Probability Distributions

Random Variable

- A random variable *x* takes on a defined set of values with different probabilities.
  - For example, if you roll a die, the outcome is random (not fixed) and there are 6 possible outcomes, each of which occur with probability one-sixth.
  - For example, if you poll people about their voting preferences, the percentage of the sample that responds "Yes on Proposition 100" is a also a random variable (the percentage will be slightly differently every time you poll).

- Roughly, <u>probability</u> is how frequently we expect different outcomes to occur if we repeat the experiment over and over ("frequentist" view)

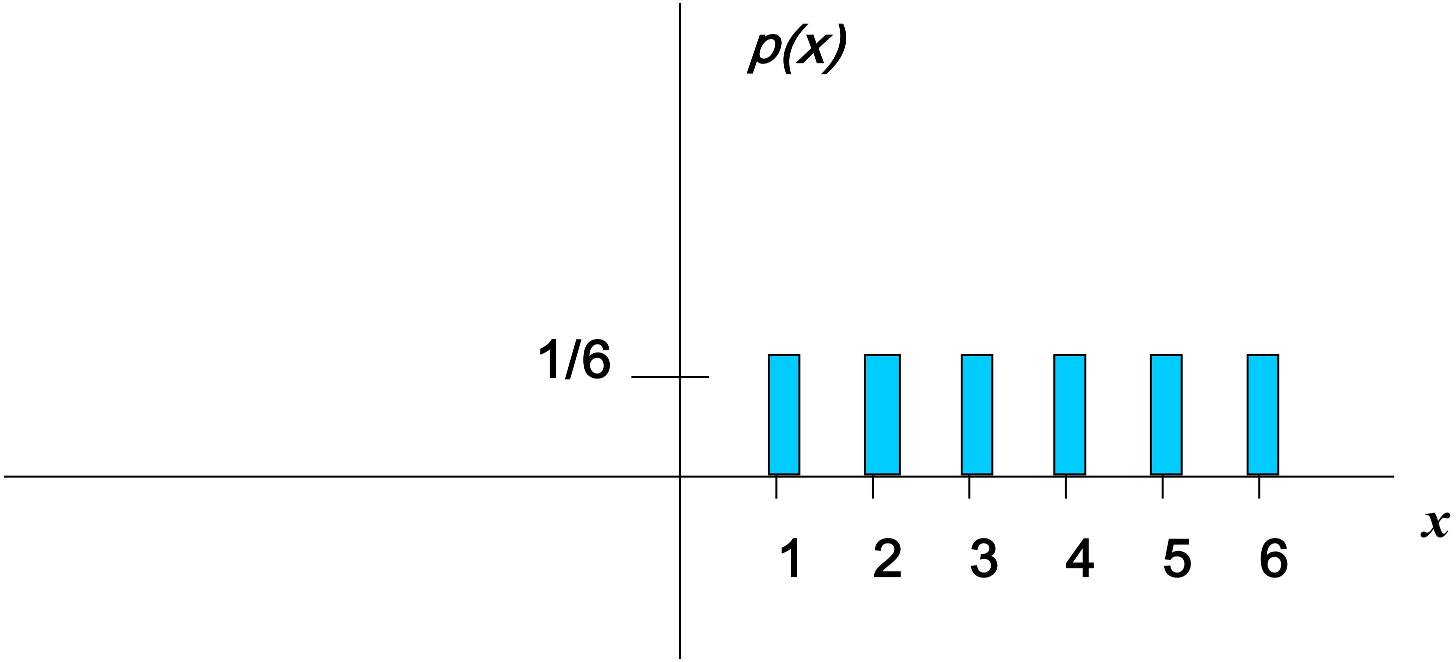
Random variables can be discrete or continuous

- **Discrete** random variables have a countable number of outcomes
  - Examples: Dead/alive, treatment/placebo, dice, counts, etc.
- **Continuous** random variables have an infinite continuum of possible values.
  - Examples: blood pressure, weight, the speed of a car, the real numbers from 1 to 6.

Probability functions

- A probability function maps the possible values of x against their respective probabilities of occurrence, $p(x)$
- $p(x)$ is a number from 0 to 1.0.
- The area under a probability function is always 1.

Discrete example: roll of a die

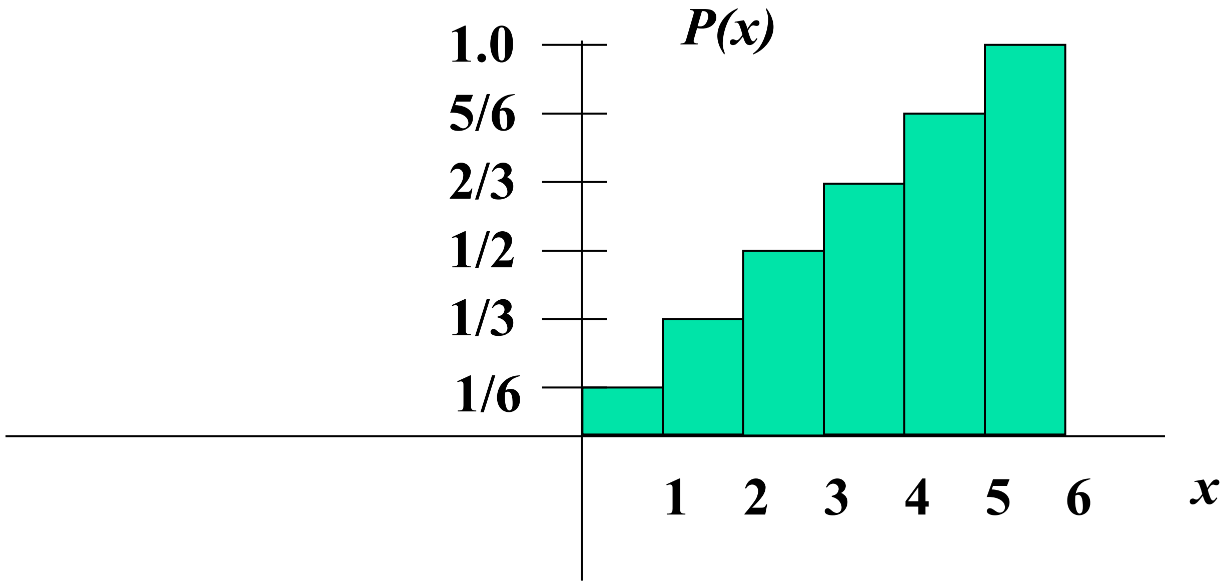


$$\sum_{\text{all } x} P(x) = 1$$

Probability mass function (pmf)

x	$p(x)$
1	$p(x=1)=1/6$
2	$p(x=2)=1/6$
3	$p(x=3)=1/6$
4	$p(x=4)=1/6$
5	$p(x=5)=1/6$
6	$p(x=6)=1/6$
	1.0

Cumulative distribution function (CDF)



Cumulative distribution function

x	$P(x \leq A)$
1	$P(x \leq 1) = 1/6$
2	$P(x \leq 2) = 2/6$
3	$P(x \leq 3) = 3/6$
4	$P(x \leq 4) = 4/6$
5	$P(x \leq 5) = 5/6$
6	$P(x \leq 6) = 6/6$

Examples

1. What's the probability that you roll a 3 or less?

$P(x \leq 3) = 1/2$

2. What's the probability that you roll a 5 or higher?

$P(x \geq 5) = 1 – P(x \leq 4) = 1-2/3 = 1/3$

Practice Problem

Which of the following are probability functions?

a. $f(x)=.25$ for x=9,10,11,12

b. $f(x)= (3-x)/2$ for x=1,2,3,4

c. $f(x)= (x^2+x+1)/25$ for x=0,1,2,3

Answer (a)

a. *f(x)=.25* for x=9,10,11,12

x	*f(x)*
9	**.25**
10	**.25**
11	**.25**
12	**<u>.25</u>**

1.0

Yes, probability function!

Answer (b)

b. *f(x)= (3-x)/2* for x=1,2,3,4

x	***f(x)***
1	**(3-1)/2=1.0**
2	**(3-2)/2=.5**
3	**(3-3)/2=0**
4	**(3-4)/2=-.5**

Though this sums to 1, you can't have a negative probability; therefore, it's not a probability function.

Answer (c)

c. $f(x) = (x^2+x+1)/25$ for x=0,1,2,3

x	f(x)
0	1/25
1	3/25
2	7/25
3	13/25

24/25

Doesn't sum to 1. Thus, it's not a probability function.

Practice Problem:

- The number of ships to arrive at a harbor on any given day is a random variable represented by *x*. The probability distribution for *x* is:

x	10	11	12	13	14
P(x)	.4	.2	.2	.1	.1

Find the probability that on a given day:

a. exactly 14 ships arrive $p(x=14) = .1$

b. At least 12 ships arrive $p(x \geq 12) = (.2 + .1 + .1) = .4$

c. At most 11 ships arrive $p(x \leq 11) = (.4 + .2) = .6$

Practice Problem:

You are lecturing to a group of 1000 students. You ask them to each randomly pick an integer between 1 and 10. Assuming, their picks are truly random:

- What's your best guess for how many students picked the number 9?

  Since $p(x=9)$ = 1/10, we'd expect about 1/10th of the 1000 students to pick 9. 100 students.

- What percentage of the students would you expect picked a number less than or equal to 6?

  Since $p(x \leq 6)$ = 1/10 + 1/10 + 1/10 + 1/10 + 1/10 + 1/10 =.6 60%

Important discrete distributions in epidemiology…

- Binomial
  - Yes/no outcomes (dead/alive, treated/untreated, smoker/non-smoker, sick/well, etc.)
- Poisson
  - Counts (e.g., how many cases of disease in a given area)

Continuous case

- The probability function that accompanies a continuous random variable is a continuous mathematical function that integrates to 1.
- The probabilities associated with continuous functions are just areas under the curve (integrals!).
- Probabilities are given for a range of values, rather than a particular value (e.g., the probability of getting a math SAT score between 700 and 800 is 2%).

Continuous case

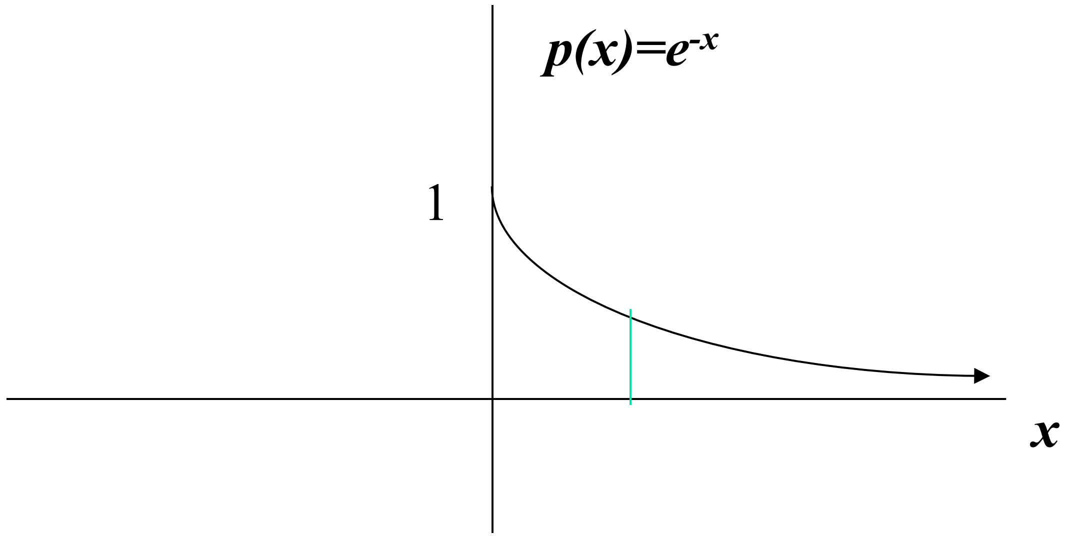
- For example, recall the negative exponential function (in probability, this is called an "exponential distribution"):

$$f(x) = e^{-x}$$

- This function integrates to 1:

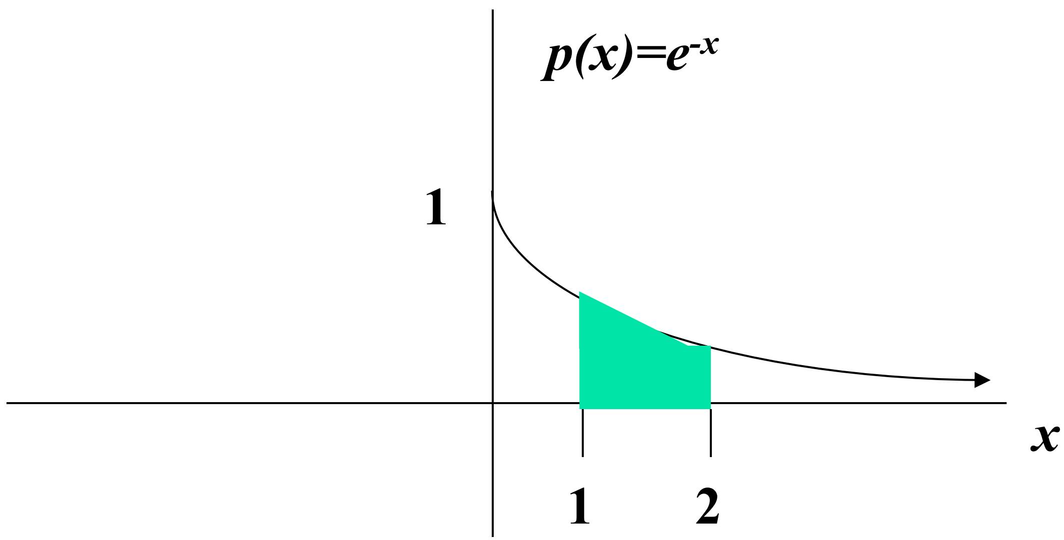
$$\int_{0}^{+\infty} e^{-x} = -e^{-x} \Big|_{0}^{+\infty} = 0 + 1 = 1$$

Continuous case: "probability density function" (pdf)



The probability that *x* is any exact particular value (such as 1.9976) is 0; we can only assign probabilities to possible ranges of x.

For example, the probability of *x* falling within 1 to 2:



$$P(1 \le x \le 2) = \int_1^2 e^{-x} = -e^{-x} \Big|_1^2 = -e^{-2} - -e^{-1} = -.135 + .368 = .23$$

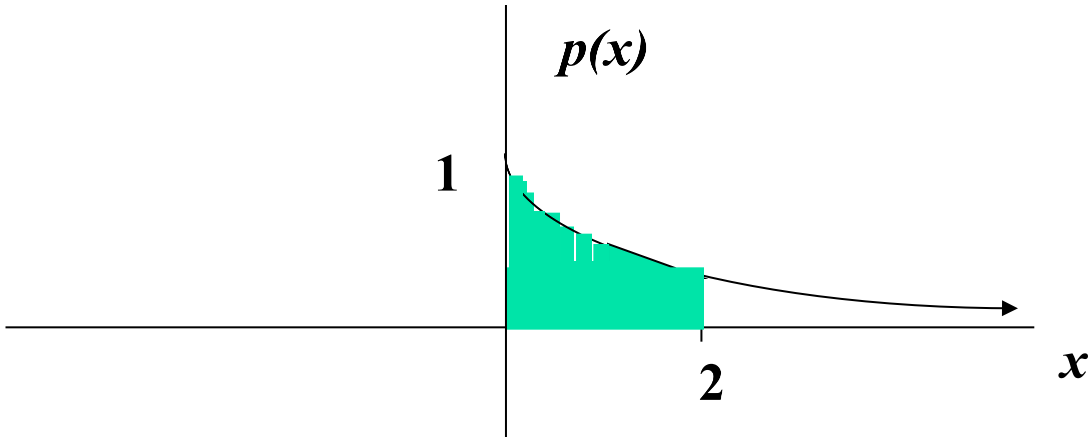
Cumulative distribution function

As in the discrete case, we can specify the “cumulative distribution function” (CDF):

The CDF here = P($x \leq$A)=

$$\int_0^A e^{-x} = -e^{-x} \Big|_0^A = -e^{-A} - -e^0 = -e^{-A} + 1 = 1 - e^{-A}$$

Example

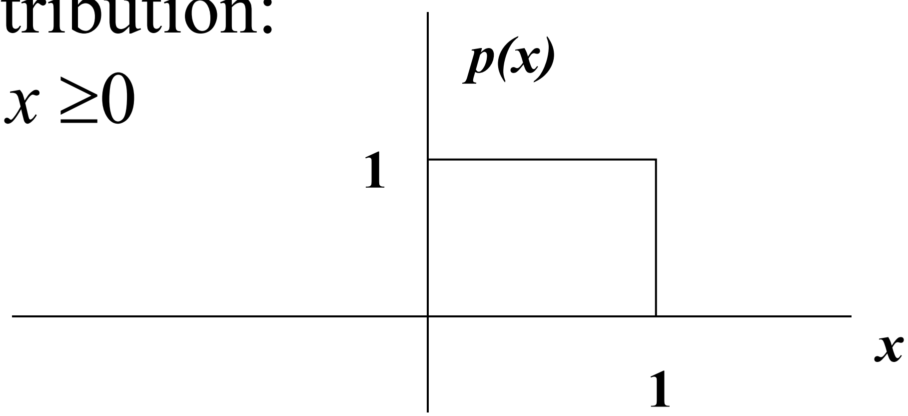


$$P(x \le 2) = 1 - e^{-2} = 1 - .135 = .865$$

Example 2: Uniform distribution

The uniform distribution: all values are equally likely

The uniform distribution:
$f(x) = 1$, for $1 \geq x \geq 0$

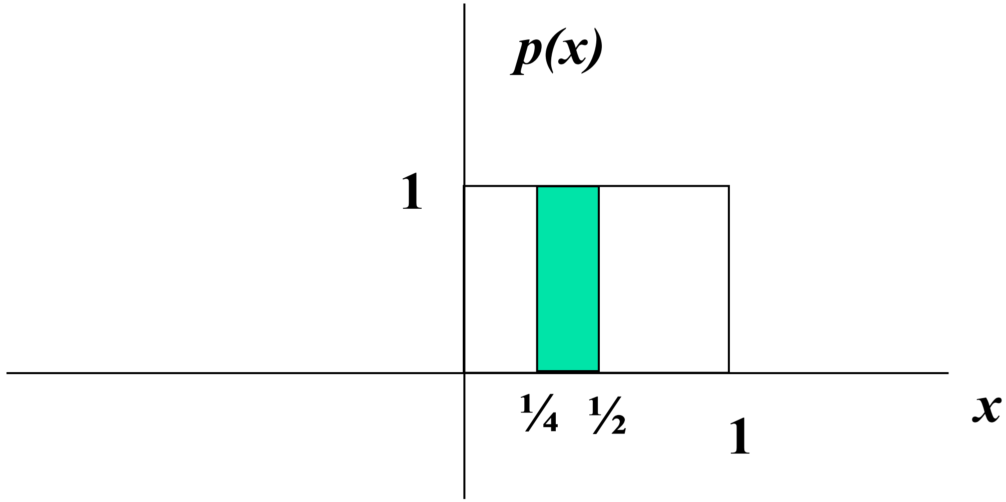


We can see it's a probability distribution because it integrates to 1 (the area under the curve is 1):

$$\int_0^1 1 = x \Big|_0^1 = 1 - 0 = 1$$

Example: Uniform distribution

What's the probability that x is between ¼ and ½?



P(½ ≥*x*≥ ¼)= ¼

Practice Problem

4. Suppose that survival drops off rapidly in the year following diagnosis of a certain type of advanced cancer. Suppose that the length of survival (or time-to-death) is a random variable that approximately follows an exponential distribution with parameter 2 (makes it a steeper drop off):

$$\text{probability function}: p(x = T) = 2e^{-2T}$$

$$[note: \int_0^{+\infty} 2e^{-2x} = -e^{-2x} \Big|_0^{+\infty} = 0 + 1 = 1]$$

What's the probability that a person who is diagnosed with this illness survives a year?

Answer

The probability of dying within 1 year can be calculated using the cumulative distribution function:

Cumulative distribution function is:

$$P(x \le T) = -e^{-2x} \Big|_0^T = 1 - e^{-2(T)}$$

The chance of surviving past 1 year is: *P(x≥1) = 1 – P(x≤1)*

$$1 - (1 - e^{-2(1)}) = .135$$

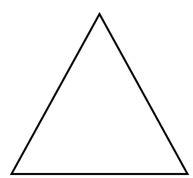
Expected value, or mean

- Expected value is just the weighted average or mean (μ) of random variable x.
- Imagine placing the masses $p(x)$ at the points X on a beam; the balance point of the beam is the expected value of x.

Example: expected value

- Recall the following probability distribution of ship arrivals:

x	10	11	12	13	14
P(x)	.4	.2	.2	.1	.1



$$\sum_{i=1}^{5} x_i\, p(x) = 10(.4) + 11(.2) + 12(.2) + 13(.1) + 14(.1) = 11.3$$

Expected value, formally

Discrete case:

$$E(X) = \sum_{\text{all } x} x_i p(x_i)$$

Continuous case:

$$E(X) = \int_{\text{all } x} x_i p(x_i) dx$$

Empirical Mean is a special case of Expected Value…

Sample mean, for a sample of n subjects: =

$$\bar{X} = \frac{\sum_{i=1}^{n} x_i}{n} = \sum_{i=1}^{n} x_i \left(\frac{1}{n}\right)$$

The probability (frequency) of each person in the sample is 1/n.

Expected value, formally

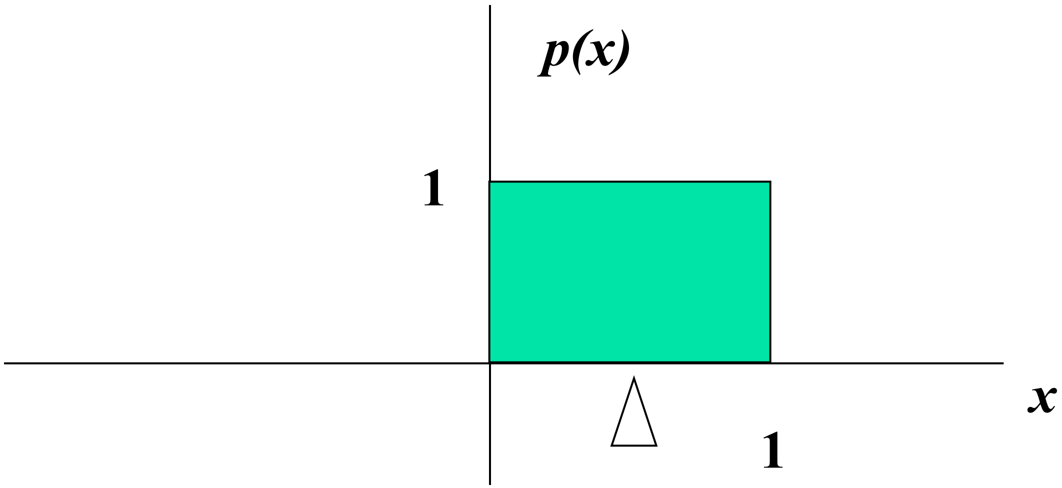
Discrete case:

$$E(X) = \sum_{\text{all } x} x_i p(x_i)$$

Continuous case:

$$E(X) = \int_{\text{all } x} x_i p(x_i) dx$$

Extension to continuous case: uniform distribution



$$E(X) = \int_0^1 x(1)dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

Symbol Interlude

- $E(X) = \mu$
  - these symbols are used interchangeably

Expected Value

- Expected value is an extremely useful concept for good decision-making!

Example: the lottery

- The Lottery (also known as a tax on people who are bad at math…)
- A certain lottery works by picking 6 numbers from 1 to 49. It costs $1.00 to play the lottery, and if you win, you win $2 million after taxes.

- *If you play the lottery once, what are your expected winnings or losses?*

Lottery

Calculate the probability of winning in 1 try:

$$\frac{1}{\binom{49}{6}} = \frac{1}{\frac{49!}{43!6!}} = \frac{1}{13{,}983{,}816} = 7.2 \text{ x } 10^{-8}$$

"49 choose 6"

Out of 49 numbers, this is the number of distinct combinations of 6.

The probability function (note, sums to 1.0):

x$	*p(x)*
-1	.999999928
+ 2 million	*7.2 x 10*$^{-8}$

Expected Value

The probability function

x$	*p(x)*
-1	.999999928
+ 2 million	7.2×10^{-8}

Expected Value

E(X) = P(win)*$2,000,000 + P(lose)*-$1.00

$= 2.0 \times 10^{6} * 7.2 \times 10^{-8} + .999999928\,(-1) = .144 - .999999928 = -\$.86$

Negative expected value is never good!

You shouldn't play if you expect to lose money!

Gambling (or how casinos can afford to give so many free drinks…)

A roulette wheel has the numbers 1 through 36, as well as 0 and 00. If you bet \$1 that an odd number comes up, you win or lose \$1 according to whether or not that event occurs. If random variable X denotes your net gain, X=1 with probability 18/38 and X= -1 with probability 20/38.

$E(X) = 1(18/38) – 1 (20/38) = -\$.053$

On average, the casino wins (and the player loses) 5 cents per game.

The casino rakes in even more if the stakes are higher:

$E(X) = 10(18/38) – 10 (20/38) = -\$.53$

If the cost is \$10 per game, the casino wins an average of 53 cents per game. If 10,000 games are played in a night, that's a cool \$5300.

**A few notes about Expected Value as a mathematical operator:

If c= a constant number (i.e., not a variable) and X and Y are any random variables…

- $E(c) = c$
- $E(cX)=cE(X)$
- $E(c + X)=c + E(X)$
- $E(X+Y)= E(X) + E(Y)$

$E(c) = c$

$E(c) = c$

- Example: If you cash in soda cans in CA, you always get 5 cents per can.
- Therefore, there's no randomness. You always expect to (and do) get 5 cents.

$E(cX)=cE(X)$

$E(cX)=cE(X)$

Example: If the casino charges \$10 per game instead of \$1, then the casino expects to make 10 times as much on average from the game (See roulette example above!)

$E(c + X) = c + E(X)$

- $E(c + X) = c + E(X)$
  - Example, if the casino throws in a free drink worth exactly \$5.00 every time you play a game, you always expect to (and do) gain an extra \$5.00 regardless of the outcome of the game.

$E(X+Y) = E(X) + E(Y)$

$E(X+Y) = E(X) + E(Y)$

Example: If you play the lottery twice, you expect to lose: -\$.86 + -\$.86.

Variance/standard deviation

"The average (expected) squared distance (or deviation) from the mean"

$$\sigma^2 = Var(x) = E[(x-\mu)^2] = \sum_{\text{all } x} (x_i - \mu)^2 p(x_i)$$

***We square because squaring has better properties than absolute value. Take square root to get back linear average distance from the mean (="standard deviation").*

Variance, formally

Discrete case:

$$Var(X) = \sigma^2 = \sum_{\text{all } x} (x_i - \mu)^2 p(x_i)$$

Continuous case:

$$Var(X) = \sigma^2 = \int_{-\infty}^{\infty} (x_i - \mu)^2 p(x_i) dx$$

Symbol Interlude

- $Var(X) = \sigma^2$
  - these symbols are used interchangeably

Handy calculation formula!

Handy calculation formula (if you ever need to calculate by hand!):

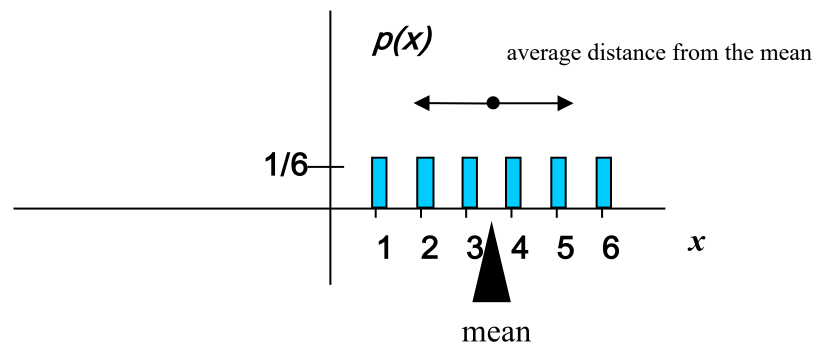
$$Var(X) = \sum_{\text{all } x} (x_i - \mu)^2 p(x_i) = \sum_{\text{all } x} x_i^{\,2} p(x_i) - (\mu)^2$$

Intervening algebra!

$$= E(x^2) - [E(x)]^2$$

For example, what's the variance and standard deviation of the roll of a die?

x	$p(x)$
1	$p(x=1)=1/6$
2	$p(x=2)=1/6$
3	$p(x=3)=1/6$
4	$p(x=4)=1/6$
5	$p(x=5)=1/6$
6	$p(x=6)=1/6$
	1.0



$$E(x)=\sum_{\text{all } x} x_i\, p(x_i)=(1)(\frac{1}{6})+2(\frac{1}{6})+3(\frac{1}{6})+4(\frac{1}{6})+5(\frac{1}{6})+6(\frac{1}{6})=\frac{21}{6}=3.5$$

$$E(x^2)=\sum_{\text{all } x} x_i^{\,2}\, p(x_i)=(1)(\frac{1}{6})+4(\frac{1}{6})+9(\frac{1}{6})+16(\frac{1}{6})+25(\frac{1}{6})+36(\frac{1}{6})=15.17$$

$$\sigma_x^2=Var(x)=E(x^2)-[E(x)]^2=15.17-3.5^2=2.92$$

$$\sigma_x=\sqrt{2.92}=1.71$$

$Var(c) = 0$

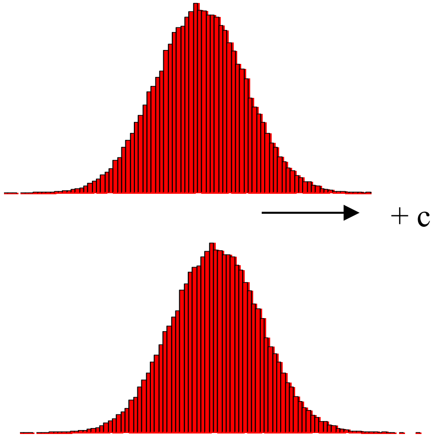
$Var(c) = 0$

Constants don't vary!

Var (c+X)= Var(X)

Var (c+X)= Var(X)

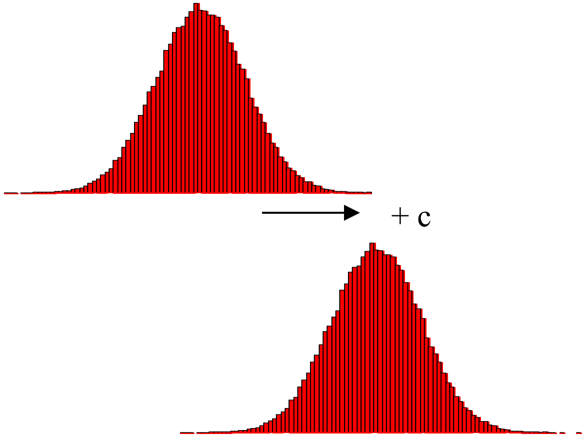
Adding a constant to every instance of a random variable doesn't change the variability. It just shifts the whole distribution by c. If everybody grew 5 inches suddenly, the variability in the population would still be the same.



Var (c+X)= Var(X)

Var (c+X)= Var(X)

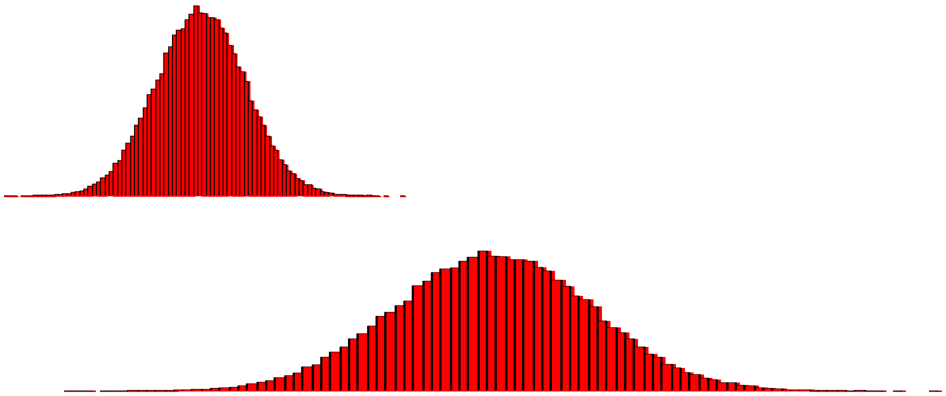
Adding a constant to every instance of a random variable doesn't change the variability. It just shifts the whole distribution by c. If everybody grew 5 inches suddenly, the variability in the population would still be the same.



$Var(cX) = c^2 Var(X)$

$Var(cX) = c^2 Var(X)$

Multiplying each instance of the random variable by c makes it c-times as wide of a distribution, which corresponds to c^2 as much variance (deviation squared). For example, if everyone suddenly became twice as tall, there'd be twice the deviation and 4 times the variance in heights in the population.



$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ ***ONLY IF X and Y are independent!!!!!!!!***

With two random variables, you have more opportunity for variation, unless they vary together (are dependent, or have covariance): $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y)$

Practice Problem

Find the variance and standard deviation for the number of ships to arrive at the harbor (recall that the mean is 11.3).

x	10	11	12	13	14
$P(x)$	.4	.2	.2	.1	.1

Answer: variance and std dev

x^2	100	121	144	169	196
$P(x)$	.4	.2	.2	.1	.1

$$E(x^2) = \sum_{i=1}^{5} x_i^{\ 2} p(x_i) = (100)(.4) + (121)(.2) + 144(.2) + 169(.1) + 196(.1) = 129.5$$

$$Var(x) = E(x^2) - [E(x)]^2 = 129.5 - 11.3^2 = 1.81$$

$$stddev(x) = \sqrt{1.81} = 1.35$$

Interpretation: On an average day, we expect 11.3 ships to arrive in the harbor, plus or minus 1.35. This gives you a feel for what would be considered a usual day!

Practice Problem

You toss a coin 100 times. What's the expected number of heads? What's the variance of the number of heads?

Answer: expected value

Intuitively, we'd probably all agree that we expect around 50 heads, right?

Another way to show this→

Think of tossing 1 coin. E(X=number of heads) = (1) P(heads) + (0)P(tails)

$\therefore$E(X=number of heads) = 1(.5) + 0 = .5

If we do this 100 times, we're looking for the sum of 100 tosses, where we assign 1 for a heads and 0 for a tails. (these are 100 "independent, identically distributed (i.i.d)" events)

$E(X_1 + X_2 + X_3 + X_4 + X_5 \ldots.. + X_{100}) = E(X_1) + E(X_2) + E(X_3) + E(X_4) + E(X_5) \ldots.. + E(X_{100}) =$

$100\ E(X_1) = 50$

Answer: variance

What's the variability, though? More tricky. But, again, we could do this for 1 coin and then use our rules of variance.

Think of tossing 1 coin.

E(X^2=number of heads squared) = 1^2 P(heads) + 0^2 P(tails)

$\therefore E(X^2) = 1(.5) + 0 = .5$

$Var(X) = .5 - .5^2 = .5 - .25 = .25$

Then, using our rule: Var(X+Y)= Var(X) + Var(Y) (coin tosses are independent!)

$Var(X_1 + X_2 + X_3 + X_4 + X_5 \ldots\ldots + X_{100}) = Var(X_1) + Var(X_2) + Var(X_3) + Var(X_4) + Var(X_5) \ldots\ldots + Var(X_{100}) =$

$100\ Var(X_1) = 100\ (.25) = 25$

SD(X)=5

Interpretation: When we toss a coin 100 times, we expect to get 50 heads plus or minus 5.