Solution & marking scheme

Name:
Roll No.:

MSO201: PROBABILITY & STATISTICS
Quiz 2; Full Marks-20

[1] The joint p.d.f. of X_1 and X_2 is given by

$$f(x_1, x_2) = \frac{1}{4}\left(\frac{1}{2\pi\,|\Sigma_1|^{\frac{1}{2}}}\, e^{-\frac{1}{2}x'\Sigma_1^{-1}x}\right) + \frac{3}{4}\left(\frac{1}{2\pi\,|\Sigma_2|^{\frac{1}{2}}}\, e^{-\frac{1}{2}x'\Sigma_2^{-1}x}\right);$$

$$\boldsymbol{x} = (x_1, x_2)' \in \mathbb{R}^2;\ \Sigma_1 = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix};\ \Sigma_2 = \begin{pmatrix} 1 & -0.5 \\ -0.5 & 1 \end{pmatrix}.$$

Prove or disprove the following statements:

(a) $X_i \sim N(0,1)$ for $i = 1, 2$.

(b) $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2$.

(c) X_1 and X_2 are independent.

(d) $Cov(X_1, X_2) = \frac{1}{4}$.

Useful Information: If $\boldsymbol{X} \sim N_p(\boldsymbol{\mu},\ \Sigma)$ with $\Sigma > 0$, then p.d.f. of $\boldsymbol{X}$ is

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{p}{2}}\,|\Sigma|^{\frac{1}{2}}}\, e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)};\ \boldsymbol{x} \in \mathbb{R}^p$$

10 marks (2+2+2+4)

[2] $X_1, \ldots, X_n$ is i.i.d. random sample from $U(0,1)$ distribution.

(a) Find C such that $\frac{n}{\sum_{i=1}^{n} X_i^4} \xrightarrow{p} C$, as $n \to \infty$.

(b) Find $\lim_{n\to\infty} P\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 \le \frac{1}{3}\right)$.

10 marks (4+6)

(1)
(a) $f(x_1, x_2) = \frac{1}{4} f_1(x_1, x_2) + \frac{3}{4} f_2(x_1, x_2)$

$f_1(x_1, x_2)$: p.d.f. of $N_2(0, \Sigma_1)$

$f_2(x_1, x_2)$: p.d.f. of $N_2(0, \Sigma_2)$

Marginal of X_1 : $\int_{-\infty}^{\infty} f(x_1, x_2)\,dx_2 = \frac{1}{4}\int_{-\infty}^{\infty} f_1(x_1, x_2)\,dx_2 + \frac{3}{4}\int_{-\infty}^{\infty} f_2(x_1, x_2)\,dx_2$

$= \frac{1}{4}(\text{p.d.f. of } N(0,1)) + \frac{3}{4}(\text{p.d.f. of } N(0,1))$

($\because$ marginals of N_p are N_1)

$\Rightarrow X_1 \sim N(0,1)$

Marginal of X_2 : $\int_{-\infty}^{\infty} f(x_1, x_2)\,dx_1 = \text{p.d.f. of } N(0,1)$

$\Rightarrow X_2 \sim N(0,1)$

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(a) is proved.

(b) $f(x_1, x_2)$ is not p.d.f. of N_2

$$\Rightarrow \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \nsim N_2$$ (b) is disproved 2

(c) $$\underbrace{f(x_1, x_2) \neq \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_1^2}\right)}_{f(x_1)} \underbrace{\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_2^2}\right)}_{f(x_2)}$$ 2

$\Rightarrow X_1$ & X_2 are not independent

(c) is disproved

(d) $EX_1 = 0$ & $EX_2 = 0$

$$\Rightarrow Cov(X_1, X_2) = E X_1 X_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2)\, dx_1\, dx_2$$

$$= \frac{1}{4} \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} x_1 x_2 f_1(x_1, x_2)\, dx_1\, dx_2 + \frac{3}{4} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_2(x_1, x_2)\, dx_1\, dx_2$$

4 $$= \underbrace{\frac{1}{4} \underset{Cov(X_1,X_2) \text{ for } f_1}{(0.5)} + \frac{3}{4} \underset{Cov(X_1,X_2) \text{ for } f_2}{(-0.5)}} = -\frac{1}{4}$$ (d) is disproved

(2)

$X_1, \ldots, X_n$ i.i.d $U(0,1)$

(a) $X_1^4, \ldots, X_n^4$ i.i.d. with $E X_1^4 = \int_0^1 x^4 dx = \frac{1}{5}$

WLLN $\Rightarrow \frac{1}{n}\sum_1^n X_i^4 \xrightarrow{p} \frac{1}{5}$ as $n \to \infty$ 2

$\Rightarrow \frac{n}{\sum_1^n X_i^4} \xrightarrow{p} 5$ as $n \to \infty$ 2

(b) $X_1^2, \ldots, X_n^2$ are i.i.d with $E X_1^2 = \frac{1}{3}$

& $V X_1^2 = E X_1^4 - (E X_1^2)^2$

$= \frac{4}{45}$

By CLT,

$\sqrt{n}\left(\frac{1}{n}\sum_1^n X_i^2 - \frac{1}{3}\right) \xrightarrow{L} N\left(0, \frac{4}{45}\right)$ — 2.

So $P\left(\frac{1}{n}\sum_1^n X_i^2 \le \frac{1}{3}\right)$

$= P\left(\frac{1}{n}\sum X_i^2 - \frac{1}{3} \le 0\right)$

$= P\left(\frac{\sqrt{n}\left(\frac{1}{n}\sum X_i^2 - \frac{1}{3}\right)}{\sqrt{\frac{4}{45}}} \le 0\right)$

$\to \Phi(0) = \frac{1}{2}$ as $n \to \infty$ 4