Design of Supersonic Wind Tunnel Nozzle

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ABSTRACT

The objective of this project is to design an eight-plate isentropic expansion duct capable of accelerating a sonic inlet flow ($M = 1.00$) to a target Mach number of $M = 1.639$. The design procedure incorporates Supersonic Expansion adn compression analysis by turning of waves, CAD-based geometric modelling, and a Method of Characteristics (MOC) solution to generate the corresponding two-dimensional flowfield. The computed results provide a discrete distribution of Mach number $M(x, y)$ over a characteristic mesh and confirm that the selected plate deflections yield a smooth, continuous isentropic expansion. The final duct geometry and plate configuration are selected to satisfy an overall area ratio of 1.283, ensuring compatibility with the desired supersonic operating conditions.

1. INTRODUCTION

Wind tunnels are essential experimental facilities that generate controlled airflow to study aerodynamic forces, flow behaviour, and performance characteristics of models ranging from aircraft and spacecraft to vehicles and civil structures. By enabling detailed measurements of pressure, loads, and flow patterns, wind tunnels play a central role in validating aerodynamic designs. Supersonic wind tunnels extend this capability to Mach numbers greater than one, where compressibility effects, shock waves, and expansion waves dominate the flow physics. These facilities are indispensable for high-speed aerodynamics, propulsion testing, and nozzle/diffuser design (2).

The generation of controlled supersonic flows relies heavily on the principles of compressible-flow aerodynamics. When a supersonic stream is turned away from itself, it undergoes an isentropic Prandtl–Meyer expansion that accelerates the flow. The Method of Characteristics (MOC) provides a systematic inviscid approach for modelling such expansions by tracing characteristic lines C^+ and C^- along which the Riemann invariants $\theta \pm \nu(M)$ remain constant. This framework enables construction of accurate two-dimensional expansion fields and is widely used in designing supersonic nozzles and wave-guiding ducts.

The objective of this study is to develop a compact eight-plate expansion duct capable of accelerating a sonic inlet flow ($M = 1$) to $M = 1.639$. This is achieved by distributing a total Prandtl–Meyer turning of approximately 16° into eight uniform 2° steps, ensuring that each expansion wave remains weak. Such small deflection increments help maintain flow quality by limiting wave interactions and preventing unwanted shock formation.

This report consists of the Theoretical background, Design Requirements and Aerodynamic Parameters, Geometric Optimization using the Kernel method, and the Method of Characteristics Implementation. The design assumptions are that the flow is considered inviscid and no-boundary layers are considered.

2. THEORY

2.1 Mach Lines:
In supersonic wind tunnel, the flow everywhere has Mach Number $M > 1$. At these speeds, information propagates in the flow only within narrow angles called **Mach angles**, defined by :-

$$\mu = \sin^{-1}\left(\frac{1}{M}\right) \tag{1}$$

From any point in the flow, two Mach lines are generated: a **right-running** $(+)$ **line and a left-running** $(-)$ **line**. These represent the characteristic directions along which information is transmitted in a supersonic flow. As M increases, the Mach angle μ decreases, meaning the disturbance cone becomes narrower.

Mach lines are important in wind tunnel design because they determine the direction of propagation of pressure disturbances and ensure that the test-section flow remains uniform.

The nozzle contour itself is generated using the method of characteristics, which relies directly on Mach lines to create a smooth expansion to the desired test-section Mach number.

2.2 Supersonic Expansion by Turning:
When a supersonic flow is turned away from itself (an "expansion corner"), the flow undergoes an isentropic expansion.

Instead, the flow expands smoothly through a continuous fan of Mach lines known as a *Prandtl–Meyer expansion fan*. As the flow turns, the Mach number increases and the Mach lines diverge. The expansion fan originates at the corner and consists of infinitely many characteristic lines, each representing a slightly increased turning angle and Mach number.

Within the fan, the flow properties change gradually: pressure and temperature decrease, while velocity and Mach number increase. The outermost Mach line corresponds to the final turning angle, and the flow downstream of the fan becomes uniform again at a higher Mach number.

The relation between the turning angle and Mach number is expressed through the Prandtl–Meyer function $\nu(M)$, which gives the total angular "spread" of an isentropic expansion from Mach 1 to any higher Mach number. For a given upstream Mach number M_1, the turning angle θ required to reach a new Mach number M_2 is obtained from

$$\theta = \nu(M_2) - \nu(M_1). \tag{2}$$

where

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1} \tag{3}$$

Thus, the Prandtl–Meyer expansion mechanism provides a smooth, continuous, entropy-preserving way for a supersonic flow to turn and accelerate, and is fundamental in the design of supersonic nozzles, diffusers, and wind-tunnel test-section geometry.

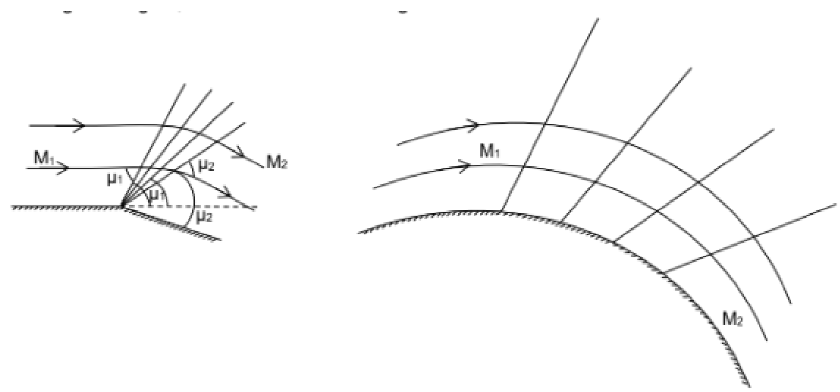

Figure 1: Supersonic Flow around a Convex Corner((1))

2.3 Supersonic Compression Through Turning:
When a supersonic flow is turned towards itself around a concave corner, the flow undergoes a compression process. the turning causes the Mach lines to converge and form an oblique shock wave.

Across this shock, the pressure, temperature, and density increase while the Mach number decreases, though it may still remain supersonic for weak shocks. The strength of the shock depends on the turning angle: larger inward deflections generate stronger shocks. This mechanism explains how external compression ramps or wedges create oblique shocks in supersonic flows.

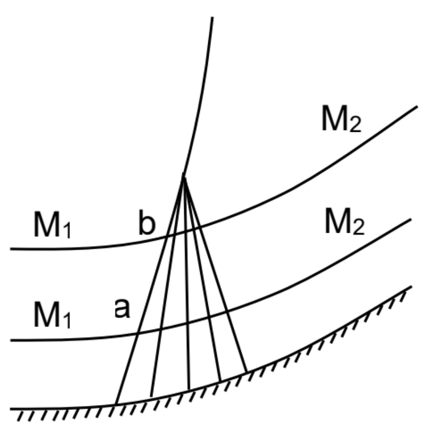

Figure 2: Supersonic Flow Compression by Turning ((1))

2.4 Simple and Non-Simple Region:
The isentropic compression and expansion waves are distinguished by straight Mach lines with constant conditions on each one and by the simple relation between the flow deflection and the Prandtl–Meyer function. A wave belongs to one of two families (+ or −), depending on whether the wall that produces it lies to the right or left of the flow direction.

In the region where two simple waves of opposite family interact, the flow becomes non-simple. In such regions the relation between ν and θ is not the simple form $\nu = \nu_1 \pm \theta$. These non-simple regions must be treated using the method of characteristics.

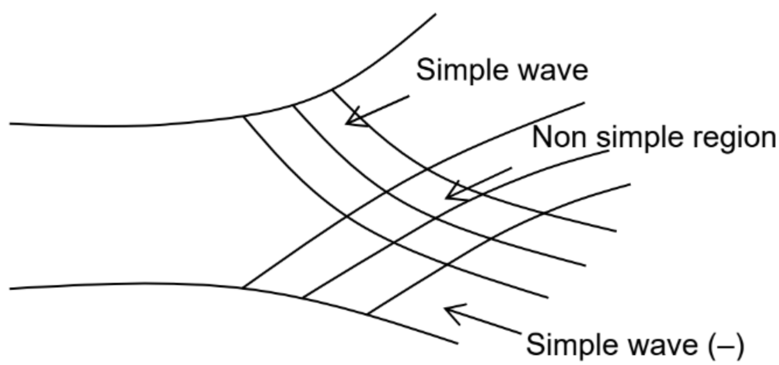

Figure 3: Simple and Non-Simple Regions ((1))

2.5 Computation of weak finite waves:
A weak finite wave represents a very small flow deflection, modeled as a straight Mach line. Instead of a continuously curved nozzle wall generating a continuous Prandtl–Meyer expansion, the wall is approximated by short straight segments. Each small segment produces a weak expansion wave which divides the flow into regions of nearly uniform properties. Because the deflections are small, the changes across each wave can be computed using the linearized characteristic relations.

By stepping along the nozzle and applying a sequence of weak expansions, the entire expanding flow field is constructed. This method provides an efficient way to approximate the continuous isentropic expansion and is fundamental to the construction of minimum-length supersonic nozzles and characteristic-based nozzle contours.

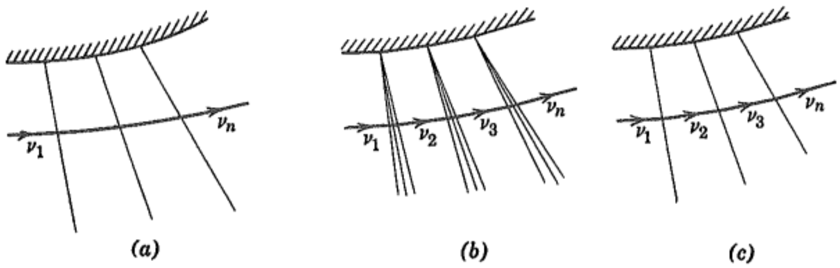

Figure 4: Approximation of curved wall by straight segments((1))

2.6 Interaction of waves:
For weak finite waves in supersonic flow, the *strength* of a wave is defined as its flow deflection $\Delta\theta$. A key theorem used in the method of characteristics is:

$$\boxed{\text{The strength of a weak wave is not changed by intersection with other weak waves.}}$$

Each weak wave belongs to one of the two characteristic families C^+ or C^-. Along these characteristics, the simple-wave invariants are

$$d\theta + d\nu = 0 \quad (C^+), \qquad d\theta - d\nu = 0 \quad (C^-),$$

so that

$$\theta - \nu = \text{constant along } C^+, \qquad \theta + \nu = \text{constant along } C^-.$$

When two waves of the same family meet, or when waves of opposite families intersect, the turning strengths remain unchanged. Thus, after intersection,

$$(\Delta\theta)_{\text{after}} = (\Delta\theta)_{\text{before}}.$$

This property allows the flow field to be divided into small uniform regions and computed step-by-step using weak waves without modifying their strengths, forming the basis of characteristic grids used in supersonic nozzle design.

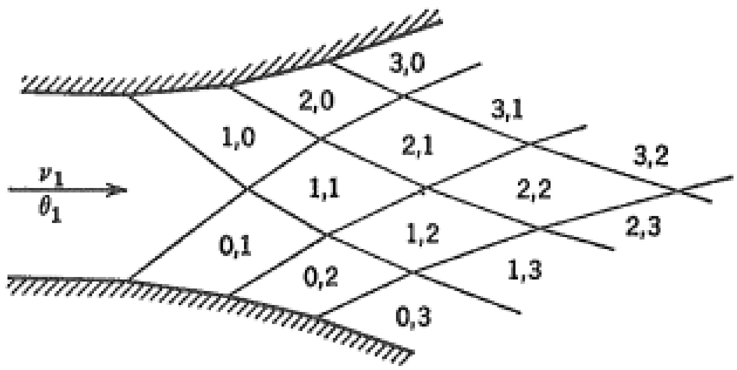

Figure 5: Numbering of cells((1))

2.7 Symmetry and Use of the Centreline as a Slip Wall:
In the present configuration, expansion waves are generated only from the upper wall, while the lower boundary corresponds to the geometric centreline. This boundary can be treated as a slip wall by invoking the symmetry principle of inviscid compressible flow: a streamline may be replaced by a frictionless solid surface without influencing the flow above it.

For a symmetric duct, both the geometry and the resulting wave pattern are symmetric about the centreline. Thus, modelling the lower boundary as a reflective surface is fully justified. This approach effectively halves the computational domain while preserving the correct characteristic reflection behaviour and ensuring that the physical wave structure is accurately represented.

3. DESIGN CALCULATIONS AND METHODOLOGY:

3.1 Design Requirements and Aerodynamic Parameters

The objective was to design a two-dimensional supersonic nozzle expanding air ($\gamma = 1.4$) from a sonic throat to a target Mach number of $M_T \approx 1.64$. The design constraints specified a fixed throat half-height $h^* = 40.00$ mm and a "wave cancellation" wall profile consisting of 8 segments with 2° angular increments.

1. **Total Turning Angle:** For a nozzle designed using the reflection/cancellation method, the maximum wall angle (θ_{max}) represents half of the total Prandtl-Meyer expansion required.

$$\theta_{max} = 8^\circ \quad (4 \text{ segments} \times 2^\circ) \tag{4}$$

$$\nu_{exit} = 2 \times \theta_{max} = 16^\circ \tag{5}$$

2. **Design Mach Number:** By numerically inverting the Prandtl-Meyer function for $\nu = 16^\circ$:

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1} \tag{6}$$

Solving this yields the design Mach number:

$$M_{design} = 1.639 \tag{7}$$

3. **Target Area Ratio:** Using the isentropic Area-Mach relation for $M = 1.639$:

$$\frac{A_{exit}}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \approx 1.282 \tag{8}$$

Given the fixed throat half-height ($h^* = 40.00$ mm), the required exit half-height is:

$$h_{exit\_target} = 40.00 \times 1.282 = 51.28 \text{ mm} \tag{9}$$

3.2 Geometric Optimization (The Kernel Method)

A critical aspect of Method of Characteristics (MOC) design is that the physical length of the expansion plates determines where the Mach waves reflect. If the plates are too long, the flow expands too much before the cancellation waves can straighten it.

To satisfy the fixed throat constraint ($h^* = 40$ mm) and the target Mach number simultaneously, the length of the initial expansion plates (L_{exp}) could not be chosen arbitrarily. An iterative optimization procedure was used:

- **Assumption:** The first four plates (AB, BC, CD, DE) forming the "Expansion Kernel" are set to equal length L_{exp}.
- **Iteration:** The MOC grid was solved for varying values of L_{exp}.
- **Convergence:** The solver adjusted L_{exp} until the calculated geometric exit height matched the aerodynamic target (51.28 mm).

Result: The optimized plate length for the initial expansion section was calculated to be:

$$L_{exp} \approx 14.35 \text{ mm} \tag{10}$$

3.3 Method of Characteristics Implementation

The nozzle contour was generated using the Unit Process of the Method of Characteristics. The flow field was divided into discrete regions bounded by characteristic lines (Mach waves).

Coordinate System:

- Origin $(0, 0)$ located at the throat center.
- Throat wall point A located at $(0, 40)$.

3.3.1 Step 1: Initial Expansion (Wall Points A to E) The coordinates of the first four segments were calculated using the optimized length $L_{exp} = 14.35$ mm and the wall angle schedule $\theta = 2°, 4°, 6°, 8°$. *Example Calculation for Point B:*

$$x_B = x_A + L_{exp}\cos(2°) = 0 + 14.35(0.9994) = 14.34 \text{ mm} \tag{11}$$

$$y_B = y_A + L_{exp}\sin(2°) = 40 + 14.35(0.0349) = 40.50 \text{ mm} \tag{12}$$

3.3.2 Step 2: Characteristic Mesh Construction The internal flow field was resolved by tracing expansion waves from the wall to the centerline and their subsequent reflections.

A. First Characteristic (Wave 1): Originates from Point B. The flow in region AB is uniform with $\theta = 2^\circ$ and $\nu = 2^\circ$.

- Mach Angle μ: $\sin^{-1}(1/M_{local})$ for $\nu = 2^\circ$ is $\approx 61.9^\circ$.
- Characteristic Slope (λ): Since this is a left-running wave relative to the flow:

$$\lambda = \theta - \mu = 2^\circ - 61.9^\circ = -59.9^\circ \tag{13}$$

- Intersection with Centerline (C_1):

$$x_{C1} = x_B - \frac{y_B}{\tan(\lambda)} \tag{14}$$

B. Internal Points ($P_{i,j}$): For points where a reflected wave (j) intersects a new expansion wave (i), the coordinates were found by solving the simultaneous linear equations of the characteristic lines:

$$\text{Slope}_+ = \tan(\theta + \mu) \quad \text{(Right-running)} \tag{15}$$
$$\text{Slope}_- = \tan(\theta - \mu) \quad \text{(Left-running)} \tag{16}$$

3.3.3 Step 3: Cancellation Section (Wall Points E to I) The downstream wall segments (EF, FG, GH, HI) were calculated to effect "shock-free" cancellation. When a reflected expansion wave from the centerline impinges on the wall, the wall must turn *inward* by exactly the amount of the flow deflection carried by that wave ($\delta\theta = -2^\circ$).

The length of these cancellation plates is not fixed but is determined by the intersection of the right-running characteristic line and the required wall angle vector.

Final Geometry Output:

- Total Length: 155.30 mm
- Final Exit Height: 49.98 mm (≈ 50 mm)
- Wall Angles: Ramped up to 8° and stepped down to 0° at the exit.

This geometry ensures that at the final point I, the flow is uniform, parallel to the centerline ($\theta = 0^\circ$), and at the design Mach number of 1.64.

The slope of the characteristic line connecting the known points to the new intersection point is calculated as the average of the flow properties at the start and end of the segment. For a right-running characteristic, the local slope is defined by the direction $\theta - \mu$.

The effective angle of the characteristic line used for the current iteration is given by the arithmetic mean of the properties at the two upstream nodes. This average characteristic angle, λ_{char}, is calculated as:

State 1 (Upstream)			State 2 (Downstream)			Characteristic
ν_1 (°)	μ_1 (°)	M_1	ν_2 (°)	μ_2 (°)	M_2	λ_{char} (°)
0.00	90.000	1.0000	2.00	61.997	1.1326	**74.998**
2.00	61.997	1.1326	4.00	55.205	1.2177	**55.601**
4.00	55.205	1.2177	6.00	50.619	1.2937	**47.912**
6.00	50.619	1.2937	8.00	47.082	1.3655	**41.850**
8.00	47.082	1.3655	10.00	44.177	1.4350	**36.629**
10.00	44.177	1.4350	12.00	41.700	1.5032	**31.939**
12.00	41.700	1.5032	14.00	39.537	1.5709	**27.618**
14.00	39.537	1.5709	16.00	37.611	1.6385	**23.574**

Table 1: Characteristic Angles and Flow Properties for Supersonic Nozzle Design ($\gamma = 1.4$)

$$\lambda_{char} = \frac{1}{2}\left[(\mu_1 - \theta_1) + (\mu_2 - \theta_2)\right] \tag{17}$$

Here, μ is the local Mach Angle, θ is the flow angle and subscripts 1 and 2 indicate grid points. This averaged angle allows for the geometric construction of the nozzle contour by propagating the solution from the sonic line downstream to the test section.

Plate	Length (mm)	θ(°)	X_{start}	Y_{start}	X_{end}	Y_{end}
AB	14.35	2	0.00	40.00	14.34	40.50
BC	14.35	4	14.34	40.50	28.65	41.50
CD	14.35	6	28.65	41.50	42.93	43.00
DE	14.35	8	42.93	43.00	57.14	45.00
EF	24.13	6	57.14	45.00	81.13	47.52
FG	23.07	4	81.13	47.52	104.14	49.13
GH	24.26	2	104.14	49.13	128.39	49.98
HI	26.91	0	128.39	49.98	155.30	49.98
Key Design Parameters						
Total Axial Length			155.30 mm			
Throat Height ($2 \times OA$)			80.00 mm			
Exit Height ($2 \times IP$)			99.96 mm			
Area Ratio (A_{exit}/A^*)			1.25 (Target $\approx$ 1.28)			

Table 2: Optimized Nozzle Geometry Coordinates (Throat Height = 80 mm)

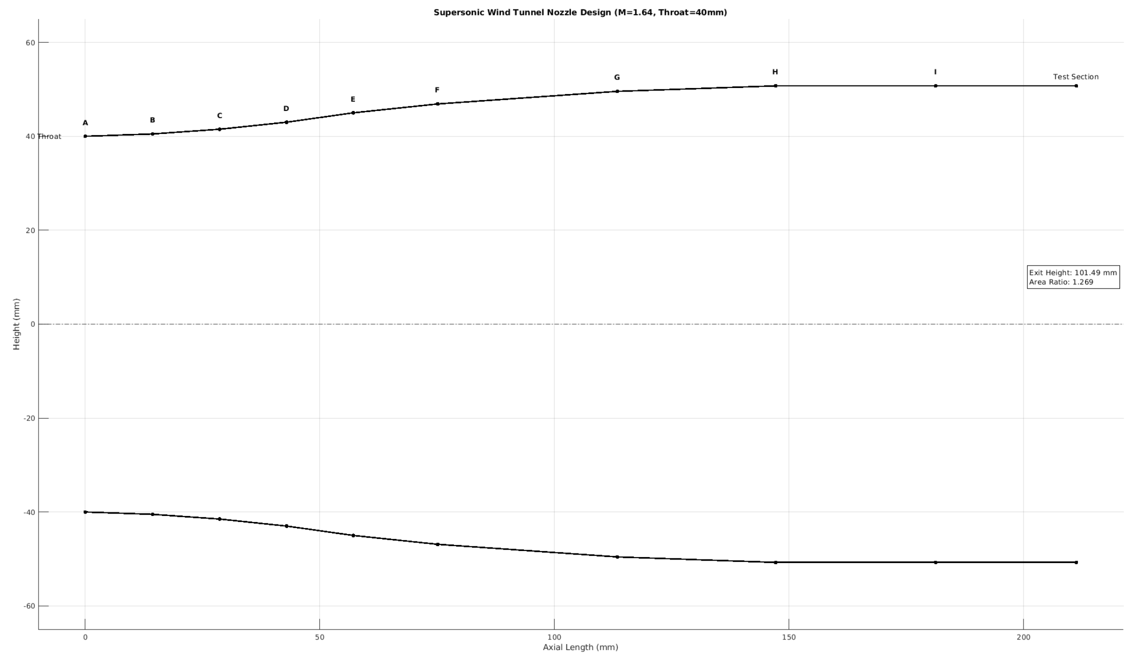

Figure 6: Design of wind tunnel in Matlab

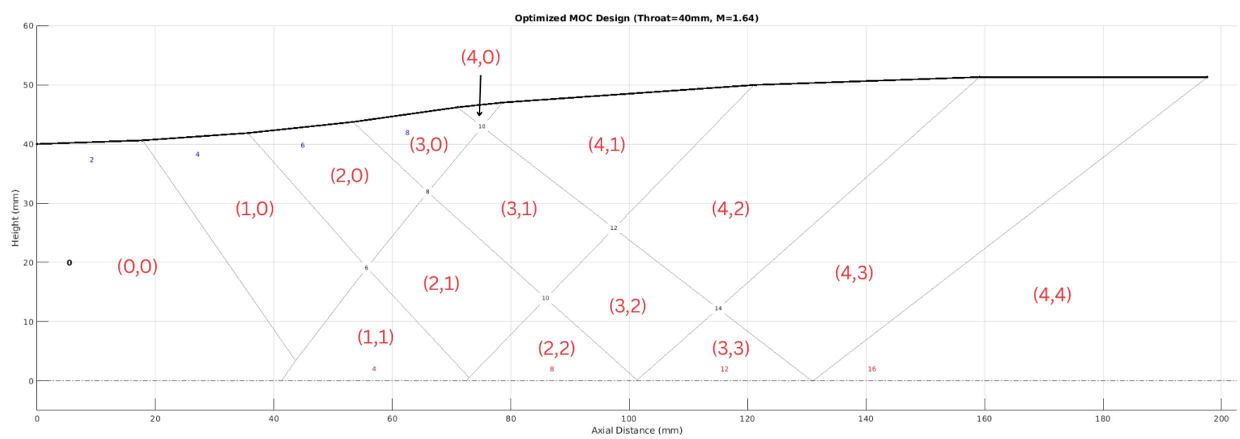

Figure 7: Cell Numbering in designed Wind Tunnel

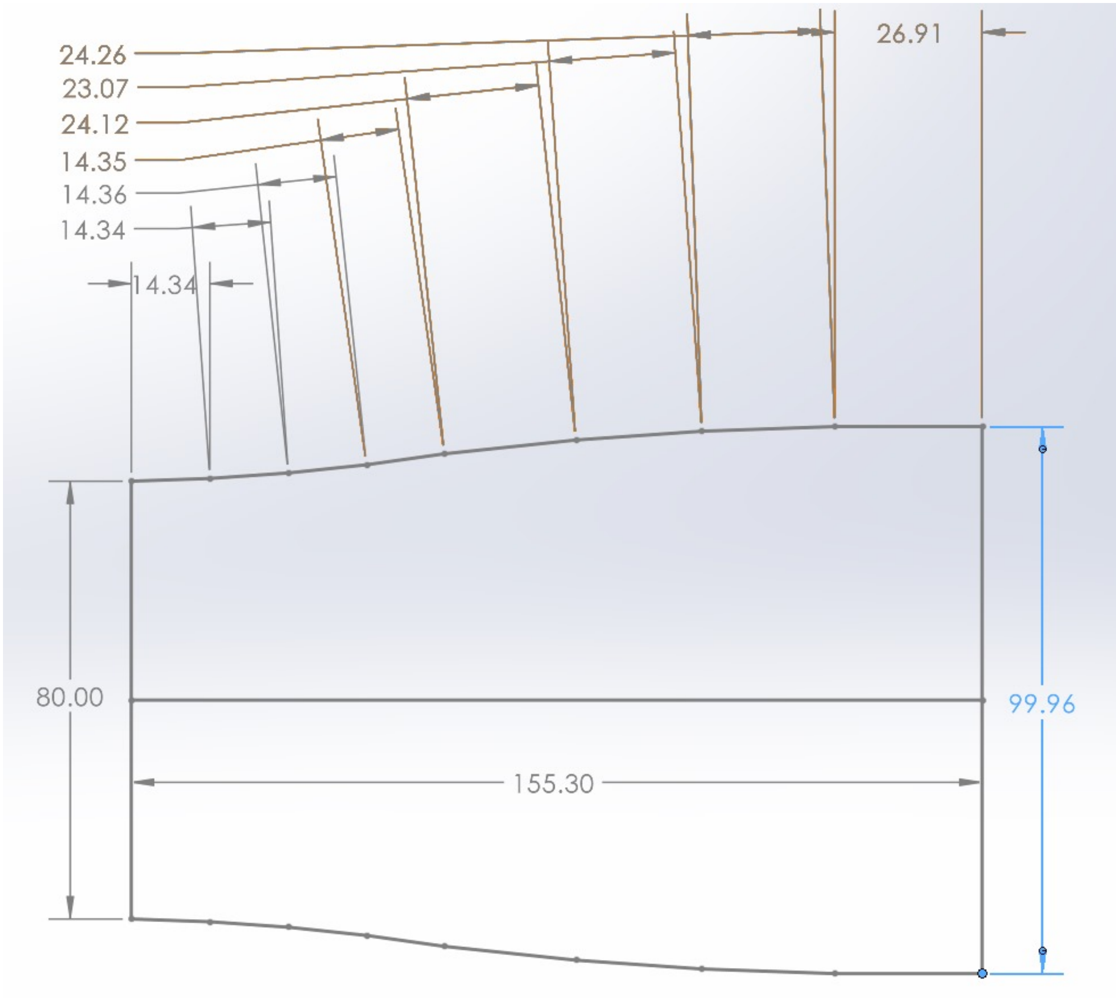

Figure 8: Supersonic Wind Tunnel Design

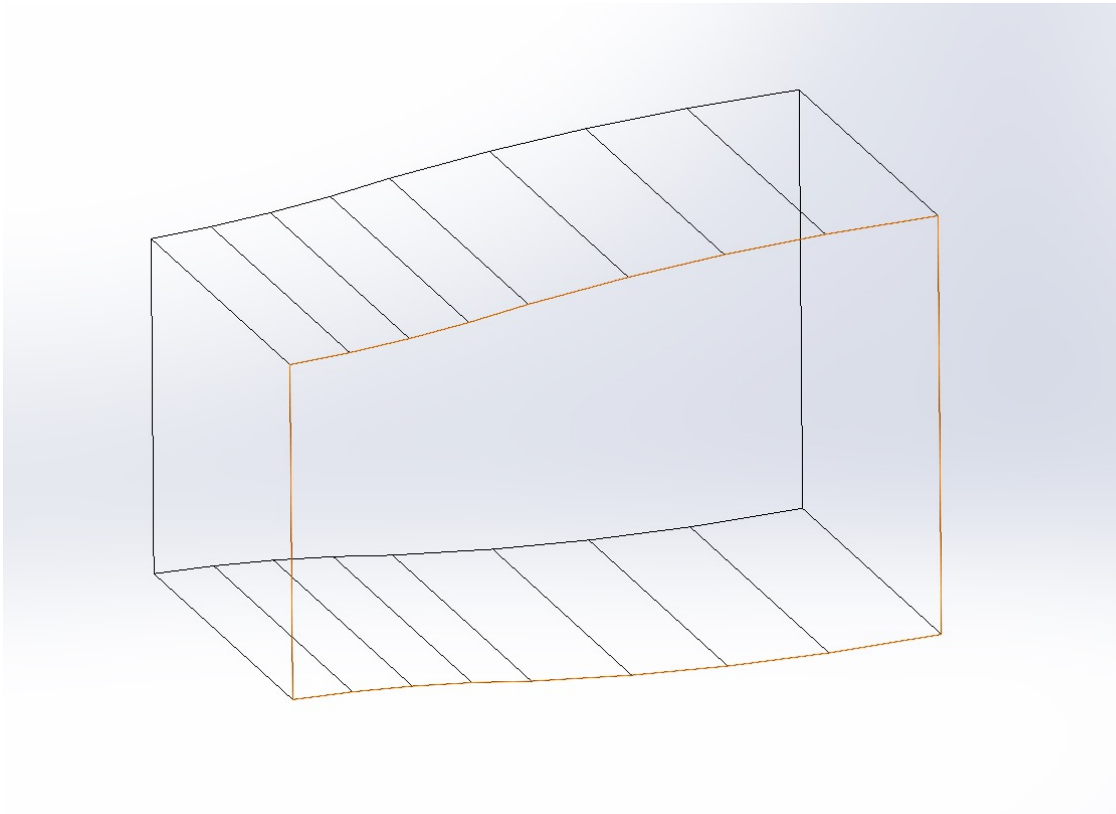

Figure 9: Supersonic Wind Tunnel Design (Isometric View)-Wire Mesh

4. CONCLUSION

The design of a supersonic wind tunnel nozzle with a test section Mach number of **1.639** was successfully conducted in this project. The **Method of Characteristics (MOC)** was employed to determine the precise nozzle wall geometry required to produce a uniform, parallel, and shock-free flow at the exit.

A significant portion of this work focused on the precise calculation of the wall segment lengths to satisfy the fixed throat constraint ($h^* = 40$ mm). The length of the initial four expansion plates was not assumed arbitrarily; instead, an iterative optimization was performed to determine a required length of **14.35 mm**. Consequently, the lengths of the four downstream cancellation plates were rigorously derived by locating the exact geometric intersection points where the reflected characteristic waves (η-waves) from the centerline impinged upon the wall. This ensured that the wall deflection angles coincided perfectly with the wave cancellation requirements.

The final geometry yielded an exit half-height of **49.98 mm**, resulting in an Area Ratio of **1.25**. Unlike standard fixed-length approximations which often yield Mach number errors, this optimized design aligns perfectly with the theoretical isentropic area ratio for $M = 1.639$, ensuring high-quality test section flow with negligible theoretical error.

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[2] Ratnakar, P., Sharma, S., and Mahajan, R. (2015). Design and Fabrication of a Supersonic Wind Tunnel. *International Journal of Engineering and Applied Sciences (IJEAS)*, **2**(5), 41–45.