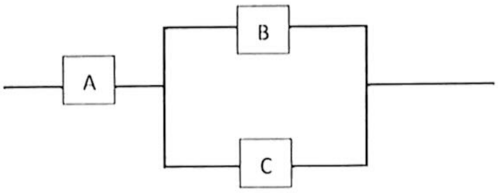
MSO201A: Probability & Statistics
Midsem Examination: Full Marks 60

[1] 3 independent components, A, B and C, of a system are connected as in the following diagram.



The system functions if the component A is functioning and at least one of B or C is functioning. The lifetime (in years) distribution of the 3 components A, B, C have the following identical exponential distribution p.d.f.

$$f(x) = \begin{cases} \frac{1}{6} e^{-\frac{x}{6}}, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find the probability that the system will function for at least 3 years.

8 marks

[2] Let the p.d.f. of a continuous random variable X be given by

$$f(x) = \begin{cases} \frac{|x|}{2}, & \text{if } -1 < x < 1 \\ \frac{x}{3}, & \text{if } 1 \le x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

Find the p.d.f. of $Y = \frac{1}{|X|}$.

9 marks

[3] Suppose $X \sim Bin(2, p);\ 0 < p < 1$. If $P(X \geq 1) = P(X \leq 1)$, then find the value of p.
$\left[\text{If } X \sim Bin(n,p), \text{then p.m.f. of } X \text{ is } P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}; x = 0,1,\ldots,n\right]$

5 marks

[4] The p.d.f. of a continuous random variable X be given by

$$f(x) = \begin{cases} 6\,x(1-x), & \text{if } 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Calculate $G(x) = P(X \leq x | X \leq 0.5)$ for $x \in \mathbb{R}$. Is $G(x)$ a distribution function?
(b) Find $E(X|X \leq 0.5)$.

10 (4+6) marks

[5] Let X be a discrete random variable with support $\mathcal{X} = \{-4, -3, -2, -1, 0, 1, 2, 3\}$ and such that $P(X = -4) = P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = P(X = 3)$ and $P(X > 0) = P(X = 0)$.
(a) Find the distribution function of $Y = X^2 + 1$.
(b) Find the m.g.f. of $Z = -Y$.

8 (6+2) marks

[6] The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{3}{8}x^2, & 0 \le x < 1 \\ \frac{24\,x - 6\,x^2 - 11}{16}, & 1 \le x < 2 \\ \frac{7}{8}, & 2 \le x < 3 \\ 1, & x \ge 3. \end{cases}$$

(a) Decompose $F(x)$ as $F(x) = \alpha\, F_d(x) + (1-\alpha)\, F_c(x)$; $F_d(x)$ and $F_c(x)$ are purely discrete and purely continuous distribution functions, respectively, and $0 < \alpha < 1$.

(b) Obtain the p.d.f. associated with $F_c(x)$.

10 (8+2) marks

[7] Let (X, Y) be a discrete random vector; possible values of X are $0,1,2,3$ and possible values of Y are $-1,1,2,3$. The joint p.m.f. of (X, Y) is given by

$$P(X = x, Y = y) = \begin{cases} \frac{1}{8}, & \text{if } (x,y) \in \{(0,-1),(1,1),(1,2),(1,3),(2,2),(3,1)\} \\ \frac{2}{8}, & \text{if } (x,y) = (2,1) \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the conditional p.m.f. of X given $Y = 1$.

(b) If $F(x,y)$ denote the joint distribution function of (X, Y), find the value of $F(1,2)$.

(c) Find $P(-1 < X \le 1, 0 < Y \le 1)$.

10 (3+3+4) marks

(1)

E_A: event that comp A functions for at least 3 yrs

E_B: . . . . . . B . . . . . 3 yrs

E_C: . . . . . . C . . . . . 3 yrs

$$P(E_A) = \frac{1}{6}\int_3^{\infty} e^{-x/6}\,dx = \underline{e^{-3/6} = e^{-1/2} = P(E_B) = P(E_C)} \quad (1)$$

Let X denote the r.v. denoting ~~the~~ life time of the system

$$\begin{aligned}
\text{Reqd prob} &= P(X \geq 3) \\
&= P(E_A \cap (E_B \cup E_C)) \\
&= P((E_A \cap E_B) \cup (E_A \cap E_C)) \\
&= \underline{P(E_A E_B) + P(E_A E_C) - P(E_A E_B E_C)} \quad (2) \\
&= P(E_A)P(E_B) + P(E_A)P(E_C) - P(E_A)P(E_B)P(E_C)
\end{aligned}$$

(using independence)

$$\begin{aligned}
&= \underline{e^{-1/2}e^{-1/2} + e^{-1/2}e^{-1/2} - e^{-1/2}e^{-1/2}e^{-1/2}} \quad (5) \\
&= 2e^{-1} - e^{-3/2} \quad (\approx 0.51)
\end{aligned}$$

(2) $$f(x) = \begin{cases} \frac{|x|}{2}, & -1 < x < 1 \\ \frac{x}{3}, & 1 \le x < 2 \\ 0, & \text{o/w} \end{cases}$$

$\mathcal{X} = (-1, 2)$

$Y = \frac{1}{|X|}$ $\qquad \mathcal{Y} = (\frac{1}{2}, \infty)$

For $x \in (-1, 0)$ $\quad y = \frac{1}{|x|} \in (1, \infty)$ $\quad y = g(x) = \frac{1}{|x|}$ is strictly monotone

$y = \underbrace{-\frac{1}{x}}_{= g_1(x)} \Rightarrow x = -\frac{1}{y} = g_1^{-1}(y)$

$$\frac{d\, g_1^{-1}(y)}{dy} = \frac{1}{y^2}$$

Also, $x \in (0, 1)$ $\quad y \in (1, \infty)$

$y = \frac{1}{x} = g_2(x)$ is strictly monotone

$x = \frac{1}{y} = g_2^{-1}(y)$

$$\frac{d\, g_2^{-1}(y)}{dy} = -\frac{1}{y^2}$$

$\Rightarrow$ For $\underline{y \in (1, \infty)}$; p.d.f. of Y is

$$f_Y(y) = f(g_1^{-1}(y)) \left| \frac{d\, g_1^{-1}(y)}{dy} \right| + f(g_2^{-1}(y)) \left| \frac{d\, g_2^{-1}(y)}{dy} \right|$$

$$= \frac{1}{2y} \cdot \frac{1}{y^2} + \frac{1}{2y} \frac{1}{y^2}$$

$$= \frac{1}{y^3} \qquad 1 < y < \infty \qquad (5)$$

For $x \in (1, 2)$ $\quad y \in (\frac{1}{2}, 1)$ and $y = \frac{1}{|x|}$ is strictly monotone

$\Rightarrow$ p.d.f. of Y for $y \in (\frac{1}{2}, 1)$ is $\qquad y = \frac{1}{x} = g^{-1}(y)$

$$f_Y(y) = f(g^{-1}(y)) \left| \frac{d\, g^{-1}(y)}{dy} \right| = \frac{1}{3y} \cdot \frac{1}{y^2}$$

$$= \frac{1}{3y^3} \qquad y \in (\frac{1}{2}, 1) \qquad (4)$$

i.e. p.d.f. of Y is

$$f_Y(y) = \begin{cases} \frac{1}{3y^3}, & \frac{1}{2} < y < 1 \\ \frac{1}{y^3}, & 1 \le y < \alpha \\ 0, & o/w \end{cases}$$

* If only one part is calculated and mentioned as p.d.f, then the answer is completely wrong and no marks can be awarded.

* The problem can be solved using d.f. based approach also. Give partial credit for steps for that method.

(3)

$X \sim Bin(2, p)$

$P(X \geq 1) = P(X=1) + P(X=2)$

$= \binom{2}{1} p(1-p) + \binom{2}{2} p^2$

$= 2p(1-p) + p^2$ — (1)

$P(X \leq 1) = P(X=0) + P(X=1)$

$= \binom{2}{0}(1-p)^2 + \binom{2}{1} p(1-p)$

$= (1-p)^2 + 2p(1-p)$ — (1)

Condition that $P(X \geq 1) = P(X \leq 1)$

$\Rightarrow 2p(1-p) + p^2 = (1-p)^2 + 2p(1-p)$

i.e. $2p = 1$

$\Rightarrow \underline{p = \frac{1}{2}}$ (3)

(4)
$$f(x) = \begin{cases} 6x(1-x), & 0 < x < 1 \\ 0, & o/w \end{cases}$$

(a)

$$G(x) = P(X \le x \mid X \le 0.5)$$
$$= \frac{P(X \le x,\ X \le 0.5)}{P(X \le 0.5)}$$

$$\text{i.e. } G(x) = \begin{cases} 0, & x < 0 \\ \dfrac{P(X \le x)}{F_X(\frac{1}{2})}, & 0 \le x \le \frac{1}{2} \\ 1, & x > \frac{1}{2} \end{cases} \quad — (*)$$

$F_X(x) = P(X \le x)$

for $x \in [0,1]$; $F_X(x) = 6\int_0^x (t - t^2)\,dt$

$$= 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)$$

$$\text{i.e. } F_X(x) = \begin{cases} 0, & x < 0 \\ 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right), & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

& $F_X(\frac{1}{2}) = 6\left(\frac{1}{8} - \frac{1}{8 \times 3}\right) = \frac{1}{2}$

$$\Rightarrow G(x) = \begin{cases} 0, & x < 0 \\ 12\left(\frac{x^2}{2} - \frac{x^3}{3}\right), & 0 \le x \le \frac{1}{2} \\ 1, & x > \frac{1}{2} \end{cases} \quad — (3)$$

$G(.)$ is non-decreasing, right cont, $G(-\infty) = 0$, $G(\infty) = 1$

$\Rightarrow G(.)$ is a d.f. (one can infer this from (*) also using the fact that $F_X(.)$ is d.f.) (1)

4

(b)

$$E(X \mid X \le \tfrac{1}{2}) = \frac{1}{F_X(\frac{1}{2})} \int_0^{1/2} 6x^2(1-x)\,dx \quad \text{— (4)}$$

$$= 12 \int_0^{1/2} (x^2 - x^3)\,dx$$

$$= 12 \left[\frac{x^3}{3} - \frac{x^4}{4} \right|_0^{1/2}$$

$$= 12 \left(\frac{1}{8 \times 3} - \frac{1}{16 \times 4} \right)$$

$$= 12 \, \frac{1}{8} \left(\frac{1}{3} - \frac{1}{8} \right) = \frac{12}{8} \times \frac{5}{24} = \underline{\frac{5}{16}} \quad (2)$$

(5)

(a) Let

$$P(X=-4) = P(X=-3) = P(X=-2) = P(X=-1)$$
$$= P(X=1) = P(X=2) = P(X=3) = p, \text{ say}$$

$$P(X>0) = 3p = P(X=0)$$

$$\Rightarrow 4p + 3p + 3p = 1 \Rightarrow p = \frac{1}{10}$$

$$\mathcal{X} = \{-4, -3, -2, -1, 0, 1, 2, 3\}$$

$$Y = X^2 + 1 \; ; \quad \mathcal{Y} = \{1, 2, 5, 10, 17\}$$

p.m.f.

$$P(Y=1) = P(X=0) = \frac{3}{10}$$
$$P(Y=2) = P(X=-1) + P(X=1) = \frac{2}{10}$$
$$P(Y=5) = P(X=-2) + P(X=2) = \frac{2}{10}$$
$$P(Y=10) = P(X=-3) + P(X=3) = \frac{2}{10}$$
$$P(Y=17) = P(X=-4) = \frac{1}{10}$$

(2)

d.f. of Y is

$$F_Y(y) = \begin{cases} 0, & y < 1 \\ \frac{3}{10}, & 1 \le y < 2 \\ \frac{5}{10}, & 2 \le y < 5 \\ \frac{7}{10}, & 5 \le y < 10 \\ \frac{9}{10}, & 10 \le y < 17 \\ 1, & y \ge 17 \end{cases}$$

Deduct 2 marks if range(s) are wrong.

(4)

(b) m.g.f. of $Z = -Y$

$$M_Z(t) = E\left(e^{tZ}\right) = E\left(e^{-tY}\right)$$

$$= \sum_y e^{-ty} P(Y=y)$$

$$= e^{-t}\,\frac{3}{10} + e^{-2t}\,\frac{2}{10} + e^{-5t}\,\frac{2}{10} + e^{-10t}\,\frac{2}{10} + e^{-17t}\,\frac{1}{10}$$

i.e. $$M_Z(t) = \frac{1}{10}\left(3e^{-t} + 2e^{-2t} + 2e^{-5t} + 2e^{-10t} + e^{-17t}\right) \quad \text{—(2)}$$

(6) $$F(x) = \begin{cases} 0, & x < 0 \\ \frac{3}{8}x^2, & 0 \le x < 1 \\ \frac{24x - 6x^2 - 11}{16}, & 1 \le x < 2 \\ \frac{7}{8}, & 2 \le x < 3 \\ 1, & x \geq 3 \end{cases}$$

$F(.)$ has jump discontinuties at

(i) $x = 1$; jump magnitude is $F(1) - F(1-)$

$$F(1) - F(1-) = \frac{24-6-11}{16} - \frac{3}{8} = \frac{1}{16}$$

(ii) $x = 2$; jump magnitude is $F(2) - F(2-)$

$$F(2) - F(2-) = \frac{7}{8} - \frac{48-24-11}{16} = \frac{1}{16}$$

(iii) $x = 3$; jump magnitude is $F(3) - F(3-) = \frac{1}{8}$

$$P(X \in \{1, 2, 3\}) = \frac{1}{16} + \frac{1}{16} + \frac{1}{8} = \frac{1}{4}$$

$$\alpha = \frac{1}{4} \; ; \; (1-\alpha) = \frac{3}{4} \quad \text{— (1)}$$

$$\Rightarrow \alpha F_d(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{16}, & 1 \le x < 2 \\ \frac{1}{8}, & 2 \le x < 3 \\ \frac{1}{4}, & x \geq 3 \end{cases}$$

$$\Rightarrow F_d(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{4}, & 1 \le x < 2 \\ \frac{1}{2}, & 2 \le x < 3 \\ 1, & x \geq 3 \end{cases} \quad \text{— (3)}$$

$$\Rightarrow (1-\alpha) F_c(x) = F(x) - \alpha F_d(x)$$

$$= \begin{cases} 0, & x < 0 \\ \frac{3}{8}x^2, & 0 \le x < 1 \\ \frac{24x - 6x^2 - 12}{16}, & 1 \le x < 2 \\ \frac{6}{8}, & 2 \le x < 3 \\ \frac{3}{4}, & x \geq 3 \end{cases}$$

i.e. $$(1-\alpha) F_c(x) = \begin{cases} 0, & x < 0 \\ \frac{3}{8}x^2, & 0 \le x < 1 \\ \frac{24x - 6x^2 - 12}{16}, & 1 \le x < 2 \\ \frac{3}{4}, & x \geq 2 \end{cases}$$

$$\Rightarrow F_c(x) = \begin{cases} 0, & x < 0 \\ x^2/2, & 0 \le x < 1 \\ \frac{4}{3}\left(\frac{3}{2}x - \frac{3}{8}x^2 - \frac{3}{4}\right), & 1 \le x < 2 \\ 1, & x \geq 2 \end{cases}$$

$$= \begin{cases} 0, & x < 0 \\ x^2/2, & 0 \le x < 1 \\ 2x - \frac{x^2}{2} - 1, & 1 \le x < 2 \\ 1, & x \geq 2 \end{cases} \quad \text{--- (4)}$$

p.d.f. Corresp to $F_c(\cdot)$

$$f_c(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & \text{o/w} \end{cases} \quad \text{--- (2)}$$

(7)

$Y \backslash X$	0	1	2	3
-1	1/8	0	0	0
1	0	1/8	2/8	1/8
2	0	1/8	1/8	0
3	0	1/8	0	0

jt p.m.f.

(a)

Marginal p.m.f. of Y

$$P(Y=-1) = \sum_x P(X=x, Y=-1) = 1/8$$

$$P(Y=1) = \sum_x P(X=x, Y=1) = \frac{4}{8} = \frac{1}{2}$$

$$P(Y=2) = \frac{2}{8}$$

$$P(Y=3) = \frac{1}{8}$$

Conditional p.m.f. of X given $Y=1$

$$P(X=0 \mid Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = 0$$

$$P(X=1 \mid Y=1) = \frac{1/8}{1/2} = \frac{1}{4}$$

$$P(X=2 \mid Y=1) = \frac{2/8}{1/2} = \frac{1}{2}$$

$$P(X=3 \mid Y=1) = \frac{1/8}{1/2} = \frac{1}{4}$$

— (3)

(b) $F(1,2) = P(X \le 1 \text{ and } Y \le 2)$

$$= \sum_{x \le 1} \sum_{y \le 2} P(X=x, Y=y)$$

$$= \sum_{x=0,1} \sum_{y=-1,1,2} P(X=x, Y=y)$$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$ — (3)

(c) $P(-1 < X \leq 1, 0 < Y \leq 1)$

$= F(1,1) - F(-1,1) - F(1,0) + F(-1,0)$ — (1)

$F(1,1) = \sum_{x \leq 1} \sum_{y \leq 1} P(X=x, Y=y) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

$F(-1,1) = 0$

$F(1,0) = \sum_{x \leq 1} \sum_{y \leq 0} P(X=x, Y=y) = \frac{1}{8}$

$F(-1,0) = 0$

$\Rightarrow P(-1 < X \leq 1, 0 < Y \leq 1) = \frac{1}{4} - 0 - \frac{1}{8} + 0$

$= \frac{1}{8}$ — (3)