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DSIP

Module 3 :- Image Transformations.

* Discrete Fourier Transform.

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}}$$

$X(k) \rightarrow$ Transformed Image.
$x(n) \rightarrow$ Input Image.
$N \rightarrow$ Matrix/Samples.

Inverse DFT (IDFT).

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi kn}{N}} \qquad 0 \leq n \leq N-1$$

DFT Kernel for $N=4$. $\Rightarrow$ $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$

Properties :-

1] Linearity
2] Periodicity
3] Convolution
4] Multiplicative
5] Time Reversal
6] Time Shift
7] Conjugation.

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* Fast Fourier Transform:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

Twiddle Factors :-

2pt

$W_2^0 = 1$

4pt

$W_4^0 = 1$

$W_4^1 = -j$

8pt

$W_8^0 = 1$

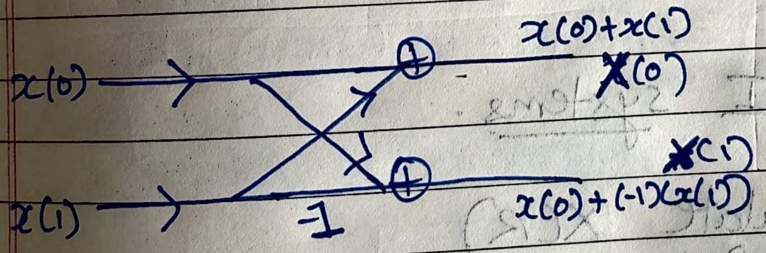
$W_8^1 = 1/\sqrt{2} - j\,1/\sqrt{2}$

$W_8^2 = -j$

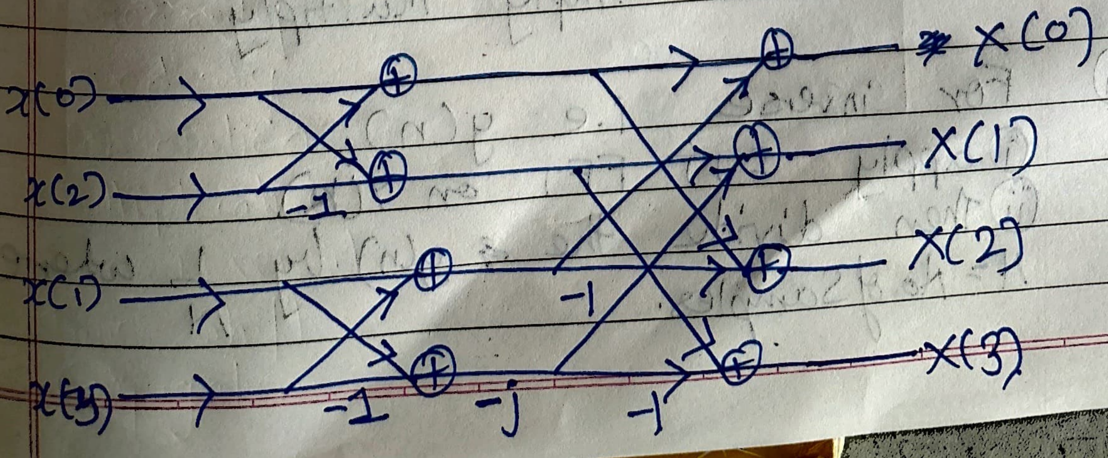
$W_8^3 = -1/\sqrt{2} - j\,1/\sqrt{2}$

FFT:- faster than DFT.
computationally efficient.
divide & conquer approach.

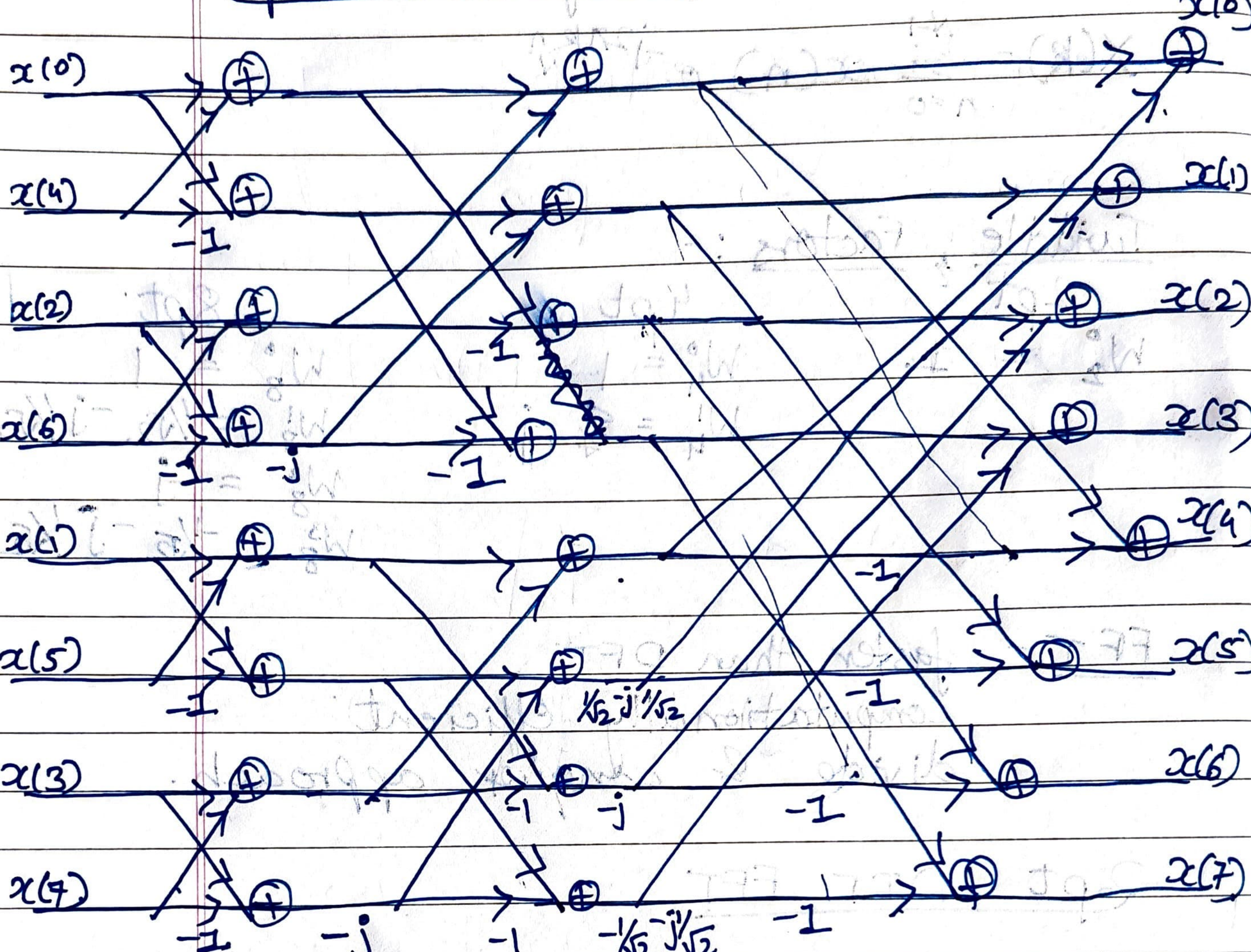
2pt DIT FFT



4pt DIT FFT



8pt DIT FFT



For LTI systems.

1. First calculate $X(K)$
2. Then $h(K)$
3. Final $Y(K) \Rightarrow X(K) \cdot h(K)$
   Simply Multiply

4. For inverse i.e $y(n)$;
   (i) apply DIT FFT on $Y(K)$
   (ii) Then divide the $y(n)$ by $\frac{1}{N}$ where N = No. of Samples.

* Need for Transformation.

i] Mathematical Convenience.

complex convolution in time domain is equal to multiplication operation in frequency domain.

ii] To extract more information.

* Unitary Transform.

Preserves the signal energy.
Packs large fraction of the energy of the image into relatively few of the transform coefficients.

A discrete linear transform is unitary if its transform matrix conforms to the unitary condition i.e $A \times A^H = I$

$A \xrightarrow{\text{conjugate}} A^* \xrightarrow{\text{transpose}} A^H$

* Orthogonal DFT

A is orthogonal if $A \cdot A^T = I$

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★ Discrete Cosine Transform (DCT)

DCT is a technique for converting a signal into elementary frequency components and is widely used in image compression.

$$X[k] = \alpha(k) \sum_{n=0}^{N-1} x(n) \cos\left\{\frac{(2n+1)\pi k}{2N}\right\}$$

$$\alpha(k) = \sqrt{\frac{1}{N}} \quad \text{if } k = 0$$

$$\alpha(k) = \sqrt{\frac{2}{N}} \quad \text{if } k \neq 0$$

IDCT

$$x(n) = \alpha(k) \sum_{k=0}^{N-1} X[k] \cos\left\{\frac{(2n+1)\pi k}{2N}\right\}$$

DCT Matrix for N = 4.

$$\begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.6532 & 0.2706 & -0.2706 & -0.6532 \\ 0.5 & -0.5 & -0.5 & 0.5 \\ 0.2706 & -0.6533 & 0.6533 & -0.2706 \end{bmatrix}$$

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Q] Compute DCT matrix for $N=4$.

$$\Rightarrow X[k] = \alpha(k) \sum_{n=0}^{N-1} x(n) \cos\left\{\frac{(2n+1)\pi k}{2N}\right\}.$$

$$\alpha(k) = \sqrt{\frac{1}{N}}, k=0 \quad ; \quad \alpha(k) = \sqrt{\frac{2}{N}}, k \neq 0$$

$N=4$.

① For $k=0$:

$$X[k] = \sqrt{\frac{1}{4}} \sum_{n=0}^{3} x(n) \cos\left\{\frac{(2n+1)\pi(0)}{8}\right\}.$$

$$X[k] = \sqrt{\frac{1}{4}} \sum_{n=0}^{3} x(n) \cos 0 \qquad \boxed{\because \cos 0 = 1}$$

$$X[k] = \sqrt{\frac{1}{4}} \sum_{n=0}^{3} x(n)$$

$$X[0] = \frac{1}{2}\left\{x(0) + x(1) + x(2) + x(3)\right\}.$$

② For $k=1$:

$$X[k] = \sqrt{\frac{2}{4}} \sum_{n=0}^{3} x(n) \cos\left\{\frac{(2n+1)\pi}{8}\right\}.$$

~~X[k]~~

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$$X[1] = \frac{1}{\sqrt{2}} \sum_{n=0}^{3} x(n) \cdot \cos\left(\frac{(2n+1)\pi}{8}\right)$$

$$X[1] = \frac{1}{\sqrt{2}} \left\{ x(0)\cos\frac{\pi}{8} + x(1)\cdot\cos\frac{3\pi}{8} + x(2)\cos\frac{5\pi}{8} + x(3)\cos\frac{7\pi}{8} \right\}.$$

$$X[1] = 0.707 \left\{ x(0)\cdot 0.9238 + x(1)\cdot 0.3827 + x(2)(-0.3827) + x(3)(-0.9238) \right\}$$

$$X[1] = 0.6532\, x(0) + 0.2706\, x(1) - 0.2706\, x(2) - 0.6532\, x(3)$$

③ For $K=2$

$$X[2] = \frac{1}{\sqrt{2}} \sum_{n=0}^{3} x(n) \cdot \cos\left(\frac{(2n+1)\pi \cdot 2}{\cancel{8}\, 4}\right)$$

$$X[2] = 0.707 \left\{ x(0)\cos\left(\frac{\pi}{4}\right) + x(1)\cos\frac{3\pi}{4} + x(2)\cos\frac{5\pi}{4} + x(3)\cos\frac{7\pi}{4} \right\}$$

$$X[2] = 0.707 \left\{ 0.707\, x(0) - 0.707\, x(1) - 0.707\, x(2) + 0.707\, x(3) \right\}$$

$$X[2] = 0.5\, x(0) - 0.5\, x(1) - 0.5\, x(2) + 0.5\, x(3)$$

④ **For $k=3$**

$$X[3] = \frac{1}{\sqrt{2}}\left\{x(0)\cos\frac{3\pi}{8} + x(1)\cos\frac{9\pi}{8} + x(2)\cos\frac{15\pi}{8} + x(3)\cos\frac{21\pi}{8}\right\}$$

$$X[3] = 0.707\{0.3827x(0) - 0.9238x(1) + 0.9238x(2) - 0.3827x(3)\}$$

$$X[3] = 0.2706x(0) - 0.6532x(1) + 0.6532x(2) - 0.2706x(3)$$

$\therefore$ DCT matrix for $N=4$ is:-

$$\begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.6532 & 0.2706 & -0.2706 & -0.6532 \\ 0.5 & -0.5 & -0.5 & 0.5 \\ 0.2706 & \cancel{0.65}-0.6532 & 0.6532 & -0.2706 \end{bmatrix}$$

☆ Symmetric & Asymmetric:

If ~~X[k]~~ Transformation matrix symmetric then:-

$$F = TfT$$

If not then :- $F = TfT'$

☆ Hadamard Transform (H·T)

$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$; $H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix}$

$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$; $H_8 = \begin{bmatrix} H_{\cancel{8}4} & H_4 \\ H_4 & -H_4 \end{bmatrix}$

$$H_8 = \begin{array}{cc} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} & \begin{matrix} 0 \\ 7 \\ 3 \\ 4 \\ 1 \\ 6 \\ 2 \\ 5 \end{matrix} \end{array}$$

- H.T is orthogonal when normalized by $\frac{1}{\sqrt{2}}$

- $X[K] = \{H(T) \cdot x(n) \cdot H(T)\}$

~~IDFT~~ Inverse :- $x(n) = \frac{1}{N}\{H(T) \cdot X(K) \cdot H(T)\}$

- H.T is symmetric.

★ **Walsh Transform** :- Symmetric

Walsh Transform is obtained from Hadamard Transform matrix by rearranging the rows in the increasing sign change order.

$$
\begin{array}{cc}
W_8 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1
\end{bmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\end{array}
$$

$F = TfT$

$f = \frac{1}{N} T F T$

* Haar Transform

Asymmetric, real, orthogonal

$$\boxed{F = H \cdot f \cdot H'}$$

Very fast transform.
Poor energy compaction for images.

$$H_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_4 = \cancel{\frac{1}{\sqrt{8}}}\ \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

$$H_8 = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

★ Slant Transform.

Unitary transform.
Designed for image coding.

$$S_4 \,(N=4) = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ a+b & a-b & -a+b & -a-b \\ 1 & -1 & -1 & 1 \\ a-b & -a-b & a+b & -a+b \end{bmatrix}$$

S_4 is orthogonal & step size is uniform

$\therefore$ values of $a = 2b$, $b = \frac{1}{\sqrt{2}}$

$a = \frac{2}{\sqrt{5}}$

$$S_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3/\sqrt{5} & 1/\sqrt{5} & -1/\sqrt{5} & -3/\sqrt{5} \\ 1 & -1 & -1 & 1 \\ 1/\sqrt{5} & -3/\sqrt{5} & 3/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$

Asymmetric Transform $\therefore F = S_4 \cdot f \cdot S_4'$

☆ KL Transform / Hotelling transform :-

De correlates the data which facilitates high degrees of compression.

Based on statistical properties of image helpful for image compression.

Data from neighbouring pixels is highly correlated, so achieving image compression without losing the quality of image is a challenge.

Steps to compute KL/Hotelling/PCA :-

1] Find mean vector & covariance matrix of x.

2] Find eigenvalues & then eigen vectors of the covariance matrix.

3] Create transformation matrix T, such that rows of T are the eigen vectors.

4] $X = T[C_x - m]$

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★ Low-Pass Filtering

Preserves low frequencies, attenuates high frequencies.

Ideal - ILPF
Butterworth (BLPF) - attenuates high frequency smoothly, less ringing effect.

$$H(u,v) = \frac{1}{1 + \left| \frac{\sqrt{k^2 + l^2}}{D_0} \right|^{2n}}$$

$D_0 \rightarrow$ cutoff frequency.

Gaussian Filter (GLPF) - blurs & smoothens properly with no edges, no ringing effect

$$H(u,v) = e^{-(u^2+v^2)/2D_0^2}$$

$D_0 \rightarrow$ cutoff freq.

Ringing Effect - Sharp cutoff frequency produce an overshoot of image features whose frequency is close to the cutoff frequencies.

★ High Pass Filtering

Preserves high frequency, attenuates low frequencies.

$$H_{HP}(u,v) = 1 - H_{LP}(u,v)$$

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★ Homomorphic Filters-

- Removes shading effects from an image.
- Enhances high frequencies
- Attenuates low frequencies but preserves fine details
- Separates image into two components:-
  1] Illumination
  2] Reflectance.

Preserves the sharp edges & features of the image

- Illumination $i(x,y)$ varies slowly & affects low frequency.
- Reflectance $r(x,y)$ varies faster & affects high frequencies.

Steps

$f(x,y) \rightarrow$ Take Ln $\rightarrow$ Apply FFT $\rightarrow$ Apply $H(u,v)$ $\rightarrow$ Take Inverse FFT $\rightarrow$ Take exp.