Quantitative Finance and Derivatives

January 5, 2016

Contents

Chapter 1

Models for asset pricing

1.1 Stochastic Processes

Stochastic processes are collections of random variables representing the evolution of some system over time. Stochastic processes can have discrete or continuous values, and can evolve over discrete or continuous time.

We have four different possible situations:

1. $X(n, \omega) : \mathbb{N} \times (\Omega, \mathcal{F}, \mathbb{P}) \to$ a subset of $\mathbb{Z}$: discrete time, discrete values. For example: a *random walk*.

2. $X(n, \omega) : \mathbb{N} \times (\Omega, \mathcal{F}, \mathbb{P}) \to$ a subset of $\mathbb{R}$: discrete time, continuous values.

3. $X(n, \omega) : \mathbb{R}^+ \times (\Omega, \mathcal{F}, \mathbb{P}) \to$ a subset of $\mathbb{Z}$: continuous time, discrete values. For example: a *Poisson process*.

4. $X(n, \omega) : \mathbb{R}^+ \times (\Omega, \mathcal{F}, \mathbb{P}) \to$ a subset of $\mathbb{R}$: continuous time, continuous values. For example: a *Brownian motion*.

1.1.1 From random walk to Brownian motion

Let

$$(X_n)_{n \geq 0}$$

be a stochastic process such that X_i, for any i, can take value 1 with probability $\mathbb{P}(X_i = 1) = \frac{1}{2}$, and -1 with probability $\mathbb{P}(X_i = -1) = \frac{1}{2}$. Let then

$S_n = X_1 + X_2 + \ldots + X_n$ be the position you are at on a line after n steps. Informally, each X_i "moves" you randomly one step to the right or one step to the left. We want to know expectation and variance of this process. Let's assume the $(X_n)_{n \geq 0}$ to be *independent and identically distributed*. We can now assert the following:

$$E\left[X_i\right] = (+1) \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0$$

$$E\left[X_i^2\right] = (+1)^2 \cdot \frac{1}{2} + (-1)^2 \cdot \frac{1}{2} = 1$$

Hence,

$$E\left[S_n\right] = \sum_{i=1}^{n} E\left[X_i\right] = 0$$

$$Var\left[S_n\right] = \sum_{i=1}^{n} E\left[X_i^2\right] = n$$

As n approaches infinity, the conditions for the *Central Limit Theorem* hold: X_is are *i.i.d.*, so we can apply

$$\frac{S_n - n \cdot E\left[X_i\right]}{\sqrt{n \cdot Var\left[X_i\right]}} = \frac{S_n}{\sqrt{n}} \underset{n \to \infty}{\sim} \mathcal{N}(0,1)$$

Consider now a non-unitary time. Suppose we move in time steps of $\delta > 0$ and in space steps of $\sqrt{\delta}$, and let's consider the process in the interval $[0,t],\ t \in \mathbb{R}^+$. Then,

$$S_t = \sum_{i=0}^{\left\lfloor \frac{t}{\delta} \right\rfloor} X_i$$

where X_i moves by $\pm\sqrt{\delta}$ with probability $\mathbb{P} = \frac{1}{2}$. Then, $E\left[S_t\right] = 0$ and

$$Var[X_i] = E^2[X_i] - E[X_i^2] = E^2[X_i] = (+\sqrt{\delta})^2 \cdot \frac{1}{2} + (-\sqrt{\delta})^2 \cdot \frac{1}{2} = \delta$$

$$Var[S_t] = \frac{t}{\delta} \cdot \delta = t$$

Let's now apply *Central Limit Theorem*:

$$\frac{S_t - \frac{t}{\delta}E[X_i]}{\sqrt{\frac{t}{\delta}Var[X_i]}} = \frac{S_t}{\sqrt{t}} \underset{t\to\infty}{\sim} \mathcal{N}(0,1).$$

So, $S_t \sim \mathcal{N}(0,t)$: this is the *Brownian motion*.

Definition 1. (Brownian motion). The stochastic process

$$(W_t)_{t\in\mathbb{R}^+} : \mathbb{R}^+ \times (\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}) \to \mathbb{R}$$

is a Brownian motion if and only if

(I) $W_0 = 0$,

(II) it is continuous,

(III) has stationary increments: the distribution doesn't depend on initial time, only on waiting time.

$$\forall t > s,\ W_t - W_s \sim \mathcal{N}(0, t-s)$$

(IV) has independent increments over disjoint intervals:

$$\forall q < r < s < t,\ (W_r - W_q) \perp (W_t - W_s).$$

□

Definition 2. (Classes of Brownian motions.)

(I) *Standard Brownian motion*, or *Wiener Process*.

(II) *Arithmetic Brownian motion*, or *Bachelier Model*.

$$dp_t = p_{t+h} - p_t = \mu dt + \sigma dW_t$$

Where μ is called *drift*, and σ the *volatility*. This is a *stochastic differential equation*, *SDE* for short: while the μdt part is *deterministic*, the

σdW_t contains a *stochastic* component, which is a standard Brownian motion. This kind of equations provides a model for the change of the price p_t over the infinitesimal time increment from t to $t+h$. The price variation could be equivalently modeled this way:

$$p_t - p_0 = \mu(t-0) + \sigma(W_t - W_0) = \mu t + \sigma W_t$$

Noting that

$$E[W_t] = 0 \quad \text{and} \quad Var[\sigma W_t] = E[(\sigma W_t)^2] = \sigma^2 E[W_t^2] = \sigma^2 t,$$

we have that, in Bachelier's model, the price $p_t = \mu t + \sigma W_t + p_0$ is a *random variable* distributed like

$$p_t \sim \mathcal{N}(p_0 + \mu t + E[W_t], \sigma^2 t) \equiv \mathcal{N}(p_0 + \mu t, \sigma^2 t).$$

(III) *Geometric Brownian motion*, or *Black-Scholes model*.

$$dp_t = \mu p_t dt + \sigma p_t dW_t$$

This model is similar to Bachelier's, but acts on a multiplicative instead of additive principle. For this model, we will need to find a solution and its distribution.

□

1.2 Itō Formula

Suppose a model for an underlying asset price dynamics is given: we know the form of dp_t; we'll assume Black-Scholes. It is now natural to assume that the price of a derivative on this asset is a *function*, let it be $f(p_t)$, of the asset price. How can we get a *stochastic differential equation* that models the derivative price variation?

Let's evaluate the *Taylor expansion* for $f(p_t)$ like in a deterministic setting.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \varepsilon$$

where ε is a remainder that goes quickly to zero. We will now discard all terms in the approximation that go to 0 faster than dt does, thus truncating the approximation to to the first order. Let now $x = p_{t+dt}$, $x_0 = p_t$ and $dp_t = p_{t+dt} - p_t$

$$f(p_{t+dt}) \approx f(p_t) + f'(p_t)(dp_t) + \frac{1}{2}f''(p_t)(dp_t)^2 + \varepsilon$$

and substitute the increment dp_t with the Black-Scholes model:

$$\begin{aligned} f(p_{t+dt}) = & f(p_t) + f'(p_t)(\mu p_t dt + \sigma p_t dW_t) + \\ & \frac{1}{2}f''(p_t)(\mu^2 p_t^2 (dt)^2 + \sigma^2 p_t^2 (dW_t)^2 + 2\mu\sigma p_t^2 dt dW_t) \end{aligned}$$

Since $E[(dW_t)^2] = dt$, we can assume $dW_t = \sqrt{dt} = (dt)^{\frac{1}{2}}$ to have order $\frac{1}{2}$. Let's analyze the orders of all the terms in the approximation:

- $\mu p_t dt$: First order: keep it.
- $\sigma p_t dW_t$: Order $\frac{1}{2}$: keep it.
- $\mu^2 p_t^2 (dt)^2$: Second order: discard it.
- $\sigma^2 p_t^2 (dW_t)^2$: First order, because $(dW_t)^2 = dt$: keep it.
- $2\mu\sigma p_t^2 dt dW_t$: Order $\frac{3}{2}$ because of $dt dW_t$: discard it.

The truncated approximation now states that

$$f(p_{t+dt}) = f(p_t) + f'(p_t)(\mu p_t dt + \sigma p_t dW_t) + \frac{1}{2}f''(p_t)(\sigma^2 p_t^2 dt).$$

The second derivative term of this equation is called *Itō correction term*. The derivative of the function with respect to time can then be computed this way:

$$df(p_t) = f(p_{t+dt}) - f(p_t) = f'(p_t)(\mu p_t dt + \sigma p_t dW_t) + \frac{1}{2}f''(p_t)(\sigma^2 p_t^2 dt)$$

Let's now separate the deterministic terms from the stochastic terms, so we can identify a *drift* and a *volatility* for the model.

$$df(p_t) = \left[f'(p_t)\mu p_t + \frac{1}{2}f''(p_t)\sigma^2 p_t^2\right]dt + f'(p_t)\sigma p_t dW_t$$

Example 1. Let $f(p_t) = \ln(p_t)$ be the price of a derivative instrument with underlying price p_t; the dynamic for the underlying follows the Black-Scholes model. Apply Itō's formula to this derivative:

$$\begin{aligned} d\ln(p_t) &= \left[\frac{1}{p_t}\mu p_t + \frac{1}{2}\left(-\frac{1}{p_t^2}\sigma^2 p_t^2\right)\right]dt + \frac{1}{p_t}\sigma p_t dW_t = \\ &= \left[\mu - \frac{1}{2}\sigma^2\right]dt + \sigma dW_t \end{aligned}$$

Considering the time interval $[0, t]$, we observe that

$$d\ln(p_t) = \ln(p_t) - \ln(p_0) = \left(\mu - \frac{1}{2}\sigma^2\right)(t-0) + \sigma(W_t - W_0)$$

and then

$$\ln\left(\frac{p_t}{p_0}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t \iff \frac{p_t}{p_0} = \mathrm{e}^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$$

which yields the solution

$$p_t = p_0 \mathrm{e}^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}.$$

Example 2. Black-Scholes model assumes the *spot risk-free interest rate* r to be constant and independent of maturity. Consider *discounting* as a function of time and asset price:

$$f(t; p_t) = \mathrm{e}^{-rt} p_t$$

The general Itō formula for a function of such a form is

$$\begin{aligned} d[f(t;p_t)] =& \frac{\partial f}{\partial p_t}dp_t + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial p_t^2}(dp_t)^2 + \underbrace{\cancel{\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2}}_{\text{order 2}} + \underbrace{\cancel{\frac{1}{2}\frac{\partial^2 f}{\partial p_t \partial t}2dt dp_t}}_{\text{order 3/2}} \\ =& \frac{\partial f}{\partial p_t}dp_t + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial p_t^2}\sigma^2 p_t^2 dt \end{aligned}$$

For the discounting function, this means

$$\begin{aligned}
d[e^{-rt}p_t] =& \mathrm{e}^{-rt} \cdot 1 \cdot dp_t + -r\mathrm{e}^{-rt} p_t dt + \cancelto{0}{\frac{1}{2} \cdot 0 \cdot \sigma^2 p_t^2 dt} \\
=& \mathrm{e}^{-rt} \left(\mu p_t dt + \sigma p_t dW_t - r p_t dt \right) \\
=& (\mu - r)\mathrm{e}^{-rt} p_t dt + \mathrm{e}^{-rt} p_t \sigma dW_t \\
d\tilde{p}_t =& (\mu - r)\tilde{p}_t dt + \sigma \tilde{p}_t dW_t
\end{aligned}$$

Hence, the distribution for a discounted asset price follows Black-Scholes model:

$$\tilde{p}_t = \tilde{p}_0 \mathrm{e}^{(\mu - r - \frac{\sigma^2}{2})t + \sigma W_t}$$

Note that the deterministic and stochastic parts were grouped together, to underline the *risk factor*.

1.3 Black-Scholes PDE for option pricing

In the context of option pricing, Black-Scholes model assumes the following:

- p_t is the underlying asset price at time t,
- r is a constant, risk-free interest rate,
- There are no transaction costs, no taxes, and no arbitrage opportunity,
- The market is liquid, and so all the instruments,
- $f(t; p_t)$ is the option price.

By Itō's lemma, we get that the option price is

$$df(t; p_t) = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial p_t} \mu p_t + \frac{1}{2} \frac{\partial^2 f}{\partial p_t^2} \sigma^2 p_t^2 \right] dt + \frac{\partial f}{\partial p_t} \sigma p_t dW_t$$

Construct a *locally* risk-free portfolio, Π_t, such that

$$\Pi_t = \begin{cases} -1 & \text{positions in options (short)} \\ \Delta_t \equiv \frac{\partial f}{\partial p_t} & \text{positions in underlying (long)} \end{cases}$$

and study the dynamics of the portfolio value by multiplying the number of positions by the dynamics for each kind of instrument (option and asset).

$$\begin{aligned}
d\Pi_t &= -1 \cdot df(t;p_t) + \frac{\partial f}{\partial p_t} dp_t \\
&= \underbrace{- \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial p_t}\mu p_t + \frac{1}{2}\frac{\partial^2 f}{\partial p_t^2}\sigma^2 p_t^2 \right] dt - \frac{\partial f}{\partial p_t}\sigma p_t dW_t}_{\text{option}} + \underbrace{\frac{\partial f}{\partial p_t}\left(\mu p_t dt + \sigma p_t dW_t\right)}_{\text{asset}} \\
&= - \left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial p_t^2}\sigma^2 p_t^2 \right] dt - \cancel{\frac{\partial f}{\partial p_t}\mu p_t dt} - \cancel{\frac{\partial f}{\partial p_t}\sigma p_t dW_t} + \cancel{\frac{\partial f}{\partial p_t}\mu p_t dt} + \cancel{\frac{\partial f}{\partial p_t}\sigma p_t dW_t} \\
&= - \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial p_t^2}\sigma^2 p_t^2 \right) dt
\end{aligned}$$

Having removed the Brownian motion, we are left without any risky term. We impose now the *no arbitrage assumption*, stating that a portfolio is risk-free if and only if its dynamics is the same of a bond, that is, it accrues interest at a constant (by assumption) rate over time.

$$d\Pi_t \equiv - \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial p_t^2}\sigma^2 p_t^2 \right) dt \overset{\text{NAA}}{=} r\Pi_t dt$$

$$-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial p_t^2}\sigma^2 p_t^2 = -rf(t;p_t) + \frac{\partial f}{\partial p_t} r p_t$$

Finally, we obtain the Black-Scholes PDE by rearranging.

$$rf(t;p_t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial p_t} r p_t + \frac{1}{2}\frac{\partial^2 f}{\partial p_t^2}\sigma^2 p_t^2$$

The objective is to establish the *fair* (or *no arbitrage*) price of an option today, that is, $f(0;p_0)$. A *final condition* can be imposed: $f(T,p_T)$, which is the *option price at maturity* or, equivalently, the *payoff*, which is known.

$$f(T; p_T) \overset{e.g.}{=} \begin{array}{ll} (P_T - K)^+ = \max\{P_T - K, 0\} & \text{European call option} \\ (K - P_T)^+ = \max\{K - P_T, 0\} & \text{European put option} \end{array}$$

European options satisfy Black-Scholes assumptions: the payoff depends *only* on the price of the underlying at time T, and it is not *path dependent*. The drift term μ does not appear in Black-Scholes PDE, and neither does in the payoff function $f(t; p_t)$; this means that the Black-Scholes option price doesn't depend on it. The drift term is strongly linked to investor's *risk aversion*: this means the option can be priced *as if* the investor is *risk neutral*.

A *risk neutral* valuation intuitively means that we are pricing in a world where every investor behaves as if he himself is risk neutral; the real world is not risk neutral, though, and investors' risk aversion is embedded in the *historical* data.

A *risk neutral* valuation of current option price, given payoff $f(T, p_T)$, is

$$f(0; p_0) = \tilde{\mathbb{E}}\left[e^{-rT} f(T; p_T)\right]$$

Where $\tilde{\mathbb{E}}$ is the expectation according to the *risk neutral probability* $\tilde{\mathbb{P}}$. We now have, in fact, two probability spaces: the first is the *historical probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, the second is the *risk neutral probability space* $(\Omega, \mathcal{F}, \tilde{\mathbb{P}}) \supseteq (\Omega, \mathcal{F}, \mathbb{P})$. To calculate the option price at any given time $t : 0 < t < T$ the information in the filtration up to time t can be used:

$$f(t; p_0) = \tilde{\mathbb{E}}\left[e^{-r(T-t)} f(T; p_t) \,|\mathcal{F}_t\right]$$

Theorem 1. If, in time $t : 0 \leq t \leq T$ the payoff is $(P_T - K)^+$, the price of an option at time t is

$$C(\tau = T - t, p_t, K, r, \sigma) = f(t; p_t) = p_t \mathcal{N}(d_1) - K\mathrm{e}^{-r\tau} \mathcal{N}(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{p_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = \frac{\ln\left(\frac{p_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}} = d_1 - \sigma\sqrt{T - t}$$

Proof. Suppose we are in a Black-Scholes world: we have one risky asset with dynamics such that $dp_t = \mu p_t dt + \sigma p_t d\tilde{W}_t$, a riskless asset with dynamics such that $dB_t = rB_t dt$, and no arbitrage opportunities. Define $\tilde{W}_t = W_t + \frac{\mu - r}{\sigma} t$ where $\mu - r$ is the *risk premium* and $\frac{\mu-r}{\sigma}$ is the *market price of risk*. Hence,

$$\begin{aligned} dp_t &= rp_t dt + \sigma p_t \left[dW_t + \frac{\mu - r}{\sigma} dt \right] \\ &= \cancel{rp_t dt} + \sigma p_t dW_t + \mu p_t dt - \cancel{rp_t dt} \\ &= \mu p_t dt + \sigma p_t dW_t \end{aligned}$$

Note that $\tilde{W}_t \sim \mathcal{N}\left(\frac{\mu-r}{\sigma}\Delta t; \Delta t\right)$ is no longer a standard Brownian motion. We want to know the option price at time $t = 0$, considering, for example, the payoff $f(T; p_T) = (p_T - K)^+$:

$$\begin{aligned} C_0 = f(0; p_0) &= \tilde{\mathbb{E}}\left[\mathrm{e}^{-rT}(p_T - K)^+\right] \\ &\overset{\text{Markov}}{=} \tilde{\mathbb{E}}\left[\mathrm{e}^{-rT}\left(p_0 \mathrm{e}^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma \tilde{W}_T} - K\right)^+\right] = (*)_1 \end{aligned}$$

Consider now that, if $X \sim \mathcal{N}(0, T)$ and $Y \sim \mathcal{N}(0, 1)$ then $X = \sqrt{T}Y$:

$$\begin{aligned} (*)_1 &= \tilde{\mathbb{E}}\left[\mathrm{e}^{-rT}\left(p_0 \mathrm{e}^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Y} - K\right)^+\right] \\ &= \int_{-\infty}^{+\infty} \mathrm{e}^{-rt}\left(p_0 \mathrm{e}^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}y} - K\right)^+ \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\frac{-y^2}{2}} dy = (*)_2 \end{aligned}$$

We compute the integral only where the payoff is positive, that is, where

$$\begin{aligned} p_0 \mathrm{e}^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}y} \geq K &\iff \mathrm{e}^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}y} \geq \frac{K}{p_0} \\ &\iff \left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}y \geq \ln\frac{K}{p_0} \\ &\iff y \geq \frac{-\ln\frac{p_0}{K} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = -d_2 \end{aligned}$$

$$
\begin{aligned}
(*)_2 &= \int_{-d_2}^{+\infty} \mathrm{e}^{-rt}\left(p_0 \mathrm{e}^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma y\sqrt{T}} - K\right)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}dy \\
&= \int_{-d_2}^{+\infty} \frac{p_0 \mathrm{e}^{-\frac{\sigma^2}{2}+\sigma y\sqrt{T}-\frac{y^2}{2}}}{\sqrt{2\pi}}dy - \int_{-d_2}^{+\infty}\frac{K\mathrm{e}^{-rt-\frac{y^2}{2}}}{\sqrt{2\pi}}dy \\
&= p_0\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(\sigma\sqrt{T}+y)^2}{2}}dy - K\mathrm{e}^{-rt}\int_{-d2}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}dy \\
&\left(\text{Let } z = y+\sigma\sqrt{T} \text{ s.t. } -\infty \le z \le d_2+\sigma\sqrt{T}\right) \\
&= p_0\int_{-\infty}^{d_2+\sigma\sqrt{T}}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{z^2}{2}}dz - K\mathrm{e}^{-rt}\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}dy \\
&= p_0\mathcal{N}(d_2+\sigma\sqrt{T}) - K\mathrm{e}^{-rt}\mathcal{N}(d_2) \\
&= p_0\mathcal{N}(d_1) - K\mathrm{e}^{-rt}\mathcal{N}(d_2).
\end{aligned}
$$

1.4 Cox-Ross-Rubinstein model

This model is the equivalent, in discrete time, of Black-Scholes. CRR accepts the assumptions of one risky asset, one riskless asset and no arbitrage. The discrete time implies the underlying is modeled using a discrete probability distribution: the Binomial model.

1.4.1 Single period model

In a simple, single period model, the asset can go up in value of a percentage u with probability p or go down in value of a percentage d with probability $1-p$.

$$
S_0 \begin{cases} S_1^u = uS_0 & p \\ S_1^d = dS_0 & 1-p \end{cases} \qquad C_0 \begin{cases} C_u = (uS_0-K)^+ & p \\ C_d = (dS_0-K)^+ & 1-p \end{cases}
$$

The percentages are related like this: $u > 1+r > 1 > d \geq 0$.

Again, construct a risk-free portfolio using the underlying and the option. For the portfolio to be risk-free, its payoff must be the same for each possible state of the world.

$$\Pi = \begin{cases} -1 & \text{option (short)} \\ \Delta & \text{underlying (long)} \end{cases}$$

$$\Pi \text{ is risk-free} \iff -C_u + \Delta u S_0 = -C_d + \Delta d S_0 \iff \Delta = \frac{C_u - C_d}{S_0(u-d)}$$

So, Δ is the quantity of stock to be held so that the portfolio is risk-free. Now impose the no arbitrage assumption: assert that the risk-free portfolio in any of the two states of the world after one period must have the same payoff as the portfolio at initial time capitalized for the interest rate.

$$-C_u + \Delta u S_0 \overset{NAA}{=} (1+r)(-C_0 + \Delta S_0)$$

$$\begin{aligned}
-C_u + \frac{C_u - C_d}{\cancel{S_0}(u-d)}\cancel{S_0}u &= (1+r)\left(-C_0 + \frac{C_u - C_d}{\cancel{S_0}(u-d)}\cancel{S_0}\right) \Rightarrow \\
\frac{-C_u(u-d) + (C_u - C_d)u}{u-d} &= -C_0(1+r) + \frac{(1+r)(C_u - C_d)}{u-d} \Rightarrow \\
\frac{C_u(u-d) + (C_u - C_d)(1+r-u)}{u-d} &= C_0(1+r) \Rightarrow \\
C_0 &= \frac{1}{1+r}\left(C_u \frac{1+r-d}{u-d} + C_d \frac{-1-r+u}{u-d}\right) \Rightarrow \\
C_0 &= \frac{1}{1+r}\left(C_u p + C_d(1-p)\right) \quad \text{where} \quad p = \frac{1+r-d}{u-d}
\end{aligned}$$

Due to the previously imposed relationships between u, d and r, p respects positivity and can be used as a *risk neutral probability*.

$$\begin{aligned}
\tilde{\mathbb{E}}[S_1] = (uS_0)p + (dS_0)(1-p) = uS_0 \frac{1+r-d}{u-d} + dS_0 \frac{u-1-r}{u-d} = \\
\frac{u + ur - ud + du - d - dr}{u-d} S_0 = \frac{(u-d)(1+r)}{u-d} S_0 = (1+r)S_0
\end{aligned}$$

1.4.2 Two-period model

In a two-period CRR model, we have a *recombining tree*, that is, up-down and down-up movements yield the same result.

$$
S_0 \left\langle \begin{array}{c} uS_0 \left\langle \begin{array}{c} u^2S_0 \\ udS_0 \end{array}\right. \\ dS_0 \left\langle \begin{array}{c} duS_0 \\ d^2S_0 \end{array}\right. \end{array}\right.
\qquad
C_0 \left\langle \begin{array}{c} C_u \left\langle \begin{array}{l} C_{uu} = (u^2S_0 - K)^+ \\ C_{ud} = (udS_0 - K)^+ \end{array}\right. \\ C_d \left\langle \begin{array}{l} C_{du} = (duS_0 - K)^+ \\ C_{dd} = (d^2S_0 - K)^+ \end{array}\right. \end{array}\right.
$$

To obtain the option price C_0 in this case, the idea is to simply backtrack from the last period, calculating the discounted C_u and C_d first.

$$
\begin{aligned}
C_u \left\langle \begin{array}{l} C_{uu} = (u^2S_0 - K)^+ \\ C_{ud} = (udS_0 - K)^+ \end{array}\right. &\implies C_u = \frac{1}{1+r}(C_{uu}p + C_{ud}(1-p)) \\
C_d \left\langle \begin{array}{l} C_{du} = (duS_0 - K)^+ \\ C_{dd} = (d^2S_0 - K)^+ \end{array}\right. &\implies C_d = \frac{1}{1+r}(C_{du}p + C_{dd}(1-p))
\end{aligned}
$$

Finally, we compose these results by computing

$$
\begin{aligned}
C_0 \left\langle \begin{array}{l} C_u \\ C_d \end{array}\right. &\implies C_0 = \frac{1}{1+r}(C_u p + C_d(1-p)) \\
&= \frac{1}{(1+r)^2}\left(C_{uu}p^2 + C_{ud}p(1-p) + C_{du}p(1-p) + C_{dd}(1-p)^2\right) \\
&= \frac{1}{(1+r)^2}\left(C_{uu}p^2 + 2C_{ud}p(1-p) + C_{dd}(1-p)^2\right)
\end{aligned}
$$

1.4.3 n-period model

Generalizing to n periods, we have a $\sim \mathcal{B}(n,p)$ model; any given path on the binomial tree over the n periods can have $j : 0 \leq j \leq n$ steps up and $n-j$ steps down for the underlying's price, with a payoff of

$$(u^j d^{n-j} S_0 - K)^+$$

for an option on the underlying. Following the same reasoning as per the two- and one-period model, we compute the option price as

$$
\begin{aligned}
C_0 &= \frac{1}{(1+r)^n} \sum_{j=0}^{n} \left(u^j d^{n-j} S_0 - K\right)^+ \binom{n}{j} p^j (1-p)^{n-j} \\
&\quad \text{(let } a \text{ s.t. } \forall j \geq a \quad u^j d^{n-j} S_0 - K > 0\text{)} \\
&= \frac{1}{(1+r)^n} \sum_{j=a}^{n} \left(u^j d^{n-j} S_0 - K\right)^+ \binom{n}{j} p^j (1-p)^{n-j} \\
&= \frac{1}{(1+r)^n} \left[\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} u^j d^{n-j} S_0 - K \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \right] \\
&= \sum_{j=a}^{n} \binom{n}{j} \left(\frac{pu}{1+r}\right)^j \left(\frac{(1-p)d}{1+r}\right)^{n-j} S_0 - \frac{1}{(1+r)^n} K \mathcal{B}_{(n,p)}(j \geq a) \\
&= S_0 \mathcal{B}_{\left(n, \frac{pu}{1+r}\right)}(j \geq a) - \frac{1}{(1+r)^n} K \mathcal{B}_{(n,p)}(j \geq a).
\end{aligned}
$$

Note that $\frac{pu}{1+r}$ is a probability because it is positive by no arbitrage assumption and sums to 1 with $\frac{(1-p)d}{1+r}$. Note that the equation looks very similar to Black-Scholes, with the Binomial distribution instead of the Normal.

1.5 Martingale

Let S_1 be the stock price tomorrow. Under *risk neutral probability*, this should be equal to the stock price today capitalized for the given interest rate:

$$\tilde{\mathbb{E}}[S_1] = (1+r)S_0$$

Dividing both sides by $(1+r)$, we obtain that the expectation (under *risk neutral probability*) of the stock price tomorrow ($t=1$) is equal to the stock price today ($t=0$).

$$\tilde{\mathbb{E}}\left[\frac{1}{(1+r)^1} S_1\right] = \frac{1}{(1+r)^0} S_0 \equiv S_0$$

This is a *martingale*: in such a process, the expectations through time are constant. A martingale is a *fair game* (in game-theoretic sense), that is, it has zero drift. The stock market is not a fair game in the real world, but in a risk neutral world it is.

Definition 3. (Martingale). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The process $(M_t)_{t\geq 0}$ is a *martingale* if the expectation at a future time t given the information up to time s is equal to the expectation at time $s : 0 \leq s \leq t \leq T$.

$$\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}[M_s]$$

Equivalently,

$$\mathbb{E}[M_{s+ds}|\mathcal{F}_s] = M_s \implies \mathbb{E}[M_{s+ds}] = \mathbb{E}[M_s]$$

□

Example 3. The Brownian motion is a martingale.

Proof. Let $(W_t)_{t\geq 0}$ be a Brownian Motion in $(\Omega, \mathcal{F}, \mathbb{P})$.

$$\mathbb{E}[W_t|\mathcal{F}_s] = W_s,\ s < t \implies \mathbb{E}[W_t - W_s|\mathcal{F}_s] = 0$$

From the definition of Brownian motion, and its independent increments property, follows

$$\mathbb{E}[W_t - W_s|\mathcal{F}_s] = \mathbb{E}[W_t - W_s] \overset{\mathcal{N}(0,\sigma)}{=} 0$$

□

1.6 Pricing exotic options

The generic process of pricing an option involves discounting the option's payoff at maturity given a model for the underlying.

1.6.1 Digital call option

The payoff for a *digital call option* is H when the option is *in the money*, 0 otherwise:

$$D_T = \begin{cases} H & \text{if } p_t \geq K \\ 0 & \text{otherwise} \end{cases}$$

Let $dp_t = (r-q)p_t dt + \sigma p_t dW_t$ be the model for the underlying, $t : 0 \le t \le T$ the current time and T the maturity. The price of the digital option at time t under the risk neutral measure is

$$D_t = \tilde{\mathbb{E}}\left[\mathrm{e}^{-r(T-t)} H \mathbb{1}_{(p_T \ge K)} | \mathcal{F}_t\right] = \mathrm{e}^{-r(T-t)} H \mathbb{E}\left[\mathbb{1}_{(p_T \ge K)} | \mathcal{F}_t\right] = \cdots$$

Knowing that $\mathbb{E}[\mathbb{1}_A] = 1 \cdot \mathbb{P}(A) + 0 \cdot \mathbb{P}(A^C) = \mathbb{P}(A)$, we can replace the expectation with the risk neutral probability; then, for the Markov property of Black-Scholes model, we replace the filtration with the information known at time t, that is, the underlying price.

$$\cdots = \mathrm{e}^{-r(T-t)} H \tilde{\mathbb{P}}(p_T \ge K | \mathcal{F}_t) = \mathrm{e}^{-r(T-t)} H \tilde{\mathbb{P}}(p_T \ge K | p_t) = \cdots$$

By Itō's lemma, we can express the price at maturity p_T as

$$p_T = p_t \mathrm{e}^{\left(r-q-\frac{\sigma^2}{2}\right)(T-t)+\sigma(W_T - W_t)}$$

Also, let $Y : Y\sqrt{T-t} = W_T - W_t$. Then,

$$\begin{aligned}
\cdots &= \mathrm{e}^{-r(T-t)} H \tilde{\mathbb{P}}\left(p_t \mathrm{e}^{\left(r-q-\frac{\sigma^2}{2}\right)(T-t)+\sigma(W_T-W_t)} \ge K\right) \\
&= \mathrm{e}^{-r(T-t)} H \tilde{\mathbb{P}}\left(p_t \mathrm{e}^{\left(r-q-\frac{\sigma^2}{2}\right)(T-t)+\sigma Y\sqrt{T-t}} \ge K\right) \\
&= \mathrm{e}^{-r(T-t)} H \tilde{\mathbb{P}}\left(Y \ge -\frac{\ln \frac{p_t}{K} + \left(r - q - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \mathrm{e}^{-r(T-t)} H \tilde{\mathbb{P}}(Y \ge -d_2) \\
&= \mathrm{e}^{-r(T-t)} H \mathcal{N}(-d_2)
\end{aligned}$$

1.6.2 Asset-or-nothing call option

The payoff for an *asset-or-nothing* call option is p_T when the option is *in the money*, 0 otherwise:

$$A_T = \begin{cases} p_T & \text{if } p_t \ge K \\ 0 & \text{otherwise} \end{cases}$$

The *asset-or-nothing* option price at present time t is then computed as

$$\begin{aligned} A_t &= \tilde{\mathbb{E}}\left[\mathrm{e}^{-r(T-t)} p_T \mathbb{1}(p_T \geq K) | \mathcal{F}_t\right] \\ &= \mathrm{e}^{-r(T-t)} \tilde{\mathbb{E}}\left[p_T \mathbb{1}(p_T \geq K) | \mathcal{F}_t\right] \overset{Markov}{=} \mathrm{e}^{-r(T-t)} \tilde{\mathbb{E}}\left[p_T \mathbb{1}(p_T \geq K) | p_t\right] = \cdots \end{aligned}$$

p_T is a *random variable*; its expectation can be computed, knowing its distribution, by integrating. The indicator function means that the expectation for p_T can be computed only over the part where $p_T \geq K$, that is, past d_2.

$$\begin{aligned} \cdots &= \mathrm{e}^{-r(T-t)} \int_{-d_2}^{\infty} p_t \mathrm{e}^{\left(r-q-\frac{\sigma^2}{2}\right)(T-t)+\sigma(W_T-W_t)} \cdot \frac{\mathrm{e}^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \\ &= \int_{-d_2}^{\infty} p_t \mathrm{e}^{\left(r-r-q-\frac{\sigma^2}{2}\right)(T-t)+\sigma(W_T-W_t)} \cdot \frac{\mathrm{e}^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \\ &= p_t \mathrm{e}^{-q(T-t)} \int_{-d_2}^{\infty} \frac{\mathrm{e}^{-\frac{\sigma^2}{2}+\sigma y\sqrt{T-t}-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \\ &= p_t \mathrm{e}^{-q(T-t)} \int_{-d_2}^{\infty} \frac{\mathrm{e}^{-\frac{(\sigma\sqrt{T-t}+y)^2}{2}}}{\sqrt{2\pi}} dy = \cdots \end{aligned}$$

Now we flip the integration domain thanks to Normal distribution's symmetry property and then integrate by substituting $z : z(y) = \sigma\sqrt{T-t} + y$; the new integration domain extremes are then $z(-\infty) = -\infty$ and $z(d_2) = d_2 + \sigma\sqrt{T-t} = d_1$.

$$\begin{aligned} \cdots &= p_t \mathrm{e}^{-q(T-t)} \int_{-\infty}^{d_1} \frac{\mathrm{e}^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \\ &= p_t \mathrm{e}^{-q(T-t)} \mathcal{N}(d_1). \end{aligned}$$

1.7 Equivalence of PDEs with Risk Neutral Valuation

Do PDEs and risk neutral valuation yield the same value for the price of a derivative instrument? Suppose $f(t; p_t)$ satisfies Black-Scholes PDE:

$$\frac{\partial f}{\partial t} + rp_t\frac{\partial f}{\partial p_t} + \frac{1}{2}\frac{\partial^2 f}{\partial p_t^2}\sigma p_t^2 = rf(t;p_t)$$

We need to check if the price arising from risk neutral valuation is the same:

$$f(t;p_t) = \tilde{\mathbb{E}}\left[e^{-r(T-t)}\cdot \text{payoff}|\mathcal{F}_t\right]$$

Example 4. Given the option price f, apply Itō to get the discounted option price.

$$\begin{aligned}
d\left[f(t;p_t)\mathrm{e}^{-rt}\right] &= -r\mathrm{e}^{-rt}f dt + \mathrm{e}^{-rt}df + 0 \\
&= -r\mathrm{e}^{-rt}f dt + \mathrm{e}^{-rt}\left(\frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial p_t}rp_t dt + \frac{\partial f}{\partial p_t}\sigma p_t dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial p_t^2}\sigma^2 p_t^2 dt\right) \\
&= \mathrm{e}^{-rt}dt\left(\cancel{-rf} + \cancel{\frac{\partial f}{\partial t} + \frac{\partial f}{\partial p_t}rp_t + \frac{1}{2}\frac{\partial^2 f}{\partial p_t^2}\sigma^2 p_t^2}\right) + \mathrm{e}^{-rt}\frac{\partial f}{\partial p_t}\sigma p_t dW_t \\
&= \mathrm{e}^{-rt}\frac{\partial f}{\partial p_t}\sigma p_t dW_t
\end{aligned}$$

We now note that the discounted option price lacks a drift, and is thus a *martingale*: hence,

$$\tilde{\mathbb{E}}\left[\mathrm{e}^{-rT}f(T;p_T)|\mathcal{F}_t\right] = \mathrm{e}^{-rt}f(t;p_t)$$

and the two approaches give the same result. **TODO not clear!**

1.8 Dynamic Hedging: the Greeks

The operation of constructing a locally risk-free portfolio like in Black-Scholes approach is an *hedging strategy*. The *Greeks* are quantities, named after the fact that each of them is indicated by a different greek letter, which convey some information on the sensitivity of the price of a derivative with respect to some financial component of the model (for example, the price or the volatility of the underlying, or some other parameter). Each greek is actually a function of time, and can thus be computed at any time t, hence providing a form of *dynamic hedging*.

1.8.1 Delta

The greek *Delta* measures the *sensitivity* of the price of the derivative with respect to the underlying's price.

$$\Delta_t = \frac{\partial f(t; S_t)}{\partial S_t}$$

Example 5. Let's calculate the Delta for an European call option. The call option price is

$$C_t = S_t \mathcal{N}(d_1) - K\mathrm{e}^{-r(T-t)}\mathcal{N}(d_2) \quad \text{where} \quad \begin{cases} d_1 = \frac{\ln \frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = d_1 - \sigma\sqrt{T-t} \end{cases}$$

The Delta is the quantity of underlying I need at time t to hedge the risk in a portfolio $\Pi_t = \left\{-1 \text{ option}, \frac{\partial f(t;S_t)}{\partial S_t} \text{ underlying}\right\}$.

$$\Delta_t^C = \frac{\partial C_t}{\partial S_t} = \mathcal{N}(d_1)$$

Notice that the Delta for an European call option is always positive, which means that in the case of the portfolio Π_t the underlying will always be held in a long position.

Proof. S_t also appears in d_1 and d_2, so we can't treat $\mathcal{N}(d_1)$ and $\mathcal{N}(d_2)$ as constants; we must differentiate them w.r. S_t too. For the chain rule,

$$\frac{\partial S_t \mathcal{N}(d_1)}{\partial S_t} = \cancelto{1}{\frac{\partial S_t}{\partial S_t}}\mathcal{N}(d_1) + S_t \frac{\partial \mathcal{N}(d_1)}{\partial S_t}.$$

So,

$$\Delta_t^C = \mathcal{N}(d_1) + S_t \frac{\partial \mathcal{N}(d_1)}{\partial S_t} - K\mathrm{e}^{-r(T-t)} \frac{\partial \mathcal{N}(d_2)}{\partial S_t}$$

Compute the two partial derivatives appearing. First, the one for $\mathcal{N}(d_1)$:

$$\begin{aligned} \frac{\partial \mathcal{N}(d_1)}{\partial S_t} &= \frac{\partial}{\partial S_t} \int_{-\infty}^{d_1} \frac{\mathrm{e}^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy = \mathcal{N}'(d_1)\frac{\partial d_1}{\partial S_t} - \mathcal{N}'(-\infty)\cancelto{0}{\frac{\partial(-\infty)}{\partial S_t}} \\ &= n(d_1)\frac{\partial d_1}{\partial S_t} = n(d_1) \cdot \frac{1}{\sigma\sqrt{T-t}} \frac{1}{\frac{S_t}{\cancel{K}}} \frac{1}{\cancel{K}} = \frac{n(d_1)}{S_t \sigma \sqrt{T-t}} \end{aligned}$$

Then, the one for $\mathcal{N}(d_2)$:

$$\begin{aligned}\frac{\partial \mathcal{N}(d_2)}{\partial S_t} &= n(d_2)\frac{\partial d_2}{\partial S_t} = n(d_2)\frac{\partial (d_1 - \sigma\sqrt{T-t})}{\partial S_t} \\ &= n(d_2)\cdot\left(\frac{1}{\sigma\sqrt{T-t}}\frac{1}{\frac{S_t}{\cancel{K}}}\frac{1}{\cancel{K}}\right) = \frac{n(d_2)}{S_t\sigma\sqrt{T-t}}\end{aligned}$$

Now plug the results in the original equation for Delta:

$$\Delta_t^C = \mathcal{N}(d_1) + S_t\frac{n(d_1)}{S_t\sigma\sqrt{T-t}} - K\mathrm{e}^{-r(T-t)}\frac{n(d_2)}{S_t\sigma\sqrt{T-t}}$$

Isolate the two terms multiplying the gaussian density n:

$$\begin{aligned}
&S_t\frac{n(d_1)}{S_t\sigma\sqrt{T-t}} - K\mathrm{e}^{-r(T-t)}\frac{n(d_2)}{S_t\sigma\sqrt{T-t}} = \\
=&\frac{1}{S_t\sigma\sqrt{T-t}}\left(S_t n(d_1) - K\mathrm{e}^{-r(T-t)}n(d_2)\right) = \\
=&\frac{1}{S_t\sigma\sqrt{T-t}}\left(\frac{S_t\mathrm{e}^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} - \frac{K\mathrm{e}^{-r(T-t)-\frac{d_2^2}{2}}}{\sqrt{2\pi}}\right) = \\
=&\frac{1}{S_t\sigma\sqrt{T-t}\sqrt{2\pi}}\left(S_t\mathrm{e}^{-\frac{d_1^2}{2}} - K\mathrm{e}^{-r(T-t)-\frac{d_2^2}{2}}\right) = \\
=&\frac{1}{S_t\sigma\sqrt{T-t}\sqrt{2\pi}}\left(S_t\mathrm{e}^{-\frac{d_1^2}{2}} - K\mathrm{e}^{-r(T-t)-\frac{d_1^2}{2}-\frac{\sigma^2(T-t)}{2}+\frac{2}{2}d_1\sigma\sqrt{T-t}}\right) = \\
=&\frac{\mathrm{e}^{-\frac{d_1^2}{2}}}{S_t\sigma\sqrt{T-t}\sqrt{2\pi}}\left(S_t - K\mathrm{e}^{-r(T-t)-\frac{\sigma^2(T-t)}{2}+d_1\sigma\sqrt{T-t}}\right) = \\
=&\frac{\mathrm{e}^{-\frac{d_1^2}{2}}}{S_t\sigma\sqrt{T-t}\sqrt{2\pi}}\left(S_t - K\mathrm{e}^{\cancel{-r(T-t)-\frac{\sigma^2(T-t)}{2}}+\ln\frac{S_t}{K}+\cancel{\left(r+\frac{\sigma^2}{2}\right)(T-t)}}\right) = \\
=&\frac{\mathrm{e}^{-\frac{d_1^2}{2}}}{S_t\sigma\sqrt{T-t}\sqrt{2\pi}}\left(S_t - K\mathrm{e}^{\ln\frac{S_t}{K}}\right) = \frac{\mathrm{e}^{-\frac{d_1^2}{2}}}{S_t\sigma\sqrt{T-t}\sqrt{2\pi}}\left(S_t - K\frac{S_t}{K}\right) = \\
=&\frac{\mathrm{e}^{-\frac{d_1^2}{2}}}{S_t\sigma\sqrt{T-t}\sqrt{2\pi}}\cancel{(S_t - S_t)} = 0.
\end{aligned}$$

We have proven $S_t\frac{n(d_1)}{S_t\sigma\sqrt{T-t}} - K\mathrm{e}^{-r(T-t)}\frac{n(d_2)}{S_t\sigma\sqrt{T-t}} = 0$, so we finally get

$$\Delta_t^C = \mathcal{N}(d_1).$$

□

Informally, Delta indicates how much to buy or sell to cover your portfolio.

1.8.2 Gamma

In approximating the derivative price $f(t+dt;S_{t+dt})$ with a first-order Taylor expansion, that is, hedging with the Delta, we commit a *hedging error*, whose quantity is

$$\left(f(t;S_t)+\frac{\partial f(t;S_t)}{\partial S_t}\right)-f(t+dt;S_{t+dt})$$

This represent the quantity that is not covered by the hedging strategy. The magnitude of the hedging error depends on how much the concavity of the derivative curve is stressed. By studying the second order derivative, we can see how long it takes for the hedging error to get too large, and thus decide how often to buy and sell to rebuild the locally risk-free portfolio. The second derivative of the derivative price with respect to the underlying price is the Gamma:

$$\Gamma_t=\frac{\partial^2 f(t;S_t)}{\partial S_t^2}=\frac{\partial\Delta_t}{\partial S_t}$$

For example, for the European call, we have:

$$\Gamma_t^C=\frac{\partial C_t}{\partial S_t^2}=\frac{\partial\Delta_t^C}{\partial S_t}=\frac{\partial\mathcal{N}(d_1)}{\partial S_t}=\frac{n(d_1)}{S_t\sigma\sqrt{T-t}}.$$

1.8.3 Other Greeks

Three other important Greeks are the Vega, the Rho and the Theta.

$$\begin{aligned}\text{Vega}_t &= \tfrac{\partial f(t;S_T)}{\partial\sigma} && \text{Sensitivity w.r. to the volatility}\\ \rho_t &= \tfrac{\partial f(t;S_T)}{\partial r} && \text{Sensitivity w.r. to the risk-free rate}\\ \Theta_t &= \tfrac{\partial f(t;S_T)}{\partial t} && \text{Sensitivity w.r. to time}\end{aligned}$$

These three Greeks allow you to hedge against model misspecifications instead of risk; for example, Black-Scholes' considers both the volatility σ and the risk-free rate r as constants, and this tends not to be true in reality.

For European call options, these three Greeks are

$$\begin{aligned}
\text{Vega}_t^C &= \frac{\partial C_t}{\partial \sigma} = S_t\sqrt{T-t}\cdot n(d_1) \\
\rho_t^C &= \frac{\partial C_t}{\partial r} = K(T-t)\mathrm{e}^{-r(T-t)}\mathcal{N}(d_2) \\
\Theta_t^C &= \frac{\partial C_t}{\partial t} = -\frac{\sigma S_t n(d_1)}{2\sqrt{T-t}} - rK\mathrm{e}^{-r(T-t)}\mathcal{N}(d_2)
\end{aligned}$$

1.8.4 Put-call parity

The Greeks for put options can be calculated by being mindful of the put-call parity relationship:

$$\forall t \quad C_t + K\mathrm{e}^{-r(T-t)} = P_t + S_t$$

Or, equivalently,

$$\begin{aligned}
C_t &= P_t + S_t - K\mathrm{e}^{-r(T-t)} \\
P_t &= C_t + K\mathrm{e}^{-r(T-t)} - S_t
\end{aligned}$$

Then, the Greeks for European put options can be calculated as follows.

Put Delta

$$\Delta_t^P = \frac{\partial P_t}{\partial S_t} = \frac{\partial C_t}{\partial S_t} - \frac{\partial S_t}{\partial S_t} = \Delta_t^C - 1 = \mathcal{N}(d_1) - 1 < 0$$

For a put option, the Delta is always negative, this means that the hedging position should always be short.

Put Gamma

For a put option, the Gamma is identical to the case of a call option.

$$\Gamma_t^P = \frac{\partial^2 P_t}{\partial S_t^2} = \frac{\partial \Delta_t^P}{\partial S_t} = \frac{\partial \Delta_t^C - 1}{\partial S_t} = \frac{\partial \Delta_t^C}{\partial S_t} = \frac{n(d_1)}{S_t\sigma\sqrt{T-t}} \equiv \Gamma_t^C$$

Other Greeks for put options

$$\text{Vega}_t^P = \frac{\partial P_t}{\partial \sigma} = S_t\sqrt{T-t}\cdot n(d_1) \equiv \text{Vega}_t^C$$

$$\rho_t^P = \frac{\partial P_t}{\partial r} = -K(T-t)\mathrm{e}^{-r(T-t)}\mathcal{N}(-d_2)$$

$$\Theta_t^P = \frac{\partial P_t}{\partial t} = -\frac{\sigma S_t n(d_1)}{2\sqrt{T-t}} + rK\mathrm{e}^{-r(T-t)}\mathcal{N}(-d_2)$$

Example 6. Suppose you have to sell an European call option whose underlying, S_t, follows a Black-Scholes model. The following is known:

$$S_0 = 8 \quad K = 8 \quad \mu = 20\% \quad \sigma = 40\% \quad r = 4\% \quad T = 1 \text{ year}$$

(1). Determine the number of underlying to buy/sell to hedge this short position.

We want to create a *delta-neutral portfolio*, that is, a portfolio where $\Delta = 0$. This way, risk is removed since the rate of change of the portfolio value with respect to the price variation of the asset is zero.

We know that

$$\Delta_0^C = \mathcal{N}(d_1) = \mathcal{N}\left(\frac{\ln\frac{8}{8} + \left(0.04 + \frac{0.4^2}{2}\right)\cdot 1}{0.4\sqrt{1}}\right) = \mathcal{N}(0.3) = 0.618$$

We can now impose the Delta for the portfolio $\pi = -1$ call option$+x$ stocks to be zero and solve the resulting equation for x to get the number of stocks the portfolio needs to have.

$$\Delta^\pi = -1\cdot\Delta_0^C + x\cdot\overbrace{\Delta_0^S}^{\text{always 1}} = 0 \implies x = \frac{\Delta_0^C}{\Delta_0^S} = \frac{0.618}{1} = 0.618$$

This means that 0.618 units of stock must be bought to hedge one unit of short call option. From this, the Delta for the put option with same strike and maturity as the call can be computed by put-call parity:

$$\Delta_0^P = \Delta_0^C - 1 = 0.618 - 1 = -0.382$$

(2). Having a call option on the same underlying, but with strike $K' = 12$, construct a portfolio which is Delta- and Gamma-neutral.

$$\Delta_0^{C'} = \mathcal{N}(d_1') = \mathcal{N}\left(\frac{\ln\frac{8}{12} + \left(0.04 + \frac{0.4^2}{2}\right)\cdot 1}{0.4\sqrt{1}}\right) = \mathcal{N}(-0.71) = 0.238$$

$$\Gamma_0^C = \frac{n(d_1)}{\sigma S_0\sqrt{T-t}} = \frac{\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{-(0.3)^2}{2}}}{0.4\cdot 8\sqrt{1}} = 0.12$$

$$\Gamma_0^{C'} = \frac{n(d_1')}{\sigma S_0\sqrt{T-t}} = \frac{\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{-(-0.71)^2}{2}}}{0.4\cdot 8\sqrt{1}} = 0.097$$

To have a Gamma-neutral portfolio, we must solve for y the equation

$$\Gamma_0^\pi = -1\cdot\Gamma_0^C + y\Gamma_0^{C'} + 0.618\cdot\overbrace{\Gamma_0^S}^{\text{always } 0} = 0 \implies y = \frac{\Gamma_0^C}{\Gamma_0^{C'}} = 1.237$$

This means we need to buy 1.237 units of the second call option; but, doing this, the portfolio may no longer be Delta-neutral, since $\Delta^\pi = 1.237\Delta_0^{C'} \neq 0$. At this point, we start from the Gamma-neutral portfolio of options and make it Delta-neutral again by imposing

$$-1\cdot\Delta_0^C + 1.237\cdot\Delta_0^{C'} + x\Delta_0^S = 0 \implies x = \Delta_0^C - 1.237\Delta_0^{C'} = 0.324$$

Finally, selling one unit of the first call option, buying 1.237 units of the second call option and 0.324 units of underlying grants a Delta- and Gamma-neutral portfolio.

1.9 More Exotic options

American, Asian, Lookback, Barrier.

Chapter 2

Interest rate models

Black-Scholes asset and derivative pricing models assume the interest rate for the considered time interval to be constant. In reality, interest rate is subject to change in a stochastic way, similar to what happens for underlyings.

2.0.1 Fundamental models

The following are some of the models used for modeling interest rate evolution. Generally, the deterministic factor indicates *mean reversion*, whereas the stochastic factor is a volatility similar to Bachelier or Black-Scholes models.

Vašiček model

$$dr_t = a(b - r_t)dt + \sigma dW_t$$

Cox-Ingersoll-Ross model

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t$$

Dothran model

$$dr_t = ar_t dt + \sigma r_t dW_t$$

Ho-Lee model

$$dr_t = \Theta(t)dt + \sigma dW_t$$

Hull-White model

$$dr_t = a(t)(b(t) - r_t)dt + \sigma(t)dW_t$$

Generalized CIR model

$$dr_t = a(t)(b(t) - r_t)dt + \sigma(t)\sqrt{r_t}dW_t$$

2.0.2 Vašiček model

Theorem 2. (Vašiček model is Gaussian)

Proof. Use Itō's Lemma to study the dynamics of $e^{as}r_s$ where r_s follows a Vašiček model. Note that the model can be written equivalently as $dr_t = -a(r_t - b)dt + \sigma dW_t$.

$$\begin{aligned} d(e^{as}r_s) &= ae^{as}r_s ds + e^{as}dr_s = ae^{as}r_s ds + e^{as}\left(-a(r_s-b)ds + \sigma dW_s\right) \\ &= \underbrace{e^{as}ab}_{\text{determ.}}\, ds + \underbrace{e^{as}\sigma}_{\text{determ.}}\, dW_s \end{aligned}$$

Now compute the value of the function in the increment $[0, t]$ and, consequently, the value of the process at time t.

$$\begin{aligned} e^{at}r_t - e^{a\cdot 0}r_0 = \int_0^t abe^{as}ds + \int_0^t e^{as}\sigma dW_s \implies \\ e^{at}r_t = r_0 + [be^{as}]_0^t + \int_0^t e^{as}\sigma dW_s \implies \\ r_t = e^{-at}r_0 + be^{-at}(e^{at}-1) + e^{-at}\int_0^t e^{as}\sigma dW_s \end{aligned}$$

The remaining integral is with respect of a function of time and a stochastic variable, and is thus a *stochastic integral*; we study its distribution.

Let h be a deterministic function, such that we can have $\int_0^t h(s)dW_s$. For example, consider the *step function*

$$h(s) = \sum_{i=0}^{n-1} h_i \mathbb{1}_{[t_i, t_{i+1}]}(s) \quad \text{s.t.} \quad \int_0^t h(s)dW_s \approx \sum_{i=0}^{n-1} h_i \left(W_{t_i+1} - W_{t_i}\right)$$

Each of the Brownian motion increments in the summation is thus distributed like

$$W_{t_{i+1}} - W_{t_i} \sim \mathcal{N}(0, t_{i+1} - t_i) \equiv \mathcal{N}(0, h_i^2(t_{i+1} - t_i))$$

From the stability of the Gaussian distribution and the independence of Brownian increments follows

$$X(W) = \int_0^t h(s)dW_s \sim \mathcal{N}\left(0, \sum_{i=0}^{n-1} h_i^2(t_{i+1} - t_i)\right) \approx \mathcal{N}\left(0, \int_0^t h^2(s)ds\right)$$

2.0.3 Stochastic interest rates

When interest rates are considered as being stochastic, the zero-coupon bond price $P(t,T)$ at time t with maturity T becomes a stochastic process, varying across the *term structure of interest rates* $\mathcal{T} = [T_1, T_2]$:

$$\left((P(t,T))_{t\in[0,T]}\right)_{T\in\mathcal{T}}$$

The zero-coupon bond behaves like a derivative instrument whose underlying is the spot interest rate. The classical approach to stochastic bond pricing is to give an exogenous model for the spot interest rate $(r_t)_{t\in[0,T]}$ and, under no arbitrage assumption, derive $(P(t,T))_{t\in[0,T]}$. Two conditions imposed for this are:

- $P(T,T) = 1$
- $P(t_1,T) < P(t_2,T) \quad \text{for all} \quad t_1 < t_2$

It has been proved that, under no arbitrage assumption, the discounted stock price and the discounted european option price are *martingales* with respect to the risk neutral measure in $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$:

$$\begin{aligned}
dS_t &= rS_t dt + \sigma S_t dW_t \implies \\
d(\mathrm{e}^{-rt}S_t) &= -r\mathrm{e}^{-rt}S_t dt + \mathrm{e}^{-rt}dS_t \\
&= -r\mathrm{e}^{-rt}S_t dt + \mathrm{e}^{-rt}\left(rS_t dt + \sigma S_t dW_t\right) \\
&= \mathrm{e}^{-rt}S_t\sigma dW_t \\
d(\mathrm{e}^{-rt}f(t;S_t)) &= \frac{\partial f}{\partial S_t}\mathrm{e}^{-rt}\sigma S_t dW_t
\end{aligned}$$

Let now $\hat{P}(t,T)$ and $P(t,T)$ be, respectively, the discounted bond price and the bond price, assume (r_t) to be such that $dr_t = \mu(r_t)dt + \sigma(r_t)dW_t$. Hence,

$$\hat{P}(t,T) = \mathrm{e}^{-\int_0^T r(s)ds} P(t,T)$$

(Note that, if r_t is supposed constant, $\mathrm{e}^{-\int_0^T r(s)ds} = rt$). Impose now $\hat{P}$ to be a martingale:

$$\tilde{\mathbb{E}}\left[\hat{P}(T,T)\middle|\mathcal{F}_t\right] = \hat{P}(t,T) \quad \forall t < T$$

$$\tilde{\mathbb{E}}\left[\mathrm{e}^{-\int_0^T r(s)ds}\hat{P}(T,T)\middle|\mathcal{F}_t\right] = \mathrm{e}^{-\int_0^T r(s)ds}\hat{P}(t,T)$$

$$\mathrm{e}^{\int_0^t r(s)ds}\tilde{\mathbb{E}}\left[\mathrm{e}^{-\int_0^T r(s)ds}\hat{P}(T,T)\middle|\mathcal{F}_t\right] = \hat{P}(t,T)$$

The part $\mathrm{e}^{\int_0^t r(s)ds}$ is known at time t, and can thus be put inside the expectation; moreover, $P(T,T) = 1$ by definition:

$$P(t,T) = \tilde{\mathbb{E}}\left[\mathrm{e}^{\int_0^t r(s)ds - \int_0^T r(s)ds} \cdot 1\middle|\mathcal{F}_t\right] = \tilde{\mathbb{E}}\left[\mathrm{e}^{-\int_t^T r(s)ds}\middle|\mathcal{F}_t\right]$$

The bond price is the expectation of the payoff 1 discounted by the correct factor, given the Brownian filtration at time t.

2.0.4 Variance and covariance for Vašiček model

It has been proved that Vašiček model is $\sim \mathcal{N}(\cdot,\cdot)$, and that the spot rate at time t is given by

$$r_t = r_0\mathrm{e}^{at} + b(1-\mathrm{e}^{-at}) + \sigma\mathrm{e}^{-at}\int_0^t \mathrm{e}^{as}dW_s$$

It can be proven that the stochastic part is Gaussian:

$$\sigma\mathrm{e}^{-at}\int_0^t \mathrm{e}^{as}dW_s \sim \mathcal{N}\left(0, \int_0^t \mathrm{e}^{2as}ds\right).$$

With h deterministic,

$$\int_0^t h(s)dW_s \approx \sum_{i=0}^{n-1} h_i \cdot (W_{t_{i+1}} - W_{t_i})$$

and we can compute the variance as

$$\mathbb{E}\left[\left(\sum_{i=0}^{n-1} h_i \left(W_{t_{i+1}} - W_{t_i}\right)\right)^2\right] =$$

$$= \mathbb{E}\left[\sum_{i=0}^{n-1} h_i^2 \left(W_{t_{i+1}} - W_{t_i}\right)^2\right] + \mathbb{E}\left[\sum_{i\neq j} h_i h_j \left(W_{t_{i+1}} - W_{t_i}\right)\left(W_{t_{j+1}} - W_{t_j}\right)\right] =$$

$$= \mathbb{E}\left[\sum_{i=0}^{n-1} h_i^2 \left(W_{t_{i+1}} - W_{t_i}\right)^2\right] + \sum_{i\neq j} h_i h_j \cancelto{0}{\mathbb{E}\left[\left(W_{t_{i+1}} - W_{t_i}\right)\left(W_{t_{j+1}} - W_{t_j}\right)\right]}$$

$$= \mathbb{E}\left[\sum_{i=0}^{n-1} h_i^2 \left(W_{t_{i+1}} - W_{t_i}\right)^2\right] = \sum_{i=0}^{n-1} h_i^2 \mathbb{E}\left[\left(W_{t_{i+1}} - W_{t_i}\right)^2\right]$$

$$= \sum_{i=0}^{n-1} h_i^2 (t_{i+1} - t_i) \approx \int_0^t h^2(s)ds$$

The expectation for the spot rate is

$$\begin{aligned}
\mathbb{E}\left[r_t\right] &= \mathbb{E}\left[r_0 \mathrm{e}^{at} + b(1 - \mathrm{e}^{-at}) + \sigma \mathrm{e}^{-at}\int_0^t \mathrm{e}^{as} dW_s\right] \\
&= r_0 \mathrm{e}^{at} + b(1 - \mathrm{e}^{at}) + \cancelto{0}{\mathbb{E}\left[\sigma \mathrm{e}^{-at}\int_0^t \mathrm{e}^{as} dW_s\right]} \\
&= r_0 \mathrm{e}^{at} + b(1 - \mathrm{e}^{at})
\end{aligned}$$

This means Vašiček is $\sim \mathcal{N}(r_0 \mathrm{e}^{at} + b(1 - \mathrm{e}^{at}), \cdot)$. Now, compute the *autocovariance* (note: $\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$):

$$\begin{aligned}
\mathrm{Cov}(r_t, r_{t+h}) &= \mathrm{Cov}\left(\sigma \mathrm{e}^{-at}\int_0^t \mathrm{e}^{as} dW_s, \sigma \mathrm{e}^{-a(t+h)}\int_0^{t+h} \mathrm{e}^{as} dW_s,\right) \\
&= \mathbb{E}\left[\sigma \mathrm{e}^{-at}\int_0^t \mathrm{e}^{as} dW_s \cdot \sigma \mathrm{e}^{-a(t+h)}\int_0^{t+h} \mathrm{e}^{as} dW_s\right] - 0 \\
&= \sigma^2 \mathrm{e}^{-at-a(t+h)}\mathbb{E}\left[\int_0^t \mathrm{e}^{as} dW_s(\omega) \cdot \int_0^{t+h} \mathrm{e}^{as} dW_s(\omega) \cdot\right] \\
&= \sigma^2 \mathrm{e}^{-at-a(t+h)}\mathbb{E}\left[\int_0^t \mathrm{e}^{as} dW_s(\omega) \cdot \left(\int_0^t \mathrm{e}^{as} dW_s(\omega) + \int_t^{t+h} \mathrm{e}^{as} dW_s(\omega)\right)\right]
\end{aligned}$$

$$= \sigma^2 \mathrm{e}^{-at-a(t+h)} \left(\mathbb{E}\left[\int_0^t \left(\mathrm{e}^{as} dW_s(\omega) \right)^2 \right] + \cancelto{0}{\mathbb{E}\left[\int_0^t \mathrm{e}^{as} dW_s(\omega) \cdot \int_t^{t+h} \mathrm{e}^{as} dW_s(\omega) \right]} \right)$$

$$= \sigma^2 \mathrm{e}^{-at-a(t+h)} \mathbb{E}\left[\int_0^t \mathrm{e}^{2as} dW_s(\omega) \right] = \sigma^2 \mathrm{e}^{-at-a(t+h)} \int_0^t \mathrm{e}^{2as} ds$$

$$= \sigma^2 \mathrm{e}^{-2at-ah} \left[\frac{1}{2a} \mathrm{e}^{2av} \right]_0^t = \sigma^2 \mathrm{e}^{-2at-ah} \cdot \frac{\mathrm{e}^{2at}-1}{2a}$$

Finally, we can assert that, with $h \to 0$, Vašiček model is distributed like

$$\sim \mathcal{N}\left(r_0 \mathrm{e}^{-at} + b(1-\mathrm{e}^{-at}), \frac{\sigma^2 \mathrm{e}^{-2at}(\mathrm{e}^{2at}-1)}{2a} \right)$$

and the price of a zero-coupon bond depends on the parameters

$$P(t,T;\alpha) \quad \alpha = (a, b, \sigma).$$

The theoretical term structure curve can be fitted with the observed data:

$$(P(0,T;\alpha))_{T \in \mathcal{T}} \overset{\text{fit}}{\equiv} (P^*(0,T;\alpha))_{T \in \mathcal{T}}$$

This is very underdetermined; instead of using a model as simple as regular Vašiček or Cox-Ingersoll-Ross, we can use the generalization of models such as Hull-White,

$$dr_t = a(t)(b(t) - r_t)dt + \sigma(t) dW_t$$

where parameters in the vector α are functions of time, yielding an infinite class of parameters.