Example 1. Find the inverse Laplace transform of

(a) $\frac{s^2+1}{s^3+3s^2+2s}$ $\quad$ $(b)\ \frac{1}{s^3-a^3}$

(a) Here the denominator $= s(s^2+3s+2) = s(s+1)(s+2)$

$$\therefore \ Let \ \frac{s^2+1}{s^3+3s^2+2s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

Multiplying both sides by $s(s+1)(s+2)$, we obtain

$$s^2+1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

Putting $s=0$, $\quad 1 = 2A \ \Rightarrow A = \frac{1}{2}$ …(i)

Putting s=-1, 2 = -B $\Rightarrow B = -2$ …(ii)

Putting $s = -2, 5 = 2C \ \Rightarrow C = \frac{5}{2}$ …(iii)

$$\therefore \ L^{-1}\left[\frac{s^2+1}{s(s+1)(s+2)}\right] = \frac{1}{2}.L^{-1}\left[\frac{1}{s}\right] - 2\,L^{-1}\left[\frac{1}{s+1}\right] + \frac{5}{2}\,L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{2} - 2\,e^{-t} + \frac{5}{2}\,e^{-2t}$$

(b) Here the denominator, $= (s-a)(s^2+as+a^2)$

$$\therefore \ Let \ \frac{1}{(s-a)(s^2+as+a^2)} = \frac{A}{s-a} + \frac{Bs+C}{s^2+as+a^2}$$

Multiplying both sides by $(s-a)(s^2+as+a^2)$, we obtain

$$1 = A(s^2+as+a^2) + (Bs+C)(s-a)$$

Equating s^2 coefficients, 0=A+B …(i)

Equating s coefficients, 0=aA-aB+C …(ii)

Equating constant terms, 1=$a^2A - aC$ …(iii)

Solving (i), (ii), (iii) we obtain, $A = \frac{1}{3}a^2$,$B = -\frac{1}{3}a^2$, $C = -\frac{2}{3}a$ …(iv)

$$L^{-1}\left[\frac{1}{s^3-a^3}\right] = A.L^{-1}\left[\frac{1}{s-a}\right] + B\,L^{-1}\left[\frac{s}{s^2+as+a^2}\right] + CL^{-1}\left[\frac{s}{s^2+as+a^2}\right]$$

$$= Ae^{at} + BL^{-1}\left[\frac{\left(s+\frac{a}{2}\right)}{\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right]$$

$$+\left(C-\frac{a}{2}B\right)L^{-1}\left[\frac{1}{\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2}\right]$$

$$= A\,e^{at} + B.e^{-\frac{a}{2}t}.\cos(\frac{\sqrt{3}a}{2}t)$$

$$+\left(C-\frac{a}{2}B\right)\ \frac{2}{\sqrt{3}\,a}\ .e^{-\frac{at}{2}}.\sin(\frac{\sqrt{3}\,at}{2})$$

$$= \frac{e^{at}}{3\,a^2} - \frac{e^{-\frac{a}{2}t}}{3a^2}.\cos\left(\frac{\sqrt{3}a}{2}t\right) - \frac{2}{\sqrt{3}\,a}\ .e^{-\frac{at}{2}}.\sin(\frac{\sqrt{3}\,at}{2})$$

$$= \frac{e^{at}}{3\,a^2} - \frac{e^{-\frac{a}{2}t}}{3a^2}\left[\cos\left(\frac{\sqrt{3}a}{2}t\right) + \sqrt{3}.\sin\left(\frac{\sqrt{3}\,at}{2}\right)\right]$$

Example 2. Find the inverse Laplace transform of

$(a)\,\frac{s^3}{s^4-16}$ $\qquad (b)\,\frac{s}{s^4+s^2+1}$

(a) Here the denominator $= (s^2+4)(s^2-4)$

$$\therefore\ Let\,\frac{s^3}{(s^2+4)(s^2-4)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2-4}$$

$Multiplying\ both\ sides\ by\ (s^2+2^2)(s^2-2^2),$ we obtain,

$s^3 = (As+B)(s^2-4) + (Cs+D)(s^2+4)$

$Equating\ the\ coefficient\ \ s^3$gives, A+C=1

$Equating\ the\ coefficient\ \ s^2$gives, B+D=0

$Equating\ the\ coefficient\ \ s$ gives, -4A+4C=0

$Equating\ the\ constant\ terms$ gives, -4B+4D=0

$Solving\ we\ obtain, A = C = \frac{1}{2}, B = D = 0$

$$\therefore L^{-1}\left[\frac{s^3}{s^4-16}\right] = \frac{1}{2}\ L^{-1}\left[\frac{s}{s^2+4}\right] + \frac{1}{2}\ L^{-1}\left[\frac{s}{s^2-4}\right]$$
$$= \frac{1}{2}(\cos 2t + \cos h\ 2t)$$

(b) Here $\frac{s}{s^4+s^2+1} = \frac{s}{(s^2+s+1)(s^2-s+1)} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2-s+1}$

$$\Rightarrow s = (As+B)(s^2-s+1)(Cs+D)(s^2+s+1)$$

$Equating\ the\ coefficient\ \ s^3$gives, A+C=0

$Equating\ the\ coefficient\ \ s^2$gives, -A+B+C+D=0

$quating\ the\ coefficient\ \ s$ gives, A-B+C+D=1

$Equating\ the\ constant\ terms$ gives, B+D=0

$Solving\ we\ obtain, A = C = 0, B = -\frac{1}{2}, \qquad D = \frac{1}{2}$

$$L^{-1}\left[\frac{s}{s^4+s^2+1}\right] = -\frac{1}{2}\ L^{-1}\left[\frac{1}{s^2+s+1}\right] + \frac{1}{2}\ L^{-1}\left[\frac{1}{s^2-s+1}\right]$$
$$= \frac{1}{2}\ L^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right] + \frac{1}{2}\ L^{-1}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right]$$
$$= -\frac{1}{2}\frac{2}{\sqrt{3}}\ e^{-\frac{t}{2}}.\sin\left(\frac{\sqrt{3}\ t}{2}\right) + \frac{1}{2}\frac{2}{\sqrt{3}}\ e^{\frac{t}{2}}.\sin\left(\frac{\sqrt{3}\ t}{2}\right)$$
$$= \frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)\sin h\ \left(\frac{t}{2}\right)$$

***<u>Some Special Results on Laplace Transform</u>*:**

I. L.T. of Derivatives:

(a) If $L[f(t)] = \bar{f}\ (s)\ \ and\ f'(t)$ be continuous then

$$L[f'(t)] = s\bar{f}\ (s) - f(0)$$

$$Proof: Here\ L[f'(t)] = \int_0^\infty e^{-st}\ f'(t)dt = 0 - f(0) + s\bar{f}\ (s)$$
$$= s\bar{f}\ (s) - f(0)$$

Hence the result is proved.

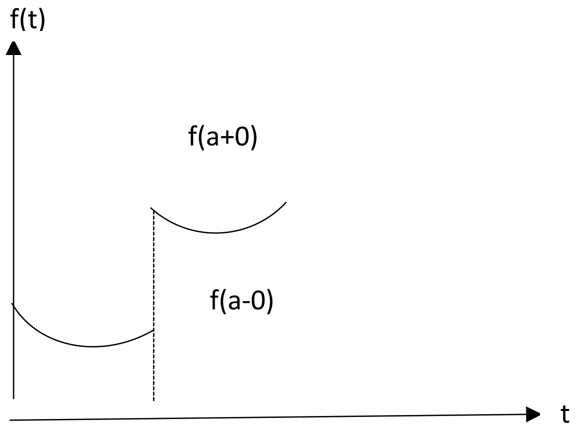


Fig. 1

(b) If the function f(t) be piecewise continuous (say an ordinary discontinuity at t=a), then

$$L[f'(t)] = s\bar{f}\,(s) - f(0) - [f(a+0) - f(a-0)]e^{-as}$$

(c) If f(t) and its first (n-1) derivatives be continuous then

$$L[f^n(t)] = s^n\,\bar{f}\,(s) - s^{n-1}f(0) - s^{n-2}f'(0) \ldots - f^{n-1}(0)$$

Here applying repeatedly the integration by parts and assuming

$\lim_{t\to\infty} e^{-st}.f^m(t) = 0,\ m = 0,1,2,\ldots,(n-1)$, we obtain the above result.

Example 1. Find $L[\sin^2 t]$

Let $f(t) = \sin^2 t$

$f'(t) = 2\sin t\cos t = \sin 2t$

We have, $L[f'(t)] = s\bar{f}\,(s) - f(0)$

$\Rightarrow\ L[sin2\,t] = s\bar{f}\,(s) - 0$

$$\Rightarrow\ \frac{2}{s^2+4} = s\bar{f}\,(s)\ \Rightarrow\ \bar{f}\,(s) = \frac{2}{s(s^2+4)}$$

Example 2. Find $L[t\cos t]$

Let $f(t) = tcos\ t$
$f'(t) = \cos t - t\sin t$
$f''(t) = -2\sin t - t\cos t$
$f(0) = 0, \qquad f'(0) = 1$

$\therefore L[f''(t)] = s^2\bar{f}\,(s) - s\,f(0) - f'(0)$
$\Rightarrow L[-2\sin t - t\cos t] = s^2\bar{f}\,(s) - 1$
$\Rightarrow\ -2\,L[\sin t] - L[tcos\ t] = s^2\bar{f}\,(s) - 1$

$$\Rightarrow\ -2\frac{1}{s^2+1}-\bar{f}\ (s)=s^2\bar{f}\ (s)-1$$
$$\Rightarrow\ \bar{f}\ (s)=\frac{s^2-1}{(s^2+1)^2}$$

II. L.T of Integration:

$$If\ L[f(t)]=\ \bar{f}\ (s), then\ L\left[\int_0^t f(u)du\right]=\frac{\bar{f}\ (s)}{s}$$

Alt. $L^{-1}\left[\frac{\bar{f}\ (s)}{s}\right]=\int_0^t f(u)du$

Proof: Let $\phi(t)=\ \int_0^t f(u)du$

$$\Rightarrow \phi'(t)=f(t)\ and\ \phi(0)=0$$

$$L[\phi'(t)]=s\ \bar{\phi}(s)-\phi(0)\Rightarrow L[f(t)]=s\bar{\phi}(s)$$
$$\Rightarrow\ \bar{\phi}(s)=\frac{L[f(t)]}{s}=\frac{\bar{f}\ (s)}{s}\ \Rightarrow L\left[\int_0^t f(u)du\right]=\frac{\bar{f}\ (s)}{s}$$

Example 3. Find $L[\int_0^t e^{-t}\cos t\ dt]$

Let $f(t)=e^{-t}\cos t$

$$\because L[\cos t]=\frac{s}{s^2+1}$$
$$L[f(t)]=\frac{s+1}{(s+1)^2+1}=\frac{s+1}{s^2+2s+2}$$

Hence $L\left[\int_0^t e^{-t}\cos t\ dt\right]=\frac{s+1}{s(s^2+2s+2)}$

Example 4. Find $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]$

We know $L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{1}{a}\sin at$

Then $L^{-1}\left[\frac{1}{s\ \ (s^2+a^2)}\right]=\int_0^t\frac{1}{a}\sin au\ du=\frac{1}{a^2}(1-\cos at)$

Again, $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]=\int_0^t\frac{1}{a^2}(1-\cos au)\ du=\frac{1}{a^2}\ (t-\frac{\sin at}{a})$

Example 5. Find $L^{-1}\left[\frac{s-a}{s^2(s^2+a^2)}\right]$

Now, $L^{-1}\left[\frac{s-a}{s^2(s^2+a^2)}\right]=L^{-1}\left[\frac{1}{s(s^2+a^2)}-\frac{a}{s^2(s^2+a^2)}\right]$

$$=L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]-L^{-1}\left[\frac{a}{s^2(s^2+a^2)}\right]$$

$=\frac{1}{a^2}\ (1-\cos at)-\frac{1}{a}\left(t-\frac{\sin at}{a}\right)$

III. Multiplication by t^n

If $L[f(t)] = \bar{f}\ (s)$, then

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\left[\bar{f}\ (s)\right], \qquad where\ n = 1, 2, 3, \ldots$$

Proof: We have $\int_0^\infty e^{-st} f(t)dt = \bar{f}\ (s)$= L{f(t)}

Differentiate both sides w.r.t. s using Leibnitz Rule for differentiation under integral sign we obtain,

$$\int_0^\infty \frac{\partial}{\partial s}(e^{-st})f(t)dt = \frac{d}{ds}\ [\bar{f}(s)]$$

$$\Rightarrow \int_0^\infty -t\ e^{-st}.f(t)dt = \frac{d}{ds}[\bar{f}\ (s)]$$

$$\int_0^\infty e^{-st}[t\ f(t)]dt = -\frac{d}{ds}\ [\bar{f}\ (s)]$$

$$i.e. L\{t\ f(t)\} = -\frac{d}{ds}\ [\bar{f}\ (s)]$$

$$Hence\ the\ result\ is\ true\ for\ n = 1$$

Let it be true for n=m

$$\therefore \int_0^\infty e^{-st}[t^m\ f(t)]dt = (-1)^m . \frac{d^m}{ds^m}[\bar{f}\ (s)]$$

Differentiate both sides w.r.t. s using Leibnitz Rule for differentiation, we obtain,

$$\int_0^\infty -t.e^{-st}[t^m\ f(t)]dt = (-1)^m . \frac{d^{m+1}}{ds^{m+1}}[\bar{f}\ (s)]$$

$$\Rightarrow \int_0^\infty e^{-st}[t^{m+1}\ f(t)]dt = (-1)^{m+1} . \frac{d^{m+1}}{ds^{m+1}}[\bar{f}\ (s)]$$

This shows that the result is true for n=m+1

It is true for n=1. It implies that it is true for n=1+1 = 2, 2+1 =3,… and so on.

∴ The result is true for all positive integral values of n.

$Note: From\ the\ above\ result, we\ can\ also\ obtain\ the\ following:$

$$L^{-1}\left[(-1)^n \frac{d^n}{ds^n}\left[\bar{f}\ (s)\right]\right] = t^n f(t), \qquad n = 1, 2, 3, \ldots$$

Example 6. Find $L[tcos\ t]$

Let $f(t) = \cos t$

$$L[f(t)] = \frac{s}{s^2+1}$$

$$\therefore L[tcos\ t] = (-1)\ \frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{(s^2+1)^2}$$

Example 7. Find $L[t^2 \sin h(at)]$

Let $f(t) = \sin hat, \quad L[f(t)] = \frac{a}{s^2-a^2}$

$$L[tf(t)] = (-1).\ \frac{d}{ds}\left(\frac{a}{s^2-a^2}\right) = \frac{2as}{(s^2-a^2)^2}$$

$$L[t^2f(t)] = (-1)\ \frac{d}{ds}\left(\frac{2as}{(s^2-a^2)^2}\right)$$

$$= -\frac{(s^2-a^2).2a - 2as\ .2\ (s^2-a^2).2s}{(s^2-a^2)^4}$$

$$= \frac{2a\ (3s^2+a^2)}{(s^2-a^2)^3}$$

IV. Division by t

If $L[f(t)] = \bar{f}(s)$, then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(p)\ dp$

Proof: We have, $\bar{f}(s) = \int_0^\infty f(t)e^{-st}\ dt$

Integrating from $s\ to\ \infty$, we obtain

$$\int_s^\infty \bar{f}(p)\ dp = \int_s^\infty \left[\int_0^\infty f(t)e^{-pt}\ dt\right] dp$$

$$= \int_0^\infty \left[\int_s^\infty e^{-pt}\ dp\right] f(t)dt \quad \text{(Change the order of integrating)}$$

$$= \int_0^\infty \left[\frac{e^{-pt}}{-t}\right]_s^\infty f(t)dt$$

$$= \int_0^\infty e^{-st}.\frac{f(t)}{t} dt$$

$$= L\left[\frac{f(t)}{t}\right]$$

Note. $L^{-1}\left[\int_s^{\infty} \bar{f}(p)\, dp\right] = \frac{f(t)}{t}$

Example 10. Prove that $L\left[\frac{\sin t}{t}\right] = \tan^{-1}\left(\frac{1}{s}\right)$ and hence find $L\left[\frac{\sin at}{t}\right]$.

Let $f(t) = \sin t$ and $L[f(t)] = \frac{1}{s^2+1}$

$$\therefore \quad L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \frac{ds}{s^2+1} = [\tan^{-1} s]_s^{\infty} = \frac{\pi}{2} - \tan^{-1} s = \tan^{-1}\left(\frac{1}{s}\right)$$

Also, $L\left[\frac{\sin at}{t}\right] = a.L\left[\frac{\sin at}{at}\right] = a.\frac{1}{a}\tan^{-1}\left(\frac{1}{\frac{s}{a}}\right)$ (by scale change property)

$$= \tan^{-1}\left(\frac{a}{s}\right)$$

Example 4: (problem on the property ‘division by t’)

Prove that, $\int_0^{\infty} \frac{e^{-at}-e^{-bt}}{t}\, dt = \ln\left({}^{b}/_{a}\right)$

Solution:

Let, $f(t) = e^{-at} - e^{-bt}$

$$L[f(t)] = L[e^{-at}] - L[e^{-bt}] = \frac{1}{s+a} - \frac{1}{s+b}$$

Now, $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty}\left[\frac{1}{s+a} - \frac{1}{s+b}\right] ds = \left[\ln\left(\frac{s+a}{s+b}\right)\right]_s^{\infty} = 0 - ln\left(\frac{s+a}{s+b}\right)$

$$\Rightarrow \int_0^{\infty} e^{-st}\,.\left(\frac{e^{-at} - e^{-bt}}{t}\right) dt = ln\left(\frac{s+b}{s+a}\right)$$

Now, put $s = 0$ in both sides and hence the result is obtained.

Convolution Theorem:

If $L^{-1}\left[\bar{f}(s)\right] = f(t)$ *and* $L^{-1}[\bar{g}(s)] = g(t)$ then

$$L^{-1}\left[\bar{f}(s).\bar{g}(s)\right] = \int_0^t f(u).g(t-u)\, du = f(t) * g(t)$$

Alternative form,

If $L[f(t)] = \bar{f}(s)$ *and* $L[g(t)] = \bar{g}(s)$ then

$$L\left[\int_0^t f(u).g(t-u)\,du\right] = \bar{f}(s).\bar{g}(s)$$

Proof:

Let, $\varphi(t) = \int_0^t f(u).g(t-u)\,du$

Now, $L[\varphi(t)] = \int_0^\infty e^{-st}.\left\{\int_0^t f(u).g(t-u)\,du\right\}dt =$
$\int_0^\infty \int_0^t e^{-st}.\ f(u).g(t-u)\,du\,dt$ __(i)

The region of integration of (i) is given as the shaded region.

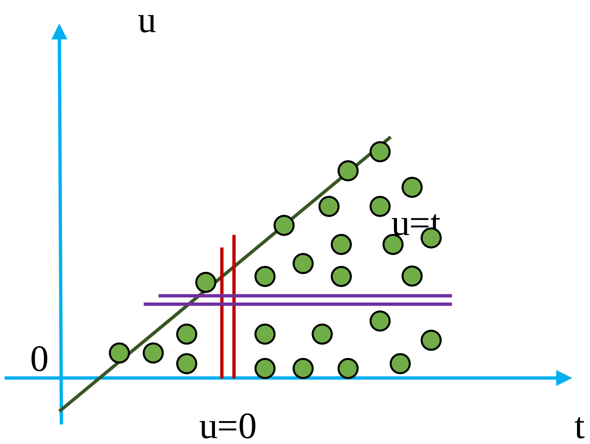
(Slide) $t : 0 \to \infty$

(Strip) $u : 0 \to t$

On changing the order of integration we get,

(Slide) $u : 0 \to \infty$

(Strip) $t : u \to \infty$



Therefore,

$$L[\varphi(t)] = \int_0^\infty \int_0^t e^{-st}.\ f(u).g(t-u)du\,dt$$

$$= \int_0^\infty e^{-su}.\ f(u)\left\{\int_u^\infty e^{-s(t-u)}.g(t-u)\,dt\right\}du$$

$$= \int_0^\infty e^{-su}.\ f(u)\left\{\int_0^\infty e^{-sp}.g(p)\,dp\right\}du$$ (Let, $(t-u) = p\,, dt = dp$)

$$= \int_0^\infty e^{-su} \,.\, f(u)\, du \,.\, \bar{g}(s) = \bar{f}(s).\bar{g}(s)$$

$$\Rightarrow L^{-1}\left[\bar{f}(s).\bar{g}(s)\right] = \int_0^t f(u).g(t-u)\, du$$

Note that Convolution * is commutative i.e., $f(t) * g(t) = g(t) * f(t)$. But, for some problem one side leads to tedious calculations. Then, it is better to use the other side.

Example 5:

Apply Convolution theorem to evaluate $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$.

Solution:

Here, $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left[\frac{s}{(s^2+b^2)}\right] = \cos at * \cos bt = \int_0^t \cos au \,.\cos b\,(t-u)du$

$$= \frac{1}{2}\int_0^t [\cos\{(bt + (a-b)u\} + \cos\{(-bt + (a+b)u\}]\, du$$

$$= \frac{1}{2}\left[\frac{\sin(bt + (a-b)u)}{a-b} + \frac{\sin(-bt + (a+b)u}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{2a\,.\sin at}{a^2-b^2} - \frac{2b\,.\sin bt}{a^2-b^2}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}$$