CS 2022 : DATA STRUCTURES & ALGORITHMS

Lecture 8: Graphs (Introduction & Searching)

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Outline

- Terminology
- Graph Representations
  - Adjacency List
  - Adjacency Matrix
- Searching in Graphs
  - Breath First search (BFS)
  - Depth First search (DFS)

Learning Outcomes

- After successfully studying contents covered in this lecture, students should be able to,
  - explain the terminology used in the graphs
  - explain the different graph representation mechanisms and their characteristics
  - explain graph search strategies and their characteristics

Introduction

Terminology

- Graph G = (V, E)
  - V = set of vertices
  - E = set of edges $\subseteq (V \times V)$
  - $|E|$ - Number of Edges
    - $|E| \le |V|^2$
  - $(u, v) \in E$: vertex **v** is **adjacent** to vertex **u**

Types of Graphs

- Undirected
  - Edges are undirected
- Directed
  - Edges are directed
- Weighted
  - Edges have weights

Terminology Cont…

- Directed Graphs
  - In-degree of a vertex
  - Out- degree of a vertex
- Undirected/Directed Graphs
  - Degree of a vertex
- Path Between Two Vertices
  - Path is simple if no vertex is repeated (no circle)

Terminology Cont…

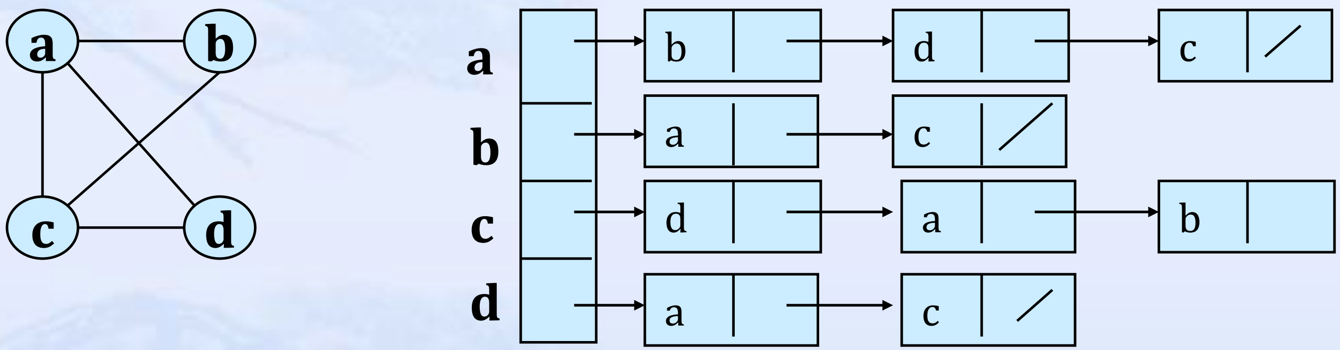
- Path Cost of a Weighted Graph
- Vertex **v** is Reachable from Vertex **u**
  - There is a path from u to v
- Length of a Path
  - Number of edges in path
- Subgraph

Terminology Cont…

- Dense
  - Edges dense ($|E| \approx |V|^2$)
- Sparse
  - Edges sparse ($|E| << |V|^2$)
- Connected/Strongly Connected
  - There is a path between every pair of vertices
  - $|E| \geq |V| - 1$.
  - If $|E| = |V| - 1$, then G is a tree

Graph Representation

- Adjacency Lists



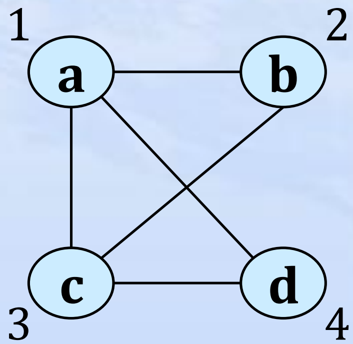
- One List Per Vertex
- For u $\in$ V, Adj[u] Consists of all Vertices Adjacent to u

Adjacency Lists

- Space Requirement $\Theta(V+E)$
  - Space-efficient, when a graph is sparse
- Determining If an Edge $(u,v) \in G$ Is Not Efficient.
  - Have to search in u's adjacency list
    - $\Theta(\text{degree}(u))$ time
    - $\Theta(V)$ in the worst case

Graph Representation

- Adjacency Matrices
  - A - $|V| \times |V|$ Matrix
  - $A[i, j] = a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$



	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

Adjacency Matrix

- Space: $\Theta(V^2)$
  - Not memory efficient for sparse graphs
- Time:
  - To list all vertices adjacent to u: $\Theta(V)$
  - To determine if $(u, v) \in E$: $\Theta(1)$
- Can Store Weights Instead of Bits for Weighted Graphs

Searching in Graphs

Searching in Graphs

- Searching a Graph
  - Systematically follow the edges of a graph to visit the vertices of the graph
  - Used to discover the structure of a graph
- Standard Graph-Searching Algorithms
  - Breadth-first Search (BFS)
  - Depth-first Search (DFS)

Breadth-First Search

- Expands the Frontier Between the Discovered and Undiscovered Vertices Uniformly Across the Breadth of the Frontier
  - A vertex is “discovered” the first time it is encountered during the search
  - A vertex is “finished” if all vertices adjacent to it have been discovered

Breadth-First Search

- Colors the vertices to keep track of progress
  - White – Undiscovered
  - Gray – Discovered but not finished
  - Black – Finished

Breadth-First Search

- Input : Graph G = (V, E) and **source vertex** s
- Output:
  - d[v] = distance (smallest # of edges, or shortest path) from s to v, for all v $\in$ V
    - d[v] = ∞ if v is not reachable from s
  - π[v] = u such that (u, v) is last edge on shortest path s   v.
    - u is v's predecessor
  - Builds breadth-first tree with root s that contains all reachable vertices

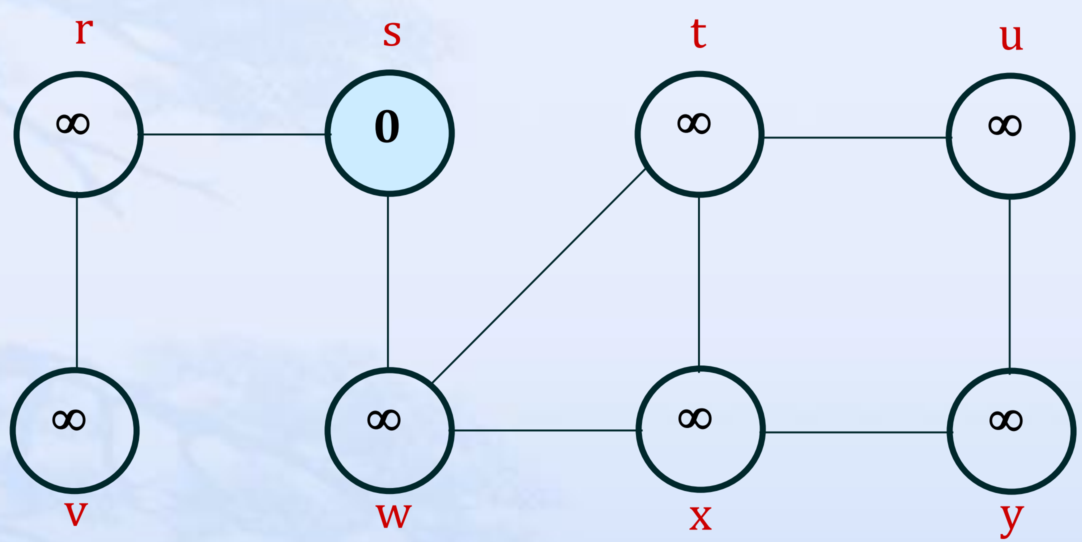
Breadth-First Search

```
BFS(G,s)
for each vertex u in V[G] –  {s}
    do color[u] ← white
         d[u] ← ∝
         π[u] ← nil
color[s] ← gray
d[s] ← 0
 π[s] ← nil
Q ← Φ
enqueue(Q,s)
while Q ≠ Φ
    do u ← dequeue(Q)
      for each v in Adj[u]
        do if color[v] = white
          then color[v] ← gray
               d[v] ← d[u] + 1
               π[v] ← u
               enqueue(Q,v)
      color[u] ← black
```

white: undiscovered
gray: discovered
black: finished

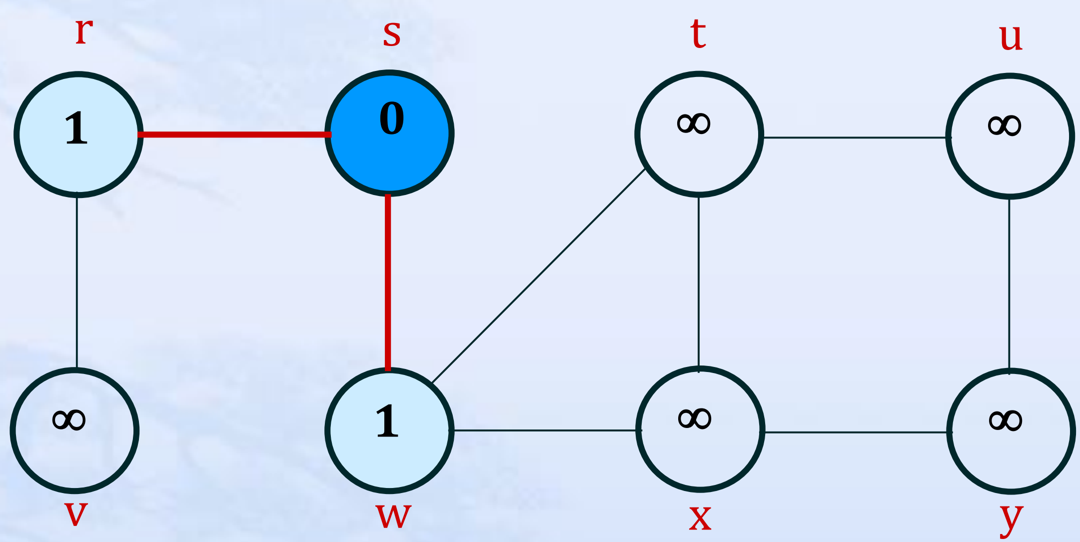
Q: a queue of discovered vertices
color[v]: color of v
d[v]: distance from s to v
π[v]: predecessor of v

BFS Example



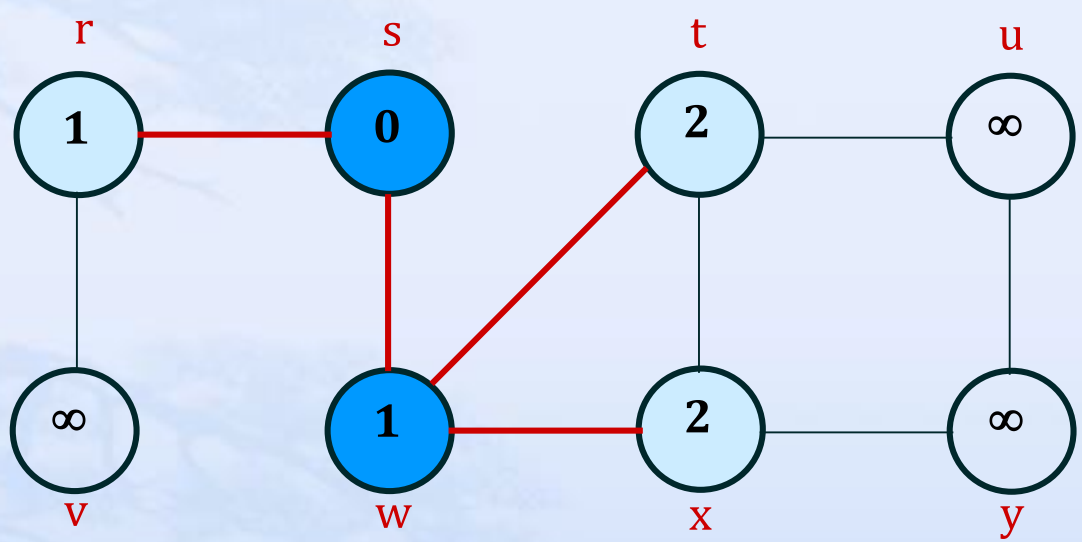
Q: s
0

BFS Example



Q: w r
1 1

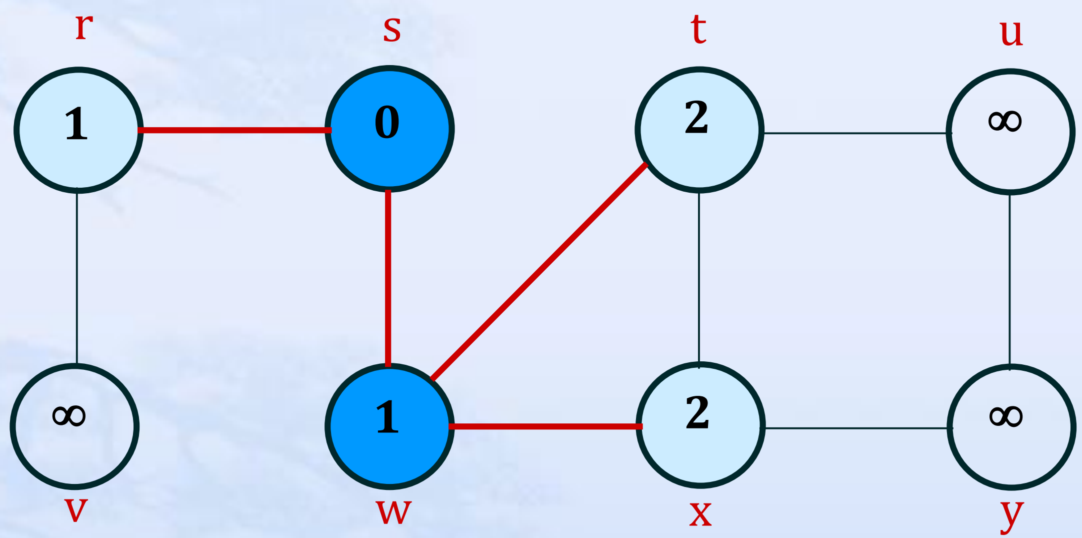
BFS Example



Q: r t x

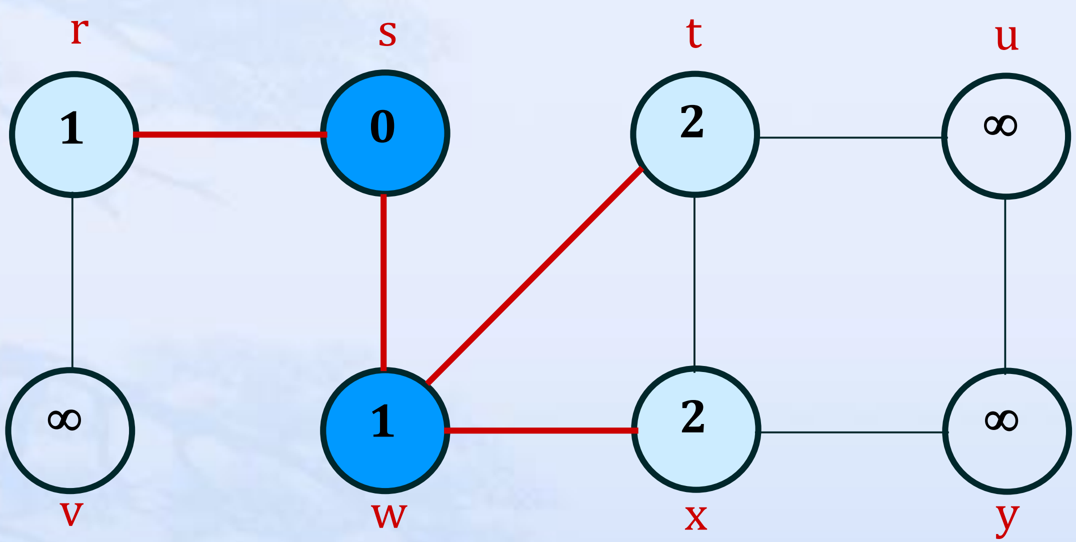
1 2 2

BFS Example



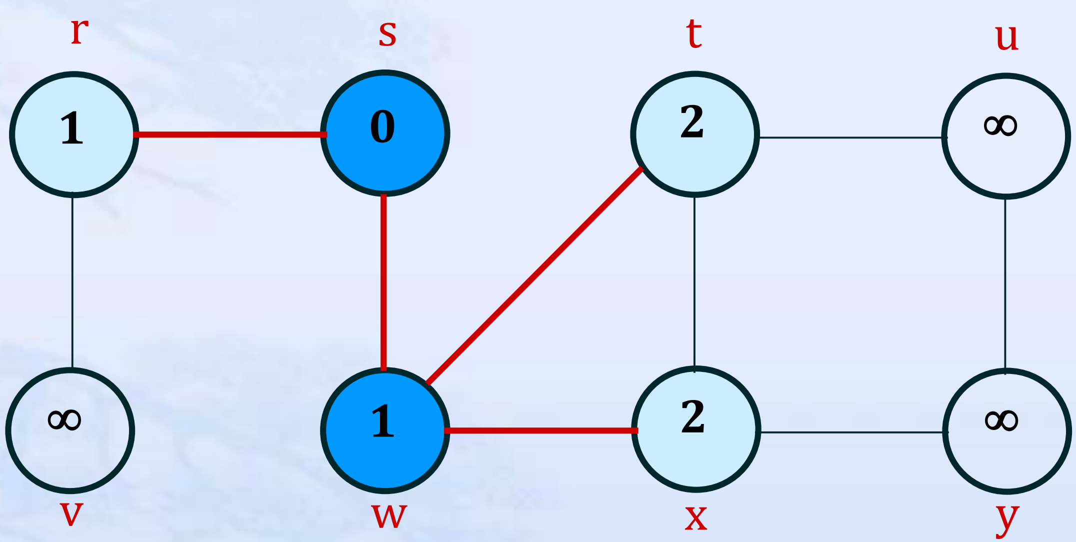
Q: r t x
1 2 2

BFS Example



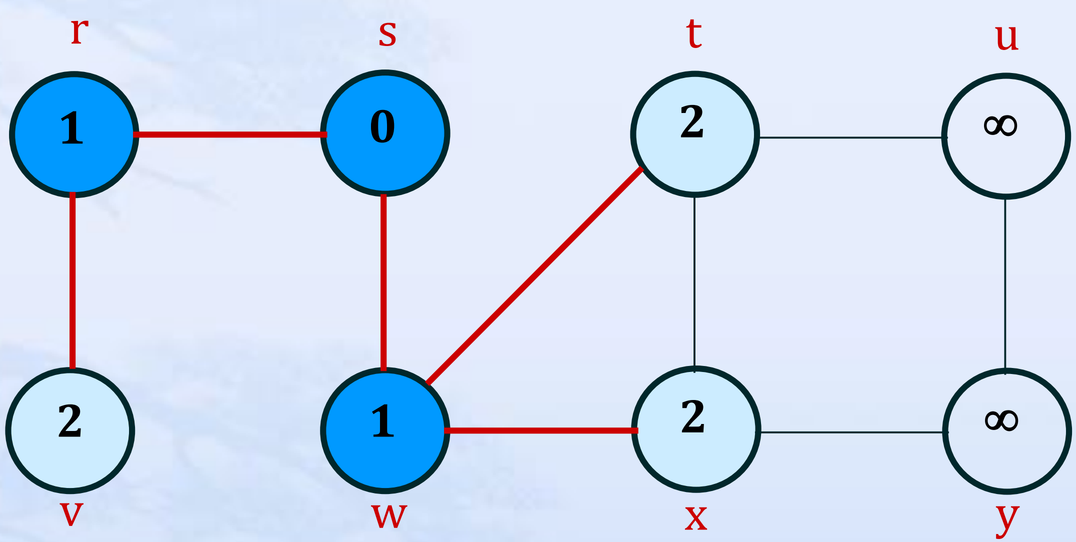
Q:	r	t	x
	1	2	2

BFS Example



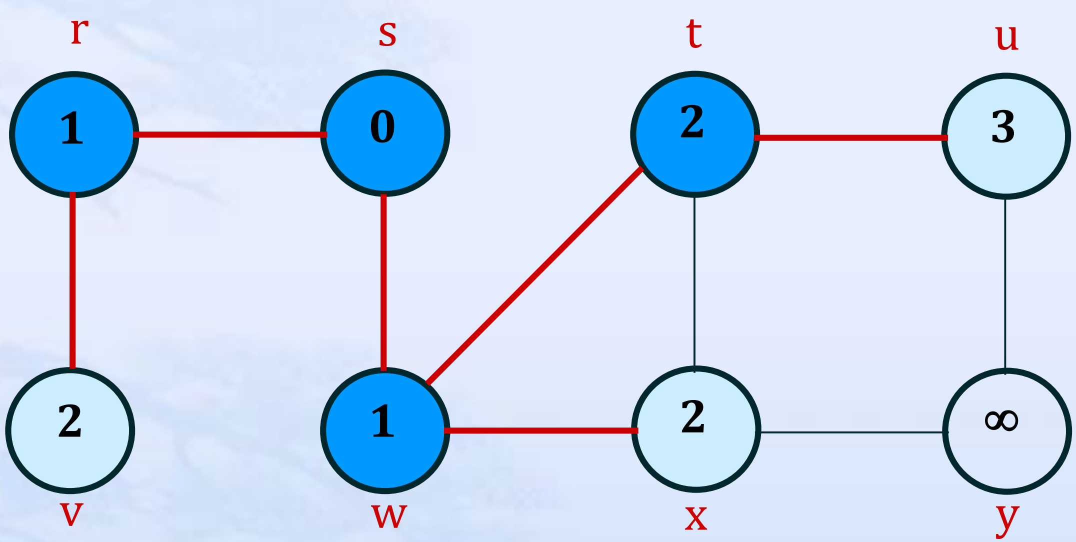
Q: r t x
1 2 2

BFS Example



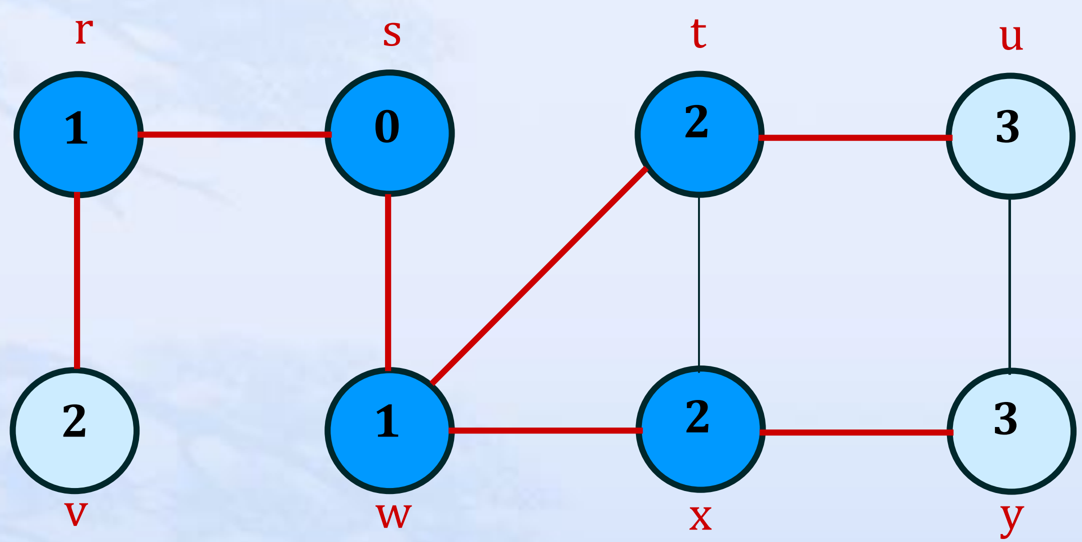
Q: t x v
2 2 2

BFS Example



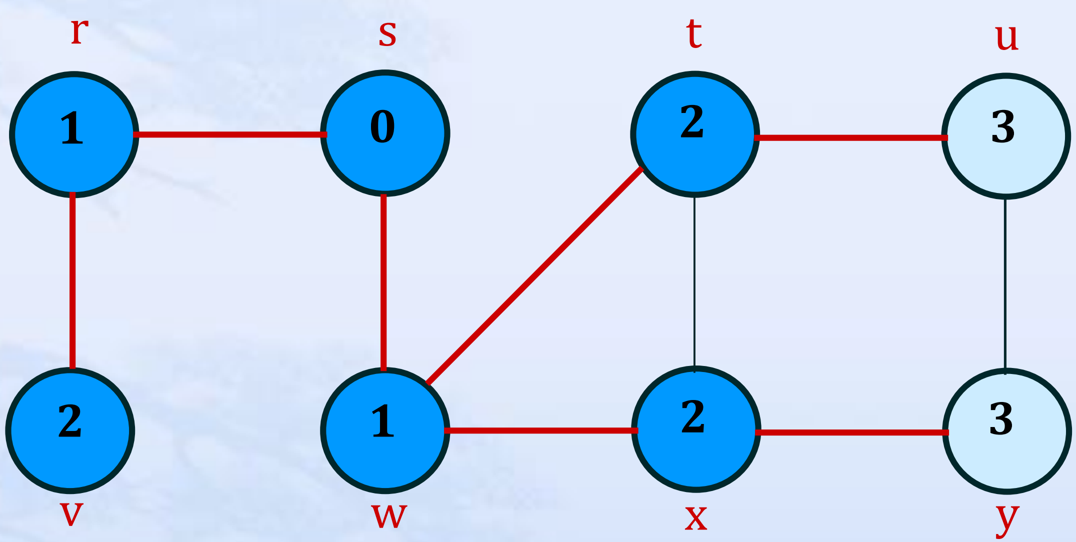
Q: x v u
2 2 3

BFS Example



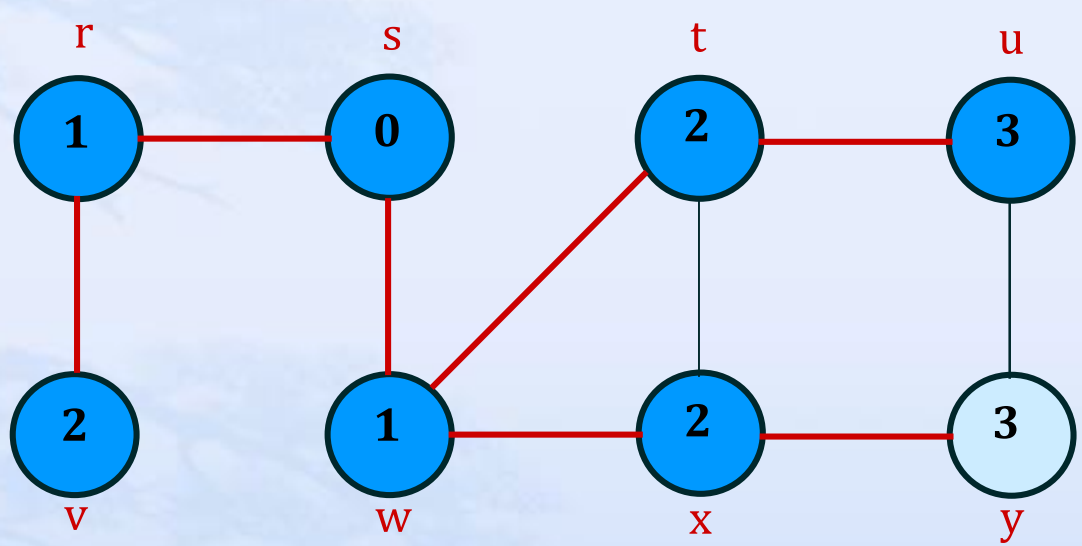
Q:	v u y
	2 3 3

BFS Example



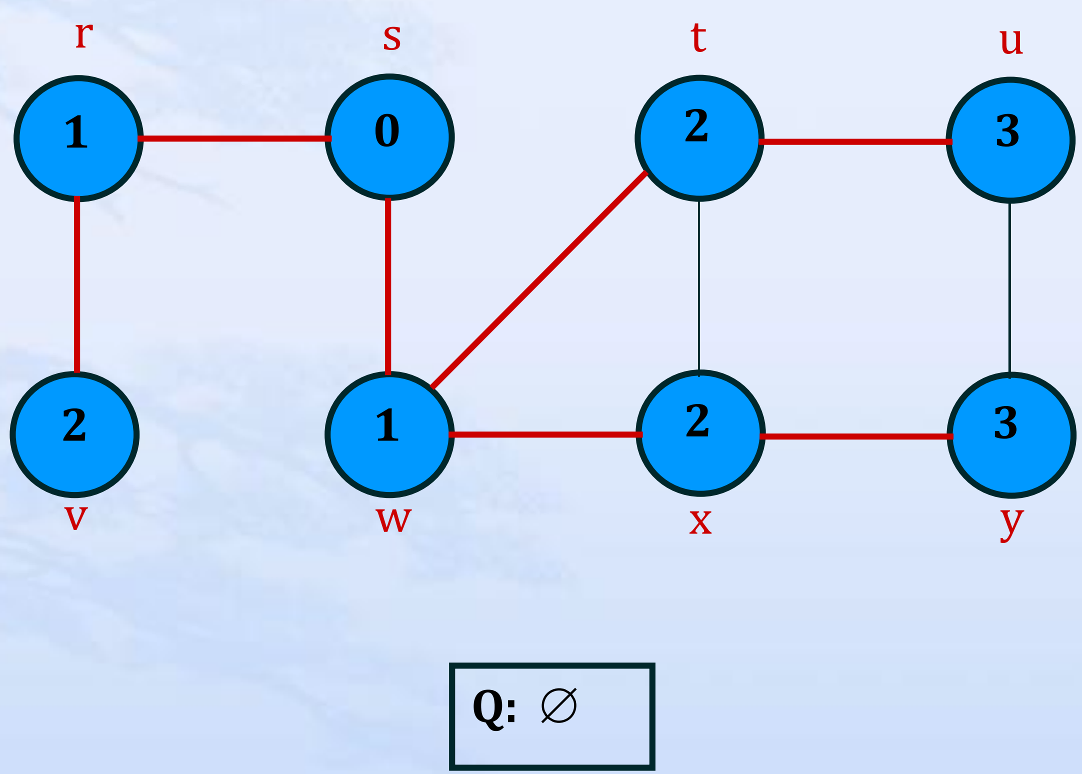
Q: u y
3 3

BFS Example



Q: y
3

BFS Example



Analysis of BFS

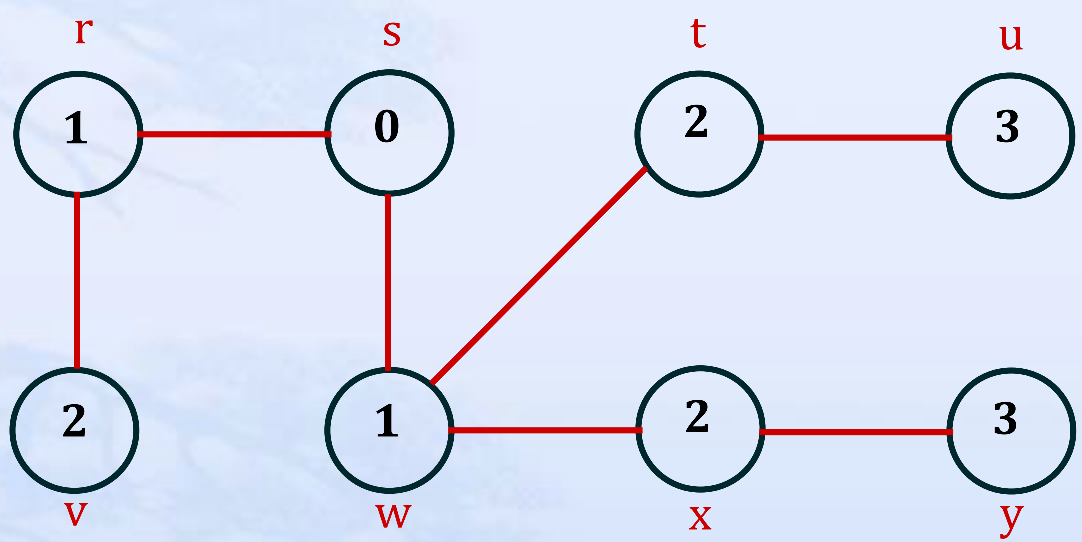
- Initialization takes O(V)
- Traversal Loop
  - After initialization, each vertex is enqueued and dequeued at most once, and each operation takes O(1)
    - So, total time for queuing is O(V)
  - The adjacency list of each vertex is scanned at most once
    - The sum of lengths of all adjacency lists is $\Theta(E)$
- Summing up over all vertices => total running time of BFS is O(V+E), linear in the size of the adjacency list representation of graph

Breadth-first Tree

- For a graph $G = (V, E)$ with source **s**, the **predecessor subgraph** of G is $G_\pi = (V_\pi, E_\pi)$ where
  - $V_\pi = \{v \in V : \pi[v] \neq \text{NIL}\} \cup \{s\}$
  - $E_\pi = \{(\pi[v], v) \in E : v \in V_\pi - \{s\}\}$
- The predecessor subgraph G_π is a **breadth-first tree** if:
  - V_π consists of the vertices reachable from s and
  - For all $v \in V_\pi$, there is a unique simple path from s to v in G_π that is also a shortest path from s to v in G.
- The edges in E_π are called **tree edges**

  $|E_\pi| = |V_\pi| - 1$

BF Tree Example



BF Tree

Depth-first Search

- Idea
  - Search as deep as possible first
- Mechanism
  - Explore edges out of the most recently discovered vertex **v**
  - When all edges of **v** have been explored, backtrack to explore other edges leaving the vertex from which **v** was discovered (its predecessor)

Depth-first Search

- Mechanism Cont…
  - Continue until all vertices reachable from the original source are discovered
  - If any undiscovered vertices remain, then one of them is chosen as a new source and search is repeated from that source

Depth-first Search

- **Input:** G = (V, E), No source vertex given!
- **Output:**
  - 2 **timestamps** on each vertex. Integers between 1 and 2|V|.
    - d[v] = **discovery time** (v turns from white to gray)
    - f [v] = **finishing time** (v tuns from gray to black)
  - π[v] : predecessor of v = u, such that v was discovered during the scan of u's adjacency list
- Uses the Same Coloring Scheme for Vertices as BFS

Depth-first Search

DFS(*G*)

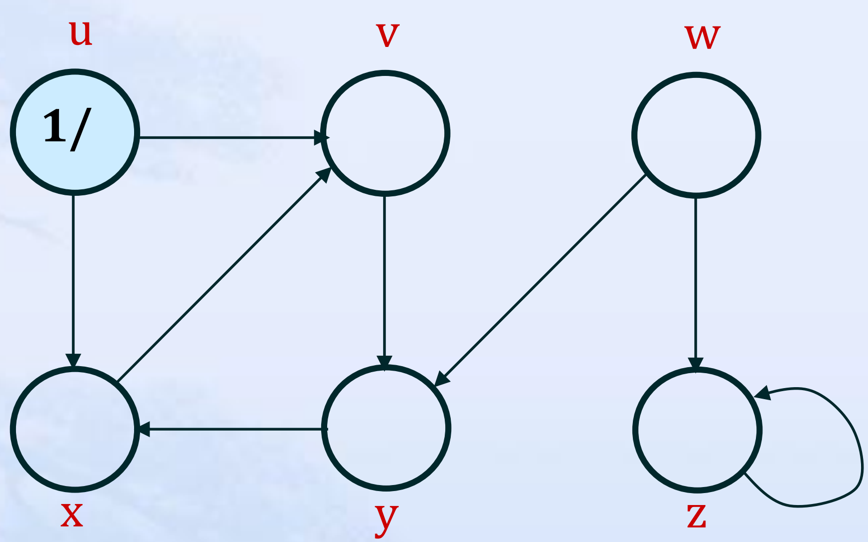
```
for each vertex u ∈ V[G]
    do color[u] ← white
       π[u] ← NIL
time ← 0
for each vertex u ∈ V[G]
    do if color[u] = white
       then DFS-Visit(u)
```

Uses a global timestamp ***time***.

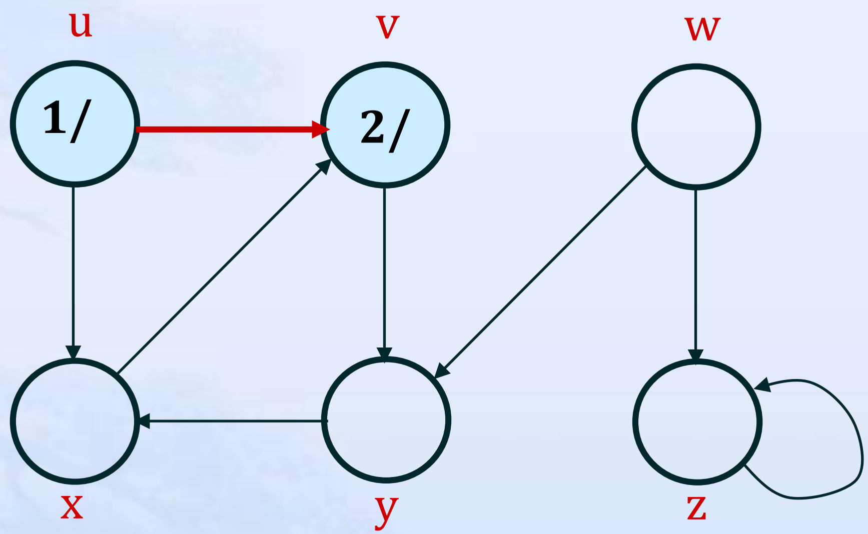
DFS-Visit(*u*)

```
color[u] ← GRAY   ∇ White vertex u has been discovered
time ← time + 1
d[u] ← time
for each v ∈ Adj[u]
    do if color[v] = WHITE
          then π[v] ← u
               DFS-Visit(v)
color[u] ← BLACK      ∇ Blacken u; it is finished.
time ← time + 1
f[u] ← time
```

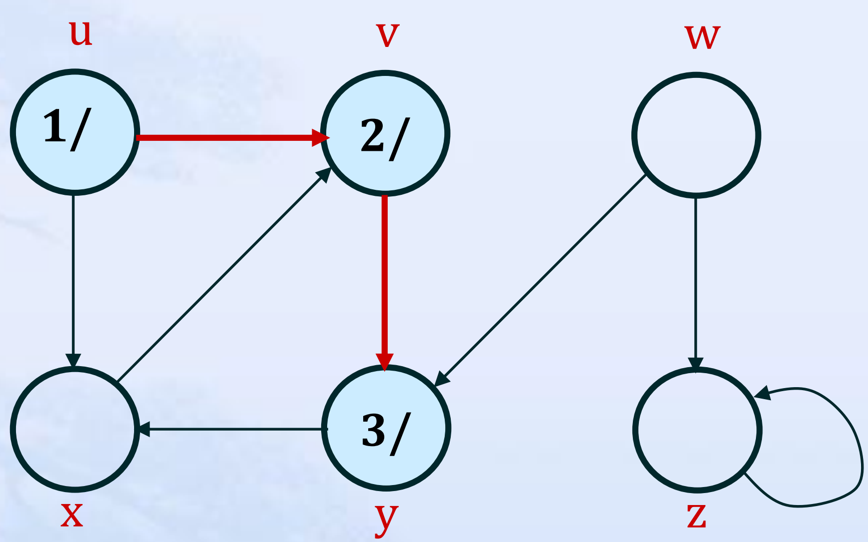
DFS Example



DFS Example



DFS Example



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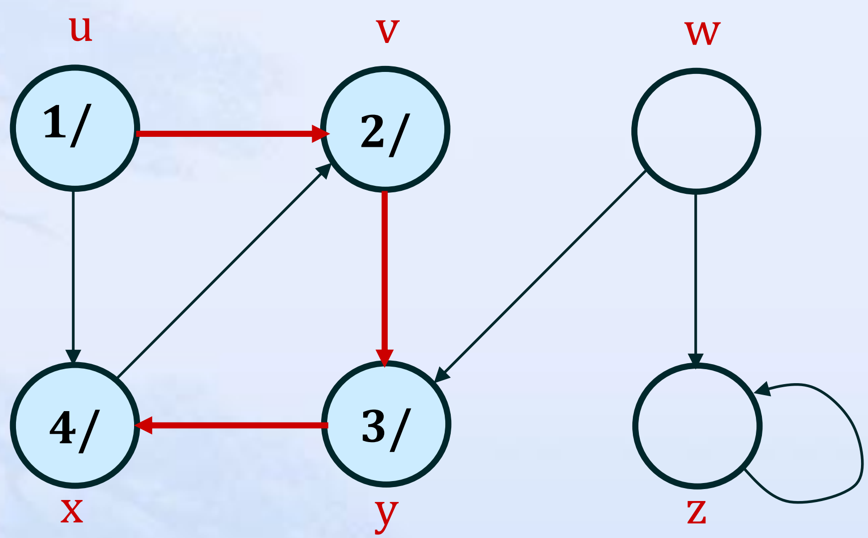

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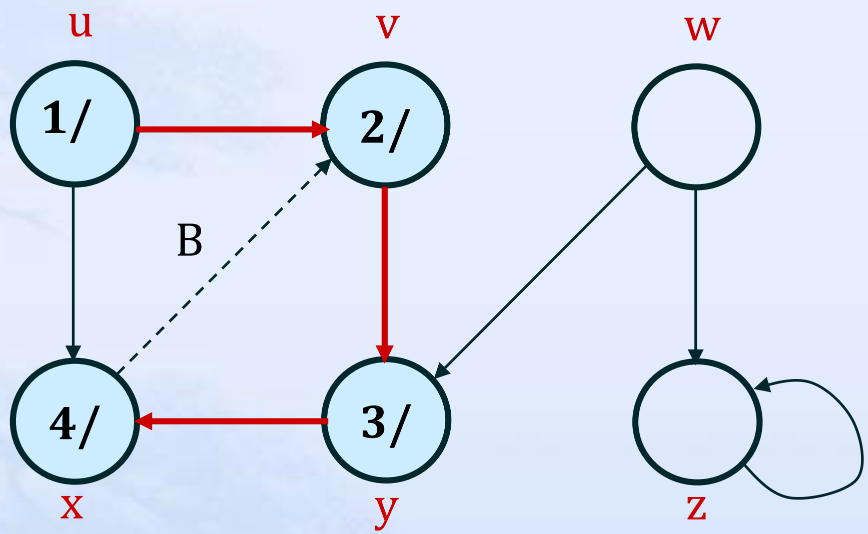
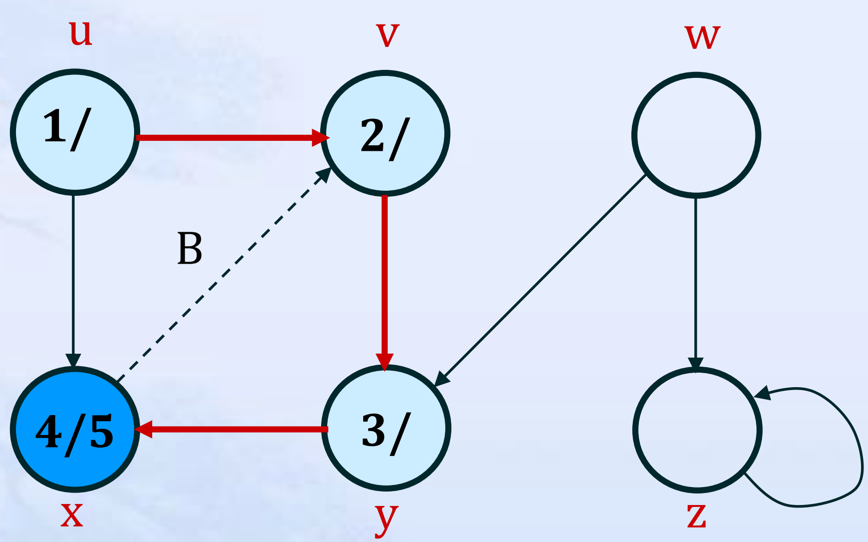
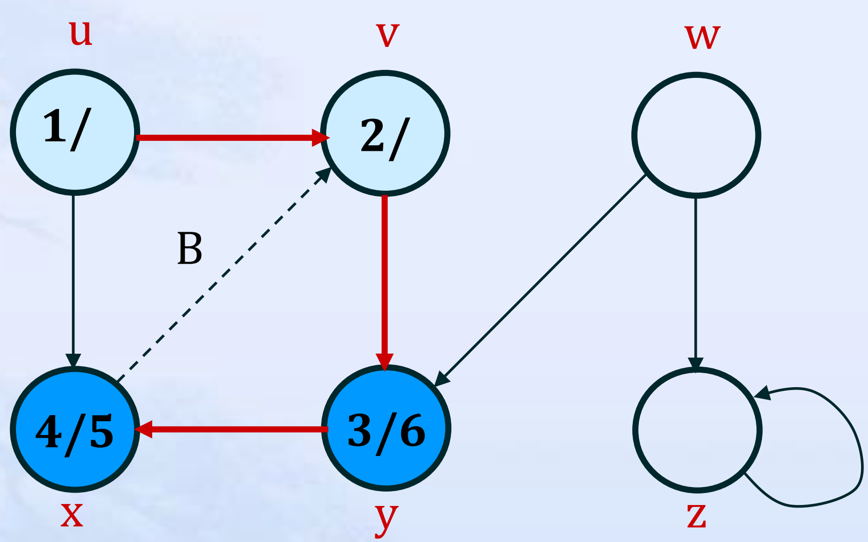


DFS Example



DFS Example



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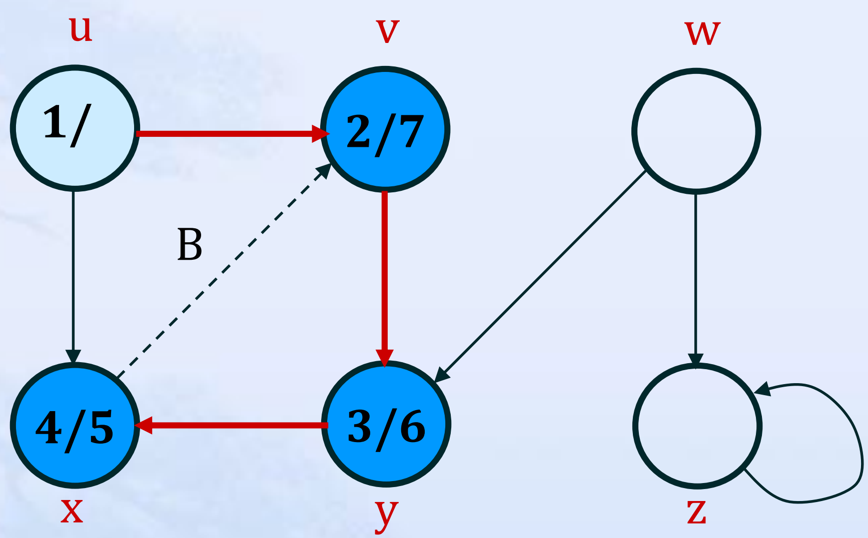
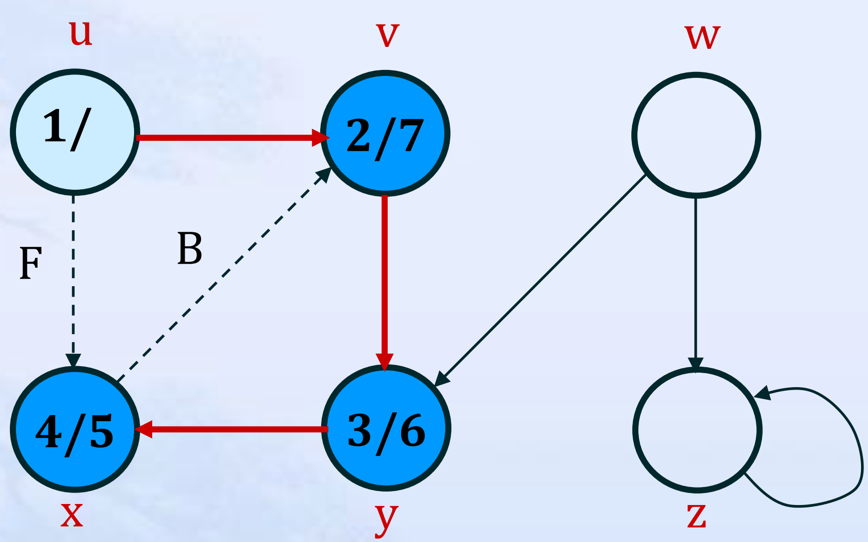
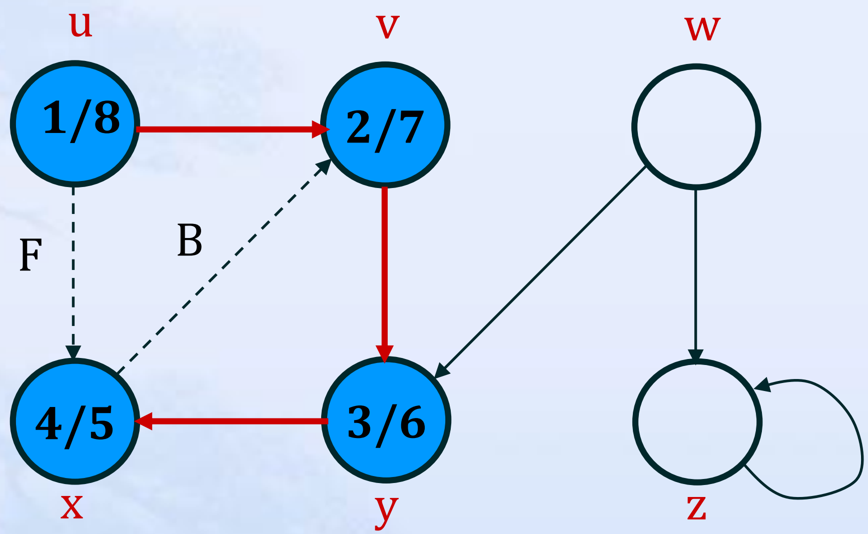


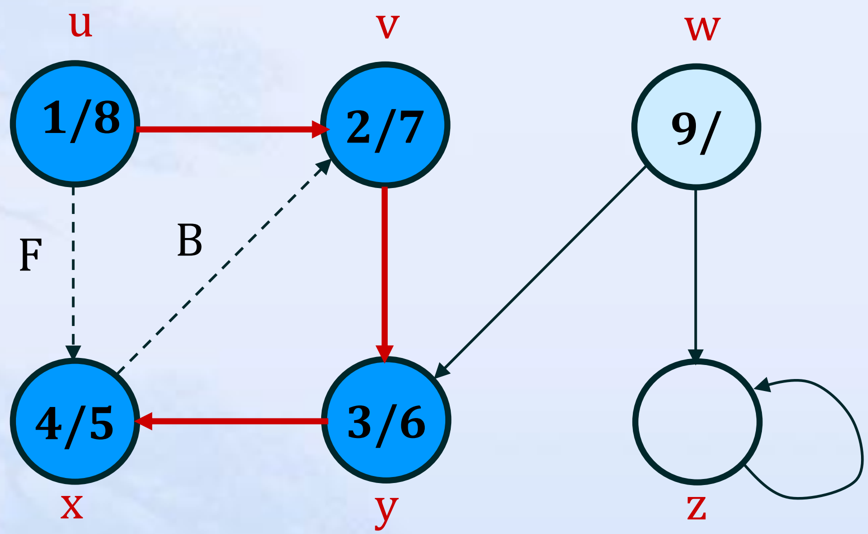
DFS Example



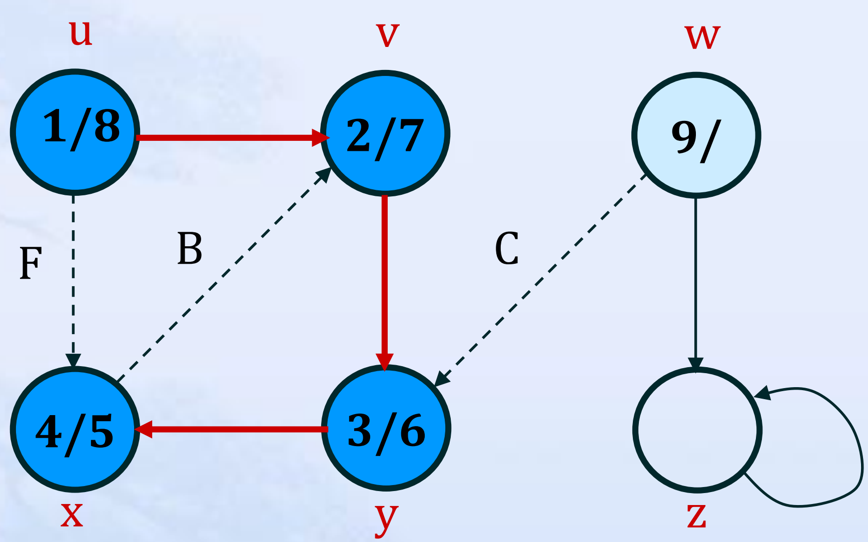
DFS Example



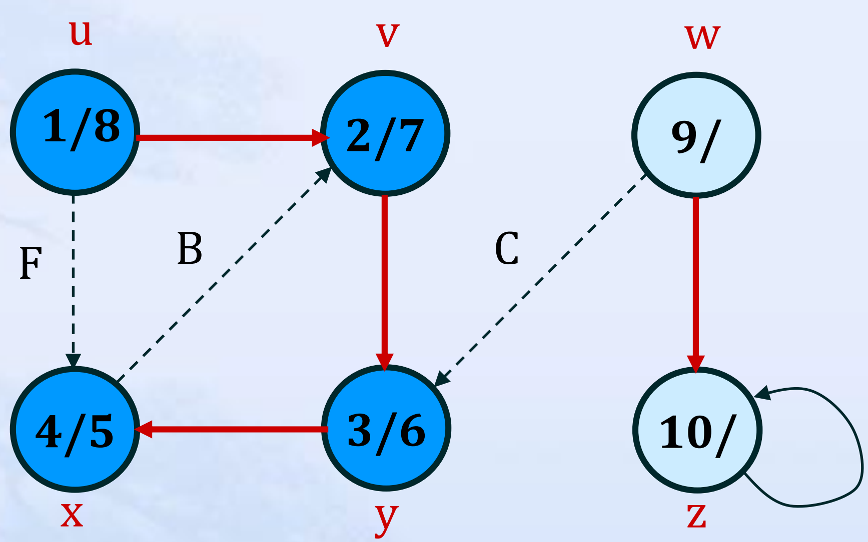
DFS Example



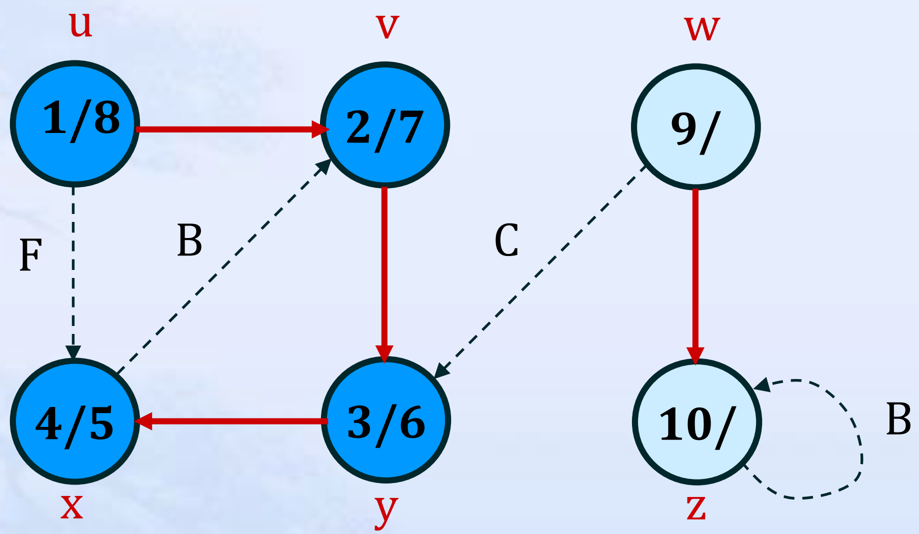
DFS Example



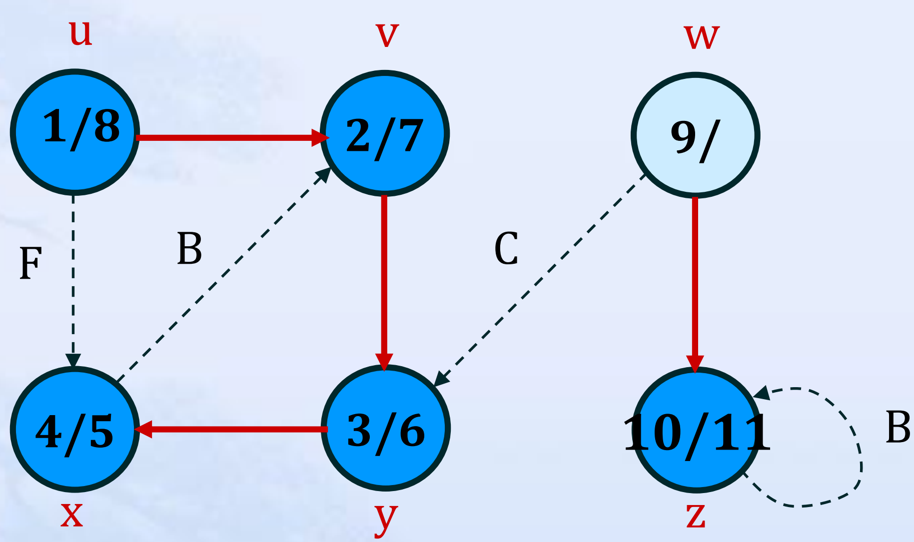
DFS Example



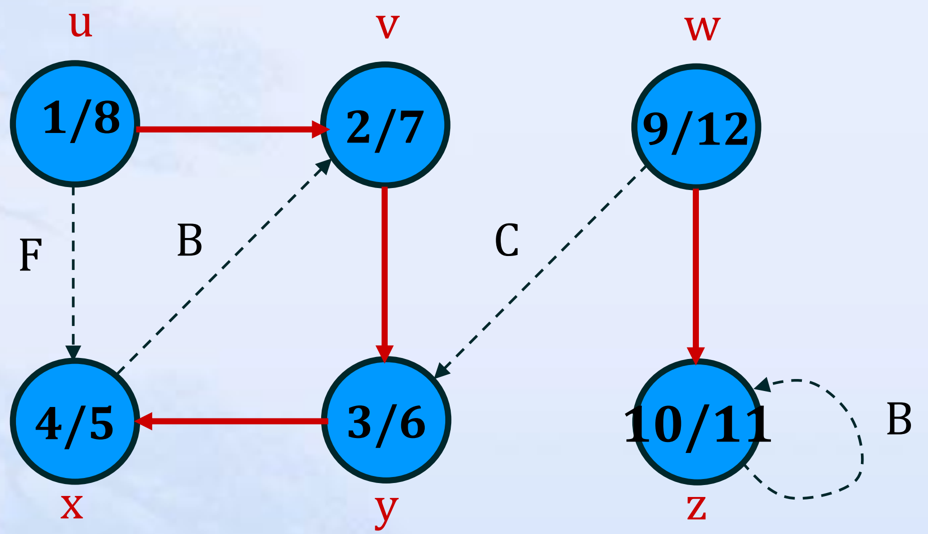
DFS Example



DFS Example



DFS Example



Analysis of DFS

- Loops on lines 1-2 & 5-7 take $\Theta(V)$ time, excluding time to execute DFS-Visit
- DFS-Visit is called once for each white vertex $v \in V$ when it's painted gray the first time.
- Lines 3-6 of DFS-Visit is executed $|Adj[v]|$ times. The total cost of executing DFS-Visit is $\sum_{v \in V} |Adj[v]| = \Theta(E)$
- Total running time of DFS is $\Theta(V+E)$

Parenthesis Theorem

Theorem 22.7

For all u, v, exactly one of the following holds:

1. $d[u] < f[u] < d[v] < f[v]$ or $d[v] < f[v] < d[u] < f[u]$ and neither u nor v is a descendant of the other.
2. $d[u] < d[v] < f[v] < f[u]$ and v is a descendant of u.
3. $d[v] < d[u] < f[u] < f[v]$ and u is a descendant of v.

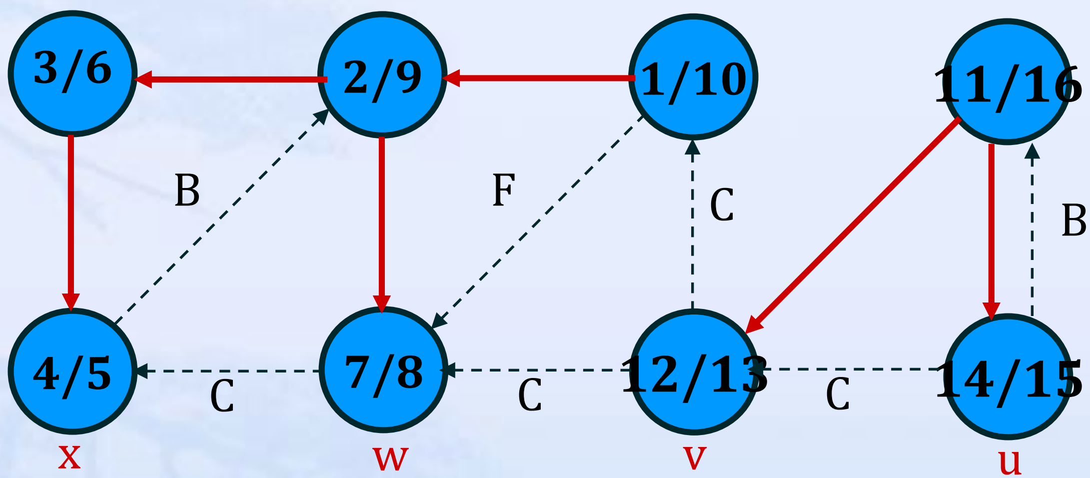
So $d[u] < d[v] < f[u] < f[v]$ cannot happen.

- Like parentheses:
  - OK: ( ) [ ] ([ ]) [( )]
  - Not OK: ([ )] [( ])

Corollary

v is a proper descendant of u if and only if $d[u] < d[v] < f[v] < f[u]$

Parenthesis Theorem Example



(s (z (y (x x) y) (w w) z) s) (t (v v) (u u) t)

Depth-First Forest (Trees)

- The Predecessor Subgraph of DFS is $G_\pi = (V, E_\pi)$ where $E_\pi = \{(\pi[v], v) : v \in V \text{ and } \pi[v] \neq \text{NIL}\}$
  - How does it differ from that of BFS?
    - *Depth-first forest* composed of several *depth-first trees*
  - The edges in E_π are called *tree edges*
  - Forest
    - An acyclic graph G that may be disconnected

WHITE-PATH THEOREM

Theorem 22.9

v is a descendant of u if and only if at time $d[u]$, there is a path $u \rightsquigarrow v$ consisting of only white vertices. (Except for u, which was *just* colored gray.)

SELF STUDYING

Self Studying

- Reading Assignment
  - Chapter 22
    - 22.1: Representation of Graphs
    - 22.2: Breadth First Search
    - 22.3: Depth First Search

References

[1] T.H. Cormen, C.E. Leiserson, R.L. Rivest and C. Stein, *Introduction to Algorithms*, 3rd Ed. Cambridge, MA, MIT Press, 2009.

[2] Lecture slides available at http://www.cs.unc.edu/~plaisted/comp550/19-graph1.ppt

Note: Slides a modified version of [2]