

# Quantitative Finance and Derivatives

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# Chapter 1

## Models for asset pricing

### 1.1 Stochastic Processes

Stochastic processes are collections of random variables representing the evolution of some system over time. Stochastic processes can have discrete or continuous values, and can evolve over discrete or continuous time.

We have four different possible situations:

1.  $X(n, \omega) : \mathbb{N} \times (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow$  a subset of  $\mathbb{Z}$  : discrete time, discrete values. For example: a *random walk*.
2.  $X(n, \omega) : \mathbb{N} \times (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow$  a subset of  $\mathbb{R}$  : discrete time, continuous values.
3.  $X(n, \omega) : \mathbb{R}^+ \times (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow$  a subset of  $\mathbb{Z}$  : continuous time, discrete values. For example: a *Poisson process*.
4.  $X(n, \omega) : \mathbb{R}^+ \times (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow$  a subset of  $\mathbb{R}$  : continuous time, continuous values. For example: a *Brownian motion*.

#### 1.1.1 From random walk to Brownian motion

Let

$$(X_n)_{n \geq 0}$$

be a stochastic process such that  $X_i$ , for any  $i$ , can take value 1 with probability  $\mathbb{P}(X_i = 1) = \frac{1}{2}$ , and  $-1$  with probability  $\mathbb{P}(X_i = -1) = \frac{1}{2}$ . Let then

$S_n = X_1 + X_2 + \dots + X_n$  be the position you are at on a line after  $n$  steps. Informally, each  $X_i$  "moves" you randomly one step to the right or one step to the left. We want to know expectation and variance of this process. Let's assume the  $(X_n)_{n \geq 0}$  to be *independent and identically distributed*. We can now assert the following:

$$E[X_i] = (+1) \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0$$

$$E[X_i^2] = (+1)^2 \cdot \frac{1}{2} + (-1)^2 \cdot \frac{1}{2} = 1$$

Hence,

$$E[S_n] = \sum_{i=1}^n E[X_i] = 0$$

$$Var[S_n] = \sum_{i=1}^n E[X_i^2] = n$$

As  $n$  approaches infinity, the conditions for the *Central Limit Theorem* hold:  $X_i$ s are *i.i.d.*, so we can apply

$$\frac{S_n - n \cdot E[X_i]}{\sqrt{n \cdot Var[X_i]}} = \frac{S_n}{\sqrt{n}} \underset{n \rightarrow \infty}{\sim} \mathcal{N}(0, 1)$$

Consider now a non-unitary time. Suppose we move in time steps of  $\delta > 0$  and in space steps of  $\sqrt{\delta}$ , and let's consider the process in the interval  $[0, t]$ ,  $t \in \mathbb{R}^+$ . Then,

$$S_t = \sum_{i=0}^{\lfloor \frac{t}{\delta} \rfloor} X_i$$

where  $X_i$  moves by  $\pm\sqrt{\delta}$  with probability  $\mathbb{P} = \frac{1}{2}$ . Then,  $E[S_t] = 0$  and

$$Var[X_i] = E^2[X_i] - E[X_i^2] = E^2[X_i] = (+\sqrt{\delta})^2 \cdot \frac{1}{2} + (-\sqrt{\delta})^2 \cdot \frac{1}{2} = \delta$$

$$\text{Var}[S_t] = \frac{t}{\delta} \cdot \delta = t$$

Let's now apply *Central Limit Theorem*:

$$\frac{S_t - \frac{t}{\delta} E[X_i]}{\sqrt{\frac{t}{\delta} \text{Var}[X_i]}} = \frac{S_t}{\sqrt{t}} \underset{t \rightarrow \infty}{\sim} \mathcal{N}(0, 1).$$

So,  $S_t \sim \mathcal{N}(0, t)$ : this is the *Brownian motion*.

**Definition 1. (Brownian motion).** The stochastic process

$$(W_t)_{t \in \mathbb{R}^+} : \mathbb{R}^+ \times (\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \rightarrow \mathbb{R}$$

is a Brownian motion if and only if

- (I)  $W_0 = 0$ ,
- (II) it is continuous,
- (III) has stationary increments: the distribution doesn't depend on initial time, only on waiting time.

$$\forall t > s, W_t - W_s \sim \mathcal{N}(0, t - s)$$

- (IV) has independent increments over disjoint intervals:

$$\forall q < r < s < t, (W_r - W_q) \perp (W_t - W_s).$$

□

**Definition 2. (Classes of Brownian motions.)**

- (I) *Standard Brownian motion*, or *Wiener Process*.
- (II) *Arithmetic Brownian motion*, or *Bachelier Model*.

$$dp_t = p_{t+h} - p_t = \mu dt + \sigma dW_t$$

Where  $\mu$  is called *drift*, and  $\sigma$  the *volatility*. This is a *stochastic differential equation*, *SDE* for short: while the  $\mu dt$  part is *deterministic*, the

$\sigma dW_t$  contains a *stochastic* component, which is a standard Brownian motion. This kind of equations provides a model for the change of the price  $p_t$  over the infinitesimal time increment from  $t$  to  $t + h$ . The price variation could be equivalently modeled this way:

$$p_t - p_0 = \mu(t - 0) + \sigma(W_t - W_0) = \mu t + \sigma W_t$$

Noting that

$$E[W_t] = 0 \quad \text{and} \quad \text{Var}[\sigma W_t] = E[(\sigma W_t)^2] = \sigma^2 E[W_t^2] = \sigma^2 t,$$

we have that, in Bachelier's model, the price  $p_t = \mu t + \sigma W_t + p_0$  is a *random variable* distributed like

$$p_t \sim \mathcal{N}(p_0 + \mu t + E[W_t], \sigma^2 t) \equiv \mathcal{N}(p_0 + \mu t, \sigma^2 t).$$

(III) *Geometric Brownian motion, or Black-Scholes model.*

$$dp_t = \mu p_t dt + \sigma p_t dW_t$$

This model is similar to Bachelier's, but acts on a multiplicative instead of additive principle. For this model, we will need to find a solution and its distribution.

□

## 1.2 Itô Formula

Suppose a model for an underlying asset price dynamics is given: we know the form of  $dp_t$ ; we'll assume Black-Scholes. It is now natural to assume that the price of a derivative on this asset is a *function*, let it be  $f(p_t)$ , of the asset price. How can we get a *stochastic differential equation* that models the derivative price variation?

Let's evaluate the *Taylor expansion* for  $f(p_t)$  like in a deterministic setting.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \varepsilon$$

where  $\varepsilon$  is a remainder that goes quickly to zero. We will now discard all terms in the approximation that go to 0 faster than  $dt$  does, thus truncating the approximation to the first order. Let now  $x = p_{t+dt}$ ,  $x_0 = p_t$  and  $dp_t = p_{t+dt} - p_t$

$$f(p_{t+dt}) \approx f(p_t) + f'(p_t)(dp_t) + \frac{1}{2}f''(p_t)(dp_t)^2 + \varepsilon$$

and substitute the increment  $dp_t$  with the Black-Scholes model:

$$f(p_{t+dt}) = f(p_t) + f'(p_t)(\mu p_t dt + \sigma p_t dW_t) + \frac{1}{2}f''(p_t)(\mu^2 p_t^2 (dt)^2 + \sigma^2 p_t^2 (dW_t)^2 + 2\mu\sigma p_t^2 dt dW_t)$$

Since  $E[(dW_t)^2] = dt$ , we can assume  $dW_t = \sqrt{dt} = (dt)^{\frac{1}{2}}$  to have order  $\frac{1}{2}$ . Let's analyze the orders of all the terms in the approximation:

- $\mu p_t dt$ : First order: keep it.
- $\sigma p_t dW_t$ : Order  $\frac{1}{2}$ : keep it.
- $\mu^2 p_t^2 (dt)^2$ : Second order: discard it.
- $\sigma^2 p_t^2 (dW_t)^2$ : First order, because  $(dW_t)^2 = dt$ : keep it.
- $2\mu\sigma p_t^2 dt dW_t$ : Order  $\frac{3}{2}$  because of  $dt dW_t$ : discard it.

The truncated approximation now states that

$$f(p_{t+dt}) = f(p_t) + f'(p_t)(\mu p_t dt + \sigma p_t dW_t) + \frac{1}{2}f''(p_t)(\sigma^2 p_t^2 dt).$$

The second derivative term of this equation is called *Itô correction term*. The derivative of the function with respect to time can then be computed this way:

$$df(p_t) = f(p_{t+dt}) - f(p_t) = f'(p_t)(\mu p_t dt + \sigma p_t dW_t) + \frac{1}{2}f''(p_t)(\sigma^2 p_t^2 dt)$$

Let's now separate the deterministic terms from the stochastic terms, so we can identify a *drift* and a *volatility* for the model.

$$df(p_t) = \left[ f'(p_t)\mu p_t + \frac{1}{2}f''(p_t)\sigma^2 p_t^2 \right] dt + f'(p_t)\sigma p_t dW_t$$

**Example 1.** Let  $f(p_t) = \ln(p_t)$  be the price of a derivative instrument with underlying price  $p_t$ ; the dynamic for the underlying follows the Black-Scholes model. Apply Itô's formula to this derivative:

$$\begin{aligned} d\ln(p_t) &= \left[ \frac{1}{p_t}\mu p_t + \frac{1}{2} \left( -\frac{1}{p_t^2}\sigma^2 p_t^2 \right) \right] dt + \frac{1}{p_t}\sigma p_t dW_t = \\ &= \left[ \mu - \frac{1}{2}\sigma^2 \right] dt + \sigma dW_t \end{aligned}$$

Considering the time interval  $[0, t]$ , we observe that

$$d\ln(p_t) = \ln(p_t) - \ln(p_0) = \left( \mu - \frac{1}{2}\sigma^2 \right) (t - 0) + \sigma(W_t - W_0)$$

and then

$$\ln\left(\frac{p_t}{p_0}\right) = \left( \mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t \iff \frac{p_t}{p_0} = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

which yields the solution

$$p_t = p_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

**Example 2.** Black-Scholes model assumes the *spot risk-free interest rate*  $r$  to be constant and independent of maturity. Consider *discounting* as a function of time and asset price:

$$f(t; p_t) = e^{-rt} p_t$$

The general Itô formula for a function of such a form is

$$\begin{aligned} d[f(t; p_t)] &= \frac{\partial f}{\partial p_t} dp_t + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial p_t^2} (dp_t)^2 + \underbrace{\frac{1}{2} \frac{\partial^2 f}{\partial t^2} (dt)^2}_{\text{order 2}} + \underbrace{\frac{1}{2} \frac{\partial^2 f}{\partial p_t \partial t} 2dt dp_t}_{\text{order 3/2}} \\ &= \frac{\partial f}{\partial p_t} dp_t + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial p_t^2} \sigma^2 p_t^2 dt \end{aligned}$$



For the discounting function, this means

$$\begin{aligned}
 d[e^{-rt}p_t] &= e^{-rt} \cdot 1 \cdot dp_t + -re^{-rt}p_t dt + \cancel{\frac{1}{2} \cdot 0 \cdot \sigma^2 p_t^2 dt} \rightarrow 0 \\
 &= e^{-rt} (\mu p_t dt + \sigma p_t dW_t - rp_t dt) \\
 &= (\mu - r)e^{-rt}p_t dt + e^{-rt}p_t \sigma dW_t \\
 d\tilde{p}_t &= (\mu - r)\tilde{p}_t dt + \sigma \tilde{p}_t dW_t
 \end{aligned}$$

Hence, the distribution for a discounted asset price follows Black-Scholes model:

$$\tilde{p}_t = \tilde{p}_0 e^{(\mu - r - \frac{\sigma^2}{2})t + \sigma W_t}$$

Note that the deterministic and stochastic parts were grouped together, to underline the *risk factor*.

### 1.3 Black-Scholes PDE for option pricing

In the context of option pricing, Black-Scholes model assumes the following:

- $p_t$  is the underlying asset price at time  $t$ ,
- $r$  is a constant, risk-free interest rate,
- There are no transaction costs, no taxes, and no arbitrage opportunity,
- The market is liquid, and so all the instruments,
- $f(t; p_t)$  is the option price.

By Itô's lemma, we get that the option price is

$$df(t; p_t) = \left[ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial p_t} \mu p_t + \frac{1}{2} \frac{\partial^2 f}{\partial p_t^2} \sigma^2 p_t^2 \right] dt + \frac{\partial f}{\partial p_t} \sigma p_t dW_t$$

Construct a *locally* risk-free portfolio,  $\Pi_t$ , such that

$$\Pi_t = \begin{cases} -1 & \text{positions in options (short)} \\ \Delta_t \equiv \frac{\partial f}{\partial p_t} & \text{positions in underlying (long)} \end{cases}$$

and study the dynamics of the portfolio value by multiplying the number of positions by the dynamics for each kind of instrument (option and asset).

$$\begin{aligned} d\Pi_t &= -1 \cdot df(t; p_t) + \frac{\partial f}{\partial p_t} dp_t \\ &= - \underbrace{\left[ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial p_t} \mu p_t + \frac{1}{2} \frac{\partial^2 f}{\partial p_t^2} \sigma^2 p_t^2 \right] dt - \frac{\partial f}{\partial p_t} \sigma p_t dW_t}_{\text{option}} + \underbrace{\frac{\partial f}{\partial p_t} (\mu p_t dt + \sigma p_t dW_t)}_{\text{asset}} \\ &= - \left[ \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial p_t^2} \sigma^2 p_t^2 \right] dt - \cancel{\frac{\partial f}{\partial p_t} \mu p_t dt} - \cancel{\frac{\partial f}{\partial p_t} \sigma p_t dW_t} + \cancel{\frac{\partial f}{\partial p_t} \mu p_t dt} + \cancel{\frac{\partial f}{\partial p_t} \sigma p_t dW_t} \\ &= - \left( \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial p_t^2} \sigma^2 p_t^2 \right) dt \end{aligned}$$

Having removed the Brownian motion, we are left without any risky term. We impose now the *no arbitrage assumption*, stating that a portfolio is risk-free if and only if its dynamics is the same of a bond, that is, it accrues interest at a constant (by assumption) rate over time.

$$d\Pi_t \equiv - \left( \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial p_t^2} \sigma^2 p_t^2 \right) dt \stackrel{\text{NAA}}{=} r \Pi_t dt$$

$$-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial p_t^2} \sigma^2 p_t^2 = -r f(t; p_t) + \frac{\partial f}{\partial p_t} r p_t$$

Finally, we obtain the Black-Scholes PDE by rearranging.

$$r f(t; p_t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial p_t} r p_t + \frac{1}{2} \frac{\partial^2 f}{\partial p_t^2} \sigma^2 p_t^2$$

The objective is to establish the *fair* (or *no arbitrage*) price of an option today, that is,  $f(0; p_0)$ . A *final condition* can be imposed:  $f(T, p_T)$ , which is the *option price at maturity* or, equivalently, the *payoff*, which is known.

$$f(T; p_T) \stackrel{\text{e.g.}}{=} \begin{array}{ll} (P_T - K)^+ = \max\{P_T - K, 0\} & \text{European call option} \\ (K - P_T)^+ = \max\{K - P_T, 0\} & \text{European put option} \end{array}$$

European options satisfy Black-Scholes assumptions: the payoff depends *only* on the price of the underlying at time  $T$ , and it is not *path dependent*. The drift term  $\mu$  does not appear in Black-Scholes PDE, and neither does in the payoff function  $f(t; p_t)$ ; this means that the Black-Scholes option price doesn't depend on it. The drift term is strongly linked to investor's *risk aversion*: this means the option can be priced *as if* the investor is *risk neutral*.

A *risk neutral* valuation intuitively means that we are pricing in a world where every investor behaves as if he himself is risk neutral; the real world is not risk neutral, though, and investors' risk aversion is embedded in the *historical data*.

A *risk neutral* valuation of current option price, given payoff  $f(T, p_T)$ , is

$$f(0; p_0) = \tilde{\mathbb{E}} [e^{-rT} f(T; p_T)]$$

Where  $\tilde{\mathbb{E}}$  is the expectation according to the *risk neutral probability*  $\tilde{\mathbb{P}}$ . We now have, in fact, two probability spaces: the first is the *historical probability space*  $(\Omega, \mathcal{F}, \mathbb{P})$ , the second is the *risk neutral probability space*  $(\Omega, \mathcal{F}, \tilde{\mathbb{P}}) \supseteq (\Omega, \mathcal{F}, \mathbb{P})$ . To calculate the option price at any given time  $t : 0 < t < T$  the information in the filtration up to time  $t$  can be used:

$$f(t; p_t) = \tilde{\mathbb{E}} [e^{-r(T-t)} f(T; p_T) | \mathcal{F}_t]$$

**Theorem 1.** If, in time  $t : 0 \leq t \leq T$  the payoff is  $(P_T - K)^+$ , the price of an option at time  $t$  is

$$C(\tau = T - t, p_t, K, r, \sigma) = f(t; p_t) = p_t \mathcal{N}(d_1) - K e^{-r\tau} \mathcal{N}(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{p_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}} \\ d_2 &= \frac{\ln\left(\frac{p_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}} = d_1 - \sigma\sqrt{T - t} \end{aligned}$$

*Proof.* Suppose we are in a Black-Scholes world: we have one risky asset with dynamics such that  $dp_t = \mu p_t dt + \sigma p_t d\tilde{W}_t$ , a riskless asset with dynamics such that  $dB_t = rB_t dt$ , and no arbitrage opportunities. Define  $\tilde{W}_t = W_t + \frac{\mu-r}{\sigma}t$  where  $\mu - r$  is the *risk premium* and  $\frac{\mu-r}{\sigma}$  is the *market price of risk*. Hence,

$$\begin{aligned} dp_t &= rp_t dt + \sigma p_t \left[ dW_t + \frac{\mu-r}{\sigma} dt \right] \\ &= \cancel{rp_t dt} + \sigma p_t dW_t + \mu p_t dt - \cancel{rp_t dt} \\ &= \mu p_t dt + \sigma p_t dW_t \end{aligned}$$

Note that  $\tilde{W}_t \sim \mathcal{N}\left(\frac{\mu-r}{\sigma}\Delta t; \Delta t\right)$  is no longer a standard Brownian motion. We want to know the option price at time  $t = 0$ , considering, for example, the payoff  $f(T; p_T) = (p_T - K)^+$ :

$$\begin{aligned} C_0 &= f(0; p_0) = \tilde{\mathbb{E}} \left[ e^{-rT} (p_T - K)^+ \right] \\ &\stackrel{\text{Markov}}{=} \tilde{\mathbb{E}} \left[ e^{-rT} \left( p_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma \tilde{W}_T} - K \right)^+ \right] = (*)_1 \end{aligned}$$

Consider now that, if  $X \sim \mathcal{N}(0, T)$  and  $Y \sim \mathcal{N}(0, 1)$  then  $X = \sqrt{T}Y$ :

$$\begin{aligned} (*)_1 &= \tilde{\mathbb{E}} \left[ e^{-rT} \left( p_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma \sqrt{T}Y} - K \right)^+ \right] \\ &= \int_{-\infty}^{+\infty} e^{-rt} \left( p_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma \sqrt{T}y} - K \right)^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = (*)_2 \end{aligned}$$

We compute the integral only where the payoff is positive, that is, where

$$\begin{aligned} p_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma \sqrt{T}y} &\geq K \iff e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma \sqrt{T}y} \geq \frac{K}{p_0} \\ &\iff \left(r - \frac{\sigma^2}{2}\right)T + \sigma \sqrt{T}y \geq \ln \frac{K}{p_0} \\ &\iff y \geq \frac{-\ln \frac{p_0}{K} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma \sqrt{T}} = -d_2 \end{aligned}$$

$$\begin{aligned}
(*)_2 &= \int_{-d_2}^{+\infty} e^{-rt} \left( p_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma y \sqrt{T}} - K \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= \int_{-d_2}^{+\infty} \frac{p_0 e^{-\frac{\sigma^2}{2}T + \sigma y \sqrt{T} - \frac{y^2}{2}}}{\sqrt{2\pi}} dy - \int_{-d_2}^{+\infty} \frac{K e^{-rt - \frac{y^2}{2}}}{\sqrt{2\pi}} dy \\
&= p_0 \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sigma \sqrt{T} + y)^2}{2}} dy - K e^{-rt} \int_{-d_2}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&\quad \left( \text{Let } z = y + \sigma \sqrt{T} \text{ s.t. } -\infty \leq z \leq d_2 + \sigma \sqrt{T} \right) \\
&= p_0 \int_{-\infty}^{d_2 + \sigma \sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - K e^{-rt} \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= p_0 \mathcal{N}(d_2 + \sigma \sqrt{T}) - K e^{-rt} \mathcal{N}(d_2) \\
&= p_0 \mathcal{N}(d_1) - K e^{-rt} \mathcal{N}(d_2).
\end{aligned}$$

## TODO Itō table and stuff

### 1.4 Cox-Ross-Rubinstein model

This model is the equivalent, in discrete time, of Black-Scholes. CRR accepts the assumptions of one risky asset, one riskless asset and no arbitrage. The discrete time implies the underlying is modeled using a discrete probability distribution: the Binomial model.

#### 1.4.1 Single period model

In a simple, single period model, the asset can go up in value of a percentage  $u$  with probability  $p$  or go down in value of a percentage  $d$  with probability  $1 - p$ .

$$S_0 \begin{cases} S_1^u = uS_0 & p \\ S_1^d = dS_0 & 1 - p \end{cases} \quad C_0 \begin{cases} C_u = (uS_0 - K)^+ & p \\ C_d = (dS_0 - K)^+ & 1 - p \end{cases}$$

The percentages are related like this:  $u > 1 + r > 1 > d \geq 0$ .

Again, construct a risk-free portfolio using the underlying and the option. For the portfolio to be risk-free, its payoff must be the same for each possible state of the world.

$$\Pi = \begin{cases} -1 & \text{option (short)} \\ \Delta & \text{underlying (long)} \end{cases}$$

$$\Pi \text{ is risk-free} \iff -C_u + \Delta u S_0 = -C_d + \Delta d S_0 \iff \Delta = \frac{C_u - C_d}{S_0(u - d)}$$

So,  $\Delta$  is the quantity of stock to be held so that the portfolio is risk-free. Now impose the no arbitrage assumption: assert that the risk-free portfolio in any of the two states of the world after one period must have the same payoff as the portfolio at initial time capitalized for the interest rate.

$$-C_u + \Delta u S_0 \stackrel{NAA}{=} (1 + r)(-C_0 + \Delta S_0)$$

$$\begin{aligned} -C_u + \frac{C_u - C_d}{S_0(u - d)} S_0 u &= (1 + r) \left( -C_0 + \frac{C_u - C_d}{S_0(u - d)} S_0 \right) \Rightarrow \\ \frac{-C_u(u - d) + (C_u - C_d)u}{u - d} &= -C_0(1 + r) + \frac{(1 + r)(C_u - C_d)}{u - d} \Rightarrow \\ \frac{C_u(u - d) + (C_u - C_d)(1 + r - u)}{u - d} &= C_0(1 + r) \Rightarrow \\ C_0 &= \frac{1}{1 + r} \left( C_u \frac{1 + r - d}{u - d} + C_d \frac{-1 - r + u}{u - d} \right) \Rightarrow \\ C_0 &= \frac{1}{1 + r} (C_u p + C_d(1 - p)) \quad \text{where} \quad p = \frac{1 + r - d}{u - d} \end{aligned}$$

Due to the previously imposed relationships between  $u, d$  and  $r, p$  respects positivity and can be used as a *risk neutral probability*.

$$\begin{aligned} \tilde{\mathbb{E}}[S_1] &= (u S_0)p + (d S_0)(1 - p) = u S_0 \frac{1 + r - d}{u - d} + d S_0 \frac{u - 1 - r}{u - d} = \\ &= \frac{u + ur - ud + du - d - dr}{u - d} S_0 = \frac{(u - d)(1 + r)}{u - d} S_0 = (1 + r) S_0 \end{aligned}$$

### 1.4.2 Two-period model

In a two-period CRR model, we have a *recombining tree*, that is, up-down and down-up movements yield the same result.

$$S_0 \left\langle \begin{array}{c} uS_0 \\ dS_0 \end{array} \right\rangle \left\langle \begin{array}{c} u^2S_0 \\ udS_0 \\ duS_0 \\ d^2S_0 \end{array} \right\rangle \quad C_0 \left\langle \begin{array}{c} C_u \\ C_d \end{array} \right\rangle \left\langle \begin{array}{c} C_{uu} = (u^2S_0 - K)^+ \\ C_{ud} = (udS_0 - K)^+ \\ C_{du} = (duS_0 - K)^+ \\ C_{dd} = (d^2S_0 - K)^+ \end{array} \right\rangle$$

To obtain the option price  $C_0$  in this case, the idea is to simply backtrack from the last period, calculating the discounted  $C_u$  and  $C_d$  first.

$$\begin{aligned} C_u \left\langle \begin{array}{c} C_{uu} = (u^2S_0 - K)^+ \\ C_{ud} = (udS_0 - K)^+ \end{array} \right\rangle &\implies C_u = \frac{1}{1+r}(C_{uu}p + C_{ud}(1-p)) \\ C_d \left\langle \begin{array}{c} C_{du} = (duS_0 - K)^+ \\ C_{dd} = (d^2S_0 - K)^+ \end{array} \right\rangle &\implies C_d = \frac{1}{1+r}(C_{du}p + C_{dd}(1-p)) \end{aligned}$$

Finally, we compose these results by computing

$$\begin{aligned} C_0 \left\langle \begin{array}{c} C_u \\ C_d \end{array} \right\rangle &\implies C_0 = \frac{1}{1+r}(C_up + C_d(1-p)) \\ &= \frac{1}{(1+r)^2} (C_{uu}p^2 + C_{ud}p(1-p) + C_{du}p(1-p) + C_{dd}(1-p)^2) \\ &= \frac{1}{(1+r)^2} (C_{uu}p^2 + 2C_{ud}p(1-p) + C_{dd}(1-p)^2) \end{aligned}$$

### 1.4.3 $n$ -period model

Generalizing to  $n$  periods, we have a  $\sim \mathcal{B}(n, p)$  model; any given path on the binomial tree over the  $n$  periods can have  $j : 0 \leq j \leq n$  steps up and  $n - j$  steps down for the underlying's price, with a payoff of

$$(u^j d^{n-j} S_0 - K)^+$$

for an option on the underlying. Following the same reasoning as per the two- and one-period model, we compute the option price as

$$\begin{aligned}
C_0 &= \frac{1}{(1+r)^n} \sum_{j=0}^n (u^j d^{n-j} S_0 - K)^+ \binom{n}{j} p^j (1-p)^{n-j} \\
&\quad (\text{let } a \text{ s.t. } \forall j \geq a \quad u^j d^{n-j} S_0 - K > 0) \\
&= \frac{1}{(1+r)^n} \sum_{j=a}^n (u^j d^{n-j} S_0 - K)^+ \binom{n}{j} p^j (1-p)^{n-j} \\
&= \frac{1}{(1+r)^n} \left[ \sum_{j=a}^n \binom{n}{j} p^j (1-p)^{n-j} u^j d^{n-j} S_0 - K \sum_{j=a}^n \binom{n}{j} p^j (1-p)^{n-j} \right] \\
&= \sum_{j=a}^n \binom{n}{j} \left( \frac{pu}{1+r} \right)^j \left( \frac{(1-p)d}{1+r} \right)^{n-j} S_0 - \frac{1}{(1+r)^n} K \mathcal{B}_{(n,p)}(j \geq a) \\
&= S_0 \mathcal{B}_{(n, \frac{pu}{1+r})}(j \geq a) - \frac{1}{(1+r)^n} K \mathcal{B}_{(n,p)}(j \geq a).
\end{aligned}$$

Note that  $\frac{pu}{1+r}$  is a probability because it is positive by no arbitrage assumption and sums to 1 with  $\frac{(1-p)d}{1+r}$ . Note that the equation looks very similar to Black-Scholes, with the Binomial distribution instead of the Normal.

## 1.5 Martingale

Let  $S_1$  be the stock price tomorrow. Under *risk neutral probability*, this should be equal to the stock price today capitalized for the given interest rate:

$$\tilde{\mathbb{E}}[S_1] = (1+r)S_0$$

Dividing both sides by  $(1+r)$ , we obtain that the expectation (under *risk neutral probability*) of the stock price tomorrow ( $t = 1$ ) is equal to the stock price today ( $t = 0$ ).

$$\tilde{\mathbb{E}} \left[ \frac{1}{(1+r)^1} S_1 \right] = \frac{1}{(1+r)^0} S_0 \equiv S_0$$

This is a *martingale*: in such a process, the expectations through time are constant. A martingale is a *fair game* (in game-theoretic sense), that is, it has zero drift. The stock market is not a fair game in the real world, but in a risk neutral world it is.



**Definition 3. (Martingale).** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. The process  $(M_t)_{t \geq 0}$  is a *martingale* if the expectation at a future time  $t$  given the information up to time  $s$  is equal to the expectation at time  $s$  :  $0 \leq s \leq t \leq T$ .

$$\mathbb{E}[M_t | \mathcal{F}_s] = \mathbb{E}[M_s]$$

Equivalently,

$$\mathbb{E}[M_{s+ds} | \mathcal{F}_s] = M_s \implies \mathbb{E}[M_{s+ds}] = \mathbb{E}[M_s]$$

□

**Example 3.** The Brownian motion is a martingale.

*Proof.* Let  $(W_t)_{t \geq 0}$  be a Brownian Motion in  $(\Omega, \mathcal{F}, \mathbb{P})$ .

$$\mathbb{E}[W_t | \mathcal{F}_s] = W_s, \quad s < t \implies \mathbb{E}[W_t - W_s | \mathcal{F}_s] = 0$$

From the definition of Brownian motion, and its independent increments property, follows

$$\mathbb{E}[W_t - W_s | \mathcal{F}_s] = \mathbb{E}[W_t - W_s] \stackrel{\mathcal{N}(0, \sigma)}{=} 0$$

□

## 1.6 Pricing exotic options

The generic process of pricing an option involves discounting the option's payoff at maturity given a model for the underlying.

### 1.6.1 Digital call option

The payoff for a *digital call option* is  $H$  when the option is *in the money*, 0 otherwise:

$$D_T = \begin{cases} H & \text{if } p_t \geq K \\ 0 & \text{otherwise} \end{cases}$$

Let  $dp_t = (r - q)p_t dt + \sigma p_t dW_t$  be the model for the underlying,  $t : 0 \leq t \leq T$  the current time and  $T$  the maturity. The price of the digital option at time  $t$  under the risk neutral measure is

$$D_t = \tilde{\mathbb{E}} \left[ e^{-r(T-t)} H \mathbb{1}_{(p_T \geq K)} | \mathcal{F}_t \right] = e^{-r(T-t)} H \mathbb{E} \left[ \mathbb{1}_{(p_T \geq K)} | \mathcal{F}_t \right] = \dots$$

Knowing that  $\mathbb{E}[\mathbb{1}_A] = 1 \cdot \mathbb{P}(A) + 0 \cdot \mathbb{P}(A^C) = \mathbb{P}(A)$ , we can replace the expectation with the risk neutral probability; then, for the Markov property of Black-Scholes model, we replace the filtration with the information known at time  $t$ , that is, the underlying price.

$$\dots = e^{-r(T-t)} H \tilde{\mathbb{P}}(p_T \geq K | \mathcal{F}_t) = e^{-r(T-t)} H \tilde{\mathbb{P}}(p_T \geq K | p_t) = \dots$$

By Itô's lemma, we can express the price at maturity  $p_T$  as

$$p_T = p_t e^{\left(r - q - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)}$$

Also, let  $Y : Y\sqrt{T-t} = W_T - W_t$ . Then,

$$\begin{aligned} \dots &= e^{-r(T-t)} H \tilde{\mathbb{P}} \left( p_t e^{\left(r - q - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)} \geq K \right) \\ &= e^{-r(T-t)} H \tilde{\mathbb{P}} \left( p_t e^{\left(r - q - \frac{\sigma^2}{2}\right)(T-t) + \sigma Y \sqrt{T-t}} \geq K \right) \\ &= e^{-r(T-t)} H \tilde{\mathbb{P}} \left( Y \geq -\frac{\ln \frac{p_t}{K} + \left(r - q - \frac{\sigma^2}{2}\right)(T-t)}{\sigma \sqrt{T-t}} \right) \\ &= e^{-r(T-t)} H \tilde{\mathbb{P}}(Y \geq -d_2) \\ &= e^{-r(T-t)} H \mathcal{N}(-d_2) \end{aligned}$$

### 1.6.2 Asset-or-nothing call option

The payoff for an *asset-or-nothing* call option is  $p_T$  when the option is *in the money*, 0 otherwise:

$$A_T = \begin{cases} p_T & \text{if } p_t \geq K \\ 0 & \text{otherwise} \end{cases}$$

The *asset-or-nothing* option price at present time  $t$  is then computed as

$$\begin{aligned} A_t &= \tilde{\mathbb{E}} \left[ e^{-r(T-t)} p_T \mathbb{1}(p_T \geq K) | \mathcal{F}_t \right] \\ &= e^{-r(T-t)} \tilde{\mathbb{E}} \left[ p_T \mathbb{1}(p_T \geq K) | \mathcal{F}_t \right] \stackrel{Markov}{=} e^{-r(T-t)} \tilde{\mathbb{E}} \left[ p_T \mathbb{1}(p_T \geq K) | p_t \right] = \dots \end{aligned}$$

The indicator function means, again, that the expectation for  $p_T$  can be computed only over the part where  $p_T \geq K$ , that is, past  $d_2$ .

$$\begin{aligned} \dots &= e^{-r(T-t)} \int_{-d_2}^{\infty} p_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T-t)+\sigma(W_T-W_t)} \cdot \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \\ &= \int_{-d_2}^{\infty} p_t e^{\left(r-r-q-\frac{\sigma^2}{2}\right)(T-t)+\sigma(W_T-W_t)} \cdot \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \\ &= p_t e^{-q(T-t)} \int_{-d_2}^{\infty} \frac{e^{-\frac{\sigma^2}{2}+\sigma y\sqrt{T-t}\frac{y}{2}}}{\sqrt{2\pi}} dy \\ &= p_t e^{-q(T-t)} \int_{-d_2}^{\infty} \frac{e^{-\frac{(\sigma\sqrt{T-t}+y)^2}{2}}}{\sqrt{2\pi}} dy = \dots \end{aligned}$$

Now we flip the integration domain thanks to Normal distribution's symmetry property and then integrate by substituting  $z : z(y) = \sigma\sqrt{T-t} + y$ ; the new integration domain extremes are then  $z(-\infty) = -\infty$  and  $z(d_2) = d_2 + \sigma\sqrt{T-t} = d_1$ .

$$\begin{aligned} \dots &= p_t e^{-q(T-t)} \int_{-\infty}^{d_1} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \\ &= p_t e^{-q(T-t)} \mathcal{N}(d_1). \end{aligned}$$