INVERSE L.T. USING METHOD OF PARTIAL FRACTIONS

If $L[f(t)] = \bar{f}(s)$ then $L^{-1}[\bar{f}(s)] = f(t)$ is said to be inverse Laplace transform.

Some inverse L.T

$\bar{f}(s)$	$f(t) = L^{-1}[\bar{f}(s)]$
$\frac{1}{s}$	1
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}, \quad n = 1,2,3,..$
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$
$\frac{s}{s^2-a^2}$	$\cosh at$
$\frac{1}{(s-a)^2+b^2}$	$\frac{1}{b}e^{at}\sin bt$
$\frac{s-a}{(s-a)^2+b^2}$	$e^{at}\cos bt$
$\frac{1}{(s-a)^n}$	$e^{at}.\frac{t^{n-1}}{(n-1)!}$

Inverse L.T follows the properties namely linearity property, first shifting property, changes of scale, changes of scale shifting.

If $\bar{f}(s)$ is rational algebraic function then we have to express $\bar{f}(s)$ in terms of partial fractions in order to find the inverse L.T.

Partial Fractions:

1. **When the denominator contains factors, real, linear and none repeated**

To each non-repeated linear factor say (x-a), there corresponds a partial fraction of the form $\frac{A}{x-a}$

where A is constant and the given function can be expressed as a sum of these partial fractions and the unknown constants can be determined either by equating the coefficients of different powers of x or by considering this as identity which is true for all value of x.

Example 1: $\frac{5x}{x^2-3x+2}$

Solun. Using partial fraction

$$\frac{5x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

To determine the coefficients of non-repeated linear factors x-1 and x-2, we write

$5x = A(x-2) + B(x-1)$

Put $x = 1$ we get $5 = -A \Rightarrow A = -5$

Put $x = 2$ we get $B = 10$

Hence $\frac{5x}{x^2-3x+2} = \frac{-5}{x-1} + \frac{10}{x-2}$

2. **When the denominator contains factors, real, linear but some are repeated**

   To each factor type $(x-a)^p$ we consider $\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_p}{(x-a)^p}$

   Where $A_1, A_2, \ldots A_p$ are constants.

Example 2. $\frac{3x^2}{(x+1)^2(x+2)}$

Let $\frac{3x^2}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$

$3x^2 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$

Put $x = -1$ we get $B = 3$

Put $x = -2$ we get $C = 12$

Equating the coefficient of x^2

$3 = A + C \Rightarrow A = 3 - C \Rightarrow A = 3 - 12 \Rightarrow A = -9$

So $\frac{3x^2}{(x+1)^2(x+2)} = \frac{-9}{x+1} + \frac{3}{(x+1)^2} + \frac{12}{x+2}$

3. **When the denominator contains factors, real, quadratic but none repeated**

To each non-repeated quadratic factors such as $x^2 + px + q (q \neq 0)$ there corresponds a partial fraction of the form $\frac{Ax+B}{x^2+px+q}$

Example 3. $\frac{4x}{(x+1)(x^2+9)}$

Let $\frac{4x}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$

$4x = A(x^2+9) + (Bx+C)(x+1)$

Put $x = -1$ then $A = -\frac{2}{5}$

Equating coefficients of x^2 $0 = A + B \Rightarrow B = -A \Rightarrow B = \frac{2}{5}$

Equating the constant terms $0 = 9A + C \Rightarrow C = -9A \Rightarrow C = \frac{18}{5}$

Hence $\frac{4x}{(x+1)(x^2+9)} = \frac{-2}{5(x+1)} + \frac{2x+18}{5(x^2+9)}$

4. **When the denominator contains factors, real, quadratic but some are repeated**

To each quadratic factor $(x^2+px+q)^r$ repeated r times there will be r partial fractions of the form $\frac{A_1x+B_1}{x^2+px+q}+\frac{A_2x+B_2}{(x^2+px+q)^2}+\cdots+\frac{A_rx+B_r}{(x^2+px+q)^r}$

where $A_1, A_2, \ldots, A_r$ and $B_1, B_2, \ldots, B_r$ are constants

Note 1. For case I, the following rule can be also applied i.e., the partial fraction corresponding in the non-repeated linear factor x-a in the denominator put x=a everywhere in the given function except in the factor (x-a)

Ex: $\frac{4x+5}{(x-1)^2(x+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{4(-2)+5}{(-2-1)^2(x+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}-\frac{1}{3(x+2)}$

$\frac{5x+3}{(x-1)(x^2+2x+5)}=\frac{5.1+3}{(x-1)(1+2.1+5)}+\frac{Ax+B}{x^2+2x+5}=\frac{1}{x-1}+\frac{Ax+B}{x^2+2x+5}$

Note 2. If the degree of the polynomial of the numerator is greater than the degree of denominator then we have to apply division rule first and then resolve into partial fraction.

Ex: $\frac{x^4+2x+1}{x^2+5x+6}=(x^2-5x+19)-\frac{63x+113}{x^2+5x+6}=(x^2-5x+19)-\frac{63x+113}{(x+2)(x+3)}$

Note 3. If the degree of the polynomial of both numerator and denominator are same then we have to apply division rule first and resolve into partial fraction

Ex: $\frac{2x^2+x+1}{x^2+5x+6}=2-\frac{9x+11}{x^2+5x+6}=2+\frac{7}{(x+2)}-\frac{16}{x+3}$

Example 4. Find the inverse LT of

a. $\frac{s^2+1}{s^3+3s^2+2s}$

b. $\frac{1}{s^3-a^3}$

Solution:

a. $s^3+3s^2+2s=s(s+1)(s+2)$

Let $\frac{s^2+1}{s^3+3s^2+2s}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$

$$s^2+1=A(s+1)(s+2)+Bs(s+2)+Cs(s+1)$$

Put $s = 0, A = \frac{1}{2}$

Put $s = -1, B = -2$

Put $s = -2, C = \frac{5}{2}$

$$\therefore L^{-1}\left[\frac{s^2+1}{s^3+3s^2+2s}\right] = L^{-1}\left[\frac{1}{2s}\right] - 2L^{-1}\left[\frac{1}{s+1}\right] + \frac{5}{2}L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{2} - 2e^{-t} + \frac{5}{2}e^{-2t}$$

b. $s^3 - a^3 = (s-a)(s^2 + as + a^2)$

Let $\frac{1}{s^3-a^3} = \frac{A}{s-a} + \frac{Bs+C}{s^2+as+a^2}$

$1 = A\left((s^2 + as + a^2)\right) + (Bs + C)(s - a)$

Equating coefficient of s^2, $0 = A + B$

Equating coefficient of s, $0 = aA - aB + C$

Equating constant term, $1 = a^2A - ac$

Solving above equations we get $A = \frac{1}{3a^2}, B = -\frac{1}{3a^2}, C = -\frac{2}{3a}$

So $L^{-1}\left[\frac{1}{s^3-a^3}\right] = AL^{-1}\left[\frac{1}{s-a}\right] + BL^{-1}\left[\frac{s}{s^2+as+a^2}\right] + CL^{-1}\left[\frac{1}{s^2+as+a^2}\right]$

$$= Ae^{at} + BL^{-1}\left[\frac{s+\frac{a}{2}}{\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right] + (C - \frac{a}{2}B)L^{-1}\left[\frac{1}{\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right]$$

$$= Ae^{at}\ \ Be^{-\frac{a}{2}t}\cos\left(\frac{\sqrt{3}a}{2}t\right) + \left(C - \frac{a}{2}B\right)\frac{2}{\sqrt{3}a}e^{-\frac{a}{2}t}\sin\left(\frac{\sqrt{3}a}{2}t\right)$$

$$= \frac{1}{3a^2}e^{at} - \frac{1}{3a^2}e^{-\frac{a}{2}t}\cos\left(\frac{\sqrt{3}a}{2}t\right) - \frac{1}{\sqrt{3}a^2}e^{-\frac{a}{2}t}\sin\left(\frac{\sqrt{3}a}{2}t\right)$$

$$= \frac{1}{3a^2}e^{at} - \frac{1}{3a^2}e^{-\frac{a}{2}t}\left[\cos\left(\frac{\sqrt{3}a}{2}t\right) - \sqrt{3}\sin\left(\frac{\sqrt{3}a}{2}t\right)\right]$$