Quantitative Finance and Derivatives

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Chapter 1

Models for asset pricing

1.1 Underlying processes

1.1.1 Stochastic Processes

Stochastic processes are collections of random variables representing the evolution of some system over time. Stochastic processes can have discrete or continuous values, and can evolve over discrete or continuous time.

We have four different possible situations:

1. $X(n, \omega) : \mathbb{N} \times (\Omega, \mathcal{F}, \mathbb{P}) \to$ a subset of $\mathbb{Z}$: discrete time, discrete values. For example: a *random walk*.

2. $X(n, \omega) : \mathbb{N} \times (\Omega, \mathcal{F}, \mathbb{P}) \to$ a subset of $\mathbb{R}$: discrete time, continuous values.

3. $X(n, \omega) : \mathbb{R}^+ \times (\Omega, \mathcal{F}, \mathbb{P}) \to$ a subset of $\mathbb{Z}$: continuous time, discrete values. For example: a *Poisson process*.

4. $X(n, \omega) : \mathbb{R}^+ \times (\Omega, \mathcal{F}, \mathbb{P}) \to$ a subset of $\mathbb{R}$: continuous time, continuous values. For example: a *Brownian motion*.

From random walk to Brownian motion

Let

$$(X_n)_{n \geq 0}$$

be a stochastic process such that X_i, for any i, can take value 1 with probability $\mathbb{P}(X_i = 1) = \frac{1}{2}$, and -1 with probability $\mathbb{P}(X_i = -1) = \frac{1}{2}$. Let then $S_n = X_1 + X_2 + \ldots + X_n$ be the position you are at on a line after n steps. Informally, each X_i "moves" you randomly one step to the right or one step to the left. We want to know expectation and variance of this process. Let's assume the $(X_n)_{n\geq 0}$ to be *independent and identically distributed*. We can now assert the following:

$$E\left[X_i\right] = (+1) \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0$$

$$E\left[X_i^2\right] = (+1)^2 \cdot \frac{1}{2} + (-1)^2 \cdot \frac{1}{2} = 1$$

Hence,

$$E\left[S_n\right] = \sum_{i=1}^{n} E\left[X_i\right] = 0$$

$$Var\left[S_n\right] = \sum_{i=1}^{n} E\left[X_i^2\right] = n$$

As n approaches infinity, the conditions for the *Central Limit Theorem* hold: X_is are *i.i.d.*, so we can apply

$$\frac{S_n - n \cdot E\left[X_i\right]}{\sqrt{n \cdot Var\left[X_i\right]}} = \frac{S_n}{\sqrt{n}} \underset{n\to\infty}{\sim} \mathcal{N}(0,1)$$

Consider now a non-unitary time. Suppose we move in time steps of $\delta > 0$ and in space steps of $\sqrt{\delta}$, and let's consider the process in the interval $[0,t],\ t \in \mathbb{R}^+$. Then,

$$S_t = \sum_{i=0}^{\left\lfloor \frac{t}{\delta} \right\rfloor} X_i$$

where X_i moves by $\pm\sqrt{\delta}$ with probability $\mathbb{P} = \frac{1}{2}$. Then, $E\left[S_t\right] = 0$ and

$$Var[X_i] = E^2[X_i] - E[X_i^2] = E^2[X_i] = (+\sqrt{\delta})^2 \cdot \frac{1}{2} + (-\sqrt{\delta})^2 \cdot \frac{1}{2} = \delta$$

$$Var[S_t] = \frac{t}{\delta} \cdot \delta = t$$

Let's now apply *Central Limit Theorem*:

$$\frac{S_t - \frac{t}{\delta}E[X_i]}{\sqrt{\frac{t}{\delta}Var[X_i]}} = \frac{S_t}{\sqrt{t}} \underset{t\to\infty}{\sim} \mathcal{N}(0,1).$$

So, $S_t \sim \mathcal{N}(0,t)$: this is the *Brownian motion*.

Definition 1. (Brownian motion). The stochastic process

$$(W_t)_{t\in\mathbb{R}^+} : \mathbb{R}^+ \times (\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}) \to \mathbb{R}$$

is a Brownian motion if and only if

1. $W_0 = 0$,
2. it is continuous,
3. has stationary increments:

$$\forall t > s,\ W_t - W_s \sim \mathcal{N}(0, t-s)$$

4. has independent increments over disjoint intervals:

$$\forall q < r < s < t,\ (W_r - W_q) \perp (W_t - W_s).$$

1.1.2 Itô Formula