Simple Linear Regression

Contents

Definition

Simple linear regression aims to find a linear relationship to describe the correlation between an independent and possibly dependent variable. The regression line can be used to predict or estimate missing values, this is known as *interpolation*.

Least Squares Regression Line, LSRL

The calculation is based on the *method of least squares*. The idea behind it is to minimise the sum of the vertical distance between all of the data points and the line of best fit.

Consider these attempts at drawing the line of best fit, they all look like they could be a fair line of best fit, but in fact Diagram 3 is the most accurate as the regression line has been calculated using the least squares regression line.

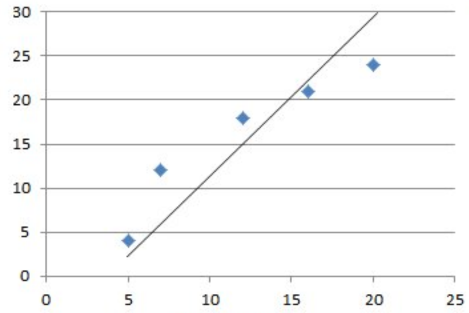


Diagram 1

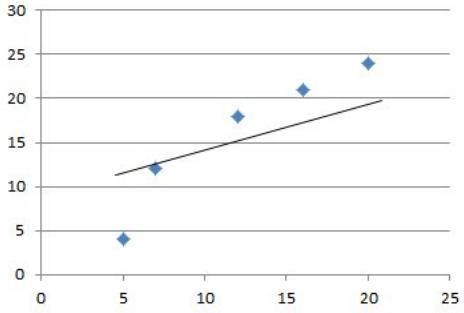


Diagram 2

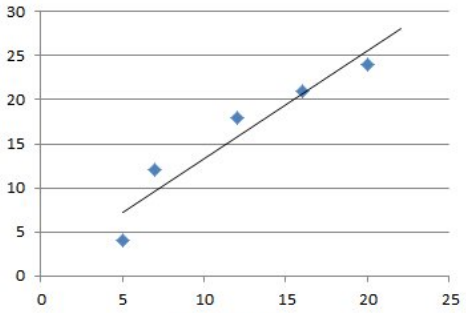


Diagram 3

The equation of the least squares regression line is

$$\hat{y} = a + bx$$

where:

- $\hat{y}$ is the predicted value of y,
- $a = \bar{y} - b\bar{x}$,
- $b = \dfrac{S_{xy}}{S_{xx}} = \dfrac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} = \dfrac{\sum(xy) - \frac{\sum x \sum y}{n}}{\sum(x^2) - \frac{(\sum x)^2}{n}}$,
- $\bar{x} = \dfrac{\sum x}{n}$,
- $\bar{y} = \dfrac{\sum y}{n}$,

Note: The underlying statistical model here is that there is a linear relation between the variables, say $y = a' + b'x$, and so we should regard the equation that we obtain using the method above as resulting in an estimate for the true equation. For this reason many authorities write $y = a + bx + \epsilon$ to emphasise this point. A further discussion on the nature of the error ϵ is not appropriate here, but is covered in the references below.

Worked Examples

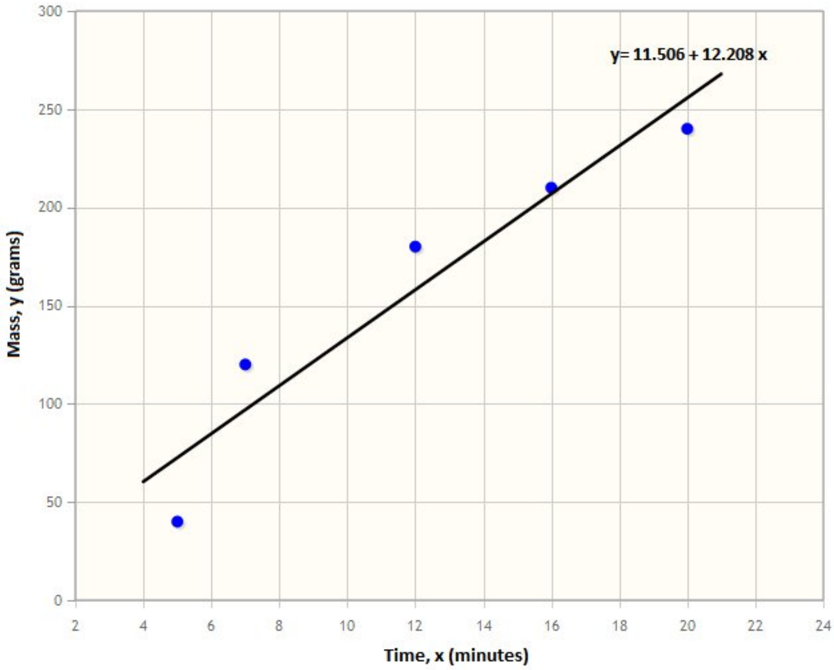
Example 1

Consider the example below where the mass, y (grams), of a chemical is related to the time, x (seconds), for which the chemical reaction has been taking place according to the table:

Time, x (seconds)	5	7	12	16	20
Mass, y (grams)	40	120	180	210	240

Find the equation of the regression line.

Solution



To work out the regression line the following values need to be calculated: $a = \bar{y} - b\bar{x}$ and $b = \dfrac{S_{xy}}{S_{xx}}$. The easiest way of calculating them is by using a table.

Start off by working out the mean of the independent and dependent variables.

$$\begin{aligned}\bar{x} &= \frac{\sum x}{n} \\ &= \frac{5+7+12+16+20}{5} \\ &= \frac{60}{5} \\ &= 12, \\ \bar{y} &= \frac{\sum y}{n} \\ &= \frac{40+120+180+210+240}{5} \\ &= \frac{790}{5} \\ &= 158.\end{aligned}$$

x_i	y_i	$x_i - \bar{x}$	$y_i - \bar{y}$	$(x_i - \bar{x})(y_i - \bar{y})$	$(x_i - \bar{x})^2$
5	40	$5-12=-7$	$40-158=-118$	$-7\times -118 = 826$	$-7^2 = 49$
7	120	$7-12=-5$	$120-158=-38$	$-5\times -38 = 190$	$-5^2 = 25$
12	180	$12-12=0$	$180-158=22$	$0\times 22 = 0$	$0^2 = 0$
16	210	$16-12=4$	$210-158=52$	$4\times 52 = 208$	$4^2 = 16$
20	240	$20-12=8$	$240-158=82$	$8\times 82 = 656$	$8^2 = 64$
$\sum x = 60$	$\sum y = 790$			$\sum (x_i - \bar{x})(y_i - \bar{y}) = 1880$	$\sum (x_i - \bar{x})^2 = 154$

Now calculate b

$$\begin{aligned}b &= \frac{S_{xy}}{S_{xx}} \\ &= \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} \\ &= \frac{1880}{154} = 12.20779...\end{aligned}$$

$$= 12.208 \text{ (3.d.p.)}$$

and calculate a

$$\begin{aligned} a &= \bar{y} - b\bar{x} \\ &= 158 - 12.208 \times 12 \\ &= 11.506... \\ &= 11.506 \text{ (3.d.p.)}. \end{aligned}$$

So the equation of the regression line is: $\hat{y} = a + bx = 11.506 + 12.208x$.

Example 2

To see how students' reaction skills have improved over a year, eight students took a reactions test at the start of the year and at the end of the year. These are their scores:

Student	Liam	Felicity	Adian	Mel	Leroy	Vic	Lawrie	Louise
First Test, x	56	75	61	61	67	72	62	61
Second Test, y	21	39	34	21	32	24	29	24

Find the equation of the regression line given that:

$$\sum x = 515, \ \sum y = 224, \ \sum x^2 = 33441, \ \sum y^2 = 6576 \ and \ \sum xy = 14590.$$

Solution

We know that the equation of the least squares regression line is

$$\hat{y} = a + bx.$$

As we have been given some summed values we are going to use $b = \frac{S_{xy}}{S_{xx}} = \frac{\sum(xy) - \frac{\sum x \sum y}{n}}{\sum(x^2) - \frac{(\sum x)^2}{n}}$.

$$\begin{aligned} b &= \frac{S_{xy}}{S_{xx}} \\ &= \frac{\sum(xy) - \frac{\sum x \sum y}{n}}{\sum(x^2) - \frac{(\sum x)^2}{n}} \\ &= \frac{14590 - \frac{515 \times 224}{8}}{33441 - \frac{515^2}{8}} \\ &= 0.590534... \\ &= 0.590 \text{ (3.d.p.)} \end{aligned}$$

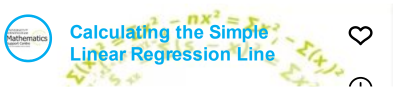
To find a we need to first work out the mean of x and y.

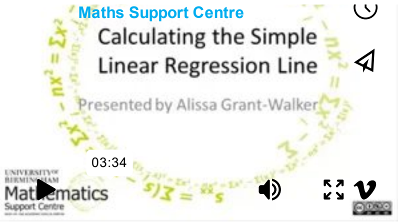
$$\begin{aligned} \bar{x} &= \frac{\sum x}{n} = \frac{515}{8} = 64.375, \\ \bar{y} &= \frac{\sum y}{n} = \frac{224}{8} = 28 \\ a &= \bar{y} - b\bar{x} \\ &= 28 - (0.590 \times 64.375) \\ &= -10.015631... \\ &= -10.016 \text{ (3.d.p.)} \end{aligned}$$

So the equation of our regression line is $\hat{y} = -10.106 + 0.590x$.

Video Example

Alissa Grant-Walker presents a video on working out the linear regression line.




Maths Support Centre
Calculating the Simple Linear Regression Line
Presented by Alissa Grant-Walker
03:34
UNIVERSITY OF BIRMINGHAM
Mathematics Support Centre

Interpreting the Regression Line

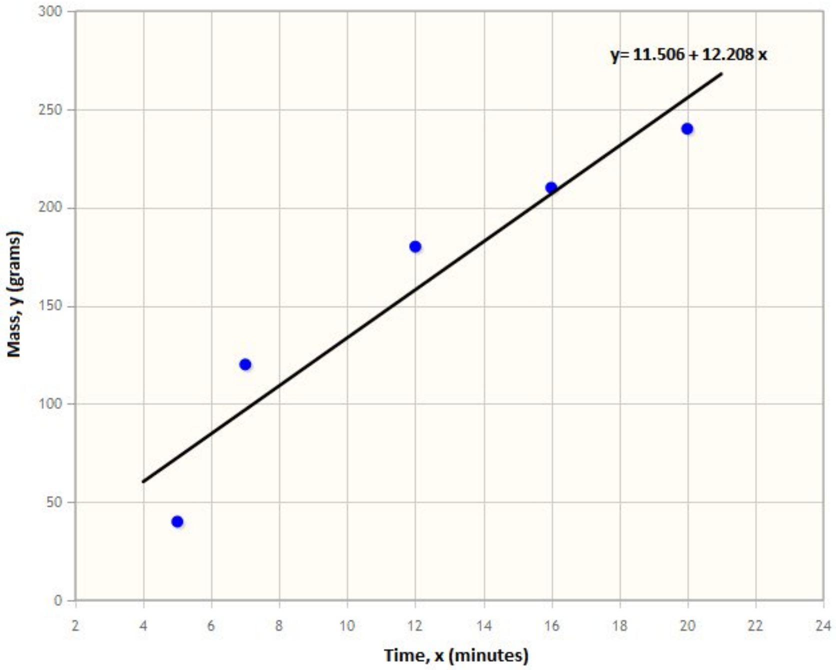
The simple linear regression line, $\hat{y} = a + bx$, can be interpreted as follows:

- $\hat{y}$ is the predicted value of y,
- a is the intercept and predicts where the regression line will cross the y-axis,
- b predicts the change in y for every unit change in x.

We can also use the equation of the regression line for finding approximate values for missing data.

Note: Using this to estimate outside the range of your data is unreliable.

Worked Example



Using the data from the last worked example about the mass of a chemical as time increases, we worked out the equation of the regression line to be $\hat{y} = 11.506 + 12.208x$. We can interpret this as for every 1 minute increase in time the mass of the chemical increases by 12.208 grams. The equation also tells us that when no time has passed, (when x is zero), the initial mass of the chemical is 11.506 grams.

Example 1

What is the mass of the chemical after ten seconds has passed?

Solution 1

Take your equation and enter the value of time $x = 10$ and calculate $\hat{y}$.

$$\hat{y} = 11.506 + 12.208 \times 10 = 133.586.$$

This means that after 10 seconds of our experiment has passed, the mass of the chemical will be 133.586 grams. Check this value against a scatter plot of our data to see if this answer is reasonable.

Example 2

By how much does the chemical increase in weight in five seconds?

Solution 2

For every minute increase in time the mass of the chemical increases by 12.208 grams. Multiply 12.208 grams by 5 to find the increase in weight of the chemical in 5 seconds.

$$12.208 \times 5 = 61.40 \text{ (grams)}.$$

Example 3

How much time does it take for the weight of the chemical to increase by 50 grams?

Solution 3

We know that for every minute increase in time the mass of the chemical increases by 12.208 grams, this also means it takes $\frac{1}{12.208}$ seconds for the chemical to increase by 1 gram. To find the time taken for the chemical to increase in weight by 50 grams we need to multiply $\frac{1}{12.208}$ by 50.

$$\frac{1}{12.208} \times 50 = 4.096 \text{ (3 d.p.)}.$$

Workbook

This workbook produced by HELM is a good revision aid, containing key points for revision and many worked examples.

- Regression and correlation

Test Yourself

- 's test on regression

Test yourself: Numbas test on linear regression

External Resources

- Simple Linear Regression at
- Worked Example of Linear Regression at
- Wonnacott, T and Wonnacott, T (1990). Introductory Statistics. 5th ed. USA: John Wiley & Sons. p355-514
- Regression line calculator online at easycalculation.com

See Also

- Residuals
- Multiple Regression