Random number generation

Quantitative Risk Management project work

Silvia Baldisserotto, Maysa Jahanbani,
Claudia Pesci, Phan Ho Tan Phat,
Andrea Venuta

Università degli Studi di Firenze - Finance and Risk Management

Random numbers

- Computer-generated numbers are *pseudo-random*: deterministic and predictable
- *Quasi-random* numbers prevent potential lack of equidistributedness
- **Definition.** *(Sample).* A sequence of number is called a *sample from the distribution* F if the numbers are independent realizations of a random variable with distribution function F
- Uniform deviates: samples from $\sim \mathcal{U}[0,1]$
- Normal deviates: samples from $\sim \mathcal{N}(0,1)$
- Drawing uniform deviates is the basis of random number generation

Linear congruential generators

- N_0 is chosen arbitrarily (called the *seed*)
- $N_i = (aN_{i-1} + b) \bmod M$ for $i > 0$, then

$$U_i = \frac{N_i}{M}, \quad U_i \in [0, 1)$$

- Suitability of the numbers U_i depends on how a, b, M are chosen

Linear congruential generators: properties

- Numbers N_i are periodic, with period $\leq M$:
  there are at most M different numbers in the class modulo M
- Examples:
  - If $N = 0$, b can't be 0, otherwise $N_i = 0$ will repeat itself
  - If $a = 0$, generator settles down on $N_n = N_0 + nb$
- Numbers are distributed "evenly" if we have exactly M different numbers in a generator with modulo M, or
- Each grid point on a *mesh* on $[0, 1]$ with size $\frac{1}{M}$ is occupied once

Quality of generators

Requirements:

1. Large period: small set of numbers makes the outcome easier to predict (choose M as large as possible)
2. Statistical tests to verify that the distribution is the intended one
   - Comparison of sample mean and variance μ, σ^2 with desired values
   - Correlation between sample values
   - Quality of approximation of the distribution
3. Distribution in higher dimensional spaces: lattice structure

Random vectors and lattice structure

- Sequences of random numbers can be arranged in m-dimensional vectors
- The vectors lie on a number of parallel $(m-1)$-dimensional hyperplanes
- The ideal condition is that the number of parallel hyperplanes is maximized: number of hyperplanes is a measure of equidistributedness
- Family of parallel lines in the (U_{i-1}, U_i)-plane

$$z_0 U_{i-1} + z_1 U_i = c + \frac{z_1 b}{M} \quad \text{where} \quad c := N_{i-1} \frac{z_0 + a z_1}{M} - z_1 k$$

for each tuple (z_0, z_1) and for all cs.

Random vectors and lattice structure

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Inversion and transformation methods

- Inversion and transformation methods generate numbers distributed according to an arbitrary distribution from uniformly distributed samples

Inversion method

Theomem. *(inversion)* Suppose $U \sim \mathcal{U}[0,1]$, and F continuous strictly increasing distribution. Then, $F^{-1}(U)$ is a sample from F.

Proof.

$$\mathbb{P}(U \leq \xi) = \xi,\ 0 < \xi < 1$$

$$\mathbb{P}(F^{-1}(U) \leq x) = \mathbb{P}(U \leq F(x)) = F(x).$$

Exponential distribution:

$$F(x) = 1 - e^{-\lambda x}$$

$$F^{-1}(x) = \frac{1}{\lambda}\log(x)$$

Cauchy distribution:

$$F(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}$$

$$F^{-1}(x) = \tan\left(\pi\left(x - \frac{1}{2}\right)\right)$$

Transformation method

Theomem. If X is a $r.v. \sim F(x)$, and $h : S \to B,\ S, B \cup \mathbb{R}$ strictly monotonous, then:

$Y := h(X)$ is a $r.v.$ with distribution

$$F_Y(y) = F(h^{-1}(y))h' > 0$$
$$F_Y(y) = F(h^{-1}(y))h' > 0$$

Exponential distribution:

If h^{-1} absolutely continuous for almost all y, density of h is

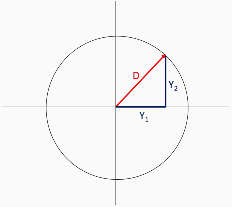
$$f(h^{-1}(y)) \left| \frac{dh^{-1}(y)}{dy} \right|$$

Cauchy distribution:

Box-Muller method

- Generate $x_1, x_2 \sim \mathcal{U}(0,1)$ random numbers
- Derive
$$h_1(x_1,x_2) := y_1 = \sqrt{-2\log x_1}\cos 2\pi x_2$$
$$h_2(x_1,x_2) := y_2 = \sqrt{-2\log x_1}\sin 2\pi x_2$$
- y_1 and y_2 will be i.i.d. $\sim \mathcal{N}(0,1)$

Box-Muller method



$$y_1 = D\cos\omega, \quad y_2 = D\sin\omega$$

$$\text{where } D = \sqrt{-2\log x_1}, \quad \omega = 2\pi x_2$$

$$h^{-1}(x_1, x_2) = \begin{cases} x_1 = \exp\left\{-\frac{y_1^2+y_2^2}{2}\right\} \\ x_2 = \frac{1}{2\pi}\arctan\frac{y_2}{y_1} \end{cases}$$

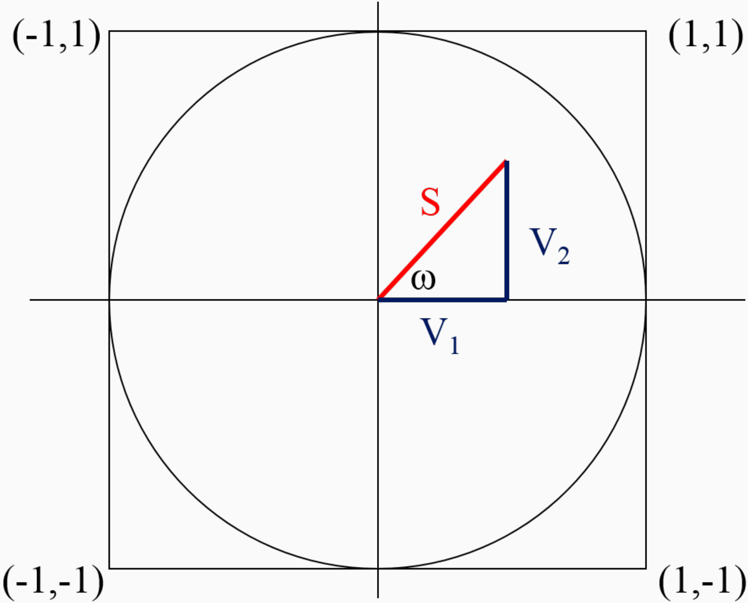
$$|\text{Jacobian}| = \det\begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{pmatrix} = \left[\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y_1^2}{2}\right)\right]\cdot\left[\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y_2^2}{2}\right)\right]$$

is the density of the bivariate standard normal distribution because it's the product of two univariate standard normal densities.

Polar method

1. Let $U_1, U_2 \sim \mathcal{U}(0,1)$
2. Define $V_i = 2U_i - 1$: $V_i \sim \mathcal{U}(-1,1)$
3. Define $S = V_1^2 + V_2^2$
4. If and only if $S \leq 1$, then define
$$Y = \sqrt{\frac{-2\ln S}{S}}$$
5. $$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} V_1 Y \\ V_2 Y \end{pmatrix} \quad \text{and} \quad X_1, X_2 \text{ i.i.d. } \sim \mathcal{N}(0,1)$$

Polar method



Correlated bivariate random variables

$$\mathbf{Z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad z_1, z_2 \sim \mathcal{N}(0,1) \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

1. Calculate the Cholesky decomposition $AA^T = \Sigma$

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a^2 & ab \\ ab & b^2 + c^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$$\rightarrow A = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2(1-\rho^2)^{\frac{1}{2}} \end{pmatrix}$$

2. Calculate $\mathbf{Z} \sim \mathcal{N}(0, \mathbb{I}_2)$
3. $\mu + A\mathbf{Z} \sim \mathcal{N}(\mu, \Sigma)$ has the desired distribution.

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mu + \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2(1-\rho^2)^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \mu + \begin{pmatrix} \sigma_1 z_1 \\ \rho\sigma_2 z_1 + \sigma_2(1-\rho^2)^{\frac{1}{2}} z_2 \end{pmatrix}$$

Implementations - Linear Congruential Generator

```
function [ rn ] = LCG( x )

  if(nargin == 0)
    x = 1;
  end

  rn = zeros(x,1);

  for i = 1:x
    rn(i) = LCGstep();
  end

end
```

```
function [ rnStep ] = LCGstep()

  persistent seed;
  M = 244944;
  a = 1597;
  b = 51749;

  if(isempty(seed))
    seed = 0;
  end

  seed = mod(seed * a + b, M);

  rnStep = seed / M;

end
```

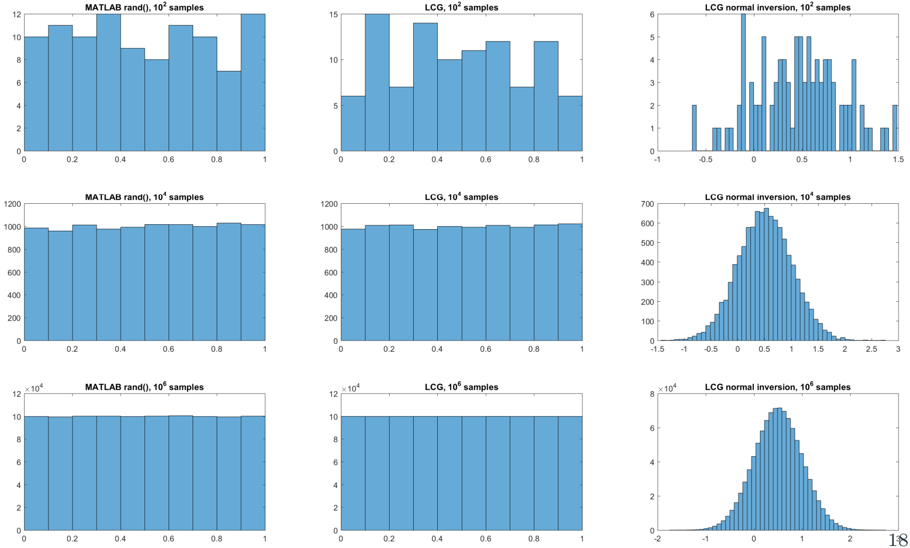
Implementations - Box-Muller method

```
function [ Z ] = BoxMuller( x )

  if(nargin == 0)
    x = 1;
  end

  U = rand(x, 2);

  theta = 2 .* pi .* U(:, 2);
  rho   = sqrt( -2 .* log( U(:, 1) ) );

  Z = [ rho .* cos(theta), rho .* sin(theta) ];

end
```

Implementations - Marsaglia polar algorithm

```
function [ Z ] = Marsaglia( x )

  if(nargin == 0)
    x = 1;
  end

  Z = zeros(x,2);

  for i = 1 : x
    W = 1;  V = [ 1, 1 ];
    while not (W < 1)
      V = 2 * rand(1, 2) - 1;
      W = V(1) .^ 2 + V(2) .^ 2;
    end

    Z(i, :) = V .* sqrt(-2 * log(W) / W);
  end
end
```

Plots - Univariate methods



Plots - Bivariate methods