Rutherford α (Alpha) scattering :- 9/3/2021

Assumptions :-

1. The entire positive charge and mass of the atom is concentrated in a small region of space is called as nucleus.

2. The electrons revolving around the nucleus in stationary orbits.

3. The mass of the nucleus is heavier than that of α - particles

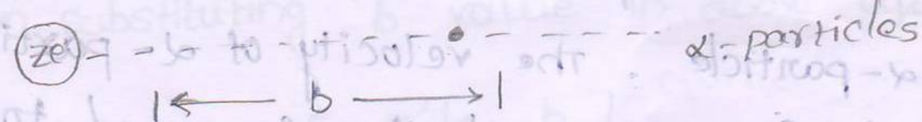
4. Both the nucleus and α - particle are considered to be point charge.

5. The interaction between nucleus and α - particle is due to Electrostatic repulsive force.

Let us consider a nucleus of positive charge 'ze', stationary at a point "G". Here 'z' refers to the atomic number and 'e' refers to the magnitude of charge 'm', velocity 'v_0' and charge '+ze' is projected towards the nucleus in a direction xy. As an alpha (α) particle approaches closer to the nucleus it experiences an electrostatic repulsion force and deflected from its actual path to a hyperbolic path pcp'

Expression for distance of closest approach:-

Let us consider the case in which α-particle is projected directly towards the centre of the nucleus. The α-particle can move up to a certain distance (b) from the nucleus at which the α-particle ~~temp~~ stops for a while and bounces back in reverse direction. In this case the scattering angle $\phi = 180°$ & Impact parametre $p = 0$.



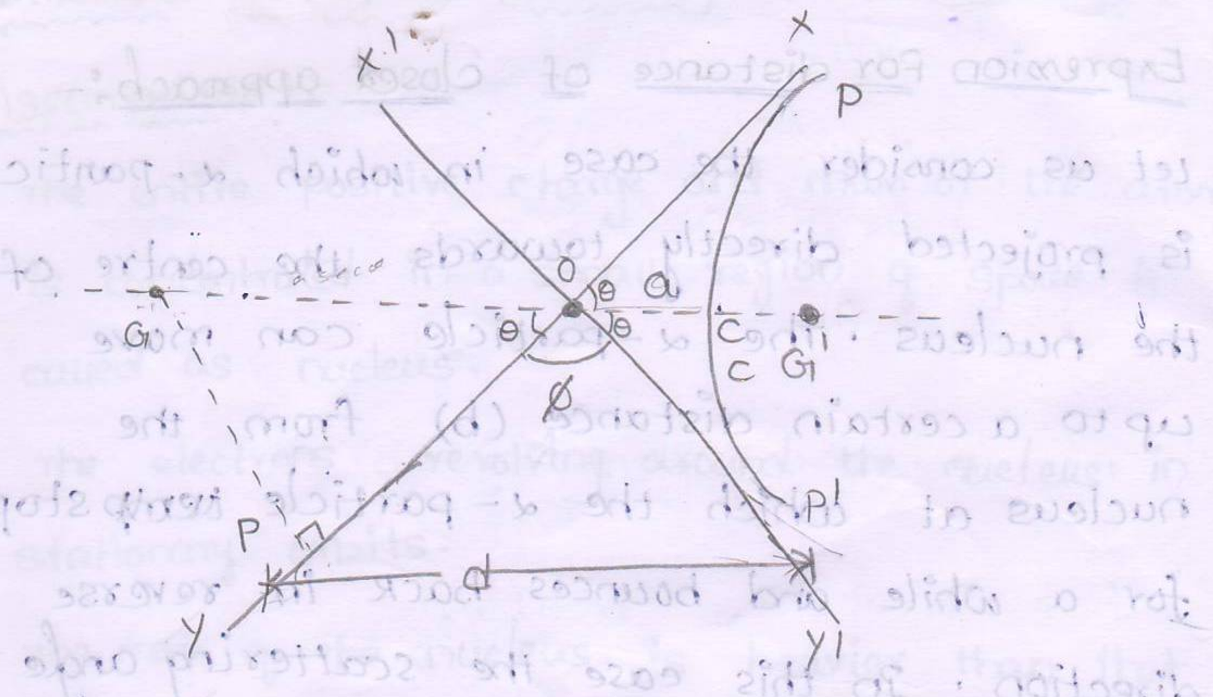
From law of conservation of energy

$$\frac{1}{2} m v_0^2 = \frac{2Ze^2}{4\pi\varepsilon_0 b}$$

$$\boxed{b = \frac{Ze^2}{\pi\varepsilon_0 m v_0^2}} \quad \text{— (1)}$$

Expression for scattering angle ϕ:-

Let us consider the general case in which the α-particle moving towards nucleus with velocity v_0 along xy as shown in the fig. As a particle approaches closer to the nucleus, it experiences Electrostatic repulsion and results deviation of path into hyperbolic, pcp'.



In the above figure. XY and X'Y' are the asymptotes of hyperbola which represents initial and final direction of α-particle. The velocity of α-particle gets decreased from p to c and then increased to initial values from c to p'. The interaction b/w nucleus and α-particles is elastic collision in which both angular momentum and energy of the α-particle are conserved.

Initial angular momentum of α-particles at

Initial angular momentum of α-particle at $\underline{p = mv_0 p}$. Angular momentum of vertex $\underline{c = Mvd}$. From law of conservation of angular momentum we can express velocity of α-particle at "c" as

$$\boxed{mv_0 p = mvd}$$

$$\boxed{v = \frac{v_0 p}{d}} \rightarrow (2)$$

Initial energy of α-particle at $P = \frac{1}{2}mv_0^2$

Energy of α-particle at vertex $C = \frac{1}{2}mv^2 + \frac{2Ze^2}{4\pi\varepsilon_0 d}$

From law of conservation of energy

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{Ze^2}{2\pi\varepsilon_0 d}$$

$$v_0^2 = v^2 + \frac{Ze^2}{\pi\varepsilon_0 md}$$

$$v^2 = v_0^2 - \frac{Ze^2}{\pi\varepsilon_0 md}$$

$$v^2 = v_0^2\left[1 - \frac{Ze^2}{\pi\varepsilon_0 mv_0^2 d}\right]$$

on substituting 'b' value in above equation we get

$$v^2 = v_0^2\left[1 - \frac{b}{d}\right]$$

from eqn (2) we can substitute $v = \frac{v_0 P}{d}$

$$\frac{\cancel{v_0^2} P^2}{d^2} = \cancel{v_0^2}\left[1 - \frac{b}{d}\right]$$

$$P^2 = d(d-b) \rightarrow (3)$$

From law of Hyperbola

eccentricity $e = \frac{OG'}{OC}$ (or) $\frac{GO}{OC}$

$$e = \frac{GO}{OC} = \frac{c}{a}$$

$$\frac{GO}{OC} = \sec\theta$$

$$GO = a\sec\theta$$

we have $GC = d = GO + OC$

$= a\sec\theta + a$

From ΔGNO

$$\frac{p}{a\sec\theta} = \sin\theta \times \sec\theta \times \frac{1}{\cos\theta}$$

$$a = p\cos\cancel{t}\theta$$

Now $GC = d = p\cot\theta \cancel{\times} \sec\theta + p\cot\theta$

$$= p\left[\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right]$$

$$= p\left[\frac{\cancel{2}\cos^2\theta/2}{\cancel{2}\sin\theta/2\cos\theta/2}\right]$$

$$= p\,\frac{\cos\theta/2}{\sin\theta/2}$$

$$d = p\cot\theta/2$$

substitute above value in eq ③ we get

$$p^2 = p\cot\theta/2\,[p\cot\theta/2 - b]$$

$$\frac{p}{\cot\theta/2} = [p\cot\theta/2 - b]$$

$$b = p\cot\theta/2 - p\tan\theta/2$$

$$b = p\left[\frac{\cos\theta/2}{\sin\theta/2} - \frac{\sin\theta/2}{\cos\theta/2}\right]$$

$$b = p\left[\frac{\cos^2\theta/2 - \sin^2\theta/2}{\sin\theta/2\,\cos\theta/2}\right]$$

$$b = p\left[\frac{2(\cos\theta)}{\sin\theta}\right]$$

$b = 2p\,\cancel{\cos\theta}\cot\theta$

$\cot\theta = \dfrac{b}{2p}$

we know that $\theta = \dfrac{180-\phi}{2}$

$= \dfrac{\pi}{2} - \dfrac{\phi}{2}$

$\cot\left(\dfrac{\pi}{2} - \dfrac{\phi}{2}\right) = \dfrac{b}{2p}$

$\tan\left(\dfrac{\phi}{2}\right) = \dfrac{b}{2p}$

$\dfrac{\phi}{2} = \tan^{-1}\left(\dfrac{b}{2p}\right) \Rightarrow \phi = 2\tan^{-1}\left(\dfrac{b}{2p}\right)$

$$\boxed{\phi = 2\tan^{-1}\left(\dfrac{ze^2}{2\pi\varepsilon_0 m v_0^2 p}\right)}$$

= x =

*Motion of Rocket :-

Rocket is an example of system of variable mass. Motion of rocket is mainly based on Newton's third law of motion.

Rocket consists of a fuel chamber and an oxidation chamber. At the tail of Rocket it has a combustion chamber at which the fuel (liquid or solid) burnt. The pressure inside the chamber goes on increasing results expelling out of hot gas with exhaust velocities through the nozzel.

Let us consider a Rocket of mass M and v be the velocity of Rocket at any instant of time from lab frame.

the velocity of gas jet with respect to the rocket is the force on jet is equal to the time rate of change in momentum of jet and expressed as

$$\frac{dm}{dt}(v_{rel}) \quad (or) \quad \frac{dm}{dt}(v-u)$$

Then force is acts as upward thurst on Rocket.

Thus thrust on rocket $T = \frac{dM}{dt}(v-u)$

The downward gravitational force on Rocket $= mg$

The Net external force on Rocket is expressed as

$$F_{ext} = \frac{dM}{dt}(v-u) - Mg \rightarrow (1)$$

from Newton's second law, External force on system of variable mass is given by

$$F_{ext} = \frac{d}{dt}(P) \quad \boxed{P = mv}$$

$$F_{ext} = \frac{d}{dt}(Mv)$$

$$F_{ext} = v\frac{dM}{dt} + M\frac{dv}{dt} \rightarrow (2)$$

Both equations (1) & (2) represents the external force on Rocket, Hence we get

$$\frac{dM}{dt}(v-u) - Mg = v\frac{dM}{dt} + M\frac{dv}{dt}$$

$$-u\frac{dM}{dt} - Mg = M\frac{dv}{dt}$$

Dividing above equation with 'M' on both sides

$$-\frac{U}{M}\frac{dM}{dt} - g = \frac{dv}{dt}$$

$$-\frac{U}{M}\,dM - g\,dt = dv$$

~~P.O.B~~: Integrating above eqn with in the limits v_0 to v

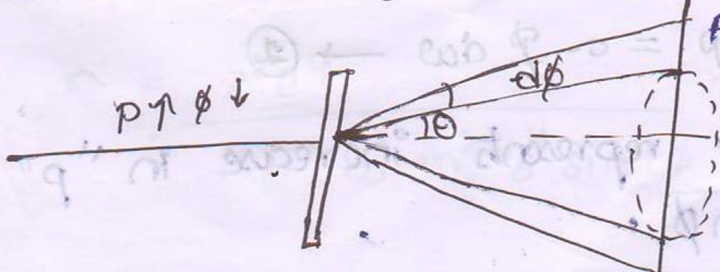
$$-U\int_{M_0}^{M}\frac{dM}{M} - g\int_0^t dt = \int_{v_0}^{v} dv$$

$$-U\left[\log M\right]_{M_0}^{M} - g\left[t\right]_0^t = \left[v\right]_{v_0}^{v}$$

$$v - v_0 = U\left[\log M_0 - \log M\right] - gt$$

$$\boxed{v = v_0 + U\log\frac{M_0}{M} - gt}$$

* Rutherford scattering cross - section :- arc = radius



$\pi(r+h)^2 - \pi r^2$

$\pi(h^2 + 2hr)$

$r\,d\phi$

$r\sin\phi$

Let us consider a beam of α-particles incident normally on a thin gold foil of thickness E. The scattered particles are detected by means of scintillations produced by them on flurocent screen.

Let "n" be the number of atom per unit volume of scattered. Let us suppose Q α-particles are incident normally on scattered.

The probable no. of α-particles which are coming towards the nucleus at a distance of impact parametre "p" can be exprresed as

$$\boxed{\pi p^2 Qnt \;\; (or) \;\; \pi p^2 I}$$

where I is intensity of incident α-particles.

Now the ~~parabola~~ probable no. of α-particle which are having impact parametre ~~as~~ in b/w p & p+dp can be expressed as

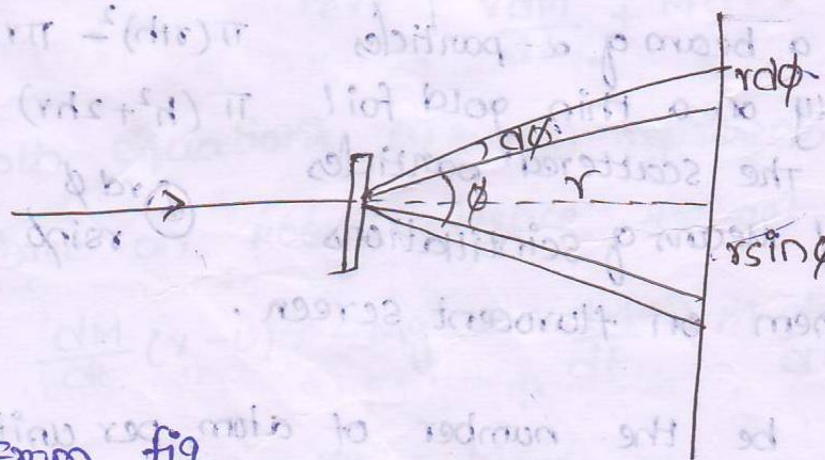
$$d(\pi p^2 Qnt)$$

$$2\pi pQnt \; dp \;\; (or) \;\; 2\pi p I dp$$

This number is equal to the no. of α-particles scattered into solid angle ~~dtheta~~ ($d\omega$) between ϕ & $\phi + d\phi$

Hence $\quad -2\pi p I \, dp = \sigma I \, d\omega \rightarrow (1)$

where '- ve' sign represents increase in "p" causes decrease in ϕ.



From fig

$$d\omega = \frac{dA}{r^2}$$

dA is the stiking area of screen

$$dA = 2\pi r^2 \sin\phi \, d\phi$$

Now solid angle $d\omega = 2\pi \sin\phi \, d\phi$

From eq^n (1) the scattering cross section is expressed as

$$\sigma = \frac{-p\,dp}{\sin\phi \cdot d\phi}$$

we know that

$$p = \frac{ze^2}{2\pi\varepsilon_0 m v_0^2} \cot\frac{\phi}{2}$$

$$dp = \frac{ze^2}{2\pi\varepsilon_0 m v_0^2}\left(-\frac{1}{2}\operatorname{cosec}^2\frac{\phi}{2}\, d\phi\right)$$

Now scattering cross section

$$\sigma = \frac{\left(\frac{ze^2}{2\pi\varepsilon_0 m v_0^2}\right)^2 \frac{1}{2}\operatorname{cosec}^2\frac{\phi}{2}\cot\frac{\phi}{2}\,\cancel{d\phi}}{\sin\phi \cdot \cancel{d\phi}}$$

$$\sigma = \frac{z^2 e^4}{16\pi^2 \varepsilon_0^2 m^2 v_0^4 \sin^4\frac{\phi}{2}}$$

Rutherford scattering formula:-

The no. of α-particles striking the fluorosent screen in unit solid angle is given by

$$2\pi p Qnt\, dp$$

These α-particle striking the screen of area dA in given by

$$dA = 2\pi r^2 \sin\phi \, d\phi$$

The no. of α-particles striking the screen per unit area is expressed as

$$N = \frac{2\pi p Q n t \, dp}{dA}$$

$$N = \frac{\cancel{2\pi} p \, dp \, Q n t}{\cancel{2\pi} r^2 \sin\phi \, d\phi}$$

By substituting p and d dp values we get

$$N = \frac{Q N T \, z^2 e^4}{16\pi^2 \varepsilon_0^2 m^2 v_0^4 r^2 \sin^4 \frac{\phi}{2}}$$

This represents the rutherford scattering formula

x

Mechanical properties of Rigid bodies

Rigid body :-

A body which does not undergo any change in its shape or size under any external forces acted upon it.

Rigid body is an ideal concept. It can not be realized in practice. In general all solid bodies are considered on Rigid bodies. In solid bodies the distance between any of constituent problem will remain unchanged under all conditions.

In general there are two types of motion.

i, Translatoly motion

ii, Rotatory motion

i) Translatory motion :-

A body which is said to be in translatory motion when each particle of the body undergoes same displacement in the same direction.

Ex:- Motion of scale on smooth inclined plane.

ii, Rotatory Motion :-

A body which is said to be an rotatory motion when every particle moves in circular path with its centre we lie on a fixed straight line called axis of rotation.

The axis of rotation may exist inside (or) outside the body.

Ex:- Motion of a fan

Motion of fly wheel etc.

→ A Rigid body may have both translatory and rotatory motion simultaneously at the same time

→ The motion of a rigid body can be described by the combination of rotatory motion and Translatory motion.

Ex:- Motion of spherical ball on a smooth inclined plane. where the centre of mass expriences translatory motion rest of the particles rotates about centre of mass.

5m/ Rotational kinematic Relations:-

Let us consider a particle "P" in a rigid body the motion of the particle can be described by two co-ordinates r & θ where 'r' is position vector of particle, 'θ' is the angular orientation from a fixed point.

Angular displacement:-

Let the particle move from P_1 to P_2 the angular displacement can be expressed as

$$\theta = \frac{|P_2 - P_1|}{r}$$

$$\theta = \frac{arc}{r}$$

It can be measured in radius.

Angular velocity :-

It is the time rate of change in angular displacement of the particle. let the particle 'p' initally at P_1 makes θ_1 angle at t_1 time, after some instant it makes θ_2 angle at P_2. The angular displacement by the particle with in the time interval $(t_2 - t_1)$ can be caused as its angular velocity.

$$\omega = \frac{|\theta_2 - \theta_1|}{|t_2 - t_1|}$$

$$\omega = \frac{\Delta\theta}{\Delta t}$$

This represents the ~~av~~ average angular velocity the instantaneous angular velocity can be measured with in the limits of $\Delta t \to 0$

$$\omega_{inst} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

$$\omega_{inst} = \frac{d\theta}{dt}$$

It can be measured in radius/sec

Angular acceleration α :-

It is the time rate of change in angular velocity of the particle "p" initially having angular velocity ω_1 at time t_1. After some instant the angular velocity ~~with~~ changes to ω_2 and t_2. The change in angular velocity with in the time interval $(t_2 - t_1)$ can be called angular acceleration

$$\alpha = \frac{|\omega_2 - \omega_1|}{|t_2 - t_1|}$$

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

This represents the ~~avg~~ average angular acceleration. The instaneous angular acceleration can be measured with in the limit $\Delta t \rightarrow 0$.

$$\alpha_{inst} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\omega}{\Delta t}$$

$$\alpha_{inst} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

It is measured in radian/sec^2

Angular momentum :-

Let us consider a particle of a rigid body which rotatory about a fixed axis with angular velocity "ω" the particle having the position vector $\vec{r}$ from the fixed point. The angular momentum of the particle may be defined as the vector product of position vector and the linear momentum of the particle. It can be represented by $\vec{L}$

$$\vec{L} = \vec{r} \times \vec{P}$$

$$\vec{L} = m(\vec{r} \times \vec{V})$$

* It can be measured in kgm^2/sec
* It can also be defined as the moment of linear momentum.

Torque :- (49)

Torque of the particle may be defined as the vector product of position vector and amount of force applied for designed angular acceleration.

$$\vec{T} = \vec{r} \times \vec{F}$$

It can be measured in N-M. It can also be defined as the moment of force. It can also be defined as the time rate of change in angular momentum of the particle.

$$T = \frac{d\vec{L}}{dt}$$

It represents the instaneous torque.

Equation motion of Rigid body :-

Let us consider the case of Rigid body which is rotating with angular velocity "ω". about a fixed axis passing through the point "O". Let us "i th" particle at P of mass m_i having position vector $\vec{r_i}$ from the fixed point 'O'.

Each particle moves in circular path and its centre lie on axis of rotation the linear momentum of the particle directed along the tangent of circular path. The angular momentum of the particle can be defined as the vector product of position vector and linear momemtum vector. Let $\vec{L_i}$ be the angular momentum of the particle then

$$\vec{l_i} = \vec{r_i} \times \vec{P_i}$$

$$\vec{l_i} = \vec{r_i} \times \overrightarrow{m_i \times v_i}$$

$$\vec{l_i} = m_i \left[\vec{r_i} \times (\vec{v_i}) \right] \quad [v_i = \omega \times r_i]$$

$$\vec{l_i} = m_i \left[\vec{r_i} \times (\omega \times \vec{r_i}) \right]$$

Now, The angular momentum of entire rigid body can be expressed as the summation of individual angular momentum of all consistent particle about the axis of rotation.

$$\vec{L} = \sum_{i=1}^{n} \vec{l_i}$$

$$= \sum_{i=1}^{n} m_i \left[\vec{r_i} \times (\vec{\omega} \times \vec{r_i}) \right]$$

$$\vec{L} = \sum_{i=1}^{n} m_i \left[(\vec{r_i} \cdot \vec{r_i}) \vec{\omega} + (\vec{\omega} \cdot \vec{r_i}) \right]$$

w.k.t $\vec{\omega} \cdot r_i = 0$, when the axis of rotation coincides with principal axis of Inertia.

Hence, $\vec{L} = \sum_{i=1}^{n} m_i \left[r_i^2 \vec{\omega} \right]$

$$\vec{L} = \left[\sum_{i=1}^{n} m_i r_i^2 \right] \omega$$

$$\vec{L} = I \, \vec{\omega} \rightarrow (1)$$

where I is moment of Inertia of rigid body.

It can be defined as the amount of torque required for desired angular acceleration.

let us differentiate Eqn ① we get

$$\frac{d\vec{L}}{dt} = I \cdot \frac{d\vec{\omega}}{dt}$$

$$\frac{d\vec{L}}{dt} = I \cdot \vec{\alpha} \rightarrow (2)$$

$\boxed{(uv)' = uv' + vu'}$

we know that, $\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$

$$= \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} \quad \left[\because \frac{d\vec{r}}{dt} \times \vec{p} = 0\right]$$

$$= \vec{r} \times \vec{F}$$

From the definition of torque $\vec{T} = \vec{r} \times \vec{F}$

$$\frac{d\vec{L}}{dt} = \vec{T} \rightarrow (3)$$

Eqn (2) & eq (3) represents same relation

Hence $\vec{T} = I\vec{\alpha}$

This represents the eqn of motion of rotating rigid body.

10m

** **Euler Equation / Oiler Equations:-**

The eqn of motion of a rigid rotating body in an inertial frame of reference derived by the relation between torque and angular acceleration of the body. This relation is expressed as

$$\boxed{\vec{T} = I\vec{\alpha}}$$

this is the case in which the axis of rotation of the rotating body co-incides with the principle axis of inertia.

If the reference frame considered to be placed inside the rotating body It constitutes a non-inertial frame. Now, we have to find the eqn of motion for a rigid rotating body in this non-inertial frame of ~~refines~~ reference. To Simply this problem we have to express a relation between space co-ordinates and body co-ordinates. This relation is expressed below as

$$\left(\frac{d-}{dt}\right)_{space} = \left(\frac{d-}{dt}\right)_{body} + \omega \times \text{—} \quad (1)$$

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L} \quad (2)$$

we know that Angular momentum vector of a symmetric vector body about a fixed axis of rotation

$$\vec{L} = L_x \hat{i} + L_y \hat{j} + L_z \hat{k}$$

$$\vec{L} = I_x \omega_x \hat{i} + I_y \omega_y \hat{j} + I_z \omega_z \hat{k}$$

Differentiating w.r.t to time, we get

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = \frac{d}{dt}\left[I_x \omega_x \hat{i} + I_y \omega_y \hat{j} + I_z \omega_z \hat{k}\right]_{body} + I_x \omega_x \frac{d\hat{i}}{dt} + I_y \omega_y \frac{d\hat{j}}{dt} + I_z \omega_z \frac{d\hat{k}}{dt} \quad (3)$$

we know that

$$\frac{d\vec{r}}{dt} = \vec{v} = \vec{\omega} \times \vec{r} =$$

$$\Rightarrow \frac{di}{dt} = \omega \times \vec{i}$$

$$\Rightarrow \frac{dj}{dt} = \omega \times \vec{j}$$

$$\Rightarrow \frac{dk}{dt} = \omega \times \vec{k}$$

sub these values in eq (3), we get

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = \frac{d}{dt}\left[I_x \frac{d\omega_x}{dt}\hat{i} + I_y \frac{d\omega_y}{dt}\hat{j} + I_z \frac{d\omega_z}{dt}\hat{k}\right]_{body}$$

$$+ \omega \times \left[I_x \omega_x \hat{i} + I_y \omega_y \hat{j} + I_z \omega_z \hat{k}\right]$$

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \omega \times \vec{L}$$

let Torque $\vec{T} = T_x\hat{i} + T_y\hat{j} + T_z\hat{k}$

$$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}, \quad \vec{L} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$$

$$|\vec{\omega} \times \vec{L}| = \begin{vmatrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ L_x & L_y & L_z \end{vmatrix}$$

$$|\vec{\omega} \times \vec{L}| = [\omega_y L_z - \omega_z L_y]\hat{i} + j[\omega_z L_x - \omega_x L_z] + \hat{k}[\omega_x L_y - \omega_y L_x]$$

$$\vec{T} = \left(\frac{d\vec{L}}{dt}\right) = \left[I_x \frac{d\omega_x}{dt} + \omega_y L_z - \omega_z L_y\right]\hat{i} + \left[I_y \frac{d\omega_y}{dt} + \omega_z L_x - \omega_x L_z\right]\hat{j} + \hat{k}\left[I_z \frac{d\omega_z}{dt} + \omega_x L_y - \omega_y L_x\right]$$

$$T_x = I_x \frac{d\omega_x}{dt} + \omega_y L_z - \omega_z L_y$$

$$T_y = I_y \frac{d\omega_y}{dt} + \omega_z L_x - \omega_x L_z$$

$$T_z = I_z \frac{d\omega_z}{dt} + \omega_x L_y - \omega_y L_x$$

These three set of equations are called as Eulers equation.

Applications of Euler Equation :-

Law of conservation of energy :-

If $T = 0 \Rightarrow T_x = T_y = T_z = 0$

$$I_x \frac{d\omega_x}{dt} + \omega_y L_z - \omega_z L_y = 0 \rightarrow (1)$$

$$I_y \frac{d\omega_y}{dt} + \omega_z L_x - \omega_x L_z = 0 \rightarrow (2)$$

$$I_z \frac{d\omega_z}{dt} + \omega_x L_y - \omega_y L_x = 0 \rightarrow (3)$$

Multiply equations (1), (2) & (3) with $\omega_x, \omega_y, \omega_z$ respectively and adding results.

$$I_x \frac{d\omega_x}{dt}\omega_x + \omega_x \cancel{\omega_y} L_z - \omega_x \cancel{\omega_z} L_y + I_y \omega_y \frac{d\omega_y}{dt}$$
$$+ \omega_y \cancel{\omega_z} L_x - \omega_y \cancel{\omega_x} L_z + I_z \omega_z \frac{d\omega_z}{dt} + \omega_x \cancel{L_y} \omega_z$$
$$- \omega_y \cancel{\omega_z} L_x = 0$$

$$I_x \omega_x \frac{d\omega_x}{dt} + I_y \omega_y \frac{d\omega_y}{dt} + I_z d\omega_z \frac{d\omega_z}{dt} = 0$$

we know that $x \frac{dx}{dt}$ can be written as $\frac{d}{dt}\left(\frac{x^2}{2}\right)$

$$\frac{1}{2}\left[\frac{d}{dt}\left(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2\right)\right] = 0$$

$K.E = \frac{1}{2}mv^2$

$$\frac{1}{2}\left[\frac{d}{dt}(E)\right] = 0 \quad \Rightarrow E = \text{constant}$$

$I\omega^2 = 2K$ $\boxed{\frac{1}{2}I\omega^2 = K}$

Law of conservation of Angular momentum :-

If $\tau = 0 \Rightarrow \tau_x = \tau_y = \tau_z = 0$

$$I_x \frac{d\omega_x}{dt} + \omega_y L_z - \omega_z L_y = 0 \rightarrow (1)$$

$$I_y \frac{d\omega_y}{dt} + \omega_z L_x - \omega_x L_z = 0 \rightarrow (2)$$

$$I_z \frac{d\omega_z}{dt} + \omega_x L_y - \omega_y L_x = 0 \rightarrow (3)$$

Multiply Equations (1), (2), & (3) with $I_x\omega_x$, $I_y\omega_y$, $I_z\omega_z$ respectively and adding results

$$I_x^2 \frac{d\omega_x}{dt}\omega_x + I_x\omega_x\omega_y L_z - I_x\cancel{\omega_z} L_y + I_y^2\omega_y\frac{d\omega_y}{dt} + I_y\omega_y\cancel{\omega_z}L_x - I_y\omega_y\omega_x L_z + I_z^2\omega_z\frac{d\omega_z}{dt} + I_z\omega_x L_y\omega_z - I_z\omega_y\cancel{\omega_z}L_x = 0$$

$$I_x^2\omega_x\frac{d\omega_x}{dt} + I_y^2\omega_y\frac{d\omega_y}{dt} + I_z^2\omega_z\frac{d\omega_z}{dt} = 0$$

we know that $x\frac{dx}{dt}$ can be written as $\frac{d}{dt}\left(\frac{x^2}{2}\right)$

$$\frac{1}{2}\left[\frac{d}{dt}\left(I_x^2\omega_x^2 + I_y^2\omega_y^2 + I_z^2\omega_z^2\right)\right] = 0$$

$$\frac{1}{2}\left[\frac{d}{dt}(L^2)\right] = 0 \Rightarrow L = \text{constant}$$

10M

Angular momentum and Inertia tensor :-

Let us consider a rigid body rotating about a fixed axis with angular velocity 'ω'. The equation of motion of the rigid body can be expressed is a simple relation b/w Torque and angular acceleration. In this relation we have considered that the axis of rotation is coincides with the principle axis of Inertia

i.e $\vec{\omega} \cdot \vec{r} = 0$

Let us consider the case in which the axis of rotation doesnot coincide with the principle axis of Inertia, where $\vec{\omega} \cdot \vec{r} \neq 0$.

The angular velocity and angular momentum have components along three fixed co-ordinate

we know that, Angular momentum vector

$$\boxed{\vec{L} = \vec{r} \times \vec{P}}$$

$$\vec{L} = \sum_{i=1}^{n} m_i r_i \times v_i \qquad [v_i = \omega \times r_i]$$

$$\vec{L} = \sum_{i=1}^{n} m_i [r_i \times (\omega \times r_i)] \qquad [a \times b \times c = bAC - CAB]$$

$$\vec{L} = \sum_{i=1}^{n} m_i [(r_i \cdot r_i)\omega - (r_i \omega) r_i] \rightarrow ①$$

In the special case $\vec{\omega} \cdot \vec{r_i} = 0$, where axis of rotation coincides with principle axis of Inertia.

But in the general form $\vec{\omega} \cdot \vec{r_i} \neq 0$.

In the component form,

position vector $\vec{r_i} = x\hat{i} + y\hat{j} + z\hat{k}$

Angular vector $\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$.

Angular momentum $\vec{L} = L_x \hat{i} + L_y \hat{j} + L_z \hat{k}$

$$\vec{r_i} \cdot \vec{r_i} = x^2 + y^2 + z^2 = r^2$$

$$\vec{r_i} \cdot \vec{\omega} = (x\hat{i} + y\hat{j} + z\hat{k})(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k})$$

$$\vec{r_i} \cdot \vec{\omega} = x\omega_x + y\omega_y + z\omega_z$$

Substitute above values in eqn ①.

$$\vec{L} = \sum_{i=1}^{n} m_i \left[(x^2+y^2+z^2)(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}) - (x\omega_x + y\omega_y + z\omega_z)(x\hat{i} + y\hat{j} + z\hat{k}) \right]$$

$$\vec{L} = \sum_{i=1}^{n} m_i \begin{bmatrix} x^2\omega_x \hat{i} + x^2 \omega_y \hat{j} + x^2 \omega_z \hat{k} + y^2 \omega_x \hat{i} + \\ y^2 \omega_y \hat{j} + y^2 \omega_z \hat{k} + z^2 \omega_x \hat{i} + \\ z^2 \omega_y \hat{j} + z^2 \omega_z \hat{k} \end{bmatrix}$$

$$- \begin{bmatrix} x^2\omega_x \hat{i} + xy\omega_x \hat{j} + x\omega_x z\hat{k} + y\omega_y x\hat{i} + \\ y^2 \omega_y \hat{j} + y \cdot \omega_y z\hat{k} + xz\omega_z \hat{i} + \\ y \cdot z\omega_z \hat{j} + z^2 \omega_z \hat{k} \end{bmatrix}$$

$$\vec{L} = \sum_{i=1}^{n} \Big(\cancel{x^2\omega_x \hat{i}} + x^2\omega_y \hat{j} + x^2 \omega_z \hat{k} + y^2 \omega_x \hat{i} + \cancel{y^2 \omega_y \hat{j}} + y^2 \omega_z \hat{k} + z^2 \omega_x \hat{i} + z^2 \omega_y \hat{j} + \cancel{z^2 \omega_z \hat{k}} - \cancel{x^2 \omega_x \hat{i}} - xy\omega_x \hat{j} - x\omega_x z\hat{k} - y\omega_y x\hat{i} - \cancel{y^2\omega_y \hat{j}} - y \cdot \omega_y z\hat{k} - x \cdot z\omega_z \hat{i} -$$

$$y \cdot z\omega_z \hat{j} - z^2 \cancel{\omega_z} \hat{k} \Big]$$

$$\vec{L} = \sum_{i=1}^{n} m_i \Big[(y^2\omega_x + z^2\omega_x - x\cdot y\,\omega_y - x\cdot z\,\omega_z)\hat{i} + (x^2\omega_y + z^2\omega_y - x\cdot y\,\omega_x - y\cdot z\,\omega_z)\hat{j} + (x^2\omega_z + y^2\omega_z - xz\,\omega_x - y\cdot z\,\omega_y)\hat{k} \Big]$$

we know that

$$\vec{L} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$$

$$L_x = \sum_{i=1}^{n} m_i \Big[(y^2\omega_x + z^2\omega_x) - xy\cdot\omega_y - x\cdot z\cdot\omega_z \Big]$$

$$L_y = \sum_{i=1}^{n} M_i \Big[x^2\omega_y + z^2\omega_y - xy\,\omega_x - yz\cdot\omega_z \Big]$$

$$L_z = \sum_{i=1}^{n} M_i \Big[x^2\cdot\omega_z + y^2\cdot\omega_z - xz\cdot\omega_x - yz\cdot\omega_y \Big]$$

Then eqn can be express in matrix from

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} M_i(y^2+z^2) & -\sum_{i=1}^{n} M_i\,xy & -\sum_{i=1}^{n} M_i\,xz \\ -\sum_{i=1}^{n} M_i\,xy & \sum_{i=1}^{n} M_i(x^2+z^2) & -\sum_{i=1}^{n} M_i\,yz \\ -\sum_{i=1}^{n} M_i\,xz & -\sum_{i=1}^{n} M_i\,yz & \sum_{i=1}^{n} M_i(x^2+y^2) \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$\to r^2 - x^2$

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} (r^2-x^2) & -\sum_{i=1}^{n} M_i\,xy & -\sum_{i=1}^{n} m_i\,xz \\ -\sum_{i=1}^{n} M_i\,xy & \sum_{i=1}^{n} M_i(x^2+y^2) & -\sum_{i=1}^{n} m_i\,yz \\ -\sum_{i=1}^{n} M_i\,xz & -\sum_{i=1}^{n} M_i\,yz & \sum_{i=1}^{n} M_i(x^2-z^2) \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$$

where I_{xx}, I_{xy} ..... etc are the co-efficients of inertia. I_{xx}, I_{yy}, I_{zz} are the principles of inertia and rest of the elements are product of inertia.

$$I_{xy} = I_{yx}$$

$$I_{zx} = I_{xz}$$

$$I_{zy} = I_{yz}$$

Inertia tension $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

It is ~~rec~~ a matrix of rank "2" & called as tensor. A rank 2 matrix have a components A rank 1 matrix have 3 components and called as vector.

A rank 0 matrix have 1 component and called as scalar

For symmetric body all of-diagonal elements will be zero.

i.e $I_{xy} = I_{yx} = 0$

$I_{yz} = I_{zy} = 0$

$I_{xz} = I_{zx} = 0$

$$\therefore I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}.$$

10m

** Symmetric top :-

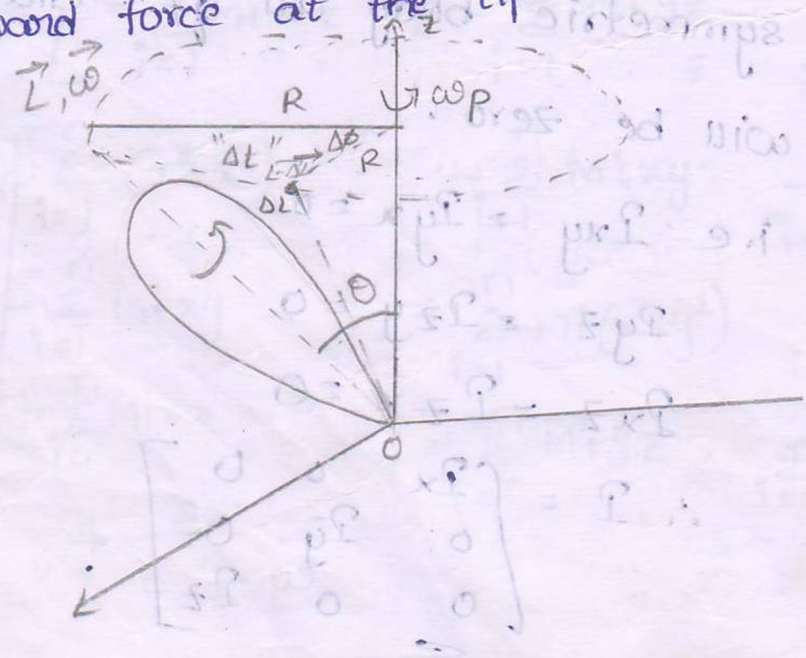
Any symmetric body spinning about it axis of symmetry which is fixed at one point, it is called as top (or) symmetric top.

Processional motion of symmetric top in gravitational feild :-

Let us consider a symmetry top which is spinning about its axis of symmetry with angular velocity $\vec{\omega}$. Its tip lie on point "O" which is origin of the reference inertial frame. The angular velocity vector $\vec{\omega}$ and angular momentum vector $\vec{L}$ directed along axis of rotation. Let "m" be the mass of symmetric top. The force acted upon the symmetric top are

(i) Downward force due to gravitational pull (mg) which exerts a torque on the top.

(ii) upward force at the tip excerts no torque.



The Net torque exacted on the symmetric top by gravitational force is expressed as

$$\vec{T} = \vec{r} \times \vec{F}$$

where $\vec{r}$ is the position vector of centre of mass 'c' from point 'O'.

The magnitude of Torque is given by

$$|T| = r \cdot F \sin(180 - \theta)$$

$$|T| = mgr\sin\theta \longrightarrow (1)$$

Due to this net torque there is a change in angular momentum along direction of torque. Let $\vec{L}$ be the angular momentum of the ~~torque~~ top at any instant of time. After some time interval 'Δt' the angular momentum vector changes to $(L - \Delta L)$. This change is along circular path above fixed vertical axis 'OZ'. The torque can be expressed as

$$T = \frac{\Delta L}{\Delta t} \longrightarrow (2)$$

The precessional velocity of the angular momentum vector about vertical axis '$\vec{OZ}$' is

$$\omega_p = \frac{\Delta \phi}{\Delta t}$$

from fig $\Delta \phi = \dfrac{\Delta L}{R}$

and $R = L\sin\theta$

Now, $$\omega_p = \frac{\Delta L}{L\sin\theta \cdot \Delta t}$$

From eq(2)

$$\omega_p = \frac{\tau}{L\sin\theta}$$

we know that Torque exerted on the top

$$\tau = mgr\sin\theta$$

Hence, precessional angular velocity

$$\omega_p = \frac{mgr\cancel{\sin\theta}}{L\cancel{\sin\theta}}$$

$$\boxed{\omega_p = \frac{mgr}{L}}$$

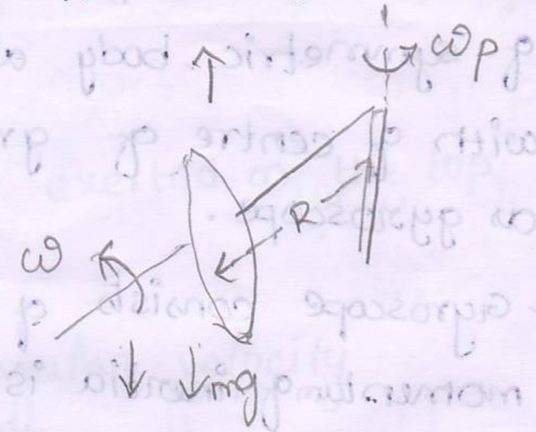
* Precessional velocity is independent of angle b/w axis of rotation and vertical axis
* The precessional angular velocity is inversly propotional to the angular momentum of the symmetric top.

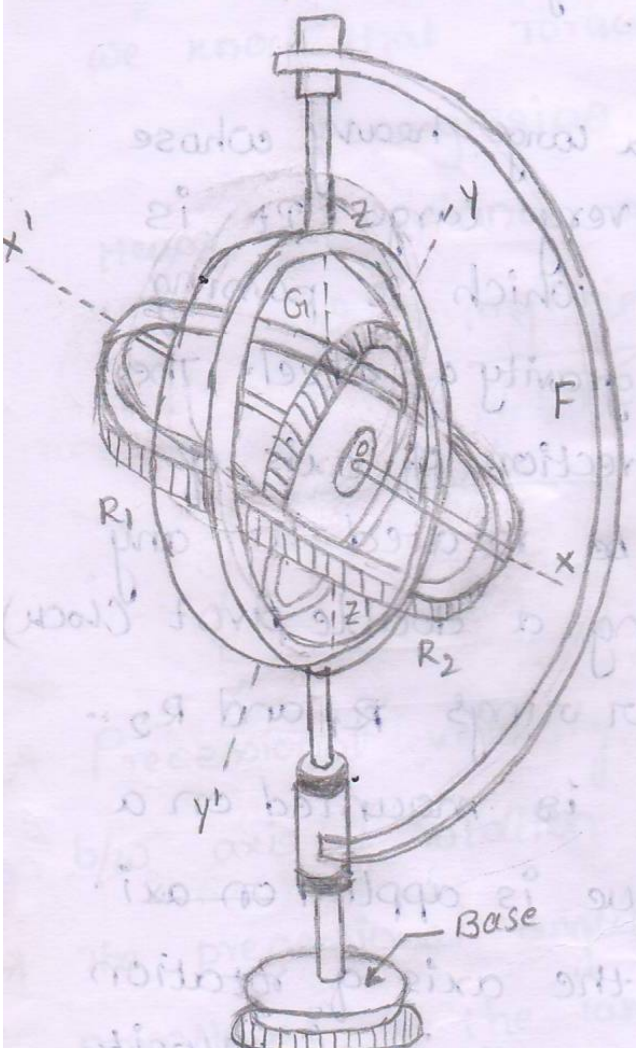
Gyroscope :-

→ A symmetric top is a symmetric body spinning about its axis which is fixed at one point. If the fixed point, about which the spinning of symmetric body ahead its axis is coincides with of centre of gravity then it is called as gyroscope.

→ Gyroscope consists of a large heavy whose momentum of inertia is very large. It is spinning about axial which is passing through the centre of gravity of wheel. The axial represents the direction of axis of rotation. The axial can be rotated in any of the direction by using a double pivot (lock) provided by ~~to~~ 2 circular rings R_1 and R_2. The whole arrangement is mounted on a frame "F". If any torque is applied ~~on axi~~ ~~along~~ perpendicular to the axis of rotation it causes a change in precessional velocity. If ~~the wheel is~~ angular momentum of the wheel is large, then there is less precession we can observe. So the gyroscope can be charcterised by stablity of axis of rotation

→ The principle of gyroscope is mostly use in construction of gyro compass which is used in submarines, shops or aeroplanes to intex the geographical North of the earth. The gyrocompass is constructed in such away that its ~~axis~~ always sets itself with its axis parllel to the axis of earth.





The torque due to gravitational field

$$\tau = mgR$$

we know that

precessional velocity

$$\omega_p = \frac{\tau}{I\omega}$$

$$\omega_p = \frac{mgR}{I\omega}$$

$$\omega_p = \frac{\cancel{m}gR}{\cancel{m}K^2\omega} \quad (1)$$

$$\omega_p = \frac{gR}{K^2\omega}$$

The time period of precession $T = \frac{2\pi}{\omega_p}$

$$T = \frac{2\pi K^2\omega}{gR}$$

* when the precessional motion starts at high rate the axle moves upward or raises.
* when the precessional motion starts at low rate the axle moves downward falls.
* The ossciliation of the axile along up and down is called nutation. so that the gyroscope has three motions (1) Rotation
(2) precession
(3) Nutation

Precession of spinning atom in magnetic feild :-

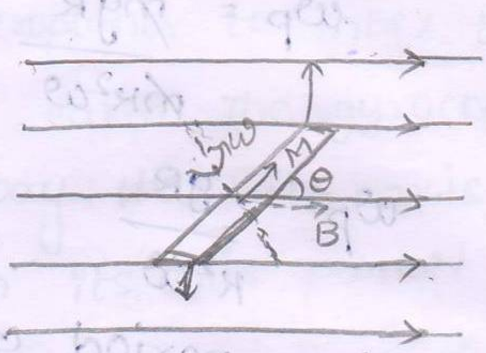
Atoms behave like a tiny magnets and having magnetic moments along the direction of spinning consider a spinning atom placed in a magnetic feild of induction 'B'. The torque acting upon the spinning atom can be expressed as the vector product of magnetic momentum vector and external magnetic induction vector.

$$\tau = M \times B$$

$$\tau = MB\sin\theta \rightarrow ①$$

$$\omega_p = \frac{MB\cancel{\sin\theta}}{L\cancel{\sin\theta}}$$

$$\boxed{\omega_p = \frac{MB}{L}}$$



we know that the precessional velocity axis of rotation of the atom about an external magnetic feild

$$\omega_p = \frac{\tau}{L\sin\theta}$$

From eq ① we get

$$\omega_p = \frac{MB\cancel{\sin\theta}}{L\cancel{\sin\theta}}$$

$$\boxed{\omega_p = \frac{MB}{L}}$$

* The precessional velocity of the spinning atom is directly propotional to the external magnetic induction.
* the precessional velocity of the spinning atom is inversly propotional to the angular momentum.

Precession of equinoxes :-

we know that the plane of equitorial of earth and the plane of orbit of earth making 23.5° angle with each other. Those two planes are intersect at two points A & B; point "A" is known as vernal eqcinox while point 'B' is known as automnal Equinox. vernal equinox observed on 21^{st} march while automnal equinox observed on 22^{nd} september. The line joining these two points A & B is called as line of equinoxes on equinoxes day, day time is equal to night time.

It is well known fact that, the earth is not a perfect sphere. The gravitational force due to sun & moon on earth are not equal. Hence a torque is acted upon axis of the earth causes precession of time of equinoxes.

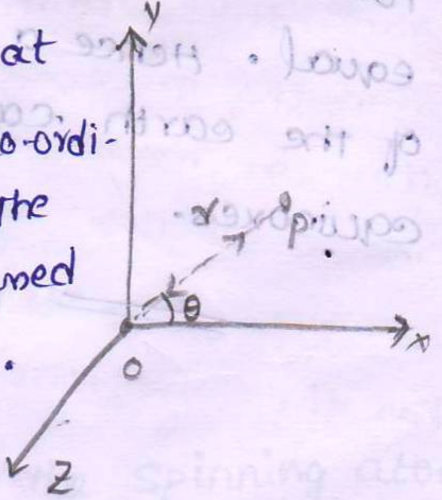
x

Unit-2 Central Force 25/3/21

5M MOTION IN CENTRAL FORCE FIELDS:-

Central force :-

central force is defined as a force which always acting upon a body or a particle towards or away from a fixed point. This ~~con~~ magnitude of central force depends upon the distance of the particle from the fixed point. This fixed point is called as central force.



* consider a centre of force lie at point 'o' which is origin of the co-ordinate system in inertial frame. The position of the particle p expressed in polar co-ordinates 'r and θ'.

* The central force acting upon the particle is expressed as '$\vec{F}$' mathematically the central force is expressed as $\vec{F} = f(r)\hat{r}$ where $f(r)$ is the magnitude of central force and is a function of distance of the particle from the fixed point. where $\hat{r}$ is unit vector along radial direction.

Examples of central force

Ex :-1

A gravitational force on a body by another stationary body is a central force. consider two bodies of masses m_1 and m_2 which are seperated with a distance 'r'. The gravitational attraction force on m_1 by mass m_2 is expressed as

$$\vec{F} = -\frac{G m_1 m_2}{r^2} \hat{r}$$

where (-ve) sign represents the force is attractive

$$\vec{F} = \frac{c}{r^2} \hat{r} \quad \text{where } c = -G m_1 m_2$$

$$\therefore \vec{F} \propto \frac{1}{r^2} \hat{r}$$

$$\text{where } f(r) = \frac{1}{r^2}$$

The gravitational force between sun and revolving earth is a central force.

~~An elect~~

Ex-2

An electrostatic force on a charged particle by another stationary charged particle is a central force

consider two particles of a charges q_1 and q_2 which are seperated with a distance 'r'. The electrostatic force between these two charges can be expressed as

$$\vec{F} = \frac{K q_1 q_2}{r^2} \hat{r}$$

$$\vec{F} = \frac{c}{r^2} \hat{r}$$

where $c = K q_1 q_2$

$$\vec{F} \propto \frac{1}{r^2} \hat{r}$$

where $F(r) = \frac{1}{r^2}$

Ex-3

The Elastic force in a stretched spring of spring constant (k) is a central force. it is expressed as

$$F = -kx$$

where x is distance b/w stretched and unstretched position.

Main features of central force :-

1) the general form of central force is represented by

$$F = \hat{r} f(r),$$

where $f(r)$ is a function of distance r of the particle from the fixed point and $\hat{r}$ is a unit vector along the radius vector 'r' of the particle with respect to that fixed point.

2) central force is a conservative force i,e the work done by the force in moving a particle from one point to another is independent of the path followed.

3) under a central force, the torque acting on the particle is always zero.

4) under a central force, the angular momentum of the particle remains constant.

5, under a central force, the areal velocity of the particle remains constant.

6) The central force is attractive when $f(r) < 0$, i,e, negative and repulsive when $f(r) > 0$, i,e, positive.

10m * *Equation of a motion of a particle under central force :-

consider a particle moving under a central force $\left[\vec{F}\left(F(r)\hat{r}\right)\right]$. The force is always acting towards (or) away from a fixed point - this fixed point is called as central force. Let $\left[\vec{r}(r.\hat{r})\right]$ be the radius vector of the particle from centre of force.

The force acting up on the particle is expressed as $\vec{F} = m\vec{a_r}$.

where "ar" is an radial acceleration of the particle.

$$\vec{F} = m\vec{r}$$

$$\vec{F} = m(\ddot{r} - r\dot{\theta}^2)\hat{r} \rightarrow (1)$$

From definition of central force

$$\vec{F} = F(r).\hat{r} \longrightarrow (2)$$

eq (1) and (2) represents the same

Hence $m(\ddot{r} - r\dot{\theta}^2) = F(r)$

$$\ddot{r} - r\dot{\theta}^2 = \frac{F(r)}{m}$$

$$\ddot{r} - r\dot{\theta}^2 = P$$

where P is force per unit mass

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = P \rightarrow (3)$$

Let $r = \frac{1}{u}$

D.w.r.t

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt}$$

$$\frac{dr}{dt} = -r^2 \frac{du}{dt}$$

$$\frac{dr}{dt} = -r^2 \frac{du}{d\theta}\left(\frac{d\theta}{dt}\right)$$

we know that $r^2\dot{\theta} = h$

$$\frac{dr}{dt} = -\cancel{r^2}\frac{h}{\cancel{r^2}}\frac{du}{d\theta}$$

$$\frac{dr}{dt} = -h\frac{du}{d\theta}$$

D.w.r.t

$$\frac{d^2r}{dt^2} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right)$$

$$\frac{d^2r}{dt^2} = -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

we know that

$$\dot{\theta} = \frac{h}{r^2}$$

$$\frac{d^2r}{dt^2} = -h \times \frac{h}{r^2} \times \frac{d^2u}{d\theta^2}$$

$$\frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\theta^2}$$

Eliminate 'r' by substitute above value in eq ③

we get

$$-h^2u^2\frac{d^2u}{d\theta^2} - r\left(\frac{h}{r^2}\right)^2 = P$$

$$-h^2u^2\frac{d^2u}{d\theta^2} - \cancel{r}\frac{h^2}{r^{\cancel{4}3}} = P$$

$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{h^2}{r^3} = P$$

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = P$$

$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{P}{h^2u^2}} \quad \therefore P = \frac{f(r)}{m}$$

This represents the eqn of the motion of a particle under a central feild.

10m

*Conservative nature of central forces

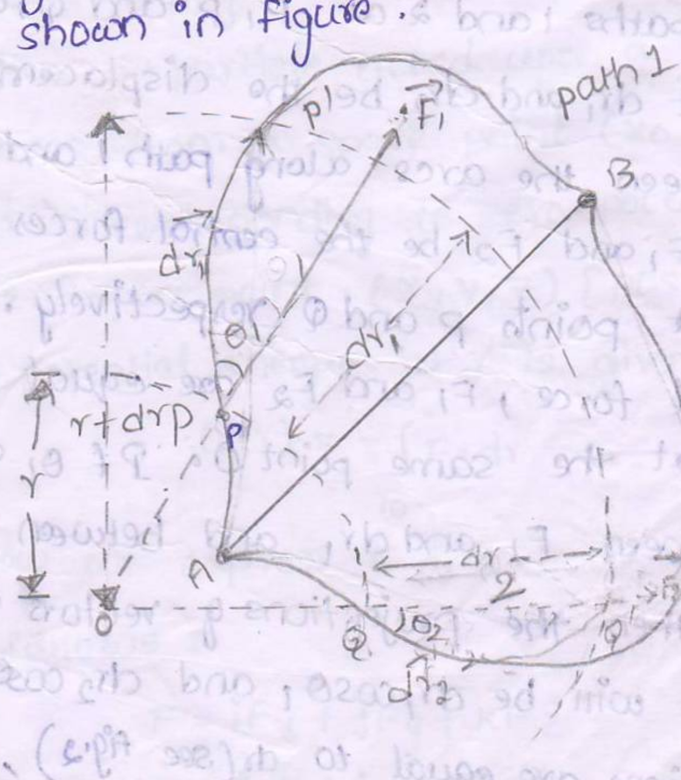
conservative force :-

A force is said to be conservative when the work done by the force in moving a particle from a point A to a point B is independent of the path followed between A and B is the same for all the paths.

The work done depends only on the particle's initial and final positions. In addition, the work done by a conservative force along a closed path is zero.

Explanation :-

consider a particle is taken from point A to point B through the path APP'B (path 1) or AQQ'B (path 2) as shown in figure.



The amount of work done by a force F is given by

$$\int_A^B F \cdot dr = \int_A^B A \cdot dr$$

If the work done along the two paths is the same, then the force is known as conservative. Thus, for a conservative force.

$$\underset{\text{path } 1}{\int_A^B F \cdot dr} = \underset{\text{path } 11}{\int_A^B F \cdot dr}$$

Central force is a conservative force :-

Now we shall prove that central force is a conservative force. consider two points A and B connected by two arbitary paths 1 and 2, as shown in fig (2). let a point particle move from point A to point B along any path under a central force which is directed away from a point 'O'. Taking 'O' as the centre, draw two ~~acrs~~ arcs of radii r and r + dr respectively. these are shown by dotted lines in fig. (2). these arcs of the circles cut the paths 1 and 2 at P, P' and Q, Q' respectively. As shown, let dr_1 and dr_2 be the displacements of the particle between the arcs along path 1 and path 2 respectively. Let F_1 and F_2 be the central forces acting on the particle at points P and Q respectively. By the defnition of central force, F_1 and F_2 are equal because they are acting at the same point O. If θ_1 and θ_2 be the angles between F_1 and dr_1 and between F_2 and dr_2 respectively, then the projections of vectors dr_1 and dr_2 on F_1 and F_2 will be $dr_1 \cos\theta_1$ and $dr_2 \cos\theta_2$. As these projections are equal to dr (see fig. 2), hence

$$dr_1 \cos\theta_1 = dr_2 \cos\theta_2$$

The work done in moving the particle from point A to point B through I is given by

$$W(A \rightarrow B) = \underset{(\text{path I})}{\int_A^B F_1 \cdot dr_1} = \underset{(\text{path 1})}{\int_A^B F \cdot dr}$$

The work done in moving the particle from point B to point A through path II is given by

$$W(B \rightarrow A) = \underset{(\text{path II})}{\int_B^A F_2 \cdot dr_2} = \underset{(\text{path II})}{\int_B^A F \cdot dr}$$

$$\left[\because \int_A^B F_1 \cdot dr_1 = \int_A^B F_2 \cdot dr_2 = \int_A^B F \cdot dr\right]$$

$$W(A \rightarrow B) = -W(B \rightarrow A)$$

(or)

$$W(A \rightarrow B) + W(B \rightarrow A) = 0$$

Thus, the total work around the closed path $A \rightarrow B \rightarrow A$ is zero.

5M
* Conservative force as a negative gradient of potential Energy :-

When a particle acted upon a conservative force 'F' moves from a space point (x_0, y_0, z_0) [position vector 'r_0' corresponding to zero potential energy] to another space point (x, y, z) [position vector r], then potential energy at 'r' is given by

$$U(r) = -\int_{r_0}^{r} F \cdot dr \longrightarrow ①$$

Now we express F and dr in rectangular co-ordinates as follows :

$$F = iF_x + jF_y + kF_z$$

$$dr = i\,dx + j\,dy + k\,dz$$

$$F \cdot dr = (iF_x + jF_y + kF_z) \cdot (idx + jdy + kdz)$$

$$= F_x dx + F_y dy + F_z dz \rightarrow (2)$$

$$U(r) = -\int_{r_0}^{r} F \cdot dr = \int_{x_0}^{x} F_x dx - \int_{y_0}^{y} F_y dy - \int_{z_0}^{z} F_z dz$$

$$= \int U(x, y, z)$$

Differeniating the equation partially with respect to x, y, z we get

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}$$

$$F = iF_x + jF_y + kF_z = i\frac{\partial U}{\partial x} - j\frac{\partial U}{\partial y} - k\frac{\partial U}{\partial z}$$

$$= -\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)U$$

$$\boxed{F = -\nabla U = -\text{grad}\,U}$$

Thus the conservative force is equal to the negative gradient of potential energy.

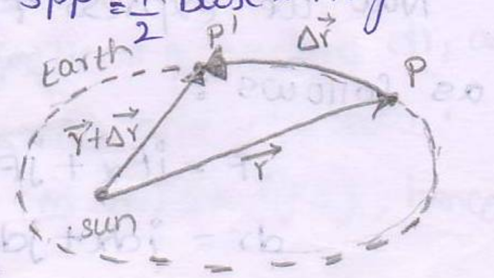
5m

*Areal velocity under central Force :-

Here we shall prove that the areal velocity of a particle moving under a central force remains constant. For this purpose we consider the motion of the earth around the sun as shown in fig. Let at any instant, 'r' be the radius vector of the from position P to position P' where the radius vector is $r + \Delta r$. If ΔA be the area are swept out by radius vector in time interval Δt, then

$$\Delta A = \text{area of triangle } SPP' = \frac{1}{2} \text{ base} \times \text{height}$$

$$\Delta A = \frac{1}{2} r \times \Delta r$$



Dividing both sides Δr, we get

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r \times \frac{\Delta r}{\Delta r}$$

(or)

$$\frac{dA}{dt} = \frac{1}{2} r \times \frac{dr}{dt} \quad (\text{where } \Delta t \to 0)$$

$$= \frac{1}{2} r \times v$$

$$= \frac{1}{2m} r \times mv$$

$$= \frac{L}{2m}$$

we know that under central force, angular momentum remains constant. Hence,

$$\frac{dA}{dt} = \text{constant}$$

(or)

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

we know that that transverse acceleration of the particle under central force feild is zero.

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

$$\frac{d}{dt}(r^2 \cdot \dot{\theta}) = 0$$

$$r^2 \dot{\theta} = \text{constant} = h$$

$$\text{Hence } \frac{dA}{dt} = \frac{h}{2} = \text{constant}$$

~~we know th.~~

$\frac{dA}{dt}$ is known as areal velocity. so areal velocity under central force remains constant. i,e the radius vector sweeps out equal areas in equal time.

** 10m.
# Kepler's laws:-

Following are the three kepler's laws about the motion of a planet:

Frist law: This law is known as the law of elliptical orbit. This law gives the shape of the orbits of a planet around the sun. According to this law, the path of a planet is an elliptical orbit around the sun, with sun at one of its foci.

Second law:- this law is known as the law of areas. This law gives the relationship between the orbital speed of the planet and its distance from the sun. According to this law, the radius vector, drawn from the sun to a planet, sweeps out equal areas in equal time, i.e., its areal velocity is constant.

Third law:- This law is known as harmonic law. This gives a relationship between the size of the orbit of a planet and its time of revolution. According to this law, the square of time period of the planet round the sun is proportional to the cube of the semi-major axis of its orbit.

Derivation for Kepler's first law:-

consider the case of a planet of mass "m" rotating round the sun of mass "M" in a path of radius "r". According to Newton's law of gravitational the gravitation force b/w sun & planet is given by

$$\vec{F} = F(r)\hat{r} = -\frac{GMm}{r^2}\hat{r}$$

$$P = \frac{F(r)}{m} = \frac{-GM}{r^2}$$

equation of motion under central force feild (gravitation)

$$\frac{d^2u}{d\theta^2} + u = \frac{-P}{h^2u^2}$$

where p is force per unit mass $\left(p = \frac{F(r)}{m}\right)$. substitute p value in above equation we get,

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2u^2r^2} \qquad \left(u = \frac{1}{r}\right)$$

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}$$

let $GM = \mu$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}$$

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2} = 0$$

It can be written as

$$\frac{d^2}{d\theta^2}\left(u - \frac{\mu}{h^2}\right) + \left(u - \frac{\mu}{h^2}\right) = 0 \qquad \frac{d}{dt}(\text{const}) = 0$$

let $u - \frac{\mu}{h^2} = x$ $\qquad \frac{\mu}{h^2} = 0$

$$\frac{d^2x}{d\theta^2} + x = 0$$

where x is the solution for giving second order D.E & it is in the form of

$$x = A\cos(\theta - \theta_0).$$

$$u - \frac{\mu}{h^2} = A\cos(\theta - \theta_0)$$

$$u = \frac{\mu}{h^2} + A\cos(\theta - \theta_0)$$

$$\frac{1}{r} = u = \frac{1 + \frac{Ah^2}{\mu}\cos(\theta-\theta_0)}{\frac{h^2}{\mu}}$$

we know that

$$\frac{1}{r} = \frac{1+e\cos(\theta-\theta_0)}{l}$$

comparing above two equation we get

$$e = \frac{Ah^2}{\mu}$$

$$\mu = \frac{Ah^2}{e}$$

$$l = \frac{h^2}{\mu}$$

If $e=1 \rightarrow$ the path followed by the sun is parabola

$e=0 \rightarrow$ circle

$e<1 \rightarrow$ ellipse

$e>1 \rightarrow$ hyperbola.

The total energy of the planet is given by $E = K.E + P.E$

$$E = \frac{1}{2}mr^2\dot{\theta}^2 + U(r) \qquad U(r) = -\frac{GMm}{r}$$

$$E = \frac{1}{2}mr^2\dot{\theta}^2 - \frac{GMm}{r}$$

$$E = \frac{1}{2}\frac{(mr^2)}{mr^2}(mr^2\dot{\theta}^2) - \frac{GMm}{r}$$

$$E = \frac{J^2}{2mr^2} - \frac{GMm}{r} \cdot \therefore [J^2 = m^2r^4\dot{\theta}^2]$$

$$\frac{J^2}{2m}\left(\frac{1}{r^2}\right) - GMm\left(\frac{1}{r}\right) - E = 0$$

$$\frac{J^2}{2m}(U^2) - GMm(U) - E = 0$$

$\therefore \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$\frac{1}{r} = U = \frac{GMm \pm \sqrt{G^2M^2m^2 + \frac{\cancel{4}^{2} J^2 E}{\cancel{2m}}}}{\frac{\cancel{2}J^2}{\cancel{2m}}}$$

$$\frac{1}{r} = \frac{GMm + \sqrt{G^2M^2m^2\left(1+\frac{2J^2E}{G^2M^2m^3}\right)}}{\frac{J^2}{m}}$$

$$\frac{1}{r} = \frac{GMm^2}{J^2}\left[1+\sqrt{1+\frac{2J^2E}{G^2M^2m^3}}\right]$$

We know that $GM = \mu$

$$\frac{1}{r} = \frac{\mu m^2}{J^2}\left[1+\sqrt{1+\frac{2J^2E}{G^2M^2m^3}}\right] \rightarrow (1)$$

$$\frac{1}{r_{min}} = \frac{\mu}{h^2}\left[1+\sqrt{1+\frac{2J^2E}{G^2M^2m^3}}\right]$$

$$\frac{J^2}{m^2} = r^4\dot{\theta}^2 = (r^2\dot{\theta})^2 = h^2$$

$$\frac{1}{r_{min}} = \frac{1+\frac{Ah^2}{\mu}}{\frac{h^2}{\mu}} \Rightarrow \frac{\mu}{h^2}\left(1+\frac{Ah^2}{\mu}\right) \rightarrow (2)$$

min $\cos\theta = 0$

(1) and (2) represents the same

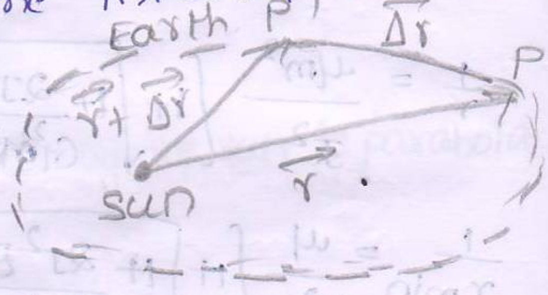
$$\frac{\cancel{\mu}}{h^2}\left(1+\frac{Ah^2}{\mu}\right) = \frac{\cancel{\mu}}{h^2}\left[1+\sqrt{1+\frac{2J^2E}{G^2m^2m^3}}\right]$$

$$\boxed{e = \frac{Ah^2}{\mu} = \sqrt{1+\frac{2J^2E}{G^2m^2m^3}}}$$

The energy of the planet is always negative. Hence eccentricity E will be less that 1 $(E<1)$. This represents the path followed by the ellipse sun is at one of its focii.

Derivation for kepler's second law :-

The areal velocity of a particle moving under a central force remains constant. For this purpose we consider the motion of the earth around the sun. let any instant, 'r' be the radius vector of the earth with respect to sun. let in a short interval of time Δt, the earth move from position p to position p' where the radius vector is $r + \Delta r$. if ΔA be the area swept out by radius vector in time interval Δt, then.



ΔA = area of Triangle

$SPP' = \frac{1}{2}$ base x height

$$\Delta A = \frac{1}{2} r \times \Delta r$$

Dividing both sides Δt, we get

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r \times \frac{\Delta r}{\Delta t} \quad (\text{or}) \quad \frac{dA}{dt} = \frac{1}{2} r \times \frac{dr}{dt}$$

(where $\Delta t \rightarrow 0$)

$$= \frac{1}{2} r \times v \Rightarrow \frac{1}{2m} r \times mv \quad [\because mvr = L]$$

$$\Rightarrow \frac{L}{2m}$$

we know that under central force, angular momentum remains constant. Hence

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

$$= \frac{1}{2} r^2$$

w.k.T transverse acceleration of the particle under force feild is zero $r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$

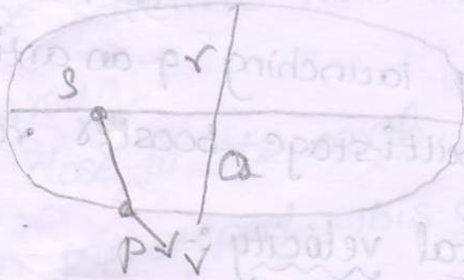
$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \Rightarrow r^2\dot{\theta} = \text{const} = h.$$

Kepler's third law :-

The time period of planet is defined as the ~~radia~~ area swept by the radius vector of the planet around sun in one complete rotation per areal velocity.

$$\text{Time period } T = \frac{\text{area swept in one rotation}}{\text{areal velocity}}$$

$$T = \frac{\pi ab}{h/2}$$



$$T = \frac{2\pi ab}{h}$$

S.O.B.S

$$T^2 = \frac{4\pi^2 a^2 b^2}{h^2} \rightarrow ①$$

semi latus rectum $l = \frac{h^2}{\mu}$

$$l = \frac{b^2}{a}$$

$$b^2 = \frac{a h^2}{\mu}$$

sub 'b^2' in eq ①

$$T^2 = \frac{4\pi^2 a^2 \cdot a h^2}{\mu h^2}$$

$$\boxed{T^2 \propto a^3}$$

Motion of satellites :-

1) Any relatively small body moving round another relatively massive body is primary called as satelite and its closed repetitive path is called as orbit.

2) For example, moon revolves round the earth and hence moon is a satelite of the earth.

3) Now-a-days artificial satelites are also put into orbits round the earth. The satelities move round the earth under the action of gravitation attraction (central force).

4) The launching of an artificial satelite is done by means of multi-stage booster rockets.

Orbital velocity :-

The orbital velocity (v) is defined as the velocity which the satelite must aquire to rotate around the earth in circular orbit of radius r. To derive an exprension of orbital velocity, let m be the mass and 'v' be the velocity of the satelite which ~~rolves~~ revolves around the earth in an orbit of radius 'r'.

the centripetal force is (mv^2/r). The gravitational force between the earth and the satellite is $\left(\frac{GMm}{r^2}\right)$, where M is the mass of the earth. The gravitational force supplies the required centripetal force. Thus

$$\frac{\cancel{m}v^2}{\cancel{r}} = G\frac{M\cancel{m}}{r^{\cancel{2}}}$$

$$v^2 = \frac{GM}{r}$$

$$v = \sqrt{\frac{GM}{r}} = \sqrt{g'r} \quad \because \left[\frac{GM}{r^2} = g'\right] \rightarrow (1)$$

where g' is acceleration due to gravity at the latitude of the satelite. In terms of acceleration due to gravity g at earth's surface, we have

$GM = gR^2$, where R is radius of the earth

$$v = R\sqrt{\left(\frac{g}{r}\right)} \quad \therefore [r = R+h]$$

$$v = R\sqrt{\left[\frac{g}{R+h}\right]} \rightarrow (2)$$

where h is the height of the satellite from the earth's surface.

wroug

when the satellite is very close to the earth, the orbital velocity is 8 km per second. This can be divided in the following way.

Period of Revolution :-

Let T be the period of revolution. Then

$$T = \frac{2\pi r}{v}$$

$$= \frac{2\pi r}{\sqrt{(g'r)}}$$

$$T = 2\pi\sqrt{\left(\frac{r}{g'}\right)}$$

Now, we derive the expression of time period in terms of R, h and g.

$$T = \frac{2\pi r}{v}$$

$$T = \frac{2\pi(r+h)}{v}$$

substituting the value of v from eq (2). we get

$$T = \frac{2\pi (R+h)^{3/2}}{R\sqrt{g}}$$

Time period of a satellite revolving near the surface of the earth.

if $h << R$

Hence $T = \frac{2\pi R^{3/2}}{R\sqrt{g}}$

$$= \frac{2\pi \sqrt{R}}{\sqrt{g}}$$

$$\Rightarrow 2\pi\sqrt{\frac{R^3}{GM}}$$

Geo-centric (stationary) satellite :

* The time period of a satellite increases with increasing distance r. Thus for some orbital radius, the time period T will be exactly equal to the time period of rotation of the earth i,e, 24 hours.
* For geo-stationary satellite, the plane of the orbit must coincide with the equitorial plane.
* It should be remembered that the orbital plane must pass through the centre of the earth.
* when the Geo-centric satellite is slightly inclined to the equator, it will appear to

move back and forth in the north - South direction.

* Now-a-days, The geo-stationary satellites are extensively is used in global communication system. They are called communication satellites.

5m
** Global positioning system :-

A global positioning system is also known as GPS.

It is a system of satelites used to help in navigation on earth, in the air and ~~in~~ warm water. A GPS unit takes ~~radial sigr~~ radio signals from the satellites. These radio signals consists information about the position and time of the satellite. A GPS receiver can subtract the current time & the time at which the signal was sent. The difference in time multiplied by the speed of light gives the distance of satellite from the GPS system. A cheap GPS unit is used in mobile phones & very expensive GPS is used in air lines.

Physiological Effects of Astronauts :-

The environmental conditions experienced by humans in outerspace is different from normal conditions. The necessary needs are breathable air and drinkable water for life support.

The group of devices that allow human beings to survive in outerspace.

-P)

i) life support system.
ii) system maintaining temprature and pressure within acceptable limits.
iii) sheilding against harmful external influence.

unit-3 Relativestics Mechanics

~~The theory~~

Theory of relativity:-

The theory which deals with relativity of motion and rest of a body or system of bodies is called as theory of relativity.

The theory of relativity is divided into 2 parts

1) special theory of relativity

2) General theory of relativity.

* The general theory of relativity deals with objects and system which are speeding up or slowing down with respect to one another.

* The special theory of relativity deals with objects and systems which are moving at a constant speed with respect one another.

* The special theory of relativity was proposed by Albert Einstin in the year 1905.

This theory is generalisation of newton mechanics.

Frames of Reference (Inertial and non-inertial)

In order to consider the idea of frame of reference, consider a ball thrown vertically upwards from the train moving with uniform velocity. the observer from the train observes that the ball moves up and down, i,e, in a straight line.

An observer on the ground observes a

parabolic path of the motion of the same ball. The reason is that the ball as observed from the ground seems to have a horizontal component of velocity equal to the velocity of the train and a vertical component of velocity equal to the velocity of the ball.

So the same event is described in different ways by different observes situated in different systems so the motion is relative. Thus the motion of the body has no meaning unless it is described with respect to some well-defined co-ordinate system. the co-ordinate system is known as frame of reference.

They are two types of frames of reference:

1) Inertial or unaccelerated frames.

2) Non-inertial or accelerated frames.

i) Inertial frames:

A frame of reference is said to be inertial when bodies in this frame obey Newton's law of inertia and other laws of Newtonian mechanics. In this frame, a body not acted upon by an external force, is at rest or moves with a constant velocity

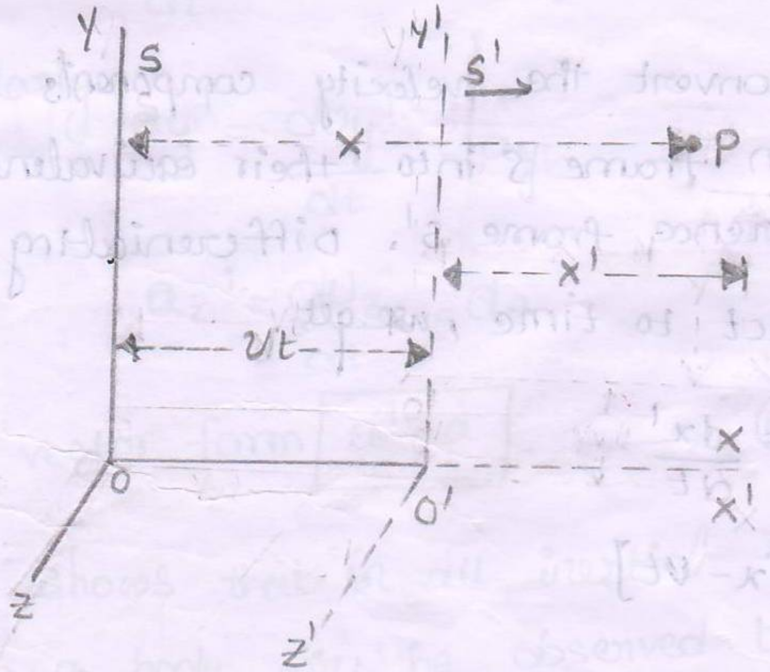
ii) Non-inertial frames:

A frame of reference is said to be non-inertial frame when a body, not acted upon by any external force, is accelerated. In this frame, the newton's laws are not valid.

Galilean Transformations (Relativity)

The Galilean transformations are used to transform the co-ordinates of position and time from one inertial frame to the other.

Suppose we have two frames of reference S and S' and let the velocity of S' relative to S be v. Consider an event happening at P at any particular time. Let the co-ordinates of P with respect to S be x, y, z, r and with respect to S' be x', y', z', r'. Let us choose our axes so that X and X' are parllel to v. Let Y' and Z' be parllel to Y and Z respectively.



we have $x = x' + vt$ or $x' = x - vt$

As there is no relative motion along Y and Z axes, we have

$$y' = y$$

$$z' = z$$

Time is independent of space cordinate system, $(t' = t)$

∴ The four equations are

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t \rightarrow ①$$

These equations are called as Galilean transformations. The inverse Galilean transformat can be

$$x = x' + vt$$
$$y = y'$$
$$z = z'$$
$$t = t' \rightarrow ②$$

let us convert the velocity components as measured in frame S into their equivalent component in the reference frame S'. Differentiating eq ① with respect to time, we get

$$u_x' = \frac{dx'}{dt}$$

$$\frac{d}{dt}[x - vt]$$

$$\frac{dx}{dt} - v$$

$$\therefore u_x' = u_x - v$$

similarly, $u_y' = \frac{d}{dt} y'$

$$\frac{dy}{dt} = u_y$$

$$u_z' = \frac{dz'}{dt}$$

$$\Rightarrow \frac{dz}{dt} = u_z$$

In vector form $\boxed{u' = u - v}$ $\longrightarrow$ (3)

The acceleration components can be obtained by differentiating velocity equations with respect to time and hence

$$a_x' = \frac{du_x'}{dt}$$

$$= \frac{d}{dt}[u_x - v]$$

$$\Rightarrow \frac{du_x}{dt} = a_x$$

similarly $a_y' = \frac{du_y'}{dt} = a_y$

$$a_z' = \frac{du_z}{dt} = a_z$$

In vector form $\boxed{a' = a}$ $\longrightarrow$ (4)

This shows that in all inertial reference frames a body will be observed to have the same acceleration.

Galilean invariance :-

According to Galilean transformation, some quantities like velocity are changed while some quantities like mass, length, time, acceleration etc, are not changed. The quantities or the laws that do not change under a transformation (here say Galilean transformation) are called Invariants.

Invariance of Newton's law :

According to Newton's law,

$$F = m\frac{d^2x}{dt^2}$$

In Galilean frame of reference, we have

$$x' = x - vt \; , y' = y \; , z' = z \text{ and } t' = t$$

Differentiating it, we get

$$\frac{dx'}{dt} = \frac{dx}{dt} - v \quad \because dt' = dt$$

and

$$\frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2} \quad [\text{as } v \text{ is constant}]$$

In newtonian mechanics, force and mases are absolute Quantities, i,e $m' = m$ and $F' = F$.

In system S, $F = m\dfrac{d^2x}{dt^2}$

In system S', $F' = m'\dfrac{d^2x'}{dt^2}$

$$\Rightarrow m\frac{d^2x}{dt^2} = F$$

Hence, the second law is changed in these two systems & Newton's law is invarient under Galilean transformations.

Invariance of conservation of momentum :-

consider the collision of two particles.

let m_1 and m_2 be the mases of two particles. In frame s, let u_1 and u_2 be the velocities before collision and v_1 and v_2, the velocities after collision respectively. Now

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 \longrightarrow (1)$$

In the same collision is observed in another frame of reference s' which is moving with velocity v with respect to frame s_1, then according to Galilean transformations, we have

$$u_1' = u_1 - v \qquad v_1' = v_1 - v$$

and

$$u_2' = u_2 - v \qquad v_2' = v_2 - v \longrightarrow (2)$$

where u_1', u_2' and v_1', v_2' are velocities before and after collision in frame s' respectively.

sub these values from eq (2) in eq (1), we get

$$m_1(u_1'+v) + m_2(u_2'+v) = m_1(v_1'+v) + m_2(v_2'+v)$$

$$m_1 u_1' + \cancel{m_1 v} + m_2 u_2' + \cancel{m_2 v} = m_1 v_1' + \cancel{m_1 v} + m_2 v_2' + \cancel{m_2 v}$$

$$m_1 u_1' + m_2 u_2' = m_1 v_1' + m_2 v_2' \longrightarrow (3)$$

Eq (3) represents the law of conservation of momentum as observed in moving frame s'.

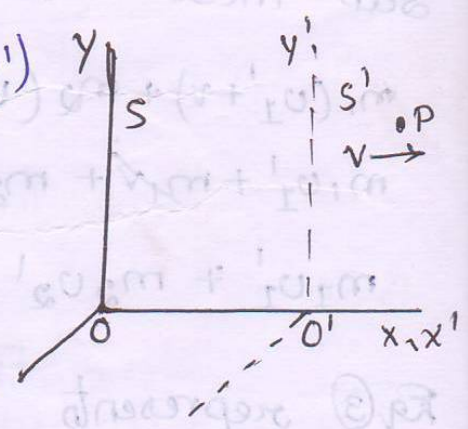
This shows that the law of conservation of momentum remains invariant under Galilean transformation.

** Lorentz Transformation equation of space and time :

M.V.V (TOM)

The invariance of speed of light in all inertial system implies that Galilean velocity transformation do not hold and need to be replaced, the laws of mechanics which we are invarient under Galilean transformations. also need to be modified without making any particular postulates lorentz algebracially searched for that transformation which may replace the Galilean tranformation to give results in agreement with Michelson - Morley experiment.

. consider two inertial systems S and S' as shown in fig. The system S' is moving with a velocity v with respect to S in the positive direction of for convience, let us suppose that the axis of two co-ordinate systems coincide at $t = t' = 0$. let a pulse of light be generated at time $t = 0$ at the origin which grows in the space. Now consider the situation when the pulse reaches at point P. let (x, y, z, t) and (x', y', z', t') be the positions and time co-ordinate of P measured by observes O and O' in frames S and S' respectively.



when the pulse is observed from S, we have velocity of light $= \frac{\text{distance}}{\text{time}}$

$$\frac{(x^2+y^2+z^2)^{1/2}}{c} = t$$

$$x^2+y^2+z^2=c^2t^2$$
$$x^2+y^2+z^2-c^2t^2=0 \rightarrow (1)$$

when the pulse is observed from s', then we have

$$\frac{(x'^2+y'^2+z'^2)^{1/2}}{c}=t'$$

$$x'^2+y'^2+z'^2=c^2t'^2$$

$$x'^2+y'^2+z'^2-c^2t'^2=0 \rightarrow (2)$$

$$x^2+\cancel{y^2}+\cancel{z^2}-c^2t^2=x'^2+\cancel{y'^2}+\cancel{z'^2}-c^2t'^2$$

$$x^2-c^2t^2=x'^2-c^2t'^2 \rightarrow [c \text{ is constant}]$$

$$y=y' \text{ and } z=z' \rightarrow (3)$$

From eq (1) and (2), using eqn (3) we have

$$x^2-c^2t^2=x'^2-c^2t'^2 \rightarrow (4)$$

The transformation between x and x' can be represented by the simple relationship

$$x'=K(x-vt) \rightarrow (5)$$

where K is the independent of x and t

If we suppose that the system s is moving relative to s' with velocity $(-v)$ along $(+x)$ direction (because motion is relative), then

$$x=K(x'+vt') \rightarrow (6)$$

substituting the value of x' from eqn (5) in eq (6) we have

$$x=K(K(x-vt)+vt')$$

$$\frac{x}{K}=K(x-vt)+vt'$$

$$\frac{x}{K}-K(x-vt)=vt'$$

$$t'=\frac{x}{vK}-\frac{Kx}{v}+Kt$$

$$t' = k\left[\frac{x}{v}\left[\frac{1}{k^2}-1\right]+t\right] \rightarrow (7)$$

substituting the value of x' from eq (5) and t' for eq (7) in eq (4), we get

$$k^2 - c^2t^2 = [k(x-vt)]^2 - c^2\left[\frac{x}{vk} - \frac{kx}{v} + kt\right]^2$$

$$k^2 - c^2t^2 - k^2x^2 - k^2v^2t^2 + 2x^2vtk + c^2\left[\frac{x}{vk} - \frac{kx}{v} + kt\right]^2 \rightarrow (8)$$

This is an identify and hence the co-efficients of various powers of x and t must vanish speed

Equating the co-efficients of t^2 to zero, we get

$$-c^2 - k^2v^2 + c^2k^2 = 0 \rightarrow (9)$$

$$k^2(c^2 - v^2) = c^2$$

$$k^2 = \frac{c^2}{c^2 - v^2}$$

$$k^2 = \frac{\cancel{c^2}/\cancel{c^2}}{\cancel{c^2}/\cancel{c^2} - \frac{v^2}{c^2}}$$

$$k^2 = \frac{1}{1-\frac{v^2}{c^2}}$$

$$\boxed{k = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}} \rightarrow (10)$$

substituting the value k from eq 10, we have lorentz transformation for space i,e;

$$\boxed{x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}} \rightarrow (A)$$

similarly, substituting the values of k in eq (7), we have the lorentz transformation for time, i.e;

$$t' = \frac{1}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}\left[t - \frac{x}{v}\left\{1-\frac{c^2-v^2}{c^2}\right\}\right]$$

$$t' = \frac{1}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}\left[t - \frac{xv^{\cancel{2}}}{\cancel{v}c^2}\right]$$

$$\boxed{t' = \frac{t-\left(\frac{vx}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}}} \rightarrow (B)$$

Eqn (A) and (B) are called as Lorentz transformation equations.

when $v <<< c$ (low velocity) i,e by Newtonian or classical mechanics, it is obvious from eq (A) & (B) that

$$x' = x - vt \quad \text{and} \quad t' = t$$

This for low values of v, Lorentz transformation approaches to Galilean.

Lorentz inverse transformations:-

If we assume that the system S is moving with velocity $-v$ relative to S' along positive directions of x, then the Lorentz transformation equations can be expressed as:

$$x = \frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}}, \quad y = y', \quad z = z' \quad \text{and}$$

$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}} \quad \text{---} (C)$$

These are known as inverse Lorentz transformations equations.

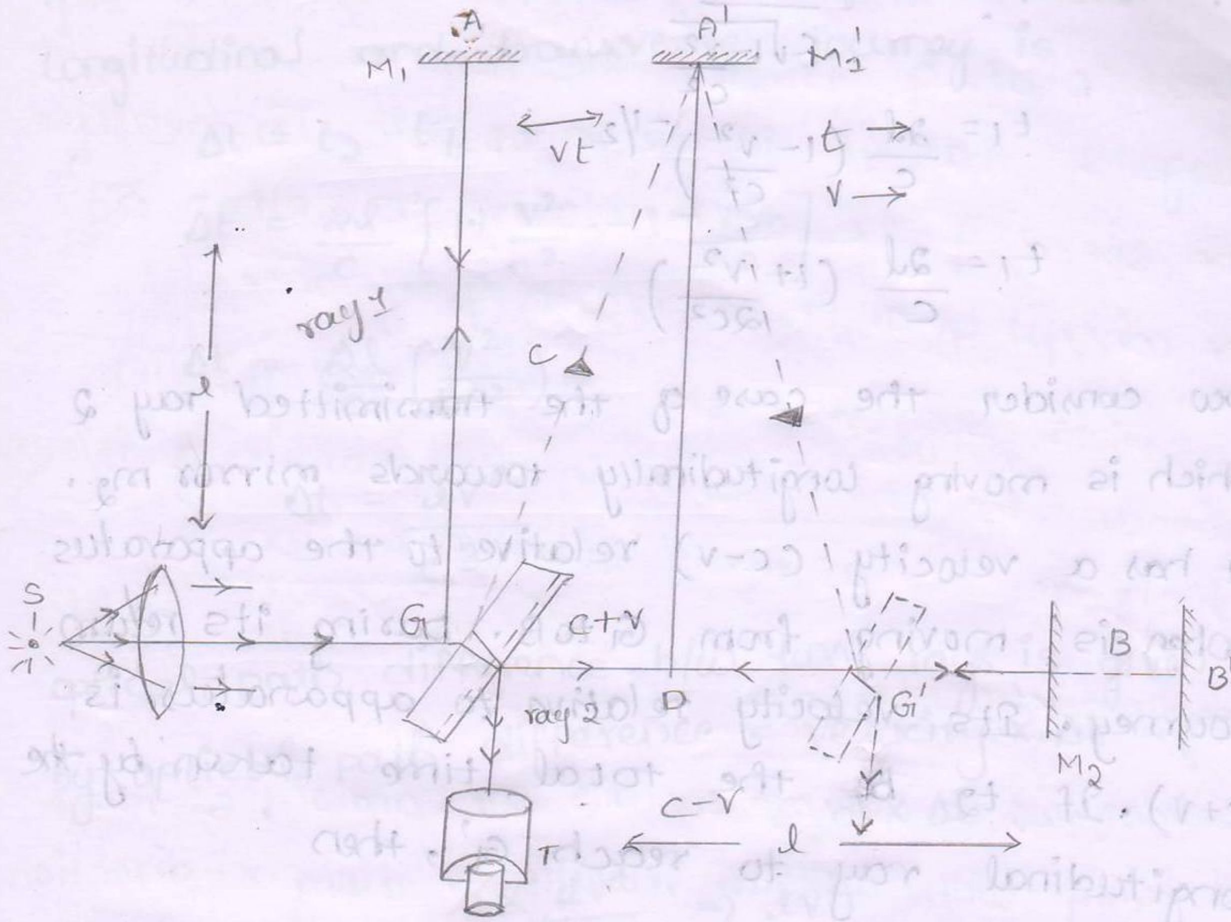
10m

**Michelson - Mosley experiment :-

** The experimental arrangement consists of a monochromatic extended source S, a collimating lens L, the semisilvered glass plate G inclined at an angle 45° to the beam. ~~It is divided~~ This glass plate divided the beam into the parts, one being reflected from the semisilvered surface G giving rise to ray 1 which travels towards mirror M_1 and the other being transmitted giving rise to ray 2 which travel towards mirror M_2. the two rays fall normally on mirrors M_1 and M_2 : The two rays fall normally on mirrors M_1 and M_2 respectively and are reflected back along their orginal paths. the reflected rays again meet at the semisilvered surface of glass plate G and enter the telescope where interference pattern is obtained.

If the apparatus is at rest in ether, the two reflected rays would take equal time to return the glass plate G. Let us suppose that the direction of motion of earth is in the direction of the initial beam. Due to the motion of the earth, the optical paths transversed by both the rays are not the same

Let the two mirrors M_1 and M_2 be at an equal distance l from the glass plate G. Further let c and v be the velocities of light and apparatus on earth respectively. It is obvious from fig. That the reflected ray of the ray from G to A' and back will be GA'G'. ~~From $\Delta GA'G$~~



From $GA'D$

$$(GA')^2 = (AA')^2 + (A'D)^2$$

If t be the time taken by the ray to move from G to A' then from fig. we have

$$(ct)^2 = l^2 + (vt)^2$$

$$c^2t^2 - v^2t^2 = l^2$$

$$t^2(c^2 - v^2) = l^2$$

$$t^2 = \frac{l^2}{c^2 - v^2}$$

$$t^2 = \frac{l^2}{c^2\left(1 - \frac{v^2}{c^2}\right)}$$

$$t = \frac{l}{c\sqrt{1 - \frac{v^2}{c^2}}}$$

If t_1 be the time taken by the ray to travel the whole path $GA'G'$, then

$$t_1 = 2t = \frac{2l}{c\sqrt{1-\frac{v^2}{c^2}}}$$

$$t_1 = \frac{2l}{c}\left(1-\frac{v^2}{c^2}\right)^{-1/2}$$

$$t_1 \doteq \frac{2l}{c}\left(1+\frac{v^2}{2c^2}\right)$$

Now consider the case of the transmitted ray 2 which is moving longitudinally towards mirror m_2. It has a velocity $(c-v)$ relative to the apparatus when is moving from G to B. During its return journey: Its velocity relative to apparatus is $(c+v)$. If t_2 be the total time taken by the longitudinal ray to reach G', then

$$t_2 = \frac{l}{c-v} + \frac{l}{c+v}$$

$$t_2 = \frac{l(c+v)+l(c-v)}{(c-v)(c+v)}$$

$$t_2 = \frac{2lc}{c^2-v^2}$$

$$t_2 = \frac{2lc}{c^2\left(1-\frac{v^2}{c^2}\right)}$$

$$\Rightarrow \frac{2l}{c\left(1-\frac{v^2}{c^2}\right)}$$

$$\Rightarrow \frac{2l}{c}\left(1-\frac{v^2}{c^2}\right)^{-1}$$

$$t_2 = \frac{2l}{c}\left(1+\frac{v^2}{c^2}\right)$$

Thus the difference in time of travel of longitudinal and transverse journey is

$$\Delta t = t_2 - t_1$$

$$\Delta t = \frac{2l}{c}\left[1+\frac{v^2}{c^2} - 1 - \frac{v^2}{c^2}\right]$$

$$\Delta t = \frac{2l}{c}\left(\frac{v^2}{2c^2}\right)$$

$$\Delta t = \frac{lv^2}{c^3}$$

optical path difference b/w two rays is given by optical path difference = velocity × Δt

$$= c \times \Delta t$$

$$\cancel{c} \times \frac{lv^2}{c^{\cancel{3}}} \Rightarrow \frac{lv^2}{c^2}$$

If λ is the wavelength of light used, then path difference in terms of wavelength $= \frac{lv^2}{c^2\lambda}$

michaelson and morley performed the experiment in two steps i,e firstly by the setting shown in fig. and secondly by turning the apparatus through 90°. when the apparatus was turned through 90°. The positions of two mirrors are changed. Now the path difference is in opposite directions. i,e the path difference is $\frac{-lv^2}{\lambda c^2}$ wavelength

the resultant path difference now becomes

$$\frac{lv^2}{\lambda c^2} - \left(\frac{-lv^2}{\lambda c^2}\right) = \frac{2lv^2}{\lambda c^2} \text{ wavelength.}$$

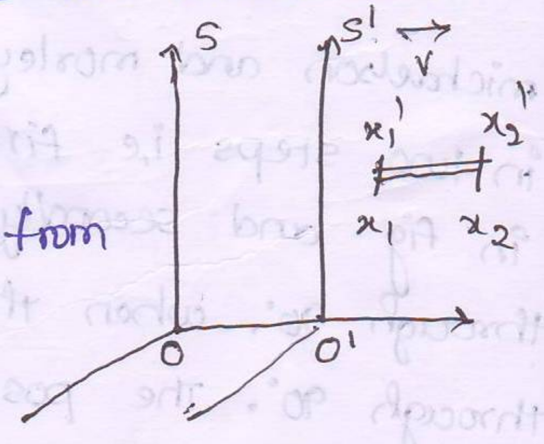
michelson and morley could observe a shift about 0.01 of fringe. The negative result suggests that it is impossible to measure the speed of the earth relative to ether or the speed of a fixed frame of referance.

the concept of a fixed frame of referance.

This suggests the speed of light in vaccum is the same in all frames of referance which are in uniform relative motion.

Length contraction:-

consider two co-ordinate systems s and s'. s' frame is moving with uniform velocity 'v' along x-direction. let O and O' are the observers from s and s' frames respectively. let a linear object of length 'l' which is observed from both the observers O and O'. The length of the rod measured from observe O is



$$l = x_2 - x_1$$

the length of the rod measured from observe O' is

$$l' = x_2' - x_1'$$

From lorentz transformation, we know that

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Hence $l' = x_2' - x_1'$

$$= \frac{x_2 - vt}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{x_1 - vt}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{x_2 - \cancel{vt} - x_1 + \cancel{vt}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{x_2 - x_1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$l' = \frac{l}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$l = l'\sqrt{1-\frac{v^2}{c^2}}$$

Thus the length of the rod is contracted by the factor $\sqrt{1-\frac{v^2}{c^2}}$.

* when $v \geq c$, the length of the rod becomes zero or imaginary.
* when $v << c$, then the length of the rod remains same from all observers.

Time dilation :-

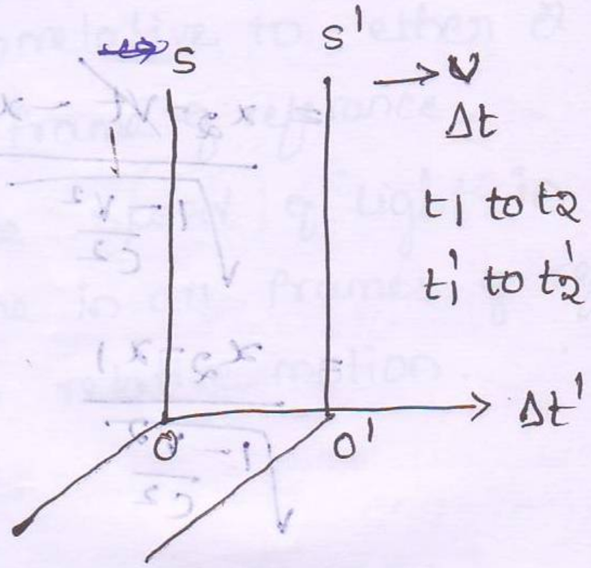
consider two systems s and s^1 let s^1 be moving with a velocity v w.r.t s in positive x-direction. suppose a clock is situated in the system s gives a signal at an interval Δt

$$\Delta t = t_2 - t_1$$

If the interval is observed by an observour in system S' as $\Delta t' = t_2' - t_1'$ $\rightarrow$ (1)

From lorentz transformations, we have

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$



Hence

$$t_1' = \frac{t_1 - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t_2' = \frac{t_2 - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

sub t_1' and t_2' in eq (1)

$$\Delta t' = \frac{t_2 - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t_1 - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Delta t' = \frac{t_2 - \cancel{\frac{vx}{c^2}} - t_1 + \cancel{\frac{vx}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{t_2 - t_1}{\sqrt{1-\frac{v^2}{c^2}}}$$

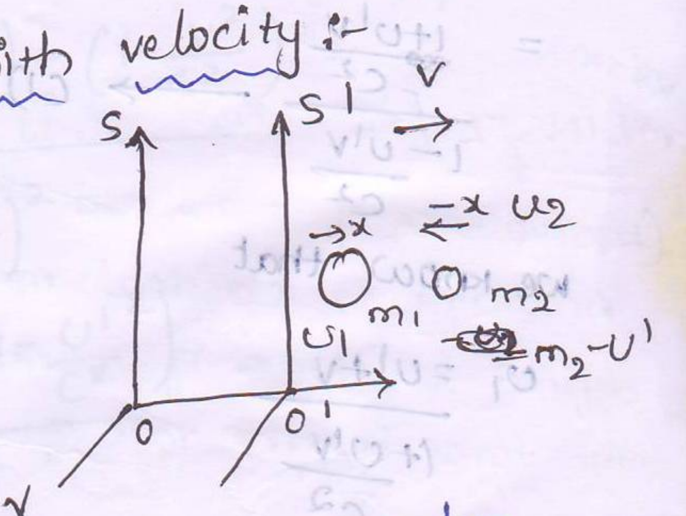
$$\Delta t' = \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}}$$

* Variation of mass with velocity

10m According to Newtonian mechanics the mass of a body does not change with velocity. But According to einstein the mass of a body the motion is different from the mass of the body is at rest.

consider two systems S and S'. let S' be moving with a velocity 'v' w.r.t S in positive x-direction. let two bodies of masses m_1 and m_2 be travelling with velocities u' and $-u'$ w.r.t S' suppose the two bodies collid at a particular instant of time. After its collision the two bodies comes to rest w.r.t S'. The velocities of these bodies w.r.t frame S is given by

variation of mass with velocity :-



$$u_1 = \frac{u'+v}{1+\frac{u'v}{c^2}}$$

$$u_2 = \frac{-u'+v}{1-\frac{u'v}{c^2}}$$

let us apply law of conservation of momentum

$$m_1 u_1 + m_2 u_2 = m_1 v + m_2 v$$

$$m_1\left(\frac{u'+v}{1+\frac{u'v}{c^2}}\right) + m_2\left(\frac{-u'+v}{1-\frac{u'v}{c^2}}\right) = m_1 v + m_2 v$$

$$m_1\left[\frac{u'+v}{1+\frac{u'v}{c^2}}-v\right]=m_2\left[v-\frac{-u'+v}{1-\frac{u'v}{c^2}}\right]$$

$$\frac{m_1}{m_2}=\frac{v-\frac{-u'+v}{1-\frac{u'v}{c^2}}}{\frac{u'+v}{1+\frac{u'v}{c^2}}-v}$$

$$\frac{m_1}{m_2}=\frac{\dfrac{v-\cancel{\frac{u'v^2}{c^2}}+u'-v}{}}{\dfrac{\left(u'+v-\cancel{v}-\cancel{\frac{u'v^2}{c^2}}\right)}{1+\frac{u'v}{c^2}}}$$

$$=\frac{1+\frac{u'v}{c^2}}{1-\frac{u'v}{c^2}}\longrightarrow(1)$$

we know that

$$u_1=\frac{u'+v}{1+\frac{u'v}{c^2}}$$

$$u_1^2=\frac{(u_1+v)^2}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$\frac{u_1^2}{c^2}=\frac{(u'+v)^2}{c^2\left(1+\frac{u'v}{c^2}\right)^2}$$

$$1-\frac{u_1^2}{c^2} = 1-\frac{1}{c^2}\left[\frac{(u'+v)^2}{\left(1+\frac{u'v}{c^2}\right)^2}\right]$$

$$1-\frac{u_1^2}{c^2} = \frac{\left(1+\frac{u'v}{c^2}\right)^2 c^2-(u'+v)^2}{c^2\left(1+\frac{u'v}{c^2}\right)^2}$$

$$1-\frac{u_1^2}{c^2} = \frac{\cancel{c^2}\left[\left(1+\frac{u'v}{c^2}\right)^2-\frac{(u'+v)^2}{c^2}\right]}{\cancel{c^2}\left[1+\frac{u'v}{c^2}\right]^2}$$

$$1-\frac{u_1^2}{c^2} = \frac{1+\frac{u'^2v^2}{c^4}+\cancel{\frac{2u'v}{c^2}}-\frac{u'^2}{c^2}-\frac{v^2}{c^2}-\cancel{\frac{2u'v}{c^2}}}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$1-\frac{u_1^2}{c^2} = \frac{\frac{u'^2}{c^2}\left(\frac{v^2}{c^2}-1\right)+\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$1-\frac{u_1^2}{c^2} = \frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$\left(1+\frac{u'v}{c^2}\right)^2 = \frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{1-\frac{u_1^2}{c^2}}$$

$$1+\frac{u'v}{c^2} = \left[\frac{\left(1-\frac{v^2}{c^2}\right)\cdot\left(1-\frac{u'^2}{c^2}\right)}{1-\frac{u_1^2}{c^2}}\right]^{1/2}$$

$$1-\frac{u'v}{c^2} = \left[\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u_1^2}{c^2}\right)}{1-\frac{u_2^2}{c^2}}\right]^{1/2}$$

$$\frac{m_1}{m_2} = \frac{1+\frac{u'v}{c^2}}{1-\frac{u'v}{c^2}} = \frac{\left(1-\frac{u_2^2}{c^2}\right)^{1/2}}{\left(1-\frac{u_1^2}{c^2}\right)^{1/2}}$$

let $m_2 = m_0 \Rightarrow u_2 = 0$

$$\frac{m}{m_0} = \frac{1}{\sqrt{1-\frac{u_1^2}{c^2}}}$$

$(u_1 = v)$

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

5M

EINSTEIN's Mass energy relation :-

(Equivalence of mass and energy).

According to classical mechanics, the energy is defined in terms of work (Force × distance) and the force is the rate of change of momentum,

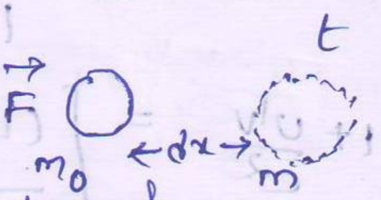
$F = \frac{dp}{dt}$ $\{p = mv\}$

hence

$$F = \frac{d}{dt}(mv)$$

According to the theory of relativity, the mass as well as velocity are variable, thus

$$F = m\frac{dv}{dt} + v\frac{dm}{dt}$$



when a particle is displaced through a distance dx by the application of a force, F, the increase in kinetic energy dk is given by

$$dK = F dx$$

$$\cancel{dK = m \cdot \frac{dv}{dx} + v \cdot \frac{dm}{dt}}$$

$$\therefore dK = m\frac{dv}{dt} \cdot dx + v \cdot \frac{dm}{dt} dx$$

$$dK = mv dv + v^2 dm \rightarrow ① \quad \frac{dx}{dt} = v$$

The variation of mass with velocity is given by

$$m = \frac{m_0}{\sqrt{\left[1 - \frac{v^2}{c^2}\right]}}$$

S.O.B.S

$$m^2 = \frac{m_0^2}{1 - \frac{v^2}{c^2}}$$

$$m^2 - \frac{m^2 v^2}{c^2} = m_0^2$$

$$\frac{m^2c^2 - m^2v^2}{c^2} = m_0^2$$

$$m^2c^2 - m^2v^2 = m_0^2 c^2$$

$$m^2c^2 - m^2v^2 - m_0^2c^2 = 0$$

Differentiate w.r.t time t

$$2mc^2 \cdot dm - 2mv^2 dm - 2vm^2 dv - m_0^2 c^2(0) = 0$$

$$2mc^2 \cdot dm - 2mv^2 dm - 2vm^2 dv = 0$$

$$c^2 \cdot dm - v^2 dm - vm dv = 0$$

$$c^2 dm = v^2 dm + vm dv \rightarrow (2)$$

From ① & ② we get

$$dK = c^2 dm$$

now consider that the body is at rest initially and by the application of force it acquires a velocity v. The mass of the body increases from m_0 to m. The total kinetic energy acquired by the body is given by

$$\int_0^K dk = \int_{m_0}^{m} c^2 dm$$

$$[K]_0^K = c^2 [m]_{m_0}^{m}$$

$$K = c^2(m-m_0)$$

$$K = c^2 m - c^2 m_0$$

$$K + c^2 m_0 = c^2 m$$

Here K is ~~a~~ initial kinetic energy

$m_0 c^2$ is initial potential energy

$$T.E = K.E + P.E$$

$$\boxed{E = mc^2}$$

** 5M

Postulates of special theory of relativity:-

⇒ In 1905 Albert Einstein gave two important conclusions based on ether wind theory. These are known as postulates of special theory of relativity.

1) All physical laws are the same in all inertial frame of reference which are moving with constant velocity w.r.t each other.

2) The speed of light in vaccam is same in every inertial frame.

17/4/2021 UNIT - IV

UNDAMPED, DAMPED AND FORCED OSCILLATIONS

Periodic motion :-

Any motion which repeats itself in equal interval time.

Ex :- The motion of the earth around the sun!

Simple Harmonic Motion :-

It is defined as the motion of an oscillatory particle which is acted upon by a restoring force which is directly propotional to the displacement but opposite to its direction. $F \propto -x$

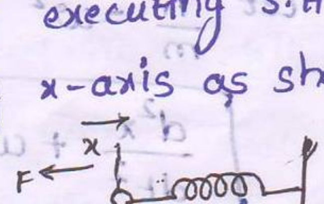
$\cancel{F = Kx}$

Simple Harmonic oscillator :-

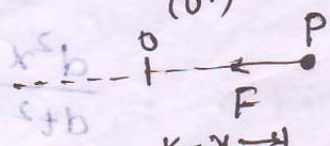
When a particle or body moves such that its acceleration is always directed towards a fixed point and varies directly as it distance from that point, the particle or body is said to execute S.H.M. The particle or body executing simple harmonic motion is called a simple oscillator.

Equation of motion of a simple oscillator :-

Consider a particle p of mass m executing S.H.M. about equilibrium position O along x-axis as shown in fig. By definition, the force under which the particle is oscillating is propotional to its displacement directed towards the mean position.



(or)



Let x be the displacement of P from O at any instant. The instantaneous Force acting upon P is given by

$F \propto -x$ (or)

$F = -kx \rightarrow (1)$

where k is proportionality factor which represents the force per unit displacement. The negitive sign is used to show that the force F is opposite to the displacement.

According to Newton's second law, the restoring force on mass m produces an acceleration $\frac{d^2x}{dt^2}$ in the mass, so that

Force = mass × acceleration.

$F = ma$

$F = m\frac{d^2x}{dt^2} \rightarrow (2)$

From eq (1) & (2)

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Let $\frac{k}{m} = \omega^2$

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \rightarrow (3)$$

sub. the value of $\frac{d^2x}{dt^2}$ in eq (3)

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} \qquad \left[\frac{dx}{dt} = v\right]$$

$$= \frac{dv}{dx} \cdot \frac{dx}{dt}$$

$$= v\frac{dv}{dx}$$

$$v\frac{dv}{dx} + \omega^2 x = 0$$

$$v\frac{dv}{dx} = -\omega^2 x$$

$$v\,dv = -\omega^2 x\,dx \rightarrow (4)$$

I. O. B. S

$$\int v\,dv = \int -\omega^2 x\,dx$$

$$\frac{v^2}{2} = \frac{-\omega^2 x^2}{2} + c \quad (c = \text{constant})$$

$$x = A \Rightarrow v = 0$$

$$0 = \frac{-\omega^2 A^2}{2} + c$$

$$c = \frac{\omega^2 A^2}{2}$$

$$\frac{v^2}{2} = \frac{-\omega^2 x^2}{2} + \frac{\omega^2 A^2}{2}$$

$$v^2 = \omega^2 (A^2 - x^2)$$

$$\boxed{v = \omega\sqrt{A^2 - x^2}} \rightarrow (5)$$

let $v = \frac{dx}{dt}$, eq (5) can be written as

$$\frac{dx}{dt} = \omega\sqrt{A^2 - x^2}$$

$$\frac{dx}{\sqrt{A^2 - x^2}} = \omega\,dt \rightarrow (6)$$

let $x = A\sin\theta$

$$dx = A\cos\theta \cdot d\theta$$

$$\frac{A\cos\theta\,d\theta}{\sqrt{A^2 - A^2\sin^2\theta}} = \omega\,dt$$

$$\frac{A\cos\theta\, d\theta}{A\sqrt{1-\sin^2\theta}} = \omega dt$$

$$\frac{A \cancel{\cos\theta}\, d\theta}{A \cancel{\cos\theta}} = \omega dt$$

$$d\theta = \omega dt \quad \rightarrow (7)$$

$$\int d\theta = \int \omega dt$$

$$\theta = \omega t + \phi \quad \rightarrow (8) \quad (\phi \text{ is constant})$$

~~displacement of simple Harmonic oscillation~~

~~$x = A\sin(\omega t + \phi)$~~

Charcteristics of a simple harmonic motion :-

Displacement :- The displacement of any particle at any instant executing S.H.M is given by

$$x = a\sin(\omega t + \phi)$$

The maximum displacement from the mean position is called amplitude. Here the amplitude is a.

velocity :

The velocity v of the oscillating particle can be obtained by differeniating eq (8) thus

$$v = \frac{dx}{dt} = A\omega\cos(\omega t + \phi) = \omega\sqrt{A^2 - x^2}$$

At the mean position i,e at $x = 0$, the velocity is maximum (ωA). so $V_{max} = \omega A$. The velocity is zero at extreme positions.

periodic time :-

Time taken for one complete oscillation is called as periodic time and is denoted by T.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{K}}$$

Frequency :-

The number of oscillations made in one second is called as frequency and is denoted by n or v. Hence

$$n \text{ or } v = \frac{1}{T} = \frac{\omega}{2\pi}$$

$$\frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

Phase :-

The angle $(\omega t + \phi)$ is called the phase of vibration. phase of a body executing S.H.M at any instant represents its state as regard its position and direction at that instant.

Epoch :-

The value of phase when $t = 0$ is called the phase or epoch. In our case ϕ is the epoch.

Types of vibrations :-

1. **Free vibrations :-**

when a body is capable of vibration is displaced from its position of equilibrium and then released, it begins to vibrate the amplitude of vibration remains constant (same) at infinite time. these type of vibration are called as free vibration. The frequency of the body under free vibrations is called as Natural frequency.

2. Damped vibrations :-

In general, the body is made to vibrate, the vibrations will gradually decreased and die away after a due course of time. This is due to the fact that the vibrating body is always subjected in external frictional force is air resistance. such type of vibrations are called as damped vibrations. During damping vibrations, the time period increases; and Amplitude gets decreased.

3) Forced vibrations :-

when a body is subjected to external periodic force. The body initially vibrates with the natural frequency and then vibrate with frequency of periodic force. these vibrations are called as Forced vibrations.

when frequency of periodic ~~time~~ force is equal to the natural frequency of the body, the amplitude increases rapidly, then the body is said to be in resonance.

10M ***

Damped Harmonic oscillator :-

· consider the case of a body under a damped oscillator. The force of the body depends upon two factors

i, The restoring force of the body is directly propotional to the displacement from the mean position and directed towards mean position.

ii, The restoreing force of the body is directly propotional to the external frictional force which is directly propotional to velocity of the body but in opposite direction.

$$F_R = -\mu x - r\frac{dx}{dt} \rightarrow (1)$$

where μ and r are the propotionality constant

According to Newtons law

$$\text{Force}(F) = m\frac{d^2x}{dt^2} \longrightarrow (2)$$

From eq ① & ② we, have

$$m\frac{d^2x}{dt^2} = -\mu x - r\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = -\frac{\mu}{m}x - \frac{r}{m}\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{\mu}{m}x = 0$$

$$\boxed{\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = 0} \rightarrow ③$$

This represents D.E of the Damped harmonic oscillation.

Let 'x' be the solution of above D.E

$$x = Ae^{\alpha t}$$

$$\frac{dx}{dt} = A\alpha e^{\alpha t}$$

$$\frac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t}$$

Substituting above values in eq. 3, we get

$$A\alpha^2 e^{\alpha t} + 2bA\alpha e^{\alpha t} + \omega^2 A e^{\alpha t} = 0$$

$$A e^{\alpha t}[\alpha^2 + 2b\alpha + \omega^2] = 0$$

$Ae^{\alpha} \neq 0$, & $\alpha^2 + 2b\alpha + \omega^2 = 0$

$$\alpha = \frac{-2b \pm \sqrt{4b^2 - 4\omega^2}}{2}$$

$$\therefore \alpha = -b \pm \sqrt{b^2 - \omega^2}$$

$$'x' = A_1 e^{(-b + \sqrt{b^2-\omega^2})t} + A_2 e^{(-b - \sqrt{b^2-\omega^2})t}$$

Case-i (over damping)

If $b^2 >> \omega^2$, in this case the term $\sqrt{b^2-\omega^2}$ is real. The terms $[-b + \sqrt{b^2-\omega^2}]$ and $[-b - \sqrt{b^2-\omega^2}]$ both are negetive hence the amplitude of osciuator decay to zero exponentially. The rate of decay mainly depends upon the factor $[-b + \sqrt{b^2-\omega^2}]$. This type of motion of the osciuator is called as over damping or dead beat.

Ex: This type of motion is observed in the case of a simple pendulum displaced from its mean position and then relaxed in thick oil.

case (ii) - critical damping

If $b^2 = \omega^2$, i,e $b^2 - \omega^2 = 0$. This is not satisfied the differential equation of damped harmonic oscillation. Here $\sqrt{b^2-\omega^2} \neq 0$ but equal to small value let it be 'h'. Here $h \to 0$

$$x = A_1 e^{(-b+h)t} + A_2 e^{(-b-h)t}$$

$$x = e^{-bt}\left[A_1 e^{ht} + A_2 e^{-ht}\right]$$

$$x = e^{-bt}\left[(1+ht)A_1 + A_2(1-ht)\right]$$

$$x = e^{-bt}\left[(A_1+A_2) + (A_1-A_2)ht\right]$$

$$x = (p+qt)e^{-bt} \to (1)$$

Here $p = (A_1+A_2)$ and $q = h(A_1-A_2)$.

The eqn represents a possible form of solution. That has t increases, the factor $(p+qt)$ increases but the factor e^{-bt} decreases. hence in this case the particle tends to acquire its position of equilibrium much rapidly than in case 1. such a motion is called critical damped motion.

This type of motion is exhibited by many pointers in instrument such as voltmeter, ammeter, etc,

Case-(iii) under-damped motion

when $b^2 << \omega^2$. In this case $\sqrt{(b^2-\omega^2)}$ is imaginary. Let us write

$$\sqrt{(b^2-\omega^2)} = i\sqrt{(\omega^2-b^2)} = i\beta$$

where $\beta = \sqrt{(\omega^2-b^2)}$ and $i = \sqrt{(-1)}$

Eq (3) now becomes

$$x = A_1 e^{(-b+i\beta)t} + A_2 e^{(-b-i\beta)t}$$

$$= e^{-bt}\left[A_1 e^{i\beta t} + A_2 e^{-i\beta t}\right]$$

$$= e^{-bt}\left[A_1(\cos\beta t + i\sin\beta t) + A_2(\cos\beta t - i\sin\beta t)\right]$$

$$= e^{-bt}\left[(A_1+A_2)\cos\beta t + i(A_1 - A_2)\sin\beta t\right]$$

$$= e^{-bt}\left[a\sin\phi\cos\beta t + a\cos\phi\sin\beta t\right]$$

where $a\sin\phi = (A_1+A_2)$ and $a\cos\phi = i(A_1-A_2)$.

$$= e^{-bt} a\sin(\beta t + \phi)$$

$$= ae^{-bt}\sin\left[\sqrt{(\omega^2-b^2)}\,t+\phi\right] \quad ①$$

This equation represents the simple harmonic motion with amplitude ae^{-bt} and time period

$$\boxed{T = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{(\omega^2-b^2)}}}$$

The amplitude of the motion is continuously decreasing owing to the factor e^{-bt}. The decay of the amplitude depends upon the damping co-efficient b. It is called "under-damped" motion.

The example of this type of motion is the motion of a pendulum in air.

5m

** Logarithmic decrement :-

Logarithmic decrement measures the rate at which the amplitude dies away. The amplitude of damped harmonic oscillator is given by

$$x = Ae^{-bt}\left[e^{(\sqrt{b^2-\omega^2})t} + e^{(-\sqrt{b^2-\omega^2})t}\right]$$

Amplitude $= Ae^{-bt}$

Let $a_1, a_2, a_3, \ldots$ be the amplitudes at time $t = T, 2T, 3T, \ldots$ respectively, where T = period of oscillation then at $t = T \Rightarrow a_1 = Ae^{-bT}$

$t = 2T \Rightarrow a_2 = Ae^{-2bT}$

$t = 3T \Rightarrow a_3 = Ae^{-3bT}$

Ratio -

$$\frac{a}{a_1} = \frac{A}{Ae^{-bT}} = e^{bT}$$

$$\frac{a_1}{a_2} = \frac{\cancel{A}e^{-bT}}{\cancel{A}e^{-2bT}} = e^{bT}$$

$$\frac{a_2}{a_3} = \frac{\cancel{A}e^{-2bT}}{\cancel{A}e^{-3bT}} = e^{bT}$$

From eqn's we get

$$\frac{a}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = \text{-----} = e^{bT}$$

Apply 'log' on both sides

$$\log_e \frac{a}{a_1} = \log_e \frac{a_1}{a_2} = \log_e \frac{a_2}{a_3} = \text{---} = bT$$

Thus, logarithmic decrement is defined as the natural logarithm of the ratio between two successive maximum amplitudes which are seperated by one period.

Relaxation time :- (5m)

The relaxation time is defined as the time taken for the total mechanical energy to decay (1/e) of its original value.

The mechanical energy of damped harmonic oscillation is given by

$$E = \frac{1}{2} \alpha^2 \mu e^{-2bt}$$

let $E = E_0$ whe $t = 0$,

$$\therefore E_0 = \frac{1}{2} \alpha^2 \mu$$

$$E = E_0 e^{-2bt} \rightarrow (2)$$

let T be the relaxation time ; i,e at $t = T$.

$E = \frac{E_0}{e}$. making this situation in eq (2), we get

$$\left(\frac{\cancel{E_0}}{E}\right) = \cancel{E_0} e^{-2bT}$$

$$e^{-1} = e^{-2bT}$$

$$-1 = -2bT$$

$$\boxed{T = \frac{1}{2b}}$$

$$E = \frac{1}{2} m v^2$$

$$= \frac{1}{2} m (ae^{-bt})^2 \omega^2$$

$$= \frac{1}{2} \cancel{m} a^2 e^{-2bt} \cdot \frac{\mu}{\cancel{m}}$$

$$\Rightarrow \cancel{-}1 = \cancel{-}2bt$$

$$\Rightarrow \boxed{T = \frac{1}{2b}} \rightarrow (3)$$

From (2) & (3), we get

$$E = E_0 e^{-t/T}$$

The Expression of power dissipation can be written as $p = \frac{E}{T}$

5m
**
Quality factor :-

The quality factor is defined as 2π times the ratio of the energy stored in the system to the energy lost per period.

$$Q = 2\pi \frac{\text{Energy of the oscillation}}{\text{Energy lost per one period}}$$

$$= 2\pi \frac{E}{p.T}$$

$$= 2\pi \frac{\cancel{E}}{\frac{\cancel{E}}{T}.T}$$

$$T = \frac{2\pi}{\omega}$$

$$\omega = \frac{2\pi}{T}$$

$$Q = 2\pi \frac{T}{T}$$ $E = E_0 e^{-2bt}$ $\left(P = \frac{E}{t}\right.$

power displacem $P = \frac{E}{T}$

$$\boxed{Q = \tau\omega}$$

ω = angular frequency

10m

***

Forced harmonic oscillator :-

Forced vibrations can be defined as the vibrations in which the body vibrates with a frequency other than its natural frequency under the action of an external periodic force. The forces acting upon the particle are.

i) A restoring force propotional to the displacement but oppositly directed is given by

$$f_r \propto -x$$

$$F_r = -\mu x$$

ii) A frictional force propotional to the velocity but oppositly directed is given by

$$F_f \propto -\frac{dx}{dt}$$

$$F_f = -r\frac{dx}{dt}$$

iii) An external periodic force which is given by $F_{ex} = F\sin(Pt)$

The total force acting upon the particle is given by $F_t = F_r + F_f + F_{ex}$

$$F_t = -\mu x - r\frac{dx}{dt} + F\sin(Pt) \rightarrow (1)$$

According to Newton's law of motion, force acted upon a mass m may be defined as the product of mass and its instantaneous acceleration.

$$F_t = m\frac{d^2x}{dt^2} \rightarrow (2)$$

eqn ① & ② represents the same hence

$$m\frac{d^2x}{dt^2} = -\mu x - r\frac{dx}{dt} + F\sin(pt)$$

$$\frac{d^2x}{dt^2} = -\frac{\mu}{m}x - \frac{r}{m}\frac{dx}{dt} + \frac{F}{m}\sin(pt)$$

$$\frac{d^2x}{dt^2} + \frac{\mu}{m}x + \frac{r}{m}\frac{dx}{dt} = \frac{F}{m}\sin(pt)$$

put $\frac{\mu}{m} = \omega^2$; $\frac{r}{m} = 2b$; $\frac{F}{m} = f$

$$\boxed{\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = f\sin pt} \rightarrow ③$$

This eqn represents differential eqn of the forced harmonic oscillation.

Let 'x' be the solution of above differential eqn and it is in the below form

$$x = A\sin(pt - \theta)$$

D. w.r.t

$$\frac{dx}{dt} = A\cos(pt-\theta) \times p \quad \text{(Again D. w.r.t)}$$

$$\frac{d^2x}{dt^2} = -A\sin(pt-\theta) \times p \times p$$

substituting these values in eq ③ we get

$-Ap^2 \sin(pt-\theta) + 2b\,Ap\cos(pt-\theta) +$

$\omega^2 A\sin(pt-\theta) = f\sin pt$

$= f\sin((pt-\theta)+\theta)$

$-Ap^2\sin(pt-\theta) + 2bAp\cos(pt-\theta) + \omega^2 A\sin(pt-\theta) =$

$f[\sin(pt-\theta)\cos\theta + \cos(pt-\theta)\sin\theta]$

comparing co-efficients of $\sin(pt-\theta)$ & $\cos(pt-\theta)$ on both sides, we have

$f\cos\theta = -Ap^2 + A\omega^2$

$f\cos\theta = A(\omega^2 - p^2) \rightarrow (4)$

$f\sin\theta = 2bAp \rightarrow (5)$

on squaring & adding given eqn

$f^2 = A^2(\omega^2-p^2)^2 + 4b^2A^2p^2$

$f = A\sqrt{(\omega^2-p^2)^2 + 4b^2p^2}$

$$A = \frac{f}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}}$$

Displacement of forced harmonic oscillator

$$x = \frac{f}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}}\sin(pt-\theta)$$

Dividing eqn (5) with eq(4) we get

$$\frac{f\sin\theta}{f\cos\theta} = \frac{2bAp}{A(\omega^2-p^2)}$$

$$\tan\theta = \frac{2bp}{\omega^2-p^2}$$

$$\boxed{\theta = \tan^{-1}\left(\frac{2bp}{\omega^2-p^2}\right)}$$

case-i, when driving force frequency is low -

$p << \omega$

Amplitude $(A) = \dfrac{f}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}}$

$$A \approx \frac{f}{\sqrt{(\omega^2)^2}}$$

$$A \approx \frac{f}{\omega^2}$$

$$\theta = \tan^{-1}\left(\frac{2bp}{\omega^2-p^2}\right)$$

$$\theta = \tan^{-1}(0)$$

$$\boxed{\theta = 0^\circ}$$

case-ii,

when $p = \omega$

$$A = \frac{f}{\sqrt{4b^2\omega^2}}$$

$$A = \frac{f}{2b\omega}$$

$$\theta = \tan^{-1}(\alpha)$$

$$\boxed{\theta = \frac{\pi}{2}}$$

case-iii,

when $p >> \omega$

$$A = \frac{f}{\sqrt{p^4+4b^2p^2}}$$

$$A = \frac{f}{p\sqrt{p^2+4b^2}}$$

$\theta = \tan^{-1}(-0)$

$\boxed{\theta = \pi}$

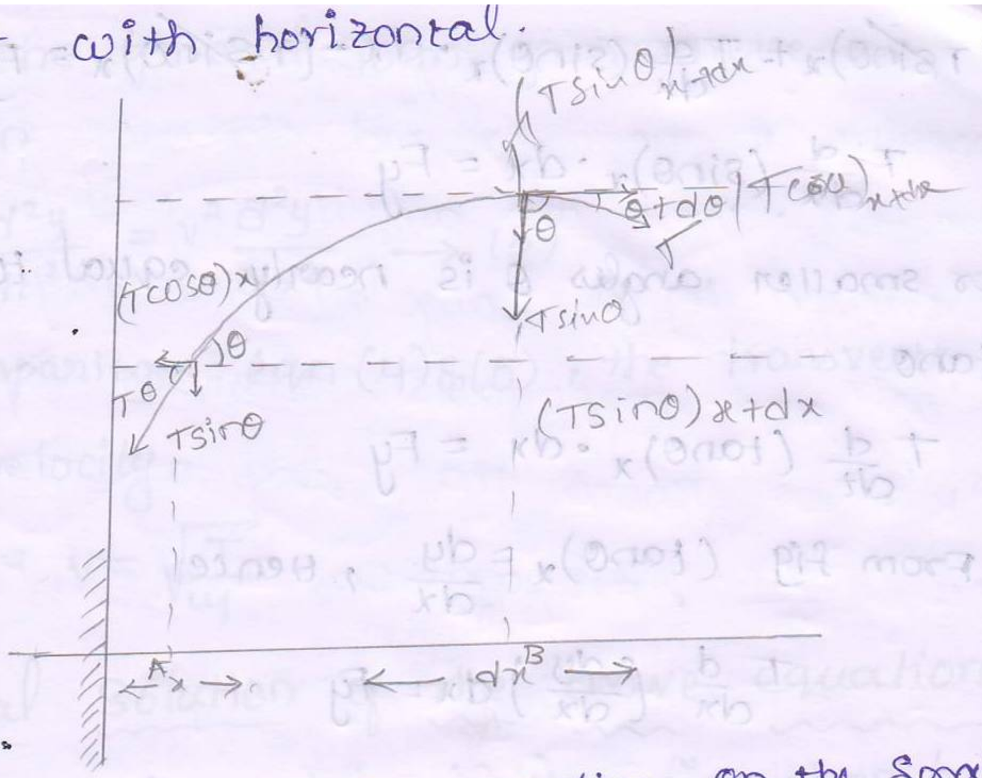
Unit - V

-: VIBRATING STRINGS :-

The vibrations in a material system or complex in nature. It can be simplified by considering the case of Transverse Vibrations in a string. suppose a string is placed between two rigid supports with tension when the string is flucked then the disturbance appeared is not localised but moves towards left or right in the form of wave. when a wave is propagating in a medium the particles move from their equilibrium position and make simple harmonic motion. If the particle displacement is parllel to the wave motion then longitudinal waves are established. If particle displacement is perpendicular to the wave motion then tranverse waves are established in the string

*Propogation of Transverse waves in a string :

Consider a string of uniform cross section(α), Length 'l' Linear density 'μ' and tension (T) is fixed between two rigid supports. let the string is stretched along x - direction. let $y(x)$ $y(x,t)$ is the displacement perpendicular to x - direction. let us consider a small element δx. when the string is stretched θ and (θ + $d\theta$) are the angles between edges of the segment with horizontal.



The Net transverse force acting on the Small segment is equal to the difference between the perpendicular components of tension at the both end points

$$F_y = (T\sin\theta)_{x+dx} - (T\sin\theta)_x \rightarrow (1)$$

Here $(T\sin\theta)_x$ is component of tension at x and $(T\sin\theta)_{x+dx}$ is component of tension at $x+dx$.

From Taylor's expansion

$$f(x+dx) = f(x) + \frac{f'(x)}{1!} \cdot dx + \frac{f''(x)}{2!} \cdot (dx)^2 + \cdots$$

similarly

$$T(\sin\theta)_{x+dx} = T\left[(\sin\theta)_x + \frac{d}{dx}(\sin\theta)_x \times dx + \cdots\right]$$

neglecting higher order terms

$$T(\sin\theta)_{x+dx} = T\left[(\sin\theta)_x + \frac{d}{dx}(\sin\theta)_x \times dx\right]$$

sub above value in eq (1), we have

$$(T\cancel{\sin\theta})_x + T\frac{d}{dx}(\sin\theta)_x \cdot dx - (T\cancel{\sin\theta})_x = F_y$$

$$T\frac{d}{dx}(\sin\theta)_x \cdot dx = F_y$$

For smaller angles θ is nearly equal to $\sin\theta = \tan\theta$

$$T\frac{d}{dt}(\tan\theta)_x \cdot dx = F_y$$

From fig $(\tan\theta)_x = \frac{dy}{dx}$, Hence

$$T\frac{d}{dx}\left(\frac{dy}{dx}\right)dx = F_y$$

$$T\frac{d^2y}{dx^2} \cdot dx = F_y \rightarrow (2)$$

According to Newton's second law, the force acting upon the segment AB is given by

$$F_y = (dm)\frac{d^2y}{dt^2}$$

Here dm is the mass of the infinitismal part of string

$$F_y = \mu\, dx \frac{d^2y}{dt^2} \longrightarrow (3)$$

Eqn (2) & (3) represents the same values hence

$$T\frac{d^2y}{dx^2} \cdot \cancel{dx} = \mu\, \cancel{dx} \cdot \frac{d^2y}{dt^2}$$

$$\boxed{\frac{d^2y}{dt^2} = \frac{T}{\mu}\frac{d^2y}{dx^2}} \rightarrow (4)$$

This second order differential equation represents the transverse wave motion in the string

From the general form of Transverse wave equation

$$\frac{d^2y}{dt^2} = v^2 \frac{d^2y}{dx^2} \rightarrow (5)$$

by comparing eqn (4) & (5), the transverse wave velocity

$$v = \sqrt{\frac{T}{m}}$$

General solution of the wave equation:

The general wave equation is given by

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

The solution of above differential equation is depends on x and t.

The solution is a periodic function and can be expressed as ~~Asin~~ asin, acos or any other periodic function

$$y = A\sin(\omega t - kx) \text{ (or)}$$

$$y = A\sin(\omega t + kx)$$

$$y = A\cos(\omega t - kx) \text{ (or)}$$

$$y = A\cos(\omega t + kx)$$

$$y = e^{i(\omega t - kx)} \text{ (or)}$$

$$y = e^{i(\omega t + kx)}$$

The change of sign represents the wave travelling in the forward direction or backward direction. There are amplitude which is represented by 'A', angular frequency represented by 'ω' and propogation constant represented by 'K'. We know that Angular frequency $(\omega) = \frac{2\pi}{T}$, propagation constant $k = \frac{2\pi}{\lambda}$

Now the solution becomes

$$y = A\sin(\omega t - kx)$$

$$= A\sin\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x\right)$$

$$\Rightarrow A\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

$$\Rightarrow A\sin\frac{2\pi}{\lambda}\left(\frac{t\lambda}{T} - x\right) \quad \left[v = \frac{\lambda}{T}\right]$$

$$\boxed{y \Rightarrow A\sin\frac{2\pi}{\lambda}(vt - x)}$$

similarly it can be expressed as.

$$y = A\cos\frac{2\pi}{\lambda}(vt - x)$$

(or)

$$y = e^{i\frac{2\pi}{\lambda}(vt - x)}$$

similarly the positive directions solutions will be

$$y = A\sin\frac{2\pi}{\lambda}(vt + x) \text{ (or)}$$

$$y = A\cos\frac{2\pi}{\lambda}(vt + x) \text{ (or)}$$

$$y = e^{i\frac{2\pi}{\lambda}(vt + x)}$$

10m ** Imp
Modes of vibration in the stretched string Fixed at both ends :-

Let us consider a stretched string of linear density 'μ', length 'l', and tension 't', is fixed at two rigid supports. Let the string is flucked the transverse wave is generated in the string which is called as progressive ~~wak~~ wave let it be moving along positive(x)direction. on reflection at the rigid support the reflected waves propagating opposite to the progressive wave i,e along negative x-direction. The super position of these two waves gives stationary waves with definite frequencies. ~~let~~

Let y_1 and y_2 be the displacements of the particle due to progressive wave and reflecting waves and can be expressed as

$$y_1 = Ae^{i(\omega t + kx)}$$

$$y_2 = Be^{i(\omega t - kx)}$$

From principle of super position the resultant displacement of the particle due to progressive wave and reflected wave is given by

$$y = y_1 + y_2$$

$$y = Ae^{i(\omega t + kx)} + Be^{i(\omega t - kx)}$$

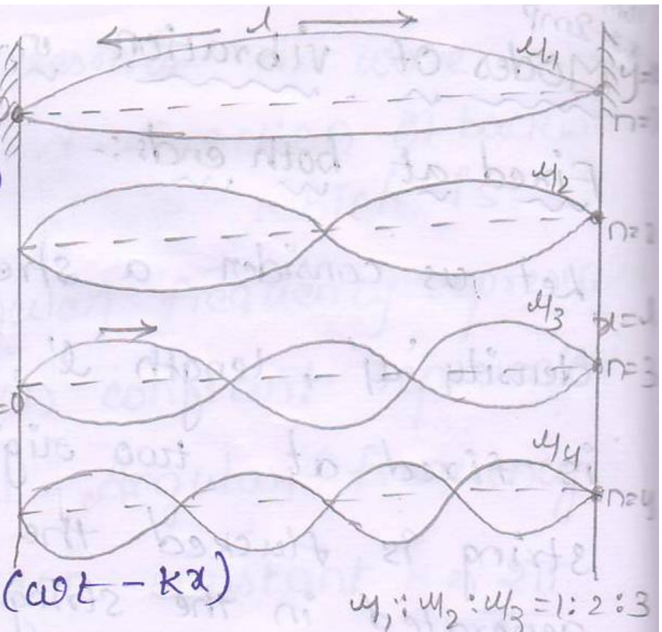
Let us apply the first Boundary condition at $x = 0$; $y = 0$

$$y = Ae^{i\omega t} + Be^{i\omega t} = 0$$

$$e^{i\omega t}[A+B] = 0$$

$$e^{\omega t} \neq 0 \; ; \; A+B = 0$$

$$A = -B$$



Hence

$$y = Ae^{i(\omega t + kx)} - Ae^{i(\omega t - kx)}$$

$$= Ae^{i\omega t}\left[e^{ikx} - e^{-ikx}\right]$$

$$= Ae^{i\omega t}[2i\sin kx]$$

Let us apply the second boundary condition at

$$x = l, \; y = 0$$

$$2iAe^{i\omega t}[\sin kl] = 0$$

$$2iAe^{i\omega t} \neq 0 \Rightarrow \sin kl = 0$$

$$\boxed{kl = n\pi} \rightarrow (1)$$

We know that

$$\omega = \frac{2\pi}{\lambda} = 2\pi \nu$$

$$k = \frac{2\pi}{\lambda} = \frac{T\omega}{\lambda} \qquad \left[\frac{\lambda}{T} = v\right]$$

$$k = \frac{\omega}{v}$$

sub the eqn (1)

$$\frac{\omega l}{v} = n\pi$$

$$\frac{2\cancel{\pi}\nu l}{v} = n\cancel{\pi}$$

$2 \nu l = n v$

n = Integer or number

$$\boxed{\nu_n = \frac{nv}{2l}}$$

v = velocity

l = length of the string

ν_n = frequency of n^{th} node

5m

Overtones and harmonics :-

Let us consider overtones and harmonics the case of a string fixed at both ends and flucked it. the progressive waves are generated in the string and which are gets reflected at the ends. The progressive waves and reflected waves interfere to produce stationary waves. The frequency of vibration will depend upon no. of nodes.

⇒ when the string is flucked at middle, it vibrates with nodes at the ends and antinode at the middle. This condition is known as fundamental vibrations or first harmonic vibrations. The frequency 'ν_1' is given by

$$\nu_n = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$$

⇒ when the string is flucked at one fourth of its length the string vibrates in two segments with one more node and antinode between the fixed ends

$$\nu_2 = \frac{2}{2l}\sqrt{\frac{T}{\mu}} = 2\nu_1$$

This is called second harmonic or 1^{st} overtone

⇒ when the string vibrates in 3 segments, the frequency of vibration is given

$$\nu_3 = \frac{3}{2l}\sqrt{\frac{T}{\mu}} \Rightarrow \cancel{3\nu} \; 3\nu_1$$

This is called third harmonic and second overtone.

In this case we have: $\nu_1 : \nu_2 : \nu_3 = 1:2:3:$ ___

5m *** **Laws of transverse vibrations :-**

* **First law :-**

The frequency of fundamental vibration is inversly propotional to the length of the string when tension and mass per unit length of constant

$$\boxed{\nu \propto \frac{1}{l}}$$

* **Second law :-**

The frequency of fundamental vibration is directly propotional to square root of tension when length and mass per unit length of constant

$$\boxed{\nu \propto \sqrt{T}}$$

* **Third law :-**

The frequency of fundamental vibration is inversly propotional to square root of mass per unit length of the wire when length and tension are constant

$$\boxed{\nu \propto \frac{1}{\sqrt{m}}}$$

This is upto 26.04.2021

UNIT - IVa → 2 Coupled oscillations

UNIT - V - 2 Ultrasonics pending.