FOURIER SERIES

Find the Fourier series for the following functions.

FOURIER EXPANSION OF $f(x)$ IN THE INTERVAL $(0, 2\pi)$

1. $f(x)=x^2$ *in* $(0, 2\pi)$ Hence deduce that $\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}\ldots\ldots$

[Ans: $f(x)=\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx-4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$]

2. $f(x)=e^{-x}, 0<x<2\pi$ & $f(x+2\pi)=f(x)$ Hence deduce the value of $\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$

[Ans: $f(x)=\frac{1-e^{-2\pi}}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}$]

3. $f(x)=x\sin x$ in the interval $0\le x\le 2\pi$. Hence deduce that $\sum_{n=1}^{\infty}\frac{1}{n^2-1}=\frac{3}{4}$

[Ans: $f(x)=-1-\frac{1}{2}\cos x+\sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx+\pi\sin x$]

4. $f(x)=\sqrt{1-\cos x}$ *in* $(0, 2\pi)$ Hence deduce that $\sum_{n=1}^{\infty}\frac{1}{4n^2-1}=\frac{1}{2}$

5. $f(x)=x, \quad 0<x\le\pi$
$\quad =2\pi-x, \quad \pi\le x<2\pi$ Hence deduce that $\frac{\pi^2}{96}=\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}\ldots\ldots$

[Ans: $f(x)=\frac{\pi}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^2}\cos nx$]

6. $f(x)=x$ *in* $(0, 2\pi)$ [Ans: $f(x)=\pi-2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$]

7. $f(x)=\frac{3x^2-6x\pi+2\pi^2}{12}$ *in* $(0, 2\pi)$ Hence deduce that $\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\ldots\ldots$

[Ans: $f(x)=\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$]

8. $f(x)=\left(\frac{\pi-x}{2}\right)$ in the interval $0\le x\le 2\pi$ Also deduce that $\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\ldots\ldots$

9. $f(x)=1, \quad 0<x\le\pi$
$\quad =2-\frac{x}{\pi}, \quad \pi\le x<2\pi$

[Ans: $f(x)=\frac{3}{4}-\frac{2}{\pi^2}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\cdots\right]+\frac{1}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 2x}{2}+\cdots\right]$]

10. $f(x)=2x \quad in \quad (0, 2\pi)$ Also find a_4 & b_{10}. [Ans: $f(x)=2\pi-4\sum_{n=1}^{\infty}\frac{\sin nx}{n}$, 0, −0.4]

11. $f(x)=\cos px, \quad in \quad (0, 2\pi)$ where p is not an integer.

12. $f(x) = kx, 0 \le x \le 2\pi$ .Also find a_4&b_{10}.

13. $f(x) = e^{2x}$in $(0,2\pi)$

14. $f(x) = e^{-2x}$in $(0,2\pi)$

FOURIER EXPANSION OF $f(x)$ IN THE INTERVAL $(-\pi, \pi)$

15.state the value of $f(x)$ at $x=0$ if $f(x) = \begin{cases} -\pi, -\pi < x < 0 \\ x, 0 < x < \pi \end{cases}$ and hence deduce that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

[Ans: $f(x)=-\frac{\pi}{4}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n-1]}{n^2}\cos nx+\sum_{n=1}^{\infty}\frac{[1-2(-1)^n]}{n}\sin nx$]

16. $\begin{aligned} f(x)&=1/2, & -\pi<x<0 \\ &=x/\pi, & 0<x<\pi \end{aligned}$ Hence deduce that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

[Ans: $f(x)=\frac{1}{2}-\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin 2nx}{n}$]

17. $\begin{aligned} f(x)&=-x-\pi, & -\pi\le x\le 0 \\ &=x+\pi, & 0\le x\le \pi \end{aligned}$ [Ans: $f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}+4\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)}$]

18. $\begin{aligned} f(x)&=0, & -\pi\le x\le 0 \\ &=x, & 0\le x\le \pi \end{aligned}$,

Hence, deduce that i) $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}\cdots\cdots$ ii) $\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\cdots\cdots$

19.Obtain Fourier Series for $f(x)=e^{-|x|}, -\pi\le x\le\pi$

20. . $\begin{aligned} f(x)&=0, & -\pi\le x\le 0 \\ &=\sin x, & 0\le x\le \pi \end{aligned}$, Hence, deduce that i) $\frac{1}{2}=\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}\cdots\cdots$

ii) $\frac{1}{4}(\pi-2)=\frac{1}{1\cdot3}-\frac{1}{3\cdot5}+\frac{1}{5\cdot7}-\cdots\cdots$[Ans: $f(x)=\frac{1}{\pi}+\frac{\sin x}{2}-\frac{2}{\pi}\left[\frac{\cos 2x}{4\cdot1^2-1}+\frac{\cos 4x}{4\cdot2^2-1}+\cdots\right]$]

21. It is given that for $-\pi<x<\pi$, $x^2=\frac{\pi^2}{3}+4\sum(-1)^n\frac{\cos nx}{n^2}$

Using Parsvel's identity prove that $\sum\frac{1}{n^4}=\frac{\pi^4}{90}$.

22. $f(x)=1+\frac{2x}{\pi}, \quad -\pi\le x\le 0$

$\quad =1-\frac{2x}{\pi}, \quad 0\le x\le \pi$

, Deduce that $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}\ldots\ldots$

[Ans: $f(x)=\sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}[1-(-1)^n]\cos nx$]

23. $f(x)=x+\frac{\pi}{2}, \quad -\pi<x<0$

$\quad =\frac{\pi}{2}-x, \quad 0<x<\pi$

, Hence, Deduce that
i) $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\ldots\ldots$
ii) $\frac{\pi^4}{96}=\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}\ldots\ldots$

[Ans: $f(x)=\sum_{n=1}^{\infty}\frac{2}{\pi n^2}[1-(-1)^n]\cos nx$]

24. Prove that $\sinh ax=\frac{2}{\pi}\sinh a\pi\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot n}{n^2+a^2}\sin nx\right]$

25. $f(x)=x\cos x, \quad in \quad (-\pi,\pi)$ [Ans: $f(x)=-\frac{1}{2}\sin x+2\sum_{n=2}^{\infty}(-1)^n\cdot\frac{n}{n^2-1}\sin nx$]

26. $f(x)=x+x^2, \quad in \quad (-\pi,\pi)$. Hence deduce that i) $\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\ldots\ldots$ ii) $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\ldots\ldots$

27. $f(x)=\cos px, \quad in \quad (-\pi,\pi)$. Where p is not an integer. Hence, prove that

$\cot p\pi=\frac{2p}{\pi}\left[\frac{1}{2p^2}-\frac{1}{p^2-1^2}+\frac{1}{p^2-2^2}-\frac{1}{p^2-3^2}+\cdots\right]$ And deduce that $\cos\theta=\frac{1}{\theta}-\sum_{n=1}^{\infty}\frac{2\theta}{n^2\pi^2-\theta^2}$

Also deduce that $\frac{1}{2}-\frac{\pi\sqrt{3}}{18}=\frac{1}{9\cdot1^2-1}+\frac{1}{9\cdot2^2-1}+\frac{1}{9\cdot3^2-1}+\cdots$

28. $f(x)=|\sin x|, \quad in \quad (-\pi,\pi)$ [Ans: $f(x)=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+\frac{\cos 6x}{35}+\cdots\right]$]

29. $f(x)=|x|, \quad in \quad (-\pi,\pi)$ Hence deduce that $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\ldots\ldots$

[Ans: $f(x)=\frac{\pi}{2}-\frac{\pi}{2}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right]$]

30. $f(x)=x\sin x, \quad in \quad (-\pi,\pi)$. Hence deduce that $\frac{1}{4}(\pi-2)=\frac{1}{1\cdot3}-\frac{1}{3\cdot5}+\frac{1}{5\cdot7}-\ldots\ldots$

[Ans: $f(x)=1-\frac{1}{2}\cos x-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx$]

31. $f(x)=\frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}}, \quad in \quad (-\pi,\pi)$ [Ans: $f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot n}{n^2+a^2}\cdot\sin nx$]

32. $f(x)=\frac{x(\pi^2-x^2)}{12}, \quad in \quad (-\pi,\pi)$ [Ans: $f(x)=\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-\cdots$]

33. $f(x)=0, \quad -\pi \le x \le 0$

$\quad = x^2, \quad 0 \le x \le \pi$

34 .If $x^2 = \frac{\pi}{3} + 4\sum(-1)^n \cdot \frac{\cos nx}{n^2}$ *for* $-\pi < x < \pi$, *prove that* $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$

35. $f(x)=\sin x, \quad in \quad (-\pi, \pi)$

36. $f(x)=\frac{\pi^2}{12}-\frac{x^2}{4}, \quad in \quad (-\pi, \pi)$ [Ans: $f(x)=\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\cdots$]

37. $f(x)=x, \quad -\pi < x < 0$

$\quad = 0, \quad 0 < x < \pi/2$

$\quad = x-\pi/2, \quad \pi/2 < x < \pi$

38. $f(x)=x^2, \quad in \quad (-\pi, \pi)$

39. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 < x > \pi \end{cases}$

40. $f(x) = x\cos x \;\; in \; (-\pi, \pi)$

41. $f(x) = \cosh px \;\; in \; (-\pi, \pi)$, p is not an integer

42. $f(x) = \frac{x(\pi-x)(\pi+x)}{12} \;\; in \; (-\pi, \pi)$

43. $f(x) = x|x|, -\pi \le x \le \pi$

44. $f(x) = e^{-|x|}, -\pi \le x \le \pi$

FOURIER EXPANSION OF $f(x)$ IN THE INTERVAL $(0, 2l)$

45. $f(x)=x^2$, *in* $(0, a)$ Hence deduce that $\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\cdots\cdots$

[Ans: $f(x)=\frac{a^2}{3}+\sum_{n=1}^{\infty}\frac{a^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{a}\right)-\sum_{n=1}^{\infty}\frac{a^2}{n\pi}\sin\left(\frac{n\pi x}{a}\right)$]

46. $f(x)=2x-x^2, 0 \le x \le 3$ [Ans: $f(x)=\sum_{n=1}^{\infty}\frac{-9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right)+\sum_{n=1}^{\infty}\frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)$]

47. $f(x)=\pi x, \quad 0 < x < 1$

$\quad = 0, \quad 1 < x < 2$,

[Ans: $f(x)=\frac{\pi}{4}-\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^2}\cos n\pi x-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x$]

48. $f(x)=\pi x, \quad 0 \le x \le 1$
$= \pi(2-x), \quad 1 \le x \le 2$, with period 2, show that $f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x$

49. $f(x)=\pi x, \quad 0<x<1$
$=0, \quad x=1$
$=\pi(2-x), \quad 1<x<2$, Hence show that $\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\cdots\cdots$ [Ans: $f(x)=\frac{\pi}{4}+2\sum_{n=1}^{\infty}(-1)^{n+1}\sin n\pi x$]

50. $f(x)=3kx/l, \quad 0<x<(l/3)$
$=3k(l-2x)/l, \ (l/3)<x<(2l/3)$,
$=\pi(2-x), \quad (2l/3)<x<l$
[Ans: $\frac{9k}{\pi^2}\sum\frac{1}{n^2}\sin\frac{2n\pi}{3}\cdot\sin\frac{2n\pi x}{l}$]

51. If $x^2=\frac{4l^2}{3}+\frac{4l^2}{\pi^2}\sum\frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right)-\frac{4l^2}{\pi}\sum\frac{1}{n}\sin\left(\frac{n\pi x}{l}\right)$ *in* $0<x<2l$, find the sum of the series

$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\cdots\cdots$ [Ans: $\frac{\pi^2}{6}$]

52. $f(x)=kx$ in the interval $0 \le x \le 2$. Hence deduce that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

53. Find Fourier series to represent $f(x) = 2x - x^2$ in (0,3) and prove that $\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\cdots\cdots\cdots$

54. $f(x) = 2-\frac{x^2}{2}$ in $0 \le x \le 2$

FOURIER EXPANSION OF $f(x)$ IN THE INTERVAL $(-l,l)$

55. $f(x)=0, \quad -c<x<0$
$=a, \quad 0<x<c$, [Ans: $f(x)=\frac{a}{2}+\frac{2a}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{c}+\frac{1}{3}\sin\frac{3\pi x}{c}+\cdots\right]$]

56. $f(x)=-x, \quad -1<x<0$
$=x, \quad 0<x<1$, [Ans: $f(x)=1-\frac{4}{\pi^2}\sum\frac{1}{(2n-1)^2}\cos n\pi x$]

57. $f(x)=x, \quad -1<x<0$
$=x+2, \quad 0<x<1$, [Ans: $f(x)=1+\frac{2}{\pi}\sum_{n=1}^{\infty}\left[1-2(-1)^n\right]\sin n\pi x$

58. $f(x)=|x|, \quad -2<x<2$, Hence deduce that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{\pi^4}{96}$

[Ans: $f(x)=1-\frac{8}{\pi^2}\sum\frac{1}{(2n-1)^2}\cos\left[\frac{(2n-1)\pi x}{2}\right]$]

59. $f(x)=1-x^2, \quad -1<x<1$, [Ans: $f(x)=\frac{2}{3}-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$

60. $f(x)=\sin ax, \quad -l<x<l$, [Ans: $f(x)=2\pi\sin al\sum\frac{(-n)(-1)^n}{n^2\pi^2-a^2l^2}\sin\frac{n\pi x}{l}$]

61. $f(x)=x-x^2, \quad -1<x<1,$ [Ans: $f(x)=-\frac{1}{3}-\frac{4}{\pi^2}\sum\frac{(-1)^n}{n^2}\cos n\pi x-\frac{2}{\pi}\sum\frac{(-1)^n}{n}\sin n\pi x$]

62. $f(x)=a^2-x^2, \quad -a<x<a,$ [Ans: $f(x)=\frac{2a^2}{3}+\frac{4a^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{a}-\frac{1}{2^2}\cos\frac{2\pi x}{a}+\frac{1}{3^2}\cos\frac{3\pi x}{a}-\ldots\ldots\right]$]

63. $f(x)=x^2, \quad -1<x<1,$ [Ans: $f(x)=\frac{1}{3}-\frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\pi x-\frac{1}{2^2}\cos 2\pi x+\frac{1}{3^2}\cos 3\pi x-\cdots\right]$]

64. $f(x)=9-x^2, \quad -3<x<3,$ [Ans: $f(x)=6+\frac{36}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{3}-\frac{1}{2^2}\cos\frac{2\pi x}{3}+\frac{1}{3^2}\cos\frac{3\pi x}{3}-\ldots\ldots\right]$]

65. $f(x)=x-x^3$ *in* $(-1,1)$ [Ans: $f(x)=-\frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n\sin n\pi x}{n^3}$]

66. $f(x)=x-x^2$ *in* $(-1,1)$ [Ans: $f(x)=-\frac{1}{3}-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n\cos n\pi x}{n^2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\sin n\pi x}{n}$]

67. $f(x)=x^2-2, \; -2\le x\le 2$ [Ans: $f(x)=-\frac{2}{3}-\frac{16}{\pi^2}\left[\cos\frac{\pi x}{2}-\frac{1}{4}\cos\pi x+\frac{1}{9}\cos\frac{3\pi x}{2}\cdots\right]$]

68. $f(x)=\begin{cases}0, & -2<x<-1\\ 1+x, & -1<x<0\\ 1-x, & 0<x<1\\ 0, & 1<x<2\end{cases}$ [Ans: $f(x)=\frac{1}{4}+\sum\frac{4}{n^2\pi^2}\left(1-\cos\frac{n\pi}{2}\right)\cdot\cos\left(\frac{n\pi x}{2}\right)$]

69. $f(x)=e^{-x}, \quad (-a,a)$

[Ans: $f(x)=\frac{\sinh a}{a}+2a\sinh a\sum\frac{(-1)^n}{a^2+n^2\pi^2}\cos\frac{n\pi x}{a}+2\pi\sinh a\sum\frac{(-1)^{n+1}\cdot n}{a^2+n^2\pi^2}\sin\frac{n\pi x}{a}$]

70. $f(x)=|x|, \quad -1<x<1$

71. $f(x)=\begin{cases}0, & -2<x<-1\\ k, & -1<x<1\\ 0, & 1<x<2\end{cases}$

72. $f(x)=\begin{cases}0, & -5<x<0\\ 7, & 0<x<5\end{cases}$ period of the function is 10.

73. $f(x)=\begin{cases}0, & -2<x<0\\ x+5, & 0<x<2\end{cases}$

74. $f(x)=1-x^2$ *in* $(-1,1)$ hence find $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots$

75. $f(x)=\begin{cases}-\sin\frac{\pi x}{k}, & -k<x<0\\ \sin\frac{\pi x}{k}, & 0<x<k\end{cases}$

76. $f(x)=\begin{cases}2(x-4), & -4<x<0\\ 2(x+4), & 0<x<4\end{cases}$

77. $f(x) = x^2 - 2$ on $(-2,2)$

HALF RANGE SERIES

78.Obtain half range sine series for $f(x) = x, \quad 0 < x < \pi/2$; $= \pi - x, \quad \pi/2 < x < \pi$. Hence find $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$.

[Ans: $f(x) = \sum \frac{4}{\pi} \frac{\sin(n\pi/2)}{n^2} \cdot \sin nx$]

79.Find half range cosine series for $f(x) = x, \quad (0, 2)$. Using Parsvel's identity, deduce that

i) $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} \cdots\cdots$ ii) $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$.

80.Obtain the expression of $f(x) = x(\pi - x), 0 < x < \pi$ as a half-range cosine series. Hence, show that i) $\frac{\pi^2}{6} = \sum_1^{\infty} \frac{1}{n^2}$

ii) $\frac{\pi^2}{12} = \sum_1^{\infty} \frac{(-1)^{n+1}}{n^2}$ iii) $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$.

[Ans: $f(x) = \frac{\pi^2}{6} - \left[\frac{1}{1^2}\cos 2x + \frac{1}{2^2}\cos 4x + \frac{1}{3^2}\cos 6x + \cdots\right]$

81.Show that if $0 < x < \pi$, $\cos x = \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{m}{4m^2 - 1} \sin 2mx$

82.Expand $f(x) = lx - x^2, 0 < x < l$ in a half range i) cosine series, ii) sine series.

Hence from sine series deduce that $\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$

[Ans: i) $f(x) = \frac{l^2}{6} - \frac{4l^2}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{4^2}\cos\frac{4\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \cdots\right]$

ii) $f(x) = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l} + \cdots\right]$]

83.Find half range cosine series for $f(x) = \begin{cases} x, & 0 < x < (\pi/2) \\ \pi - x, & (\pi/2) < x < \pi \end{cases}$

[Ans: $f(x) = \frac{\pi}{4} - \frac{8}{\pi}\left[\frac{1}{2^2}\cos 2x + \frac{1}{6^2}\cos 6x + \frac{1}{10^2}\cos 10x + \cdots\right]$]

84.Prove that in the interval $0 < x < \pi$, $\frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi}\left[\frac{\sin x}{a^2+1} - \frac{2\sin 2x}{a^2+4} + \frac{3\sin 3x}{a^2+9} - \cdots\right]$

85.Obtain half-range sine series for $f(x) = x(2-x)$ in $0 < x < 2$ and hence find $\sum \frac{1}{n^6} = \frac{\pi^6}{945}$

86.Obtain half range sine series for $f(x) = \begin{cases} (1/4) - x, & 0 < x < (1/2) \\ x - (3/4), & (1/2) < x < 1 \end{cases}$

[Ans: $f(x)=\left(\frac{1}{\pi}-\frac{4}{\pi^2}\right)\sin\pi x+\left(\frac{1}{3\pi}-\frac{4}{3^2\pi^2}\right)\sin 3\pi x+\left(\frac{1}{5\pi}-\frac{4}{5^2\pi^2}\right)\sin 5\pi x+\cdots$]

87.Obtain half-range cosine series for $f(x)=x$ *in* $0<x<l$. Hence deduce that $\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\cdots=\frac{\pi^4}{1440}$

88.Obtain half-range cosine series for $f(x)=\begin{cases}kx, & 0<x<(l/2)\\ l-x, & (l/2)<x<l\end{cases}$

Hence, deduce that i) $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\cdots\cdots$ ii) $\frac{\pi^4}{96}=\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}\cdots\cdots$

[Ans: $f(x)=\frac{kl}{4}-\frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l}+\frac{1}{6^2}\cos\frac{6\pi x}{l}+\frac{1}{10^2}\cos\frac{10\pi x}{l}+\cdots\right]$]

89.Find half range sine series of period $2l$ for $f(x)=\begin{cases}\frac{2x}{l}, & 0<x<(l/2)\\ \frac{2}{l}(l-x), & (l/2)<x<l\end{cases}$

[Ans: $f(x)=\frac{8}{\pi^2}\sum\frac{1}{n^2}\sin\frac{n\pi}{2}\cdot\sin\frac{n\pi x}{l}$]

90.Obtain sine series for $f(x)=\begin{cases}mx, & 0<x\le(\pi/2)\\ m(\pi-x), & (\pi/2)\le x<\pi\end{cases}$ [Ans: $f(x)=\frac{4m}{\pi}\left[\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\cdots\right]$]

91.Obtain half range cosine series for $f(x)=\sin\left(\frac{\pi x}{l}\right)$ *in* $0<x<l$.

[Ans: $f(x)=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{1}{1\cdot 3}\cos\frac{2\pi x}{l}+\frac{1}{3\cdot 5}\cos\frac{4\pi x}{l}+\cdots\right]$]

92.Obtain half-range cosine series for $f(x)=(x-1)^2$ *in* $0<x<1$.Hence, find $\sum_{n=1}^{\infty}\frac{1}{n^2}$ & $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$

[Ans: $f(x)=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi x}{n^2}$]

93.Find HRSSfor $f(x)=\begin{cases}\frac{2x}{3}, & 0\le x\le\frac{\pi}{3}\\ \frac{\pi-x}{3}, & \frac{\pi}{3}\le x\le\pi\end{cases}$

[Ans: $f(x)=\frac{\sqrt{3}}{\pi}\left[\frac{1}{1^2}\sin x+\frac{1}{2^2}\sin 2x-\frac{1}{4^2}\sin 4x-\frac{1}{5^2}\sin 5x+\cdots\right]$]

94.Obtain the half range sine series for $f(x)=x(\pi-x), 0<x<\pi$ Hence, find $\sum_{n=1}^{\infty}\frac{(-1)^3}{(2n-1)^3}$

95.Show that in the interval $0<x<\pi$, $\sin x=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos 2x}{2^2-1}+\frac{\cos 4x}{4^2-1}+\cdots\right]$

96.Obtain half-range sine series for $f(x)=x^2$ *in* $0<x<3$.

97.Obtain HRCS for $f(x)=x(2-x)$ *in* $0<x<2$ [Ans: $f(x)=\frac{2}{3}-\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{[1+(-1)^n]}{n^2}\cos\left(\frac{n\pi x}{2}\right)$]

98. Find half range cosine series for $f(x)=\begin{cases} kx & 0<x<l/2 \\ 0 & l/2<x<l \end{cases}$ Hence deduce that $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$

99. Find half range cosine series for f(x) = x on (0,2) hence deduce that $\frac{\pi^4}{90}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$

COMPLEX FORM OF FOURIER SERIES

Obtain complex form of Fourier series for the following functions:

100. $f(x)=e^{ax}$ in $(-\pi,\pi)$ where a is not an integer [Ans: $f(x)=\sum_{-\infty}^{\infty}\frac{(-1)^n \sinh a\pi\cdot(a+in)}{\pi(a^2+n^2)}e^{inx}$]

101. $f(x)=e^{ax}$ in $(-l,l)$, where a is not an integer. [Ans: $f(x)=\sum_{-\infty}^{\infty}\frac{(-1)^n \sinh al\cdot(al+in\pi)}{(a^2l^2+n^2\pi^2)}e^{in\pi x/l}$]

102. $f(x)=\cosh ax+\sinh ax$ in $(-l,l)$ [Ans: $f(x)=\sum_{-\infty}^{\infty}\frac{(-1)^n \sinh al\cdot(al+in\pi)}{(a^2l^2+n^2\pi^2)}e^{in\pi x/l}$]

103. $f(x)=\cosh ax$ in $(-l,l)$ [Ans: $f(x)=\sum_{-\infty}^{\infty}\frac{(-1)^n al \sinh al}{(a^2l^2+n^2\pi^2)}e^{in\pi x/l}$]

104. $f(x)=\sin ax$ in $(-\pi,\pi)$, where a is not an integer. [Ans: $f(x)=\frac{\sin a\pi}{\pi i}\sum(-1)^n\cdot\frac{n}{(a^2-n^2)}\cdot e^{inx}$]

105. $f(x)=e^{ax}$ in $(0,a)$ [Ans: $f(x)=\left(e^{a^2}-1\right)\sum_{-\infty}^{\infty}\frac{e^{2in\pi x/a}}{(a^2-2in\pi)}$]

106. $f(x)=e^{ax}$ in $(-1,1)$ [Ans: $f(x)=\sum_{-\infty}^{\infty}\frac{(-1)^n \sinh a\cdot(a+in\pi)}{(a^2+n^2\pi^2)}e^{in\pi x}$]

107. $f(x)=\cosh 3x+\sinh 3x$ in $(-\pi,\pi)$ [Ans: $f(x)=\sum_{-\infty}^{\infty}\frac{(-1)^n \sinh 3\pi\cdot(3+in)}{(9+n^2)\pi}e^{inx}$]

108. $f(x)=0, \quad 0<x<l$
$\quad = a, \quad l<x<2l$
[Ans: $f(x)=\frac{a}{2}+\frac{ai}{\pi}\left[\left(e^{u}-e^{-u}\right)+\frac{1}{3}\left(e^{3u}-e^{-3u}\right)+\cdots\right]$ where $u=\frac{i\pi x}{l}$]

109. $f(x)=e^{-x}$ in $(-\pi,\pi)$ and (-1, 1). [Ans: $f(x)=\sum_{-\infty}^{\infty}\frac{(-1)^n(1-in\pi)\sinh 1}{1+n^2\pi^2}e^{in\pi x}$]

110. $f(x)=\cos ax$ in $(-\pi,\pi)$, where a is not an integer. [Ans: $f(x)=\frac{a\sin a\pi}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n}{(a^2-n^2)}e^{inx}$]

111. $f(x)=e^{ax}$ in $(0,2\pi)$ where a is not an integer.

112. $f(x)=2x$ in $(0,2\pi)$ [Ans: $f(x)=\sum_{-\infty}^{\infty}\frac{2i}{n}e^{inx}$]

113. $f(x)=\cosh 2x+\sinh 2x$ in $(-5,5)$ [Ans: $f(x)=\sum_{-\infty}^{\infty}\frac{(-1)^n \sinh 10\cdot(10+in\pi)}{(100+n^2\pi^2)}e^{in\pi x/5}$]

114. $f(x)=\cosh 2x+\sinh 2x$ in $(-2,2)$ [Ans: $f(x)=\sum_{-\infty}^{\infty}\frac{(-1)^n \sinh 4\cdot(4+in\pi)}{(16+n^2\pi^2)}e^{in\pi x/2}$]

115. $f(x)=1, \quad 0<x<1$
$\quad = 0, \quad 1<x<2$

116. $f(x)=\cosh ax$ in $(-\pi,\pi)$ [Ans: $f(x)=\frac{a\sinh a\pi}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n}{(a^2+n^2)}e^{inx}$]

117. $f(x)=x^2+x \quad in \quad (-\pi,\pi)$

118. $f(x)=e^{-ax} \quad in \quad (-2,2)$ where a is not an integer.

119. $f(x) = \cos hx \; in \; (-1,1)$

120. If $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ and $f(x) = f(x+2) \; for \; all \; x$

And hence prove that $f(x) = \frac{1}{2} + \frac{2}{\pi}\{\frac{\sin(\pi x)}{1} + \frac{\sin(3\pi x)}{3} + \frac{\sin(5\pi x)}{5} + \cdots ...\}$

FOURIER INTEGRAL

121. Express the function $f(x)=\begin{cases} -e^{kx}, & for\; x<0 \\ e^{-kx}, & for\; x>0 \end{cases}$ as Fourier Integral and prove that

$$\int_0^\infty \frac{\omega \sin \omega x}{\omega^2+k^2}\, d\omega = \frac{\pi}{2} e^{-kx} \quad if\; x>0, k>0$$

122. Using Fourier Cosine integral prove that $e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{(\omega^2+2)}{(\omega^4+4)} \cdot \cos \omega x \, d\omega$

123. Find Fourier Integral for $f(x)=\begin{cases} 1-x^2 & for\; |x| \le 1 \\ 0 & for\; |x|>1 \end{cases}$ [Ans: $f(x)=\frac{4}{\pi}\int_0^\infty \frac{\sin\omega + \omega\cos\omega}{\omega^3} \cdot \cos \omega x \, d\omega$]

124. Express the function $f(x)=\begin{cases} \sin x, & 0<x\le \pi \\ 0, & x<0, x>\pi \end{cases}$ as Fourier Integral. and prove that

$$f(x)=\frac{1}{\pi}\int_0^\infty \frac{\sin \omega x + \cos[\omega(\pi - x)]}{1-\omega^2}\, d\omega$$ Hence deduce that $$\int_0^\infty \frac{\cos(\omega\pi/2)}{1-\omega^2}\, d\omega = \frac{\pi}{2}$$

125. Find Fourier Integral representation for $f(x)=\begin{cases} e^{ax} & x\le 0, \; a>0 \\ e^{-ax} & x \ge 0, \; a>0 \end{cases}$

Hence show that $\int_0^\infty \frac{\cos \omega x}{\omega^2+a^2}\, d\omega = \frac{\pi}{2a}e^{-ax}, \quad x>0, a>0$

126. Find Fourier Sine integral representation for $f(x)=\frac{e^{-ax}}{x}$ [Ans: $f(x)=\frac{2}{\pi}\int_0^\infty \sin \omega x \cdot \tan^{-1}(\omega/a)\, d\omega$]

127. Find Fourier Cosine integral for $f(x)=\begin{cases} 1-x^2 & 0\le x\le 1 \\ 0 & x>1 \end{cases}$ Hence evaluate $\int_0^\infty \frac{\sin\omega - \omega\cos\omega}{\omega^3} \cos\frac{\omega}{2}\, d\omega$

[Ans: $\frac{3\pi}{16}$] Hint :put $x=1/2$

D1. Express the function $f(x)=\begin{cases} \sin x, & 0<x\le\pi \\ 0, & x<0, x>\pi \end{cases}$ as Fourier sine Integral and evaluate $\int_0^\infty \frac{\sin \omega x \cdot \sin \pi\omega}{1-\omega^2}\, d\omega$

D2. Find Fourier sine integral of $f(x)$ where $f(x)=\begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$

D3. Express $f(x)=e^{-kx}\ (k>0)$ as Fourier Sine and Cosine Integral and show respectively that

i) $\int_0^\infty \frac{\omega \sin \omega x}{k^2+\omega^2} d\omega = \frac{\pi}{2} e^{-kx}$ ii) $\int_0^\infty \frac{\omega \cos \omega x}{k^2+\omega^2} d\omega = \frac{\pi}{2k} e^{-kx}$

D4. Find Fourier Integral representation for $f(x) = \begin{cases} x, & 0<x<a \\ 0, & x>a \end{cases}$ and $f(-x)=f(x)$

FOURIER TRANSFORM

D5. Find Fourier Transform of $f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$

D6. Find Fourier Transform of $f(x) = \begin{cases} 1-|x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$, hence show that $\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$

D7. Find Inverse Fourier Transform of Φ (s) defined as Φ (s = $\begin{cases} 1+s^2, & |s| \le 1 \\ 0, & |s| > 1 \end{cases}$

D8. Find Fourier Cosine Transform of e^{-ax} , $a>0$

D9. Find Fourier Sine Transform of e^{-ax} , $a>0$, hence find $F_S(xe^{-ax})$ and $F_S \frac{e^{-ax}}{x}$. Also deduce the value of $\int_0^\infty \frac{\sin sx}{x} dx$.

D10. Find Fourier Cosine Transform of $f(x) = \begin{cases} \cos x, & 0<x<a \\ 0, & x>a \end{cases}$

D11. Find $f(x)$ if $\int_0^\infty f(x) \sin sx\, dx = e^{-as}$

D12. Find Inverse Fourier Cosine Transform of $\frac{1}{1+s^2}$

D13. Find Fourier Sine Transform and Fourier Cosine Transform of f(x) if $f(x) = \begin{cases} 1, & 0<x<1 \\ 0, & x>1 \end{cases}$

D14. Find $f(x)$ if $\int_0^\infty f(x) \cos sx\, dx = \frac{\sin s}{s}$