event that there is co-edu. in college

$P(A \cap E_i) = P(E_i) \times P(A/E_i)$.

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Event E_i ①	Prior Prob. $P(E_i)$ ②	Condition Prob $P(A/E_i)$ ③	Joint Prob $P(A \cap E_i)$ ④ = ②③	Posterior Prob $P(E_i/A)$ ⑤ = ④/$P(A)$
E_1	4/9	0.3	2/15	0.261
E_2	2/9	0.5	1/9	0.217
E_3	3/9	0.8	~~8/30~~ 4/15	0.522
		~~P(A)~~	Σ $\boxed{P(A) = 0.51}$ ①	②

Q) Factory produces certain type of outputs by 3 type of machines. Respective daily produc^n are as follows:

M_1 : 3000 units.	past experience shows that
M_2 : 2500 units.	1% of the output by M_1 is
M_3 : 4500 units	defective & 1.2% & 2% by

M_2 & M_3 resp. are defective. Item is drawn at random & found to be defective, what is the prob. that it comes from M_1, M_2 & M_3?

→ ~~Event~~ → Items produced by M_i, A = Item selected is Defective.

E_i	$P(E_i)$	$P(A/E_i)$	$P(A \cap E_i)$ ↳ $P(E_i) \times P(A/E_i)$	$P(E_i/A)$ ↳ $P(A \cap E_i)/P(A)$
E_1	~~3/10~~ 3000/10000	0.01	0.003	0.2
E_2	2500/10000	0.012	0.003	0.2
E_3	4500/10000	0.02	0.009	0.6
			$P(A) = 0.015$	

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Q) There are 2 bags ① & ②, ① contains n white & 2 black balls, ② contains ~~n black &~~ 2 white & n black. One of the bags is selected at random & 2 balls are drawn from it w/o replacement, both the balls drawn are white & prob. that bag ① was used to draw the balls is 6/7. Find value of n.

→ E_1: Bag ① was selected.
E_2: Bag ② was selected.
A: Both the balls drawn are white.

$P(E_1) = 1/2$, $P(E_2) = 1/2$

$P(E_1/A) = 6/7$

$P(A/E_1) = \frac{^nC_2}{^{n+2}C_2}$ ⟶ n white + 2 Black.

$= \frac{n!}{2!(n-2)!} \Big/ \frac{(n+2)!}{2!\,(n+2-2)!}$

$= \frac{n(n-1)\cancel{(n-2)!}}{\cancel{2}\cdot\cancel{(n-2)!}} \Big/ \left(\frac{(n+2)(n+1)\cancel{n!}}{\cancel{2}\cancel{n!}}\right)$

$= \frac{n(n-1)}{(n+1)(n+2)}$

$P(A/E_2) = \frac{^2C_2}{^{(n+2)}C_2} = 1 \Big/ \frac{(n+2)!}{2!\,n!}$

$= \frac{2\cancel{n!}}{(n+2)(n+1)\cancel{n!}} = \frac{2}{(n+1)(n+2)}$

By Baye's thm.

$P(E_1/A) = \frac{P(E_1)\cdot P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$

$= \frac{\cancel{\frac{1}{2}}\cdot\frac{n(n-1)}{\cancel{(n+1)(n+2)}}}{\cancel{\frac{1}{2}}\cdot\frac{n(n-1)}{\cancel{(n+1)(n+2)}} + \cancel{\frac{1}{2}}\frac{2}{\cancel{(n+1)(n+2)}}} = \frac{n(n-1)}{n(n-1)+2} = \frac{6}{7}$ ↳ given.

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$\Rightarrow 7n(n-1) = 6[n(n-1)+2]$

$7n^2 - 7n = 6(n^2 - n + 2)$

$\cancel{n^2 - 6n - 2 = 0}$

$n^2 - n - 12 = 0$

$\boxed{n = 4}$

Q) Contents of urns are as follows.

I $\rightarrow$ 1 W, 2 B, 3 R.

II $\rightarrow$ 2 W, 1 B, 1 R.

III $\rightarrow$ 4 W, 5 B, 3 R.

1 urn is ~~josen~~ chosen at random & 2 balls drawn from it are W & R. What is prob. that they come from urn I, II, & III resp.?

$\rightarrow$ E_1: Urn I is selected.

E_2: Urn II is selected

E_3: — " — III " — " —

A: 2 Balls selected are ~~white & Red~~. W & R.

$P(E_1) = P(E_2) = P(E_3) = 1/3$.

$P(A/E_1) = \dfrac{{}^1C_1 \times {}^3C_1}{{}^6C_2} = \dfrac{1 \times 3}{15} = \dfrac{1}{5}$

$P(A/E_2) = \dfrac{{}^2C_1 \times {}^1C_1}{\cancel{{}^6C_2}\ {}^4C_2} = \dfrac{2 \times 1}{6} = \dfrac{1}{3}$

$P(A/E_3) = \dfrac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2} = \dfrac{\cancel{4} \times 3}{\cancel{66}\ 11} = \dfrac{2}{11}$

$P(E_1/A) = P(E_1) \times P(A/E_1) = \dfrac{1}{3} \times \dfrac{1}{5} = \dfrac{1}{15}$.

$P(E_2/A) = P(E_2) \times P(A/E_2) = \dfrac{1}{3} \times \dfrac{1}{3} = \dfrac{1}{9}$

$P(E_3/A) = P(E_3) \times P(A/E_3) = \dfrac{1}{3} \times \dfrac{2}{11} = \dfrac{2}{33}$

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Q) In answering a Ques^n on a M.C. Test, student either knows the ans. or he guesses the ans. P is the Prob that he knows the ans. And $(1-P)$ is the prob. that he guesses the ans.
Assume that student who guesses the answ. will be correct has prob. 1/5. 5 is the no. of M.C. alternatives. What is the prob. that student knew the ans. & he answered it correctly?

→ E_1: Student knows the ~~ans~~ correct ans.
E_2: 〃 guesses 〃
A: Student gets the correct ans.
$P(E_1) = P$
$P(E_2) = 1 - P$
$P(A/E_2) = 1/5$
$P(A/E_1) = 1$ → student knows the ans.

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* Random variable is a $func^n$ which assigns a real number to an outcome of an experiment.
  - Discrete R.V. → PMF
  - Continuous R.V. → PDF

Eg) → $S = \{(HHH), (HHT), (HTH), (THH), (HTT), (THT), (TTH), (TTT)\}$.

1-D R.V.

X = no. of Tails

$X = 0 \Rightarrow \{(HHH)\}$

$X = 1 \Rightarrow \{(HHT), (HTH), (THH)\}$

$X = 2 \Rightarrow \{(HTT), (THT), (TTH)\}$.

$X = 3 \Rightarrow \{(TTT)\}$.

* **Two Dimensional Random Variable**

Let X & Y be two R.V. defined on the same sample space S. Then the $func^n$ (X, Y) that assigns a point in $\mathbb{R}^2$ [$(X, Y) \in \mathbb{R}^2$ → $\mathbb{R}$ = Set of real numbers.]

$\mathbb{R}^2 \Rightarrow \underbrace{\mathbb{R} \times \mathbb{R}}_{\text{Cross}}$ $\qquad = \underbrace{\mathbb{P}}_{\text{Rational}} \cup \underbrace{\mathbb{Q}}_{\text{Irrational}}$

is called 2-D Random variable.

* **Two-Dimesional or Joint Probability Mass $func^n$ (PMF):**

If (X, Y) is 2-D discrete R.V. with each possible outcome (x_i, y_j) w/ probability P_{ij} satisfying the conditions:

① $P(x_i, y_j) \geq 0, \forall i, j$.

② $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} P(x_i, y_j) = 1$

then P is called Joint Prob. of Mass $func^n$ of ~~Random~~ (X, Y) & Prob. distribution of (X, Y) is given by ⟶

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$x \backslash y$	y_1	y_2	...	y_m	Total
x_1	P_{11}	P_{12}	...	P_{1m}	P_1
x_2	P_{21}	P_{22}	...	P_{2m}	P_2
x_3	⋮	⋮		⋮	⋮
⋮	⋮	⋮		⋮	⋮
x_n	P_{n1}	P_{n2}		P_{nm}	P_n
Total	P_1'	P_2'	...	P_m'	1

Marginal PMF: If we find Probabilities $P_1, P_2 \dots P_n$ of $x_1, x_2, \dots x_n$ only, irrespective of values taken by y; Distribution so obtained is called Marginal Prob. Distribution of X

$$P_1 = \sum_{j=1}^{m} P_{1j} \rightarrow \text{This is for } x=1, \text{ similarly } P_2 = \sum_{j=1}^{m} P_{2j} \dots$$

Similar for Marginal Prob. Distribuⁿ of Y.

$$P_1 = \sum_{i=1}^{n} P_{i1}$$

* Conditional Probability Distribution:

It is given by ~~Prob~~ $P(X=x_i / Y=y_i) = \dfrac{P(X=x_i, Y=y_i)}{P(Y=y_i)}$

(Q) Joint Probability Distribution of X & Y is given by

$$P(X=x, Y=y) = \frac{x+3y}{24} \longrightarrow x = 1,2 \ \& \ y = 1,2.$$

Find PMF & Marginal Prob. Distbuⁿ of X & Y.

→ Prob. dist. of (x, y) is given by:

$Y \backslash X$	1	2	Total
1	1/6	5/24	3/8
2	7/24	1/3	15/24
Total	11/24	13/24	1

Marginal Prob. distribuⁿ of Y.

→ PMF

Marginal ~~PMF~~ prob. dist. of X

$$^nC_r = \frac{n!}{r!(n-r)!}$$

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Marginal Prob. Dist. of X & Y:

X	P(X)
1	11/24
2	13/24

Y	P(Y)
1	3/8
2	15/24

Q) Three balls are drawn at random w/o repetiⁿ from box containing 2 white, 4 black, 3 Red balls.
If x denotes no. of white balls drawn
y denotes no. of red balls drawn.
i) Find joint prob. distⁿ of x & y.
ii) a) $P(x \leq 1)$ (b) $P(x \leq 1, y \leq 2)$.
(c) $P(y \leq 2 / x \leq 1)$ (d) $P(x+y \leq 2)$
iii) Marginal distribuⁿ of x.
iv) Marginal distribⁿ of y.
v) Condⁿ prob. of x given $y=1$.
vi) Condⁿ prob. of y given $x=2$.
vii) Check whether x, y are independent.

(i)

X \ Y	0	1	2	3	Total
0	1/21	3/14	1/7	1/84	5/12
1	3/21	2/7	1/14	0	1/2
2	1/21	1/28	0	0	1/12
Total	5/21	15/28	3/14	1/84	1

(i) $P(X=0, Y=0) = P(3 \text{ black balls})$

$$= \frac{^4C_3}{^9C_3} = \frac{4}{84} = \frac{1}{21}$$

(ii) $P(X=0, Y=1) = P(1 \text{ red}, 2 \text{ black})$

$$= \frac{^3C_1 \times {^4C_2}}{^9C_3} = \frac{3 \times 6}{84} = \frac{18}{84}$$

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$P(X=0, Y=2) = P(2\text{red}, 1\text{black})$

$= \frac{{}^3C_2 \times {}^4C_1}{{}^9C_3} = \frac{3\times4}{84} = \frac{12}{84} = \frac{1}{7}$

$P(X=0, Y=3) = P(3\text{red balls})$

$= \frac{{}^3C_3}{{}^9C_3} = \frac{1}{84}$

$P(X=1, Y=0) = P(1\text{ white}, 2\text{ black})$

$= \frac{{}^2C_1 \times {}^4C_2}{{}^9C_3} = \frac{2\times6}{84} = \frac{1}{7}$

$P(X=1, Y=1) = P(1W, 1B, 1R)$

$= \frac{{}^2C_1 \times {}^3C_1 \times {}^4C_1}{{}^9C_3} = \frac{2\times3\times4}{84} = \frac{24}{84} = \frac{2}{7}$

$P(X=1, Y=2) = P(1W, 2R)$

$= \frac{{}^2C_1 \times {}^3C_2}{{}^9C_3} = \frac{2\times3}{84} = \frac{6}{84} = \frac{1}{14}$

$P(X=1, Y=3) = \cancel{P(1W,} ? 0$

$P(X=2, Y=0) = P(2W, 1B)$

$= \frac{{}^2C_2 \times {}^4C_1}{{}^9C_3} = \frac{4}{84} = \frac{1}{21}$

$P(X=2, Y=1) = P(2W, 1R)$

$= \frac{{}^2C_2 \times {}^3C_1}{{}^9C_3} = \frac{3}{84}$

$P(X=2, Y=2) = 0$

$P(X=2, Y=3) = 0$

(iii) Marginal Distribuⁿ of X; (iv) Marginal Distriboⁿ of Y

X	0	1	2
P(X)	5/12	1/2	1/12

Y	0	1	2	3
P(Y)	5/21	15/28	3/14	1/84.

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ii) a) $P(X \le 1) = P(X=0) + P(X=1)$

$= \frac{5}{12} + \frac{1}{2} = \frac{11}{12}$

b) $P(X \le 1, Y \le 2)$

$= \sum_{j=0}^{2} P(X=0, Y=j) + \sum_{j=0}^{2} P(X=1, Y=j)$

$= \left(\frac{1}{21} + \frac{3}{14} + \frac{1}{7}\right) + \left(\frac{1}{7} + \frac{2}{7} + \frac{1}{14}\right) = \frac{19}{21}$

c) $P(Y \le 2 / X \le 1) = \frac{P(Y \le 2, X \le 1)}{P(X \le 1)}$

$= \frac{19/21}{(11/12)} = \frac{76}{77}$

d) $P(X+Y \le 2) = \sum_{j=0}^{2} P(X=0, Y=j) + \sum_{j=0}^{1} P(X=1, Y=j)$

$+ P(X=2, Y=0)$

$= \frac{17}{42} + \frac{3}{7} + \frac{1}{21} = \frac{37}{42}$

v) $P(X=0 / Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{3/14}{15/28} = \frac{2}{5}$

$P(X=1 / Y=1) = \frac{2/7}{15/28} = \frac{8}{15}$

$P(X=2 / Y=1) = \frac{1/28}{15/28} = \frac{1}{15}$

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vii) X & Y are independent if,

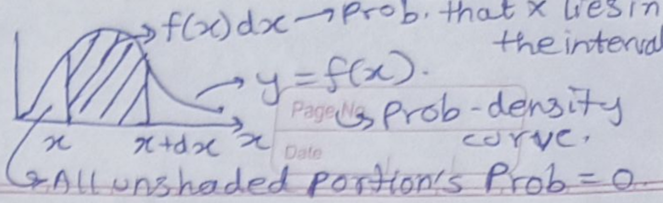
$$P(X,Y) = P(X)\,P(Y)$$

$$P(0,1) = 3/14.$$

$$\left.\begin{array}{l} P(0) = 5/12 \\ P(1) = 15/28 \end{array}\right\} \quad P(0)\,P(1) \neq P(0,1) \neq \frac{3}{24}.$$

$$\frac{5}{12} \times \frac{15}{28}$$

Hence, X & Y are not Independent.



* 2-D Continuous Probability Distribution

Let (X, Y) be 2-D continuous Random variable and let $f_{XY}(x,y)$ be a func^n of (x,y) such that

① $f_{XY}(x,y) \geq 0$

② $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

③ $\int_{a}^{b}\int_{c}^{d} f_{XY}(x,y)\,dx\,dy = P(a \leq X \leq b,\ c \leq Y \leq d)$

then $f_{XY}(x,y)$ is called <u>two dimensional Prob. Density func^n.</u>

*** Marginal Prob. Density func^n (PDF) of X is given by:**

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy \rightarrow f_X(x) \geq 0, \quad \int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

Marginal PDF of Y is given by:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx \rightarrow f_Y(y) \geq 0, \quad \int_{-\infty}^{\infty} f_Y(y)\,dy = 1.$$

Q) 2-D Random variable (X, Y) has a joint PDF

$$f_{XY}(x,y) = \begin{cases} 15e^{-3x-5y} & ; x>0, y>0 \\ 0 & , \text{otherwise.} \end{cases}$$

Find: ⓐ i) $P(1<X<2,\ 0.2<Y<0.3)$

ii) $P(X<2,\ Y>0.2)$

ⓑ Marginal PDF of X & Y.

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→ ⓐ i) $P(1<x<2,\ 0.2<y<0.3)$

$$\Rightarrow \int_{y=0.2}^{0.3}\int_{x=1}^{2} 15^{-3x-5y}\,dx\,dy$$

$$= 15\int_{y=0.2}^{0.3}\left[\frac{e^{-3x-5y}}{-3}\right]_1^2 dy.$$

$$= 15\int_{y=0.2}^{0.3}\left[\frac{e^{-6-5y}}{-3}-\frac{e^{-3-5y}}{-3}\right] dy$$

$$= \cancel{15}\left[\frac{e^{-6-1.5}}{\cancel{15}}-\frac{e^{-3-1}}{\cancel{15}}\right] - \left[-e^{-6-1}+e^{-3-1.5}\right]$$

$$= e^{-7.5}+e^{-4}-e^{-7}-e^{-4.5}$$

ii) $P(0<x<2,\ 0.2<y<\infty)$

$$\Rightarrow \int_{y=0.2}^{\infty}\int_{x=0}^{2} 15e^{-3x-5y}\,dx\,dy.$$

$$\Rightarrow 15\left[\int_{y=0.2}^{\infty} e^{-5y}dy \int_{x=0}^{2} e^{-3x}dx\right]$$

$$\cancel{15}\left[\frac{-e^{-1}}{\cancel{-5}}\times\left(\frac{e^{-6}}{\cancel{-3}}-\frac{e^{0}}{\cancel{-3}}\right)\right]$$

$$e^{-1}\times(e^{-6}-1)$$

$$\cancel{-e^{-7}+e^{-1}=0.367}.$$

$$e^{-7}+e^{-1}=0.367.$$

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(b) Marginal PDF of X:

$$f_X(x) = \int_0^\infty f_{XY}(x,y)\,dy.$$

$$= \int_0^\infty 15e^{-3x-5y}\,dy.$$

$$= 15e^{-3x}\int_0^\infty e^{-5y}\,dy.$$

$$= 15e^{-3x}\left[0 - \frac{e^0}{-5}\right]$$

$$= 3e^{-3x}$$

Marginal PDF of Y:

$$f_Y(y) = \int_0^\infty f_{XY}(x,y)\,dx.$$

$$= 15e^{-5y}\int_0^\infty e^{-3x}\,dx.$$

$$= 15e^{-5y}\left[0 - \frac{e^0}{-3}\right]$$

$$= 5e^{-5y}.$$

9) Given $f_{XY}(x,y) = \begin{cases} cx(x-y), & 0<x<2 \;\&\; -x \leq y < x. \\ 0, & o/w. \end{cases}$
c = constant.

1. Evaluate c.
2. Find $f_X(x)$
3. Find $f_{Y/X}(y/x)$
4. Find $f_Y(y)$.

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Q) Find the prob. distribuⁿ of no. of Heads when a fair coin is tossed 4 times. Also find P.D. of sum of the nos. appearing on toss of 2 unbiased dice.

→ Total no. of outcomes $= 2^4 = 16$.

$\therefore S = \{(HHHH), (HHHT), (HHTH), (HTHH), (THHH),$
$(HHTT), (HTTH), (TTHH), (HTHT), (THTH), (THHT),$
$(HTTT), (THTT), (TTHT), (TTTH), (TTTT)\}$.

Random var.

Let X = no. of heads

x:	0	1	2	3	4
$P(X=x_i)$:	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

→ Prob. Distribuⁿ.

$\Sigma P(X=x_i) = 1$

2 Dice are thrown,

Total no. of outcomes $= 6^2 = 36$.

$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

X = Sum of the nos. on the 2 dice.

x:	2	3	4	5	6	7	8	9	10	11	12
$P(X=x_i)$:	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$\Sigma P(X=x_i) = 1$

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Q) A random var. X has a ~~joint P.D~~ following P.D.

X:	0	1	2	3	4	5	6	7
$P(X=x_i)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$
$\hookrightarrow$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{10^2}$	$\frac{2}{10^2}$	$\frac{17}{10^2}$

① Find k.

② Find $P\left(\frac{1.5 < x < 4.5}{x > 2}\right)$

③ Smallest value of λ for which $P(x \leq \lambda) > 1/2$

$\longrightarrow$ ① $k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1 = \Sigma P(X = x_i)$

$\Rightarrow 9k + 10k^2 = 1.$

$10k^2 + 9k - 1 = 0.$

$k = \frac{1}{10} = 0.1.$

② $P(A/B) = \frac{P(A \cap B)}{P(B)}.$

$\therefore P\left(\frac{1.5 < x < 4.5}{x > 2}\right) = \frac{P((1.5 < x < 4.5) \cap (x > 2))}{P(x > 2)}$

$= \frac{P(2 < x < 4.5)}{P(x > 2)}$

$= \frac{P(x = 3) + P(x = 4)}{\sum_{x_i = 3}^{7} P(X = x_i)}$

$= \frac{2/10 + 3/10}{2/10 + 3/10 + 1/10^2 + \frac{2}{10^2} + \frac{17}{10^2}}$

$= \frac{5/\cancel{10}}{\frac{7\cancel{0}}{\cancel{10^2}}} = \frac{5}{7} \Rightarrow ②$

③ $P(X \leq 3) = 5/10 = 1/2 \not> 1/2.$

$\therefore$ We must have $\lambda > 3.$

$\therefore$ we take $\lambda = 4.$

$\therefore P(x \leq 4) = \frac{8}{10} = \frac{4}{5} > \frac{1}{2}$ ✓

$\therefore$ Smallest value of λ satisfying the condn is 4.

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* Expectation & Variance of Random var. X.

Let X be a discrete Random var. taking values $x_1, x_2, \dots, x_n$ w/ the probabilities $P_1, P_2, \dots, P_n$, then, Expectation of X is given by,

$$E(X) = \sum_{i=1}^{n} x_i P_i$$

& Variance of X is given by,

$$V(X) = E(X^2) - [E(X)]^2$$

where, $E(X^2) = \sum_{i=1}^{n} x_i^2 \cdot P_i$

Q) A Random var. X has following P.D:

X:	1	2	3	4	5	6	7
$P(X=x_i)$:	k	$2k$	$3k$	k^2	k^2+k	$2k^2$	$4k^2$.

Find: ① k
② $P(X<5)$
③ $P(X>5)$
④ $P\left(\frac{X<5}{2<X\le 6}\right)$
⑤ $P\left(\frac{X=4}{3\le X\le 4}\right)$
⑥ $E(X)$
⑦ $V(X)$

→ ① $\sum P_i = 1 \Rightarrow 8k^2 + 7k = 1$
$\boxed{k = 1/8}$

② $P(X<5) = \sum_{i=1}^{4} P(X=x_i) = 6k + k^2 = \frac{6}{8} + \frac{1}{64} = \frac{49}{64}.$

③ $P(X>5) = P(X=6) + P(X=7) = 6k^2 = \frac{6}{64}$

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(4) $P\left(\frac{X<5}{2<X\le 6}\right) = \frac{P[(X<5)\cap(2<X\le 6)]}{P(2<X\le 6)}$

$= \frac{P[2<X<5]}{P(2<X\le 6]}$

$= \frac{P(X=3)+P(X=4)}{\sum_{i=3}^{5} P(X=x_i)}$

$= \frac{3k+k^2}{4k+4k^2} = \frac{\frac{3}{8}+\frac{1}{64}}{\frac{4}{8}+\frac{4}{64}} = \frac{\frac{25}{64}}{\frac{36}{64}} = \frac{25}{36}$

(5) $P\left(\frac{X=4}{3\le X\le 4}\right) = \frac{P[(X=4)\cap(3\le X\le 4)]}{P(3\le X\le 4)}$

$= \frac{P(X=4)}{P(3\le X\le 4)} = \frac{k^2}{3k+k^2} = \frac{1/64}{\frac{3}{8}+\frac{1}{64}} = \frac{1/64}{25/64}$

$= 1/25$

(6) $E(X) = \sum_{i=1}^{n} x_i p_i$

$= 1(k)+2(2k)+3(3k)+4(k^2)+5(k^2+k)+6(2k^2)$
$+7(4k^2)$

$= k+4k+9k+4k^2+5k^2+5k+12k^2+28k^2$

$= 49k^2+19k$

$= \frac{49}{64}+\frac{19}{8} = \frac{152+49}{64} = \boxed{\frac{201}{64} = E(X)}$

(7) $V(X) = E(X^2) - [E(X)]^2$

$E(X^2) = 1(k)+2^2(2k)+3^2(3k)+4^2(k^2)+5^2(k^2+k)+6^2(2k^2)$
$+7^2(4k^2)$

$= k+8k+27k+16k^2+25k^2+k+72k^2+196k^2$

$= 309k^2+61k = \frac{309}{64}+\frac{61}{8} = \frac{309+488}{64} = \frac{797}{64}$

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$$V(X) = \frac{797}{64} - \left(\frac{201}{64}\right)^2$$

$$= \frac{797 \times 64 - (201)^2}{64^2}$$

$$= \frac{51008 - 40401}{4096} = \frac{10607}{4096} = \underline{\underline{2.5895}}$$

Q) A R.V. X takes values;

X : -2 -1 0 1 2

Such that, $P(X>0) = P(X=0) = P(X<0)$

$P(X=-2) = P(X=-1)$

$P(X=1) = P(X=2)$.

Obtain Prob. distribu^n of X & find (1) $P\left(\frac{-1 \le X \le 1}{-2 \le X \le 0}\right)$

(2) $P\left(\frac{X=1}{0 \le X \le 2}\right)$

→ ~~Let $P(X=0) = Q$.~~

~~$\therefore P(X>0) = k = P(X=-2) + P(X=-1)$~~

~~=~~

Let $P(X=1) = k = P(X=2) = P(X=-1) = P(X=-2)$

$\therefore P(X>0) = \cancel{P(X=0)}\ P(X=1) + P(X=2)$

$= k + k = 2k$.

$\therefore P(X<0) = P(X=0) = P(X>0) = 2k$.

$\therefore \Sigma P_i = 1$.

$\therefore k + k + 2k + k + k = 6k = 1 \Rightarrow \boxed{k = \frac{1}{6}}$

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$\therefore P(X=-1) = P(X=-2) = k = \frac{1}{6}.$

$P(X=1) = P(X=2) = k = \frac{1}{6}.$

$P(X=0) = 2k = \frac{2}{6} = \frac{1}{3}.$

① $P\left(\frac{-1 \le X \le 1}{-2 \le X \le 0}\right) = \cancel{P[\ldots]}$

$= \frac{P[(-1 \le X \le 1) \cap (-2 \le X \le 0)]}{P(-2 \le X \le 0)}$

$= \frac{P[(-1 \le X \le 0)]}{P(-2 \le X \le 0)}$

$= \frac{P(X=-1) + P(X=0)}{\sum_{i=-2}^{0} P(X=x_i)}$

$= \frac{1/6 + 1/3}{1/6 + 1/6 + 1/3} = \frac{3/6}{4/6} = \frac{3}{4}.$

② $P\left(\frac{X=1}{0 \le X \le 2}\right) = \frac{P[(X=1) \cap (0 \le X \le 2)]}{P(0 \le X \le 2)}$

$= \frac{P(X=1)}{\sum_{i=0}^{2} P(X=x_i)} = \frac{1/6}{1/3 + 1/6 + 1/6} = \frac{1/6}{4/6} = \frac{1}{4}$

* Cumulative function of ~~of~~ a discrete R.V. X.

Let X be a discrete R.V. taking values $x_1, x_2, \ldots x_n$
Define a func^n F such that $F(x_i) = P(X \le x_i)$

i.e $F(x_1) = P(x_1)$

$$0 \le F(x) \le 1 \qquad \left\{ \begin{aligned} F(x_2) &= P(x_1) + P(x_2) \\ &\vdots \\ F(x_{n-1}) &= P(x_1) + P(x_2) + \cdots + P(x_{n-1}) \\ F(x_n) &= 1. \end{aligned} \right.$$

This 'F' is called cumulative Distribution function ~~(cdf)~~ (CDF)

Q) A R.V. X takes values 1, 2, 3, 4, such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$
Find PMF & CDF.

→ Let $P(X=3) = k$

$\therefore P(X=1) = \frac{k}{2}, \; P(X=2) = \frac{k}{3}, \; P(X=4) = \frac{k}{5}.$

$\because \Sigma p_i = 1.$

$\therefore k + \frac{k}{2} + \frac{k}{3} + \frac{k}{5} = \frac{30k + 15k + 10k + 6k}{30}$

$$\boxed{k = \frac{30}{61}}$$

$\therefore$ X :	1	2	3	4	
$P(X=x_i)$:	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$	⇒ PMF
CDF ⇒ $F(x)$:	$\frac{15}{61}$	$\frac{25}{61}$	$\frac{55}{61}$	$\frac{61}{61} = 1$	

Also, $P(1 < X < 3) = P(X=2) = 10/61$

$P(X \le 3) = 55/61 = F(3)$

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* **PDF of continuous R.V.**

Continuous func^n $y = f(x)$ such that:
1. $f(x)$ is integrable
2. $f(x) \geq 0$
3. $\int_a^b f(x)\,dx = 1. \longrightarrow x \in [a,b]$
4. $\int_\alpha^\beta f(x)\,dx = P(\alpha \leq x \leq \beta)$

then this f is called Probability Density func^n (PDF) of a R.V. X.

Q) Continuous R.V. X has following PDF
$f(x) = kx^2$; $0 \leq x \leq 2$
1. Find k
2. $P(0.2 \leq x \leq 0.5)$
3. $P\left(\frac{X \geq 3/4}{X \geq 1/2}\right)$

$\rightarrow$ Since $f(x)$ is a PDF,

$$\int_0^2 kx^2 = 1.$$

$$k\left[\frac{x^3}{3}\right]_0^2 = 1.$$

$$k\left[\frac{8}{3}\right] = 1 \Rightarrow \boxed{k = \frac{3}{8}} \longrightarrow (1)$$

2. $P(0.2 \leq x \leq 0.5) = \int_{0.2}^{0.5} \frac{3}{8} x^2\,dx.$

$$= \frac{3}{8}\left[\frac{x^3}{3}\right]_{0.2}^{0.5} = \frac{\cancel{3}}{8} \times \left[\left[\frac{0.125}{\cancel{3}}\right] - \left[\frac{0.008}{\cancel{3}}\right]\right]$$

$$= 0.014$$

(3) $P\left(\frac{X \geq 3/4}{X \geq 1/2}\right)$

Let A : $x \geq 1/2$
B : $x \geq 3/4$ $\Big\}$ $P(B/A) = \frac{P(A \cap B)}{P(A)}$

$$= \frac{P(x \geq 3/4)}{P(x \geq 1/2)}$$

Now, $P(A) = \int_{1/2}^{2} \frac{3}{8} x^2 dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_{0.5}^{2}$

$$= \frac{\cancel{3}}{8}\left[\frac{8}{\cancel{3}} - \frac{0.125}{\cancel{3}}\right]$$

$$= \frac{7.875}{8} = 0.984.$$

$P(B) = \int_{3/4}^{2} \frac{3}{8} x^2 dx = \frac{3}{8}\left[\frac{x^3}{\cancel{3}}\right]_{3/4}^{2} = \frac{1}{8}\left[8 - \frac{27}{64}\right]$

$$= \cancel{\frac{3}{8}\left[\frac{8}{3} - \frac{1.33}{3}\right]} = 0.959.$$

$$= \cancel{7.83}\ \cancel{0.833}.$$

$\therefore$ $P(B/A) = \cancel{\frac{0.984}{0}}\ \frac{0.833}{0.984} =$

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Q) Find the value of k such that

Find $P(x \le 1.5)$

$f(x) = \begin{cases} kx, & 0 \le x \le 1 \\ k, & 1 \le x \le 2 \\ k(3-x), & 2 \le x \le 3 \end{cases}$

$\rightarrow \int_0^3 f(x)\,dx = 1.$

$\Rightarrow \int_0^1 kx\,dx + \int_1^2 k\,dx + \int_2^3 k(3-x)\,dx = 1.$

$k\left[\frac{x^2}{2}\right]_0^1 + k[x]_1^2 + \cancel{3k - 3x}\, 3k[x]_2^3 - k\left[\frac{x^2}{2}\right]_2^3 = 1.$

$\frac{k}{2} + 2k - k + 3k[1] - k\left[\frac{9}{2} - \frac{4}{2}\right] = 1.$

$\frac{k}{2} + k + 3k - \frac{5}{2}k = 1.$

$2k = 1$

$\boxed{k = \frac{1}{2}}$

$P(x \le 1.5) = \int_0^{1.5} f(x)\,dx = \int_0^1 \frac{1}{2}x\,dx + \int_1^{1.5} \frac{1}{2}\,dx$

$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}[x]_1^{1.5}$

$= \frac{1}{2}\left[\frac{1}{2}\right] + \frac{1}{2}\cdot\frac{1}{2}$

$= \frac{1}{2}$

Q) A continuous R.V. X, has PDF $f(x) = A + Bx$. $0 \le x \le 1$

If mean of the ~~dist~~ R.V. X is $1/3$.
Find A & B.

→ $f(x) = A + Bx$, $0 \le x \le 1$.
we have, $\int_0^1 f(x)\,dx = 1$.

$= \int_0^1 (A + Bx)\,dx = 1$.

$= \left[Ax + \frac{Bx^2}{2}\right]_0^1 = 1$.

$A + \frac{B}{2} = 1$ — (1).

* In continuous distribution;
$E(X) = \text{Mean}(X) = \int_a^b x f(x)\,dx$.

$\therefore E(X^2) = \int_a^b x^2 f(x)\,dx$.
$\therefore V(X) = E(X^2) - [E(X)]^2$

→ Mean(X) $= E(X) = \int_0^1 x f(x)\,dx = \frac{1}{3}$ ⇒ given.

$\int_0^1 x(A + Bx)\,dx = 1/3$

$\left[A\frac{x^2}{2} + B\frac{x^3}{3}\right]_0^1 = 1/3$

$\frac{A}{2} + \frac{B}{3} = \frac{1}{3}$

$\frac{3A + 2B}{6} = \frac{1}{3}$ ⇒ $3A + 2B = 2$ — (2).

$\therefore A = 2, B = -2$.

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(1) Find k if $f(x) = \begin{cases} kx^2(1-x^3), & 0 \le x \le 1. \\ 0, & o/w. \end{cases}$

i. is PDF.

Find: (1) $P(0 \le x \le 1/2)$

(2) Mean.

(3) $V(x)$

$\longrightarrow \int_0^1 kx^2(1-x^3)\,dx = 1.$

$k\int_0^1 (x^2 - x^5)\,dx = 1.$

$k\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = 1;$

$k\left[\frac{1}{3} - \frac{1}{5}\right] = 1 \Rightarrow k\left[\frac{3}{18}\right] = 1.$

$\boxed{k = \frac{18}{3}} \Rightarrow \boxed{6 = k}$

(1) $P(0 \le X \le 1/2) \Rightarrow \int_0^{1/2} 6(x^2 - x^5)\,dx.$

$= 6\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^{1/2}$

$= 6\left[\frac{0.125}{3} - \frac{0.015}{6}\right]$

$= 6\left[0.041 - \cancel{0.00}0.0025\right]$

$= \underline{0.261}$

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② $Mean(x) = E(x) = \int_0^1 x f(x)\,dx.$

$$= \int_0^1 x \cdot 6[x^2 - x^5]\,dx$$

$$= 6\int_0^1 [x^3 - x^6]\,dx$$

$$= 6\left[\frac{x^4}{4} - \frac{x^7}{7}\right]_0^1$$

$$= 6\left[\frac{1}{4} - \frac{1}{7}\right] = 6\left[\frac{3}{28}\right] = \boxed{\frac{9}{14}} \quad -②$$

③ $E(x^2) = \int_0^1 x^2 \cdot 6(x^2 - x^5)\,dx.$

$$= 6\int_0^1 [x^4 - x^7]\,dx$$

$$= 6\left[\frac{x^5}{5} - \frac{x^8}{8}\right] = 6\left[\frac{1}{5} - \frac{1}{8}\right]$$

$$= 6\left[\frac{3}{40}\right] = \boxed{\frac{9}{20}}$$

$$V(x) = \frac{9}{40} - \left(\frac{9}{14}\right)^2 = \frac{9}{40} - \frac{81}{196}$$

$$= \cancel{0.225 - 0.413}$$

$$= 0.45 - 0.413$$

$$\boxed{V(x) = 0.037.} \quad -③$$

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* CDF of continuous Prob. Distribution:

$F(x) = P(X \le x)$
$= \int_{-\infty}^{x} f(t)\,dt \quad , -\infty < x < \infty$

is called cumulative Distribution function or CDF of R.V. X.

Properties:

① $0 \le F(x) \le 1$
② $x_1 \le x_2 \Rightarrow F(x_1) \le F(x_2)$
③ $F'(x) = f(x) \ge 0 \longrightarrow \frac{d}{dx}F(x) = f(x)$
④ $P(a \le x \le b) = F(b) - F(a)$

Q) CDF of continuous R.V. X is given by

$$F(x) = \begin{cases} 0 & ; x < 0 \\ x^2 & , 0 \le x \le 1 \\ 1 & ; x > 1 \end{cases}$$

① Find PDF.
② $P(\frac{1}{2} \le x \le \frac{4}{5})$

→ ① $F'(x) = f(x) = \begin{cases} 0 & , x < 0 \\ 2x & , 0 \le x \le 1 \\ 0 & , x > 1 \end{cases}$

② $P(\frac{1}{2} \le x \le \frac{4}{5}) = F(\frac{4}{5}) - F(\frac{1}{2})$
$= \left(\frac{4}{5}\right)^2 - \left(\frac{1}{2}\right)^2$
$= \frac{16}{25} - \frac{1}{4} = \frac{64-25}{100} = \frac{39}{100} = 0.39$

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Q) Find C.DF for following PDF

$$f(x) = \begin{cases} \frac{1}{2}x^2 e^{-x}, & 0 \le x \le \infty \\ 0, & o/w \end{cases}$$

→ $$F(x) = \int_0^x f(x)\,dx = \int_0^x \frac{1}{2}x^2 e^{-x}\,dx.$$

$$= \frac{1}{2}\left[x^2(-e^{-x}) - 2x(e^{-x}) + 2(-e^{-x})\right]_0^x$$ (succesive integration)

$$\underline{\underline{F(x) = 1 - e^{-x}\left(\frac{x^2}{2} + x + 1\right)}}$$

* Binomial Distribution:

Let X be a R.V. that follows a Binomial Distribution such that: ~~$P(X=r) = {}^nC_r p^r q^{n-r}$~~

$$\boxed{P(X=r) = {}^nC_r\, p^r q^{n-r}}$$

p = prob. of success.
q = prob. of failure. $= 1-p$

$\Rightarrow \underline{p+q=1}$

p = constant
n & p = Parameters of binomial distribution.

Properties:

① Mean $= E(X) = \sum x_i p_i = np$
n = no. of trials.

② Variance of $X = V(X) = npq$

③ If x_1 & x_2 are two binomial variates w/ parameter (n_1, p_1) & (n_2, p_2) respectively.
i.e.: $X_1 \sim B(n_1, p_1)$
$X_2 \sim B(n_2, p_2)$
then $X_1 + X_2$ will follow a Binomial Distribution ~~if $p_1 = p_2$~~ w/ parameters (n_1+n_2, p)

$$X_1 + X_2 \sim B(n_1+n_2, p)$$
where $p = p_1 = p_2$

④ If Mode is the value of X which has maximum probability.
Case (i): When $(n+1)p \in \mathbb{Z}$ say k.
Integer
2 modes: $k, k-1$
Case (ii): If $(n+1)p \notin \mathbb{Z}$, then mode = Integral part of $(n+1)p$.
Integer
If 34.4 Integral Part

Q) If X is binomially distributed w/ $E(X) = 2$ & $V(X) = 4/3$.
Find Prob. Distribuⁿ of X and prob. of atleast ~~1~~ one success.

→ $E(X) = 2$, $V(X) = 4/3$.
$= np$ $\qquad = npq$.

$$\therefore \frac{npq}{np} = q = \frac{4}{3\times 2} \Rightarrow \boxed{q = \frac{2}{3}}$$

$$\therefore \boxed{p = \frac{1}{3}}$$

$$np = 2 \Rightarrow n \times \frac{1}{3} = 2.$$

$$\therefore \boxed{n = 6}$$

$$\therefore P(X = x) = {}^nC_x \, p^x \, q^{n-x}$$

X :	0	1	2	3	4	5	6
$P(X=x)$:	${}^6C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^6$	${}^6C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^5$	${}^6C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4$	${}^6C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3$	${}^6C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2$	${}^6C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^1$	${}^6C_6\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^0$
	$\frac{64}{729}$	$\frac{192}{729}$	$\frac{240}{729}$	$\frac{160}{729}$	$\frac{60}{729}$	$\frac{12}{729}$	$\frac{1}{729}$

$$P(\text{at least one success}) = 1 - P(\text{no success})$$
$$= 1 - \frac{64}{729}$$
$$= \frac{665}{729}$$

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Q) Prob. that man aged 60 will live upto 70 is 0.65. What is the prob. that out of 10 such men at 60 atleast 7 will live upto 70.

→ p = man lives upto 70 = 0.65

$n = 10$

$q = 1-p = 1-0.65 = 0.35$.

$$P(X=x) = {}^nC_x\, p^x q^{n-x} = {}^{10}C_x (0.65)^x (0.35)^{10-x}$$

0.5)

$$P(X \geq 7) = P(X=7,8,9,10)$$

$$= \sum_{x=7}^{10} {}^{10}C_x (0.65)^x (0.35)^{10-x}$$

$$= {}^{10}C_7 (0.65)^7 (0.35)^{10-7} + {}^{10}C_8 (0.65)^8 (0.35)^{10-8} + {}^{10}C_9 (0.65)^9 (0.35)^{10-9} + {}^{10}C_{10} (0.65)^{10} (0.35)^{10-10}$$

$$= 120(0.049)(0.042) + 45(0.031)(0.122) + 10(0.020)(0.35) + 1(0.013)(1)$$

$$= 0.246 + 0.170 + 0.07 + 0.013$$

$$\doteq 0.499$$

1/4

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Q) Find p, if $n=6$, $9P(X=4) = P(X=2)$

Q) Comms. Sys. consists of n-components each of which has P="p". The total sys. will func^n effectively if atleast 1/2 of its component are functioning. For what value of 'p', 5 component sys. is more likely to be effective than a 3 component sys.

→ For Bin. Dist. we have $P(X=x) = {}^nC_x p^x q^{n-x}$

$P(5 \text{ comp. sys. will work eff.}) = P(X=3,4,5)$

X = no. of components

$= {}^5C_3 p^3 q^2 + {}^5C_4 p^4 q + {}^5C_5 p^5 q^0$

$= 10p^3q^2 + 5p^4q^1 + p^5$

$P(3 \text{ comp. eff.}) = P(X=2,3)$

$= {}^3C_2 p^2 q + {}^3C_3 p^3 q^0 = 3p^2q + p^3$.

5 comp. sys. will work more eff. than 3 comp. if,

$P(X=5) > P(X=3)$

$10p^3(1-p)^2 + 5p^4(1-p) + p^5 > 3p^2(1-p) + p^3$

$10p^3 - 20p^4 + 10p^5 + 5p^4 + p^5 > 3p^2 - 3p^3 + p^3$

$-3p^2 + 12p^3 - 15p^4 + 6p^5 > 0$

$3p^2(4p - 5p^2 + 2p^3 - 1) > 0$

$3p^2(p-1)(2p^2 - 3p + 1) > 0$

1	2	-5	4	-1
	↓	2	-3	1
	2	-3	1	0

$3p^3(p-1)(2p-1)(p-1) > 0$

$2p - 1 > 0$ i.e $\underline{p > 1/2}$ → Only then 5 comp. sys. will work more effectively than 3 comp.

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Q) It has been claimed that in 60% of all solar grid installation, utility bill is reduced by atleast 1/3rd. What are the probabilities that the utility bill will be reduced by atleast 1/3rd in
① 4 of 5 installations.
② Atleast 4 of 5 installations.

→ Let X = no. of installations. ⇒ R.V.
Given: $p = 60\% = 0.6$
$\therefore q = 0.4$
If $n = 5$, $x = 4$

① $P(X=4) = {}^5C_4 (0.6)^4 (0.4)$
$= 0.26$
② $P(X \geq 4) = {}^5C_4 (0.6)^4 (0.4) + {}^5C_5 (0.6)^5 (0.4)^0$
$= 0.26 + 0.077$
$= 0.337.$

Q) Prob that any moment, 1 Telephone line out of 10 will be busy is 0.2. ~~what is the prob that~~
① What is prob. that 5 lines are busy.
② Find Expected no. of busy lines. & also find prob. of this expected no.
③ Find Prob. that all lines are busy.

Q) 7 Dice are thrown 729 times. How many times do you expect atleast 4 dice to show 3 or 5.

→ X = No. of dice showing 3 or 5.
$n = 7$, $p = 1/6 + 1/6 = 2/6 = 1/3$
$q = 2/3$

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$$P(X \geq 4) = P(X=4) + P(X=5) + P(X=6) + P(X=7)$$
$$= {}^7C_4 (1/3)^4 (2/3)^3 + {}^7C_5 (1/3)^5 (2/3)^2 + {}^7C_6 (1/3)^6 (2/3)$$
$$+ {}^7C_7 (1/3)^7 (2/3)^0$$
$$= \frac{379}{3^7} = \frac{379}{2187}$$

$\therefore$ Expected no. of times getting 3 or 5 (atleast 4 dice) out of 729 throws $= 729 \times P(X \geq 4)$
$$= \frac{276291}{3^7}$$
$$= 126.33$$

Q) Let X & Y be independent binomial variates w/ parameters,
$n_1 = 6, p_1 = 1/2, n_2 = 4, p_2 = 1/2$ respectively.
Evaluate: $P(X+Y=3)$

→ We have: $X+Y \sim B(n_1+n_2, p)$ ⇒ binomial Distribution. (English ⇒ X+Y follows a)

$$X+Y \sim B(10, 1/2)$$

Let $X+Y = Z$
$$\therefore P(Z=3) = {}^nC_3\, p^3 q^{n-3}$$
$$= {}^{10}C_3 \left(\frac{1}{3}\right)^3 \left(\frac{1}{2}\right)^7 = 120(\cancel{0.037}\ 0.125)(0.007)$$
$$= \cancel{0.037}\ 0.105$$

$$P(Z \geq 3) = 1 - P(Z < 3)$$
$$= 1 - [P(Z=0) + P(Z=1) + P(Z=2)]$$
$$= 0.945$$

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Q) 3 Fair coins are tossed 3000 times. Find the frequencies of distribuⁿ as heads & tails & tabulate the results. Also calculate mean & std. deviaⁿ of the distribution.

→ $n=3,\ p=1/2$.
X : no. of heads.
$N = 3000$

X :	0	1	2	3
$P(X=x)$:	0.125	0.375	0.375	0.125
$f = NP$:	375	1125	1125	375

$$P(X=x) = {}^nC_x\, p^x q^{n-x}$$
$$= {}^3C_x (0.5)^x (0.5)^{3-x}$$
$$= 0.125$$

$$\text{Mean} = np = \frac{3}{2}.$$

$$\text{Variance} = npq = 3 \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}.$$

$$\text{Std. deviation} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

* Fitting of Binomial Distribuⁿ:

Q) 7 coins are tossed & no. of heads obtained is noted. Experiment is repeated 128 times & following distribution is obtained.

- ~~N=128, n=7.~~

No. of heads :	0	1	2	3	4	5	6	7	Total
observed Freq :	7	6	19	35	30	23	7	1	= 128

→ Fit a Binomial dist. if,

① The coins are unbiased.

② If the nature of the coins is not known.

→ ① $p = 1/2$, $q = 1/2$, $n = 7$, $N = 128$.

$$P(X=x) = {}^nC_x p^x q^{n-x}$$

$$= {}^7C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{7-x}$$

~~x: P(X=x)~~

X : No. of heads.

X :	0	1	2	3	4	5	6	7	
$P(X=x)$:	$\frac{1}{2^7}$	$\frac{7}{2^7}$	$\frac{21}{2^7}$	$\frac{35}{2^7}$	$\frac{35}{2^7}$	$\frac{21}{2^7}$	$\frac{7}{2^7}$	$\frac{1}{2^7}$	
Expected Np = Freq : ($128 = 2^7$)	$\frac{\cancel{2^7}}{\cancel{2^7}}$	$\frac{7\times\cancel{2^7}}{\cancel{2^7}}$	$\frac{21\times\cancel{2^7}}{\cancel{2^7}}$	$\frac{35\times\cancel{2^7}}{\cancel{2^7}}$	$\frac{35\times\cancel{2^7}}{\cancel{2^7}}$	$\frac{21\times\cancel{2^7}}{\cancel{2^7}}$	$\frac{7\times\cancel{2^7}}{\cancel{2^7}}$	$\frac{1\times\cancel{2^7}}{\cancel{2^7}}$	= 128.

② $$\bar{x} = \frac{\Sigma x_i f_i}{\Sigma f_i} = \frac{6 + 2(19) + 3(35) + 4(30) + 5(23) + 6(7) + 7}{128}$$

$$= \frac{6 + 38 + 105 + 120 + 115 + 42 + 7}{128}$$

$$= \frac{433}{128} = 3.38.$$

but $\bar{x} = np = 3.38$

$$p = \frac{3.38}{7} = 0.48$$

$\therefore q = 0.52$

$$P(X=x) = {}^nC_x p^x q^{n-x}$$
$$= {}^7C_x (0.48)^x (0.52)^{7-x}$$

~~Solve~~ calculate.

$$= {}^7C_0 (0.48)^0 (0.52)^7 = 0.0102$$
$${}^7C_1 (0.48) (0.52)^6 = 0.0094$$
$${}^7C_2 (0.48)^2 (0.52)^5 = 0.183$$
$${}^7C_3 (0.48)^3 (0.52)^4 = 0.283$$
$${}^7C_4 (0.48)^4 (0.52)^3 = 0.261$$
$${}^7C_5 (0.48)^5 (0.52)^2 = 0.144$$
$${}^7C_6 (0.48)^6 (0.52)^1 = 0.044$$
$${}^7C_7 (0.48)^7 (0.52)^0 = 0.0058$$

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★ Poisson Distribution:

It is a limiting case of Binomial Distribution under the condition n is sufficiently large ($n \to \infty$) and p is sufficiently small ($p \to 0$) but $np = m$ is finite.

PDF:

A R.V. X follows a poisson distribution w/ parameter 'm' (mean of the distribution)

if $P(X=x) = \frac{e^{-m} m^x}{x!}$; $x = 0, 1, 2, \ldots$

Properties:

① Mean of the distribution, $m = np$

② Variance $\& V(X) = m$.

③ If $X_1, X_2, \ldots, X_n$ are n independent Poisson variates w/ parameters $m_1, m_2, \ldots, m_n$, then,

$Y = X_1 + X_2 + \cdots + X_n$

is also a Poisson variate w/ parameter $m_1 + m_2 + \cdots + m_n$

but, $Y = a_1X_1 + a_2X_2 + a_3X_3 + \cdots + a_nX_n$.

$Y = X_1 - X_2 - X_3 - \cdots - X_n$.

are not Poisson variates

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Q) Mean of a Poisson Distribuⁿ is 4.
Find ~~prob.~~ $P(m-2\sigma < X < m+2\sigma)$.

→ $m = 4 \Rightarrow Var = 4$.
$\sigma = \sqrt{Var} = 2$.

$P(X=x) = \frac{e^{-m} m^x}{x!} = \frac{e^{-4} 4^x}{x!}$

$\therefore P(0 < x < 8) = P(X = 1 \text{ to } 7)$

$= \sum_{x=1}^{7} \frac{e^{-4} 4^x}{x!}$

$= e^{-4}\left[\frac{4}{1} + \frac{16}{2} + \frac{64}{6} + \frac{256}{24} + \frac{1024}{120} + \frac{4096}{720} + \frac{16384}{5040}\right]$

$= 0.018[50.806]$

$= 0.914$.

Q) A car hire firm has 2 cars. No. of demand for a car on each day follows a poisson distribution ~~Find the probability Distribuⁿ~~ w/ mean 1.5.
Find the prob. that
1. Niether car is used.
2. Some demand is refused.

→ X = No. of demands
$m = 1.5$

$P(X=x) = \frac{e^{-1.5}(1.5)^x}{x!}$

1. $P(X=0) = \frac{e^{-1.5}(1.5)^0}{0!} = 0.223$

② $P(X>2) = 1 - P(X \le 2)$
$= 1 - [P(X=0) + P(X=1) + P(X=2)]$
$= 1 - [0.223 + \cancel{0.223}\ 0.3345 + 0.250]$
$= 0.192$

Q) If X_1, X_2, X_3 are 3 poisson variates w/ parameter $m_1 = 1$, $m_2 = 2$, $m_3 = 3$.
Find: ① $P[(X_1 + X_2 + X_3) \ge 3]$
② $P[(X_1 = 1)]$ given that $X_1 + X_2 + X_3 = 3$

→ $z = X_1 + X_2 + X_3$ is a poisson variate w/ parameter $m = 6 = (m_1 + m_2 + m_3)$

$\therefore P(Z = z) = \dfrac{e^{-6} 6^{z}}{z!}$

$\therefore P(Z \ge 3) = 1 - P(Z < 3)$
$= 1 - [P(Z=0) + P(Z=1) + P(Z=2)]$
$= 1 - [0.002 + 0.014 + 0.0446]$
$= \cancel{0.9}\ 1 - [0.606]$
$= 0.9393$

② $P[(X_1 = 1) / X_1 + X_2 + X_3 = 3]$

$$= \frac{P[(X_1 = 1) \cap (X_1 + X_2 + X_3 = 3)]}{P(X_1 + X_2 + X_3 = 3)}$$

$$= \frac{P[\underbrace{(X_1 = 1)}_{A} \cap \underbrace{(X_2 + X_3 = 2)}_{B}]}{P(X_1 + X_2 + X_3 = 3)}$$

$$= \frac{P(X_1 = 1) \times P(X_2 + X_3 = 2)}{P(X_1 + X_2 + X_3 = 3)}$$

$$= \frac{\left(\dfrac{e^{-1} 1^{1}}{1!}\right) \times \left(\dfrac{e^{-5} 5^{2}}{2!}\right)}{\left(\dfrac{e^{-6} 6^{3}}{3!}\right)}$$

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Q) No. of accidents in a year of Taxi drivers in a city follows poisson distribu^n w/ mean = 3. Out of 1000 Taxi drivers find approx. no. of drivers w/ ① No accidents in a year
② More than 3 accidents in a year.

→ X = No. of accidents
$N = 1000$
$$P(X=x) = \frac{e^{-3}3^x}{x!}$$

① P(No accidents) $= P(X=0)$
$$= \frac{e^{-3} \cdot 3^0}{0!} = 0.0497$$

∴ Expected No. of drivers w/ no accidents
$= 1000 \times 0.0497$
$= 50$

② P(more than 3 accidents) $= P(X>3)$
$= 1 - P(X \leq 2)$
$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$
$= 1 - [0.049 + 0.147 + 0.2205 + \cancel{0.447}\ 0.2205]$
$= \cancel{0.1425}\ 0.363$

Expected no. of Drivers $= 1000 \times P(X>3)$
$= 363$.

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Q) In a certain factory turning out blades there is a small chance : 1/500 for any blade to be defective blades are supplied in a packet of 10. Use poisson distⁿ to calculate approx. no. of packets containing
1. No defective.
2. 1 defective.
3. 2 Defective

Blades in a consignment of 10,000 packets.

→ Let X = No. of Defective blades.

$N = 10{,}000$

$n = 10$

$p = 1/500$

$m = np = 10 \times \frac{1}{500} = 0.02.$

$$P(X = x) = \frac{e^{-m} m^x}{x!} = \frac{e^{-0.02}(0.02)^x}{x!}.$$

1. $P(X=0) = \frac{e^{-0.02}(0.02)^0}{0!} = 0.9802 \Rightarrow 9802$ ↓ No. of packets w/ no defective.
2. $P(X=1) = 0.0196.$
3. $P(X=2) = 0.0001$

Whenever p is very small, & n is very large,
go w/ poisson distribu^n.

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Q) A Insurance company found that only 0.01% of the popula^n meets w/ an accident each year. If its 1000 policy holders were selected, what is the probability that,
(1) No more than 2 of its clients meets w/ an accident next year.

→ $p = 0.01\% = 0.0001$
$n = 1000$
$m = np = 0.0001 \times 1000$
$= 0.1$

$P(x = x) = \frac{e^{-0.1}(0.1)^x}{x!}$

$P(x \leq 2) = P(x=0) + P(x=1) + P(x=2)$
$= 0.9048 + 0.0904 + 0.0045$
$= 0.9997$

Q) Fit a poisson distribu^n w/ the following data:

No. of deaths:	0	1	2	3	4	Total
Freq:	123	59	14	3	1	$\Sigma f_i = 200 = N$

→ $m = \frac{\Sigma x_i f_i}{\Sigma f_i} = 0.5$

$\therefore P(X=x) = \frac{e^{-m} m^x}{x!} = \frac{e^{-0.5}(0.5)^x}{x!}$

$P(X=0) = 0.606$
$P(X=1) = 0.303$
$P(X=2) = 0.075$
$P(X=3) = 0.012$
$P(X=4) = 0.001$

Expected freq: ~~0.606~~ ~~0.303~~ ~~0.075~~ ~~0.012~~ ~~0.001~~
NP : 121 61 15 2 1

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Q) Using poisson distribuⁿ, find approx. Value of
$^{300}C_2 (0.02)^2 (0.98)^{298} + {}^{300}C_3 (0.02)^3 (0.98)^{297}$?

→ We observe that, these probabilities are prob. of binomial distribuⁿ w/,

$\left.\begin{array}{l} n = 300 \\ p = 0.02 \end{array}\right\} m = np = 6$

$\therefore P(X=x) = \dfrac{e^{-6}(6)^x}{x!}$

$\therefore P(X=2) = \dfrac{e^{-6}(6)^2}{2!} = \dfrac{0.002 \times 36}{2}$

$= 0.044$

$P(X=3) = \dfrac{e^{-6}(6)^3}{3!} = \dfrac{0.002 \times 216}{6}$

$= 0.072$

$\Rightarrow P(X=2) + P(X=3) = 0.044 + 0.072$

$= 0.116.$

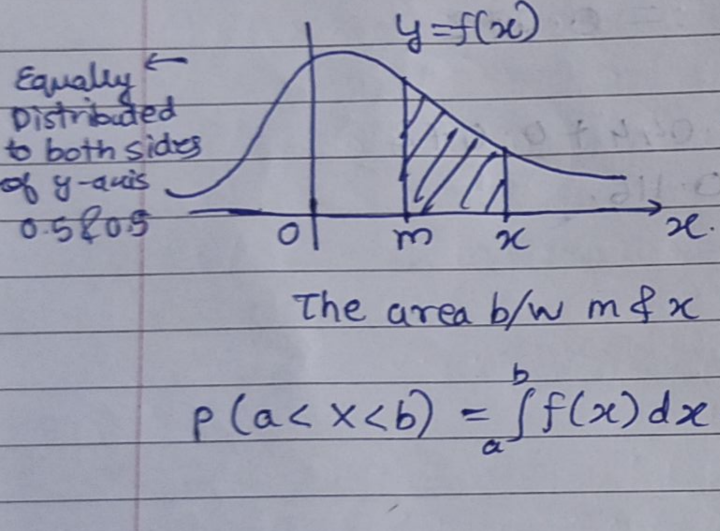
* <u>Normal Distribution:</u>
A continuous R.V. X is set to follow normal distⁿ w/ parameters m (mean) & σ (~~va~~ std. deviaⁿ). if its Prob. density funcⁿ (PDF) is given by,

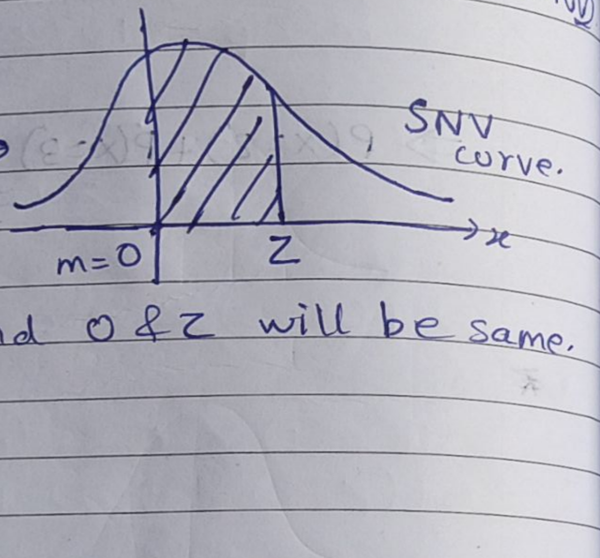
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \times e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

i.e. $X \sim N(m, \sigma)$ ~~follows~~.

<u>Note:</u>
(1) If $X \sim N(m, \sigma)$ then $Z = \frac{X-m}{\sigma} \sim N(\overset{m}{0}, \overset{\sigma}{1})$

↳ standard Normal Variate (SNV)





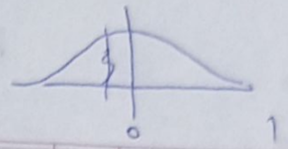
The area b/w m & x and 0 & Z will be same.

$$P(a < X < b) = \int_a^b f(x)\,dx.$$

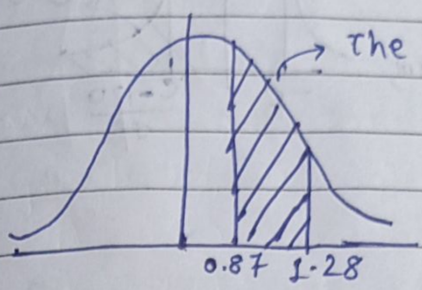
(2) Let $X_1, X_2, \ldots, X_n$ be normal variates w/ parameters $m_1, m_2, \ldots, m_n$ & $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, then.

$$Y = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$$

is also a normal variate w/ parameters,
$m = a_1 m_1 + a_2 m_2 + \cdots + a_n m_n$ &
$\sigma = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + \cdots + a_n^2 \sigma_n^2$



Q) Find $P(0.87 < Z < 1.28)$

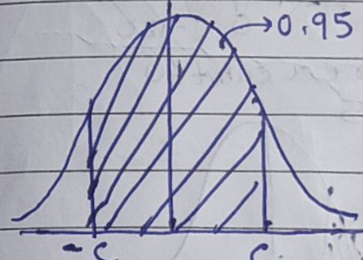


$$\therefore P(0.87 < Z < 1.28) = P(0 < Z < 1.28) - P(0 < Z < 0.87)$$
$$= 0.3997 - 0.3078$$
$$= 0.0919.$$

Q) Find the value of such that,

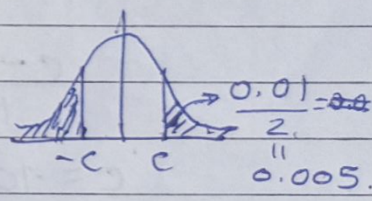
① $P(-c < Z < c) = 0.95$
② $P(|Z| \geq c) = 0.01$
③ $P(x > c) = 0.02$
④ $P(x < c) = 0.05$
$m = 120, \sigma = 10.$

→



① $\therefore P(0 < Z < c) = \dfrac{0.95}{2}$
$= 0.475$
$\therefore c = 1.96$ (from table).

② $P(|Z| > c) = 0.01$
i.e $P(Z > c, Z < -c) = 0.01$
$\therefore P(Z > c) = 0.005.$



$P(0 < Z < c) = 0.5 - P(Z > c)$
$= 0.5 - 0.005$
$= 0.495$
$c = 2.58$ } Table (0.4951)

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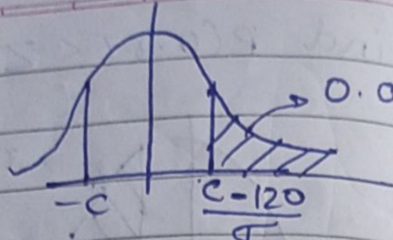
③ $P(x > c) = 0.02$



To convert it into normal form,

$$P\left(\frac{x-m}{\sigma} > \frac{c-m}{\sigma}\right) = 0.02$$

$$P\left(z > \frac{c-120}{10}\right) = 0.02$$

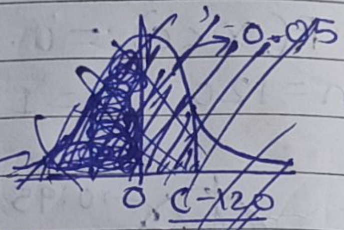
$$P\left(0 < z < \frac{c-120}{10}\right) = 0.5 - 0.02$$
$$= 0.48$$

$$\therefore \frac{c-120}{10} = 2.06$$
$$c = 140.6$$

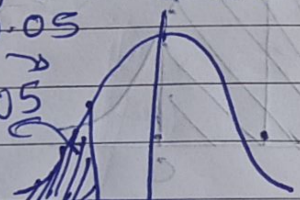
④ $P(x < c) = 0.05$

$$P\left(\frac{x-m}{\sigma} < \frac{c-m}{\sigma}\right) = 0.05$$

$$P\left(z < \frac{c-m}{\sigma}\right) = 0.05$$



$\cancel{\frac{c-120}{10} =}$ since that big area can't ~~be equal to 0.05~~
we take



$$P\left(0 < z < \frac{c-120}{10}\right) = 0.5 - 0.05$$
$$= 0.495$$

$$\frac{c-120}{10} = -1.65$$ -ve coz the point is in -ve x-axis.

$$c = 103.5$$

Q) Monthly salary X is normaly distributed b/w m = ₹3000 & std. devia^n σ ₹250. What should be the min. salary of the worker so that he belongs to top 5% workers?

→ $m = 3000$, $\sigma = 250$

Let X : Monthly salary of a worker.

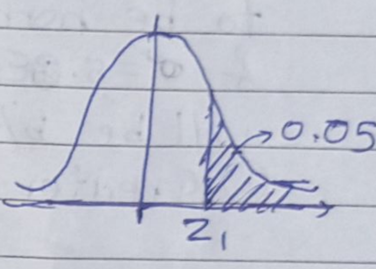
Let x_1 be min. salary of top 5% workers.

SNV = $Z = \frac{X - m}{\sigma} = \frac{X - 3000}{250}$

$P(X > X_1) = 5\%$

$\Rightarrow P\left(\frac{X - m}{\sigma} > \frac{X_1 - m}{\sigma}\right) = 5\%$

$P(Z > Z_1) = 0.05$



$P(0 < Z < Z_1) = 0.5 - 0.05$

$= 0.45$

$\therefore Z_1 = 1.65 = \frac{X_1 - 3000}{250}$

$X_1 = 3412.5$

Q) Diameter of tops of the cans are normally distributed w/ std. dev^n 0.05 cm. At what mean diameter of the m/c should be set so that not more than 5% of the tops can have diameter exceeding 3 cm.

→ X : Diameter of the top.

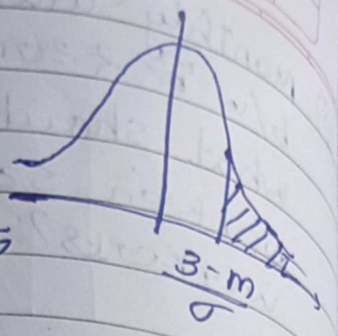
$\sigma = 0.05$

$P(X > 3) = 5\%$

→ →

$$P\left(\frac{X-m}{\sigma} > \frac{3-m}{\sigma}\right) = 0.05$$

ie, $P\left(z > \frac{3-m}{\sigma}\right) = 0.05.$



$\therefore P\left(0 < z < \frac{3-m}{\sigma}\right) = 0.5 - 0.05 = 0.45$

$\frac{3-m}{0.05} = 1.65 \Rightarrow \underline{m = 2.917}$

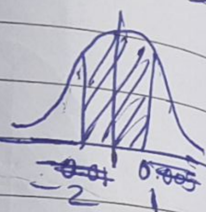
Q) Marks obtained by 1000 students are found to be normally distributed w/ mean = 70 & $\sigma = 5$. Estimate ① no. of students whose marks will be b/w 60 & 75
② Greater than or equal to 75.

→ X: Marks of the student.

① $P(60 < X < 75) \Rightarrow P\left(\frac{60-70}{5} < \frac{X-70}{5} < \frac{75-70}{5}\right)$

$P(-2 < z < 1)$

$\cancel{P(-0.001 < z < 0.005)}$

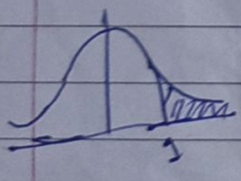


$\Rightarrow P(-2 < z < 0) + P(0 < z < 1)$

$\cancel{0.0040 +}$

$0.4772 + 0.3413 = \underline{0.8185}$

② $P(75 \le X) = P\left(\frac{X-70}{5} \geq \frac{75-70}{5}\right)$



$= P(z \geq 1) = 0.5 - 0.3413$
$= \underline{0.1587}$

$= P(0 < z < 1)$

Q If X_1 & X_2 are 2 normal variates w/ mean $m_1 = 30$ & $m_2 = 25$ & $\sigma_1^2 = 16$ (variance) $\sigma_2^2 = 12$. and $Y = 3X_1 - 2X_2$

Find $P(60 \le Y \le 80)$

→ $Y = a_1 X_1 + a_2 X_2$. → Normal variate

$\therefore a_1 = 3,\ a_2 = -2.$

$m = a_1 m_1 + a_2 m_2$

$= 3(30) + (-2)(25) = 40$

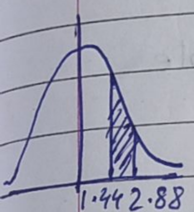
$\sigma^2 = 9 \times 16 + (-2)^2 \times 12$

$= 144 + 48 = 192.$

$\therefore \sigma = 8\sqrt{3}$

$\therefore P(60 \le Y \le 80) = P\left(\frac{60-40}{8\sqrt{3}} \le Z \le \frac{80-40}{8\sqrt{3}}\right)$

$= P(1.44 \le Z \le 2.88)$



$= \cancel{P(0 \le Z \le 2.88)}$.

$= P(0 \le Z \le 2.88) - P(0 \le Z \le 1.44)$

$= 0.4980 - 0.4261$

$= \underline{0.0729}$.

Q) Two independent R.V. X & Y are distributed normally w/ mean & std. dev^n as $(52, 3)$ & $(50, 2)$ respectively. Find the prob. that randomly pair of values of X & Y will differ by 1.7 or more

→ $P(|X - Y| \ge 1.7)$

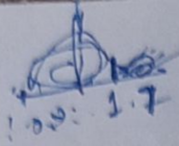
Let $X - Y = U$

$\therefore P(|U| \ge 1.7) \Rightarrow$ mean of $U = 52 - 50 = 2$ ← $1(52) + (-1)(50)$

var. of $U = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2$

$= 9 + 4 = 13$

$\therefore \sigma$ of $U = \sqrt{13}$



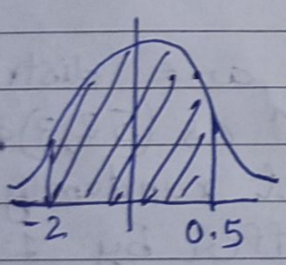
$$\therefore P(|U| \geq 1.7) = 1 - P(|U| \leq 1.7)$$
$$= 1 - P(-1.7 \leq U \leq 1.7)$$
$$= 1 - P\left(\frac{-1.7-2}{\sqrt{13}} \leq Z \leq \frac{1.7-2}{\sqrt{13}}\right)$$
$$= 1 - P(-1.03 \leq Z \leq -0.08)$$
$$= 1 - P(-1.03 < Z < 0) - P(-0.08 < Z < 0)$$
$$= 1 - (0.3485 - 0.0319)$$
$$= \cancel{1 - 0.6196}\ 1 - (0.3166)$$
$$= \cancel{0.6834} = 0.6834$$

Q) The incomes of group of 10,000 persons are normally distributed w/ mean = 520 & s.d = 60.
① Find no. of persons having income b/w 400 & 550
② The lowest income of richest 500 people

→ X: Income of the person.

① $P(400 < x < 550) = P\left(\frac{400-520}{60} < \frac{x-520}{60} < \frac{550-520}{60}\right)$

$$= P\left(\frac{-120}{60} < Z < \frac{1}{2}\right)$$

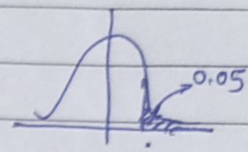


$$= P(-2 < Z < 0.5)$$
$$= P(-2 \geq Z) + P(Z \leq 0.5)$$
$$= 0.4772 + 0.1915$$
$$= \cancel{0.2857}\ 0.6687.$$

(2) X_1 : lowest income of richest 500 people.

$$P(X \geq X_1) = \frac{500}{10000} = 0.05$$

$$P(0 \le Z \le z_1) = 0.5 - 0.05$$
$$= 0.45.$$



$\cancel{\therefore x - 520}$

$$z_1 = 1.65.$$

$$\frac{X_1 - 520}{60} = 1.65$$

$\boxed{X_1 = 619}$ $\longrightarrow$ min. salary of richest 500 people.

5/3/24 MODULE-1 continued.

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* Exponential Distribution:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , \text{otherwise.} \end{cases}$$

$$\text{Mean} = \frac{1}{\lambda} = E(x)$$

$$\text{Var} = E(x^2) - [E(x)]^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Note: $\int_K^{\infty} f(x)\,dx = P(x > k) = e^{-\lambda k}$

Q) The milage of cars with certain kind of ~~time~~ tire is a R.V. having an expo. distri. w/ ~~me~~ mean $= 1/\lambda = 40000$ km.

Find the prob. that one of these tires will last.

i) Atleast 20,000 km.

ii) Atmost 30,000 km.

→ x = mileage

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{40000} \cdot e^{-\frac{1}{40000}x}$$

i) $P(x \geq 20000) = \int_{20000}^{\infty} f(x)\,dx = e^{-\frac{1}{40000} \times 20000}$

$= e^{-0.5}$

ii) $P(x \leq 30000) = \int_0^{30000} f(x)\,dx = 1 - e^{-0.75}$

$= 0.5270$

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Q) The no. of kms. that a car can run before its battery wears out is expo. distributed w/. an avg. value of 10,000 km. The owner desires to take 5000 km trip. What is prob. that he'll be able to complete the trip w/o having to change the battery.

→ Assume that car has been used for some time.

X = no. of km.

$\text{mean} = \frac{1}{\lambda} = 10{,}000$

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{10000} \cdot e^{-\frac{1}{10000}x}$$

$$P(X \geq 5000) = \int_{5000}^{\infty} f(x)\,dx = e^{-\frac{1}{10000} \times 5000}$$

$$= e^{-0.5}$$

Q) Amount of time that a watch will run w/o having to be reset is a R.V. having Exp. dist. with mean = 120 days. Find the prob. that the watch will have to be set.

i) In less than 24 days.

ii) Not have to be reset in atleast 180 days.

→ X = Time the watch will run w/o reset.

$\text{mean} = \frac{1}{\lambda} = 120$

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{120} e^{-\frac{1}{120}x}$$

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i) $\cdot P(X<24) = \int_0^{24} f(x)\,dx = 0.1813$

ii) $P(X \geq 180) = \int_{180}^{\infty} f(x)\,dx = e^{-\frac{1}{120}\times 180} = 0.2231.$

★ <u>Memory Less Property of Exeponential Distribution.</u>

If X is exponentially distributed, then

$$P(X > s+t / \underbrace{X > S}_{\text{Forgets this event.}}) = P(X > t) \quad \forall s,t > 0.$$

Q) The length of a showere on an island in rainy season has a exp. dist. w/ parameter 2. (Time being measured in minutes). ~~What~~ i

(i) What is the prob. that shower will last more than 3 mins?

(ii) If shower has already lasted for 2 min. what is the prob. that it will last for atleast 1 more minute?

→ X = Length of shower.

$\lambda = 2$

$f(x) = 2e^{-2x}$

i) $P(X > 3) = \int_3^{\infty} f(x)\,dx = e^{-2\times 3} = e^{-6}$

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ii) $X > 2 \to$ given.

$$P(X > 1+2 / X > 2) = P(X > 1)$$
$$= \int_1^\infty f(x)dx = e^{-2}$$

---

* Uniform Distribution: Discrete → same as binomial but here prob. is allways fix; Continuous

A discrete R.V. X is set to follow a Uniform dist. if ~~is~~ its PMF is given by,

- $P(X=x) = \frac{1}{n}$; $x = 1, 2, 3, \ldots, n$.
- Mean $= \frac{n+1}{2}$
- Var $= \frac{n^2-1}{12}$

Q) Fair Die is rolled & X denotes no. appeared on the top of the die. Find the prob. that,
i) Even no. appears.
ii) No. appears < 3.
iii) Compute mean & Var. of X.

→ X = A number appears on the top.

i) $P(X = 2, 4, 6) = P(X=2) + P(X=4) + P(X=6)$
$$= 1/6 + 1/6 + 1/6 = \frac{1}{2}.$$

ii) $P(X = 1, 2) = P(X=1) + P(X=2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

iii) Mean $= \frac{7}{2}$, Var $= \frac{35}{12}$.

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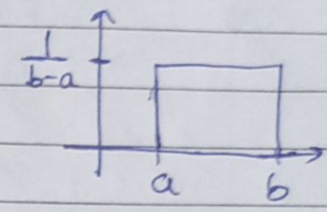
★ Continuous Uniform Distribution.

A continuous R.V. X is set to follow a uniform distribution, over an interval $[a,b]$ if its PDF is given by
$\Rightarrow X \sim U[a,b]$

PDF $\Rightarrow$ • $f(x) = \begin{cases} k & , a \leq x \leq b \\ 0 & , \text{otherwise.} \end{cases}$

$\because \int_a^b k\,dx = 1 \Rightarrow \boxed{k = \frac{1}{b-a}}$



• Mean $= \frac{b+a}{2}$.

• Var $= \frac{(b-a)^2}{12}$

Note: For any sub interval, $[c,d] \subset$ of $[a,b]$
$P(c \leq x \leq d) = \frac{d-c}{b-a}$

Q) If X is uniformly dist. in the interval $[-2,2]$
$X \sim U[-2,2]$.
Find ① $P(X<1)$
② $P(|X-1| \geq 1/2)$

$\rightarrow$ ① $X < 1 \Rightarrow X \in [-2,1] \subset [-2,2]$
$\therefore P(X<1) = P(-2 < x < 1) = \frac{1-(-2)}{2-(-2)} \cancel{= \frac{3}{4}}$

$= \frac{3}{4}$.

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(2) $|x-1| \geq 1/2 \Rightarrow x \geq 3/2$ & $x \leq 1/2$.

$x \in [3/2, 2]$ $\quad$ $x \in [-2, 1/2]$

$\therefore P(|x-1| \geq 1/2) = P(\frac{3}{2} \leq x \leq 2) + P(-2 \leq x \leq 1/2)$

$$= \frac{2 - 3/2}{2-(-2)} + \frac{1/2 - (-2)}{2-(-2)}$$

$$= \frac{1/2}{4} + \frac{5/2}{4} = \frac{1}{8} + \frac{5}{8}$$

$$= 3/4.$$

Q) A question is given to 30 participants & time allowed to answer it is 25 sec. Find the Prob. the participants respond within 6 sec.

→ Interval of ~~succesful event~~ Total distribution $\Rightarrow [0, 25] = [a, b]$.

$\therefore f(x) = \frac{1}{25-0} = \frac{1}{25}$

Interval of succesful event $\Rightarrow [0, 6] = [c, d]$

$P(x < 6) = \int_0^6 f(x)\,dx = \frac{6}{25}$

$\therefore$ No. of participants $= 30 \times \frac{6}{25}$.

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Q) Suppose RV. x. is taken ~~for~~ from 690 to 850. in uniform distribuⁿ. Find the prob. that the no. N is > 790.

→ Interval of dist → $[690, 850] = [a, b]$.

Interval of succesful event ⇒ $[790, 850] = [c, d]$.

$$\therefore P(790 \leq x \leq 850) = \cancel{790}\ \cancel{80}\ \cancel{\frac{850-790}{850-}}$$
$$= \frac{850-790}{850-690} = \frac{60}{160} = \frac{3}{8}$$

Q) Man w/ n keys wants to open the door by trying the keys independently & randomly one by one. Find the mean & var of no. of trials if unsuccesful keys are kept aside.

7/3/24 # MODULE-3: SAMPLING THEORY

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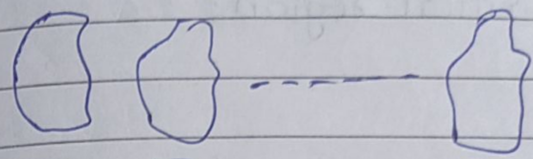
★ **Large-Sampling.** $(n > 30)$ μ σ Population μ = mean σ = std. deviaⁿ.

Suppose there are various samples:

S_1 S_2 S_n



Mean ⇒ $\bar{x}_1$ $\bar{x}_2$ $\bar{x}_n$

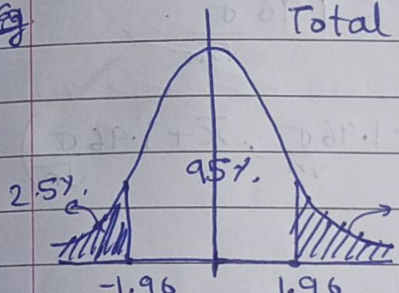
$\bar{X}$: $\bar{x}_1$ $\bar{x}_2$ $\bar{x}_3$ - - - - $\bar{x}_n$

Theorem: If $\bar{X}$ is mean of each sample of size 'n' from the population with mean(μ) & std. deviaⁿ(σ) then $\bar{X}$ is normally distributed w/ mean (μ) & std. deviaⁿ ($\sigma/\sqrt{n}$) ~~and~~ ⇒ $\bar{X} \sim N(\mu, \sigma/\sqrt{n})$

and $z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ is SNV (std. normal variate)

with mean = 0 & std. deviaⁿ = 1.

Eg Total shaded area = 5%.



We know that for SNV, 95% area under the curve lies b/w ±1.96.

99% area lies between ±2.58.

99.73% area lies b/w ±3

In other representaⁿ; $P(|z| > 1.96) = 0.05$ → Beyond ±1.96.

$P(|z| < 1.96) = 0.95$ → B/w ±1.96.

Beyond ← $P(|z| > 2.58) = 0.01$

B/w ← $P(|z| < 2.58) = 0.99$

$P(|z| > 3) = 0.0027$ → Beyond ±3

$P(|z| < 3) = 0.9973$. → B/w ±3

* **Levels of significance: & Critical Region:**

Acceptance Region

Critical Region

Levels marked by the Probabilities 0.05 or 0.01 which decide the significance of the event are called level of significance (5%, 1%), and the corresponding region is called critical region.

~~The limits within~~

* **Confidence Limits & Confidence level:**

The limits within which we expect z to lie, with specified probabilities (~~0.95~~, 0.99) are called confidence limits & Region b/w confidence limits is called confidence level (95%, 99%)

* **Interval Estimation:**

We know that at 5% LoS $|z| < 1.96$.

$$\therefore \left|\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right| < 1.96$$

$$= |\bar{x}-\mu| < 1.96\frac{\sigma}{\sqrt{n}}$$

$$= |\mu-\bar{x}| < 1.96\frac{\sigma}{\sqrt{n}}$$

$$\Rightarrow \mu \in \left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

Sample std. devia^n $\Rightarrow s$
Popula^n std. devia^n $\Rightarrow \sigma$

s

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Q) ~~Samp~~ ~~Me~~ Measurements of weights of samples of 200 ball bearings has a mean$(\bar{x})$ = 0.824 N, and Std. devia^n(s) of 0.042 N.
Find 95% confidence limits for the mean weight of all ball bearings.

→ $n = 200$
$\bar{x} = 0.824$ N.
$s = 0.042$ N.

$$\hat{\sigma}_{\bar{x}} = \frac{s}{\sqrt{n}} = \frac{0.042}{\sqrt{200}} = 0.003$$

At 5% LOS,

$$\therefore \mu \in (\bar{x} - 1.96\,\hat{\sigma}_{\bar{x}},\ \bar{x} + 1.96\,\hat{\sigma}_{\bar{x}})$$

$$\mu \in (0.824 - 1.96(0.003),\ 0.824 + 1.96(0.003))$$

$$\underline{\mu \in (0.8182,\ 0.830).}$$

Q) A sample of size 65 was drawn in the process of estimating the mean annual income of 950 families. The mean & std. devia^n of sample were found to be ₹4730 & ₹765 respectively.
Find 98% confidence interval for the popula^n mean.

→ $n = 65$, $N = 950$
$\bar{x} = 4730$, $s = 765$.

$$\frac{n}{N} = \frac{65}{950} = 0.068$$

$$\sqrt{\frac{N-n}{N-1}} = \text{Finite population multiplier}$$

$$= \sqrt{\frac{950-65}{950-1}} = \sqrt{\frac{885}{949}} = 0.965.$$

$$\therefore \hat{\sigma}_{\bar{x}} = \frac{s}{\sqrt{n}} \times \sqrt{\frac{N-n}{N-1}} = \frac{765}{\sqrt{65}} \times 0.965 = 91.686.$$

↳ Multiply when popula^n is finite.

At 2% LOS.

$$\mu \in (\bar{x} - 2.33\, \hat{\sigma}_{\bar{x}},\ \bar{x} + 2.33\, \hat{\sigma}_{\bar{x}})$$

$$\mu \in (4730 - 2.33(91.686),\ 4730 + 2.33(91.686))$$
$$\mu \in (4516.37,\ 4943.62)$$

---

* <u>Testing of Hypothesis:</u>
In the process of ~~te~~sting the Hypothesis, we commit errors which are as follows:

i) <u>Type-1 Error:</u>
If the true Hypo. is rejected, then its called Type-1.

ii) <u>Type-2 Error:</u>
If false Hypo. is accepted, its called Type-2 Error.

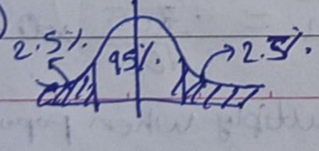
Q) A Random sample of 50 items, gives the mean $\bar{x} = 6.2$ & variance (s^2) 10.24. Can it be regarded as drawn from a normal population ~~between~~ with mean 5.4 at 5% LoS.

→ $n = 50,\ \bar{x} = 6.2,\ s^2 = 10.24,\ \mu = 5.4$
$s = 3.2$

Null Hypothesis = $H_0 : \mu = 5.4$
Alternate Hypo. = $H_a : \mu \neq 5.4$

$$|z| = \left| \frac{\bar{x} - \mu}{s/\sqrt{n}} \right| = \left| \frac{6.2 - 5.4}{3.2/7.07} \right| = \frac{0.8}{0.452} = 1.769 \simeq 1.77$$

(Two tailed Test)



when comparing b/w 2 products, go w/ one tailed test. → • Left one tailed, Right one tailed

$Z_{cal} \Rightarrow -ve$ | $Z_{cal} = -ve$
$Z_\alpha \Rightarrow -ve$ | $Z_\alpha = +ve$.

At 5% LoS.

$Z_\alpha = 1.96$.

$\therefore Z_{calculated} < Z_\alpha$.

$\therefore$ We accept H_0

which means sample is drawn from the normal population w/ mean 5.4.

Q) A tyre company claims that the lives of tyres have mean 42,000 kms w/ s.d. 4,000 km. The change in the produc^n process is believed to result in better product. A test sample of 81 new tyres has a ~~test~~ mean life of 42,500 kms. Test at 5% LoS that the new product is significantly better than the old one.

→ $n = 81$, $\mu = 42000$ (population), $\sigma = 4000$, $\bar{x} = 42500$ (sample)

$H_0 : \mu = 42000$ (Despite change in process, there is no change)

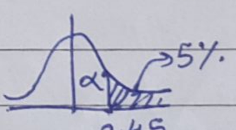
$H_a : \mu > 42000$ (There is significant change in product)

$$Z = \left| \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} \right| = \left| \frac{42500 - 42000}{4000/\sqrt{81}} \right|$$

$$= \left| \frac{500}{444.44} \right| = 1.125.$$

At 5% LoS.

$Z_\alpha = 1.65$



$\therefore Z_{cal} < Z_\alpha$.

$\therefore$ We accept H_0 ⇒ There is no significant change in the product.

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Distribution of Difference b/w means.

* If $\bar{x}_1$ & $\bar{x}_2$ are means of samples of sizes n_1 & n_2 populations with population means μ_1 & μ_2 and populaⁿ std. deviation σ_1 & σ_2, then,

$\bar{x}_1 - \bar{x}_2$ follows normal distribution w/ mean $\mu_1 - \mu_2$ and std. deviation $\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

$\therefore$ SNV is $Z = \dfrac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$

↳ S.E. (standard error)

Case-1: If $\mu_1 = \mu_2$ & $\sigma_1 = \sigma_2 = \sigma$ (known)

then $Z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$, $SE = \sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$

Case-2: If $\sigma_1 \neq \sigma_2$ (unknown)

& $\sigma_1^2 = S_1^2$ S = unbiased estimate of

$\sigma_2^2 = S_2^2$ population of std-deviaⁿ.

then, $S.E. = \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$

Case-3: If $\sigma_1 = \sigma_2 = \sigma$ (unknown)

then $S.E. = \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$

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Q) Two samples are drawn from 2 different population

	Size	Mean	S.D.
I	400 (n_1)	124 ($\overline{x_1}$)	14 (S_1)
II	250 (n_2)	120 ($\overline{x_2}$)	12 (S_2)

Find 95% confidence limits for difference b/w means

→ $\overline{x_1} - \overline{x_2} = 4$

$$S.E. = \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$

$$= \sqrt{\frac{(14)^2}{400} + \frac{(12)^2}{250}}$$

$$= \sqrt{\frac{196}{400} + \frac{144}{250}} = 1.03$$

$$\mu_1 - \mu_2 \in \left((\overline{x_1} - \overline{x_2}) - 1.96 \times S.E., (\overline{x_1} - \overline{x_2}) + 1.96 \times S.E\right)$$

$$\mu_1 - \mu_2 \in (1.977, 6.022)$$

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Q) Means of samples of 2 sizes 1000 & 2000 resp. are 67.50 & 68 inches. Can the samples be regarded as drawn from same popula^n of std. devia^n 2.5 inches ~~at 5% LoS~~

→ $\overline{x}_1 = 67.50$, $\overline{x}_2 = 68$, ~~$\sigma = 2.5$~~ s.d. = 2.5

Let $H_0 : \mu_1 = \mu_2$
$H_a : \mu_1 \neq \mu_2$

Now $|z|_{cal} = \left| \dfrac{\overline{x}_1 - \overline{x}_2}{\text{Std. Error}} \right|$

$$S.E. = 2.5\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} = 2.5\sqrt{\frac{2+1}{2000}}$$
$$= 0.096.$$

$\therefore |z|_{cal} = 5.16$ → Now here at 5% & 1% LOS, it will still be not in the region.

$\therefore$ At 0.27% LoS
$Z_\alpha = 3$.
$\because Z_{cal} > Z_\alpha$.
$\therefore$ We reject H_0 & Accept H_a.
$\therefore$ Samples are not drawn from same population.

Q) The avg. marks ~~so~~ scored by 32 boys is 72 w/ s.d. 8. While that of 36 girls is 70 w/ sd. 6. Test at 1% LoS whether the boys performed better than girls.

→ $n_1 = 32$, $\overline{x}_1 = 72$, $s_1 = 8$
$n_2 = 36$, $\overline{x}_2 = 70$, $s_2 = 6$

$H_0 : \mu_1 = \mu_2$
$H_a : \mu_1 > \mu_2$ (Boys perform better)
↳ If we take $\mu_1 \neq \mu_2$, we need ~~to~~ to take 2 tailed test.

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$$S.E. = \sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}} = 1.732$$

$$|z|_{cal} = \left|\frac{\bar{x}_1-\bar{x}_2}{S.E.}\right| = 1.15.$$

$\therefore$ At 1% LOS,

$Z_\alpha = 2.33.$

$\because Z_{cal} < Z_\alpha.$

$\therefore$ Accept H_0 $\Rightarrow$ Boys don't perform better than girls.

Q) 2 Samples drawn from different populations gives the following result.

	Size	Mean	S.D
Sample I	125 (n_1)	340 ($\bar{x}_1$)	25 (s_1)
Sample II	150 (n_2)	380 ($\bar{x}_2$)	30 (s_2)

Test hypothesis at 5% LOS difference of mean of 2 popula^n is 35.

$\rightarrow$ $H_0: \mu_1-\mu_2 = 35$

$H_a: \mu_1-\mu_2 \neq 35$

$$S.E. = \sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}} = \sqrt{\frac{(25)^2}{125}+\frac{(30)^2}{150}}$$

$$\therefore Z = \frac{|\bar{x}_1-\bar{x}_2|-(\mu_1-\mu_2)}{S.E.} = 1.5.$$

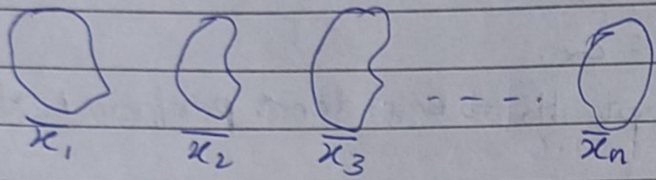
At 5% LOS $Z_\alpha = 1.96$

$\therefore Z_{cal} < Z_\alpha.$

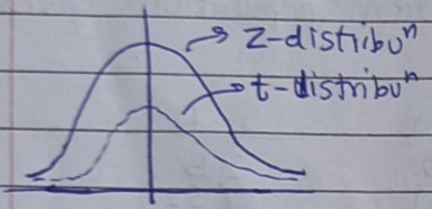
H_0 accepted.

* Small Sampling

If we take large no. of sample of small size ($n \leq 30$) & draw the frequency curve of mean of each sample then we see that this curve will not follow normal Distribution, it will follow Student's t-distribution



$\bar{x} = \bar{x}_1 \; \bar{x}_2 \; \bar{x}_3 \; \bar{x}_4 \dots \bar{x}_n$



$\bar{x} \sim N(\mu, \sigma/\sqrt{n})$

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

- **Properties:**

① Student t-distribuⁿ is symmetrical w/ mean 0.

② Variance of t-distribuⁿ is greater than 1. & it will tend to 1 as no. of degree of freedom, ∴ size of sample becomes large.

- **Degree of freedom:**

The no. of values which are free to choose.

* **Formula:** ($n < 30$) → Then we don't follow t-distribution, we follow z-dist.

① If σ = known (parent populaⁿ), $Z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$

② If σ = unknown, ⓐ $t = \dfrac{\bar{x} - \mu}{S/\sqrt{n}}$, $S = \sqrt{\dfrac{\Sigma(x_i - \bar{x})^2}{n-1}}$

ⓑ $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n-1}}$; $s = \sqrt{\dfrac{\Sigma(x_i - \bar{x})^2}{n}}$

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* Interval Estimation

Q) A sample of size n $=10$ has mean 40 & s.d. 10. Construct 99% confidence interval for population mean.

→ $n=10$, $s=10$, $\bar{x}=40$.

$SE = \frac{s}{\sqrt{n-1}} = \frac{10}{\sqrt{10}} = 3.33$

At 99% confidence level & $(n-1) = 10-1 = 9$, d.o.f.

$t_\alpha = 3.250$.

$\therefore \mu \in (\bar{x} - 3.25\, SE, \bar{x} + 3.25\, SE)$

$\mu \in (29.16, 50.83)$

* Testing the Hypothesis:

Sample size	σ-known	σ-unknown
$n \geq 30$	z test	z test
$n < 30$	z test	t-test

Use one-tailed

Q) A soap manufacturing company distribute soaps in retail store before advertisement, the mean sale per week per shop was 140 dozen. After the advertisement, a sample of 26 shops was taken & mean sale was 147 dozens w/ s.d = 16. Can you consider ~~a~~ advertisement effective?

→ Given: $\mu = 140$, $n = 26$
$\bar{x} = 147$, $s = 16$.

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$H_0: \mu = 140$ (Ad is not effective)
$H_a: \mu > 140$

$$t_{cal} = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{147-140}{16/\sqrt{26-1}} = \frac{7\sqrt{25}}{16}$$
$$= 2.190$$

A 5% LoS & n-1 = 25 dof.
↳ one tailed, so check for 0.10 in table.
$t_\alpha = 1.708$.
$\therefore t_{cal} > t_\alpha$.
$\therefore H_a$ is accepted ⇒ Ad was effective.

Q) 9 items of sampling have following values. Does the mean of these 9 items differ significantly from assumed population mean 47.5.

→

x	$(x_i-\bar{x})^2$
45	16.81
47	4.41
50	0.81
52	8.41
48	1.21
47	4.41
49	0.01
53	15.21
51	3.61
$\Sigma x = 442$	$\Sigma(x_i-\bar{x})^2 = 54.889$

↳ $\bar{x} = \frac{442}{9} = 49.1$

$\therefore s = \sqrt{\frac{\Sigma(x_i-\bar{x})^2}{n}} = 2.472$

$\frac{s}{\sqrt{n-1}} = 0.87$

$H_0: \mu = 47.5$ (Doesn't differ)
$H_a: \mu \neq 47.5$

$$|t_{cal}| = \left|\frac{\bar{x}-\mu}{s/\sqrt{n-1}}\right| = \frac{49.1-47.5}{0.87}$$
$$= 1.83$$

At 5% LoS & 8 dof.
$t_\alpha = 2.306$.
$\therefore t_{cal} < t_\alpha \Rightarrow H_0$ is accepted.
$\therefore$ It doesn't differ significantly.

4

* ## Linear Programming Problem (LPP)

Simplex method:

Consider $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ —— (1)
(objective funcⁿ)

subject to, $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \; (\geq b_1)$
$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \; (\geq b_2)$
$\vdots$
$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \; (\geq b_m)$ (2)
(constraints)

& $x_1, x_2, x_3, \ldots x_n \geq 0$ —— (3)
(Non-negativity condition)

Definitions:
(1)

Large ? 1 populaⁿ
& small } 2 populaⁿ

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★ Testing the difference b/w the means.

Case-1: Independent Samples

If The sample size n_1+n_2-2 is small
$\Rightarrow (n_1+n_2-2) < 30$.

then the unbiased estimate of common populaⁿ std. deviaⁿ is given by,

$$S_P = \sqrt{\frac{\Sigma(x_{i1}-\bar{x}_1)^2 + \Sigma(x_{i2}-\bar{x}_2)^2}{n_1+n_2-2}}$$

$\underbrace{\bar{x}_1, \bar{x}_2}_{\text{means}}$
$\underbrace{x_{i1} \& x_{i2}}_{\text{Readings}}$

If the unbiased estimates of 2 populaⁿ std. deviaⁿ are given by,

$$S_1 = \sqrt{\frac{\Sigma(x_{i1}-\bar{x}_1)^2}{n_1-1}}, \quad S_2 = \sqrt{\frac{\Sigma(x_{i2}-\bar{x}_2)^2}{n_2-1}}$$

then,
$$S_P = \sqrt{\frac{S_1^2(n_1-1)+S_2^2(n_2-1)}{n_1+n_2-2}}$$

If std. deviations of samples are given by,

$$s_1 = \sqrt{\frac{\Sigma(x_{i1}-\bar{x}_1)^2}{n_1}}, \quad s_2 = \sqrt{\frac{\Sigma(x_{i2}-\bar{x}_2)^2}{n_2}}$$

then, $S_P = \sqrt{\dfrac{s_1^2 n_1 + s_2^2 n_2}{n_1+n_2-2}}$

$\therefore$ Std. Error, $S.E. = S_P\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}$

and $t_{cal} = \dfrac{\bar{x}_1-\bar{x}_2}{S.E.}$, under the condⁿ $\mu_1 = \mu_2$

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Q) If 2 independent random samples of sizes, 15 & 8 have means & popula^n std devia^n. respectively, $\overline{x_1} = 980$, $\overline{x_2} = 1012$, $\sigma_1 = 75$, $\sigma_2 = 80$. Test $\mu_1 = \mu_2$ at 5% LoS

→ $n_1 = 15$, $n_2 = 8$

$n_1 + n_2 - 2 = 15 + 8 - 2 = 21 < 30$ ∴ small sample.

$H_0 : \mu_1 = \mu_2$.

$S.E. = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = 34.27$ $\quad H_a : \mu_1 \neq \mu_2$.

$|Z_{cal}| = \left| \frac{\overline{x_1} - \overline{x_2}}{SE} \right| = 0.93$

At 5% LoS $Z_\alpha = 1.96$.

∴ $Z_{cal} < Z_\alpha \Rightarrow \mu_1 = \mu_2$.

∴ we accept H_0.

Q) Sample of 8 students of 16 years each shown up systolic Blood pressure of 118.4 mm of Hg with std. devia^n 12.17 mm of Hg. While a sample of 10 students of 17 yrs, shows mean systolic B.P of 121 mm of Hg w/ std. devia^n 12.88mm of Hg. The investigator feels that, BP is related to age. Do you think that the data provides enough reasons to support investigator's feelings at 5% LoS.

→ $n_1 = 8$, $n_2 = 10$

$\overline{x_1} = 118.4$, $\overline{x_2} = 121$

$s_1 = 12.17$, $s_2 = 12.88$

$H_0 : \mu_1 = \mu_2 \Rightarrow$ BP is not related to age.

$H_a : \mu_1 \neq \mu_2$

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$$S_p = \sqrt{\frac{s_1^2 n_1 + s_2^2 n_2}{n_1+n_2-2}}$$

$$= \sqrt{\frac{148.108(8) + 165.89(10)}{16}} = 13.33$$

$$S.E. = S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} = 6.32.$$

$$|t_{cal}| = \frac{\bar{x}_1 - \bar{x}_2}{S.E.} = 0.41.$$

At 5% LoS & $n_1+n_2-2 = 16$ dof.

$\therefore t_\alpha = 2.12$

$\therefore t_{cal} < t_\alpha$.

we accept Ho.

$\therefore$ The BP doesn't depend upon age and the data provided doesn't support the investigators feelings.

Q) Means of 2 random samples of sizes 9 & 7 are 196.42 & 198.82 resp. Sum of the squares of deviation from the mean are 26.94 & 18.73 resp. Can the samples be considered to be drawn from the same population

→ $n_1 = 9, \quad n_2 = 7$

$\bar{x}_1 = 196.42, \quad \bar{x}_2 = 198.82.$

$\Sigma(x_{i1} - \bar{x}_1)^2 = 26.94, \quad \Sigma(x_{i2} - \bar{x}_2)^2 = 18.73.$

$H_0 : \mu_1 = \mu_2$ (samples ~~belong~~ drawn from same popul^n)

$H_a : \mu_1 \neq \mu_2$

$$S_p = \sqrt{\frac{\Sigma(\ )^2 + \Sigma(\ )^2}{n_1+n_2-2}} = 1.81$$

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$$S.E. = S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} = 0.91$$

$$t_{cal} = \frac{\bar{x}_1 - \bar{x}_2}{S.E} = -2.64 \Rightarrow |t| = 2.64.$$

~~At 5~~

At 5% LoS & $n_1+n_2-2 = 14$ dof.

$t_\alpha = 2.145$

$\therefore t_{cal} > t_\alpha \Rightarrow$ We Reject H_0.

$\therefore$ ~~At~~ Can't be drawn from same populaⁿ.

Q) 6 Pigs injected w/ 0.5 mg of medication took on a avg. 15.4 sec to fall asleep. w/ a unbiased std. deviaⁿ, 2.2 sec. While 6 other pigs injected w/ 1.5 mg of medicaⁿ took on an avg. 11.2 sec to fall asleep w/ an unbiased std. deviaⁿ 2.6 sec. Use 5% LOS to test that difference in the doses have no effect.

→ $n_1 = 6,\ n_2 = 6$

$\bar{x}_1 = 15.4,\ \bar{x}_2 = 11.2$

$S_1 = 2.2,\ S_2 = 2.6.$

$H_0: \mu_1 = \mu_2$ (No effect).

$H_a: \mu_1 \neq \mu_2.$

$$S_p = \sqrt{\frac{S_1^2(n_1-1)+S_2^2(n_2-1)}{n_1+n_2-2}} = 2.408$$

$$SE = S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} = 1.3911$$

$$|t_{cal}| = \left|\frac{\bar{x}_1-\bar{x}_2}{SE}\right| = 3.02$$

5% LoS & $n_1+n_2-2 = 10$ dof,

$t_\alpha = 2.28$

$\because t_{cal} > t_\alpha$, H_a is accepted.

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Q) The heights of 6 sailors in ~~the~~ inches are : x_1, while heights of 10 soldiers are : x_2. Discuss the soldiers on an avg are taller than sailors.

x_1	$(x_1-\overline{x_1})^2$	x_2	$(x_2-\overline{x_2})^2$
63	25	61	46.24
65	9	62	33.64
68	0	65	7.84
69	1	66	3.24
71	9	69	1.44
72	16	69	1.44
		70	4.84
		71	10.24
		72	17.64
		73	27.04
$\Sigma x_1 = 408$	$\Sigma = 60$	$\Sigma x_2 = 678$	$\Sigma = 153.6$

$\overline{x_1} = 68$, $\overline{x_2} = 67.8$.

$H_0 : \mu_1 = \mu_2$ (Not taller)

$H_a : \mu_2 > \mu_1$ ~~(Sailor~~ (taller).

$$S_p = \sqrt{\frac{\Sigma(x_1-\overline{x_1})^2 + \Sigma(x_2-\overline{x_2})^2}{n_1+n_2-2}} = 3.9$$

$$S.E. = S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}} = 3.9\sqrt{\frac{1}{6}+\frac{1}{10}} = 3.9\sqrt{\frac{5+3}{30}}$$

$$= 3.9\sqrt{\frac{8}{30}}$$

$$= 2.013.$$

$t_{cal} = \dfrac{\overline{x_1}-\overline{x_2}}{SE} = 0.099$, → one tailed → 10% on table

$t_\alpha = 1.761$ at 14 dof & 5% LoS

$\therefore t_\alpha > t_{cal}$.

$\therefore \mu_1 = \mu_2$

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* Case-2 - Dependendent Samples

Q) Certain injection are administrated to certain patients resulted in the following changes in B.P. Can it be concluded that the injecⁿ will be in general effective in increase in B.P.

x	$(x-\bar{x})^2$
5	5.85
2	0.336
8	29.37
-1	12.816
3	0.176
0	6.656
6	11.69
-2	20.97
1	2.496
5	5.85
0	6.656
4	2.016
$\Sigma x = 31$	104.9
$\bar{x} = 2.58$	

$H_0 : \mu = 0$

$H_a : \mu \neq 0$

$$s = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n}}$$

$$= \sqrt{\frac{104.9}{12}} = \sqrt{8.74} = 2.95$$

$$|t_{cal}| = \frac{\bar{x} - \mu}{s/\sqrt{n-1}}$$

$$= \frac{2.58}{2.95/\sqrt{11}} = \frac{2.58 \times \sqrt{11}}{2.95}$$

$$= 2.9$$

At 5% LOS & 11 dof

$t_\alpha = 2.201$

$t_{cal} > t_\alpha$

$\therefore H_a$ is accepted.

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9) 10 school boys given a test in statistics & scores were recorded. They were given a month's coaching & 2nd test was recorded. Test if the marks given below give evidence to the fact that students were benefitted by coaching.

T_1 :	70	68	56	75	80	90	68	75	56	58
T_2 :	68	70	52	73	75	78	80	92	54	55

$(T_2 - T_1)$	$(x-\bar{x})^2$
−2	4.41
2	3.61
−4	16.81
−2	4.41
−5	26.01
−12	146.41
12	141.61
17	285.61
−2	4.41
−3	9.61
$\Sigma = 1$	$\Sigma = 642.9$

$\bar{x} = \frac{1}{10} = 0.1$

$H_0 : \mu = 0$

$H_a : \mu \neq 0$

$s = \sqrt{\frac{\Sigma (x_i - \bar{x})^2}{n}} = 8.01$

$t_{cal} = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{0.1 - 0}{8.01/\sqrt{3}}$

$= 0.037.$

At 5% LOS & 9 dof,

$t_\alpha = 2.262.$

$\therefore t_{cal} < t_\alpha.$

$\therefore H_0$ is accepted.

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* χ^2 distribution (chi-square Distribution)

It is used to study non-parametric tests,
Eg: Whether a die is biased or unbiased.
In χ^2 we assume that there is no change i.e. not bias.

$$\chi^2 = \Sigma\left[\frac{(O-E)^2}{E}\right]$$, O : Observed frequency.
E : Expected frequency.

- **Uses:**
1) To test the independence of attributes.
2) To test goodness of fit.
3) To test discrepencies b/w O & E
4) To test equality of several proportions.
5) To test hypothesis about σ^2.

- **Conditions for χ^2 test:**
1) $N > 50$ (No. of observa^n)
2) Frequency of every cell > 5
3) No. of classes n must not be too small or too large
$4 \le n \le 16$.

- **Yate's Correction:** $$\chi^2 = \Sigma\left[\frac{\{|O-E| - 0.5\}^2}{E}\right]$$
↳ only for 2×2

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I) Test Independence of Attributes

Q) Two batches of 12 animals each are given test of inoculation. One batch was inoculated & the other was not. The no. of dead & survived animals are given in the table. Can the inoculation be regarded as effective against the disease at 5% LOS?

	Dead	Surviving	Total
Inoculated	2 (5)	10 (7)	12
Not-Inocul.	8 (5)	4 (7)	12
Total	10	14	24

Ho: There is no associaⁿ b/w inoculaⁿ & death.
Ha: There is association.

O	E	$\lvert O-E\rvert-0.5$	$\{\lvert O-E\rvert-0.5\}^2/E$
2	$\frac{10\times12}{24}=5$	2.5	1.25
10	5	2.5	1.25
8	7	2.5	0.84
4	7	2.5	0.84

$\chi^2_{cal} = 4.92$

$DOF = (r-1)(c-1) = (2-1)(2-1) = 1.$

$\chi^2_{table} = 3.81$

$\chi^2_{cal} > \chi^2_{table}$.

∴ Ha is accepted.

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Q) The following table given the result of opinion pole for the parties A, B, C. Test whether the age and choice of parties are independant.

Age	A	B	C	Total
		parties		
20 - 35	25	20	25	70
35 - 50	20	25	35	80
Above 50	25	25	30	80
Total	70	70	90	230

O	E	$\lvert O-E \rvert$	$(O-E)^2$	$(O-E)^2/E$
25	21	4	16	0.76
20	21	1	1	0.04
25	28	3	9	0.32
20	24	4	16	0.66
25	24	1	1	0.041
35	32	3	9	0.28
25	24	1	1	0.04
25	24	1	1	0.04
30	32	2	4	0.125
				$\chi^2 = 2.305$

$DOF = (3-1)(3-1) = 4$

$\chi^2_{table} = 9.488$ at 5% LoS

$\chi^2_{table} < \chi^2_{cal}$

$\therefore$ H_0 accepted.

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II) Goodness of Fit

Q) The following table gives the no. of accidents in a city during week. Find whethere the accidents are uniformly distributed over a week.

Day:	Sun	Mon	Tue	Wed	Thur	Fri	Sat	Total
Freq:	13	15	9	11	12	10	14	84

Ho: Accidents are equally distributed occur all days.

Ha: Accidents do not occur equally.

$$x^2 = \Sigma \frac{(O-E)^2}{E} = \frac{(13-12)^2}{12} + \frac{(15-12)^2}{12} + \frac{(9-12)^2}{12} + \frac{(11-12)^2}{12} + \frac{(12-12)^2}{12} + \frac{(10-12)^2}{12} + \frac{(14-12)^2}{12}$$

$$= \frac{1}{12}[1+9+9+1+0+4+4] = \frac{28}{12} = 2.33$$

At $\alpha = 5\%$ LoS, dof $= n-1 = 7-1 = 6$.

$X^2_{table} = 12.59 > X^2_{cal}$

$\therefore$ Ho accepted.

Accident occurs equally on all work days.

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Q) Theory predicts that proportion of beans in 4 group A B C D ⇒ 9 : 3 : 3 : 1, in an experiment among 1600 beans. The no. in groups is 882, 313, 287, 118. Does the experiment result support the theory?

→ H_0: The propor^n of beans in 4 groups is 9 : 3 : 3 : 1
H_a: Propor^n is not 9:3:3:1.

Sum of proportions ⇒ 9+3+3+1 = 16.

No. of beans in group A will be : $\frac{9 \times 1600}{16} = 900$

" " " " " B " " : $\frac{3 \times 1600}{16} = 300$

" " " " " C " " : $\frac{3 \times 1600}{16} = 300$

" " " " " D " " : $\frac{1 \times 1600}{16} = 100$

$$\chi^2 = \frac{\Sigma(O-E)^2}{E} = \frac{(882-900)^2}{900} + \frac{(313-300)^2}{300} + \frac{(287-300)^2}{300} + \frac{(118-100)^2}{100}$$
$$= 4.72.$$

Dof = 4 − 1 = 3 , LOS = 5%.

$\chi^2 = 7.81$

$\chi^2_{cal} < \chi^2_{table}$.

∴ Accept H_0.

Class freq. < 5, then combine w/ ~~near~~ next one

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Q)

O	E	$(O-E)^2/E_i$
$13\begin{cases}1\\12\end{cases}$	$17\begin{cases}2\\15\end{cases}$	$\frac{(13-17)^2}{17} = 0.94$
66	66	0
220	210	0.47
495	484	0.25
792	799	0.06
924	943	0.38
792	799	
495	484	
220	210	
66	66	
$13\begin{cases}12\\1\end{cases}$	$17\begin{cases}15\\2\end{cases}$	

$$\chi^2 = \sum \frac{(O-E)^2}{E} = \underline{3.84}$$

Apply χ^2 test

H_0: Fit is good.
H_a: Fit is not good.

Originally there were 13 classes, since they are reduced by 2, by grouping twice, so dof. gets reduced by 2. ~~Also~~
Also 3 constraints are introduced like mean std. devia^n & total freq. of original data,
$\therefore$ dof further gets reduced by 3.
$\therefore$ dof $\Rightarrow 13-2-3 = \underline{\underline{8}}$.

At 5% LoS & 8 dof,
$\chi^2_{table} = 15.51$
$\therefore \chi^2_{cal} < \chi^2_{table} \Rightarrow H_0$ Accepted.

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Q) The following mistakes per page were observed in a book

No. of mistakes per page:	0	1	2	3	4
No. of Pages:	211	90	19	5	0

Fit a poisson distribuⁿ & test the goodness of fit.

→ H_0: Fit is good.
H_a: Fit is not good.

$$\text{mean} = \frac{\sum f_i x_i}{\sum f_i} = \cancel{\frac{143}{496}} = \frac{143}{325} = 0.44 = m.$$

$$P(X=x) = \frac{e^{-m} m^x}{x!} = \frac{e^{-0.44}(0.44)^x}{x!}$$

$P(X=0) = 0.644 \Rightarrow 209$
$P(X=1) = 0.283 \Rightarrow 92$
$P(X=2) = \cancel{0.124}\ 0.062 \Rightarrow 20$
$P(X=3) = \cancel{0.054}\ 0.009 \Rightarrow 3$
$P(X=4) = \cancel{0.024}\ 0.001 \Rightarrow 1$

x	O	E	$(O-E)^2/E$
0	211	209	0.019
1	90	92	0.043
2	19 (24)	20 (24)	~~0~~
3	5 (5)	3 (4)	0
4	0	1	~~0~~

$\chi^2 = 0.062$

~~Note~~
$\sum f_i x_i$ & $\sum f_i$ will be same in both.
$\therefore$ dof $= 5-2-2 = 1$.
$\therefore$ At 5% LOS & 1 dof
$\chi^2_{table} = 3.84$.
$\chi^2_{cal} < \chi^2_{table}$.
$\therefore$ Accept H_0.

Hypothesis concerning several proportions, ~~sample of 3 st~~

1) Sample of 3 shipments A, B, C of defective items gave the following results:

	Ship-A	Ship B	Ship-C	Total
Defective	5	8	9	22
Non-def.	35	42	51	128
Total	40	50	60	150

Test whether the proportion of defective items is same in the 3 shipments at 5% LoS.

→ H_0: Proportion is same. $(P_1 = P_2 = P_3)$
H_a: Proportion is not same $(P_1 \neq P_2 \neq P_3)$

O	E	$(O-E)^2/E$
5	6	0.167
8	7	0.142
9	9	0
35	34	~~χ^2 = 0.309~~ 0.029
42	43	0.026 ~~At 5% LoS & dof~~
51	51	0

$\Sigma = 0.36$

MODULE-4: OPTIMIZATION TECHNIQUES

★ **Linear Programming problem (LPP)**

Simplex method:

Consider $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ — ① (objective func^n).

Subject to,
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \ (\geq b_1)$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \ (\geq b_2)$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \ (\geq b_m)$$
② (constraints)

& $x_1, x_2, x_3, \ldots x_n \geq 0$ — ③ (Non-negativity condition)

Definitions

Any set of values $x_1, x_2, \ldots x_n$, which satisfies constraints given in ②, is called solution of LPP.

Any sol^n which satisfies the ~~constraints~~ non-negativity restrictions given in ③, is called feasible solution.

Any feasible sol^n which maximises or minimises the objective func^n in ①, is called optimal ~~feasible~~ sol^n

If the constraints of LPP are $\leq$ type, ~~then~~
i.e., $\sum_{j=1}^{n} a_{ij}x_j \leq b_i \ (i = 1, 2, \ldots k)$

then we add non-negative variables S_i (slack variables) so that $\sum_{j=1}^{n} a_{ij}x_j + S_i = b_i$

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If the constraints of LPP are $\geq$ type, then we subtract the variables S_i (surplus variables)

$$\sum_{j=1}^{n} a_{ij}x_j - S_i = b_i$$

* Canonical form of LPP:

The form maximise $Z = \sum_{i=1}^{n} c_i x_i$

Subject to, $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$, $i = 1, 2, \dots m$

, $x_i \geq 0$.

• Characteristics of Canonical form

1) If the objective func^n is of minimisation type, then we must have $\geq$ type inequality and ~~a~~ if it is of maximisation type, then we must have $\leq$ type inequality.

2) If the constraint is in the form of equation, then express it as an inequality.

Eg: $a_1x_1 + a_2x_2 = b \Rightarrow a_1x_1 + a_2x_2 \leq b$ &

$a_1x_1 + a_2x_2 \geq b$

3) We must have $x_i \geq 0$. (Restricted)

If any variable is unrestricted, say x_j, then write ~~x_j as~~ $x_j = x_j' - x_j''$

where x_j', x_j'' both are ≥ 0

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* **Standard form of LPP:**

In std. form, we introduce slack variables and express the objective functions as well as all constraints in the form of equalities.

i.e., Maximise $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n + 0S_1 + 0S_2 + \dots + 0S_m$

Subject to, $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + S_1 + 0S_2 + \dots + 0S_m = b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + 0S_1 + S_2 + \dots + 0S_m = b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + 0S_1 + 0S_2 + \dots + S_m = b_m$

constraints

& $x_1, x_2, x_3 \dots x_n ; S_1, S_2, \dots S_m \geq 0$

• **Characterstics of Std. form:**

1) All the constraints are expressed in the form of equations using slack variables

2) RHS of all constraints are non-negative.

3) Objective func^n should be of maximisation type

4) All decision variables $(x_1, x_2, \dots x_n)$ & slack variables $(S_1, S_2, \dots S_m)$ are non-negative

* **Definition:**

- **Basic Solution:** A sol^n obtained by setting any 'n' variables out of $(m+n)$ variables equal to zero and solving for remaing 'm' variables is called a Basic Solution.

Such 'm' variables are called Basic variables and remaining 'n' zero valued variables are called non-basic variables.

Total no. of Basic solutions $= {}^{(m+n)}C_m$

(BFS)
- **Basic Feasible Solution:** A basic sol^n which also satisfies non-negativity restriction is called BFS.

2 Types:

1) Non-Degenerate BFS: If in the ~~the~~ BFS, all 'm' values of the basic variables x_i are positive (>0) then its called NDBFS.

2) Degenerate BFS: If in the BFS, one or more values of 'm' basic variables are zero, then its called DBFS.

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Q) Convert the following LPP in std. form.

$$\text{Min } z = -3x_1 + 2x_2 - x_3$$

subject to, $x_1 - 3x_2 + 2x_3 \geq -6$

$3x_1 + 4x_3 \leq 3$

$-3x_1 + 5x_2 \leq 4$

$x_1, x_2 \geq 0$, x_3 is unrestricted.

→ x_3 is unrestricted.

Let $x_3 = x_3' - x_3''$

$$\text{Max } Z' = -Z = 3x_1 - 2x_2 + x_3 + 0S_1 + 0S_2 + 0S_3$$

Sub. to, $-x_1 + 3x_2 - 2(x_3' - x_3'') + S_1 + 0S_2 + 0S_3 = 6$

We add slack variable S_1 to convert $\leq$ to $=$.

$3x_1 + 4(x_3' - x_3'') + 0S_1 + S_2 + 0S_3 = 3$

$-3x_1 + 5x_2 + S_3 = 4$

→ $x_1, x_2, x_3', x_3'', S_1, S_2, S_3 \geq 0$.

Q) Find all basic solⁿ of the following systems, which of them are basic feasible, non-degenerate, infeasible basic & optimal basic feasible solⁿ.

$$\text{Max } Z = x_1 + 3x_2 + 3x_3$$

Sub. to, $x_1 + 2x_2 + 3x_3 = 4$.

$2x_1 + 3x_2 + 5x_3 = 7$

→ No. of variables = 3

No. of constraints = 2

$\therefore 3 - 2 = 1$ variable to be equated to 0.

No. of Basic soln	Non-Basic variable	Basic Variable	Eqn & value of Basic variable	Is the soln feasible	Is soln Degenerate	Value of z	Is soln optimal
1	$x_3=0$	x_1, x_2	$x_1+2x_2=4$ $2x_1+3x_2=7$ $\therefore x_1=2, x_2=1$	Yes	No	5	Yes
2	$x_2=0$	x_1, x_3	$x_1+3x_3=4$ $2x_1+5x_3=7$ $\therefore x_1=1, x_3=1$	Yes	No	4	No
3	$x_1=0$	x_2, x_3	$2x_2+3x_3=4$ $3x_2+5x_3=7$ $\therefore x_2=1, x_3=2$	No	No	3	No

~~No. of basic soln | Non-Basic variable | Basic Variab~~

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* Simplex Method:

Procedure:

S-1: Write the given LPP in std. form.

S-2: write initial basic feasible solution.

S-3: Make initial simplex table.

S-4: Are all the entries in z-row non-negative?
1) If Yes, then the current sol^n is optimal BFS.
2) If No, then select the most negative entry (smallest) in z-row.
The corresponding column is called key (pivot) column & corresponding variable enters in the basis.

S-5: Obtain the replacement ratio by dividing solution column by key column.

S-6: Are all the ratios infinite and/or negative?
1) If Yes, then LPP has unbounded sol^n.
2) If No, then select the minimum finite non-negative ratio. (If ratio has 0, then take that ratio whose denominator is +ve in pivot column)
The corresponding row is called key Row & corresponding variable leaves the basis.
If there is a tie, then take arbitrary minimum non-negative random number.

S-7: Mark the key element as intersection of key row & key column.

S-8: Make the new simplex table by elementary row transformation as:
1) Make key element 1 by dividing key row by key element.
2) Make all other elements of key column 0 by subtracting or adding proper multiples of new row to the old row.

S-9: Go to step-4 & continue, till you reach Step-4-1 or Step-6-1

Q)

Itera^n no.
0
x_1 enters
S_2 leaves
1
x_2 enters
S_1 Leaves
2

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Max $Z = 3x_1 + 2x_2$
sub to, $3x_1 + 2x_2 \leq 18$
$0 \leq x_1 \leq 4$
$0 \leq x_2 \leq 6$

Max $Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 + 0S_3$ — Std. form
S.T., $3x_1 + 2x_2 + S_1 + 0S_2 + 0S_3 = 18$
$x_1 + 0x_2 + 0S_1 + S_2 + 0S_3 = 4$
$0x_1 + x_2 + 0S_1 + 0S_2 + S_3 = 6$
$\to x_1, x_2, S_1, S_2, S_3 \geq 0$

Initial BFS is obtained by putting, $5-3 = 2$ variables to 0.
∴ Let $x_1 = x_2 = 0$ → non-basic variable
∴ $S_1 = 18, S_2 = 4, S_3 = 6$

Initial simplex table

This should always form an identity matrix

enters co-eff of

	Basic variable	x_1	x_2	S_1	S_2	S_3	RHS Sol^n	Ratio	
		key col.	→ S-4 (2)						
R_1	Z	-3	-2	0	0	0	0		
	S_1	3	2	1	0	0	18	$18/3 = 6$	
R_3	S_2	1*	0	0	1	0	4	$4/1 = 4$	→ leaves
R_4	S_3	0	1	0	0	1	6	$6/0 = \infty$	→ S-6 (2)
R_1'	Z	0	-2	0	3	0	12		R_1+3R_3
R_2'	S_1	0	2*	1	-3	0	6	$6/2 = 3$	R_2-3R_3 → leaves
R_3'	x_1	1	0	0	1	0	4	$4/0 = \infty$	
R_4'	S_3	0	1	0	0	1	6	$6/1 = 6$	
	Z	0	0	1	0	0	18		$R_1'+2R_2'$
R_2'	x_2	0	1	1/2	-3/2	0	3		$R_4'-R_2'$
	x_1	1	0	0	1	0	4		
	S_3	0	0	-1/2	3/2	1	3		

1st practice

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Since all the entries in z-row are non-negative, the current sol^n is optimal basic feasible sol^n

$\therefore$ We have; $x_1 = 4$, $x_2 = 3$, $\underbrace{S_1 = 0, S_2 = 0}_{\text{Non-basic variable.}}$, $S_3 = 3$.

$Z_{max} = 18$.

Since the value of non-basic variable S_2 in Z-row is 0, this LPP has an alternate solution.

It. No.	Basic variable	x_1	x_2	S_1	S_2	S_3	RHS Sol^n	Ratio	
2	Z	0	0	1	0	0	18		
S_2 enters	x_2	0	1	1/2	-3/2	0	3	-2	
S_3 leaves	x_1	1	0	0	1	0	4	4	
	S_3	0	0	-1/2	3/2	1	3	2	→ leaves
3	Z	0	0	1	0	0	18		
	x_2	0	1	0	0	1	6		~~Alternat sol^n~~
	x_1	1	0	1/3	0	-2/3	2		
	S_2	0	0	-1/3	1	2/3	2		

Alternate sol^n is: $x_1 = 2$, $x_2 = 6$, $S_1 = 0$, $S_2 = 2$, $S_3 = 0$.

$\because$ Value of NBV. S_3 in Z-row is 0, this LPP has alt. sol^n.

This LPP has infinite no. of sol^n.

To find ∞ no. of sol^n,

Let $X_1 = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 0 \\ 3 \end{pmatrix}$, $X_2 = \begin{pmatrix} 2 \\ 6 \\ 0 \\ 2 \\ 0 \end{pmatrix}$

$\therefore$ Infinite no. of non-basic optimal feasible sol^n is given by,

$$X = \lambda X_1 + (1-\lambda) X_2 \quad , \quad 0 < \lambda < 1$$

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$$x = \begin{pmatrix} 4\lambda \\ 3\lambda \\ 0 \\ 0 \\ 3\lambda \end{pmatrix} + \begin{pmatrix} 2-2\lambda \\ 6-6\lambda \\ 0 \\ 2-2\lambda \\ 0 \end{pmatrix} = \begin{pmatrix} 2+2\lambda \\ 6-3\lambda \\ 0 \\ 2-2\lambda \\ 3\lambda \end{pmatrix}$$

This is non-basic because, for basic sol^n we must have $(5-3=2)$, 2 zeroes, but we get only 1 zero in this sol^n.

Q) Solve the following LPP by simplex method

Min $Z = x_1 - 3x_2 + 3x_3$.

Sub. to, $3x_1 - x_2 + 2x_3 \leq 7$

$2x_1 + 4x_2 \geq -12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$.

$\Rightarrow$ Std. form is

Max $Z' = -Z = -x_1 + 3x_2 - 3x_3$

$\Rightarrow Z' + x_1 - 3x_2 + 3x_3 = 0$

Sub. to, $3x_1 - x_2 + 2x_3 + S_1 = 7$.

$-2x_1 - 4x_2 + S_2 = 12$

$-4x_1 + 3x_2 + 8x_3 + S_3 = 10$

$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$.

↓ coeff. of

It. no.	Basic sol^n	x_1	x_2	x_3	s_1	s_2	s_3	RHS sol^n	Ratio	
0	R_1 Z'	1	-3	3	0	0	0	0		
x_2 enters	R_2 S_1	3	-1	2	1	0	0	7	-7	
s_3 leaves	R_3 S_2	-2	-4	0	0	1	0	12	-3	
	S_3	-4	3*	8	0	0	1	10	10/3 →	
1	R_1' Z'	-3	0	11	0	0	1	10		R_1+3R_4
x_1 enters	R_2' S_1	*5/3	0	14/3	1	0	1/3	31/3	31/5 →	R_2+R_4
s_1 leaves	R_3' S_2	-22/3	0	32/3	0	1	4/3	76/3	-76/22	R_3+4R_4
	R_4 x_2 R_4'	-4/3	1	8/3	0	0	1/3	10/3	-10/4.	
Z	Z'	0	0	97/5	9/5	0	8/5	143/5		$R_1'+3R_2'$
	R_2' x_1	1	0	14/5	3/5	0	1/5	31/5		$R_3'+\frac{22}{3}R_2'$
	S_2	0	0	468/15	66/15	1	42/15	1062/15		$R_4'+\frac{4}{3}R_2'$
	x_2	0	1	96/15	12/15	0	9/15	174/15		

Since all ~~entries~~ entries in z-row is non-negative, The current sol^n optimal ~~for~~ basic feasible sol^n.

$\therefore Z'_{max} = 143/15$

$Z_{min} = -143/15.$

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Q) Minimise $z = x_1 + 2x_2 + x_3$

Sub. to, $x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \leq 1$

$\frac{3}{2}x_1 + 2x_2 + x_3 \geq 8$

$x_1, x_2, x_3 \geq 0.$

$\rightarrow$ Max $z' = -z = -x_1 - 2x_2 - x_3 - MA_2$

Sub. to, $x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 + S_1 = 1$ — (1).

$\frac{3}{2}x_1 + 2x_2 + x_3 - S_2 + A_2 = 8$ — (2)

Mult. (2) by M & add to z'

$\therefore$ Max $z' = (-1 + \frac{3}{2}M)x_1 + (-2 + 2M)x_2 + (-1 + M)x_3 - MS_2 - 8M.$

i.e, $z' + (1 - \frac{3}{2}M)x_1 + (2 - 2M)x_2 + (1 - M)x_3 + MS_2 + 8M = 0$

Sub. to, $x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 + S_1 = 1$

$\frac{3}{2}x_1 + 2x_2 + x_3 - S_2 + A_2 = 8$

$x_1, x_2, x_3, S_1, S_2, A_2 \geq 0$

<u>Initial BFS:</u>

$x_1 = 0 = x_2 = x_3 = S_2$ → We make the Slack var. that is W/ Artificial var. as zero.

$\therefore S_1 = 1, A_2 = 8$

$1-\frac{3}{2}M-(2-2M)2 \qquad -3+\frac{5}{2}M \qquad 1-M-\cancel{2}+4M.$

$1-\frac{3}{2}M-4+4M$

forms Identity matrix.

Coeff. No.

Iter. No	Basic var.	x_1	x_2	x_3	S_1	S_2	A_2	RHS Sol^n	Ratio	
0	R_1 Z'	$1-\frac{3}{2}M$	$2-2M$	$1-M$	0	M	0	$-8M$		
x_2 enters	S_1	1	$1/2$	$1/2$	1	0	0	1	2 →	
S_1 leaves	R_3 A_2	$3/2$	2	1	0	-1	1	8	4	
	Z'	$-3+\frac{5}{2}M$	0	$-1+M$	$-4+4M$	M	$\cancel{4-4M}$ 0	$-4-4M$		$R_1-(2-2M)R_2$
	R_2 x_2	2	1	1	2	0	0	2		R_3-2R_2
	A_2	$-5/2$	0	-1	-4	-1	1	4		

since all entries in z-row are non-negative, but A_2 appears w/ +ve value

(3) ∴ Given sol^n is pseudo optimal basic sol^n.

* **Duality:**

(1) Maximise $z = 2x_1 + x_2 + x_3$.

Sub. to. $x_1 + x_2 + x_3 \geq 6$

$3x_1 - 2x_2 + 3x_3 = 3$

$-4x_1 + x_3 \leq 10$

$x_1, x_3 \geq 0$, $x_2 \Rightarrow$ unrestricted.

→ Let $x_2 = x_2' - x_2''$

∴ we have,

canonical form

Max. $z = 2x_1 + \cancel{x_2 + x_3} (x_2' - x_2'') + x_3$

Sub. to, $-x_1 - (x_2' - x_2'') - x_3 \leq -6$

$3x_1 - 2(x_2' - x_2'') + 3x_3 \leq 3$

$-3x_1 + 2(x_2' - x_2'') - 3x_3 \leq -3$

$-4x_1 + x_3 \leq 10$

$x_1, x_2', x_2'', x_3 \geq 0$.

Let y_1, y_2', y_2'', y_3 be the duel ~~vat~~ variables

∴ Duel LPP is, | The RHS constants become the coeff of duel variables.

Min $W = -6y_1 + 3y_2' - 3y_2'' + 10y_3$ (coeff)

~~Sub. to,~~ (V) Here we take transpose of body matrix

Sub. to,

$-y_1 + 3y_2' - 3y_2'' - 4y_3 \geq 2$

$-y_1 - 2y_2' + 2y_2'' + 0y_3 \geq 1$

$y_1 + 2y_2' - 2y_2'' + 0y_3 \geq -1$

$-y_1 + 3y_2' - 3y_2'' + y_3 \geq 1$

$y_1, y_2', y_2'', y_3 \geq 0$

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$\therefore$ The duel LPP becomes,

Min $W = -6y_1 + 3y_2' - 3y_2'' + 10y_3$

Sub to, $-y_1 + 3y_2' - 3y_2'' - 4y_3 \geq 2.$

$-y_1 - 2y_2' + 2y_2'' + 0y_3 = 1.$

$-y_1 + 3y_2' - 3y_2'' + y_3 \geq 1.$

$y_1, y_2', y_2'', y_3 \geq 0,$ ~~Let y_2~~

$\therefore$ The No. of constraints have become no. of variable in duel. & No. of variable in primal have become no. of constraints in duel.

Let $y_2 = y_2' - y_2''$

$\therefore$ Min $W = -6y_1 + 3y_2 \cancel{- 3y_2''} + 10y_3.$

Sub. to, $-y_1 + 3y_2 - 4y_3 \geq 2$

$-y_1 - 2y_2 + 0y_3 = 1$

$-y_1 + 3y_2 + y_3 \geq 1.$

~~y_1, y_2~~ $y_1, y_3 \geq 0$, $y_2 \Rightarrow$ unrestricted.

Q) Max $Z = x_1 + 3x_2 - 2x_3 + 5x_4$.

Sub. to, $3x_1 - x_2 + x_3 - 4x_4 = 6$

$5x_1 + 3x_2 - x_3 - 2x_4 = 4$.

$x_1, x_2 \geq 0$, $x_3, x_4 \rightarrow$ unrestricted.

$\rightarrow$ Let $x_3 = x_3' - x_3''$

$x_4 = x_4' - x_4''$

Canonical form is,

Max $Z = x_1 + 3x_2 - 2(x_3' - x_3'') + 5(x_4' - x_4'')$

Sub to $3x_1 - x_2 + (x_3' - x_3'') - 4(x_4' - x_4'') = 6$.

$\cancel{5x_1 + 3x_2 - (x_3' - x_3'') - 2(x_4' - x_4'') = 4}$.

$-3x_1 + x_2 - x_3' + x_3'' + 4(x_4' - x_4'') \leq -6$.

$5x_1 + 3x_2 - x_3' + x_3'' - 2x_4' + 2x_4'' \leq 4$

$-5x - 3x_2 + x_3' - x_3'' + 2x_4' - 2x_4'' \leq -4$.

$x_1, x_2, x_3', x_3'', x_4', x_4'' \geq 0$.

Let y_1, y_2, y_3, y_4 be duel variable

$\therefore$ Duel LPP is,

Min $w = 6y_1 - 6y_2 + 4y_3 - 4y_4$

Sub. to, $3y_1 - 3y_2 + 5y_3 - 5y_4 \geq 1$

$-1y_1 + y_2 + 3y_3 - 3y_4 \geq 3$

$y_1 - y_2 - y_3 + y_4 \geq -2$ ×

$-y_1 + y_2 + y_3 - y_4 \geq 2$

$4y_1 + 4y_2 - 2y_3 + 2y_4 \geq 5$ ×

$4y_1 - 4y_2 + 2y_3 - 2y_4 \geq -5$.

We combine these to make the constraints in duel = 4 which are the no. of var. in LPP. We can eliminate anyone of the eqn in the pair.

$y_1, y_2, y_3, y_4 \geq 0$.

Duel simplex meth
Solving LPP using ·

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Let $w_1 = y_1 - y_2$, $w_2 = y_3 - y_4$.

$\text{Min } W = 6w_1 + 4w_2$.

Sub. to,
$$3w_1 + 5w_2 \geq 1$$
$$-w_1 + 3w_2 \geq 3$$
$$-w_1 + w_2 = 2.$$
$$4w_1 + 2w_2 = -5$$
w_1, w_2 unrestricted.

In primal the variables which are unrestricted, the corresponding constraints in duel are equality.
& vice versa
In primal the constraints which are equality, the corresponding variables in duel are unrestricted.

NLPP w/ more than 1 eq & ineq.

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★ Non-Linear Programming Problem:

~~① No constraint~~
② ~~1 Equality~~
~~③~~

- **NLPP w/o constraints:**

Q) Let, $f(x_1,x_2,x_3) = x_1^2+x_2^2+x_3^2-6x_1-8x_2-10x_3$.

$\frac{\partial f}{\partial x_1} = 0 \Rightarrow 2x_1-6=0$

$\frac{\partial f}{\partial x_2} = 0 \Rightarrow 2x_2-8=0$

$\frac{\partial f}{\partial x_3} = 0 \Rightarrow 2x_3-10=0$

$\left.\right\} \begin{matrix} x_1=3 \\ x_2=4 \\ x_3=5 \end{matrix}$

$\therefore x_0(3,4,5)$ is a stationary point.

Consider Hessian matrix

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x^3} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ at } x_0$$

Principal minor/Determinant of order 1

Consider $D_1 = |2| = 2$

$D_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$

$D_3 = |H| = 8$

$\because D_1, D_2, D_3$ are all +ve

x_0 is a point of minima

$\therefore Z_{min} = -50$

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Q) Obtain relative maxima or minima for following func

$f(x_1, x_2, x_3) = x_1 + 2x_3 + x_2 x_3 - x_1^2 - x_2^2 - x_3^2$

$\frac{\partial f}{\partial x_1} = 0 \Rightarrow 1 - 2x_1 = 0$

$\frac{\partial f}{\partial x_2} = 0 \Rightarrow x_3 - 2x_2 = 0$

$\frac{\partial f}{\partial x_3} = 0 \Rightarrow 2 + x_2 - x_3 = 0$

$\left.\right\} \begin{array}{l} x_1 = 1/2 \\ x_2 = 2/3 \\ x_3 = 4/3. \end{array}$

$\therefore x_0 \left(\frac{1}{2}, \frac{2}{3}, \frac{4}{3}\right)$

$H = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ at x_0

Consider $D_1 = |-2| = -2$

$D_2 = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4$

$D_3 = |H| = -6.$

$\because D_1$ & D_3 are -ve & D_2 is +ve.

x_0 is a point of maxima.

$\therefore Z_{max}$

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* NLPP w/ 1 equality constraint:

- **Langrangian Multiplier method:**

~~Optimize~~:

Q) Optimize $z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$

Sub to, $x_1 + x_2 + x_3 = 20$

$x_1, x_2, x_3 \geq 0$

→ Let $f(x_1, x_2, x_3) = z$

Let $g(x_1, x_2, x_3) = x_1 + x_2 + x_3$.

& $h(x_1, x_2, x_3) = g(x_1, x_2, x_3) - 20$.

Consider $L(x_1, x_2, x_3; \lambda) = f(x_1, x_2, x_3) - \lambda h(x_1, x_2, x_3)$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 4x_1 + 10 - \lambda(1) = 0$ — (1)

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 + 8 - \lambda = 0$ — (2)

$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 6x_3 + 6 - \lambda = 0$ — (3)

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow -h(x_1, x_2, x_3) = 0$.

i.e. $x_1 + x_2 + x_3 = 20$ — (4).

Making all the coeff of x_1, x_2, x_3 equal,

(1) $\times 3 +$ (2) $\times 6 +$ (3) $\times 2$

$\Rightarrow 12\underbrace{(x_1 + x_2 + x_3)}_{=20 \text{ from (4)}} + 90 - 11\lambda = 0$.

$\therefore \lambda = 30$.

Sub. $\lambda = 30$ in (1), (2) & (3), we get,

$x_1 = 5,\ x_2 = 11,\ x_3 = 4$

$\therefore x_0(5, 11, 4)$ is a stationary point.

Now $\Delta_4 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 f}{\partial x_3 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} - \lambda \frac{\partial^2 h}{\partial x_3^2} \end{vmatrix}$

$= \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix}$

∴ $C_2 - C_4$, $C_3 - C_4$

Try & make most terms in $R_1 = 0$ so that we need ~~to solve less det.~~

$$\begin{array}{cccc} + & - & + & - \end{array}$$

$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & -6 & -6 & 0 \end{vmatrix}$

$= -1 \begin{vmatrix} 1 & 4 & 0 \\ 1 & 0 & 2 \\ 1 & -6 & -6 \end{vmatrix} = -44$

$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -6$

∵ Δ_3, Δ_4 are −ve,

$x_0 (5, 11, 4)$ is a pt. of minima.

& $Z_{min} = f(5, 11, 4) = 281$.

$\Delta_3 = 4 \quad \Delta_4 = -12$

$Z_{max} = 35 \quad x_0 (5, 3, 2)$

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Using Lagrangian's multiplier method:

Solve:

$z = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$

Sub to, $x_1 + x_2 + x_3 = 10$

$x_1, x_2, x_3 \geq 0.$

NLPP w/ 1 inequality constraint:

Kuhn Tucker Method:

Max $Z = 2x_1^2 - 7x_2^2 + 12x_1x_2$

Sub to, $2x_1 + 5x_2 \leq 98$

$x_1, x_2 \geq 0.$

Let $f(x_1, x_2) = 2x_1^2 - 7x_2^2 + 12x_1x_2$

$h(x_1, x_2) = 2x_1 + 5x_2 - 98$

Kuhn-tucker conditions are.

$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0$

$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$

$h(x_1, x_2) \leq 0$

$\lambda h(x_1, x_2) = 0$

$\lambda \geq 0$

$\therefore$ we get

1) $4x_1 + 12x_2 - 2\lambda = 0$ — (1).
2) $-14x_2 + 12x_1 - 5\lambda = 0$ — (2)
3) $2x_1 + 5x_2 - 98 = 0$ — (3)
4) $\lambda(2x_1 + 5x_2 - 98) = 0$ — (4)
5) $x_1, x_2, \lambda \geq 0$ — (5)

Now from (4), Either $\lambda = 0$ or $2x_1 + 5x_2 - 98 = 0$

Case (1): $\lambda = 0$, then from (1) & (2)

$$\left.\begin{array}{l} 4x_1 + 12x_2 = 0 \\ -14x_1 + 12x_2 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0. \end{array}$$

$\therefore Z = 0$

$\because z = 0$, it is not a feasible sol^n

$\therefore$ we reject these values.

Case (2): $\lambda \neq 0$, $2x_1 + 5x_2 = 98$ — (6)

(1) $\times 5$ − (2) $\times 2$

$$\Rightarrow \left.\begin{array}{l} 20x_1 + 60x_2 = 0 \\ +28x_2 \mp 24x_1 = 0 \end{array}\right\} \begin{array}{l} -x_1 + 22x_2 = 0 \\ x_1 = 22x_2 \text{ — (7).} \end{array}$$

Solving (6) & (7),

$x_1 = 44, x_2 = 2.$

from (1), $\lambda = 100 > 0$

$\therefore Z_{max} = f(44, 2)$

$= 4900$

Q) Max $Z = 8x_1 + 10x_2 - x_1^2 - x_2^2$.
Sub to, $3x_1 + 2x_2 \leq 6$
$x_1, x_2 \geq 0$.

→ Let $f(x_1, x_2) = 8x_1 + 10x_2 - x_1^2 - x_2^2$
$h(x_1, x_2) = 3x_1 + 2x_2 - 6$

By conditions,

$8 - 2x_1 - 3\lambda = 0$ — ①
$10 - 2x_1 - 2\lambda = 0$ — ②
$3x_1 + 2x_2 - 6 \leq 0$ — ③.
$\lambda(3x_1 + 2x_3 - 6) = 0$ — ④.
$x_1, x_2, \lambda \geq 0$ — ⑤

Case①, If $\lambda = 0$, from ① & ②
$x_1 = 4, x_2 = 5$
Condition ③ not satisfied
∴ Reject these values.

Case②, If $\lambda \neq 0$, then, $3x_1 + 2x_2 = 6$ — ⑥.
① × 2 − ② × 3
$-2x_1 + 3x_2 - 7 = 0$ —— ⑦.
Solving ⑥ & ⑦
$x_1 = 4/13$
$x_2 = 33/13$
from ①, $\lambda = 32/13 > 0$
∴ $Z_{max} = f\left(\frac{4}{13}, \frac{33}{13}\right)$

$= \frac{277}{13}$