Exploring Stochastic and Bayesian Elements in War of Attrition

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Abstract

The purpose of this paper is to take the well-known game War of Attrition, and use probability and stochastic processes to transform it from a theoretical game to a model that can be applied and used to study a large variety of waiting games in society. Some examples of where the modified version of the game can be used have been analyzed.

What is a stochastic game? (and related concepts)

A *stochastic game* is a game with two or more players that interact repeatedly over time, where at each stage, the game is in some *state*. All players choose to make decisions simultaneously, and these decisions impact the payoffs for the players and the future states of the game according to some probability distribution. In essence, there are elements of the game that are random and not entirely predictable. A Bayesian game is one in which players have incomplete information regarding some aspects of the game. This can include information about each other, a prize, or other components. This paper will bring in both stochastic and Bayesian properties to explore multiple models of War of Attrition.

A *mixed strategy* is a decision strategy by a player where rather than choosing a single action in every stage, they randomize decisions based on some probability distribution. A mixed strategy Nash equilibrium in a game is a state where each player, though playing probabilistically, plays with a strategy that gives them the highest possible potential payoff. In such a state, no player has any incentive to deviate to another action/strategy.

In Bayesian game theory, A type space is a set that includes possibilities regarding missing or incomplete information in a game. A belief is a player's personal opinion on the likeliness of some specific type to be the true one. These concepts will be covered more comprehensively later in this paper.

War of Attrition

War of Attrition is a classic game theory game where players compete to see who can stay in the game the longest. Those who continue to stay in the game incur a cost, and the last standing player wins some preconceived prize. An important point to note is that the winner incurs all costs up to the end of the game, making the question of choosing to stay in the game or leave one of *how much the prize is worth.* In its simplest form, War of Attrition is a 2 player game with complete information (a model that will be explored soon in this paper). Applications of the game include:
- Evolutionary biology: The model explains animal conflict behaviors without physical fighting that include endurance-based displays such as prolonged staring, posturing, or vocalizations, in order to win some "prize" such as mates, food or territory.
- Political science: the model can be used to understand drawn out political movements such as strikes or wars. In cases such as the Vietnam war, opposing sides continue to incur human and economic costs, hoping the opponent will be the first to run out of resources. This strategy may also be used by governments during protests or uprisings; they may delay giving into demands with the hopes of the movement dying out over time.
- Economics: Price wars work in a manner similar to war of attrition, where companies selling similar products will lower their costs in order to drive opponents out of the market. No firm in this phase will make any profits, and the firm that can sustain the losses the longest would win the war.

The standard model

Players cannot see each other's moves, they are informed of the results of the game only after it is over. For the sake of simplicity, we also assume players have infinite resources and do not reach a stage where they cannot incur the cost. Despite this, players behave rationally because each spending has some kind of opportunity cost.

Let T be the set of all discrete time intervals (states) where players choose to make their moves.

Let $S = \{s_1, s_2\}$ be the set of players.

Let V be the prize. We assume all players have the same valuation V for the prize.

Let c_t be the cost incurred for not defecting in time interval t.

The mixed strategy must be such that the player is indifferent to the time of quitting. For this, the payoff has to be some constant K across all $t \in T$.

Consider the function $F(t)$ which outputs the cumulative probability of a player quitting by time t, and let $f(t)$ be the associated density, which can be interpreted as the probability of quitting exactly at time t.

Payoff for player s_1:

If player s_1 chooses to quit at time t, there are three possible outcomes:
Case 1: Player s_2 quits before time t. The associated payoff is

$$V - \sum_{i=1}^{t} c_i,$$

while this case occurs with probability proportional to

$$\sum_{k=1}^{t-1} f(k).$$

Case 2: Player s_2 quits exactly at time t. The associated payoff is

$$\frac{V}{2} - \sum_{i=1}^{t} c_i,$$

while this case occurs with probability proportional to

$$f(t).$$

Case 3: Player s_2 has not quit by time t. The associated payoff is

$$-\sum_{i=1}^{t} c_i,$$

while this occurs with probability propotional to

$$1 - F(t).$$

Thus, the overall expected payoff for player s_1 quitting at time t is

$$\underbrace{\left(V - \sum_{i=1}^{t} c_i\right)\left(\sum_{k=1}^{t-1} f(k)\right)}_{\text{Case 1: payoff}\times\text{probability}} \quad + \quad \underbrace{\left(\frac{V}{2} - \sum_{i=1}^{t} c_i\right) f(t)}_{\text{Case 2: payoff}\times\text{probability}} \quad + \quad \underbrace{\left(-\sum_{i=1}^{t} c_i\right)(1 - F(t))}_{\text{Case 3: payoff}\times\text{probability}}$$

Assume the cost incurred in each time interval is constant: $c_i = c \ \forall i$. Then:

$$\begin{aligned}
\text{Payoff}(t) &= (V - ct)\sum_{k=1}^{t-1} f(k) + \left(\frac{V}{2} - ct\right) f(t) + (-ct)(1 - F(t)) \\
&= V\sum_{k=1}^{t-1} f(k) + \frac{V}{2} f(t) - ct\left(\sum_{k=1}^{t-1} f(k) + f(t) + (1 - F(t))\right) \\
&= V\sum_{k=1}^{t-1} f(k) + \frac{V}{2} f(t) - ct \cdot 1 \\
&= \boxed{V\sum_{k=1}^{t-1} f(k) + \frac{V}{2} f(t) - ct}.
\end{aligned}$$

Why does a pure strategy here not work?

Intuition suggests that the best time to quit would be at time k when

$$\sum_{t=1}^{k} c_t = V,$$

that is, when the total cost incurred is equal to the value of the prize. Because both players value the prize the same amount, and are paying the same cost, none would like to pay more than the value of prize V, and would like to quit here. However, when both players quit here, they split the prize, which leads to the payoff for each player being

$$\frac{V}{2} - \sum_{t=1}^{k} c_t = -\frac{V}{2}.$$

This is a negative payoff and leads to losses! One might assume, then, that the rational time to quit is when the total cost incurred is

$$\sum_{t=1}^{k} c_t = \frac{V}{2},$$

If the other player quits as well, the prize is split, and the total payoff is 0. However, in such a case, a player has an incentive to stay longer in the game and receive the entire prize V, which leads to a positive reward for that player but no reward and therefore an overall negative reward for the other player.

Mixed strategy:

As mentioned earlier, a mixed strategy entails a strategy that does not include the same action in every state but assigns different actions probabilistic values and picks one at random. For a mixed strategy to be dominant in this game, a player must believe the payoff to be equal regardless of the time of quitting-that is, the player must be indifferent to the value of t. Recall the expected payoff at time t:

$$\text{Payoff}(t) = V\sum_{k=1}^{t-1} f(k) + \frac{V}{2} f(t) - c.$$

To ensure indifference, the payoff function must be set to some constant K.

$$\text{Payoff}(t) = K \quad \Rightarrow \quad V\sum_{k=1}^{t-1} f(k) + \frac{V}{2} f(t) = K + c$$

Solving for $f(t)$:

$$\frac{V}{2} f(t) = K + c - V\sum_{k=1}^{t-1} f(k)$$

$$f(t) = \frac{2}{V}\left(K + c - V\sum_{k=1}^{t-1} f(k)\right).$$

This defines the mixed strategy recursively.

A dry run

In our dry run, the cost c_t for time t is linear. Let us assume, further, that the cost is equal to the number of time steps taken such that when $t = 1, c = 1$, when $t = 2, c = 2$, and so on. This makes

$$C(t) = \sum_{i=1}^{t} i = \frac{t(t+1)}{2})$$

The prize V is also some constant; let us assume here that $V = 20$.
A rational player would not play beyond the time in which the cost is worth more than the prize. In this case, this occurs at the sixth time step (1 + 2 + 3 + 4 + 5 + 6 = 21 > 20). So, the probability of quitting for player s_1 has to be distributed between the first 5 time steps. For simplicity, we set K = 1 so that

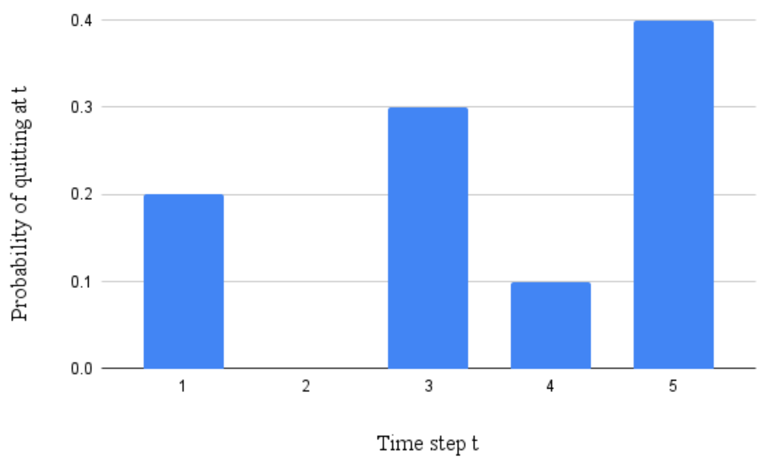
the pay-off is constant across all time steps. As defined earlier, the recursive formula for $f(t)$ is:

$$\frac{V}{2}f(t) = K + c - V\sum_{k=1}^{t-1} f(k)$$

$$f(t) = \frac{2}{V}\left(K + c - V\sum_{k=1}^{t-1} f(k)\right)$$

So, computation for the probability of quitting for all 5 time steps:

$$f(1) = \frac{1-(-1)}{10} = 0.2$$
$$f(2) = \frac{1-1.0}{10} = 0$$
$$f(3) = \frac{1-(-2)}{10} = 0.3$$
$$f(4) = \frac{1-0}{10} = 0.1$$
$$f(5) = \frac{1-(-3)}{10} = 0.4$$



Restructruring to make a Bayesian game

Consider an updated version of the game where $T = \{1, 2, \ldots, n\}$ is the set of discrete time intervals. $S = \{s_1, s_2, s_3\}$ is the set of three players. V is the common prize awarded to the last remaining player.
Each player s_i has a private cost function over time:

$$c_i(t) = a_i \cdot e^{b_i t}$$

where a_i and b_i are private parameters known only to player s_i. The player's type is defined as:

$$\theta = (a_i, b_i)$$

Type Space:
The type of each player belongs to some set Θ, the elements of which are known to all players. An example of such a set may be:

$$\Theta = \{(1, 0.1),\ (1, 0.2),\ (2, 0.1),\ (2, 0.2)\}$$

Each player knows their own type $\theta \in \Theta$, but not the types of the others. For any player s_i without information, the probability that some other player $s_j \neq s_i$ has any type is the same. So, for the example given above,

$$P(\theta = (a, b)) = 0.25 \quad \text{for each } (a, b) \in \Theta$$

Expected Payoff for Player s_1 Quitting at Time t

Each player s_i has a private cost function:

$$c_i(t) = a_i \cdot e^{b_i t}$$

Define the total cost incurred by s_1 up to time t as:

$$C_1(t) = \sum_{k=1}^{t} c_1(k)$$

Let $f_2(t)$, $f_3(t)$ be the probabilities that players s_2 and s_3 quit exactly at time t, respectively. As in the previous section, an analysis of payoffs and probabilities follows. Note that the "probabilities" below are proportional constants which reflect the magnitude of the actual probability that the case occurs, as in the previous section.

Case A: Both s_2 and s_3 quit before t

$$\text{Payoff} = V - C_1(t).$$

$$\text{Probability} = \sum_{k=1}^{t-1} \sum_{\substack{n=1 \\ n \neq k}}^{t-1} f_2(k) f_3(n).$$

Case B: One quits before t, one at t

$$\text{Payoff} = \frac{V}{2} - C_1(t).$$

$$\text{Probability} = \sum_{k=1}^{t-1} f_2(k) f_3(t) + f_2(t) \sum_{n=1}^{t-1} f_3(n).$$

Case C: Both quit at t

$$\text{Payoff} = \frac{V}{3} - C_1(t).$$

$$\text{Probability} = f_2(t) \cdot f_3(t).$$

Case D: At least one survives past t

$$\text{Payoff} = -C_1(t).$$

$$\text{Probability} = 1 - \left(\sum_{k=1}^{t-1} \sum_{\substack{n=1 \\ n \neq k}}^{t-1} f_2(k) f_3(n) + \sum_{k=1}^{t-1} f_2(k) f_3(t) + f_2(t) \sum_{n=1}^{t-1} f_3(n) + f_2(t) f_3(t) \right) = P_D,$$

where we rename the probability as P_D for notational convenience.

Overall payoff for s_1 quitting at t:

$$\begin{aligned}
\text{Payoff}_{s_1}(t) = (V - C_1(t)) \cdot & \sum_{m=1}^{t-1} \sum_{\substack{n=1 \\ n \neq m}}^{t-1} f_2(m) f_3(n) \\
& + \left(\frac{V}{2} - C_1(t) \right) \cdot \left(\sum_{m=1}^{t-1} f_2(m) f_3(t) + f_2(t) \sum_{n=1}^{t-1} f_3(n) \right) \\
& + \left(\frac{V}{3} - C_1(t) \right) \cdot f_2(t) f_3(t) \\
& + (-C_1(t) P_D)
\end{aligned}$$

$$\begin{aligned}
= {} & V F(t-1)^2 - C(t) F(t-1)^2 \\
& + V F(t-1) f(t) - 2C(t) F(t-1) f(t) \\
& + \frac{V}{3} f(t)^2 - C(t) f(t)^2 \\
& - C(t) \left(1 - F(t-1)^2 - 2F(t-1) f(t) - f(t)^2 \right)
\end{aligned}$$

Expanding the last term:

$$-C(t) + C(t) F(t-1)^2 + 2C(t) F(t-1) f(t) + C(t) f(t)^2$$

$$\begin{aligned}\text{Payoff}(t) = {} & VF(t{-}1)^2 \\ & + VF(t{-}1)f(t) \\ & + \frac{V}{3}f(t)^2 \\ & - C(t)\end{aligned}$$

$$\boxed{\text{Payoff}(t) = VF(t{-}1)^2 + VF(t{-}1)f(t) + \frac{V}{3}f(t)^2 - C(t)}.$$

Solving for the Mixed Strategy Function

Again, to ensure that the player is indifferent to the choice of quitting time, we set the expected payoff at time t equal to a constant K:

$$VF(t{-}1)^2 + VF(t{-}1)f(t) + \frac{V}{3}f(t)^2 = K + C(t)$$

Let $A = F(t{-}1)$.

$$\frac{V}{3}f(t)^2 + VAf(t) - (K + C(t) - VA^2) = 0.$$

This is a quadratic in $f(t)$ of the form:

$$af(t)^2 + bf(t) + c = 0, \quad \text{with} \quad a = \frac{V}{3}, \quad b = VA, \quad c = -(K + C(t) - VA^2).$$

Applying the quadratic formula:

$$f(t) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Substituting in the expressions for a, b, c, we get:

$$f(t) = \frac{-VA + \sqrt{V^2A^2 + \frac{4V}{3}(K + C(t) - VA^2)}}{\frac{2V}{3}}.$$

$$\boxed{f(t) = \frac{3}{2V}\left[-VF(t{-}1) + \sqrt{V^2F(t{-}1)^2 + \frac{4V}{3}\left(K + C(t) - VF(t{-}1)^2\right)}\right]}.$$

What happens when players have varying beliefs about other players?

In the previous model, we assumed each player's likeliness of having any type to be equal. This changes when we consider the perspective of a player having some belief of how another player is likely to behave. We now assume that player s_1 holds certain non-uniform beliefs about the types of players s_2 and s_3. For example, s_1 believes that s_2 is most likely to be of type $(2, 0.1)$, assigning it a probability of 0.5, while assigning probabilities of 0.1, 0.2, and 0.2 to the types $(1, 0.1)$, $(1, 0.2)$, and $(2, 0.2)$, respectively. These beliefs replace the earlier assumption of a uniform distribution over types and imply that the probability functions $f_2(t)$ and $f_3(t)$ must now be computed as expectations over these subjective type distributions.

Does it change the strategy?

When player s_1 holds non-uniform beliefs over the types of players s_2 and s_3, the quitting probabilities $f_2(t)$ and $f_3(t)$ are computed as expectations over those beliefs. Consider some common type space Θ where each type $\theta = (a, b)$ defines a cost function $c(t) = ae^{bt}$.
Let $\Pr_{s_1}(\theta)$ be the probability that s_1 assigns to a player being of type θ.

$$f_2(t) = \sum_{\theta \in \Theta} \Pr_{s_1}(\theta) \cdot f_2^{\theta}(t), \quad f_3(t) = \sum_{\theta \in \Theta} \Pr_{s_1}(\theta) \cdot f_3^{\theta}(t)$$

These are used to define:

$$F(t-1) = \sum_{k=1}^{t-1} \frac{f_2(k) + f_3(k)}{2}$$

(this is the average quitting probability of both players s_2 and s_3) which is substituted into the expected payoff function:

$$\text{Payoff}_{s_1}(t) = VF(t-1)^2 + VF(t-1)f(t) + \frac{V}{3}f(t)^2 - C_1(t)$$

Solving for $f(t)$ gives the same functional form as before:

$$f(t) = \frac{3}{2V}\left[-VF(t-1) + \sqrt{V^2F(t-1)^2 + 4V^3\left(K + C(t) - VF(t-1)^2\right)}\right].$$

Thus, while the equation for $f(t)$ remains unchanged, the values it depends on now reflect s_1's beliefs about the other players' types.

Even more hidden information and noisy signals.

Consider a further modified variation of the game where players do not all value the prize V the same amount. For each player s_i there exists some value V_i

denoting how valuable the prize is to them. Players do not know how valuable a prize is to another player; let us assume they receive some signal g_i about how likely it is for another player s_i to value the prize some certain amount. They once again only have a probability distribution of how likely it is for another player to have some certain valuation for the prize. The reason this makes this particularly interesting is because the time of quitting, as mentioned in the beginning of the paper, is mainly about how much each player *values* the prize, and how much of a cost they are willing to incur to win it.
In this final model, each player s_i has a private valuation V_i for the prize, a private type $\theta_i = (a_i, b_i)$ determining their cost function $c_i(t) = a_i e^{b_i t}$, and only receives a noisy signal g_i about other players' valuations and types. Based on this, players form beliefs over the joint distribution of opponent types and valuations.
Player s_1's total cost incurred up to time t is:

$$C_1(t) = \sum_{k=1}^{t} c_1(k) = \sum_{k=1}^{t} a_1 e^{b_1 k}.$$

Let $f_2(t), f_3(t)$ be the expected probabilities that players s_2 and s_3 quit exactly at time t, calculated as:

$$f_j(t) = \sum_{\theta,V} \Pr_{s_1}(\theta, V \mid g_1) \cdot f_j^{\theta,V}(t), \quad \text{for } j = 2, 3,$$

where $f_j^{\theta,V}(t)$ is the type- and valuation-dependent quitting probability of player s_j, and $\Pr_{s_1}(\cdot \mid g_1)$ is player s_1's belief distribution based on their signal.
Define the average cumulative quitting probability up to time $t-1$ as:

$$F(t-1) = \frac{1}{2} \sum_{k=1}^{t-1} \left[f_2(k) + f_3(k) \right].$$

We can prove the following results using analysis which is extremely similar to the previous sections. The associated payoff is as follows, which rearranges itself to give an expression for $f(t)$.

$$\text{Payoff}(t) = V_1 F(t-1)^2 + V_1 F(t-1) f(t) + \frac{V_1}{3} f(t)^2 - C_1(t).$$

$$V_1 F(t-1)^2 + V_1 F(t-1) f(t) + \frac{V_1}{3} f(t)^2 = K + C_1(t),$$

Solving for $f(t)$:

$$f(t) = \frac{3}{2V_1} \left[-V_1 F(t-1) + \sqrt{V_1^2 F(t-1)^2 + 4V_1^3 \left(K + C_1(t) - V_1 F(t-1)^2 \right)} \right].$$

Conclusion

The purpose of this paper was to transform a classic War of Attrition game into a model that can be applied to real-life scenarios. One such scenario, and the one from which the game derives its name, is the one of geopolitical standoffs and war.

Theoretical Concept	**Symbol / Function**	**Interpretation**
Player	s_i	A nation engaged in a standoff or war
Time Steps	$t \in T$	Each round of decision-making (e.g., monthly moves, escalations).
Prize	V_i	The value of winning the standoff (e.g., territory, influence).
Private Valuation	V_i	Each state values the outcome differently based on internal interests.
Cost Function	$c_i(t) = a_i e^{b_i t}$	The state's escalating cost from prolonging the standoff (economic, political, human life).
Total Cost	$C_i(t) = \sum_{k=1}^{t} c_i(k)$	Total cost incurred by continuing the standoff until time t.
Signal	g_i	Indirect cues about opponent's strength or intentions.
Belief over Opponent	$\Pr_{s_i}(\theta, V \mid g_i)$	A state's understanding of the opponent's cost tolerance and goals.
Quitting Probability	$f_j(t)$	Perceived chance that the opponent will de-escalate at time t.

Another very common application could include labor strikes and collective bargaining. A table with a comparative analysis for the same has been attatched as well.

Theoretical Concept	**Symbol / Function**	**Interpretation**
Player	s_i	Worker's union/employer engaged in strike.
Time Steps	$t \in T$	Each day of the strike.
Prize	V_i	The desired settlement — e.g., wage increases, working conditions, etc.
Private Valuation	V_i	Each side values the outcome differently (e.g., unions may prioritize pay, employers may prioritize stability).
Cost Function	$c_i(t) = a_i e^{b_i t}$	Escalating cost of prolonging the strike such as lost wages or lost profits and reputation.
Total Cost	$C_i(t) = \sum_{k=1}^{t} c_i(k)$	Total cost incurred up to day t of the strike.
Signal	g_i	Media reports, turnout numbers, management memos — noisy signals about resolve.
Belief over Opponent	$\Pr_{s_i}(\theta, V \| g_i)$	A side's belief about the opponent's financial or emotional endurance and goal intensity.
Quitting Probability	$f_j(t)$	Estimated chance the other side will settle or back down at time t.

Ultimately, adding stochastic processes and elements of randomness to a theoretical game proves to be an undertaking with useful results. While the game alone does provide a strong framework for applications in society, it misses many of the nuances and specifications that come with human behavior. The purpose of game theory, at least as the author believes it to be, is being able to use the language of mathematics in order to replicate and study how the actions of human beings - to understand large scale society, to make predictions, or for the simple purpose of wanting to know how we as a species behave. This paper (hopefully) took one step towards making that replication precise, and one step closer to understanding ourselves.

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