

23 If  $\sum a_n$  does not converge show that  $\sum |a_n|$  does not converge.

24 Find conditions which guarantee that  $a_1 + a_2 - a_3 + a_4 + a_5 - a_6 + \dots$  will converge (negative term follows two positive terms).

25 If the terms of  $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$  are rearranged into  $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \dots$ , show that this series now adds to  $\frac{1}{2} \ln 2$ . (Combine each positive term with the following negative term.)

26 Show that the series  $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{4} - \frac{1}{7} + \dots$  converges to  $\frac{1}{2} \ln 2$ .

27 What is the sum of  $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} - \frac{1}{6} + \dots$ ?

28 Combine  $1 + \dots + \frac{1}{n} - \ln n \rightarrow \gamma$  and  $1 - \frac{1}{2} + \frac{1}{3} - \dots \rightarrow \ln 2$  to prove  $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} + \frac{1}{7} - \dots = \ln 2$ .

29 (a) Prove that this alternating series converges:

$$1 - \int_1^2 \frac{dx}{x} + \frac{1}{2} - \int_2^3 \frac{dx}{x} + \frac{1}{3} - \int_3^4 \frac{dx}{x} + \dots$$

(b) Show that its sum is Euler's constant  $\gamma$ .

30 Prove that this series converges. Its sum is  $\pi/2$ .

$$\int_0^\pi \frac{\sin x}{x} dx + \int_\pi^{2\pi} \frac{\sin x}{x} dx + \dots = \int_0^\infty \frac{\sin x}{x} dx.$$

31 The cosine of  $\theta = 1$  radian is  $1 - \frac{1}{2!} + \frac{1}{4!} - \dots$ . Compute  $\cos 1$  to five correct decimals (how many terms?).

32 The sine of  $\theta = \pi$  radians is  $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \dots$ . Compute  $\sin \pi$  to eight correct decimals (how many terms?).

33 If  $\sum a_n^2$  and  $\sum b_n^2$  are convergent show that  $\sum a_n b_n$  is absolutely convergent.

**Hint:**  $(a \pm b)^2 \geq 0$  yields  $2|ab| \leq a^2 + b^2$ .

34 Verify the Schwarz inequality  $(\sum a_n b_n)^2 \leq (\sum a_n^2)(\sum b_n^2)$  if  $a_n = (\frac{1}{2})^n$  and  $b_n = (\frac{1}{3})^n$ .

35 Under what condition does  $\sum_0^{\infty} (a_{n+1} - a_n)$  converge and what is its sum?

36 For a conditionally convergent series, explain how the terms could be rearranged so that the sum is  $+\infty$ . All terms must eventually be included, even negative terms.

37 Describe the terms in the product  $(1 + \frac{1}{2} + \frac{1}{4} + \dots)(1 + \frac{1}{3} + \frac{1}{6} + \dots)$  and find their sum.

38 True or false:

(a) Every alternating series converges.

(b)  $\sum a_n$  converges conditionally if  $\sum |a_n|$  diverges.

(c) A convergent series with positive terms is absolutely convergent.

(d) If  $\sum a_n$  and  $\sum b_n$  both converge, so does  $\sum (a_n + b_n)$ .

39 Every number  $x$  between 0 and 2 equals  $1 + \frac{1}{2} + \frac{1}{4} + \dots$  with suitable terms deleted. Why?

40 Every number  $s$  between  $-1$  and  $1$  equals  $\pm \frac{1}{2} \pm \frac{1}{4} \pm \frac{1}{8} \pm \dots$  with a suitable choice of signs. (Add  $1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$  to get Problem 39.) Which signs give  $s = -1$  and  $s = 0$  and  $s = \frac{1}{3}$ ?

41 Show that no choice of signs will make  $\pm \frac{1}{3} \pm \frac{1}{9} \pm \frac{1}{27} \pm \dots$  equal to zero.

42 The sums in Problem 41 form a Cantor set centered at zero. What is the smallest positive number in the set? Choose signs to show that  $\frac{1}{3}$  is in the set.

\*43 Show that the tangent of  $\theta = \frac{1}{2}(\pi - 1)$  is  $\sin 1/(1 - \cos 1)$ . This is the imaginary part of  $s = -\ln(1 - e^i)$ . From  $s = \sum e^{in}/n$  deduce the remarkable sum  $\sum (\sin n)/n = \frac{1}{2}(\pi - 1)$ .

44 Suppose  $\sum a_n$  converges and  $|x| < 1$ . Show that  $\sum a_n x^n$  converges absolutely.

## 10.4 The Taylor Series for $e^x$ , $\sin x$ , and $\cos x$

This section goes back from numbers to functions. Instead of  $\sum a_n = s$  it deals with  $\sum a_n x^n = f(x)$ . **The sum is a function of  $x$ .** The geometric series has all  $a_n = 1$  (including  $a_0$ , the constant term) and its sum is  $f(x) = 1/(1 - x)$ . The derivatives of  $1 + x + x^2 + \dots$  match the derivatives of  $f$ . Now we choose the  $a_n$  differently, to match a different function.

The new function is  $e^x$ . All its derivatives are  $e^x$ . At  $x = 0$ , this function and its derivatives equal 1. To match these 1's, we move factorials into the denominators.

Term by term the series is

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (1)$$

$x^n/n!$  has the correct  $n$ th derivative ( $= 1$ ). **From the derivatives at  $x = 0$ , we have built back the function!** At  $x = 1$  the right side is  $1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots$  and the left side is  $e = 2.71828\dots$ . At  $x = -1$  the series gives  $1 - 1 + \frac{1}{2} - \frac{1}{6} + \dots$ , which is  $e^{-1}$ .

The same term-by-term idea works for differential equations, as follows.

**EXAMPLE 1** Solve  $dy/dx = -y$  starting from  $y = 1$  at  $x = 0$ .

**Solution** The zeroth derivative at  $x = 0$  is the function itself:  $y = 1$ . Then the equation  $y' = -y$  gives  $y' = -1$  and  $y'' = -y' = +1$ . The alternating derivatives  $1, -1, 1, -1, \dots$  are matched by the alternating series for  $e^{-x}$ :

$$y = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots = e^{-x} \text{ (the correct solution to } y' = -y\text{).}$$

**EXAMPLE 2** Solve  $d^2y/dx^2 = -y$  starting from  $y = 1$  and  $y' = 0$  (the answer is  $\cos x$ ).

**Solution** The equation gives  $y'' = -1$  (again at  $x = 0$ ). The derivative of the equation gives  $y''' = -y' = 0$ . Then  $y'''' = -y'' = +1$ . The even derivatives are alternately  $+1$  and  $-1$ , the odd derivatives are zero. This is matched by a series of even powers, which constructs  $\cos x$ :

$$y = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots = \cos x.$$

The first terms  $1 - \frac{1}{2}x^2$  came earlier in the book. Now we have the whole alternating series. It converges absolutely for all  $x$ , by comparison with the series for  $e^x$  (odd powers are dropped). The partial sums in Figure 10.4 reach further and further before they lose touch with  $\cos x$ .

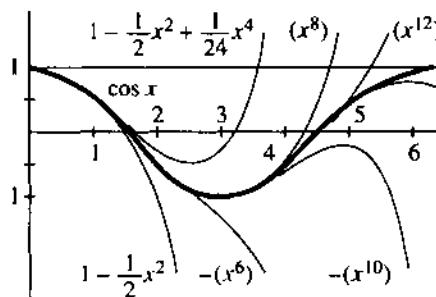


Fig. 10.4 The partial sums  $1 - x^2/2 + x^4/24 - \dots$  of the cosine series.

If we wanted plus signs instead of plus-minus, we could average  $e^x$  and  $e^{-x}$ . The differential equation for  $\cosh x$  is  $d^2y/dx^2 = +y$ , to give plus signs:

$$\frac{1}{2}(e^x + e^{-x}) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \dots \text{ (which is } \cosh x\text{).}$$

### TAYLOR SERIES

The idea of **matching derivatives by powers** is becoming central to this chapter. The derivatives are given at a basepoint (say  $x = 0$ ). They are numbers  $f(0), f'(0), \dots$ . The derivative  $f^{(n)}(0)$  will be the  $n$ th derivative of  $a_n x^n$ , if we choose  $a_n$  to be  $f^{(n)}(0)/n!$

Then the series  $\sum a_n x^n$  has the same derivatives at the basepoint as  $f(x)$ :

**10K** The *Taylor series* that matches  $f(x)$  and all its derivatives at  $x = 0$  is

$$f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3 + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

The first terms give the linear and quadratic approximations that we know well. The  $x^3$  term was mentioned earlier (but not used). Now we have *all* the terms—an “infinite approximation” that is intended to equal  $f(x)$ .

Two things are needed. First, the series must converge. Second, the function must do what the series predicts, away from  $x = 0$ . Those are true for  $e^x$  and  $\cos x$  and  $\sin x$ ; the series equals the function. We proceed on that basis.

The Taylor series with special basepoint  $x = 0$  is also called the “*Maclaurin series*.”

**EXAMPLE 3** Find the Taylor series for  $f(x) = \sin x$  around  $x = 0$ .

**Solution** The numbers  $f^{(n)}(0)$  are the values of  $f = \sin x$ ,  $f' = \cos x$ ,  $f'' = -\sin x$ , ... at  $x = 0$ . Those values are  $0, 1, 0, -1, 0, 1, \dots$ . All even derivatives are zero. To find the coefficients in the Taylor series, divide by the factorials:

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots \quad (2)$$

**EXAMPLE 4** Find the Taylor series for  $f(x) = (1+x)^5$  around  $x = 0$ .

**Solution** This function starts at  $f(0) = 1$ . Its derivative is  $5(1+x)^4$ , so  $f'(0) = 5$ . The second derivative is  $5 \cdot 4 \cdot (1+x)^3$ , so  $f''(0) = 5 \cdot 4$ . The next three derivatives are  $5 \cdot 4 \cdot 3, 5 \cdot 4 \cdot 3 \cdot 2, 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ . After that all derivatives are zero. Therefore the Taylor series *stops* after the  $x^5$  term:

$$1 + 5x + \frac{5 \cdot 4}{2!}x^2 + \frac{5 \cdot 4 \cdot 3}{3!}x^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{4!}x^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5!}x^5. \quad (3)$$

You may recognize  $1, 5, 10, 10, 5, 1$ . They are the *binomial coefficients*, which appear in Pascal’s triangle (Section 2.2). By matching derivatives, we see why  $0!, 1!, 2!, \dots$  are needed in the denominators.

There is no doubt that  $x = 0$  is the nicest basepoint. But Taylor series can be constructed around other points  $x = a$ . The principle is the same—*match derivatives by powers*—but now the powers to use are  $(x - a)^n$ . The derivatives  $f^{(n)}(a)$  are computed at the new basepoint  $x = a$ .

**The Taylor series begins with**  $f(a) + f'(a)(x - a)$ . This is the tangent approximation at  $x = a$ . The whole “infinite approximation” is centered at  $a$ —at that point it has the same derivatives as  $f(x)$ .

**10L** The *Taylor series* for  $f(x)$  around the basepoint  $x = a$  is

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2 + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n. \quad (4)$$

**EXAMPLE 5** Find the Taylor series for  $f(x) = (1+x)^5$  around  $x = a = 1$ .

**Solution** At  $x = 1$ , the function is  $(1+1)^5 = 32$ . Its first derivative  $5(1+x)^4$  is  $5 \cdot 16 = 80$ . We compute the  $n$ th derivative, divide by  $n!$ , and multiply by  $(x - 1)^n$ :

$$32 + 80(x - 1) + 80(x - 1)^2 + 40(x - 1)^3 + 10(x - 1)^4 + (x - 1)^5. \quad (5)$$

That Taylor series (which stops at  $n = 5$ ) should agree with  $(1 + x)^5$ . It does. We could rewrite  $1 + x$  as  $2 + (x - 1)$ , and take its fifth power directly. Then 32, 16, 8, 4, 2, 1 will multiply the usual coefficients 1, 5, 10, 10, 5, 1 to give our Taylor coefficients 32, 80, 80, 40, 10, 1. The series stops as it will stop for any polynomial—because the high derivatives are zero.

**EXAMPLE 6** Find the Taylor series for  $f(x) = e^x$  around the basepoint  $x = 1$ .

**Solution** At  $x = 1$  the function and all its derivatives equal  $e$ . Therefore the Taylor series has that constant factor (note the powers of  $x - 1$ , not  $x$ ):

$$e^x = e + e(x - 1) + \frac{e}{2!}(x - 1)^2 + \frac{e}{3!}(x - 1)^3 + \dots \quad (6)$$

### DEFINING THE FUNCTION BY ITS SERIES

Usually, we define  $\sin x$  and  $\cos x$  from the sides of a triangle. But we could start instead with the series. Define  $\sin x$  by equation (2). The logic goes backward, but it is still correct:

First, prove that the series converges.

Second, prove properties like  $(\sin x)' = \cos x$ .

Third, connect the definitions by series to the sides of a triangle.

We don't plan to do all this. The usual definition was good enough. But note first: There is no problem with convergence. The series for  $\sin x$  and  $\cos x$  and  $e^x$  all have terms  $\pm x^n/n!$ . The factorials make the series converge for all  $x$ . The general rule for  $e^x$  times  $e^y$  can be based on the series. Equation (6) is typical:  $e$  is multiplied by powers of  $(x - 1)$ . Those powers add to  $e^{x-1}$ . So the series proves that  $e^x = ee^{x-1}$ . That is just one example of the multiplication  $(e^x)(e^y) = e^{x+y}$ :

$$\left(1 + x + \frac{x^2}{2} + \dots\right)\left(1 + y + \frac{y^2}{2} + \dots\right) = 1 + x + y + \frac{x^2}{2} + xy + \frac{y^2}{2} + \dots \quad (7)$$

Term by term, multiplication gives the series for  $e^{x+y}$ . Term by term, differentiating the series for  $e^x$  gives  $e^x$ . Term by term, the derivative of  $\sin x$  is  $\cos x$ :

$$\frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad (8)$$

We don't need the famous limit  $(\sin x)/x \rightarrow 1$ , by which geometry gave us the derivative. The identities of trigonometry become identities of infinite series. We could even define  $\pi$  as the first positive  $x$  at which  $x - \frac{1}{6}x^3 + \dots$  equals zero. But it is certainly not obvious that this sine series returns to zero—much less that the point of return is near 3.14.

The function that will be defined by infinite series is  $e^{i\theta}$ . This is the exponential of the **imaginary number**  $i\theta$  (a multiple of  $i = \sqrt{-1}$ ). The result  $e^{i\theta}$  is a **complex number**, and our goal is to identify it. (We will be confirming Section 9.4.) The technique is to treat  $i\theta$  like all other numbers, real or complex, and simply put it into the series:

**DEFINITION**  $e^{i\theta}$  is the sum of  $1 + (i\theta) + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \dots$  (9)

Now use  $i^2 = -1$ . The even powers are  $i^4 = +1$ ,  $i^6 = -1$ ,  $i^8 = +1$ ,  $\dots$ . We are just multiplying  $-1$  by  $-1$  to get  $1$ . The odd powers are  $i^3 = -i$ ,  $i^5 = +i$ ,  $\dots$ . There-

fore  $e^{i\theta}$  splits into a *real part* (with no  $i$ 's) and an *imaginary part* (multiplying  $i$ ):

$$e^{i\theta} = \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \dots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \dots\right). \quad (10)$$

You recognize those series. They are  $\cos \theta$  and  $\sin \theta$ . Therefore:

*Euler's formula* is  $e^{i\theta} = \cos \theta + i \sin \theta$ . Note that  $e^{2\pi i} = 1$ .

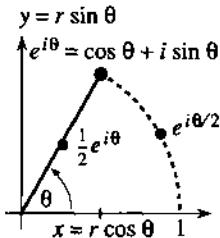


Fig. 10.5

The real part is  $x = \cos \theta$  and the imaginary part is  $y = \sin \theta$ . Those coordinates pick out the point  $e^{i\theta}$  in the "complex plane." Its distance from the origin  $(0, 0)$  is  $r = 1$ , because  $(\cos \theta)^2 + (\sin \theta)^2 = 1$ . Its angle is  $\theta$ , as shown in Figure 10.5. The number  $-1$  is  $e^{i\pi}$ , at the distance  $r = 1$  and the angle  $\pi$ . It is on the real axis to the left of zero. If  $e^{i\theta}$  is multiplied by  $r = 2$  or  $r = \frac{1}{2}$  or any  $r \geq 0$ , the result is a complex number at a distance  $r$  from the origin:

**Complex numbers:**  $re^{i\theta} = r(\cos \theta + i \sin \theta) = r \cos \theta + ir \sin \theta = x + iy$ .

With  $e^{i\theta}$ , a negative number has a logarithm. *The logarithm of  $-1$  is imaginary* (it is  $i\pi$ , since  $e^{i\pi} = -1$ ). A negative number also has fractional powers. The fourth root of  $-1$  is  $(-1)^{1/4} = e^{i\pi/4}$ . More important for calculus: *The derivative of  $x^{5/4}$  is  $\frac{5}{4}x^{1/4}$* . That sounds old and familiar, but at  $x = -1$  it was never allowed.

**Complex numbers tie up the loose ends left by the limitations of real numbers.**

The formula  $e^{i\theta} = \cos \theta + i \sin \theta$  has been called "one of the greatest mysteries of undergraduate mathematics." Writers have used desperate methods to avoid infinite series. That proof in (10) may be the clearest (I remember sending it to a prisoner studying calculus) but here is a way to start from  $d/dx(e^{ix}) = ie^{ix}$ .

**A different proof of Euler's formula** Any complex number is  $e^{ix} = r(\cos \theta + i \sin \theta)$  for some  $r$  and  $\theta$ . Take the  $x$  derivative of both sides, and substitute for  $ie^{ix}$ :

$$(\cos \theta + i \sin \theta)dr/dx + r(-\sin \theta + i \cos \theta)d\theta/dx = ir(\cos \theta + i \sin \theta).$$

Comparing the real parts and also the imaginary parts, we need  $dr/dx = 0$  and  $d\theta/dx = 1$ . The starting values  $r = 1$  and  $\theta = 0$  are known from  $e^{i0} = 1$ . Therefore  $r$  is always 1 and  $\theta$  is  $x$ . Substituting into the first sentence of the proof, we have Euler's formula  $e^{ix} = 1(\cos x + i \sin x)$ .

## 10.4 EXERCISES

### Read-through questions

The a series is chosen to match  $f(x)$  and all its b at the basepoint. Around  $x = 0$  the series begins with  $f(0) + \underline{c}x + \underline{d}x^2$ . The coefficient of  $x^n$  is e. For  $f(x) = e^x$  this series is f. For  $f(x) = \cos x$  the series is g. For  $f(x) = \sin x$  the series is h. If the signs were all positive in those series, the functions would be  $\cosh x$  and i. Addition gives  $\cosh x + \sinh x = \underline{l}$ .

In the Taylor series for  $f(x)$  around  $x = a$ , the coefficient of  $(x - a)^n$  is  $b_n = \underline{k}$ . Then  $b_n(x - a)^n$  has the same l as  $f$  at the basepoint. In the example  $f(x) = x^2$ , the Taylor coefficients are  $b_0 = \underline{m}$ ,  $b_1 = \underline{n}$ ,  $b_2 = \underline{o}$ . The series  $b_0 + b_1(x - a) + b_2(x - a)^2$  agrees with the original p. The series for  $e^x$  around  $x = a$  has  $b_n = \underline{q}$ . Then the Taylor series reproduces the identity  $e^x = (\underline{r})(\underline{s})$ .

We define  $e^x$ ,  $\sin x$ ,  $\cos x$ , and also  $e^{i\theta}$  by their series. The derivative  $d/dx(1 + x + \frac{1}{2}x^2 + \dots) = 1 + x + \dots$  translates to t. The derivative of  $1 - \frac{1}{2}x^2 + \dots$  is u. Using  $i^2 = -1$  the series  $1 + i\theta + \frac{1}{2}(i\theta)^2 + \dots$  splits into  $e^{i\theta} = \underline{v}$ . Its square gives  $e^{2i\theta} = \underline{w}$ . Its reciprocal is  $e^{-i\theta} = \underline{x}$ . Multiplying by  $r$  gives  $re^{i\theta} = \underline{y} + i\underline{z}$ , which connects the polar and rectangular forms of a A number. The logarithm of  $e^{i\theta}$  is B.

1 Write down the series for  $e^{2x}$  and compute all derivatives at  $x = 0$ . Give a series of numbers that adds to  $e^2$ .

2 Write down the series for  $\sin 2x$  and check the third derivative at  $x = 0$ . Give a series of numbers that adds to  $\sin 2\pi = 0$ .

In 3–8 find the derivatives of  $f(x)$  at  $x = 0$  and the Taylor series (powers of  $x$ ) with those derivatives.

3  $f(x) = e^{ix}$

4  $f(x) = 1/(1+x)$

5  $f(x) = 1/(1-2x)$

6  $f(x) = \cosh x$

7  $f(x) = \ln(1-x)$

8  $f(x) = \ln(1+x)$

Problems 9–14 solve differential equations by series.

9 From the equation  $dy/dx = y - 2$  find all the derivatives of  $y$  at  $x = 0$  starting from  $y(0) = 1$ . Construct the infinite series for  $y$ , identify it as a known function, and verify that the function satisfies  $y' = y - 2$ .

10 Differentiate the equation  $y' = cy + s$  ( $c$  and  $s$  constant) to find all derivatives of  $y$  at  $x = 0$ . If the starting value is  $y_0 = 0$ , construct the Taylor series for  $y$  and identify it with the solution of  $y' = cy + s$  in Section 6.3.

11 Find the infinite series that solves  $y'' = -y$  starting from  $y = 0$  and  $y' = 1$  at  $x = 0$ .

12 Find the infinite series that solves  $y' = y$  starting from  $y = 1$  at  $x = 3$  (use powers of  $x - 3$ ). Identify  $y$  as a known function.

13 Find the infinite series (powers of  $x$ ) that solves  $y'' = 2y - y$  starting from  $y = 0$  and  $y' = 1$  at  $x = 0$ .

14 Solve  $y'' = y$  by a series with  $y = 1$  and  $y' = 0$  at  $x = 0$  and identify  $y$  as a known function.

15 Find the Taylor series for  $f(x) = (1+x)^2$  around  $x = a = 0$  and around  $x = a = 1$  (powers of  $x - 1$ ). Check that both series add to  $(1+x)^2$ .

16 Find all derivatives of  $f(x) = x^3$  at  $x = a$  and write out the Taylor series around that point. Verify that it adds to  $x^3$ .

17 What is the series for  $(1-x)^5$  with basepoint  $a = 1$ ?

18 Write down the Taylor series for  $f = \cos x$  around  $x = 2\pi$  and also for  $f = \cos(x - 2\pi)$  around  $x = 0$ .

In 19–24 compute the derivatives of  $f$  and its Taylor series around  $x = 1$ .

19  $f(x) = 1/x$

20  $f(x) = 1/(2-x)$

21  $f(x) = \ln x$

22  $f(x) = x^4$

23  $f(x) = e^{-x}$

24  $f(x) = e^{2x}$

In 25–33 write down the first three nonzero terms of the Taylor series around  $x = 0$ , from the series for  $e^x$ ,  $\cos x$ , and  $\sin x$ .

25  $xe^{2x}$

26  $\cos \sqrt{x}$

27  $(1 - \cos x)/x^2$

28  $\frac{\sin x}{x}$

29  $\int_0^x \frac{\sin x}{x} dx$

30  $\sin x^2$

31  $e^{x^2}$

32  $b^x = e^{x \ln b}$

33  $e^x \cos x$

\*34 For  $x < 0$  the derivative of  $x^n$  is still  $nx^{n-1}$ :

$$\frac{d}{dx}(x^n) = \frac{d}{dx}(|x|^n e^{inx}) = n|x|^{n-1} e^{inx} \frac{d|x|}{dx}.$$

What is  $d|x|/dx$ ? Rewrite this answer as  $nx^{n-1}$ .

35 Why doesn't  $f(x) = \sqrt{x}$  have a Taylor series around  $x = 0$ ? Find the first two terms around  $x = 1$ .

36 Find the Taylor series for  $2^x$  around  $x = 0$ .

In 37–44 find the first three terms of the Taylor series around  $x = 0$ .

37  $f(x) = \tan^{-1} x$

38  $f(x) = \sin^{-1} x$

39  $f(x) = \tan x$

40  $f(x) = \ln(\cos x)$

41  $f(x) = e^{\sin x}$

42  $f(x) = \tanh^{-1} x$

43  $f(x) = \cos^2 x$

44  $f(x) = \sec^2 x$

45 From  $e^{i\theta} = \cos \theta + i \sin \theta$  and  $e^{-i\theta} = \cos \theta - i \sin \theta$ , add and subtract to find  $\cos \theta$  and  $\sin \theta$ .

46 Does  $(e^{i\theta})^2$  equal  $\cos^2 \theta + i \sin^2 \theta$  or  $\cos \theta^2 + i \sin \theta^2$ ?

47 Find the real and imaginary parts and the 99th power of  $e^{ix}$ ,  $e^{ix/2}$ ,  $e^{ix/4}$ , and  $e^{-ix/6}$ .

48 The three cube roots of 1 are 1,  $e^{2\pi i/3}$ ,  $e^{4\pi i/3}$ .

(a) Find the real and imaginary parts of  $e^{2\pi i/3}$ .

(b) Explain why  $(e^{2\pi i/3})^3 = 1$ .

(c) Check this statement in rectangular coordinates.

49 The cube roots of  $-1 = e^{i\pi}$  are  $e^{i\pi/3}$  and \_\_\_\_\_ and \_\_\_\_\_. Find their sum and their product.

50 Find the squares of  $2e^{i\pi/3} = 1 + \sqrt{3}i$  and  $4e^{i\pi/4} = 2\sqrt{2} + 2\sqrt{2}i$  in both polar and rectangular coordinates.

51 Multiply  $e^{is} = \cos s + i \sin s$  times  $e^{it} = \cos t + i \sin t$  to find formulas for  $\cos(s+t)$  and  $\sin(s+t)$ .

52 Multiply  $e^{is}$  times  $e^{-it}$  to find formulas for  $\cos(s-t)$  and  $\sin(s-t)$ .

53 Find the logarithm of  $i$ . Then find another logarithm of  $i$ . (What can you add to the exponent of  $e^{in_i}$  without changing the result?)

54 (Proof that  $e$  is irrational) If  $e = p/q$  then

$$N = p \left[ \frac{1}{e} - \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots \pm \frac{1}{p!} \right) \right]$$

would be an integer. (Why?) The number in brackets—the distance from the alternating series to its sum  $1/e$ —is less than the last term which is  $1/p!$  Deduce that  $|N| < 1$  and reach a contradiction, which proves that  $e$  cannot equal  $p/q$ .

55 Solve  $dy/dx = y$  by infinite series starting from  $y = 2$  at  $x = 0$ .

## 10.5 Power Series

This section studies the properties of a power series. When the basepoint is zero, the powers are  $x^n$ . The series is  $\sum a_n x^n$ . When the basepoint is  $x = a$ , the powers are  $(x - a)^n$ . We want to know when and where (and how quickly) the series converges to the underlying function. For  $e^x$  and  $\cos x$  and  $\sin x$  there is convergence for all  $x$ —but that is certainly not true for  $1/(1-x)$ . The convergence is best when the function is smooth.

First I emphasize that power series are not the only series. For many applications they are not the best choice. An alternative is a sum of sines,  $f(x) = \sum b_n \sin nx$ . That is a “Fourier sine series”, which treats all  $x$ ’s equally instead of picking on a basepoint. A Fourier series allows jumps and corners in the graph—it takes the rough with the smooth. By contrast a power series is terrific near its basepoint, and gets worse as you move away. The Taylor coefficients  $a_n$  are totally determined at the basepoint—where all derivatives are computed. Remember the rule for Taylor series:

$$a_n = (\text{nth derivative at the basepoint})/n! = f^{(n)}(a)/n! \quad (1)$$

A remarkable fact is the convergence in a *symmetric interval around  $x = a$* .

**10M** A power series  $\sum a_n x^n$  either converges for all  $x$ , or it converges only at the basepoint  $x = 0$ , or else it has a *radius of convergence r*:

$\Sigma a_n x^n$  converges absolutely if  $|x| < r$  and diverges if  $|x| > r$ .

The series  $\Sigma x^n/n!$  converges for all  $x$  (the sum is  $e^x$ ). The series  $\Sigma n!x^n$  converges for no  $x$  (except  $x = 0$ ). The geometric series  $\Sigma x^n$  converges absolutely for  $|x| < 1$  and diverges for  $|x| > 1$ . Its *radius of convergence is  $r = 1$* . Note that its sum  $1/(1-x)$  is perfectly good for  $|x| > 1$ —the function is all right but the series has given up. If something goes wrong at the distance  $r$ , a power series can’t get past that point.

When the basepoint is  $x = a$ , the interval of convergence shifts over to  $|x - a| < r$ . The series converges for  $x$  between  $a - r$  and  $a + r$  (symmetric around  $a$ ). We cannot say in advance whether the endpoints  $a \pm r$  give divergence or convergence (absolute or conditional). *Inside* the interval, an easy comparison test will now prove convergence.

**PROOF OF 10M** Suppose  $\Sigma a_n X^n$  converges at a particular point  $X$ . The proof will show that  $\Sigma a_n x^n$  converges when  $|x|$  is less than the number  $|X|$ . Thus convergence at  $X$  gives convergence at all closer points  $x$  (I mean closer to the basepoint 0). Proof: Since  $\Sigma a_n X^n$  converges, its terms approach zero. Eventually  $|a_n X^n| < 1$  and then

$$|a_n x^n| = |a_n X^n| |x/X|^n < |x/X|^n.$$

Our series  $\Sigma a_n x^n$  is absolutely convergent by comparison with the geometric series for  $|x/X|$ , which converges since  $|x/X| < 1$ .

**EXAMPLE 1** The series  $\Sigma nx^n/4^n$  has radius of convergence  $r = 4$ .

*The ratio test and root test are best for power series.* The ratios between terms approach  $x/4$  (and so does the  $n$ th root of  $nx^n/4^n$ ):

$$\frac{(n+1)x^{n+1}}{4^{n+1}} / \frac{nx^n}{4^n} = \frac{x}{4} \frac{n+1}{n} \text{ approaches } L = \frac{x}{4}.$$

The ratio test gives convergence if  $L < 1$ , which means  $|x| < 4$ .

**EXAMPLE 2** The sine series  $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$  has radius  $r = \infty$  (it converges everywhere).

The ratio of  $x^{n+2}/(n+2)!$  to  $x^n/n!$  is  $x^2/(n+2)(n+1)$ . This approaches  $L = 0$ .

**EXAMPLE 3** The series  $\Sigma(x-5)^n/n^2$  has radius  $r = 1$  around its basepoint  $a = 5$ .

The ratios between terms approach  $L = x - 5$ . (The fractions  $n^2/(n+1)^2$  go toward 1.) There is absolute convergence if  $|x-5| < 1$ . This is the interval  $4 < x < 6$ , symmetric around the basepoint. This series happens to converge at the endpoints 4 and 6, because of the factor  $1/n^2$ . That factor decides the delicate question—convergence at the endpoints—but all powers of  $n$  give the same *interval of convergence*  $4 < x < 6$ .

### CONVERGENCE TO THE FUNCTION: REMAINDER TERM AND RADIUS $r$

Remember that a Taylor series starts with a function  $f(x)$ . The derivatives at the basepoint produce the series. Suppose the series converges: **Does it converge to the function?** This is a question about the remainder  $R_n(x) = f(x) - s_n(x)$ , which is the difference between  $f$  and the partial sum  $s_n = a_0 + \dots + a_n(x-a)^n$ . **The remainder  $R_n$  is the error if we stop the series**, ending with the  $n$ th derivative term  $a_n(x-a)^n$ .

**10N** Suppose  $f$  has an  $(n+1)$ st derivative from the basepoint  $a$  out to  $x$ . Then for some point  $c$  in between (position not known) the remainder at  $x$  equals

$$R_n(x) = f(x) - s_n(x) = f^{(n+1)}(c)(x-a)^{n+1}/(n+1)! \quad (2)$$

The error in stopping at the  $n$ th derivative is controlled by the  $(n+1)$ st derivative.

You will guess, correctly, that the unknown point  $c$  comes from the Mean Value Theorem. For  $n=1$  the proof is at the end of Section 3.8. That was the error  $e(x)$  in linear approximation:

$$R_1(x) = f(x) - f(a) - f'(a)(x-a) = \frac{1}{2}f''(c)(x-a)^2.$$

For every  $n$ , the proof compares  $R_n$  with  $(x-a)^{n+1}$ . Their  $(n+1)$ st derivatives are  $f^{(n+1)}$  and  $(n+1)!$  The generalized Mean Value Theorem says that the ratio of  $R_n$  to  $(x-a)^{n+1}$  equals the ratio of those derivatives, at the right point  $c$ . That is equation (2). The details can stay in Section 3.8 and Problem 23, because the main point is what we want. **The error is exactly like the next term  $a_{n+1}(x-a)^{n+1}$** , except that the  $(n+1)$ st derivative is at  $c$  instead of the basepoint  $a$ .

**EXAMPLE 4** When  $f$  is  $e^x$ , the  $(n+1)$ st derivative is  $e^x$ . Therefore the error is

$$R_n = e^x - \left(1 + x + \dots + \frac{x^n}{n!}\right) = e^c \frac{x^{n+1}}{(n+1)!}. \quad (3)$$

At  $x=1$  and  $n=2$ , the error is  $e - (1 + 1 + \frac{1}{2}) \approx .218$ . The right side is  $e^c/6$ . The unknown point is  $c = \ln(.218 \cdot 6) = .27$ . Thus  $c$  lies between the basepoint  $a=0$  and the error point  $x=1$ , as required. The series converges to the function, because  $R_n \rightarrow 0$ .

In practice,  $n$  is the number of derivatives to be calculated. We may aim for an error  $|R_n|$  below  $10^{-6}$ . Unfortunately, the high derivative in formula (2) is awkward to estimate (except for  $e^x$ ). And high derivatives in formula (1) are difficult to compute. Most real calculations use only a few terms of a Taylor series. For more accuracy we move the basepoint closer, or switch to another series.

There is a direct connection between the function and the convergence radius  $r$ . A hint came for  $f(x) = 1/(1-x)$ . The function blows up at  $x = 1$ —which also ends the convergence interval for the series. Another hint comes for  $f = 1/x$ , if we expand around  $x = a = 1$ :

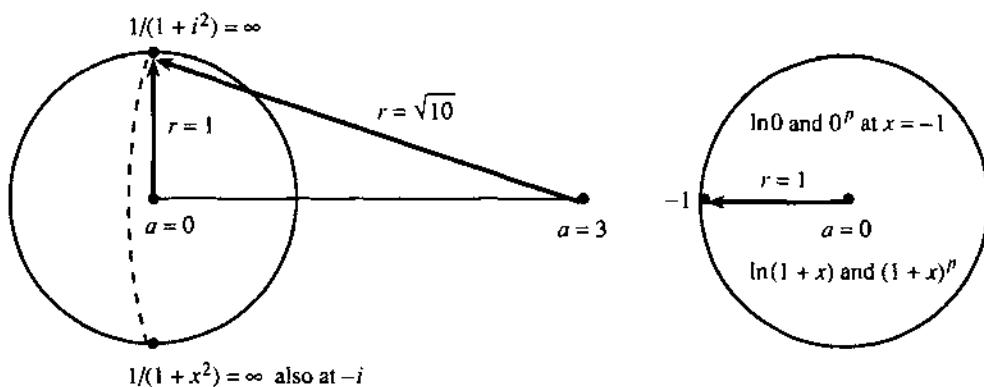
$$\frac{1}{x} = \frac{1}{1-(1-x)} = 1 + (1-x) + (1-x)^2 + \dots \quad (4)$$

This geometric series converges for  $|1-x| < 1$ . Convergence stops at the end point  $x = 0$ —exactly where  $1/x$  blows up. *The failure of the function stops the convergence of the series.* But note that  $1/(1+x^2)$ , which never seems to fail, also has convergence radius  $r = 1$ :

$$1/(1+x^2) = 1 - x^2 + x^4 - x^6 + \dots \text{ converges only for } |x| < 1.$$

When you see the reason, you will know why  $r$  is a “radius.” There is a circle, and the function fails at the edge of the circle. The circle contains complex numbers as well as real numbers. The imaginary points  $i$  and  $-i$  are at the edge of the circle. *The function fails at those points because  $1/(1+i^2) = \infty$ .*

Complex numbers are pulling the strings, out of sight. The circle of convergence reaches out to the nearest “singularity” of  $f(x)$ , real or imaginary or complex. For  $1/(1+x^2)$ , the singularities at  $i$  and  $-i$  make  $r = 1$ . If we expand around  $a = 3$ , the distance to  $i$  and  $-i$  is  $r = \sqrt{10}$ . If we change to  $\ln(1+x)$ , which blows up at  $x = -1$ , the radius of convergence of  $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$  is  $r = 1$ .



**Fig. 10.6** Convergence radius  $r$  is distance from basepoint  $a$  to nearest singularity.

### THE BINOMIAL SERIES

We close this chapter with one more series. It is the Taylor series for  $(1+x)^p$ , around the basepoint  $x = 0$ . A typical power is  $p = \frac{1}{2}$ , where we want the terms in

$$\sqrt{1+x} = 1 + \frac{1}{2}x + a_2x^2 + \dots$$

The slow way is to square both sides, which gives  $1 + x + (2a_2 + \frac{1}{4})x^2$  on the right. Since  $1 + x$  is on the left,  $a_2 = -\frac{1}{8}$  is needed to remove the  $x^2$  term. Eventually  $a_3$  can be found. The fast way is to match the derivatives of  $f = (1+x)^{1/2}$ :

$$f' = \frac{1}{2}(1+x)^{-1/2} \quad f'' = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(1+x)^{-3/2} \quad f''' = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1+x)^{-5/2}$$

At  $x = 0$  those derivatives are  $\frac{1}{2}, -\frac{1}{4}, \frac{3}{8}$ . Dividing by  $1!, 2!, 3!$  gives

$$a_1 = \frac{1}{2} \quad a_2 = -\frac{1}{8} \quad a_3 = \frac{1}{16} \quad a_n = \frac{1}{n!} \left(\frac{1}{2}\right) \left(\frac{1}{2}-1\right) \cdots \left(\frac{1}{2}-n+1\right).$$

These are the *binomial coefficients* when the power is  $p = \frac{1}{2}$ .

Notice the difference from the binomials in Chapter 2. For those, the power  $p$  was a positive integer. The series  $(1+x)^2 = 1 + 2x + x^2$  stopped at  $x^2$ . The coefficients for  $p = 2$  were  $1, 2, 1, 0, 0, 0, \dots$ . For fractional  $p$  or negative  $p$  those later coefficients are *not zero*, and we find them from the derivatives of  $(1+x)^p$ :

$$(1+x)^p = p(1+x)^{p-1} = p(p-1)(1+x)^{p-2} = f^{(n)} = p(p-1)\cdots(p-n+1)(1+x)^{p-n}.$$

Dividing by  $0!, 1!, 2!, \dots, n!$  at  $x = 0$ , the binomial coefficients are

$$1-p \quad \frac{p(p-1)}{2} \quad \dots \quad \frac{f^{(n)}(0)}{n!} = \frac{p(p-1)\cdots(p-n+1)}{n!}. \quad (5)$$

For  $p = n$  that last binomial coefficient is  $n!/n! = 1$ . It gives the final  $x^n$  at the end of  $(1+x)^n$ . For other values of  $p$ , the binomial series never stops. *It converges for  $|x| < 1$* :

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \dots = \sum_{n=0}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!} x^n. \quad (6)$$

When  $p = 1, 2, 3, \dots$  the binomial coefficient  $p!/n!(n-p)!$  counts the number of ways to select a group of  $n$  friends out of a group of  $p$  friends. If you have 20 friends, you can choose 2 of them in  $(20)(19)/2 = 190$  ways.

Suppose  $p$  is not a positive integer. What goes wrong with  $(1+x)^p$ , to stop the convergence at  $|x| = 1$ ? The failure is at  $x = -1$ . If  $p$  is negative,  $(1+x)^p$  blows up. If  $p$  is positive, as in  $\sqrt{1+x}$ , the higher derivatives blow up. Only for a positive integer  $p = n$  does the convergence radius move out to  $r = \infty$ . In that case the series for  $(1+x)^n$  stops at  $x^n$ , and  $f$  never fails.

A power series is a function in a new form. It is not a simple form, but sometimes it is the only form. To compute  $f$  we have to sum the series. To square  $f$  we have to multiply series. But the operations of calculus—derivative and integral—are easier. That explains why power series help to solve differential equations, which are a rich source of new functions. (Numerically the series are not always so good.) I should have said that the derivative and integral are easy for each separate term  $a_n x^n$ —and fortunately the convergence radius of the whole series is not changed.

If  $f(x) = \sum a_n x^n$  has convergence radius  $r$ , so do its derivative and its integral:

$$df/dx = \sum n a_n x^{n-1} \quad \text{and} \quad \int f(x) dx = \sum a_n x^{n+1}/(n+1) \text{ also converge for } |x| < r.$$

**EXAMPLE 5** The series for  $1/(1-x)$  and its derivative  $1/(1-x)^2$  and its integral  $-\ln(1-x)$  all have  $r = 1$  (because they all have trouble at  $x = 1$ ). The series are  $\sum x^n$  and  $\sum nx^{n-1}$  and  $\sum x^{n+1}/(n+1)$ .

**EXAMPLE 6** We can integrate  $e^{x^2}$  (previously impossible) by integrating every term in its series:

$$\int e^{x^2} dx = \int \left(1 + x^2 + \frac{1}{2!}x^4 + \dots\right) dx = x + \frac{x^3}{3} + \frac{1}{2!} \left(\frac{x^5}{5}\right) + \frac{1}{3!} \left(\frac{x^7}{7}\right) + \dots.$$

This always converges ( $r = \infty$ ). The derivative of  $e^{x^2}$  was never a problem.

## 10.5 EXERCISES

## Read-through questions

If  $|x| < |X|$  and  $\sum a_n X^n$  converges, then the series  $\sum a_n x^n$  also a. There is convergence in a b interval around the c. For  $\sum (2x)^n$  the convergence radius is  $r = \underline{d}$ . For  $\sum x^n/n!$  the radius is  $r = \underline{e}$ . For  $\sum (x - 3)^n$  there is convergence for  $|x - 3| < \underline{f}$ . Then  $x$  is between g and h.

Starting with  $f(x)$ , its Taylor series  $\sum a_n x^n$  has  $a_n = \underline{1}$ . With basepoint  $a$ , the coefficient of  $(x - a)^n$  is j. The error after the  $x^n$  term is called the k  $R_n(x)$ . It is equal to l where the unknown point  $c$  is between m. Thus the error is controlled by the n derivative.

The circle of convergence reaches out to the first point where  $f(x)$  fails. For  $f = 4/(2 - x)$ , that point is  $x = \underline{o}$ . Around the basepoint  $a = 5$ , the convergence radius would be  $r = \underline{p}$ . For  $\sin x$  and  $\cos x$  the radius is  $r = \underline{q}$ .

The series for  $\sqrt{1+x}$  is the r series with  $p = \frac{1}{2}$ . Its coefficients are  $a_n = \underline{s}$ . Its convergence radius is t. Its square is the very short series  $1 + x$ .

**In 1–6 find the Taylor series for  $f(x)$  around  $x = 0$  and its radius of convergence  $r$ . At what point does  $f(x)$  blow up?**

1  $f(x) = 1/(1 - 4x)$

2  $f(x) = 1/(1 - 4x^2)$

3  $f(x) = e^{1-x}$

4  $f(x) = \tan x$  (through  $x^3$ )

5  $f(x) = \ln(e + x)$

6  $f(x) = 1/(1 + 4x^2)$

**Find the interval of convergence and the function in 7–10.**

7  $f(x) = \sum_0^{\infty} \left(\frac{x-1}{2}\right)^n$

8  $f(x) = \sum_0^{\infty} n(x-a)^{n-1}$

9  $f(x) = \sum_0^{\infty} \frac{1}{n+1}(x-a)^{n+1}$

10  $f(x) = (x-2\pi) - \frac{(x-2\pi)^3}{3!} + \dots$

11 Write down the Taylor series for  $(e^x - 1)/x$ , based on the series for  $e^x$ . At  $x = 0$  the function is 0/0. Evaluate the series at  $x = 0$ . Check by l'Hôpital's Rule on  $(e^x - 1)/x$ .

12 Write down the Taylor series for  $xe^x$  around  $x = 0$ . Integrate and substitute  $x = 1$  to find the sum of  $1/n!(n+2)$ .

13 If  $f(x)$  is an even function, so  $f(-x) = f(x)$ , what can you say about its Taylor coefficients in  $f = \sum a_n x^n$ ?

14 Puzzle out the sums of the following series:

(a)  $x + x^2 - x^3 + x^4 + x^5 - x^6 + \dots$

(b)  $1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \dots$

(c)  $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots$

15 From the series for  $(1 - \cos x)/x^2$  find the limit as  $x \rightarrow 0$  faster than l'Hôpital's rule.

16 Construct a power series that converges for  $0 < x < 2\pi$ .

17–24 are about remainders and 25–36 are about binomials.

17 If the cosine series stops before  $x^8/8!$  show from (2) that the remainder  $R_7$  is less than  $x^8/8!$  Does this also follow because the series is alternating?

18 If the sine series around  $x = 2\pi$  stops after the terms in problem 10, estimate the remainder from equation (2).

19 Estimate by (2) the remainder  $R_n = x^{n+1} + x^{n+2} + \dots$  in the geometric series. Then compute  $R_n$  exactly and find the unknown point  $c$  for  $n = 2$  and  $x = \frac{1}{2}$ .

20 For  $-\ln(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + R_3$ , use equation (2) to show that  $R_3 \leq \frac{1}{4}$  at  $x = \frac{1}{2}$ .

21 Find  $R_n$  in Problem 20 and show that the series converges to the function at  $x = \frac{1}{2}$  (prove that  $R_n \rightarrow 0$ ).

22 By estimating  $R_n$  prove that the Taylor series for  $e^x$  around  $x = 1$  converges to  $e^x$  as  $n \rightarrow \infty$ .

23 (Proof of the remainder formula when  $n = 2$ )

(a) At  $x = a$  find  $R_2, R'_2, R''_2, R'''_2$ .

(b) At  $x = a$  evaluate  $g(x) = (x-a)^3$  and  $g', g'', g'''$ .

(c) What rule gives  $\frac{R_2(x) - R_2(a)}{g(x) - g(a)} = \frac{R'_2(c_1)}{g'(c_1)}$ ?

(d) In  $\frac{R'_2(c_1) - R'_2(a)}{g'(c_1) - g'(a)} = \frac{R''_2(c_2)}{g''(c_2)}$  and

$\frac{R''_2(c_2) - R''_2(a)}{g''(c_2) - g''(a)} = \frac{R'''_2(c)}{g'''(c)}$  where are  $c_1$  and  $c_2$  and  $c$ ?

(e) Combine (a-b-c-d) into the remainder formula (2).

24 All derivatives of  $f(x) = e^{-1/x^2}$  are zero at  $x = 0$ , including  $f(0) = 0$ . What is  $f(1)$ ? What is the Taylor series around  $x = 0$ ? What is the radius of convergence? Where does the series converge to  $f(x)$ ? For  $x = 1$  and  $n = 1$  what is the remainder estimate in (2)?

25 (a) Find the first three terms in the binomial series for  $1/\sqrt{1-x^2}$ .

(b) Integrate to find the first three terms in the Taylor series for  $\sin^{-1} x$ .

26 Show that the binomial coefficients in  $1/\sqrt{1-x} = \sum a_n x^n$  are  $a_n = 1 \cdot 3 \cdot 5 \cdots (2n-1)/2^n n!$

27 For  $p = -1$  and  $p = -2$  find nice formulas for the binomial coefficients.

28 Change the dummy variable and add lower limits to make  $\sum n x^{n-1} = \sum (n+1)x^n$ .

**29** In  $(1-x)^{-1} = \sum x^n$  the coefficient of  $x^n$  is the number of groups of  $n$  friends that can be formed from 1 friend (not binomial · repetition is allowed!). The coefficient is 1 and there is only one group—the same friend  $n$  times.

(a) Describe all groups of  $n$  friends that can be formed from 2 friends. (There are  $n+1$  groups.)

(b) How many groups of 5 friends can be formed from 3 friends?

**30** (a) What is the coefficient of  $x^n$  when  $1+x+x^2+\dots$  multiplies  $1+x+x^2+\dots$ ? Write the first three terms.

(b) What is the coefficient of  $x^5$  in  $(\sum x^k)^3$ ?

**31** Show that the binomial series for  $\sqrt{1+4x}$  has integer coefficients. (Note that  $x^n$  changes to  $(4x)^n$ . These coefficients are important in counting trees, paths, parentheses...)

**32** In the series for  $1/\sqrt{1-4x}$ , show that the coefficient of  $x^n$  is  $(2n)!$  divided by  $(n!)^2$ .

Use the binomial series to compute **33–36** with error less than  $1/1000$ .

**33**  $(15)^{1/4}$

**34**  $(1001)^{1/3}$

**35**  $(1.1)^{1.1}$

**36**  $e^{1/1000}$

**37** From  $\sec x = 1/[1 - (1 - \cos x)]$  find the Taylor series of  $\sec x$  up to  $x^6$ . What is the radius of convergence  $r$  (distance to blowup point)?

**38** From  $\sec^2 x = 1/[1 - \sin^2 x]$  find the Taylor series up to  $x^2$ . Check by squaring the secant series in Problem 37. Check by differentiating the tangent series in Problem 39.

**39** (Division of series) Find  $\tan x$  by long division of  $\sin x/\cos x$ :

$$\left(x - \frac{x^3}{6} + \frac{x^5}{120} \dots\right) \Big/ \left(1 - \frac{x^2}{2} + \frac{x^4}{24} \dots\right) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

**40** (Composition of series) If  $f = a_0 + a_1x + a_2x^2 + \dots$  and  $g = b_1x + b_2x^2 + \dots$  find the  $1, x, x^2$  coefficients of  $f(g(x))$ . Test on  $f = 1/(1+x)$ ,  $g = x/(1-x)$ , with  $f(g(x)) = 1-x$ .

**41** (Multiplication of series) From the series for  $\sin x$  and  $1/(1-x)$  find the first four terms for  $f = \sin x/(1-x)$ .

**42** (Inversion of series) If  $f = a_1x + a_2x^2 + \dots$  find coefficients  $b_1, b_2$  in  $g = b_1x + b_2x^2 + \dots$  so that  $f(g(x)) = x$ . Compute  $b_1, b_2$  for  $f = e^x - 1$ ,  $g = f^{-1} = \ln(1+x)$ .

**43** From the multiplication  $(\sin x)(\sin x)$  or the derivatives of  $f(x) = \sin^2 x$  find the first three terms of the series. Find the first four terms for  $\cos^2 x$  by an easy trick.

**44** Somehow find the first six nonzero terms for  $f = (1-x)/(1-x^3)$ .

**45** Find four terms of the series for  $1/\sqrt{1-x}$ . Then square the series to reach a geometric series.

**46** Compute  $\int_0^1 e^{-x^2} dx$  to 3 decimals by integrating the power series.

**47** Compute  $\int_0^1 \sin^2 t dt$  to 4 decimals by power series.

**48** Show that  $\sum x^n/n$  converges at  $x = -1$ , even though its derivative  $\sum x^{n-1}$  diverges. How can they have the same convergence radius?

**49** Compute  $\lim_{x \rightarrow 0} (\sin x - \tan x)/x^3$  from the series.

**50** If the  $n$ th root of  $a_n$  approaches  $L > 0$ , explain why  $\sum a_n x^n$  has convergence radius  $r = 1/L$ .

**51** Find the convergence radius  $r$  around basepoints  $a = 0$  and  $a = 1$  from the blowup points of  $(1 + \tan x)/(1 + x^2)$ .

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