Celestial Mechanics
beyond the boring ODEs

Vedant Shenoy

Indian Institute of Technology Bombay
Mumbai

January 13, 2020

Summary

1 **That Kepler boy and his weird circles**
   - Motivating Newton's Law

2 **apple**

That Kepler boy and his weird circles

The Laws of Kepler

1. The orbits of planets around the sun are in ellipses, with the Sun at one of the focii

2. The areal velocity of the planet is a constant

3. The square of the period of orbit of the planet in years is equal to the cube of the semi-major axis of the orbit in AU

The Laws of Kepler

1. The orbits of planets around the sun are in ellipses, with the Sun at one of the focii

2. The areal velocity of the planet is a constant

3. The square of the period of orbit of the planet in years is equal to the cube of the semi-major axis of the orbit in AU

The Laws of Kepler

1. The orbits of planets around the sun are in ellipses, with the Sun at one of the focii

2. The areal velocity of the planet is a constant

3. The square of the period of orbit of the planet in years is equal to the cube of the semi-major axis of the orbit in AU
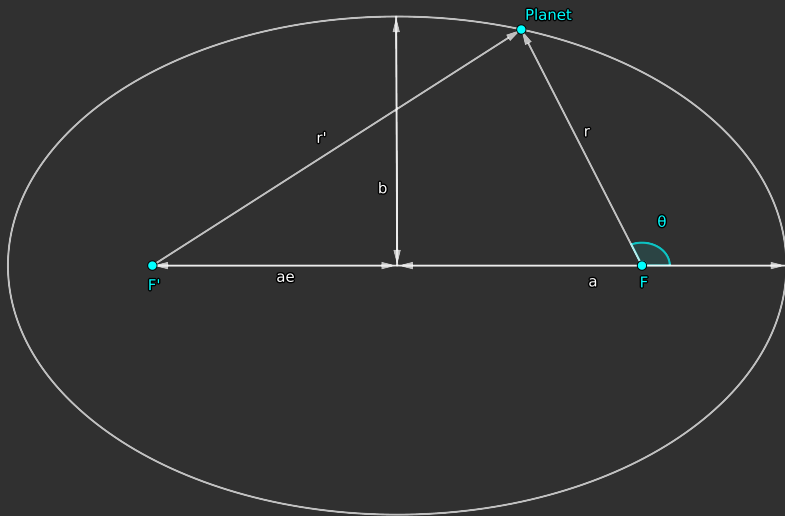
Figure: Orbit of a planet

The Laws of Kepler Reloaded

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \tag{1}$$

$$r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = \text{constant} \tag{2}$$

$$P^2 \text{ (in years)} = a^3 \text{ (in AU)}$$

The Laws of Kepler Reloaded

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} \tag{1}$$

$$r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = \text{constant} \tag{2}$$

$$P^2 \text{ (in years)} = a^3 \text{ (in AU)}$$

The Laws of Kepler Reloaded

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \tag{1}$$

$$r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = \text{constant} \tag{2}$$

$$P^2 \text{ (in years)} = a^3 \text{ (in AU)}$$

The Laws of Kepler Revolutions

$$\mathbf{F} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} \tag{3}$$

That Kepler boy and his weird circles
Motivating Newton's Law

Standing on the shoulders of giants (so not Hooke)

This is relatively simple to do with calculus

In plane polar co-ordinates, we have

$$\mathbf{r} = r\hat{\mathbf{r}}$$

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}$$

That last term in the third equation looks familiar.
$r^2\dot{\theta} = \text{constant} \implies 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = 0$, so immediately, Kepler's second law implies that the gravitational force is radial (apart from the obvious(?) conservation law)

Kepler's first law gives us

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

Differentiating with respect to time,

$$\dot{r} = -\frac{l}{(1 - e\cos\theta)^2} e\sin\theta \ \dot{\theta}$$

Now, we can use Kepler's second law ($\dot{A} = \frac{1}{2}r^2\dot{\theta} = k$) and substitute $\dot{\theta}$ in terms of r and put back Kepler's first law to substitute for the $(1 - e\cos\theta)$ term in the denominator to eventually get

$$\dot{r} = -\frac{2k}{l}e\sin\theta$$

and differentiating again and simplifying as above, we get

$$\ddot{r} = -\frac{4k}{lr^2}\left(1 - \frac{l}{r}\right)$$

Now, we can write the acceleration $\ddot{\boldsymbol{r}}$ as

$$\ddot{\boldsymbol{r}} = \left[-\frac{4k}{lr^2} \left(1 - \frac{l}{r} \right) - r \left(\frac{2k}{l} \right)^2 \right] \hat{\boldsymbol{r}} = -\frac{4k}{lr^2} \hat{\boldsymbol{r}}$$

So, we get the force to be of the form,

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}} = -\frac{4mk/l}{r^2} \hat{\boldsymbol{r}}$$
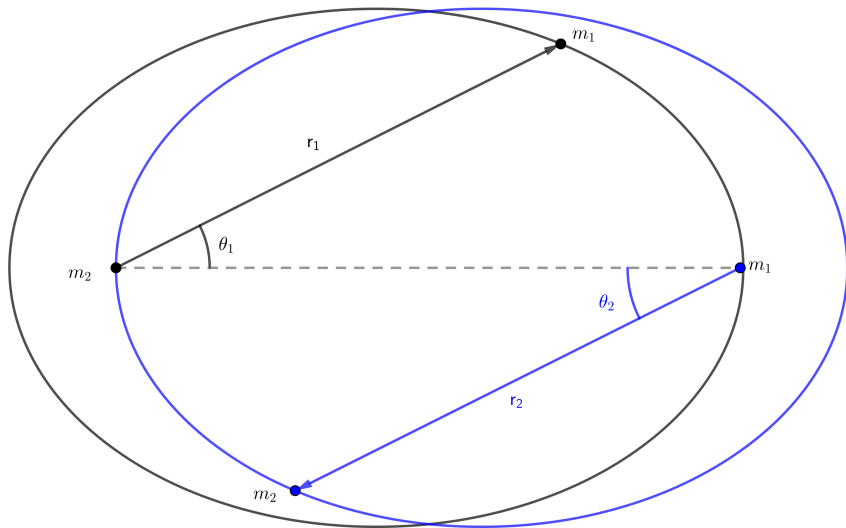
But wait...

We have more: the planet's frame of motion.
The same calculation will yield the same results, with the masses exchanged.

But wait...

We have more: the planet's frame of motion.
The same calculation will yield the same results, with the masses exchanged.

Confused?

From the frame of m_1, we can write the force $\boldsymbol{F}_{12}$ exerted by m_1 on m_2 as

$$\boldsymbol{F}_{12} = -\frac{4m_2 k_1}{l r_1^2} \hat{r_1}$$

and conversely,

$$\boldsymbol{F}_{21} = -\frac{4m_1 k_2}{l r_2^2} \hat{r_2}$$

Now, using Newton's Third Law, and accounting for the fact that $r_1 = r_2$ and $\hat{r_1} = -\hat{r_2}$, we get

$$\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$$
$$m_1 k_1 = m_2 k_2$$

i.e.

$$\frac{k_1}{k_2} = \frac{m_2}{m_1} = \alpha$$

α is a constant with respect to mass of the bodies, the distance between them and time. So, we can write the force as

$$\boldsymbol{F}_{12} = -\frac{4\alpha m_1 m_2}{r^2} \hat{\boldsymbol{r}}$$

This should make it obvious why Newton's Law looks the way that it does.

$$\mathbf{F} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} \tag{4}$$

apple

Now, we go the other way around. Let us assume Newton is right, and see what we get.

From the fact that the force is central, we immediately get the Conservation of Angular Momentum, and are able to define a plane of motion of the two bodies.

The usual derivation is straightforward and lengthy (i.e. not fun), so we shall take the scenic route by considering the quantity (for any general central force, $\mathbf{F} = f(r)\hat{\mathbf{r}}$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p} \times \mathbf{L}) = \dot{\mathbf{p}} \times \mathbf{L} = f(r)\hat{\mathbf{r}} \times (m\mathbf{r} \times \dot{\mathbf{r}})$$

$$= \frac{mf(r)}{r}(\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}))$$

$$= \frac{mf(r)}{r}(r\dot{r}\mathbf{r} - r^2\dot{\mathbf{r}})$$

$$= mf(r)r^2\left(\frac{\dot{r}\mathbf{r}}{r^2} - \frac{\dot{\mathbf{r}}}{r}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p} \times \mathbf{L}) = mf(r)r^2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right)$$

If only $f(r)\ r^2$ were a constant of some sort. Then we could have a quantity called the Laplace-Runge-Lenz vector which would have been constant with time.

We define

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}}$$

and we know that $\dot{\mathbf{A}} = 0$ from earlier (and $k = GMm$). A very good question to ask at this point would be "What is this vector?"
What is constant in the orbital motion of a planet?

We define

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}}$$

and we know that $\dot{\mathbf{A}} = 0$ from earlier (and $k = GMm$). A very good question to ask at this point would be "What is this vector?"
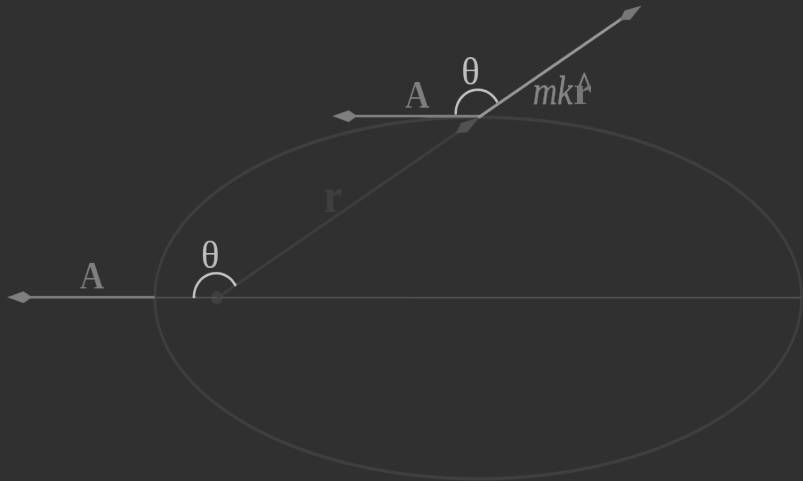What is constant in the orbital motion of a planet?

The orbit.

Figure: Wikipedia

Since we want to know the variation of θ with r, we can take a dot product:

$$\mathbf{A} \cdot \mathbf{r} = Ar\cos\theta = \mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) - mkr$$

$$\mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) = (\mathbf{r} \times \mathbf{p}) \cdot \mathbf{L} = \mathbf{L} \cdot \mathbf{L} = L^2$$

So, without any pain whatsoever, we have

$$r = \frac{L^2/mk}{1 + A/mk \cos\theta}$$

Note that $e = \frac{A}{mk}$, and so $\frac{\mathbf{A}}{mk}$ is often called the eccentricity vector.

Taking the dot product of $\mathbf{A}$ with itself gives the expression for energy in terms of the eccentricity

$$A^2 = m^2k^2 + 2mEL^2$$

$$e^2 = 1 + \frac{2L^2}{mk^2}E$$

where $E = \frac{p^2}{2m} - \frac{k}{r}$

The End