<u>Solution</u>

(1) $\Omega = \{HH, HT, TH, TT\}$

$\mathcal{F} = \{\phi, \Omega, \{HH\}, \{HT, TH, TT\}\}$

A: $X(\omega) = \begin{cases} 1, & \omega = HH \\ 0, & o/w \end{cases}$

$$X^{-1}(-\infty, x] = \{\omega : X(\omega) \le x\} = \begin{cases} \phi, & x < 0 \\ \{HT, TH, TT\}, & 0 \le x < 1 \\ \Omega & x \geq 1 \end{cases}$$

$\Rightarrow X^{-1}(-\infty, x] \in \mathcal{F} \quad \forall x \in \mathbb{R}$

$\Rightarrow X$ is a random variable

B: $Y(\omega) = \begin{cases} 0, & \omega = TT \\ 1, & \omega = TH, HT \\ 2, & \omega = HH \end{cases}$

$$Y^{-1}(-\infty, x] = \{\omega : X(\omega) \le x\} = \begin{cases} \phi, & x < 0 \\ \{TT\}, & 0 \le x < 1 \\ \{TT, TH, HT\}, & 1 \le x < 2 \\ \Omega, & x \geq 2 \end{cases}$$

i.e. $Y^{-1}(-\infty, x] = \{TT\} \quad \forall \; 0 \le x < 1$

$\{TT\} \notin \mathcal{F}$

$\Rightarrow Y$ is not a random variable

C: $\mathcal{F}_1 = \{\phi, \Omega, \{TT\}, \{HT, TH, HH\}, \{TH, HT\}, \{HH\}, \{HH, TT\}\}$

$\mathcal{F}_1$ is not closed under complementation (complement of $\{HH\} \notin \mathcal{F}_1$)

$\Rightarrow \mathcal{F}_1$ is not a σ-field

$\mathcal{F} \cap \mathcal{F}_1 = \{\phi, \Omega, \{HH\}\}$ is also not closed under complementation

$\Rightarrow \mathcal{F} \cap \mathcal{F}_1$ is not a σ-field

(2)

(a) $P(X=0 \text{ or } X=1)$

$= P(X=0) + P(X=1)$

$= \left(F(0) - F(0-)\right) + \left(F(1) - F(1-)\right)$

$= \left(\frac{1}{4} - 0\right) + \left(\frac{1}{4}\left(\frac{7}{2} - e^{-1/2}\right) - \frac{1}{4}\left(2 - e^{-1/2}\right)\right)$

$= \frac{1}{4} + \frac{3}{8} = \frac{5}{8}$

(b) $P(X=2 \text{ or } X \in [0, 1.5))$

$= P(X=2) + P(0 \le X < 1.5)$

$= \left(F(2) - F(2-)\right) + \left(F(1.5-) - F(0-)\right)$

$= \left(\frac{1}{4}(4 - e^{-1}) - \frac{1}{4}\left(\frac{7}{2} - e^{-1}\right)\right) + \left(F(1.5) - 0\right)$

$= \frac{1}{8} + \frac{1}{4}\left(\frac{7}{2} - e^{-3/4}\right)$

(c) $P(X = 0.5 \text{ or } X \in (1, 2))$

$= P(X = 0.5) + P(1 < X < 2)$

$= \left(F(0.5) - F(0.5-)\right) + \left(F(2-) - F(1)\right)$

$= 0 + \left(\frac{1}{4}\left(\frac{7}{2} - e^{-1}\right) - \frac{1}{4}\left(\frac{7}{2} - e^{-\frac{1}{2}}\right)\right)$

$= 0 + \frac{1}{4}\left(e^{-\frac{1}{2}} - e^{-1}\right)$