MATHEMATICS II

(MAC 02)

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LAPLACE TRANSFORMATION

A transformation is a mathematical device that converts a function into another function.

Laplace transformation is an integral transform that converts a function of real variable into a function of complex variable.

It is a widely used device which is particularly very effective for solving the linear differential equations – both ordinary and partial. It reduces an ordinary differential equation into an algebraic equation.

Let *f(t)* be a given function defined for all t ≥ 0. We multiply *f(t)* by e^{-st} and integrate w.r.t. t from zero to infinity. Then, if the resulting integral exists, it is a function of s, say $F(s)$:

$$\boldsymbol{F(s) = \bar{f}(s) = \int_0^\infty e^{-st} f(t) dt}$$

where t > 0 and s is complex variable.

This function $\boldsymbol{F(s)}$ is known as **Laplace transform** of the function f(t) provided the integral exist. It is also sometimes denoted by $\mathscr{L}(f)$.

L.T. OF ELEMENTARY FUNCTIONS:

I. $f(t) = 1$

$$\bar{f}(s) = \int_0^\infty e^{-st}.1dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s},\ \ s > 0$$

II. $f(t) = e^{at}$

$$\bar{f}(s) = \int_0^\infty e^{-st}.e^{at}dt = \int_0^\infty e^{-(s-a)t}dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a},\ \ s > a$$

III. $f(t) = t^n,\ n > -1$

$$\bar{f}(s) = \int_0^\infty e^{-st}.t^n dt$$

$$= \int_0^\infty e^{-k} \left(\frac{k}{s}\right)^n \frac{dk}{s} \text{ , (put } st = k \; so \; dt = \frac{dk}{s}\text{)}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-k} k^n dk$$

$$= \frac{\Gamma(n+1)}{s^{n+1}}, if \; n > -1 \; and \; s > 0$$

IV. $f(t) = \cos at$

$$\bar{f}(s) = \int_0^\infty e^{-st} . \cos at \, dt = \frac{s}{s^2+a^2}$$

V. $f(t) = \sin at$

$$\bar{f}(s) = \int_0^\infty e^{-st} . \sin at \, dt = \frac{a}{s^2+a^2}$$

VI. $f(t) = \cosh at$

$$\bar{f}(s) = \int_0^\infty e^{-st} . \cosh at \, dt = \int_0^\infty e^{-st} . \frac{e^{at}+e^{-at}}{2} dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t} dt + \int_0^\infty e^{-(s+a)t} dt\right]$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s}{s^2-a^2}$$

VII. $f(t) = \sinh at$

$$\bar{f}(s) = \int_0^\infty e^{-st} . \sinh at \, dt = \int_0^\infty e^{-st} . \frac{e^{at}-e^{-at}}{2} dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t} dt - \int_0^\infty e^{-(s+a)t} dt\right]$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2}$$

PROPERTIES OF L.T.

1. **Linearity property:**

If $L[f(t)] = \bar{f}(s)$ and $L[g(t)] = \bar{g}(s)$ then

$L\{af(t) \pm bg(t)\} = aL[f(t)] \pm bL[g(t)] = a\bar{f}(s) \pm b\bar{g}(s)$

2. **First shifting property:**

$L[f(t)] = \bar{f}(s)$ then $L[e^{at}f(t)] = \bar{f}(s-a)$

Proof:

$L[e^{at}f(t)] = \int_0^\infty e^{-st}e^{at}f(t)dt$

$= \int_0^\infty e^{-(s-a)t}f(t)dt$

$= \int_0^\infty e^{-pt}f(t)dt,$ (put $s-a=p$)

$= \bar{f}(p) = \bar{f}(s-a)$

$f(t)$	$L[e^{at}f(t)]$
t^n	$\frac{n!}{(s-a)^{n+1}}$
$\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$\cosh bt$	$\frac{s-a}{(s-a)^2-b^2}$
$\sinh bt$	$\frac{b}{(s-a)^2-b^2}$

3. **Change of scale:**

If $L[f(t)] = \bar{f}(s)$ then $L[f(at)] = \frac{1}{a}\ \bar{f}\left(\frac{s}{a}\right)$

Proof:

We know that $L[f(t)] = \bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$

$L[f(at)] = \int_0^\infty e^{-st} f(at)dt$

$= \frac{1}{a}\int_0^\infty e^{-\frac{sp}{a}} f(p)dp$ put $at = p, dt = \frac{dp}{a}$

$= \frac{1}{a}\,\bar{f}\left(\frac{s}{a}\right)$

Ex: Let $L[\sinh t] = \frac{1}{s^2-1}$ then $L[\sinh 2t] = \frac{1}{2}\frac{1}{\left(\frac{s}{2}\right)^2-1} = \frac{2}{s^2-4}$

Ex: Let $L[f(t)] = \frac{s+1}{s^2-2}$, then $L[f(3t)] = \frac{1}{3}\frac{\frac{s}{3}+1}{\left(\frac{s}{3}\right)^2-2} = \frac{s+3}{s^2-18}$

4. **Change of scale shifting:**

If $L[f(t)] = \bar{f}(s)$ then $L[e^{bt} f(at)] = \frac{1}{a}\ \bar{f}\left(\frac{s-b}{a}\right)$

For example,

$L[\cos t] = \frac{s}{s^2+1}$ then $L[e^{2t}\cos 3t] = \frac{1}{3}\frac{\frac{s-2}{3}}{(\frac{s-2}{3})^2+1} = \frac{s-2}{(s-2)^2+3^2}$

EXISTENCE CONDITIONS

L.T. does not exist for all functions. If it exists it is uniquely determined. The following are the conditions to be satisfied:

Let f(t) be the given function. If

1. f(t) is piecewise continuous on every finite interval
2. f(t) satisfy the following inequality:
   $|f(t)| \leq b.e^{at}$ for all $t \geq 0$ for some constants a and b,
   then $L[f(t)]$ exists.

   The function which satisfies the condition (2) is called of exponential order.
   For example $\cosh t < e^t\ \ \forall\, t > 0$, $\ t^n < n!\, e^t$, $(n = 0,1,2, \dots)\ \forall\, t > 0$

But $e^{t^2} > be^{at}$ whatever may be a and b. so $L\{e^{t^2}\}$ does not exist.

Similarly $\frac{1}{t}$ does not have L.T.

Examples:

1. Find the L.T of

(a) $\cos 3t \cos 4t$

(b) $\sin \sqrt{t}$

Solution:

(a) Here $\cos 3t \cos 4t = \frac{1}{2}(\cos 7t + \cos t)$

$$\therefore L[\cos 3t \cos 4t] = \frac{1}{2}(L[\cos 7t] + L[\cos t])$$

$$= \frac{1}{2}\left(\frac{s}{s^2+49} + \frac{s}{s^2+1}\right) = \frac{s(s^2+25)}{(s^2+49)(s^2+1)}$$

(b) $$L[\sin \sqrt{t}] = L\left[\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \cdots\right]$$

$$= L\left[t^{\frac{1}{2}}\right] - \frac{1}{3!}L\left[t^{\frac{3}{2}}\right] + \frac{1}{5!}\ L\left[t^{\frac{5}{2}}\right] - \cdots$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} - \frac{1}{3!}\frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} + \frac{1}{5!}\ \frac{\Gamma\left(\frac{7}{2}\right)}{s^{\frac{7}{2}}} - \cdots, \quad \left[\ L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}, if\ n > -1\ and\ s > 0\ \right]$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{\frac{3}{2}}} - \frac{1}{6}\frac{\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}}{s^{\frac{5}{2}}} + \frac{1}{120}\frac{\left(\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}\right)}{s^{\frac{7}{2}}} - \cdots \qquad [\text{since } \Gamma(n+1) = \text{n}\,\Gamma(n)\ ]$$

$$= \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-\frac{1}{4s}}$$

Example 2. Find the L.T of

a. $\cosh at \sin at$
b. $t \sin at$
c. $\cosh(5t + 2)$

Solution:

a. $\cosh at = \frac{e^{at}+e^{-at}}{2}$

$$L[\cosh at \sin at] = \frac{1}{2}(L(e^{at}\sin at) + L[e^{-at}\sin at])$$

$$= \frac{1}{2}\left(\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2}\right) = \frac{a(s^2+2a^2)}{s^4+4a^4}$$

b. $L\{te^{iat}\} = \frac{1}{(s-ia)^2}$

Now $L\{te^{iat}\} = L[t(\cos at + i\sin at)] = \frac{(s+ia)^2}{(s^2+a^2)^2}$

$$L[t\cos at] + i\,L[t\sin at] = \frac{(s^2-a^2)+2ias}{(s^2+a^2)^2}$$

Equating real and imaginary part

$$L[t\sin at] = \frac{2as}{(s^2+a^2)^2}$$

c. $\cosh(5t+2) = \frac{e^{5t+2}+e^{-(5t+2)}}{2}$

So $L\{\cosh(5t+2)\} = \frac{e^2}{2}L[e^{5t}] + \frac{e^{-2}}{2}L[e^{-5t}]$

$$= \frac{e^2}{2}.\frac{1}{s-5} + \frac{e^{-2}}{2}.\frac{1}{s+5}$$