Chapter 3

Image Transform: Frequency Domain Representation and Enhancement 10 CO3

3.1 Introduction , DFT and its properties, radix-2 algorithm(2- DFT), FFT algorithm: divide and conquer approach, Dissemination in Time(DIT)-FFT

3.2 Discrete Cosine Transform, Walsh Transform, Hadamard Transform, Haar Transform, Principal component Analysis(PCA/Hoteling Transform), Introduction to Wavelet Transform

3.3 Low Pass and High Pass Frequency domain filters: Ideal, Butterworth, Homomorphic filter

Self-Learning Topic: Discrete Sine Transform (DST)

Discrete Fourier Transform

- Discrete fourier transform for a finite sequence is given by:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{kn}{N}}$$

- Evaluating summation for finite sequence for 0 to N-1.
- k/N corresponds to frequency F
- N corresponds to time for discrete signals

Discrete Fourier Transform

- Magnitude function
- $X(k) = \sqrt{Xr^2(k)+Xi^2(k)}$

- Phase function
- Angle $X(k)=\tan^{-1}[Xi(k)/Xr(k)]$

Inverse Discrete Fourier Transform

The inverse discrete Fourier transform of X(k) is defined as

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \qquad 0 \leq n \leq N-1$$

Where K and n are in the range of 0 ,1,2……N-1 For example, if N=4, K= 0,1,2,3: N=0,1,2,3

Properties of DFT

Linearity property:

If $X_1(k)=DFT[x_1(n)]$ & $X_2(k)=DFT[x_2(n)]$, then

$$DFT[a_1x_1(n)+a_2x_2(n)]=a_1X_1(k)+a_2X_2(k)$$

Periodicity property:

If $X(k)$ is the N-point DFT of $x(n)$, then

$$X(k+N)=X(k)$$

Properties of DFT

- DFT of circular convolution of two sequences is equivalent to product of their individual DFTs.

Convolution property:

If $X_1(k) = DFT[x_1(n)]$ & $X_2(k) = DFT[x_2(n)]$, then

$$DFT[x(n) \, \text{Ⓝ} \, x_2(n)] = X_1(k)X_2(k)$$

Where Ⓝ indicates N-point circular convolution.

Properties of DFT

- DFT of product of two discrete time sequences is equivalent to circular convolution of DFTs of individual sequences scaled by factor of 1/N.

Multiplication property:

If $X_1(k) = DFT[x_1(n)]$ & $X_2(k) = DFT[x_2(n)]$, then

$$DFT[x1(n)x2(n)] = (1/N)[X1(k)\ N\ X2(k)]$$

Where N Indicates N-point circular convolution.

Properties of DFT

- Reversing the N point sequence in time is equivalent to reversing the DFT sequence.

Time reversal property:

If $X(k)$ is the N-point DFT of $x(n)$, then $DFT[x(N-n)] = X(N-k)$

Properties of DFT

- The circular time shift property of DFT says that if discrete time signal is circularly shifted in time by m units , then its DFT is multiplied by e^-j2pikm/N

Time shift property:

If X(k) is the N-point DFT of x(n), then

$$\mathcal{DFT}\{x((n-m))_N\} = X(k)\, e^{-j\frac{2\pi km}{N}}$$

Properties of DFT

- Conjugation
- DFT of complex conjugate of any sequence is equal to complex conjugate of DFT of that sequence , with sequence delayed by k samples in frequency domain.
- DFT{x(n)}=X(k)
- DFT{X*(n)}=X*(N-k)

Fast Fourier Transform

- FFT is method of computing DFT with reduced number of calculations.

- Computational efficiency is achieved if we adopt a divide and conquer approach.

- Approach is based on decomposition of N point DFT into successively smaller DFTs.

Radix-2 FFT

- N=2^m
- m is number of stages
- N point sequence is decimated into 2 point , further from a 2 point sequence 4 point DFT can be computed and so on.

Phase or Twiddle Factor

- Complex valued factor in DFT is defined as W_N

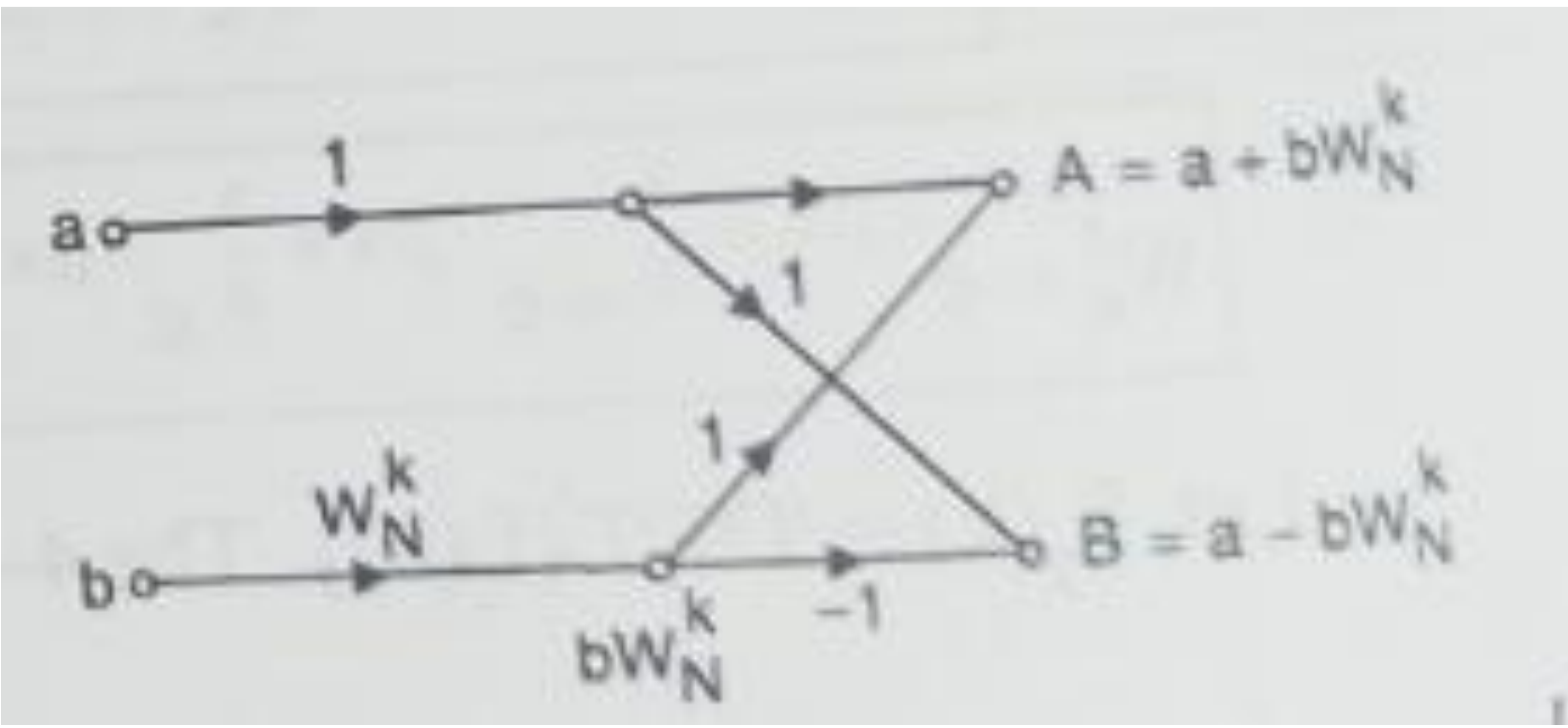
also called as twiddle factor.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{kn}{N}}$$

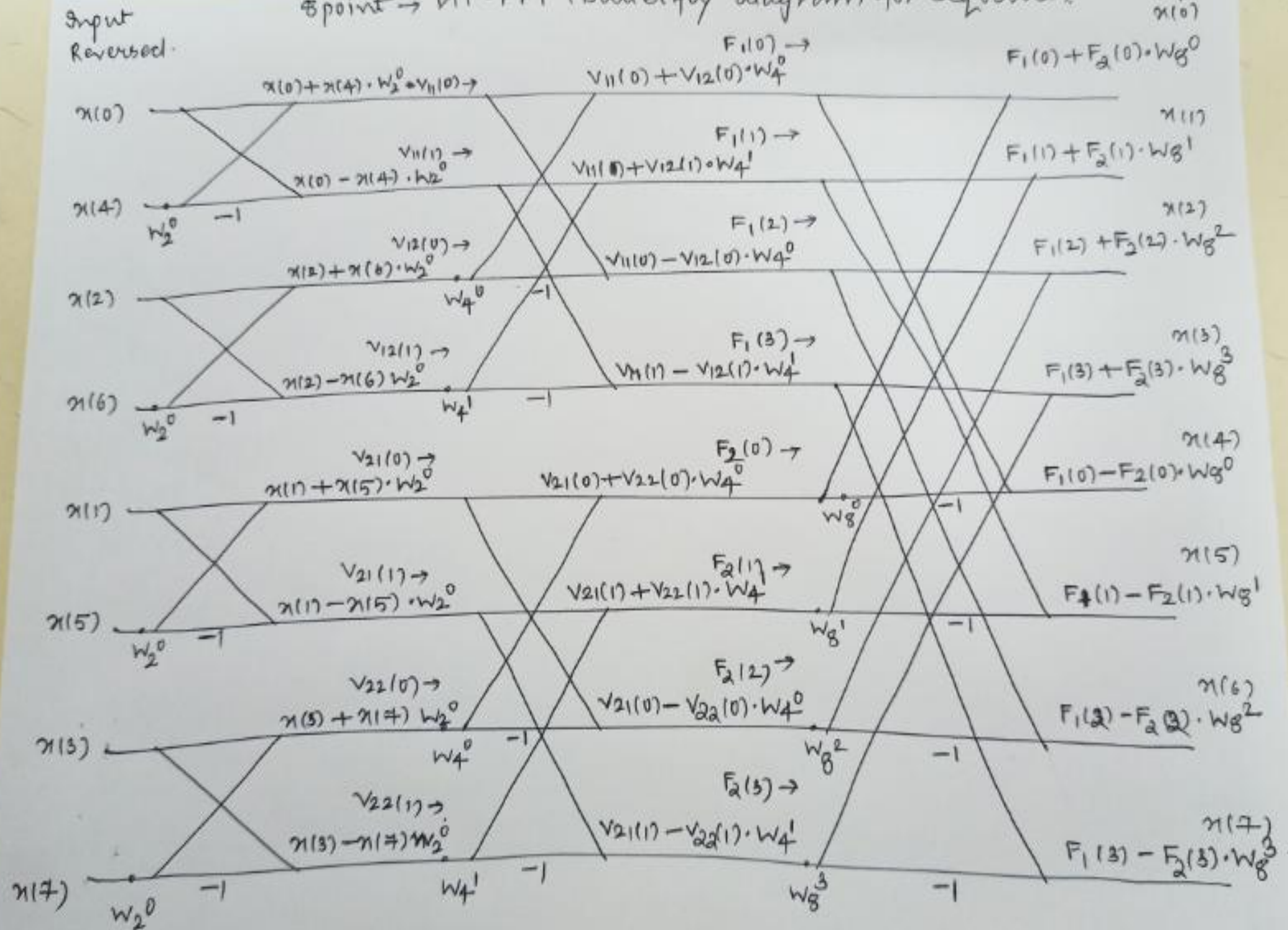
- Twiddle factor is given as
- W = e^(-j2*pi)
- Twiddle factor can be multiplied or divided by any integer.

DIT-FFT

- Steps to find DFT using Radix 2 DIT FFT:



8 point → DIT - FFT (Butterfly diagram for sequence)



2 point 4 point 8

$W_2^0 = 1$ $W_4^0 = 1$

$W_4^1 = -j$

$$W_N^k = e^{\frac{-j2\pi k}{N}}$$

$$W_2^0 = e^0 = 1$$

$$W_4^0 = e^0 = 1$$

$$W_4^1 = e^{\frac{-j2\pi(1)}{4}}$$

$e^{-j\pi/2}$ $\cos\pi/2 - j\sin\pi/2$

4 point

$W_4^0 = 1$

$W_4^1 = -j$

8 point

$W_8^0 = 1$

$W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$W_8^2 = -j$

$W_8^3 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

Table 5.2 : Comparison of Number of Comput

Number of points N	Direct Computation		Radix-2 FFT	
	Complex additions $N(N-1)$	Complex Multiplications N^2	Complex additions $N\log_2 N$	Complex Multiplications $(N/2)\log_2 N$
$4\,(=2^2)$	12	16	$4 \times \log_2 2^2 = 4 \times 2 = 8$	$\frac{4}{2} \times \log_2 2^2 = \frac{4}{2} \times 2 = 4$
$8\,(=2^3)$	56	64	$8 \times \log_2 2^3 = 8 \times 3 = 24$	$\frac{8}{2} \times \log_2 2^3 = \frac{8}{2} \times 3 = 12$
$16\,(=2^4)$	240	256	$16 \times \log_2 2^4 = 16 \times 4 = 64$	$\frac{16}{2} \times \log_2 2^4 = \frac{16}{2} \times 4 = 32$
$32\,(=2^5)$	992	1,024	$32 \times \log_2 2^5 = 32 \times 5 = 160$	$\frac{32}{2} \times \log_2 2^5 = \frac{32}{2} \times 5 = 80$
$64\,(=2^6)$	4,032	4,096	$64 \times \log_2 2^6 = 64 \times 6 = 384$	$\frac{64}{2} \times \log_2 2^6 = \frac{64}{2} \times 6 = 192$
$128\,(=2^7)$	16,256	16,384	$128 \times \log_2 2^7 = 128 \times 7 = 896$	$\frac{128}{2} \times \log_2 2^7 = \frac{128}{2} \times 7 = 448$

Image Transforms

- Image transforms are extensively used in image processing and image analysis.

- Transform is basically a mathematical tool, which allows us to move from one domain to another domain (time domain to the frequency domain).

- migrate from one domain to another domain is to perform the task at hand in an easier manner.

Image Transforms

- Image transforms are useful for fast computation of convolution and correlation.
- The transforms do not change the information content present in the signal.
- Transforms play a significant role in various image-processing applications such as image analysis, image enhancement, image filtering and image compression.

NEED FOR TRANSFORM

(i) Mathematical Convenience

Convolution in time domain $\Longleftrightarrow$ Multiplication in the frequency domain

The complex convolution operation in time domain is equal to simple multiplication operation in the frequency domain.

NEED FOR TRANSFORM

(ii) *To Extract more Information*

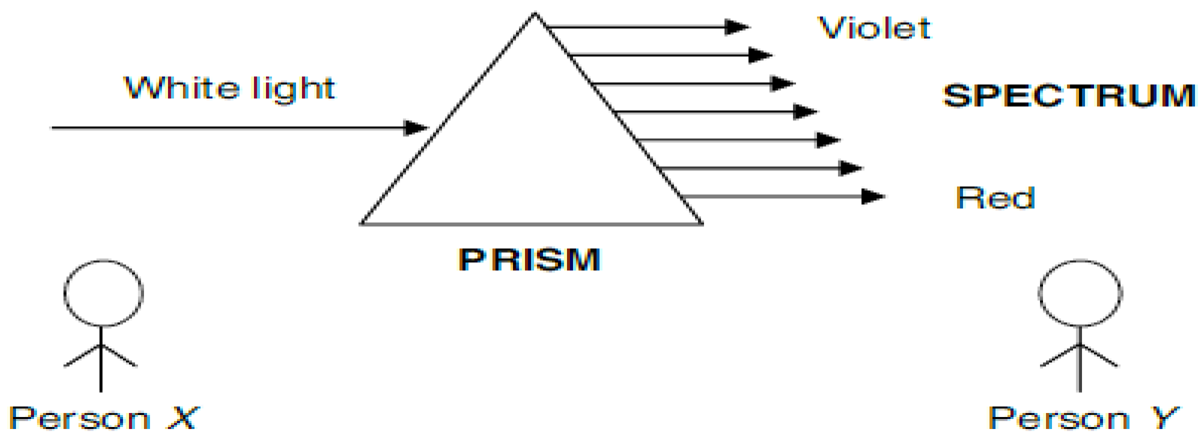


Fig. 4.1 *Spectrum of white light*

Fig. 4.2 *Concept of transformation*

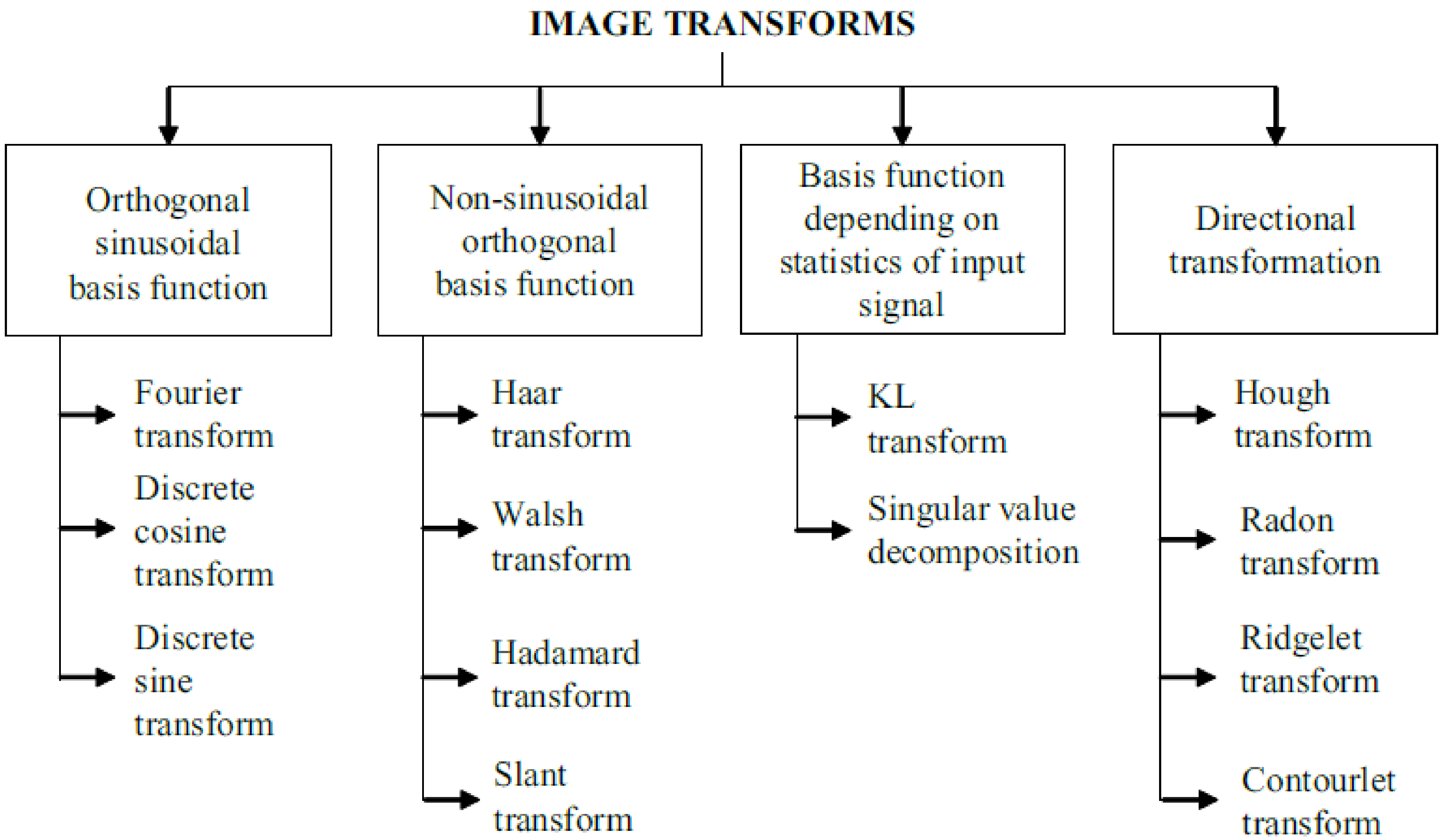

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Image Transform

1) Orthogonal sinusoidal basis function

- One of the most powerful transforms with orthogonal sinusoidal basis function is the Fourier transform.
- The transform which is widely used in the field of **image compression** is discrete cosine transform.

Image Transform

2) Non-sinusoidal orthogonal basis function

- The transforms whose basis functions are non-sinusoidal in nature.

- One of the important advantages of wavelet transform is that signals (images) can be **represented in different resolutions.**

Image Transform

3) Basis function depending on statistics of input signal

- The transform whose basis function depends on the statistics of input signal are KL transform and singular value decomposition.

- The KL transform is considered to be the best among all linear transforms with respect to **energy compaction**.

Image Transform

4) Directional transformation

- The transforms whose basis functions are effective in **representing the directional information of a signal** include Hough transform, Radon transform, Ridgelet transform and Contourlet transform.

Need for transform

- The need for transform is most of the signals or images are time domain signal (ie) signals can be measured with a function of time.
- This representation is not always best.
- For most image processing applications anyone of the mathematical transformation are applied to the signal or images to obtain further information from that signal.

Unitary Transform A discrete linear transform is unitary if its transform matrix conforms to the unitary condition

$$A \times A^{H} = I \tag{4.11}$$

where A = transformation matrix, A^{H} represents Hermitian matrix.

$$A^{H} = A^{*T}$$

I = identity matrix

When the transform matrix A is unitary, the defined transform is called unitary transform.

Orthogonal DFT

- If A is unitary and has only real elements , then it is orthogonal matrix

- A is orthogonal if A.A’ = I

Image Transform

1) Check whether the DFT matrix is unitary or not.

Solution

Step 1* Determination of the matrix *A

Finding 4-point DFT (where $N = 4$)

The formula to compute a DFT matrix of order 4 is given below.

$$X(K) = \sum_{n=0}^{3} x(n)e^{-j\frac{2\pi}{4}kn} \quad \text{where } k = 0, 1..., 3$$

1. *Finding* $X(0)$

$$X(0) = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3)$$

2. *Finding* $X(1)$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi}{2}n}$$

$$= x(0) + x(1)e^{-j\frac{\pi}{2}} + x(2)e^{-j\pi} + x(3)e^{-j\frac{3\pi}{2}}$$

$$X(1) = x(0) - jx(1) - x(2) + jx(3)$$

3. *Finding* $X(2)$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n}$$

$$= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$X(2) = x(0) - x(1) + x(2) - x(3)$$

4. *Finding* $X(3)$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j\frac{3\pi}{2}n}$$

$$= x(0) + x(1)e^{-j\frac{3\pi}{2}} + x(2)e^{-j3\pi} + x(3)e^{-j\frac{9\pi}{2}}$$

$$\boldsymbol{X}(3) = x(0) + jx(1) - x(2) - jx(3)$$

Collecting the coefficients of $X(0)$, $X(1)$, $X(2)$ and $X(3)$, we get

$$X[k] = A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Step 2 Computation of A^H

To determine A^H, first determine the conjugate and then take its transpose.

$$A \xrightarrow{\text{Conjugate}} A^* \xrightarrow{\text{Transpose}} A^H$$

Step 2a Computation of conjugate A^*

$$A^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

Step 2b Determination of transpose of A^*

$$\left(A^*\right)^T = A^{\mathrm{H}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

***Step 3* Determination of $A \times A^H$**

$$A \times A^{\mathrm{H}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 4 \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The result is the identity matrix, which shows that Fourier transform satisfies unitary condition.

Unitary Transform

- unitary transformation preserves the **signal energy**

- Most unitary transforms pack a **large fraction of the energy of the image into relatively few of the transform coefficients.**

- Few of the transform coefficients have significant values and these are the coefficients that are close to the origin

DFT of Image Matrix (2D)

Example 4.4 *Compute the 2D DFT of the 4 × 4 grayscale image given below.*

$$f[m, n] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Solution The 2D DFT of the image $f[m, n]$ is represented as $F[k, l]$.

$$F[k, l] = \text{kernel} \times f[m, n] \times (\text{kernel})^T$$

The kernel or basis of the Fourier transform for $N = 4$ is given by

$$\text{The DFT basis for } N = 4 \text{ is given by} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

DFT of Image Matrix (2D)

$$F(k,l) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$F(k,l) = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 4.5 *Compute the inverse 2D DFT of the transform coefficients given by*

$$F[k,l] = \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution The inverse 2D DFT of the Fourier coefficients $F[k, l]$ is given by $f[m, n]$ as

$$f[m,n]=\frac{1}{N^2}\times \text{kernel}\times F[k,l]\times(\text{kernel})^T$$

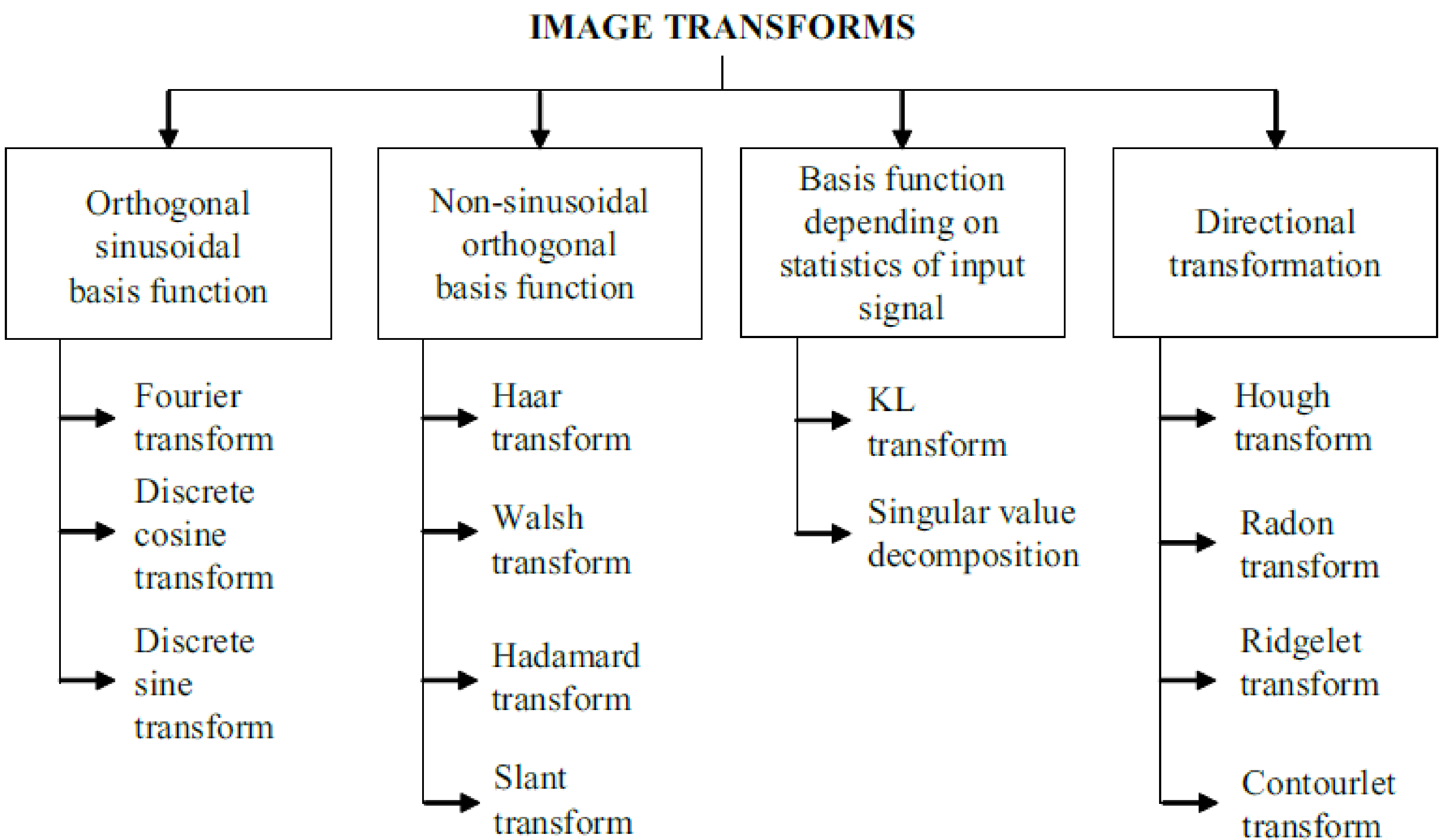
In this example, $N = 4$

$$f[m,n]=\frac{1}{16}\times\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{bmatrix}\times\begin{bmatrix}16 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\times\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{bmatrix}$$

$$f[m,n]=\frac{1}{16}\times\begin{bmatrix}16 & 0 & 0 & 0\\ 16 & 0 & 0 & 0\\ 16 & 0 & 0 & 0\\ 16 & 0 & 0 & 0\end{bmatrix}\times\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{bmatrix}$$

$$f[m,n]=\frac{1}{16}\times\begin{bmatrix}16 & 16 & 16 & 16\\ 16 & 16 & 16 & 16\\ 16 & 16 & 16 & 16\\ 16 & 16 & 16 & 16\end{bmatrix}$$

$$f[m,n]=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$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Discrete Cosine Transform (DCT)

- The discrete cosine transforms are the members of a family of real-valued discrete sinusoidal unitary transforms.

- A discrete cosine transform consists of a set of basis vectors that are sampled cosine functions.

- DCT is a technique for converting a signal into elementary frequency components and it is widely used in image compression.

Thus, the kernel of a one-dimensional discrete cosine transform is given by

$$X[k] = \alpha(k) \sum_{n=0}^{N-1} x[n] \cos\left\{\frac{(2n+1)\pi k}{2N}\right\} \text{ , where } 0 \leq k \leq N-1$$

$$\alpha(k) = \sqrt{\frac{1}{N}} \text{ if } k = 0$$

$$\alpha(k) = \sqrt{\frac{2}{N}} \text{ if } k \neq 0$$

Example 4.10 *Compute the discrete cosine transform (DCT) matrix for N = 4.*

Solution The formula to compute the DCT matrix is given by

$$X[k] = \alpha(k)\sum_{n=0}^{N-1} x[n]\cos\left\{\frac{(2n+1)\pi k}{2N}\right\}, \text{where } 0 \le k \le N-1$$

where

$$\alpha(k) = \sqrt{\frac{1}{N}} \text{ if } k = 0$$

$$\alpha(k) = \sqrt{\frac{2}{N}} \text{ if } k \neq 0$$

In our case, the value of $N = 4$. Substituting $N = 4$ in the expression of $X[k]$ we get

$$X[k] = \alpha(k)\sum_{n=0}^{3} x[n]\cos\left[\frac{(2n+1)\pi k}{8}\right]$$

Substituting $k = 0$ in Eq. (4.82), we get

$$X[0] = \sqrt{\frac{1}{4}} \times \sum_{n=0}^{3} x[n]\cos\left[\frac{(2n+1)\pi \times 0}{8}\right]$$

$$= \frac{1}{2} \times \sum_{n=0}^{3} x[n]\cos(0) = \frac{1}{2} \times \sum_{n=0}^{3} x[n] \times 1$$

$$= \frac{1}{2} \times \sum_{n=0}^{3} x[n]$$

$$= \frac{1}{2} \times \{x(0) + x(1) + x(2) + x(3)\}$$

$$X[0] = \frac{1}{2}x(0) + \frac{1}{2}x(1) + \frac{1}{2}x(2) + \frac{1}{2}x(3)$$

Substituting $k = 1$ in Eq. (4.82), we get

$$X[1] = \sqrt{\frac{2}{4}} \times \sum_{n=0}^{3} x[n] \cos\left[\frac{(2n+1)\pi \times 1}{8}\right]$$

$$= \sqrt{\frac{1}{2}} \times \sum_{n=0}^{3} x[n] \cos\left[\frac{(2n+1)\pi}{8}\right]$$

$$= \sqrt{\frac{1}{2}} \times \left\{x(0) \times \cos\left(\frac{\pi}{8}\right) + x(1) \times \cos\left(\frac{3\pi}{8}\right) + x(2) \times \cos\left(\frac{5\pi}{8}\right) + x(3) \times \cos\left(\frac{7\pi}{8}\right)\right\}$$

$$= 0.707 \times \{x(0) \times 0.9239 + x(1) \times 0.3827 + x(2) \times (-0.3827) + x(3) \times (-0.9239)\}$$

$$X(1) = 0.6532x(0) + 0.2706x(1) - 0.2706x(2) - 0.6532x(3)$$

Substituting $k = 2$ in Eq. (4.82), we get

$$X(2) = \sqrt{\frac{2}{4}} \sum_{n=0}^{3} x[n] \cos\left[\frac{(2n+1)\pi \times 2}{8}\right]$$

$$= \sqrt{\frac{1}{2}} \times \sum_{n=0}^{3} x[n] \cos\left[\frac{(2n+1)\pi}{4}\right]$$

$$= \sqrt{\frac{1}{2}} \times \left\{x(0) \times \cos\left(\frac{\pi}{4}\right) + x(1) \times \cos\left(\frac{3\pi}{4}\right) + x(2) \times \cos\left(\frac{5\pi}{4}\right) + x(3) \times \cos\left(\frac{7\pi}{4}\right)\right\}$$

$$= 0.7071 \times \{x(0) \times 0.7071 + x(1) \times (-0.7071) + x(2) \times (-0.7071) + x(3) \times (0.7071)\}$$

$$X(2) = 0.5x(0) - 0.5x(1) - 0.5x(2) + 0.5x(3)$$

Substituting $k = 3$ in Eq. (4.82), we get

$$X(3) = \sqrt{\frac{2}{4}}\sum_{n=0}^{3} x[n]\cos\left[\frac{(2n+1)\pi \times 3}{8}\right]$$

$$= \sqrt{\frac{1}{2}} \times \sum_{n=0}^{3} x[n]\cos\left[\frac{(2n+1)\times 3\pi}{8}\right]$$

$$= \sqrt{\frac{1}{2}} \times \left\{x(0)\times\cos\left(\frac{3\pi}{8}\right) + x(1)\times\cos\left(\frac{9\pi}{8}\right) + x(2)\times\cos\left(\frac{15\pi}{8}\right) + x(3)\times\cos\left(\frac{21\pi}{8}\right)\right\}$$

$$= 0.7071\times\left\{x(0)\times 0.3827 + x(1)\times(-0.9239) + x(2)\times(0.9239) + x(3)\times(-0.3827)\right\}$$

$$X(3) = 0.2706x(0) - 0.6533x(1) + 0.6533x(2) - 0.2706x(3)$$

Collecting the coefficients of $x(0)$, $x(1)$, $x(2)$ and $x(3)$ from $X(0)$, $X(1)$, $X(2)$ and $X(3)$, we get

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.6532 & 0.2706 & -0.2706 & -0.6532 \\ 0.5 & -0.5 & -0.5 & 0.5 \\ 0.2706 & -0.6533 & 0.6533 & -0.2706 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

IDCT

$$x[n] = \alpha(k) \sum_{k=0}^{N-1} X[k] \cos\left[\frac{(2n+1)\pi k}{2N}\right], 0 \leq n \leq N-1$$

Symmetric and Asymmetric

- **Transformation Matrix – T**
- **Image Matrix – f**
- **If Transformation matrix is symmetric:**
- **F = TfT**

- **If Transformation matrix is asymmetric:**
- **F = TfT’**

Hadamard Transform

- Hadamard Transform is based on Hadamard Matrix, square array having +1 and-1 entries.
- Hadamard Matrix of order of 2is given by

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

Hadamard Transform

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The Hadamard matrix of order $2N$ can be generated by Kronecker product operation:

$$H_{2N} = \begin{bmatrix} H_N & H_N \\ H_N & -H_N \end{bmatrix}$$

Substituting $N = 2$

$$H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Similarly, substituting $N = 4$

$$H_8 = \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

Hadamard Transform

- The rows and columns of the HM are orthogonal.
- For orthogonality of Vectors the dot product has to be zero.

First row $\rightarrow [1 \quad 1]$

Second row $\rightarrow [1 \; -1]$

$\therefore [1 \quad 1] \cdot [1 \; -1]' = 0$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

A is orthogonal if A.A’ = I

Hadamard Transform

- Is the Hadamard matrix Orthogonal?
- Hence Hadamard Matrix is orthogonal but we get constant 2 means that it is not normalized

Hadamard Transform

- A Normalized 2X2 Hadamard transform is done by multiplying with $\frac{1}{\sqrt{2}}$ and is given by
- A = $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

- Check if A is orthogonal?
- Is H(4) Orthogonal and normalized?

Hadamard Transform

- If x(n) is a N point 1D sequence of finite valued real numbers arranged in a column the Hadamard transform sequence is given by
- **X[K]={H(N).x(n)}**
- Inverse Hadamard of Sequence
- **x[n] = 1/N{H(N).X(K)}**

Hadamard Transform

- Compute the Hadamard transform of Sequence {1, 2, 0,3}'
- Compute the inverse Hadamard transform of Sequence {6, -4, 0,2}'

Here N = 4, H(N) is a 4 × 4 matrix in this case.

$$X[n] = [H(N) \cdot x(n)]$$

$$X[n] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}$$

$$\therefore X[n] = \begin{bmatrix} 6 \\ -4 \\ 0 \\ 2 \end{bmatrix}$$

Hadamard Transform

- Hadamard Transform of image (2D)
- Hadamard is symmetric transform
- **F = TfT = [H(N) f H(N)]**
- **f = NxN image**
- **F = Transformed Image**
- **Compute the Hadamard Transform of the Image shown below**

2	1	2	1
1	2	3	2
2	3	4	3
1	2	3	2

1	1	1	1
1	1	0	1
−1	1	−1	1
1	0	1	1

Hadamard Transform

$$\begin{bmatrix} 34 & 2 & -6 & -6 \\ 2 & 2 & 2 & 2 \\ -6 & 2 & 2 & 2 \\ -6 & 2 & 2 & 2 \end{bmatrix}$$

10	−4	0	2
−2	−4	0	−2
4	2	2	0
4	6	−2	0

An image matrix is given by $f(m,n) = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix}$. Find the 2D Hadamard transform for this image matrix.

Hadamard Transform

$$H_8 = \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix}$$

$$H_8 = \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

$$H_8 = \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix} = \left[\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{matrix}\right] \begin{matrix} 0 \\ 7 \\ 3 \\ 4 \\ 1 \\ 6 \\ 2 \\ 5 \end{matrix} \Rightarrow W(8) = \left[\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{matrix}\right] \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix}$$

Walsh Transform

$$W(8) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix},$$

Walsh Transform is obtained from Hadamard Transform matrix by rearranging the rows in the increasing sign change order.

Walsh Transform

- Walsh transform can calculated using the matrix as follows:
- **X[K]={W(N).x(n)}**
- Inverse Walsh of Sequence
- **x[n] = {W(N).X(K)}/N**

Walsh transform

- Compute the Walsh transform of Sequence {1, 2, 0,3}’
- **X[n]={W(N).x(n)}**
- W(4) $= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$ ⇨ No. of Sign Change (Sequency Ordering)

- Compute the inverse Walsh transform of Sequence {6, -4, 0,2}’

Walsh Transform

- Walsh Transform of image (2D)
- Walsh is symmetric transform
- **F = TfT = [W(N) f W(N)]**
- **f = NxN image**
- **F = Transformed Image**
- **Compute the Walsh Transform of the Image shown below**

2	1	2	1
1	2	3	2
2	3	4	3
1	2	3	2

1	1	1	1
1	1	0	1
−1	1	−1	1
1	0	1	1

Walsh Transform

Since W(4) is symmetric.

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$\therefore$ F =

34	−6	−6	2
−6	2	2	2
−6	2	2	2
2	2	2	2

Haar Transform

- Haar Transform is derived from Haar matrix.
- Expressed in matrix form as
- $F = HfH'$

Haar Transform

- For N=2 dan N=4:

$$\mathbf{Hr}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{Hr}_4 = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

- Properties of Haar Transform:
  1. Real and orthogonal: $\mathbf{Hr} = \mathbf{Hr}^*$ dan $\mathbf{Hr}^{-1} = \mathbf{Hr}^T$
  2. Very fast transform : O(N) operation on Nx1 vector.
  3. Poor energy compaction for images

Haar Transform

$$H_3 = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

The Haar transform coefficients of a n=4-point signal $x_4 = [1, 2, 3, 4]^T$ can be found as

$$y_4 = H_4 x_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

The input signal can then be perfectly reconstructed by the inverse Haar transform

$$\hat{x_4} = H_4^T y_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 5 \\ -2 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Compute the Haar transform of Sequence {1, 2, 0,3}’

$$X[n] = [\mathrm{Haar}(N) \cdot x(n)]$$

$$= \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}$$

$$X[n] = \frac{1}{\sqrt{4}} \{6, 0, -\sqrt{2}, -3, \sqrt{2}\}$$

Haar Transform

- Haar Transform of image (2D)
- Haar is asymmetric transform
- **F = [Haar(N) f Haaar(N)’]**
- **f = NxN image**
- **F = Transformed Image**
- **Compute the Haar Transform of the Image shown below**

2	1	2	1
1	2	3	2
2	3	4	3
1	2	3	2

1	1	1	1
1	1	0	1
−1	1	−1	1
1	0	1	1

Haar Transform

$$F = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix}$$

$$\times\frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

$$F = \begin{bmatrix} 8.5 & -1.5 & -0.707 & 1.414 \\ -1.5 & 0.5 & 0.7071 & 0 \\ -0.7071 & 0.7071 & 1.00 & 0 \\ 1.414 & 0.00 & 0.00 & 0.00 \end{bmatrix}$$

IMAGE ENHANCEMENT IN THE FREQUENCY DOMAIN

$$\boxed{\textbf{Convolution in spatial domain = Multiplication in the frequency domain}}$$

$$\text{Filtering in spatial domain} = f(m,n) * h(m,n)$$

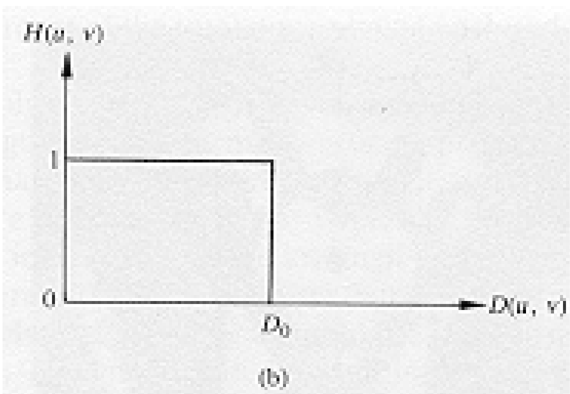
$$\text{Filtering in the frequency domain} = F(k,l) \times H(k,l)$$

f(m,n) – original image
F(k,l) – FT of original image
h(m,n) – Filtering mask
H(k,l) – FT of filtering mask

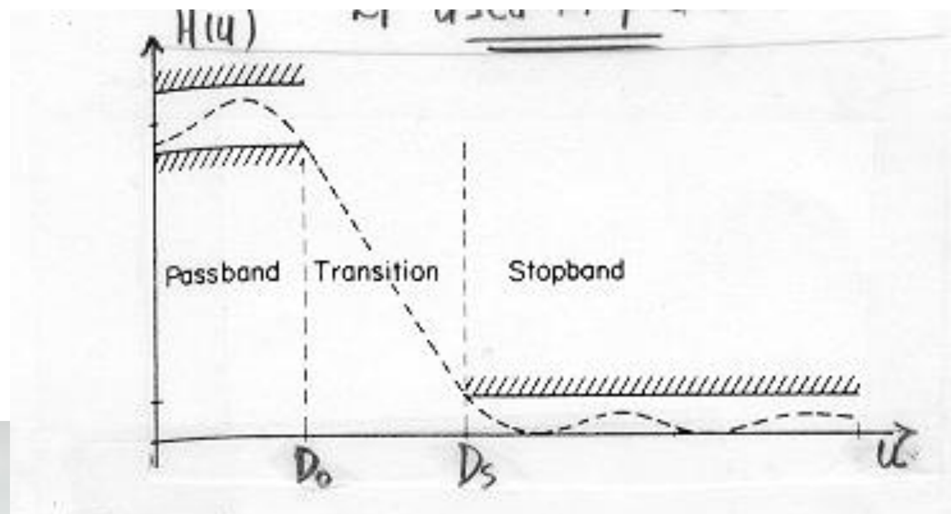
Low-pass (LP) filtering

- Preserves low frequencies, attenuates high frequencies.

ideal



in practice

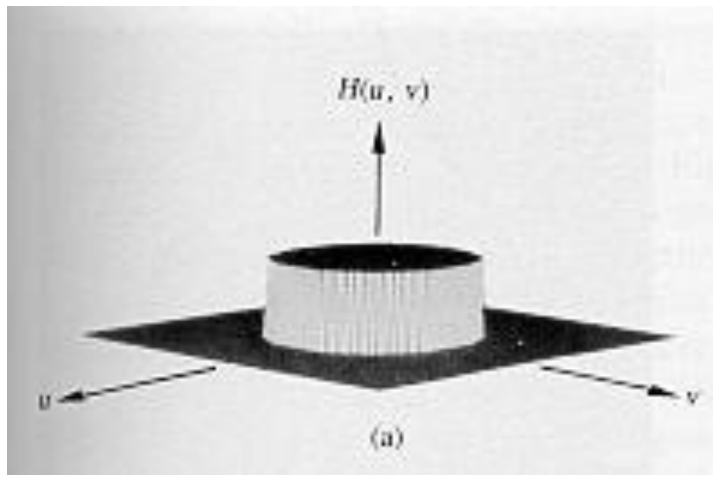


$$H(u, v) = 1; \text{ if } D(u, v) \le D_0$$
$$= 0; \text{ if } D(u, v) > D_0$$

D_0: cut-off frequency

Lowpass (LP) filtering (cont'd)

- In 2D, the cutoff frequencies lie on a circle.



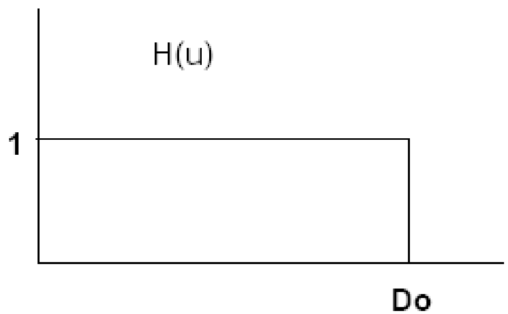
$$H(u,v)=\begin{cases}1 & \text{if } u^2+v^2\le {D_0}^2\\ 0 & \textit{otherwise}\end{cases}$$

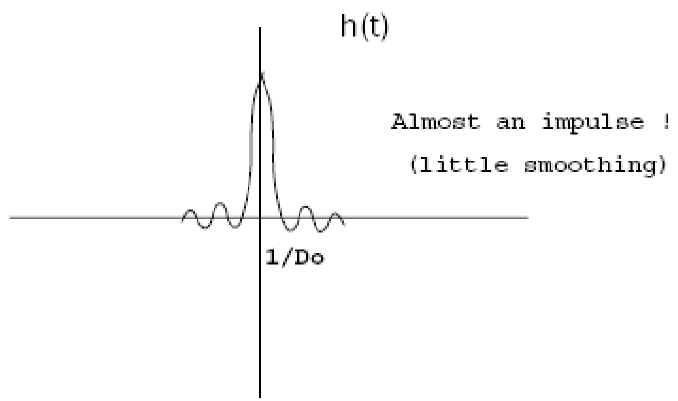
Low-pass (LP) filtering

- D0 is the cut-off frequency of the low-pass filter.
- The cut-off frequency determines the amount of frequency components passed by the filter.
- The smaller the value of D0 , the greater the number of image components eliminated by the filter.
- The value of D0 is chosen in such a way that the components of interest are passed through.

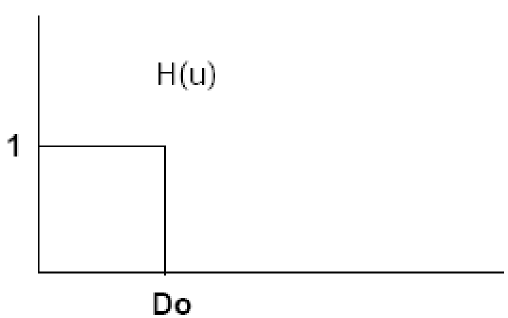
How does D_0 control smoothing? (cont'd)

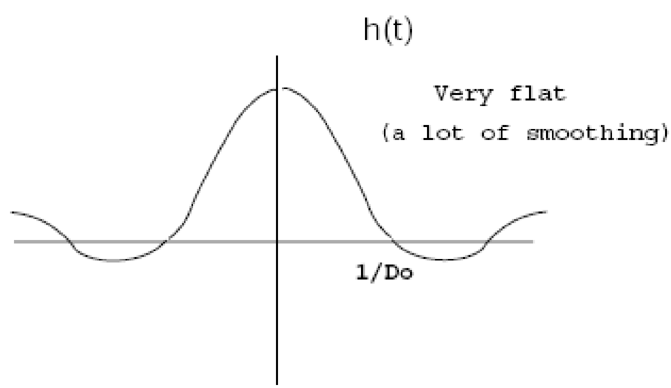
- D_0 controls the amount of blurring





r=78 (99%)

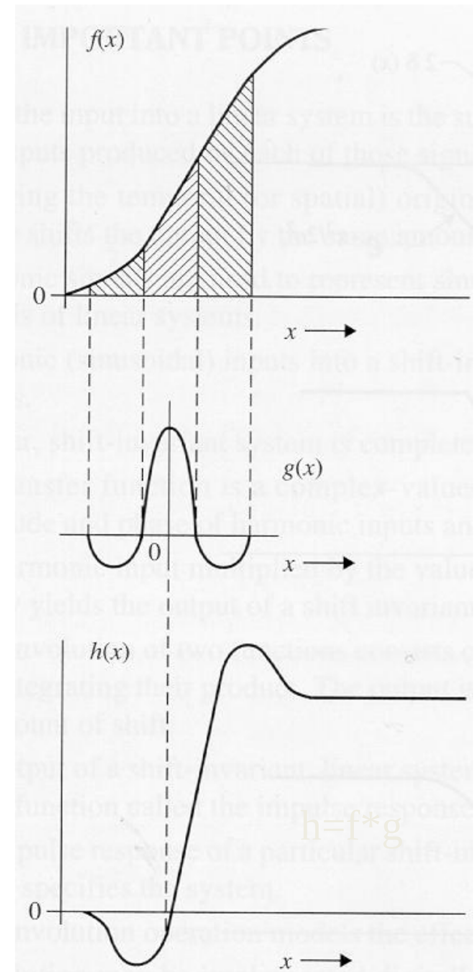


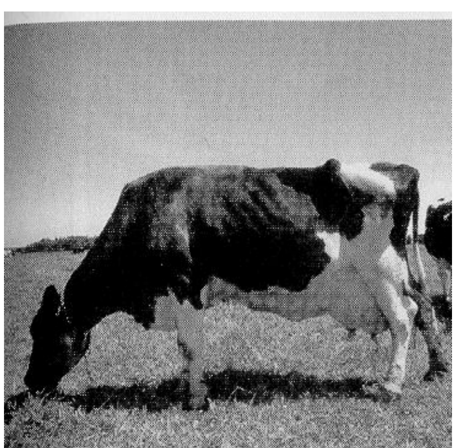


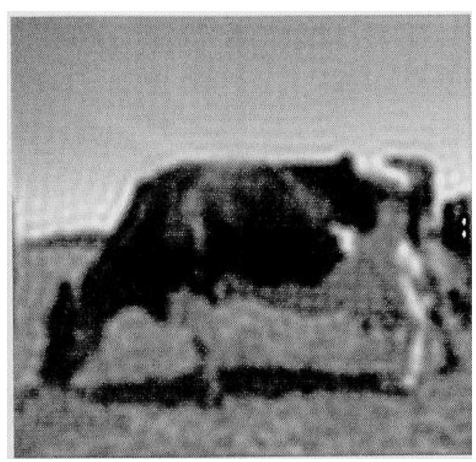
r=8 (90%)

Ringing Effect

- Sharp cutoff frequencies produce an overshoot of image features whose frequency is close to the cutoff frequencies **(ringing effect).**







Low Pass (LP) Filters

- Ideal low-pass filter (ILPF)
- Butterworth low-pass filter (BLPF)
- Gaussian low-pass filter (GLPF)

Butterworth Low-pass Filter The transfer function of a two-dimensional Butterworth low-pass filter is given by

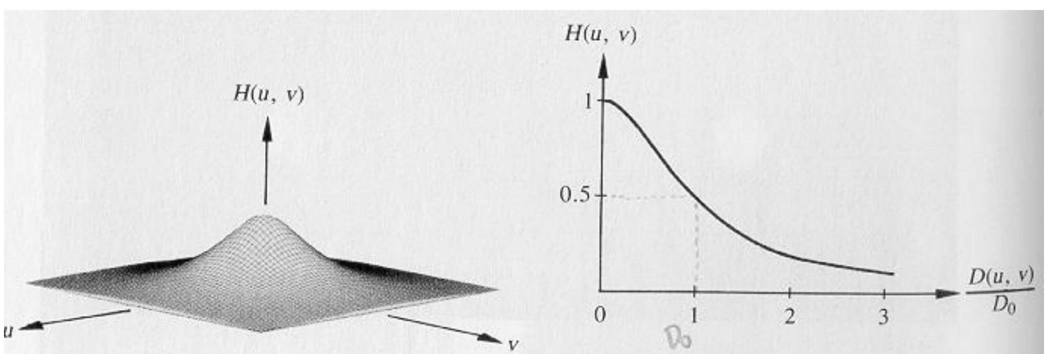
$$H(k,l)=\frac{1}{1+\left[\frac{\sqrt{k^2+l^2}}{D_0}\right]^{2n}} \tag{5.18}$$

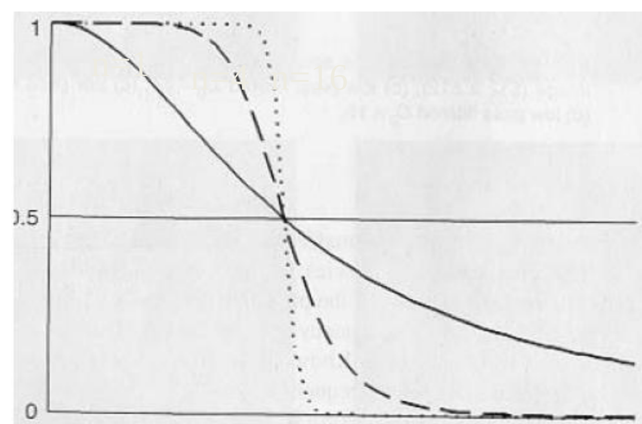
In the above expression, n refers to the filter order and D_0 represents the cut-off frequency.

Butterworth LP filter (BLPF)

- In practice, we use filters that <u>attenuate high frequencies smoothly</u> (e.g., **Butterworth** LP filter) → less ringing effect

$$H(u,v) = \frac{1}{1 + [\sqrt{u^2 + v^2}/D_0]^{2n}}$$





BLPF first order with D0 =10, 20,30

- close all;: Closes all open figures.
- clear all;: Clears all variables from the workspace.
- clc;: Clears the command window.
- n=1;: Sets the order of the Butterworth filter (first-order filter).
- Fourier Transform of the Image
- fft2(im);: Computes the 2D Fast Fourier Transform (FFT) of the image, which converts the image from the spatial domain to the frequency domain.
- fftshift(imf);: Shifts the zero-frequency component to the center of the frequency domain.

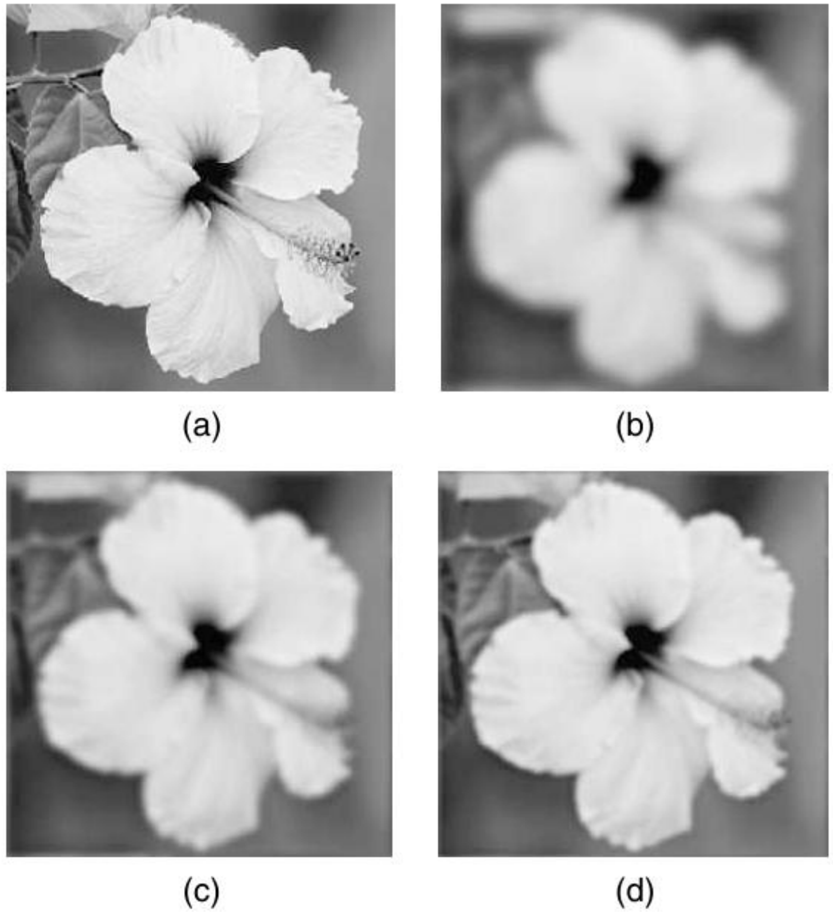
- Designing the Butterworth Low-Pass Filter
- H=zeros(co,ro);: Initializes a filter mask with zeros of the same size as the image.
- The nested for loop iterates over all pixels of the image:
- d = (i-cx).^2 + (j-cy).^ 2;
- %Computes the squared distance from the center of the frequency plane.
- H(i,j) = 1/(1+((d/fc/fc).^(2*n)));
- % Defines the Butterworth low-pass filter at each pixel. The filter attenuates frequencies higher than the cutoff frequency fc.

- Applying the Filter
- outf = imf .* H;
- % Applies the Butterworth filter to the frequency domain representation of the image by multiplying the FFT of the image (imf) with the filter mask H.
- Inverse Fourier Transform and Display
- out = abs(ifft2(outf));
- % Converts the filtered image back to the spatial domain using the Inverse Fast Fourier Transform imshow(im),title('Original Image');
- figure,imshow(uint8(out)),title('Lowpass Filtered Image');: Displays the low-pass filtered image.

MATLAB code to perform a two-dimensional Butterworth low-pass filter

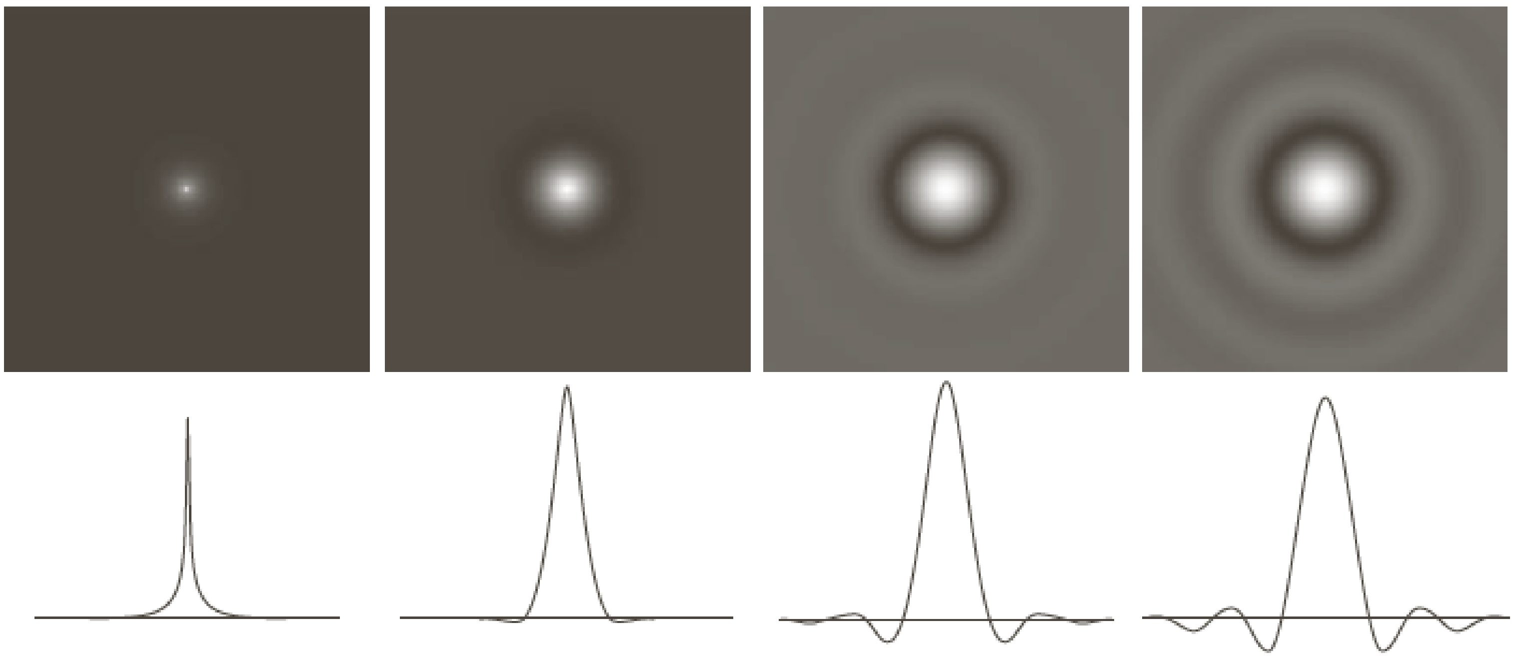
```
%This code is used to Butterworth lowpass fi lter
close all;  clear all; clc;
 im=imread('d:\work\hibiscus.tif');
 fc=20;%Cutoff frequency
 n=1;
 [co,ro] = size(im);
 cx = round(co/2); % find the centre of the image
 cy = round (ro/2); %Compute the coordinates of the image center.
 imf=fftshift(fft2(im));
 H=zeros(co,ro);
 for i = 1 : co
    for j =1 : ro
        d = (i-cx).^2 + (j-cy).^ 2;
        H(i,j) = 1/(1+((d/fc/fc).^(2*n)));
    end;
 end;
 outf = imf .* H;
 out = abs(ifft2(outf));
 imshow(im),title('Original Image'),
 f igure,imshow(uint8(out)),title('Lowpass Filterd Image')
```

(a) Original image
(b) Low-pass filtered image with D0 = 10
(c) Low-pass filtered image with D0 = 20
(d) Low-pass filtered image with D0 = 30



Spatial Representation of BLPFs

n=1 n=2 n=5 n=20

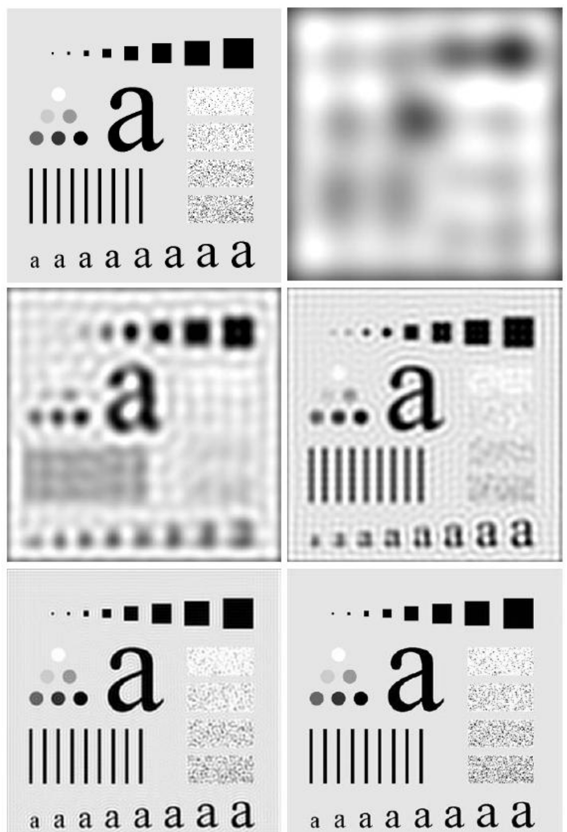


a b c d

FIGURE 4.46 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is 1000×1000 and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.

Comparison: Ideal LP and BLPF

ILPF

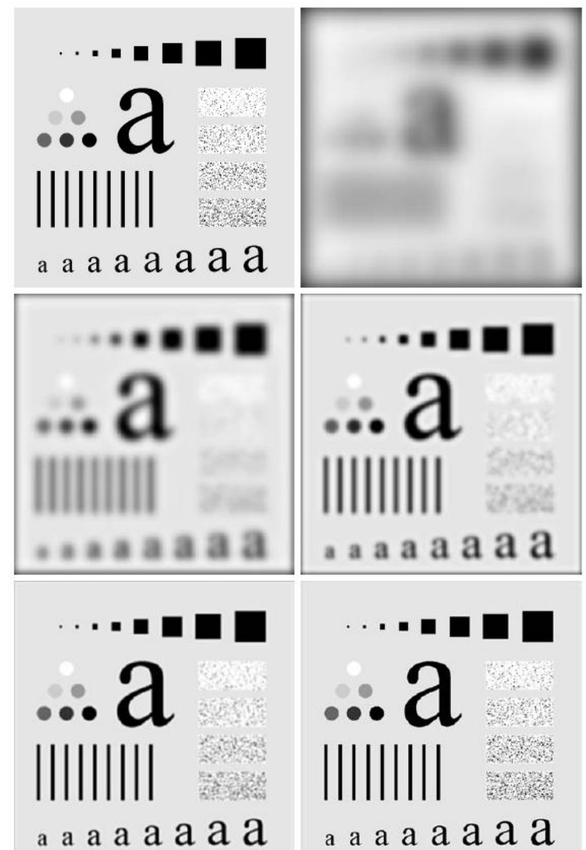


a b
c d
e f

FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

D_0=10, 30, 60, 160, 460

BLPF



a b
c d
e f

FIGURE 4.45 (a) Original image. (b)–(f) Results of filtering using BLPFs of order 2, with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Fig. 4.42.

D_0=10, 30, 60, 160, 460

n=2

Gaussian LP filter (GLPF)

Gaussian Lowpass Filters (GLPF) in two dimensions is given

$$H(u,v) = e^{-(u^2+v^2)/2\sigma^2}$$

By letting $\sigma = D_0$

$$H(u,v) = e^{-(u^2+v^2)/2D_0^2}$$

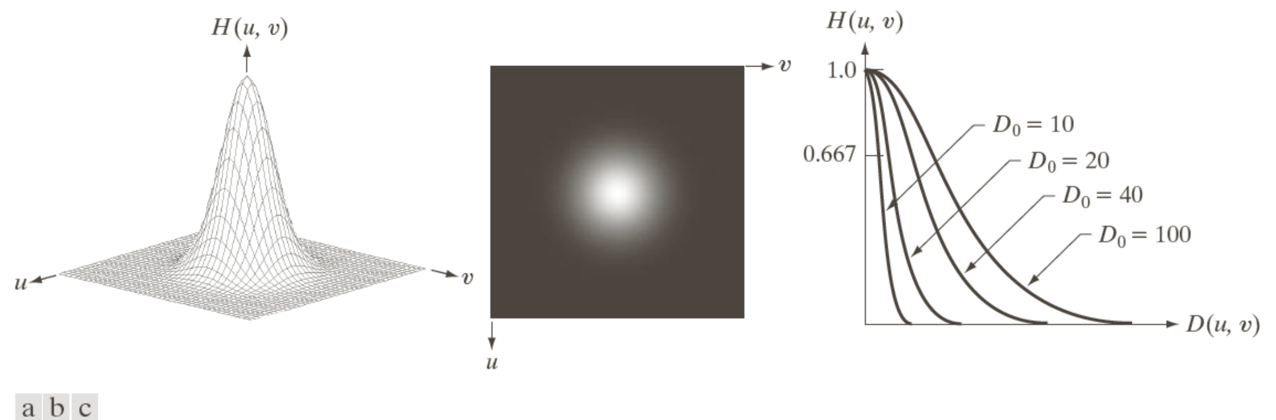
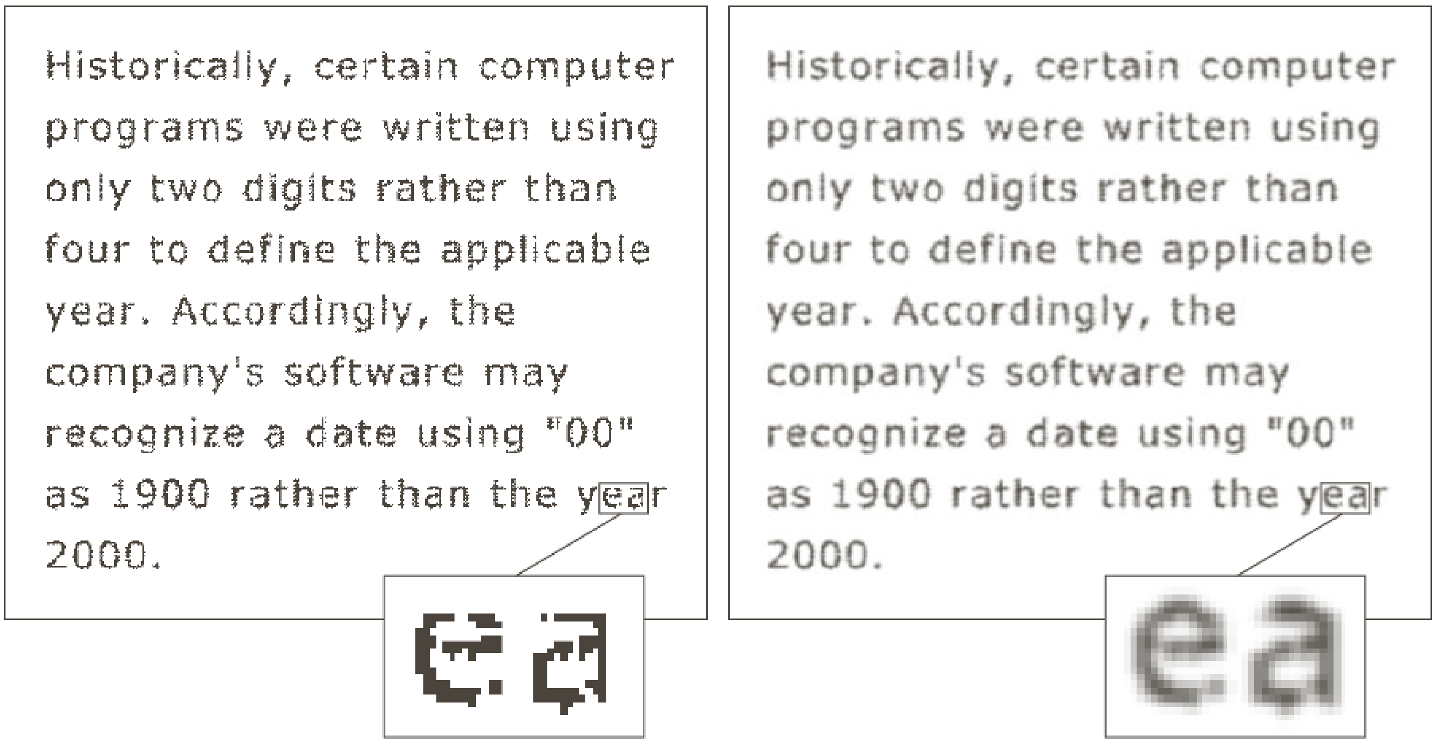


FIGURE 4.47 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

Example: smoothing by GLPF (1)



a b

FIGURE 4.49
(a) Sample text of low resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

Examples of smoothing by GLPF (2)

D_0=100

D_0=80

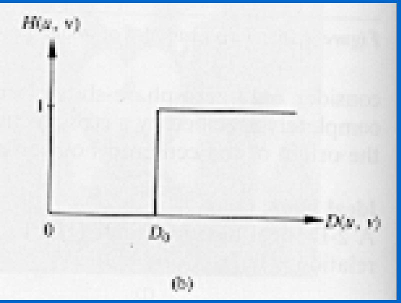
a b c

FIGURE 4.50 (a) Original image (784×732 pixels). (b) Result of filtering using a GLPF with $D_0 = 100$. (c) Result of filtering using a GLPF with $D_0 = 80$. Note the reduction in fine skin lines in the magnified sections in (b) and (c).

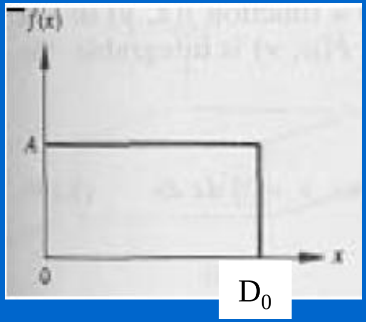
High-Pass filtering

- A high-pass filter can be obtained from a low-pass filter using:

$$H_{HP}(u,v) = 1 - H_{LP}(u,v)$$

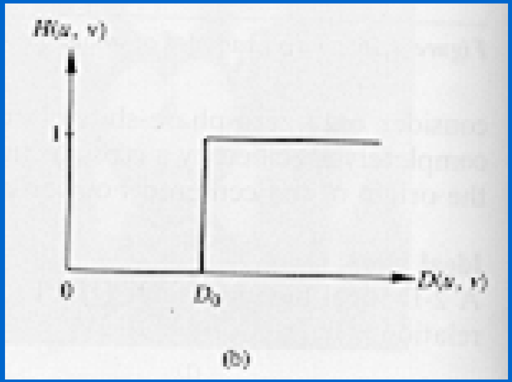


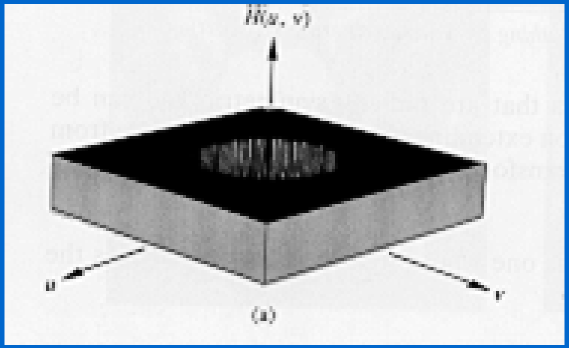
= 1 -



High-pass filtering (cont'd)

- Preserves high frequencies, attenuates low frequencies.





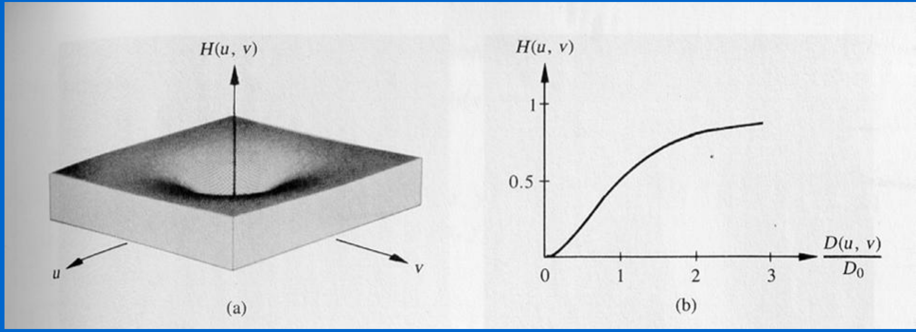
$$H(u,v) = \begin{cases} 1 & \text{if} u \geq D_0 \\ 0 & otherwise \end{cases}$$

$$H(u,v) = \begin{cases} 1 & \text{if} u^2 + v^2 \geq {D_0}^2 \\ 0 & otherwise \end{cases}$$

Butterworth high pass filter (BHPF)

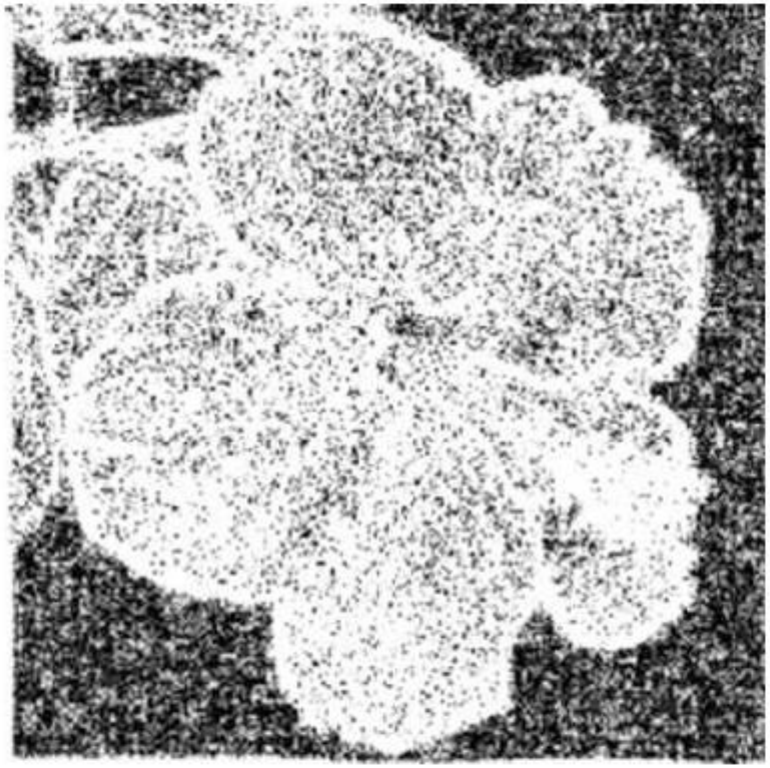
- In practice, we use filters that attenuate low frequencies smoothly (e.g., **Butterworth** HP filter) → less ringing effect

$$H(u,v) = \frac{1}{1 + [D_0/\sqrt{u^2+v^2}]^{2n}}$$



(a) (b)

output of the MATLAB code for two different cut-off frequencies D0 (a) fc = 40 and D0 = 80



MATLAB code to perform a two-dimensional Butterworth high-pass filter

- %This code is used to Butterworth highpass fi lter
- close all; clear all; clc;
- im=imread('d:\work\hibiscus.tif');
- fc=40;
- n=1;
- [co,ro] = size(im);
- cx = round(co/2); % fi nd the centre of the image
- cy = round (ro/2);
- imf=fftshift(fft2(im));
- H=zeros(co,ro);
- for i = 1 : co
- for j =1 : ro
- d = (i-cx).^2 + (j-cy).^ 2;
- if d ~= 0
- H(i,j) = 1/(1+((fc*fc/d).^(2*n)));
- end; end; end;
- outf = imf .* H;
- out = abs(ifft2(outf));
- imshow(im),title('Original Image'),fi gure,imshow(uint8(out)),title
- ('Highpass Filterd Image')
- figure,imshow(H),title('2D View of H'),fi gure,surf(H),
- title('3D View of H')

Comparison: IHPF and BHPF

IHPF

a b c

FIGURE 4.54 Results of highpass filtering the image in Fig. 4.41(a) using an IHPF with $D_0 = 30, 60$, and 160.

D_0=30,60,160

BHPF

a b c

FIGURE 4.55 Results of highpass filtering the image in Fig. 4.41(a) using a BHPF of order 2 with $D_0 = 30, 60$, and 160, corresponding to the circles in Fig. 4.41(b). These results are much smoother than those obtained with an IHPF.

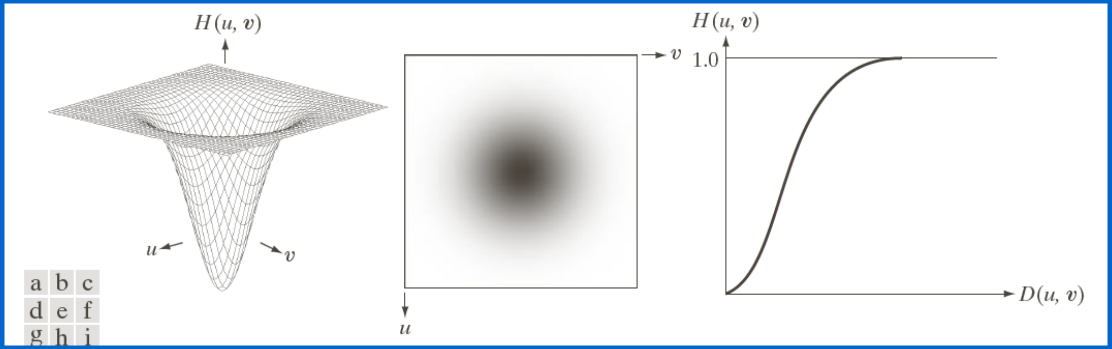
D_0=30,60,160

n=2

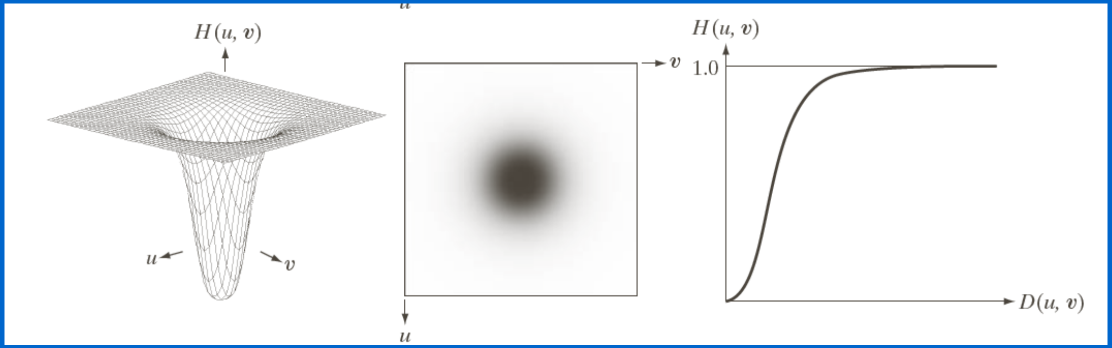
Gaussian HP filter

A 2-D Gaussian highpass filter (GHPL) is defined as

$$H(u,v) = 1 - e^{-(u^2+v^2)/2D_0^2}$$



GHPF



BHPF

Comparison: BHPF and GHPF

BHPF

a b c

FIGURE 4.55 Results of highpass filtering the image in Fig. 4.41(a) using a BHPF of order 2 with $D_0 = 30, 60,$ and 160, corresponding to the circles in Fig. 4.41(b). These results are much smoother than those obtained with an IHPF.

D_0=30,60,160

n=2

GHPF

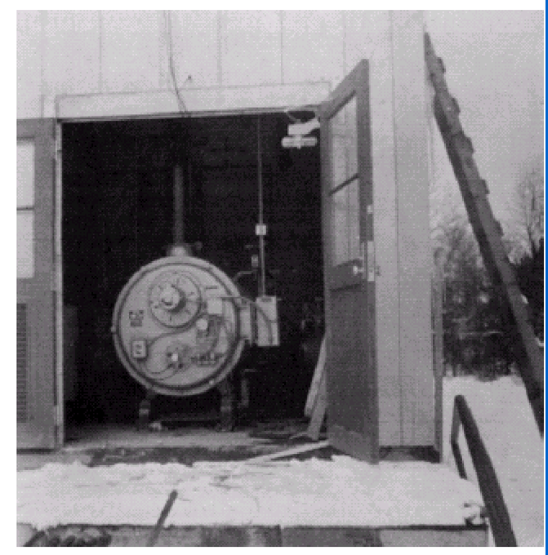
a b c

FIGURE 4.56 Results of highpass filtering the image in Fig. 4.41(a) using a GHPF with $D_0 = 30, 60,$ and 160, corresponding to the circles in Fig. 4.41(b). Compare with Figs. 4.54 and 4.55.

D_0=30,60,160

Homomorphic filtering

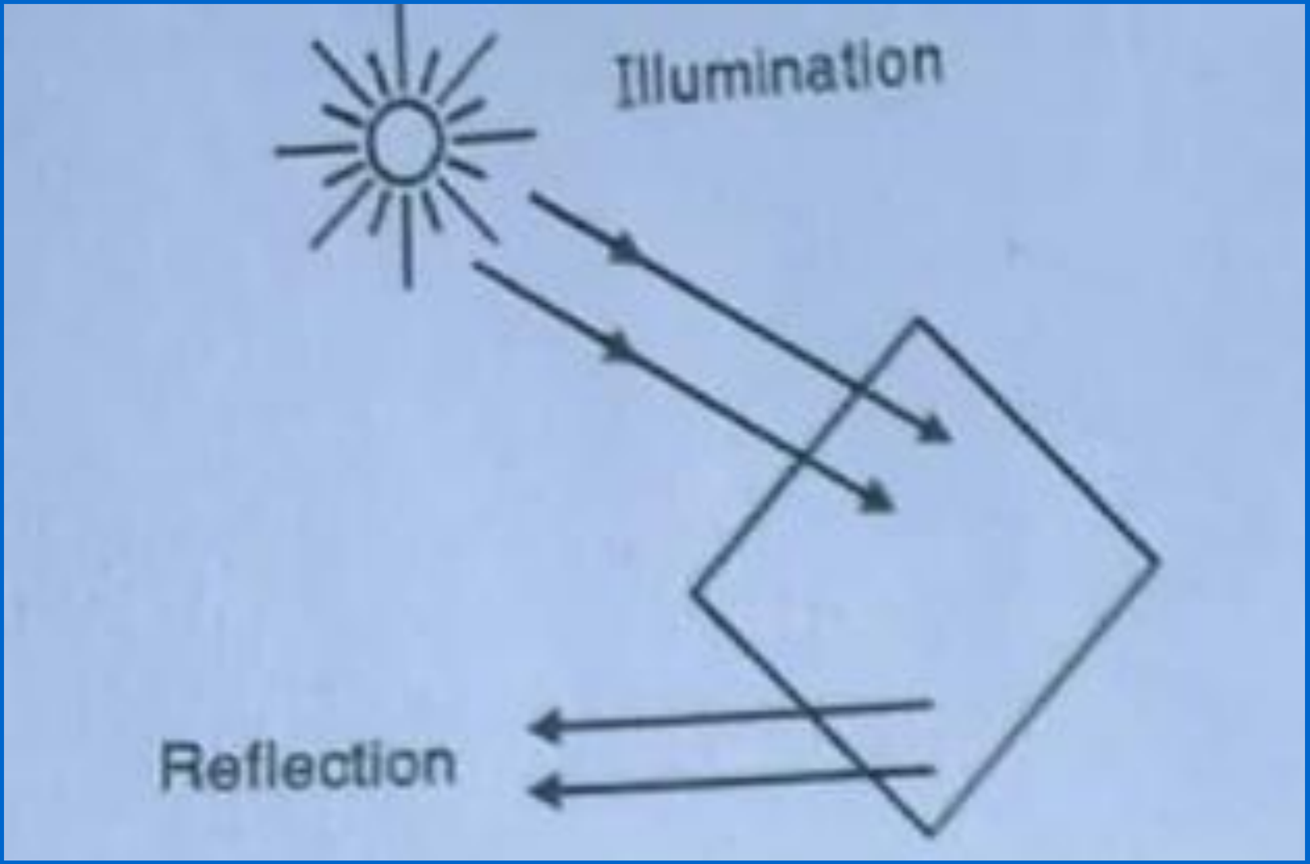
- Many times, we want to remove shading effects from an image (i.e., due to uneven illumination)
  - Enhance high frequencies
  - Attenuate low frequencies but preserve fine detail.



Homomorphic filtering (HF)

- Homomorphic filtering (HF) is a methodology that separates an image into two components: illumination and reflectance.
- Through the processing of these components, it is possible to significantly improve the contrast of the low-frequency components while preserving the edges and sharp features of the image.

Homomorphic filtering (HF)



Homomorphic Filtering (cont'd)

- Consider the following model of image formation:

$$f(x, y) = i(x, y)\, r(x, y)$$

i(x,y): illumination
r(x,y): reflection

- In general, the illumination component i(x,y) varies **slowly** and affects **low** frequencies mostly.
- In general, the reflection component r(x,y) varies **faster** and affects **high** frequencies mostly.

IDEA: separate low frequencies due to i(x,y)
from high frequencies due to r(x,y)

How are frequencies mixed together?

- Low and high frequencies from **i(x,y)** and **r(x,y)** are mixed together.

$$f(x, y) = i(x, y)\, r(x, y) \Rightarrow F(u,v) = I(u,v) * R(u,v)$$

- When applying filtering, it is difficult to handle low/high frequencies separately.

$$F(u,v)H(u,v) = [I(u,v) * R(u,v)]H(u,v)$$

Can we separate them?

- Idea:

Take the ln( ) of $f(x, y) = i(x, y)\, r(x, y)$

$$ln(f(x, y)) = ln(i(x, y)) + ln(r(x, y))$$

Steps of Homomorphic Filtering

(1) Take $ln(f(x,y)) = ln(i(x,y)) + ln(r(x,y))$

(2) Apply FT: $\mathcal{F}(ln(f(x,y))) = \mathcal{F}(ln(i(x,y))) + \mathcal{F}(ln(r(x,y)))$

or $Z(u,v) = Illum(u,v) + Refl(u,v)$

(3) Apply H(u,v)

$$Z(u,v)H(u,v) = Illum(u,v)H(u,v) + Refl(u,v)H(u,v)$$

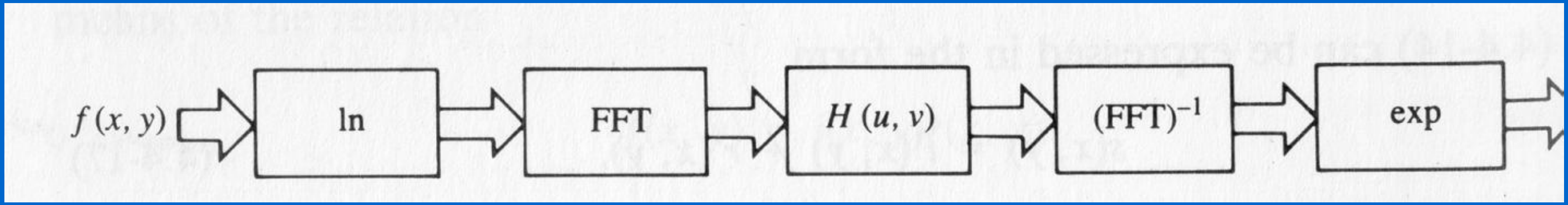
Steps of Homomorphic Filtering (cont'd)

(4) Take Inverse FT:

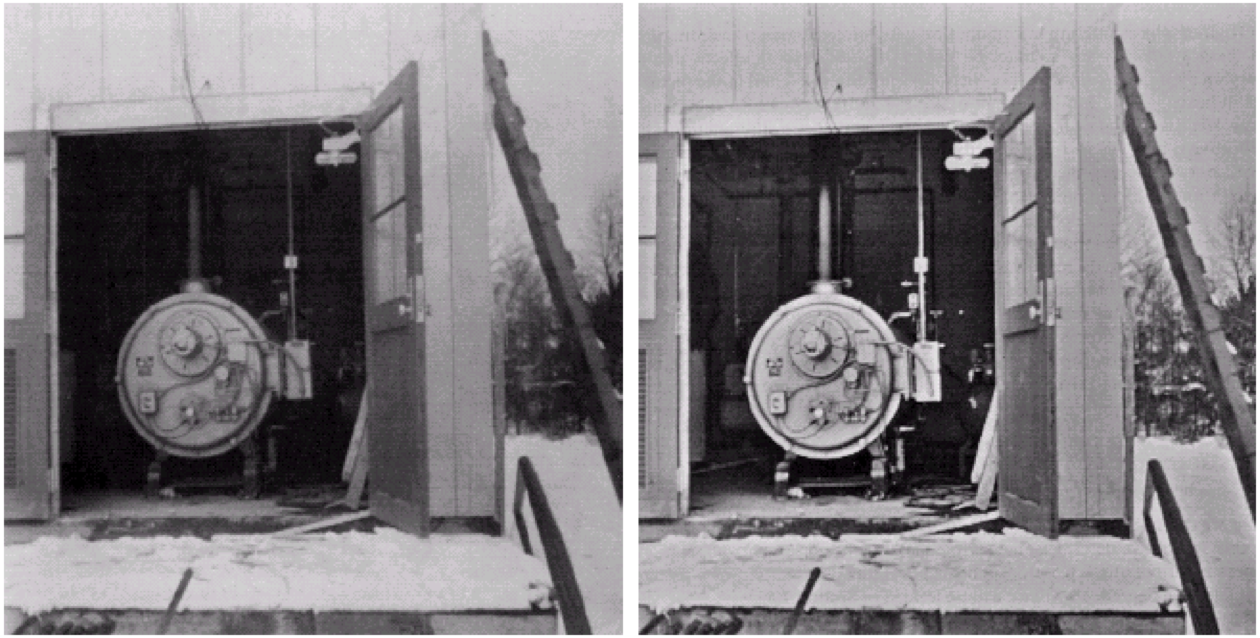
$$F^{-1}(Z(u,v)H(u,v)) = F^{-1}(Illum(u,v)H(u,v)) + F^{-1}(Refl(u,v)H(u,v))$$

or $$s(x,y) = i'(x,y) + r'(x,y)$$

(5) Take exp( ) $$e^{s(x,y)} = e^{i'(x,y)}e^{r'(x,y)}$$ or $$g(x,y) = i_0(x,y)r_0(x,y)$$



Homomorphic Filtering: Example



a b

FIGURE 4.33 (a) Original image. (b) Image processed by homomorphic filtering (note details inside shelter). (Stockham.)

Slant transform

- A new unitary transform called the slant transform, specifically designed for image coding, has been developed.
- The transformation possesses a discrete sawtoothlike basis vector which efficiently represents linear brightness variations along an image line.

Slant transform

- A unitary kernel matrix for the slant transform is obtained from 2x2 Haar or Hadamard transform.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- The slant matrix for N= 4

$$S(4) = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ a+b & a-b & -a+b & -a-b \\ 1 & -1 & -1 & 1 \\ a-b & -a-b & a+b & -a+b \end{bmatrix}$$

Slant transform

- a and b are constants to be determined subject to following conditions:

1. A step size must be uniform
2. S(4) must be orthogonal

- Lets check:
- It two columns of second row
- (a+b)-(a-b) = 2b

Slant transform

- It second and third column of second row
- $(a - b) - (-a + b) = 2a - 2b$
- step size must be uniform
- $2a - 2b = 2b$
- $a = 2b$
- Substitute in 4x4 matrix a and b constsnts

Slant transform

$$\therefore\ S(4) = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3b & b & -b & -3b \\ 1 & -1 & -1 & 1 \\ b & -3b & 3b & -b \end{bmatrix}$$

Now determine a and b values using orthogonal condition

$$\left\{\frac{1}{\sqrt{4}}[3b\ \ b\ \ -b\ \ -3b]\right\}\cdot\left\{\frac{1}{\sqrt{4}}[3b\ \ b\ \ -b\ \ -3b]^{\prime}\right\} = 1$$

Slant transform

$$\therefore\ b = \frac{1}{\sqrt{5}}$$

$$\therefore\ a = 2b$$

$$\therefore\ a = \frac{2}{\sqrt{5}}$$

$$\therefore S(4) = \frac{1}{\sqrt{4}} \begin{array}{cc} \begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{3}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{-3}{\sqrt{5}} \\ 1 & -1 & -1 & 1 \\ \frac{1}{\sqrt{5}} & \frac{-3}{\sqrt{5}} & \frac{3}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

Slant transform

- $X(n) = S(N).x(n)$
- $X(n)$ = Slant transform
- $S(N)$ = NxN Slant matrix
- $x(n)$ = 1D data of length N arranged in columns

Slant transform

- Find the slant transform of the given signal
- X(n) =[1 2 2 1]'

$$X[n] = [S(4) \cdot x(n)]$$

$$X[n] = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{3}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{-3}{\sqrt{5}} \\ 1 & -1 & -1 & 1 \\ \frac{1}{\sqrt{5}} & \frac{-3}{\sqrt{5}} & \frac{3}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$$X[n] = \frac{1}{\sqrt{4}} \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

Slant transform

- Given an image, calculate the Slant Transform
- $$\begin{matrix} 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 2 \\ 2 & 4 & 4 & 2 \\ 2 & 2 & 2 & 2 \end{matrix}$$

Slant transform

$$F = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{3}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{-3}{\sqrt{5}} \\ 1 & -1 & -1 & 1 \\ \frac{1}{\sqrt{5}} & \frac{-3}{\sqrt{5}} & \frac{3}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 2 \\ 2 & 4 & 4 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}\frac{1}{\sqrt{4}}$$

$$\times\begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{3}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{-3}{\sqrt{5}} \\ 1 & -1 & -1 & 1 \\ \frac{1}{\sqrt{5}} & \frac{-3}{\sqrt{5}} & \frac{3}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}'$$

$$D = \frac{1}{4}\begin{bmatrix} 40 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Slant transform

- Apply slant transform and DCT on the given image and compare the result
- $$\begin{matrix} 2 & 2 & 2 & 1 \\ 2 & 4 & 4 & 2 \\ 2 & 4 & 4 & 2 \\ 2 & 2 & 2 & 2 \end{matrix}$$

Karhren Loeve Transform (Hotelling Transfrom)

- By de-correlating this data more compression can be achieved.

- KL transform de-correlates the data which facilitates high degrees of compression

Karhren Loeve Transform (Hotelling Transfrom)

- Also called as Eigen Vector Transform.
- Based on statistical properties of image and has several important properties that make it useful for image processing – image compression.
- Data from neighbouring pixels is highly correlated , so achieving image compression without losing the quality of image is a challenge.

Steps to compute KL transform

To summarise, the [illegible]

(i) Find the mean vector and the covariance matrix of x.

(ii) Find the eigenvalues and then the eigenvectors of the covariance matrix.

(iii) Create the transformation matrix T, such that the rows of T (basis functions) are the eigenvectors.

(iv) $X = T\,[C_x - m]$.

[illegible] the concepts that are stated

The mean vector of f is given by the formula Equation (11.28).

$$m_f = \frac{1}{N}\sum_{k=1}^{N} f_k$$

Where N is the number of columns (in this case $N = 4$).

$$\therefore\ m_f = \frac{1}{4}[f_1 + f_2 + f_3 + f_4] = \frac{1}{4}\begin{bmatrix}4\\4\end{bmatrix}$$

Now,

$$m_f \cdot m_f' = \frac{1}{4}\begin{bmatrix}4\\4\end{bmatrix}\cdot\frac{1}{4}[4\ \ 4] = \frac{1}{16}\begin{bmatrix}16 & 16\\16 & 16\end{bmatrix}$$

Similarly,

$$\sum_{k=1}^{N} f_K \cdot f_K' = f_1 \cdot f_1' + f_2 \cdot f_2' + f_3 \cdot f_3' + f_4 \cdot f_4'$$

Now,

$$f_1 \cdot f_1' = \begin{bmatrix} 1 \\ 2 \end{bmatrix} [1 \;\; 2] = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$f_2 \cdot f_2' = \begin{bmatrix} 2 \\ -1 \end{bmatrix} [2 \;\; -1] = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

$$\mathbf{f}_3 \cdot \mathbf{f}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \quad 1] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{f}_4 \cdot \mathbf{f}_4 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} [0 \quad 2] = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

The covariance matrix of f is given by the formula Equation (11.30).

$$C_f = \frac{1}{N}\sum_{k=1}^{N} f_k \cdot f'_k - m_f \cdot m'_f$$

$$C_f = \frac{1}{N}[(f_1 \cdot f'_1 - m_f \cdot m'_f) + (f_2 \cdot f'_2 - m_f \cdot m'_f) + (f_3 \cdot f'_3 - m_f \cdot m'_f) + (f_4 \cdot f'_4 - m_f \cdot m'_f)]$$

$$\therefore C_f = \begin{bmatrix} 0.5 & -0.75 \\ -0.75 & 1.5 \end{bmatrix}$$

Eigen Value

$$|C_x - \lambda I| = 0$$

$$\left|\begin{bmatrix} 0.5 & -0.75 \\ -0.75 & 1.5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

$$\therefore \begin{vmatrix} 0.5-\lambda & -0.75 \\ -0.75 & 1.5-\lambda \end{vmatrix} = 0$$

$$(0.5-\lambda)(1.5-\lambda) - 0.5625 = 0$$

$$\therefore 0.75 - 0.5\lambda - 1.5\lambda + \lambda^2 - 0.5625 = 0$$

$$\lambda^2 - 2\lambda + 0.1875 = 0$$

$\lambda_1 = 0.0986$ and $\lambda_2 = 1.9014$.

We find the first eigenvector (v_1) corresponding to the first eigenvalue $\lambda_1 = 0.0986$

We use the equation

$$C_f v_1 = \lambda v_1$$

$$\begin{bmatrix} 0.5 & -0.75 \\ -0.75 & 1.5 \end{bmatrix} \begin{bmatrix} v_{10} \\ v_{11} \end{bmatrix} = 0.0986 \begin{bmatrix} v_{10} \\ v_{11} \end{bmatrix}$$

$$\text{where } v_1 = \begin{bmatrix} v_{10} \\ v_{11} \end{bmatrix}$$

$$\therefore\ 0.5v_{10} - 0.75v_{11} = 0.0986v_{10}$$

$$-0.75v_{10} + 1.5v_{11} = 0.0986v_{11}$$

Rearranging (a) and (b), we get,

$$0.4014v_{10} - 0.75v_{11} = 0$$

$$-0.75v_{10} + 1.4014v_{11} = 0$$

We could use either (c) or (d) to get a relationship between v_{10} and v_{11}. L
ion (c),

$$\therefore\ 0.4014v_{10} = 0.75v_{11}$$

$$\therefore\ v_{10} = 1.8684v_{11}$$

Equation (d) would have given the same result.

Therefore eigenvector v_1 corresponding to eigenvalue λ_1 is

$$v_1 = \begin{bmatrix} 1.8685 \\ 1 \end{bmatrix}$$

Similarly we calculate the second eigenvector v_2 corresponding to eigenvalue $\lambda_2 = 1.9014$. We use the same equation i.e.,

$$C_f v_2 = \lambda_2 v_2$$

$$\begin{bmatrix} 0.5 & -0.75 \\ -0.75 & 1.5 \end{bmatrix} \begin{bmatrix} v_{20} \\ v_{21} \end{bmatrix} = 1.9014 \begin{bmatrix} v_{20} \\ v_{21} \end{bmatrix}$$

$$\text{where } v_2 = \begin{bmatrix} v_{20} \\ v_{21} \end{bmatrix}$$

$$\therefore 0.5v_{20} - 0.75v_{21} = 1.9014v_{20} \qquad \text{...(e)}$$

$$-0.75v_{20} + 1.5v_{21} = 1.9014v_{21}$$ ...(f)

Rearranging (a) and (b), we get,

$$1.4014v_{20} - 0.75v_{21} = 0$$ ...(g)

$$-0.75v_{20} + 0.4014v_{21} = 0$$ ...(h)

We could use either (g) or (h) to get a relationship between v_{20} and v_{21}. Let's use Equation (g),

$$-1.4014\, v_{20} = 0.75v_{21}$$

$$\therefore\ v_{20} = -0.5352v_{21}$$

Equation (h) would have given the same result.

$$v_2 = \begin{bmatrix} -0.5352 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors are

$$v_1 = \begin{bmatrix} 1.8685 \\ 1 \end{bmatrix} \text{ and } \qquad v_2 = \begin{bmatrix} -0.5352 \\ 1 \end{bmatrix}$$

These eigenvectors need to be normalized.

$$v_{1N} = \frac{v_1}{\| v_1 \|}$$

Where v_{1N} is the normalised eigenvector and $\| \cdot \|$ is the Norm

$$v_{1N} = \frac{1}{\sqrt{(1.8685)^2 + (1)^2}} \begin{bmatrix} 1.8685 \\ 1 \end{bmatrix}$$

$$= 0.47186 \begin{bmatrix} 1.8685 \\ 1 \end{bmatrix}$$

$$\therefore \; v_{1N} = \begin{bmatrix} 0.8817 \\ 0.47186 \end{bmatrix}$$

Similarly we normalise v_2,

$$v_{2N} = \frac{v_2}{\| v_2 \|}$$

$$v_{2N} = \frac{1}{\sqrt{(-0.5352)^2 + (1)^2}} \begin{bmatrix} -0.5352 \\ 1 \end{bmatrix}$$

$$= 0.8817 \begin{bmatrix} -0.5352 \\ 1 \end{bmatrix}$$

$$\therefore \; v_{2N} = \begin{bmatrix} -0.4719 \\ 0.8817 \end{bmatrix}$$

$$\therefore \quad \mathbf{T} = \begin{bmatrix} -0.4719 & 0.8817 \\ 0.8817 & 0.4719 \end{bmatrix}$$

$$F = T[f - m_f]$$

where f is every column of the original image.

$$F_1 = T[f_1 - m_f]$$

$$F_1 = \begin{bmatrix} -0.4719 & 0.8817 \\ 0.8817 & 0.4719 \end{bmatrix} \left(\underset{f_1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} - \underset{m_f}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \right)$$

$$F_1 = \begin{bmatrix} 0.8817 \\ 0.4719 \end{bmatrix}$$

Similarly,

$$F_2 = T \cdot [f_2 - m_f] = \begin{bmatrix} -2.2352 \\ -0.0621 \end{bmatrix}$$

$$F_3 = T \cdot [f_3 - m_f] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$F_4 = T \cdot [f_4 - m_f] = \begin{bmatrix} 1.3535 \\ -0.4098 \end{bmatrix}$$

Here the image, $f = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & -1 & 1 & 2 \end{bmatrix}$ is

transformed to $F = \begin{bmatrix} 0.8817 & -2.2352 & 0 & 1.3535 \\ 0.4719 & -0.0621 & 0 & -0.4098 \end{bmatrix}$