7B Appilications of integral transforms

Unit-1 Laplace transforms – I

Defination:- Laplace transforms.

Let $f(t)$ be a function of t defined on all positive values of t. (That is $t \in (0,\infty)$) Then the Laplace transform of $f(t)$ is defined as $\int_0^\infty e^{-pt}.f(t).dt$ where p is a parameter which may be real (or) complex number and it is denoted by L

$L\{f(t)\} = \bar{f}(p)$

formula $L\{f(t)\} = \bar{f}(p) = \int_0^\infty e^{-pt}.f(t).dt$

① $L\{1\} = \frac{1}{p}$ $\qquad (\because L\{k\} = \frac{k}{p})$

$$L\{f(t)\} = \int_0^\infty e^{-pt}.f(t).dt$$

$$L(1) = \int_0^\infty e^{-pt}(1).dt$$

$$= \int_0^\infty e^{-pt}dt$$

$$= \left[\left(\frac{e^{-pt}}{-p}\right)\right]_{t=0}^{t=\infty}$$

$$= \frac{1}{-p}\left[e^{-pt}\right]_{t=0 \text{ lower limit}}^{t=\infty \text{ upper limit}}$$

$$= \frac{+1}{-p}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{1}{-p}[0-1]$$

$$= \frac{1}{-p}[-1]$$

$$= \frac{\cancel{-}1}{\cancel{-}p} = \boxed{\frac{1}{p}}$$

② $L(e^{at}) = \frac{1}{p-a}$

$$L\{f(t)\} = \int_0^\infty e^{-pt}.f(t).dt$$

put $f(t) = e^{at}$

$$L(e^{at}) = \int_0^\infty e^{-pt}.e^{at}\,dt$$

$$= \int_0^\infty e^{-pt+at}.dt$$

$$= \int_0^\infty e^{-(p-a)t}\,dt$$

$$= \left[\frac{e^{-(p-a)t}}{-(p-a)}\right]_{t=0}^{t=\infty}$$

$$= \frac{1}{-(p-a)}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{1}{-(p-a)}(0-0)$$

$$= \frac{-1}{-(p-a)} = \frac{1}{(p-a)}$$

③ $L\{e^{-at}\} = \frac{1}{(p+a)}$

put $a = -a$ In 2^{nd} form let

$$L\{e^{-at}\} = \frac{1}{p-(-a)} = \frac{1}{p+a}$$

④ $L\{\sin h\,at\} = \frac{a}{p^2-a^2}$

$$\sin hx = \frac{e^x - e^{-x}}{2}$$

$$\sin h\,at = \frac{e^{at} - e^{-at}}{2}$$

$$L\{\sin h\,at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{p-a} - \frac{1}{p+a}\right]$$

$= \frac{1}{2}\left[\frac{p+a-p+a}{(p-a)(p+a)}\right]$

$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}a}{p^2-a^2}\right]$

$= \frac{a}{p^2-a^2}$

⑤ $L\{\cosh at\} = \frac{p}{p^2-a^2}$

$\cosh x = \frac{e^x+e^{-x}}{2}$

$\cosh at = \frac{e^{at}+e^{-at}}{2}$

$L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$= \frac{1}{2}\left[\frac{p}{p-a} - \frac{p}{p+a}\right]$

$= \frac{1}{2}\left[\frac{p+\cancel{a}+p-\cancel{a}}{p^2-a^2}\right]$

$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}p}{p^2-a^2}\right]$

$= \frac{p}{p^2-a^2}$

⑥ $L\{\sin at\} = \frac{a}{p^2+a^2}$

$L\{e^{at}\} = \frac{1}{p-a}$

$L\{e^{-at}\} = \frac{1}{p+a}$

⑨ $\Gamma\frac{1}{2} = \sqrt{\pi}$

⑩ $\Gamma n = (n-1)\Gamma n-1$

⑪ $\cosh at = \frac{e^{at}+e^{-at}}{2}$

⑦ $L\{\cos at\} = \frac{1}{p^2+a^2}$

⑧ $L\{t^n\} = \frac{n!}{p^{n+1}} = \frac{\Gamma n+1}{p^{n+1}}$ Gamma of $n+1$ (Γ = Gamma)

$* \boxed{\Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx}$

Ex:-1 find $L\{7e^{2t} + 9e^{-2t} + 5\cos t + 7t^3 + 5\sin 3t + 7\sinh 2t +$
$9\cosh 3t + 3\}$

$= 7L\{e^{2t}\} + 9L\{e^{-2t}\} + 5L\{\cos t\} + 7L\{t^3\} + 5L\{\sin 3t\} +$
$7L\{\sinh 2t\} + 9L\{\cosh 3t\} + 3L\{1\}$

$= 7\left(\frac{1}{p-2}\right) + 9\left(\frac{1}{p+2}\right) + 5\left(\frac{p}{p^2+1^2}\right) + 7\left(\frac{3!}{p^2+3^2}\right) + 5\left(\frac{3}{p^2+3^2}\right) +$
$7\left(\frac{2}{p^2-2^2}\right) + 9\left(\frac{p}{p^2-3^2}\right) + 3\left(\frac{1}{p}\right)$

$= \frac{7}{p-2} + \frac{9}{p+2} + \frac{5p}{p^2+1} + \frac{42}{p^4} + \frac{15}{p^2+9} + \frac{14}{p^2-4} + \frac{9p}{p^2-9} + \frac{3}{p}$

Ex:- Find the laplace transform of

i) $(t^2+1)^2$ ii) $\sin 2t . \cos t$

iii) $\cosh^2 2t$ iv) $\sinh^3 3t$.

i) $F(t) = (t^2+1)^2 = t^4 + 1 + 2t^2$

$\therefore L\{f(t)\} = L\{t^4 + 1 + 2t^2\}$

$= L\{t^4\} + L\{1\} + L\{2t^2\}$

$= L\{t^4\} + L\{1\} + 2L\{t^2\}$

$= \frac{4!}{p^{4+1}} + \frac{1}{p} + 2\left(\frac{2!}{p^{2+1}}\right)$

$= \frac{24}{p^5} + \frac{1}{p} + \frac{4}{p^3}$

$= \frac{24 + p^4 + 4p^2}{p^5}$

ii) $F(t) = \sin 2t . \cos t$

Take $A = 2t$ and $B = t$

$\sin 2t . \cos t = \frac{\sin(2t+t) + \sin(2t-t)}{2}$

$= \frac{\sin 3t + \sin t}{2}$

$\therefore L\{f(t)\} = L\{\sin 2t \cdot \cos t\}$

$= L\{\frac{\sin 3t + \sin t}{2}\}$

$= \frac{1}{2}[L\{\sin 3t\} + L\{\sin t\}]$ $\quad [\because \sin(A+B)+\sin(A-B) = 2\sin A \cdot \cos B$

$= \frac{1}{2}[\frac{3}{p^2+3^2} + \frac{1}{p^2+1^2}]$

$= \frac{1}{2}[\frac{3}{p^2+9} + \frac{1}{p^2+1}]$

$= \frac{1}{2}[\frac{3p^2+3+p^2+9}{(p^2+9)(p^2+1)}] \Rightarrow \frac{2p^2+6}{(p^2+9)(p^2+1)}$

ii) $F(t) = \cosh^2 2t$

$= (\cosh 2t)^2$

$= [\frac{e^{2t}+e^{-2t}}{2}]^2$ $\quad [\because \cosh x = \frac{e^x+e^{-x}}{2}$

$(\frac{a}{b})^m = \frac{a^m}{b^m}$

$= \frac{(e^{2t}+e^{-2t})^2}{2^2}$ $\quad (a^m)^n = a^{m \cdot n}$

$= \frac{(e^{2t})^2+(e^{-2t})^2+2 \cdot \cancel{e^{2t}} \cdot \cancel{e^{-2t}}}{4}$

$= \frac{e^{4t}+e^{-4t}+2}{4}$

$\therefore L\{f(t)\} = L\{\cosh^2 2t\}$

$= L\{\frac{e^{4t}+e^{-4t}+2}{4}\}$

$= \frac{1}{4}[L\{e^{4t}\} + L\{e^{-4t}\} + L\{2\}]$

$= \frac{1}{4}[\frac{1}{p-4} + \frac{1}{p+4} + \frac{2}{p}]$

$= \frac{1}{4}[\frac{(p+4)(p)+(p)(p-4)+2(p-4)(p+4)}{(p-4)(p+4)(p)}]$

$= \frac{1}{4}[\frac{p^2+4p+p^2-4p+2(p^2-16)}{(p^2-16)(p)}]$

$$= \frac{1}{4}\left[\frac{4p^2-32}{(p^2-16)(p)}\right]$$

$$= \frac{\cancel{4}(p^2-8)}{\cancel{4}(p^2-16)(p)}$$

$$= \frac{p^2-8}{(p^2-16)(p)}$$

iv) $\sinh^3 3t$

$$(\sinh 3t)^3 = \left[\frac{e^{3t}-e^{-3t}}{2}\right]^3$$

$$(a-b)^3 = a^3-b^3-3ab(a-b)$$

$$= \frac{e^{9t}-e^{-9t}-3(e^{3t}-e^{-3t})}{8}$$

$$= \frac{e^{9t}-e^{-9t}-3e^{3t}+3e^{-3t}}{8}$$

$$\therefore L\{\sinh^3 3t\} = L\left\{\frac{e^{9t}-e^{-9t}-3e^{3t}+3e^{-3t}}{8}\right\}$$

$$= \frac{1}{8}\left[\frac{1}{p-a} - \frac{1}{p+a} - 3\frac{1}{p-3} + 3\frac{1}{p+3}\right]$$

$$\left(\because L\{e^{at}\} = \frac{1}{p-a},\ L\{e^{-at}\} = \frac{1}{p+a}\right)$$

$$= \frac{1}{8}\left[\frac{p+9-p+9}{(p-9)(p+9)} - 3\left(\frac{p+3-p+3}{(p-3)(p+3)}\right)\right]$$

$$= \frac{1}{8}\left[\frac{18}{p^2-81} - \frac{18}{p^2-9}\right]$$

$$= \frac{1}{8}\times 18\left[\frac{p^2-9-p^2+81}{(p^2-81)(p^2-9)}\right]$$

$$= \frac{1}{\cancel{8}}\times 81\times \overset{9}{\cancel{72}}\times\frac{1}{(p^2-81)(p^2-9)}$$

$$= \frac{162}{(p^2-81)(p^2-9)}$$

Ex:- show that $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{p}}$

$$LHS = L\left\{\frac{1}{\sqrt{\pi t}}\right\}$$

$$= L\left\{\frac{1}{\sqrt{\pi}\sqrt{t}}\right\}$$

$$= \frac{1}{\sqrt{\pi}} L\left\{\frac{1}{\sqrt{t}}\right\}$$

$$= \frac{1}{\sqrt{\pi}} L\left\{\frac{1}{t^{1/2}}\right\} \qquad \left(\because L(t^n) = \frac{\Gamma n+1}{p^{n+1}}\right)$$

$$= \frac{1}{\sqrt{\pi}} L\left\{t^{-\frac{1}{2}}\right\}$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma -\frac{1}{2}+1}{p^{-\frac{1}{2}+1}}$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma \frac{1}{2}}{p^{\frac{1}{2}}} \Rightarrow \frac{1}{\cancel{\sqrt{\pi}}} \cdot \frac{\cancel{\sqrt{\pi}}}{\sqrt{p}} \Rightarrow \boxed{\frac{1}{\sqrt{p}} = R.H.S}$$

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Ex:- using the Expansion $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots,$

show that $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2p^{3/2}} \cdot e^{-\frac{1}{4p}}$

Given that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$

$$\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \cdots\cdots$$

$$= t^{1/2} - \frac{t^{\frac{3}{2}}}{3!} + \frac{t^{\frac{5}{2}}}{5!} - \frac{t^{\frac{7}{2}}}{7!} + \cdots\cdots$$

$$L.H.S = L\{\sin\sqrt{t}\}$$

$$= L\left\{t^{\frac{1}{2}} - \frac{t^{\frac{3}{2}}}{3!} + \frac{t^{\frac{5}{2}}}{5!} - \frac{t^{\frac{7}{2}}}{7!} + \cdots\right\}$$

$$= L\{t^{\frac{1}{2}}\} - \frac{1}{3!} L\{t^{\frac{3}{2}}\} + \frac{1}{5!} L\{t^{\frac{5}{2}}\} - \frac{1}{7!} L\{t^{\frac{7}{2}}\} + \cdots\cdots \qquad \left[\because L\{t^n\} = \frac{\Gamma n+1}{p^{n+1}}\right]$$

$$= \frac{\Gamma \frac{1}{2}+1}{p^{\frac{1}{2}+1}} - \frac{1}{3!} \cdot \frac{\Gamma \frac{3}{2}+1}{p^{\frac{3}{2}+1}} + \frac{1}{5!} \cdot \frac{\Gamma \frac{5}{2}+1}{p^{\frac{5}{2}+1}} - \frac{1}{7!} \cdot \frac{\Gamma \frac{7}{2}+1}{p^{\frac{7}{2}+1}} + \cdots\cdots$$

$$= \frac{\Gamma \frac{3}{2}}{p^{\frac{3}{2}}} - \frac{1}{3!} \frac{\Gamma \frac{5}{2}}{p^{5/2}} + \frac{1}{5!} \frac{\Gamma 7/2}{p^{7/2}} - \frac{1}{7!} \frac{\Gamma 9/2}{p^{9/2}} + \cdots$$

$$= \frac{\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)}{p^{\frac{3}{2}}} - \frac{1}{3!}\frac{\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)}{p^{\frac{5}{2}}} + \frac{1}{5!}\frac{\left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right)}{p^{\frac{7}{2}}} - \frac{1}{7!}\frac{\left(\frac{9}{2}-1\right)\Gamma\left(\frac{9}{2}+1\right)}{p^{9/2}} + \cdots$$

$$= \frac{\frac{1}{2}\cdot\Gamma\frac{1}{2}}{p^{\frac{3}{2}}} - \frac{1}{3!}\frac{\frac{3}{2}\Gamma\frac{3}{2}}{p^{\frac{5}{2}}} + \frac{1}{5!}\frac{\frac{5}{2}\Gamma\frac{5}{2}}{p^{\frac{7}{2}}} - \frac{1}{7!}\frac{\frac{7}{2}\Gamma\frac{7}{2}}{p^{9/2}} + \cdots$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{p^{\frac{3}{2}}} - \frac{1}{3!}\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{p^{\frac{5}{2}}} + \frac{1}{5!}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{p^{\frac{7}{2}}} \cdots$$

$$= \frac{\sqrt{\pi}}{2p^{\frac{3}{2}}}\left[1 - \frac{1}{3!}\times\frac{\frac{3}{2}}{p} + \frac{1}{5!}\times\frac{\frac{5}{2}\times\frac{3}{2}}{p^2} - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2p^{\frac{3}{2}}}\left[1 - \frac{1}{1\times2\times\cancel{3}}\times\frac{\cancel{3}}{2p} + \frac{1}{\cancel{5}\times4\times\cancel{3}\times2\times1}\times\frac{\cancel{5}\times\cancel{3}}{2\cdot 2p^2} - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2p^{\frac{3}{2}}}\left[1 - \frac{1}{4p} + \frac{1}{2}\left(\frac{1}{16p^2}\right) - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2p^{\frac{3}{2}}}\left[1 - \frac{1}{4p} + \frac{\left(\frac{1}{4p}\right)^2}{2!} \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2p^{\frac{3}{2}}}\cdot e^{-\frac{1}{4p}} \qquad \left[\because e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \cdots\right]$$

$$\therefore \boxed{L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2p^{\frac{3}{2}}}\cdot e^{-\frac{1}{4p}}}$$

Ex:- Find the Laplace transform of the function.

$$F(t) = \begin{cases} 4, & 0 < t < 1 \\ 3, & t > 1 \end{cases}$$

By the defination,

$$L\{F(t)\} = \int_0^\infty e^{-pt}\cdot F(t)\cdot dt$$

$$= t\int_0^1 e^{-pt}. F(t)\,dt + t\int_1^\infty e^{-pt}. F(t).dt$$

$$= \int_0^1 e^{-pt}.4.dt + \int_1^\infty e^{-pt}.3.dt$$

$$= 4\left[\frac{e^{-pt}}{-p}\right]_0^1 + 3\left[\frac{e^{-pt}}{-p}\right]_1^\infty$$

$$= 4\left[\frac{e^{-p}}{-p} - \left(\frac{1}{-p}\right)\right] + 3\left[\frac{0}{-p} - \left(\frac{e^{-p}}{-p}\right)\right]$$

$$= \frac{4e^{-p}}{p} + \frac{4}{p} + \frac{3e^{-p}}{p}$$

$$= \frac{-4e^{-p}}{p} + \frac{4}{p} + \frac{3e^{-p}}{p}$$

$$= \frac{-4e^{-p} + 4 + 3e^{-p}}{p}$$

$$= \boxed{\frac{4 - e^{-p}}{p}}$$

* State and prove First shifting theorm.

Statement:- If $L\{F(t)\} = f(p)$, then $\underset{L.H.S}{L\{e^{at}.F(t)\}} = \underset{R.H.S}{f(p-a)}$

Proof:-

Given $L\{F(t)\} = f(p)$

$$L.H.S = L\{e^{at}.F(t)\}$$

$$= \int_0^\infty e^{-pt}.e^{at}.F(t).dt \qquad [\because a^m.a^n = a^{m+n}]$$

$$= \int_0^\infty e^{-pt+at}.F(t)\,dt$$

$$= \int_0^\infty e^{-(pt-at)}.F(t).dt$$

$$= \int_0^\infty e^{-(p-a)t}.F(t).dt$$

$$= f(p-a) = R.H.S \qquad \boxed{\therefore L\{e^{at}F(t)\} = f(p-a)}$$

Note:- $L\{e^{at}.F(t)\} = f(p-a)$

$= [f(p)]_{p\to p-a}$

$= [L\{F(t)\}]_{p\to p-a}$

Ex:- Find Laplace transform of

i) $e^{-t}.\cos 2t$

ii) $e^{-3t}.(2.\cos 5t - 3.\sin 5t)$

iii) $e^{-t}(3.\sin 2t - 5.\cosh 2t)$

iv) $e^{-at}.\sinh bt$

v) $e^{-at}.\cosh bt$

vi) $t.e^{2t}.\sin 3t$

vii) $(t+3)^2.e^t$

viii) $e^{-t}.\cos^2 t$

ix) $e^{3t}.\sin^2 t$

x) $e^{4t}.\sin 2t\cos t$

xi) $\cosh at.\sin bt$

xii) $e^{t}(\cos 2t + \frac{1}{2}\sinh 2t)$

xiii) $\sinh at.\sin at$

xiv) $t.\sin at$

xv) $t.\cos at$

i) $e^{-t}.\cos 2t$

Let $F(t) = \cos 2t \Rightarrow L\{F(t)\} = \frac{p}{p^2+4}$

$L\{e^{-t}.\cos 2t\} = [L\{F(t)\}]_{p\to p-a} = \left[\frac{p}{p^2+4}\right]_{p\to p+1}$

$= \frac{p+1}{(p+1)^2+4} = \frac{p+1}{p^2+2p+5}$

vi) $t.e^{2t}.\sin 3t$

Let $F(t) = t.\sin 3t$

$\therefore L\{t\sin 3t\} = L\{t.\text{ I.P of } e^{i3t}\}$

$= \text{I.P of } L\{t.e^{i3t}\}$

$= \text{I.P of } [L\{t\}]_{p\to p-3i}$

$= \text{I.P of } \left[\frac{1}{p^2}\right]_{p\to p-3i}$

$= \text{I.P of } \frac{1}{(p-3i)^2}$

$$= I.P \text{ of } \frac{1}{(P-3i)^2} \times \frac{(P+3i)^2}{(P+3i)^2}$$

$$= I.P \text{ of } \frac{(P+3i)^2}{(P^2+9)^2}$$

$$= I.P \text{ of } \frac{(P^2-9)+6Pi}{(P^2+9)^2}$$

$$= I.P \text{ of } \frac{P^2-9}{(P^2+9)^2} + i\left[\frac{6P}{(P^2+9)^2}\right]$$

$$= \frac{6P}{(P^2+9)^2}$$

$$\therefore L\{t.e^{2t}.\sin 3t\}$$

$$= L\{e^{2t}.t.\sin 3t\} = \left[\frac{6P}{(P^2+9)^2}\right]_{P\to P-2}$$

$$= \frac{6(P-2)}{((P-2)^2+9)^2}$$

$$= \frac{6(P-2)}{(P^2-4P+13)^2}$$

xiv) $t.\sin at$

$$L\{t.\sin at\} = L\{t.I.P \text{ of } e^{iat}\}$$

$$= I.P \text{ of } L\{t.e^{iat}\}$$

$$= I.P \text{ of } [L\{t\}]_{P\to P-ai}$$

$$= I.P \text{ of } \left(\frac{1}{P^2}\right)_{P\to P-ai}$$

$$= I.P \text{ of } \frac{1}{(P-ai)^2}$$

$$= I.P \text{ of } \frac{1}{(P-ai)^2} \times \frac{(P+ai)^2}{(P+ai)^2}$$

$$= I.P \text{ of } \frac{(P+ai)^2}{(P^2+a^2)^2}$$

$$= I.P \text{ of } \frac{(P^2-a^2)+i(2Pa)}{(P^2+a^2)^2}$$

$= \text{I.P of } \frac{p^2-a^2}{(p^2+a^2)^2} + i\left(\frac{2pa}{(p^2+a^2)^2}\right)$

$= \frac{2pa}{(p^2+a^2)^2}$

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* State and prove Second shifting theorm.

Statement:- If $L\{f(t)\} = f(p)$ and

$$G(t) = \begin{cases} F(t-a), & \text{if } t>a \\ 0, & \text{if } t<a \end{cases}$$

then, $L\{G(t)\} = e^{-ap}.f(p)$

proof:-

$L.H.S = L\{G(t)\}$

$= \int_0^\infty e^{-pt}.G(t).dt$

$= \int_0^a e^{-pt}.G(t)\,dt + \int_a^\infty e^{-pt}.G(t)\,dt$

$= \int_0^a e^{-pt}.0.dt + \int_a^\infty e^{-p}.F(t-a).dt$

$= 0 + \int_a^\infty e^{-pt}.F(t-a).dt$

$= \int_a^\infty e^{-pt}.F(t-a)\,dt$

Put $t-a=u$
$d(t-a)=du$
$dt-0=du$
$dt=du$
$t=a \Rightarrow a-a=u \Rightarrow 0=$
$t=\infty \Rightarrow \infty-a=u$

$= \int_0^\infty e^{-p(u+a)}.F(u).du$

$= \int_0^\infty e^{-pu-pa}.F(u)\,du$

$= \int_0^\infty e^{-pu}.e^{-pa}\,F(u)\,du$

$= e^{-pa}\int_0^\infty e^{-pu}.F(u)\,du$

$= e^{-pa}.f(p)$

$= R.H.S$

Ex:- Find the Laplace transform of $G(t)$, where

$$G(t) = \begin{cases} \cos\left(t-\frac{\pi}{3}\right), & \text{if } t > \frac{\pi}{3} \\ 0, & \text{if } t < \frac{\pi}{3} \end{cases}$$

By second shifting theorem,

$$L\{G(t)\} = e^{-ap} \cdot f(p)$$

$$= e^{-ap} \cdot L\{F(t)\} \longrightarrow (1) \text{ eq}$$

Given $F(t-a) = \cos\left(t-\frac{\pi}{3}\right)$

Here $a = \frac{\pi}{3}$

$\therefore F\left(t-\frac{\pi}{3}\right) = \cos\left(t-\frac{\pi}{3}\right)$

Let $t-\frac{\pi}{3} = t$

$\therefore F(t) = \cos t$

Sub. $F(t) = \cos t$ in eqn (1)

$$L\{G(t)\} = e^{-\frac{\pi}{3}p} \cdot L\{\cos t\}$$

$$= e^{-\frac{\pi}{3}p}\left(\frac{p}{p^2+1}\right)$$

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* state and prove change of scale property for Laplace transform.

Statement:- If $L\{F(t)\} = f(p)$, then $L\{F(at)\} = \frac{1}{a} \cdot f\left(\frac{p}{a}\right)$

Proof:-

Given $L\{F(t)\} = f(p)$

$$L.H.S = L\{F(at)\}$$

$$= \int_0^\infty e^{-pt} \cdot F(at) \cdot dt$$

[$\because$ put $at = u$, $t = \frac{u}{a}$, $dt = \frac{1}{a} \cdot du$, $t = 0 \Rightarrow u = 0$, $t = \infty \Rightarrow u = \infty$

$$= \int_0^\infty e^{-p\left(\frac{u}{a}\right)} \cdot F(u) \frac{1}{a} \cdot du$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{p}{a}\right)u} \cdot F(u)\, du$$

[$\because$ since u is a dummy variable So replace put $u = t$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{p}{a}t} \cdot F(t)\, dt$$

$= \frac{1}{a} \cdot f\left(\frac{p}{a}\right) = R.H.S$

$$\boxed{L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)}$$

Ex:- If $L\{F(t)\} = \frac{9p^2-12p+15}{(p-1)^3}$, then find $L\{F(3t)\}$. By using change of scale property.

Given $L\{F(t)\} = \frac{9p^2-12p+15}{(p-1)^3} = f(p)$

$\therefore f(p) = \frac{9p^2-12p+15}{(p-1)^3}$

$\therefore$ Here $a=3$

By using change of scale property

$L\{F(at)\} = \frac{1}{a} \cdot f(p/a)$

$\therefore L\{F(3t)\} = \frac{1}{3} \cdot f\left(\frac{p}{3}\right)$

$= \frac{1}{3}\left[\frac{9(p/3)^2 - 12(p/3)+15}{\left(\frac{p}{3}-1\right)^3}\right]$

$= \frac{9(p^2-4p+15)}{(p-3)^3}$

Ex:- Applying change of scale property, if $L\{F(t)\} = \frac{p^2-p+1}{(2p+1)^2 (p-1)}$

Then show that $L\{F(2t)\} = \frac{p^2-2p+4}{4(p+1)^2(p-2)}$

Ex:- If $L\{F(t)\} = \frac{1}{p} \cdot e^{-\frac{1}{p}}$, Then p.t $L\{e^{-t} \cdot F(3t)\} = \frac{e^{-3/p+1}}{p+1}$.

First we find $L\{F(3t)\}$ By using change of scale property

$L\{F(3t)\} = \frac{1}{3} \cdot f\left(\frac{p}{3}\right)$

$$= \frac{1}{3}\left[\frac{1}{\frac{p}{3}} \cdot e^{-\frac{1}{\frac{p}{3}}}\right] \qquad \left(\because f(p) = \frac{1}{p} e^{-\frac{1}{p}}\right)$$

$$= \frac{1}{p} \cdot e^{-\frac{3}{p}}$$

By using first shifting theorm,

$$L\{e^{at} \cdot F(t)\} = \left[L\{F(t)\}\right]_{p \to p-a}$$

$$\therefore L\{e^{-t} \cdot F(3t)\} = \left[L\{F(3t)\}\right]_{p \to p+1}$$

$$= \left[\frac{1}{p} \cdot e^{-\frac{3}{p}}\right]_{p \to p+1}$$

$$= \frac{1}{p+1} \cdot e^{-\frac{3}{p+1}}$$

$$= \frac{e^{-3/p+1}}{p+1}$$

Find the Laplace transform

(4) $e^{-at} \sinh bt$

Let $F(t) = \sinh bt$

$$L\{f(t)\} = L\{\sinh bt\}$$

$$= \frac{b}{p^2 - b^2}$$

By using 1st shifting theorm replace p by p+a

$$\therefore L\{e^{-at} \sinh bt\} = \frac{b}{(p+a)^2 - b^2}$$

(5) $e^{-at} \cosh bt$

Let $f(t) = \cosh bt$

$$L\{F(t)\} = \frac{1}{p^2 - b^2}$$

By 1st shifting theorm replace p by p+a

$$L\{e^{-at} \cosh bt\} = \frac{p+a}{(p+a)^2 - b^2}$$

(7) $(t+3)^2 e^t$

Let $F(t) = (t+3)^2$

$$L\{F(t)\} = L\{t^2+9+6t\}$$

$$= L\{t^2\} + L\{9\} + 6L\{t\}$$

$$= \frac{2!}{p^3} + \frac{9}{p} + 6 \cdot \frac{1!}{p^2}$$

$$= \frac{2}{p^3} + \frac{9}{p} + \frac{6}{p^2}$$

$$= \frac{2+9p^2+6p}{p^3}$$

By using first shifting theorm replace p by $(p-1)$

$$L\{(t+3)^2 e^t\} = \frac{2+9(p-1)^2+6(p-1)}{(p-1)^3}$$

$$= \frac{9p^2-12p+5}{(p-1)^3}$$

(8) $e^{-t} \cdot \cos^2 t$

Let $F(t) = \cos^2 t = \frac{1+\cos 2t}{2}$

$$L\{F(t)\} = \frac{1}{2}\left[L\{1\} + L\{\cos 2t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{p} + \frac{p}{p^2+4}\right]$$

By using first shifting theorm replace p by $(p+1)$

$$L\{e^{-t} \cdot \cos^2 t\} = \frac{1}{2}\left[\frac{1}{p+1} + \frac{p+1}{(p+1)^2+4}\right]$$

$$L\{F(t)\} = L\{\cos 2t\} + \frac{1}{2} L\{\sinh 2t\}$$

$$= \frac{p}{p^2+4} + \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{p^2-4}$$

By using first shifting theorm Replace p by p-1

$$L\{e^t(\cos 2t + \frac{1}{2}\sinh 2t)\} = \frac{p-1}{(p-1)^2+4} + \frac{1}{(p-1)^2-4}$$

$= \frac{p-1}{p^2-2p+5} + \frac{1}{p^2-2p-3}$

(11) $\cosh at \cdot \cos at$

$\cosh at = \frac{e^{at}+e^{-at}}{2}$

$\cosh at \cdot \cos at = \left(\frac{e^{at}+e^{-at}}{2}\right)\cos at$

$= \frac{1}{2}\left[e^{at}\cos at + e^{-at}\cos at\right)$

Let $F(t) = \cos at$

$L\{F(t)\} = \frac{p}{p^2+a^2}$

By using 1st shifting theorm Replace p by $(p-a)$

$L\{e^{at}\cos at\} = \frac{p-a}{(p-a)^2+a^2} = \frac{p-a}{p^2+a^2-2pa+a^2} = \frac{p-a}{p^2-2pa+2a^2}$

Again by 1st shifting theorm replace p by $p+a$

$L\{e^{-at}\cos at\} = \frac{p+a}{(p+a)^2+a^2} = \frac{p+a}{p^2+a^2+2pa+a^2} = \frac{p+a}{p^2+2pa+2a^2}$

$\therefore L\{\cosh at \cdot \cos at\} = \frac{1}{2}L\{e^{at}\cos at + e^{-at}\cos at\}$

$= \frac{1}{2}\left[\frac{p-a}{p^2-2pa+2a^2} + \frac{p+a}{p^2+2pa+2a^2}\right]$

$= \frac{1}{2}\left[\frac{p-a}{(p^2+2a^2)-2ap} + \frac{p+a}{(p^2+2a^2)+2ap}\right]$

$= \frac{1}{2}\frac{[(p-a)(p^2-2a^2)+2pa) + (p+a)((p^2+2a^2)-2ap]}{[(p^2+2a^2)-2ap][p^2+2a^2]+2ap}$

$= \frac{(p+ai)^2}{(p^2+a^2)^2} \Rightarrow \frac{p^2-a^2+i2ap}{(p^2+a^2)^2} \Rightarrow \frac{p^2-a^2}{(p^2+a^2)^2} + i\frac{2ap}{(p^2+a^2)^2}$

$\therefore L\{t\cdot\cos at + i\sin at\} = \frac{p^2-a^2}{(p^2+a^2)^2} + i\frac{2ap}{(p^2+a^2)^2}$

$= L\{t.\cos at\} + i L\{t.\sin at\}$

$$= \frac{p^2-a^2}{(p^2+a^2)^2} + i\frac{2ap}{(p^2+a^2)^2}$$

Equating the real and Imaginary part of in both sides.

$$L\{t.\cos at\} = \frac{p^2-a^2}{(p^2+a^2)^2}$$

$$L\{t\sin at\} = \frac{2ap}{(p^2+a^2)^2}$$

(13) $\sinh at.\sin at$

$$\sinh at = \frac{e^{at}-e^{-at}}{2}$$

$$\sinh at.\sin at = \left(\frac{e^{at}-e^{-at}}{2}\right)\sin at$$

$$= \frac{1}{2}[e^{at}\sin at - e^{-at}\sin at]$$

Let $F(t) = \sin at$

$$L\{F(t)\} = \frac{a}{p^2+a^2}$$

By using 1st shifting theorm replace p by p-a

$$= \frac{1}{2}\left[\frac{1}{p+1} + \frac{p+1}{p^2+2p+5}\right]$$

(9) $e^{3t}.\sin^2 t$

Let $F(t) = \sin^2 t = \frac{1-\cos 2t}{2}$

$$L\{F(t)\} = \frac{1}{2}[L\{1\} - L\{\cos 2t\}]$$

$$= \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+4}\right]$$

By using first shifting theorm replace p by p-3

$$L\{e^3.\sin^2 t\} = \frac{1}{2}\left[\frac{1}{p-3} - \frac{p-3}{(p-3)^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{1}{p-3} - \frac{p-3}{p^2-6p+3}\right]$$

(10) $e^{4t}\sin 2t.\cos t$

Let $F(t) = e^{4t}.\sin 2t.\cos t = \frac{\sin 3t+\sin t}{2}$

$$\therefore \left(\sin A.\cos B = \frac{\sin(A+B)+\sin(A-B)}{2}\right)$$

$$L\{F(t)\} = \frac{1}{2}[L\{\sin 3t\} + L\{\sin t\}]$$

$$= \frac{1}{2}\left[\frac{3}{p^2+9} + \frac{1}{p^2+1}\right]$$

By 1st shifting theorem replace p by (p-4)

$$L\{e^{4t} \cdot \sin 2t \cdot \cos t\} = \frac{1}{2}\left\{\frac{3}{(p-4)^2+9} + \frac{1}{(p-4)^2+1}\right\}$$

$$= \frac{1}{2}\left[\frac{3}{p^2-8p+25} + \frac{1}{p^2-8p+17}\right]$$

(12) $e^t[\cos 2t + \frac{1}{2}\sinh 2t]$

Let $F(t) = \cos 2t + \frac{1}{2}\sinh 2t$

$$= \frac{(p+ai)^2}{(p^2+a^2)^2} = \frac{p^2-a^2+i2ap}{(p^2+a^2)^2}$$

$$= \frac{p^2-a^2}{(p^2+a^2)^2} + \frac{i2ap}{(p^2+a^2)}$$

$$L\{t \cdot \cos at + i\sin at)\} = \frac{p^2-a^2}{(p^2+a^2)^2} + i\frac{2ap}{(p^2+a^2)^2}$$

$$L\{t \cdot \cos at\} + iL\{t \cdot \sin at\}$$

$$= \frac{p^2-a^2}{(p^2+a^2)^2} + i\frac{2ap}{(p^2+a^2)^2}$$

Equating the real and Imaginary part on both sides.

$$[L\{t\}\cos at] = \frac{p^2-a^2}{(p^2+a^2)^2}$$

$$L\{t \sin at\} = \frac{2ap}{(p^2+a^2)^2}$$

UNIT-II LAPLACE TRANSFORMS - II

Topics:-

→ L.T of derivatives

→ L.T of integrals

→ L.T of $t^n . F(t)$

→ L.T of $\frac{F(t)}{t}$

→ Evaluation of integrals by L.T

① L.T of derivatives :-

Formulae:- ① $L\{F'(t)\} = P.L\{F(t)\} - F(0)$

First derivative

Proof:- $LHS = L\{F'(t)\}$

$$= \int_0^\infty e^{-pt}. F'(t).dt$$

$$= \left[e^{-pt} \int F'(t)\,dt - \int\left(\frac{d}{dt}(e^{-pt})\int F'(t)\,dt\right)dt\right]_0^\infty$$

$$= \left(e^{-pt}. F(t) - \int e^{-pt}.(-P).F(t)\,dt\right)_0^\infty$$

$$\left[\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx\right]$$

$$= \left(e^{-pt}. F(t)\right)_0^\infty + P\left[\int_0^\infty e^{-pt}. F(t)\,dt\right]$$

$$= 0 - F(0) + P. L\{F(t)\} \Rightarrow P.L\{F(t)\} - F(0)$$

② $L\{F''(t)\} = P^2 L\{F(t)\} - P.F(0) - F'(0)$

Proof:- $LHS = L\{F''(t)\}$

2nd derivative

$$= \int_0^\infty e^{-pt}. F''(t).dt$$

$$= \left[e^{-pt}. \int F''(t)\,dt - \int\left(\frac{d}{dt}(e^{-pt})\int F''(t)\,dt\right)dt\right]_0^\infty$$

$$= \left(e^{-pt}. F'(t)\right)_0^\infty - \int_0^\infty e^{-pt}.(-P).F'(t)\,dt$$

$$= -F'(0) + P\int_0^\infty e^{-pt}. F'(t)\,dt$$

$$= -F'(0) + P.L\{F'(t)\}$$

$$= -F'(0) + P[PL\{F(t)\} - F(0)]$$

$= -F'(0) + p^2 L\{F(t)\} - pF(0)$

$= p^2 L\{F(t)\} - p\cdot F(0) - F'(0)$.

Ex:- Find the laplace transform of

i) e^{at} ii) $\cos at$ iii) $\sin at$

i) e^{at}

Let $F(t) = e^{at}$

$= F'(t) = e^{at}\cdot a = a\cdot e^{at}$

By first formula,

$L\{F'(t)\} = p\cdot L\{F(t)\} - F(0)$

$\therefore L\{a e^{at}\} = p\cdot L\{e^{at}\} - 1$

$= a L\{e^{at}\} = p\cdot L\{e^{at}\} - 1$

$\Rightarrow 1 = p\cdot L\{e^{at}\} - a L\{e^{at}\}$

$\Rightarrow 1 = L\{e^{at}\}\cdot (p-a)$

$\Rightarrow \frac{1}{p-a} = L\{e^{at}\}$

$\Rightarrow \boxed{L\{e^{at}\} = \frac{1}{p-a}}$

ii) $\cos at$

Let $F(t) = \cos at$

$\Rightarrow F'(t) = -\sin at\cdot (a) = -a\cdot \sin at$

$\Rightarrow F''(t) = -a\cdot \cos at\cdot a = -a^2\cdot \cos at$

By 2nd Formula,

$L\{F''(t)\} = p^2 L\{F(t)\} - p F(0) - F'(0)$

$\therefore L\{-a^2 \cos at\} = p^2 L\{\cos at\} - p(1) - 0$

$= -a^2 L\{\cos at\} = p^2 L\{\cos at\} - p$

$= p = p^2 L\{\cos at\} + a^2 L\{\cos at\}$

$= p = L\{\cos at\}\cdot (p^2+a^2)$

$= \frac{p}{p^2+a^2} = L\{\cos at\}$

$$= \boxed{L\{\cos at\} = \frac{P}{p^2+a^2}}$$

iii) sinat Let $F'(t) = \sin at$

$= \cos at$

Let $F'(t) = -\sin at \cdot a \ni F''(t) = -a^2 \cdot \sin at$

by Second Formula

$$L\{F''(t)\} = P^2 L\{F(t)\} - P(F(0) - F'(0)$$

$$= L\{-a^2 \sin at\} = P^2 L\{\sin at\} - a(1) - 0$$

$$= -a^2 L\{\sin at\} = p^2 L\{\sin at\} - a$$

$$= a = p^2 L\{\sin at\} + a^2 L\{\sin at\}$$

$$= a = L\{\sin at\}(p^2+a^2)$$

$$= \frac{a}{p^2+a^2} = L\{\sin at\}$$

* If $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2p^{3/2}} \cdot e^{-1/4p}$, then find $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$

Let $F(t) = \sin\sqrt{t}$

$$= F'(t) = \cos\sqrt{t} \cdot \frac{1}{2\sqrt{t}} = \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

By 1st formula,

$$L\{F'(t)\} = P \cdot L\{F(t)\} - F(0)$$

$$\therefore L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = P \cdot L\{\sin\sqrt{t}\} - 0$$

$$= \frac{1}{\cancel{2}} \cdot L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = P \cdot \frac{\sqrt{\pi}}{\cancel{2}p^{3/2}} \cdot e^{-\frac{1}{4}P}$$

$$= L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\cancel{P} \cdot \sqrt{\pi}}{\cancel{P} \cdot \sqrt{P}} \cdot e^{-\frac{1}{4}P}$$

$$= \sqrt{\frac{\pi}{P}} \cdot e^{-1/4P}$$

ii) Laplace transform of integrals:-

Formula:- If $L\{F(t)\} = f(P)$, then $L\left\{\int_0^t F(t)\,dt\right\}$

$$= \frac{1}{P} f(P).$$

Proof :- Let $G(t) = \int_0^t F(t)\,dt$

Then $G'(t) = \frac{d}{dt} \int_0^t F(t)\,dt$

$\Rightarrow G'(t) = F(t)$

we know that $L\{G'(t)\} = P.L\{G(t)\} - G(0)$

$\therefore L\{F(t)\} = P.L\{\int_0^t F(t)\,dt\} - 0$

$= f(P) = P.L\{\int_0^t F(t)\,dt\}$

$= \frac{f(P)}{P} = L\{\int_0^t F(t)\,dt\}$

$= \boxed{L\{\int_0^t F(t).dt = \frac{1}{P}.f(P)}$

* Find $L\{\int_0^t e^{-t}.\cos t\,dt\}$

Let $f(t) = e^{-t}.\cos t$

$= L\{F(t)\} = L\{e^{-t}.\cos t\}$

(By FST)

$= L\{\cos t\}_{P \to P+1}$

$= \frac{P}{(P^2+1)}_{P \to P+1}$

$= \frac{P+1}{(P+1)^2+1} = \frac{P+1}{P^2+2P+2} = f(P)$

$\therefore L\{\int_0^t e^{-t}.\cos t.\,dt\} = \frac{1}{P}.f(P)$

$= \frac{1}{P}.\left[\frac{P+1}{P^2+2P+2}\right]$

* Find $L\{\int_0^t t.e^{-t}.\sin t.dt\}$

Let $F(t) = t.e^{-t}.\sin t$

$= L\{F(t)\} = L\{t.e^{-t}.\sin t\}$

$= L\{e^{-t}.t.\sin t\}$

$= [L\{t.\sin t\}]_{P \to P+1}$

$= \left[\frac{2P}{(P^2+1)^2}\right]_{P \to P+1}$

$$= \frac{2(p+1)}{((p+1)^2+1)^2} \Rightarrow \frac{2p+2}{(p^2+2p+2)^2}$$

* $L\{\int_0^t e^t \cosh dt\}$

* $L\{\int_0^t \sinh dt \, dt\}$

* **Laplace transform of $t^n \cdot F(t)$:-**

Formula:- $L\{t^n \cdot F(t)\} = (-1)^n \cdot \frac{d^n}{dp^n}[f(p)]$, where $f(p) = L\{F(t)\}$

Ex:- Find i) $L\{t \cdot \sin at\}$ ii) $L\{t \cdot e^{2t} \cdot \sin 3t\}$

iii) $L\{t \cdot e^{-t} \cdot \sin 2t\}$

Sol i) $L\{t \cdot \underbrace{\sin at}_{F(t)}\} = (-1)^1 \cdot \frac{d^1}{dp^1}[L\{\sin at\}]$

$$= (-1) \cdot \frac{d}{dp}\left[\frac{a}{p^2+a^2}\right]$$

$$= -\left[\frac{(p^2+a^2)(0) - a(2p)}{(p^2+a^2)^2}\right]$$

$$= \frac{2ap}{(p^2+a^2)^2}$$

ii) $L\{t \cdot \underbrace{e^{2t} \cdot \sin 3t}_{F(t)}\}$

$$= (-1)^1 \cdot \frac{d^1}{dp^1}[L\{e^{2t} \cdot \sin 3t\}]$$

$$= (-1) \frac{d}{dp}\left[(L\{\sin 3t\})_{p \to p-2}\right]$$

$$= -\frac{d}{dp}\left(\frac{3}{p^2+9}\right)_{p \to p-2}$$

$$= -\frac{d}{dp}\left[\frac{3}{(p-2)^2+9}\right]$$

$$= -\frac{d}{dp}\left[\frac{3}{p^2-4p+13}\right]\begin{matrix}\to u\\ \to v\end{matrix}$$

$$= -\left[\frac{(p^2-4p+13)(0) - 3(2p-4)}{(p^2-4p+13)^2}\right]$$

$$= \frac{3(2p-4)}{(p^2-4p+13)^2}$$

iii) $L\{t.e^{-t}.\sin 2t\}$

$$\underset{F(t)}{\downarrow} \quad = (-1)^1 \frac{d^1}{dp^1}\left[L\{e^{-t}\sin 2t\}\right]$$

$$= -1\frac{d}{dp}\left[L\{\sin 2t\}_{p\to p+1}\right]$$

$$= -\frac{d}{dp}\left[\left(\frac{2}{p^2+4}\right)\right]_{p\to p+1}$$

$$d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$= -\frac{d}{dp}\left[\left(\frac{2}{p^2+4}\right)_{p\to p+1}\right]$$

$$= \frac{-d}{dp}\left[\frac{2}{p^2+1+2p-4p+4}\right]$$

$$= -\frac{d}{dp}\left[\frac{2}{p^2-2p+5}\right]$$

$$= -\frac{(p^2-2p+5)(0) - 2(2p+2)}{(p^2-2p+5)^2}$$

$$= \frac{4p+4}{(p^2-2p+5)^2}$$

* Laplace transform of division by t:

Formula :- $L\left\{\frac{F(t)}{t}\right\} = \int_p^{\infty} L\{F(t)\}\, dp$

Ex:- find the laplace transform of

i) $\frac{\sin t}{t}$ ii) $\frac{e^{-at} - e^{-bt}}{t}$ iii) $L\left\{\int_0^t e^{-t} \cos t\, d\right.$

i) $\frac{\sin t}{t}$

Let $F(t) = \sin t$

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_p^{\infty} L\{\sin t\}\, dp$$

$$= \int_p^{\infty} \frac{1}{p^2+1}\, dp$$

$$= \left[\tan^{-1} p\right]_p^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1} p$$

$$= \frac{\pi}{2} - \tan^{-1} p$$

$$= \cot^{-1} p \qquad (\tan^{-1}x + \cot^{-1}x) = \frac{\pi}{2}$$

ii) $\frac{e^{-at} - e^{-bt}}{t}$

Let $F(t) = e^{-at} - e^{-bt}$

$$\therefore L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \int_p^{\infty} L\{e^{-at} - e^{-bt}\}\, dp$$

$$= \int_p^{\infty} \left[\frac{1}{p+a} - \frac{1}{p+b}\right] dp$$

$$= \left[\log(p+a) - \log(p+b)\right]_p^{\infty}$$

$$= \log\left(\frac{p+a}{p+b}\right)_p^{\infty}$$

$$= \log\left(\frac{p\left(1+\frac{a}{p}\right)}{p\left(1+\frac{b}{p}\right)}\right)_p^\infty$$

$$= \log\left(\frac{1+0}{1+0}\right) - \log\left(\frac{1+\frac{a}{p}}{1+\frac{b}{p}}\right)$$

$$= 0 - \log\left(\frac{p+a}{p+b}\right)$$

$$= -\log\left[\frac{p+a}{p+b}\right]$$

$$= \log\frac{1}{\left(\frac{p+a}{p+b}\right)} \Rightarrow \log\left[\frac{p+b}{p+a}\right]$$

(iii) $L\left[\int_0^t e^{-t}\cdot \cos t\, dt\right]$

$F(t) = e^{-t}\cdot \cos t.$

$$L[F(t)] = L[e^{-t}\cdot \cos t]$$

$$= L[\cos t]_{p\to p+1}$$

$$= \left(\frac{p}{p^2+1}\right)_{p+1}$$

$$= \frac{p+1}{(p+1)^2+1}$$

$$= \frac{p+1}{p^2+1+2p+1} =$$

$$= \frac{p+1}{p^2+2p+2}.$$

Evaluation of integrals by Laplace transforms

Ex:- i) $\int_0^{\infty} t \cdot e^{-3t} \cdot dt$ ii) $\int_0^{\infty} e^{-4t} \cdot \sin 3t \cdot dt$

i) $\int_0^{\infty} t \cdot e^{-3t} \cdot dt$

$\left[\because L\{F(t)\} = \int_0^{\infty} e^{-pt} \cdot F(t) dt \right.$

$\left. \because L\{t^n\} = \frac{n!}{p^{n+1}} \right.$

$F(t) = t$

$= \left[L\{t^1\} \right]_{p=3}$

$= \left[\frac{1!}{p^{1+1}} \right] = \left(\frac{1}{p^2} \right)_{p=3}$

$= \left(\frac{1}{3^2} \right) = \frac{1}{9}$

ii) $\int_0^{\infty} e^{-4t} \cdot \sin 3t \cdot dt$

$= \left[L\{\sin 3t\} \right]_{p=4}$ $\left[\because L\{\sin at\} = \frac{a}{p^2+a^2} \right.$

$= \left[\frac{3}{p^2+3^2} \right]_{p=4}$

$= \left(\frac{3}{p^2+9} \right)_{p=4} = \frac{3}{4^2+9} \Rightarrow \frac{3}{16+9} \Rightarrow \frac{3}{25}$

Ex:- Show that i) $\int_0^{\infty} t \cdot e^{-3t} \sin t \, dt \Rightarrow \frac{3}{50}$

ii) $\int_0^{\infty} t^3 \cdot e^{-t} \cdot \sin t \, dt = 0$

i) LHS $= \int_0^{\infty} t \cdot \cancel{e^{-3t}} \cdot \sin t \, \cancel{dt}$

$= \left[L\{t \cdot \sin t\} \right]_{p=3}$

$= L\{t \cdot \text{I.P of } e^{it}\}_{p=3}$

$= \text{I.P of } L\{t \cdot e^{it}\}_{p=3}$

$= \left(\text{I.P of } (L\{t\})_{p \to p-i} \right)_{p=3}$ (By using f.s.t)

$$= \left(I.P \text{ of } \left(\frac{1}{p^2}\right)_{p \to p-i}\right)_{p=3}$$

$$= \left(I.P \text{ of } \frac{1}{(p-i)^2}\right)_{p=3}$$

$$= \left(I.P \text{ of } \frac{1}{(p-i)^2} \times \frac{(p+i)^2}{(p+i)^2}\right)_{p=3}$$

$$= \left(I.P \text{ of } \frac{(p+i)^2}{(p^2+1)^2}\right)_{p=3}$$

$$= \left(\frac{2p}{(p^2+1)^2}\right)_{p=3} = \frac{2\times 3}{(3^2+1)^2} = \frac{6}{100} = \boxed{\frac{3}{50}} \text{ R.H.S}$$

ii) L.H.S $\int_0^\infty t^3 . e^{-t} . \sin t \, dt$ -

$$= \left[L\{t^3 . \sin t\}\right]_{p=1}$$

$$= L\{t^3 . I.P \text{ of } e^{it}\}_{p=1}$$

$$= I.P \text{ of } L\{t^3 . e^{it}\}_{p=1}$$

$$= I.P \text{ of } \left(\left(L\{t^3\}\right)_{p \to p-i}\right)_{p=1}$$

$$= \left(I.P \text{ of } \left(\frac{3!}{p^4}\right)_{p \to p-i}\right)_{p=1}$$

$$= \left(I.P \text{ of } \frac{6}{(p-i)^4}\right)_{p=1}$$

$$= \left(I.P \text{ of } \frac{6}{(p-i)^4} \times \frac{(p+i)^4}{(p+i)^4}\right)_{p=1}$$

$$= \left(I.P \text{ of } \frac{6(p+i)^4}{(p^2+1)^4}\right)_{p=1}$$

$$= \left[\frac{6[4p)(p^2-1)}{(p^2+1)^4}\right]_{p=1}$$

$\therefore [\because (p+i)^2)^2 = (p^2-1)^2 - 4p^2 + 2(2pi)(p^2-1)$

$$= \frac{24(1)\,(1^2-1)}{(1^2+1)^4} = \boxed{0} \text{ R.H.S.}$$

Ex:- Evaluate $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt$ and hence show that

$$\int_0^\infty \frac{e^{-3t}-e^{-6t}}{t}\,dt = \log 2$$

Already we have

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \log\left(\frac{p+b}{p+a}\right)$$

Now, $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt$

$$= \int_0^\infty 1.\left(\frac{e^{-at}-e^{-bt}}{t}\right)dt$$

$$= \int_0^\infty e^{0t}\left[\frac{e^{-at}-e^{-bt}}{t}\right]dt$$

$$= \left(L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}\right)_{p=0}$$

$$= \left(\log\frac{p+b}{p+a}\right)_{p=0}$$

$$= \log\frac{0+b}{0+a}$$

$$= \log\frac{b}{a}$$

Take $a=3$, and $b='6'$

$$\therefore \int_0^\infty \frac{e^{-3t}-e^{-6t}}{t}\,dt \Rightarrow \log\frac{\cancel{6}\,2}{\cancel{3}} \Rightarrow \log 2$$

Bessel's function:-

Bessel function of order "n" is defined by

$$J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{t}{2}\right)^{n+2r}$$

10m
2023 Ex:- prove that $L\{J_0(at)\} = \frac{1}{\sqrt{p^2+a^2}}$

UNIT-3 **Inverse Laplace Transforms**

Def:- If $L\{f(t)\} = f(p)$ then $L^{-1}\{f(p)\} = f(t)$

Formula:-

① $L^{-1}\left\{\frac{1}{p}\right\} = 1$

② $L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at}$

③ $L^{-1}\left\{\frac{1}{p+a}\right\} = e^{-at}$

④ $L^{-1}\left\{\frac{a}{p^2+a^2}\right\} = \sin at$

⑤ $L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$

⑥ $L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{n!}$

Ex:- find the inverse Laplace transform of

① $\frac{3p}{p^2+25}$ ② $\frac{2p-5}{p^2+4}$ ③ $\frac{3p-8}{4p^2+25}$ ④ $\frac{2p+1}{p^2-4}$

⑤ $\frac{2p-5}{4p^2+25}$

① $L^{-1}\left\{\frac{3p}{p^2+25}\right\}$

$= 3 \cdot \cos 5t$

② $L^{-1}\left\{\frac{2p-5}{p^2+4}\right\}$

$= L^{-1}\left\{\frac{2p}{p^2+4}\right\} - L^{-1}\left\{\frac{5}{p^2+4}\right\}$

$= 2 \cdot \cos 2t - \frac{5}{2} \cdot \sin 2t$

③ $L^{-1}\left\{\frac{3p-8}{4p^2+25}\right\}$

$$L^{-1}\left\{\frac{3p}{4p^2+25}\right\} - L^{-1}\left\{\frac{8}{4p^2+25}\right\}$$

$$= \frac{3}{4} L^{-1}\left\{\frac{p}{p^2+\frac{25}{4}}\right\} - \frac{8}{4} L^{-1}\left\{\frac{1}{p^2+\frac{25}{4}}\right\}$$

$$= \frac{3}{4} L^{-1}\left\{\frac{p}{p^2+\left(\frac{5}{2}\right)^2}\right\} - 2L^{-1}\left\{\frac{1}{p^2+\left(\frac{5}{2}\right)^2}\right\}$$

$$= \frac{3}{4} \cdot \cos\frac{5}{2}t - \frac{2}{5/2} L^{-1}\left\{\frac{\frac{5}{2}}{p^2(5/2)^2}\right\}$$

$$= \frac{3}{4}\cos\frac{5}{2} - \frac{4}{5}\sin\frac{5}{2}t$$

(iv) $L^{-1}\left\{\frac{2p+1}{p^2+4}\right\}$

$$= L^{-1}\left\{\frac{2p}{p^2+4}\right\} + L^{-1}\left\{\frac{1}{p^2+4}\right\}$$

$$= 2\cdot\cos.2t + L^{-1}\left\{\frac{1}{p^2+4}\right\}$$

$$= 2\cdot\cos 2t + \frac{1}{2}\sin 2t$$

(v) $\frac{2p-5}{4p^2+25}$

$$L^{-1}\left\{\frac{2p}{4p^2+25}\right\} - L^{-1}\left\{\frac{5}{4p^2+25}\right\}$$

$$= \frac{2}{4}\left\{\frac{p}{p^2+\frac{25}{4}}\right\} - L^{-1}\frac{5}{4}\left\{\frac{1}{p^2+\frac{25}{4}}\right\}$$

$$= 2L^{-1}\left\{\frac{p}{p^2+\left(\frac{5}{2}\right)^2}\right\} - \frac{5}{4}L^{-1}\left\{\frac{+1}{p^2+\left(\frac{5}{2}\right)^2}\right\}$$

$$= 2\cdot\cos\frac{5}{2} + \frac{-\frac{5}{4}}{\frac{5}{2}} L^{-1}\left\{\frac{\frac{5}{2}}{p^2+\left(\frac{5}{2}\right)^2}\right\}$$

$$= 2\cdot\cos\frac{5}{2} + -\frac{1}{2}\sin 5/2$$

Ex 10m show that $L^{-1}\left\{\frac{1}{p}\cdot\sin\frac{1}{p}\right\} = t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \frac{t^7}{(7!)^2} + \cdots$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin\frac{1}{p} = \frac{1}{p} - \frac{\left(\frac{1}{p}\right)^3}{3!} + \frac{\left(\frac{1}{p}\right)^5}{5!} - \frac{\left(\frac{1}{p}\right)^7}{7!} + \cdots$$

$$= \frac{1}{p} - \frac{1}{3!\,p^3} + \frac{1}{5!\,p^5} - \frac{1}{7!\,p^7} + \cdots$$

$$\frac{1}{p}\cdot\sin\frac{1}{p} = \frac{1}{p}\left[\frac{1}{p} - \frac{1}{3!\,p^3} + \frac{1}{5!\,p^5} - \frac{1}{7!\,p^7} + \cdots\right]$$

$$= \left[\frac{1}{p^2} - \frac{1}{3!\,p^4} + \frac{1}{5!\,p^6} - \frac{1}{7!\,p^8} + \cdots\right]$$

$$L.H.S = L^{-1}\left\{\frac{1}{p}\cdot\sin\frac{1}{p}\right\}$$

$$= L^{-1}\left\{\frac{1}{p^2} - \frac{1}{3!\,p^4} + \frac{1}{5!\,p^6} - \frac{1}{7!\,p^8} + \cdots\right\}$$

$$= L^{-1}\left\{\frac{1}{p^2}\right\} - \frac{1}{3!}L^{-1}\left\{\frac{1}{p^4}\right\} + \frac{1}{5!}L^{-1}\left\{\frac{1}{p^6}\right\} - \frac{1}{7!}L^{-1}\left\{\frac{1}{p^8}\right\} + \cdots$$

$$= \frac{t^1}{1!} - \frac{1}{3!}\cdot\frac{t^3}{3!} + \frac{1}{5!}\cdot\frac{t^5}{5!} - \frac{1}{7!}\cdot\frac{t^7}{7!} + \cdots \qquad \boxed{\because L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{n!}}$$

$$= t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \frac{t^7}{(7!)^2} + \cdots \quad R.H.S$$

Ex show that $L^{-1}\left\{\frac{1}{p}\cdot\cos\frac{1}{p}\right\} = t - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2}$

we know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos\frac{1}{p} = \frac{1}{p} - \frac{\left(\frac{1}{p}\right)^2}{2!} + \frac{\left(\frac{1}{p}\right)^4}{4!} - \frac{\left(\frac{1}{p}\right)^6}{6!} + \cdots$$

$$= \frac{1}{p} - \frac{1}{2!\,p^2} + \frac{1}{4!\,p^4} - \frac{1}{6!\,p^6} + \cdots$$

$$\frac{1}{p}\cdot\cos\frac{1}{p} = \frac{1}{p}\left[\frac{1}{p} - \frac{1}{2!\,p^2} + \frac{1}{4!\,p^4} - \frac{1}{6!\,p^6} + \cdots\right]$$

$$= \left[\frac{1}{p^2} - \frac{1}{2!\,p^3} + \frac{1}{4!\,p^5} - \frac{1}{6!\,p^7} + \cdots\right]$$

$L.H.S = L^{-1}\left\{\frac{1}{P}\cdot\cos\frac{1}{P}\right\}$

$= L^{-1}\left\{\frac{1}{P^2} - \frac{1}{2!\,P^3} + \frac{1}{4!\,P^5} - \frac{1}{6!}P^7 + \cdots\right\}$

$= L^{-1}\left\{\frac{1}{P^2}\right\} - \frac{1}{2!}\left\{\frac{1}{P^3}\right\} + \frac{1}{4!}\left\{\frac{1}{P^5}\right\} - \frac{1}{6!}\left\{\frac{1}{P^7}\right\} + \cdots$

$= \frac{t^0}{1!} - \frac{1}{2!}\cdot\frac{t^2}{2!} + \frac{1}{4!}\cdot\frac{t^4}{4!} - \frac{1}{6!}\cdot\frac{t^6}{6!}$

$= 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} = R.H.S$

Finding inverse LapLace transforms By using partial fractions

Ex: Find $L^{-1}\left\{\frac{1}{(P+1)(P-2)}\right\}$

$\frac{1}{(P+1)(P-2)} = \frac{A}{(P+1)} + \frac{B}{P+2}$

$\frac{1}{\cancel{(P+1)(P-2)}} = \frac{A(P-2) + B(P+1)}{\cancel{(P+1)(P-2)}}$

$1 = A(P-2) + B(P+1)$

$P=2 \Rightarrow 1 = B(3) \Rightarrow \boxed{B = \frac{1}{3}}$

$P=-1 \Rightarrow 1 = A(-3) = \boxed{A = \frac{1}{-3}}$

$\therefore \frac{1}{(P+1)(P-2)} = \frac{\frac{1}{-3}}{P+1} + \frac{\frac{1}{3}}{P-2}$

$\therefore L^{-1}\left\{\frac{1}{(P+1)(P-2)}\right\} = L^{-1}\left\{\frac{-\frac{1}{3}}{P+1} + \frac{\frac{1}{3}}{P-2}\right\}$

$= L^{-1}\left\{\frac{-\frac{1}{3}}{P+1}\right\} + L^{-1}\left\{\frac{\frac{1}{3}}{P-2}\right\}$

$= \frac{-1}{3}L^{-1}\left\{\frac{1}{P+1}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{P-2}\right\}$

$= \frac{-1}{3}\left[e^{-1t}\right] + \frac{1}{3}\left[e^{2t}\right]$

$= -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}$

Ex ② Find $L^{-1}\left\{\frac{p^2+p-2}{p^2(p^2+2)(p^2-2)}\right\}$

Ex ③ Find $L^{-1}\left\{\frac{3p+1}{(p-1)(p^2+1)}\right\}$

$$\frac{3p+1}{(p-1)(p^2+1)} = \frac{A}{(p-1)} + \frac{Bp+C}{p^2+1}$$

$$= \frac{3p+1}{(p-1)\cancel{(p^2+1)}} = \frac{A(p^2+1)+(Bp+C)(p-1)}{(p-1)\cancel{(p^2+1)}}$$

$$= 3p+1 = A(p^2+1)+(Bp+C)(p-1)$$

$$p=1 \Rightarrow 4 = A(2) \Rightarrow \boxed{A=2}$$

p^2 coefficients $\Rightarrow 0 = A+B$

$$0 = 2+B \Rightarrow \boxed{B=-2}$$

p coefficient $= +3 = -B+C$

$$+3 = -(-2)+C$$

$$+3 = 2+C$$

$$\boxed{C=1}$$

$$\therefore \frac{3p+1}{(p-1)(p^2+1)} = \frac{2}{p-1} + \frac{-2p+1}{p^2+1}$$

$$\therefore L^{-1}\left\{\frac{3p+1}{(p-1)(p^2+1)}\right\}$$

$$= L^{-1}\left\{\frac{2}{p-1} + \frac{-2p+1}{p^2+1}\right\}$$

$$= L^{-1}\left\{\frac{2}{p-1} - \frac{2p}{p^2+1} + \frac{1}{p^2+1}\right\}$$

$$= L^{-1}\left\{\frac{2}{p+1}\right\} - L^{-1}\left\{\frac{2p}{p^2+1}\right\} + L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$$= 2 \cdot L^{-1}\left\{\frac{1}{p-1}\right\} - 2L^{-1}\left\{\frac{p}{p^2+1}\right\} + L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$$= 2 \cdot e^{t} - 2 \cdot \cos 1t + \sin 1t$$

$$= 2e^{t} - 2\cos t + \sin t$$

* Find $L^{-1}\left\{\dfrac{p^2}{(p^2+4)(p^2+25)}\right\}$

Formula :- $\dfrac{1}{(p^2+a^2)(p^2+b^2)} = \dfrac{1}{(b^2-a^2)}\left[\dfrac{1}{p^2+a^2} - \dfrac{1}{p^2+b^2}\right]$

$$\therefore \frac{1}{(p^2+4)(p^2+25)} = \frac{1}{25-4}\left[\frac{1}{p^2+4} - \frac{1}{p^2+25}\right]$$

$$= \frac{1}{21}\left[\frac{1}{p^2+4} - \frac{1}{p^2+25}\right]$$

Now, $\dfrac{p^2}{(p^2+4)(p^2+25)} = \dfrac{1}{21}\left[\dfrac{p^2}{p^2+4} - \dfrac{p^2}{p^2+25}\right]$

$$\therefore \cancel{L^{-1}\left\{\frac{p^2}{(p^2+4)(p^2+25)}\right\} = L^{-1}\left\{\frac{1}{21}\right.}$$

$$= \frac{1}{21}\left[\frac{(p^2+4)-(4)}{p^2+4} - \frac{(p^2+25)-25}{p^2+25}\right]$$

$$= \frac{1}{21}\left[\left(1-\frac{4}{p^2+4}\right) - \left(1-\frac{25}{p^2+25}\right)\right]$$

$$= \frac{1}{21}\left[\frac{25}{p^2+25} - \frac{4}{p^2+4}\right]$$

$$\therefore L^{-1}\left\{\frac{p^2}{(p^2+4)(p^2+25)}\right\} = L^{-1}\left\{\frac{1}{21}\left[\left(\frac{25}{p^2+25}\right) - \left(\frac{4}{p^2+4}\right)\right]\right\}$$

$$= \frac{1}{21}\left[L^{-1}\left\{\frac{25}{p^2+25}\right\} - L^{-1}\left\{\frac{4}{p^2+4}\right\}\right]$$

$$= \frac{1}{21}\left[\frac{\cancel{25}^{5}}{\cancel{5}}(\sin 5t) - \frac{4}{\cancel{2}}(\sin 2t)\right]$$

$$= \frac{1}{21}(5\sin 5t - 2\sin 2t)$$

* Finding inverse Laplace transform By using first shifting theorm.

Statement :- If $L^{-1}\{f(p)\} = f(t)$, then $L^{-1}\{f(p-a)\} = e^{at}.f(t)$

Ex:- ① Find $L^{-1}\left\{\dfrac{1}{p^2+2p+5}\right\}$

$$L^{-1}\left\{\frac{1}{p^2+2p+5}\right\}$$

$= L^{-1}\left\{\frac{1}{p^2+2p+1+4}\right\}$ $\left[(a+b)^2 = a^2+2ab+b^2\right.$ $\left(p^2 = 2(p)(1)+1+4 \right.$

$= L^{-1}\left\{\frac{1}{(p+1)^2+4}\right\}$

$= L^{-1}\Big\{ = e^{-1\cdot t}\cdot L^{-1}\{f(p)\}$

$= e^{-t}\cdot L^{-1}\left\{\frac{1}{p^2+4}\right\}$ $\left[\because f(p+1) = \frac{1}{(p+1)^2+4}\right.$

$= e^{-t}\cdot\frac{1}{2}\cdot\sin 2t$

$= \frac{e^{-t}\cdot\sin 2t}{2}$

Ex:-② Find $L^{-1}\left\{\frac{p+3}{p^2-10p+29}\right\}$

$= L^{-1}\left\{\frac{p+3}{p^2-10p+29}\right\}$

$= L^{-1}\left\{\frac{p+3}{p^2-10p+25+4}\right\}$

$= L^{-1}\left\{\frac{p+3}{(p-5)^2+4}\right\}$

$= L^{-1}\left\{\frac{(p-5)+8}{(p-5)^2+4}\right\}$

$= e^{5t}\cdot L^{-1}\left\{\frac{p+8}{p^2+4}\right\}$

$= e^{5t}\left[L^{-1}\left\{\frac{p}{p^2+4}\right\} + L^{-1}\left\{\frac{8}{p^2+4}\right\}\right]$

$= e^{5t}\left[\cos 2t + \frac{\cancel{8}^{4}}{\cancel{2}}\cdot\sin 2t\right]$

$= e^{5t}\left[\cos 2t + 4\sin 2t\right]$

Ex:③ Find $L^{-1}\left\{\frac{p}{p^2+4p+5}\right\}$

$= L^{-1}\left\{\frac{p}{p^2+4p+5}\right\}$

$= L^{-1}\left\{\frac{p}{p^2+4p+4+1}\right\}$

$$= L^{-1}\left\{\frac{p}{(p+2)^2+1}\right\}$$

$$= L^{-1}\left\{\frac{(p+2)-2}{(p+2)^2+1}\right\}$$

$$e^{-2t}\, L^{-1}\left\{\frac{p-2}{p^2+1}\right\}$$

$$= e^{-2t}\left[L^{-1}\left\{\frac{p}{p^2+1}\right\} - L^{-1}\left\{\frac{2}{p^2+1}\right\}\right]$$

$$= e^{-2t}\left[\cos t - 2.\sin t\right]$$

Ex:- ④ Find $L^{-1}\left\{\frac{2p+3}{p^2+2p+2}\right\}$

$$L^{-1}\left\{\frac{2p+3}{p^2+2p+2}\right\}$$

$$L^{-1}\left\{\frac{2p+3}{p^2+2p+1+1}\right\}$$

$$L^{-1}\left\{\frac{2p+3}{(p+1)^2+1}\right\}$$

$$= L^{-1}\left\{\frac{2p+2+1}{(p+1)^2+1}\right\}$$

$$= L^{-1}\left\{\frac{2(p+1)+1}{(p+1)^2+1}\right\}$$

$$= e^{-1.t}\, L^{-1}\left\{\frac{2p+1}{p^2+1}\right\}$$

$$= e^{-t}\left[L^{-1}\left\{\frac{2p}{p^2+1}\right\} + L^{-1}\left\{\frac{1}{p^2+1}\right\}\right]$$

$$= e^{-t}\left[2.\cos t + \sin t\right]$$

Ex:- Find $L^{-1}\left\{\frac{3p-2}{p^2-4p+20}\right\}$

$$3e^{2t}\cos 4t + e^{2t}\sin 4t$$

Ex: Find $L^{-1}\left\{\frac{p+2}{(p-2)^3}\right\}$

$$= L^{-1}\left\{\frac{(P-2)+4}{(P-2)^3}\right\}$$

$$= e^{2t}\, L^{-1}\left\{\frac{P+4}{P^3}\right\}$$

$$= e^{2t}\left[L^{-1}\left\{\frac{P}{P^3}\right\} + L^{-1}\left\{\frac{4}{P^3}\right\}\right]$$

$$= e^{2t}\left[L^{-1}\left\{\frac{1}{P^2}\right\} + 4L^{-1}\left\{\frac{1}{P^3}\right\}\right]$$

$$= e^{2t}\left\{\frac{t^1}{1!} + 4\cdot\frac{t^2}{2!}\right\}$$

$$= e^{2t}\left[t + 2t^2\right]$$

Ex:- Find $L^{-1}\left\{\frac{P}{(P+3)^2}\right\}$

$$L^{-1}\left\{\frac{P}{(P+3)^2}\right\}$$

$$= L^{-1}\left\{\frac{P+3-3}{(P+3)^2}\right\}$$

$$= L^{-1}\left\{\frac{P+3}{(P+3)^2}\right\} - L^{-1}\left\{\frac{3}{(P+3)^2}\right\}$$

$$= e^{3t}\left\{\frac{P+3}{P+3}\right\} - L^{-1}\left\{\frac{3}{P^3}\right\}$$

$$= e^{-3t}\left\{t + 3L^{-1}\left\{\frac{1}{P^3}\right\}\right\}$$

$$= e^{-3t}\{t + 3$$

* Finding inverse Laplace transform By using second shifting theorm

Statement:- $L^{-1}\{e^{-aP}.\,f(P)\} = G(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

where $L^{-1}\{f(P)\} = f(t)$

Ex:- Find $L^{-1}\left\{\frac{e^{-5P}}{(P-2)^4}\right\}$

Here $a=5$, $f(p) = \frac{1}{(p-2)^4}$

$$= L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{(p-2)^4}\right\}$$

$$= e^{2t} \cdot L^{-1}\left\{\frac{1}{p^4}\right\}$$

$$= e^{2t} \cdot \frac{t^3}{3!} \Rightarrow \boxed{\frac{e^{2t} \cdot t^3}{6} = F(t)}$$

$$\therefore L^{-1}\{e^{-ap} \cdot f(p)\} = G(t) = \begin{cases} \frac{e^{2(t-5)}}{6} (t-5 & t > 5 \\ 0 & t < 5 \end{cases}$$

Ex:- Find $L^{-1}\left\{\frac{e^{-2p}}{p^2+4p+5}\right\}$

Here $a=2$, $f(p) = \frac{1}{p^2+4p+5}$

$$L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{p^2+4p+5}\right\}$$

$$= L^{-1}\left\{\frac{1}{p^2+4p+4+1}\right\}$$

$$= L^{-1}\left\{\frac{1}{(p+2)^2+1}\right\}$$

$$= e^{-2t} \cdot L^{-1}\left\{\frac{1}{p^2+1}\right\} \quad \text{(By F.s.t)}$$

$$= e^{-2t} \cdot \sin t = F(t)$$

$\therefore$ Now, $L^{-1}\left\{\frac{e^{-2p}}{p^2+4p+5}\right\} = G(t) = \begin{cases} f(t-2), & t > 2 \\ 0, & t < 2 \end{cases}$

$$= \begin{cases} e^{-2(t-2)} \cdot \sin(t-2), & t > 2 \\ 0, & t < 2 \end{cases}$$

Finding inverse laplace transforms By using change of scale property

If $L^{-1}\{f(p)\} = F(t)$, then $L^{-1}\{f(ap)\} = \frac{1}{a} \cdot F\left(\frac{t}{a}\right)$

Ex:- ① If $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2}\cdot t\cdot \sin t$, find $L^{-1}\left\{\frac{8p}{(4p^2+1)^2}\right\}$

Given $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2}\cdot t\cdot \sin t$

Here $f(p) = \frac{p}{(p^2+1)^2}$, $F(t) = \frac{1}{2}\cdot t\cdot \sin t$

write "ap" for "p"

$$f(ap) = \frac{ap}{((ap)^2+1)^2} = \frac{ap}{(a^2p^2+1)^2}$$

$$\Rightarrow L^{-1}\{f(ap)\} = L^{-1}\left\{\frac{ap}{(a^2p^2+1)^2}\right\}$$

$$= \frac{1}{a}\cdot F\left(\frac{t}{a}\right) = L^{-1}\left\{\frac{ap}{(a^2p^2+1)^2}\right\}$$

$$= \frac{1}{a}\cdot\frac{1}{2}\cdot\frac{t}{a}\cdot\sin\frac{t}{a} = L^{-1}\left\{\frac{ap}{(a^2p^2+1)^2}\right\}$$

$$= \frac{t}{2a^2}\cdot\sin\frac{t}{a} = L^{-1}\left\{\frac{ap}{(a^2p^2+1)^2}\right\}$$

$$= L^{-1}\left\{\frac{ap}{(a^2p^2+1)^2}\right\} = \frac{t}{2a^2}\cdot\sin\frac{t}{a}$$

put $a=2$

$$= L^{-1}\left\{\frac{2p}{(4p^2+1)^2}\right\} = \frac{t}{8}\cdot\sin\frac{t}{2}$$

$$= 4\times L^{-1}\left\{\frac{2p}{(4p^2+1)^2}\right\} = \cancel{4}\frac{t}{\cancel{8}}\sin\frac{t}{2}$$

$$= L^{-1}\left\{\frac{8p}{(4p^2+1)^2}\right\} = \frac{t}{2}\cdot\sin\frac{t}{2}$$

Ex:-2 If $L^{-1}\left\{\frac{p^2-1}{(p^2+1)^2}\right\} = t\cdot\cos t$, Find $L^{-1}\left\{\frac{9p^2-1}{(9p^2+1)^2}\right\}$

Given $L^{-1}\left\{\frac{p^2-1}{(p^2+1)^2}\right\} = t\cdot\cos t$

Here $f(p) = \frac{p^2-1}{(p^2+1)^2}$ and $F(t) = t\cdot\cos t$

write "ap" for "p"

$$f(ap) = \frac{(ap)^2-1}{((ap)^2+1)^2} = \frac{a^2p^2-1}{(a^2p^2+1)^2}$$

$$= L^{-1}\{f(ap)\} = L^{-1}\left\{\frac{a^2p^2-1}{(a^2p^2+1)^2}\right\}$$

$$= \frac{1}{a}\cdot F\left(\frac{t}{a}\right) = L^{-1}\left\{\frac{a^2p^2-1}{(a^2p^2+1)^2}\right\}$$

$$= \frac{1}{a}\cdot\frac{t}{a}\cdot\cos\frac{t}{a} = L^{-1}\left\{\frac{a^2p^2-1}{(a^2p^2+1)^2}\right\}$$

put $a=3$

$$\frac{1}{3}\cdot\frac{t}{3}\cdot\cos\frac{t}{3} = L^{-1}\left\{\frac{9p^2-1}{(9p^2+1)^2}\right\}$$

$$= \frac{t}{9}\cdot\cos\frac{t}{3} = L^{-1}\left\{\frac{9p^2-1}{(9p^2+1)^2}\right\}$$

$$= L^{-1}\left\{\frac{9p^2-1}{(9p^2+1)^2}\right\} = \frac{t}{9}\cdot\cos\frac{t}{3}$$

Finding inverse Laplace transforms By using derivatives:

$$L^{-1}\{f^{(n)}(p)\} = (-1)^n\cdot t^n\cdot F(t)$$

Ex:- Finding the inverse Laplace transform of i) $\log\left(1+\frac{1}{p^2}\right)$ ii) $\log\left(\frac{p+3}{p+4}\right)$

i) let $f(p) = \log\left(1+\frac{1}{p^2}\right) = \log\left(\frac{p^2+1}{p^2}\right)$

$$= \log(p^2+1) - \log p^2$$

$$\therefore f'(p) = \frac{1}{p^2+1}(2p) - \frac{1}{p^2}(2p)$$

$$= f'(p) = \frac{2p}{p^2+1} - \frac{2}{p}$$

$$= L^{-1}\{f'(p)\} = L^{-1}\left\{\frac{2p}{p^2+1} - \frac{2}{p}\right\}$$

$$= (-1)^1\cdot t^1\cdot F(t) = 2L^{-1}\left\{\frac{p}{p^2+1}\right\} - 2L^{-1}\left\{\frac{1}{p}\right\}$$

$$= -t\cdot F(t) = 2\cdot\cos t - 2(1)$$

$$= F(t) = \frac{2\cos t - 2}{-t} \Rightarrow \frac{-(2\cos t - 2)}{t}$$

$$= \frac{2 - 2\cos t}{t}$$

$$= L^{-1}\{f(p)\} = \frac{2-2\cos t}{t}$$

$$= L^{-1}\{\log(1+\frac{1}{p^2})\} = \frac{2-2\cos t}{t}$$

ii) Let $f(p) = \log\left(\frac{p+3}{p+4}\right) = \log(p+3) - \log(p+4)$

$$f'(p) = \frac{1}{p+3}\cdot(1) - \frac{1}{p+4}$$

$$L^{-1}\{f'(p)\} = L^{-1}\left\{\frac{1}{p+3} - \frac{1}{p+4}\right\}$$

$$(-1)^1 \cdot t^1 \cdot F(t) = e^{-3t} - e^{-4t}$$

$$-t \cdot F(t) = e^{-3t} - e^{-4t}$$

$$F(t) = \frac{e^{-3t} - e^{-4t}}{+t}$$

$$L^{-1}\{f(p)\} = \frac{e^{-3t} - e^{-4t}}{t}$$

$$L^{-1}\{\log\left(\frac{p+3}{p+4}\right)\} = \frac{e^{-3t} - e^{-4t}}{t}$$

Ex:- Find $L^{-1}(\tan^{-1} p)$

Let $f(p) = \tan^{-1} p$

$$f'(p) = \frac{1}{1+p^2}$$

$$L^{-1}\{f'(p)\} = L^{-1}\left\{\frac{1}{1+p^2}\right\}$$

$$= (-1)^1 \cdot t^1 \cdot F(t) = \sin t$$

$$F(t) = \frac{\sin t}{-t}$$

$$L^{-1}\{f(p)\} = \frac{\sin t}{-t}$$

$L^{-1}\{Tan^{-1}p\} = -\frac{\sin t}{t}$

Ex:- Find $L^{-1}\{Cot^{-1}p\}$

Let $f(p) = Cot^{-1}p$

$f'(p) = \frac{-1}{1+p^2}$

$L^{-1}\{(f(p))'\} = L^{-1}\{\frac{-1}{1+p^2}\}$

$(-1)^1 t^1 F(t) = -\sin t$

$F(t) = \frac{\cancel{-}\sin t}{\cancel{-}t}$

$L^{-1}\{F(t)\} = \frac{\sin t}{t}$

$L^{-1}\{Cot^{-1}p\} = \frac{\sin t}{t}$

Convolution:- Let $F(t)$ and $G(t)$ be two functions, where $t>0$. The Convolution product of $F(t)$ and $G(t)$ denoted by $F(t) * G(t)$, is defined as

$$F(t) * G(t) = \int_0^t F(u) \cdot G(t-u)\, du$$

* 10m state and prove Convolution theorim.

statement:- If $L\{F(t)\} = f(p)$ and $L\{G(t)\} = g(p)$, then $L\{F(t) * G(t)\} = f(p) \cdot g(p)$

proof:-

$L.H.S = L\{F(t) * G(t)\}$

$= \int_0^\infty e^{-pt} (F(t) * G(t))\, dt$

$= \int_{t=0}^{t=\infty} e^{-pt} \left[\int_{u=0}^{u=t} F(u) \cdot G(t-u)\, du\right] dt$

By changing the order of integration,

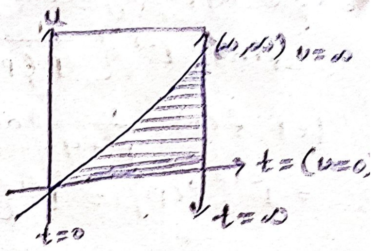
Limits :- $t : 0 \to \infty$ (c)

$u : 0 \to t$ (v)

New Limits :-

$t : u \to \infty$ (v)

$u : 0 \to \infty$ (c)



$$= \int_{u=0}^{\infty} e^{-pt}\left[\int_{t=u}^{\infty} F(u) \cdot G(t-u)\, dt\right] du$$

$$= \int_{u=0}^{\infty} e^{-pt}\left[\int_{t=u}^{\infty} e^{-pu} \cdot e^{pu} \cdot F(u) \cdot G(t-u)\, dt\right] du$$

$$= \int_{u=0}^{\infty} e^{-pu} \cdot F(u)\left[\int_{t=u}^{\infty} e^{-pt} \cdot e^{pu} \cdot G(t-u)\, dt\right] du$$

$$= \int_{u=0}^{\infty} e^{-pu} \cdot F(u)\left(\int_{t=u}^{\infty} e^{-p(t-u)} \cdot G(t-u)\, dt\right] du$$

$$= \int_{u=0}^{\infty} e^{-pu} \cdot F(u)$$

$\because$ put $t-u=v$

$d(t-u)=dv$

$dt-du=dv$

$dt-0=dv$

$dt=dv$

$t=u \Rightarrow u-u$

$\Rightarrow v=0$

$t=\infty \Rightarrow \infty-$

$$= \int_{u=0}^{\infty} e^{-pu} \cdot F(u)\left(\int_{v=0}^{v=\infty} e^{-pv} \cdot G(v)\, dv\right) du$$

$$= \int_{u=0}^{\infty} e^{-pu} \cdot F(u) \cdot \left[L\{G(t)\}\right] du$$

$$= \int_{u=0}^{\infty} e^{-pu} \cdot F(u) \cdot g(p)\, du$$

$$= g(p)\left[\int_{u=0}^{\infty} e^{-pu} \cdot F(u) \cdot du\right]$$

$$= g(p) \cdot L\{F(t)\}$$

$$= g(p) \cdot f(p)$$

$$= f(p) \cdot g(p) = R.H.S$$

$\therefore$ L.H.S = R.H.S

$\therefore L\{F(t) * G(t)\} = f(p) \cdot g(p)$

Ex:- ① Applying Convolution theorem,

find $L^{-1}\left\{\frac{1}{p(p^2+4)}\right\}$

Let $f(p) = \frac{1}{p^2+4}$ and $g(p) = \frac{1}{p}$

$= L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{p^2+4}\right\}$ | $L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{p}\right\}$

$F(t) = \frac{1}{2}\sin 2t$ | $G(t) = 1$

By using Convolution theorem,

$$L^{-1}\{f(p)\cdot g(p)\} = F(t) * G(t)$$

$$\therefore L^{-1}\left\{\frac{1}{p^2+4}\cdot\frac{1}{p}\right\} = \frac{1}{2}\sin 2t * 1$$

$$\Rightarrow L^{-1}\left\{\frac{1}{p(p^2+4)}\right\} = \int_0^t \frac{1}{2}\sin 2u \cdot 1\, du$$

$$= \frac{1}{2}\int_0^t \sin 2u\, du$$

$$= \frac{1}{2}\left[\frac{-\cos 2u}{2}\right]_0^t$$

$$= \frac{1}{4}\left[(-\cos 2t) - (-1)\right]$$

$$= \frac{1}{4}\left[-\cos 2t + 1\right]$$

$$= \frac{1}{4}\left[1 - \cos 2t\right]$$

Ex:- Using convolution theorem, Evaluate $L^{-1}\left\{\frac{1}{p(p^2+2p+2)}\right\}$

Let $f(p) = \frac{1}{p^2+2p+2}$ and $g(p) = \frac{1}{p}$

Then $L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{p^2+2p+2}\right\}$ | $L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{p}\right\} = 1$

$= L^{-1}\left\{\frac{1}{(p+1)^2+1}\right\}$ | $G(t) = 1$

$= e^{-t}L^{-1}\left\{\frac{1}{p^2+1}\right\}$

$= e^{-t}\cdot\sin t$

$\therefore F(t) = e^{-t}\sin t$

By applying convolution theorem,

$$L^{-1}\{f(p)\cdot g(p)\} = F(t) * G(t)$$

$$\therefore L^{-1}\left\{\frac{1}{p^2+2p+2}\cdot\frac{1}{p}\right\} = e^{-t}\cdot\sin t * 1$$

$$= L^{-1}\left\{\frac{1}{p(p^2+2p+2)}\right\} = e^{-t}\cdot\sin t * 1$$

$$= \int_0^t e^{-u}\cdot\sin u\cdot 1\, du$$

$$= \int_0^t e^{-u}\cdot\sin u\cdot du$$

$$= \left[\frac{e^{-u}}{(-1)^2+(1)^2}\left[-1\cdot\sin u - 1\cdot\cos u\right]\right]_0^t$$

$$= \left[\frac{e^{-u}}{2}\left[-\sin u - \cos u\right]\right]_{u=0}^{u=t}$$

$$\boxed{\begin{array}{l} \because \int e^{ax}\cdot\sin bx\, dx \\ \Rightarrow = \frac{e^{ax}}{a^2+b^2}\left[a\cdot\sin bx - b\cdot\cos bx\right] \\ \Rightarrow \because a=-1,\ b=1 \\ x=u \end{array}}$$

$$= \frac{e^{-t}}{2}\left[-\sin t - \cos t\right] - \frac{1}{2}\left[-1\right]$$

$$= \frac{e^{-t}}{2}\left[-\sin t - \cos t\right] + \frac{1}{2}$$

$$= \frac{1 - e^{-t}(\sin t + \cos t)}{2}$$

Ex:-3 Using Applying convolution theorem, find $L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\}$

$$\frac{p}{(p^2+a^2)^2} = \frac{p\times 1}{(p^2+a^2)\cdot(p^2+a^2)} = \underbrace{\left[\frac{p}{p^2+a^2}\right]}_{f(p)}\cdot\underbrace{\left(\frac{1}{p^2+a^2}\right)}_{g(p)}$$

Let $f(p) = \frac{p}{p^2+a^2}$ and $g(p) = \frac{1}{p^2+a^2}$

then $L^{-1}\{f(p)\} = L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at = F(t)$

$$L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{1}{a}\cdot\sin at = G(t)$$

By using convolution theorem,

$$L^{-1}\{f(p)\cdot g(p)\} = F(t) * G(t)$$

$$\therefore L^{-1}\left\{\frac{p}{p^2+a^2}\cdot\frac{1}{p^2+a^2}\right\} = \cos at * \frac{1}{a}\sin at$$

$$\Rightarrow L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} = \int_0^t \cos au \cdot \frac{1}{a} \cdot \sin a(t-u)\, du$$

$$= \frac{1}{a}\int_0^t \cos au \cdot \sin(at - au)\, du$$

$$= \frac{1}{2a}\int_0^t 2\cos \underset{A}{au} \cdot \sin(\underset{B}{at-au})\, du$$

$$= \frac{1}{2a}\int_0^t [\sin(\cancel{au}+at-\cancel{au}) - \sin(au - at + au)]\, du$$

$\boxed{2\cos A \cdot \sin B = \sin(A+B) - \sin(A-B)}$

$$= \frac{1}{2a}\int_0^t [\sin at - \sin(2au - at)]\, du$$

$$= \frac{1}{2a}\left(\sin at \cdot (u)_0^t - \left(-\frac{\cos(2au-at)}{2a}\right)_0^t \right) \quad ①$$

$$= \frac{1}{2a}\left[(\sin at)\cdot(t) + \frac{1}{2a}[\cos \cancel{a}t - \cos \cancel{a}t]\right]$$

$$= \frac{1}{2a}\left[t \cdot \sin at + \frac{1}{2a}(0)\right]$$

$$= \frac{1}{2a}[t \cdot \sin at + 0]$$

$$= \frac{1}{2a}\cdot t \cdot \sin at = \frac{t}{2a}\cdot \sin at$$

$$\therefore L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} = \frac{t}{2a}\cdot \sin at$$

Note:- put $a=1 \Rightarrow L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{t}{2(1)}\cdot \sin(1)t \Rightarrow \frac{t}{2}\sin t$

UNIT-4 APPLICATION OF LAPLACE TRANSFORMS

→ Solution of differential equation with constant coefficients

→ Solution of differential equation of with variable coefficients

→ Solution of integral equation

→ Converting the differential equation into integral equation and integral equation into differential equation

① Solution of differential equation with constant coefficients:

Formula :- ① $L\{y'(t)\} = p \cdot L\{y(t)\} - y(0)$

② $L\{y''(t)\} = p^2 \cdot L\{y(t)\} - p \cdot y(0) - y'(0)$

Proof :- ① L.H.S $= L\{y'(t)\}$

$$= \int_0^\infty \underset{u}{e^{-pt}} \cdot \underset{v}{y'(t)} \cdot dt$$

$\because \int uv\,dx = u\int v\,dx - \int \left(\frac{du}{dx} \int v\,dx\right)dx$

$$= \left[e^{-pt} \cdot y(t) - \int -p \cdot e^{-pt} \cdot y(t)\,dt\right]_0^\infty$$

$$= \left[e^{-pt} \cdot y(t) + p\int e^{-pt} \cdot y(t) \cdot dt\right]_0^\infty$$

$$= \left[e^{-pt} \cdot y(t)\right]_0^\infty + p\int_0^\infty e^{-pt} \cdot y(t) \cdot dt$$

$$= 0 - y(0) + p \cdot L\{y(t)\}$$

$$= -y(0) + p \cdot L\{y(t)\}$$

$$= p \cdot L\{y(t)\} - y(0) = R.H.S.$$

② L.H.S $= L\{y''(t)\}$

$$= \int_0^\infty \underset{u}{e^{-pt}} \cdot \underset{v}{y''(t)}\,dt$$

$$= \left[e^{-pt} \cdot y'(t) - \int -p \cdot e^{-pt} \cdot y'(t)\,dt\right]_0^\infty$$

$= \left(e^{-pt} \cdot y'(t)\right) + p\int e^{-pt} \cdot y'(t)\, dt\Big)_0^{\infty}$

$= \left(e^{-pt} y'(t)\right)_0^{\infty} + p\int_0^{\infty} e^{-pt} \cdot y'(t)\, dt$

$= 0 - y'(0) + p \cdot L\{y'(t)\}$

$= -y'(0) + p \cdot [p \cdot L\{y(t)\} - y(0)]$

$= -y'(0) + p^2 L\{y(t)\} - p \cdot y(0)$

$= p^2 \cdot L\{y(t)\} - p \cdot y(0) - y'(0) =$ R.H.S.

5m

Ex:- ① Solve $(D+1)y = 0$, if $y = y_0$ when $t = 0$?

Given differential equation is

$(D+1)y = 0$

$= Dy + y = 0$

$= y' + y = 0$

$= y'(t) + y(t) = 0$

Applying Laplace transform on both sides,

$L\{y'(t) + y(t)\} = L\{0\}$

$\Rightarrow L\{y'(t)\} + L\{y(t)\} = 0$

$= p \cdot L\{y(t)\} - y(0) + L\{y(t)\} = 0$

$= p \cdot L\{y(t)\} - y_0 + L\{y(t)\} = 0.$

$= L\{y(t)\}(p+1) = y_0$

$= L\{y(t)\} = \dfrac{y_0}{p+1}$

$= y(t) = L^{-1}\left\{\dfrac{y_0}{p+1}\right\}$

$= y(t) = y_0\, L^{-1}\left\{\dfrac{1}{p+1}\right\}$

$= y(t) = y_0 \cdot e^{-t}$ is the required solution

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Ex:- (2) Solve $\frac{dy}{dt}+y=1$, given $y=2$ when $t=0$.

Given differential equation is

$$\frac{dy}{dt}+y=1$$

$$\Rightarrow y'(t)+y(t)=1$$

Applying Laplace transform on both sides

$$L\{y'(t)+y(t)\}=L\{1\}$$

$$= L\{y'(t)\}+L\{y(t)\}=\frac{1}{P}$$

$$= P.L\{y(t)\}-y(0)+L\{y(t)\}=\frac{1}{P}$$

$$= P.L\{y(t)\}-2+L\{y(t)\}=\frac{1}{P}$$

$$= L\{y(t)\}(P+1)=\frac{1}{P}+2 \Rightarrow \frac{1+2P}{P}$$

$$= L\{y(t)\}=\frac{1+2P}{P(P+1)}$$

$$= y(t)=L^{-1}\left\{\frac{1+2P}{P(P+1)}\right\}$$

$$= y(t)=L^{-1}\left\{\frac{1}{P}+\frac{1}{P+1}\right\}$$

$$= y(t)=L^{-1}\left\{\frac{1}{P}\right\}+L^{-1}\left\{\frac{1}{P+1}\right\}$$

$$= \boxed{y(t)=1+e^{-t}}$$ is the required Solution

$$\frac{1+2P}{P(P+1)}=\frac{A}{P}+\frac{B}{P+1}$$

$$1+2P=A(P+1)+BP$$

P coeff $=A+B=2$

Const $=A=1$

$1+B=2 \Rightarrow B=1$

$$\therefore \frac{1+2P}{P(P+1)}=\frac{1}{P}+\frac{1}{P+1}$$

Ex:-3 Solve $(D^2+4D+3)y=e^{-t}$ given $y=1$, $\frac{dy}{dt}=1$ at $t=0$

Given differential equation is

$$(D^2+4D+3)y=e^{-t}$$

$$= D^2y+4Dy+3y=e^{-t}$$

$$= y''(t)+4y''(t)+3y(t)=e^{-t}$$

Applying L.T on both sides

$$L\{y''(t)+4y'(t)+3y(t)\}=L\{e^{-t}\}$$

$$= L\{y''(t)\}+4L\{y'(t)\}+3L\{y(t)\}=\frac{1}{P+1}$$

$$= p^2 L\{y(t)\} - p y(0) - y'(0) + 4\left(pL\{y(t)\} - y(0)\right) + 3L\{y(t)\} = \frac{1}{p+1}$$

$$= p^2 L\{y(t)\} - p(1) - 1 + 4\left(pL\{y(t)\} - 1\right) + 3L\{y(t)\} = \frac{1}{p+1}$$

$$= p^2 L\{y(t)\} - p - 1 + 4pL\{y(t)\} - 4 + 3L\{y(t)\} = \frac{1}{p+1}$$

$$= L\{y(t)\}(p^2+4p+3) = \frac{1}{p+1} + \frac{p+5}{1}$$

$$= L\{y(t)\}(p^2+4p+3) = \frac{1+(p+1)(p+5)}{p+1}$$

$$= L\{y(t)\}(p^2+4p+3) = \frac{1+p^2+6p+5}{p+1}$$

$$= L\{y(t)\}(p^2+4p+3) = \frac{p^2+6p+6}{p+1}$$

$$= L\{y(t)\} = \frac{p^2+6p+6}{(p+1)(p^2+4p+3)}$$

$$= L\{y(t)\} = \frac{p^2+6p+6}{(p+1)(p+1)(p+3)}$$

$$= L\{y(t)\} = \frac{p^2+6p+6}{(p+3)(p+1)^2}$$

$$\frac{p^2+6p+6}{(p+3)(p+1)^2} = \frac{A}{p+3} + \frac{B}{p+1} + \frac{C}{(p+1)^2}$$

$$= p^2+6p+6 = A(p+1)^2 + B(p+1)(p+3) + C(p+3)$$

$= p^2$ coeff $= A + B = 1$ — ①

$= p$ coeff $= 2A + 4B + C = 6$ — ②

$=$ Const $=$ $A + 3B + 3C = 6$ — ③

For ① - ③ $= -2B - 3C = -5$ — ④

$\Rightarrow$ ① × ② - ② $\Rightarrow -2B - C = -4$ ⑤

$\Rightarrow$ ④ - ⑤ $\Rightarrow -2C = -1$

$$\boxed{C = +\frac{1}{2}}$$

$\Rightarrow p = -3$

$9 - 18 + 6 = 4A$

$-3 = 4A \Rightarrow A = -\frac{3}{4}$

① $-\frac{3}{4} + B = 1$

$B = 1 + \frac{3}{4} = \boxed{\frac{7}{4} = B}$

$$= y(t) = L^{-1}\left\{\frac{p^2+6p+6}{(p+3)(p+1)^2}\right\}$$

$$= y(t) = L^{-1}\left\{\frac{-\frac{3}{4}}{p+3} + \frac{\frac{7}{4}}{p+1} + \frac{\frac{1}{2}}{(p+1)^2}\right\}$$

$$= y(t) = -\frac{3}{4}L^{-1}\left\{\frac{1}{p+3}\right\} + \frac{7}{4}L^{-1}\left\{\frac{1}{p+1}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{(p+1)^2}\right\}$$

$$= y(t) = -\frac{3}{4}e^{-3t} + \frac{7}{4}e^{-t} + \frac{1}{2}e^{-t}.t$$

is the required solution

Ex:④ Solve $(D^2-D-2)y = 20.\sin 2t$, given $y=-1$, $Dy=2$, when $t=0$.

Given D.E is

$$(D^2-D-2)y = 20.\sin 2t$$

$$= D^2y - Dy - 2y = 20.\sin 2t$$

$$\Rightarrow y''(t) - y'(t) - 2y(t) = 20.\sin 2t$$

Applying L.T on both sides,

$$L\{y''(t)\} - L\{y'(t)\} - 2L\{y(t)\} = 20\,L\{\sin 2t\}$$

$$\Rightarrow p^2 L\{y(t)\} - p.y(0) - y'(0) - (p.L\{y(t)\} - y(0)) - 2L\{y(t)\} = 20\left(\frac{2}{p^2+4}\right)$$

$$= p^2L\{y(t)\} - p(-1) - 2 - pL\{y(t)\} + (-1) - 2L\{y(t)\} = \frac{40}{p^2+4}$$

$$= p^2L\{y(t)\} + p - 2 - pL\{y(t)\} - 1 - 2L\{y(t)\} = \frac{40}{p^2+4}$$

$$= p^2L\{y(t)\} - pL\{y(t)\} - 2L\{y(t)\} + p - 3 = \frac{40}{p^2+4}$$

$$= L\{y(t)\}(p^2-p-2) = \frac{40}{p^2+4} - p + 3$$

$$= L\{y(t)\}(p^2-p-2) = \frac{40-(p-3)(p^2+4)}{p^2+4}$$

$$= L\{y(t)\} = \frac{40-p^3+3p^2-4p+12}{(p^2+4)(p^2-p-2)}$$

$$= L\{y(t)\} = \frac{-p^3+3p^2-4p+52}{(p^2+4)(p-2)(p+1)}$$

$$= y(t) = L^{-1}\left\{\frac{-p^3+3p^2-4p+52}{(p^2+4)(p-2)(p+1)}\right\}$$

Rough

$$\frac{p^3+3p^2-4p+52}{(p^2+4)(p-2)(p+1)} = \frac{Ap+B}{p^2+4} + \frac{C}{(p-2)} + \frac{D}{(p+1)}$$

$$= \frac{-p^3+3p^2-4p+52}{(p^2+4)\cancel{(p-2)}(p+1)} = \frac{(Ap+B)(p-2)(p+1) + C(p^2+4)(p+1) + D(p^2+4)(p-2)}{(p^2+4)\cancel{(p-2)}(p+1)}$$

$$= -p^3+3p^2-4p+52 = (Ap+B)(p-2)(p+1) + C(p^2+4)(p+1) + D(p^2+4)(p-2)$$

$$p=-1 \Rightarrow 1+3+4+52 = (Ap+B)(0) + C(p^2+4)(0) + D((-1)^2+4)(-1-2)$$

$$= 0+0-15D$$

$$60 = -15D \quad = D=4$$

$p=2 \Rightarrow -8+12-8+52 = (Ap+B)(p-2)(p+1) + C(p^2+4)(p+1) + D(p^2+4)(p-2)$

$= 0 + C(4+4)\cdot(2+1) + 0$

$= 0 + C(8)(3) + 0 + C = 24$

$48 = 24C$

$C = 2$

p^3 coeff $= -1 = A + C - D$

p^2 coeff $= 3 = -A + B + C - 8D$

p coef $= -4 = -2A - B + 4C + 2D$

Const $= 52 = -2B + 4C - 8D$

Sub C, D value is eq p^3 coeff

$-1 = A + 2 + (-4)$

$A + 2 - 4 = -1$

$A - 2 = -1$ we have

$A = 1$

Rough

$3 = -A + B + C - 2D$

$= -1 + B + 2 + 8$

$3 = B - 10$ $\boxed{B = 6}$

$$\frac{p^3+3p^2-4p-52}{(p^2+4)(p-2)(p+1)} = \frac{(1)p-6}{p^2+4} + \frac{2}{p-2} - \frac{4}{p+1}$$

$$= \frac{p}{p^2+4} - \frac{6}{p^2+4} + \frac{2}{p-2} - \frac{4}{p+1}$$

$$L\{y(t)\} = L\left\{\frac{p}{p^2+4} - \frac{6}{p^2+4} + \frac{2}{p-2} - \frac{4}{p+1}\right\}$$

$$y(t) = L^{-1}\left\{\frac{p}{p^2+4}\right\} - L^{-1}\left\{\frac{6}{p^2+4}\right\} + L^{-1}\left\{\frac{2}{p-2}\right\} - 4L^{-1}\left\{\frac{1}{p+1}\right\}$$

$= \cos 2t - \frac{6}{2}\sin 2t + 2e^{2t} - \ldots$

$= \boxed{y(t) = \cos 2t - \sin 2t + 2e^{2t} - 4e^{-t}}$

Ex: Solve $(D^2+1)y = \sin t \cdot \sin 2t$, given $y = 1$, $Dy = 0$ when $t = 0$.

$p=2 \Rightarrow -8+12-8+52 = (Ap+B)(p-2)(p+1) + C(p^2+4)(p+1) + D(p^2+4)(p-2)$

$= 0 + C(4+4)\cdot(2+1) + 0$

$= 0 + C(8)(3) + 0 + C = 24$

$48 = 24C$

$C = 2$

p^3 Coeff $= -1 = A + C - D$

p^2 Coeff $= 3 = -A + B + C - 8D$

p Coeff $: -4 = -2A - B + 4C + 2D$

Const $= 52 = -2B + 4C - 8D$

Sub C, D value is eq p^3 coeff

$-1 = A + 2 + (-4)$

$A + 2 - 4 = -1$

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$A = 1$

we have

Rough

$3 = -A + B + C - 2D$

$= -1 + B + 2 + 8$

$3 = B - 10$ $\boxed{B=6}$

$\dfrac{p^3+3p^2-4p-52}{(p^2+4)(p-2)(p+1)} = \dfrac{(1)p-6}{p^2+4} + \dfrac{2}{p-2} - \dfrac{4}{p+1}$

$= \dfrac{p}{p^2+4} - \dfrac{6}{p^2+4} + \dfrac{2}{p-2} - 4/_{p+1}$

$L\{y(t)\} = L\left\{\dfrac{p}{p^2+4} - \dfrac{6}{p^2+4} + \dfrac{2}{p-2} - \dfrac{4}{p+1}\right\}$

$y(t) = L^{-1}\left\{\dfrac{p}{p^2+4}\right\} - L^{-1}\left\{\dfrac{6}{p^2+4}\right\} + L^{-1}\left\{\dfrac{2}{p-2}\right\} - 4L^{-1}\left\{\dfrac{1}{p+1}\right\}$

$= \cos 2t - \dfrac{6}{2}\sin 2t + 2e^{2t} - 4e^{-t}$

$\boxed{y(t) = \cos 2t - 3\sin 2t + 2e^{2t} - 4e^{-t}}$

Ex:- Solve $(D^2+1)y = \sin t \cdot \sin 2t$, given $y=1$, $Dy=0$ when $t=0$.

Given D.E is $(D^2+1)y = \sin t \cdot \sin 2t$

$= D^2y + y = \sin t \cdot \sin 2t$

$= y''(t) + y(t) = \sin t \cdot \sin 2t$

$= y''(t) + y(t) = \dfrac{1}{-2}[-2\sin t \cdot \sin 2t]$

$[\because -2\sin A \sin B = \cos(A+B) - \cos(A-B)$

$= y''(t) + y(t) = \dfrac{1}{-2}[\cos(t+2t) - \cos(t-2t)]$

$= y''(t) + y(t) = \dfrac{1}{2}[\cos(t-2t) - \cos(t+2t)]$

$= y''(t) + y(t) = \dfrac{1}{2}[\cos t - \cos 3t]$

Applying L.T on both sides

$L\{y''(t)\} + L\{y(t)\} = \dfrac{1}{2}[L\{\cos t\} - L\{\cos 3t\}]$

$= p^2 \cdot L\{y(t)\} - p\cdot y(0) - y'(0) + L\{y(t)\} = \dfrac{1}{2}\left[\dfrac{p}{p^2+1} - \dfrac{p}{p^2+9}\right]$

$= p^2 L\{y(t)\} - p(1) - 0 + L\{y(t)\} = \dfrac{1}{2}\left[\dfrac{p}{p^2+1} - \dfrac{p}{p^2+9}\right]$

$= L\{y(t)\}(p^2+1) = \dfrac{1}{2}\left[\dfrac{p}{p^2+1} - \dfrac{p}{p^2+9}\right] + p$

$= L\{y(t)\} = \dfrac{1}{p^2+1}\left[\dfrac{1}{2}\left(\dfrac{p}{p^2+1}\right) - \dfrac{1}{2}\left(\dfrac{p}{p^2+9}\right)\right] + \dfrac{p}{p^2+1}$

$= L\{y(t)\} = \dfrac{1}{2}\left[\dfrac{p}{(p^2+1)^2}\right] - \dfrac{1}{2}\left[\dfrac{p}{(p^2+1)(p^2+9)}\right] + \dfrac{p}{p^2+1}$

$= L\{y(t)\} = \dfrac{1}{2}\left[\dfrac{p}{(p^2+1)^2}\right] - \dfrac{1}{2}\left[\dfrac{1}{8}\left[\dfrac{p}{p^2+1} - \dfrac{p}{\dots}\right]\right] + \dfrac{p}{\dots}$

$= L\{y(t)\} = \frac{1}{2}\left(\frac{p}{(p^2+1)^2}\right) - \frac{1}{16}\left(\frac{p}{p^2+1}\right) + \frac{1}{16}\left(\frac{p}{p^2+9}\right) + \frac{p}{p^2+1}$

$= L\{y(t)\} = \frac{1}{2}\left(\frac{p}{(p^2+1)^2}\right) + \frac{1}{16}\left(\frac{p}{p^2+9}\right) + \frac{15p}{p^2+1}$

$= y(t) = L^{-1}\left\{\frac{1}{2}\left(\frac{p}{(p^2+1)^2}\right) + \frac{1}{16}\left(\frac{p}{p^2+9}\right) + \frac{15p}{16(p^2+1)}\right\}$

$= y(t) = \frac{1}{2}L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} + \frac{1}{16}L^{-1}\left\{\frac{p}{p^2+9}\right\} + \frac{15}{16}L^{-1}\left\{\frac{p}{p^2+1}\right\}$

$= y(t) = \frac{1}{2}\left(\frac{t}{2(1)}\sin 1.t\right) + \frac{1}{16}\cos 3t + \frac{15}{16}\cos t$

$\left(\because L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} = \frac{t}{2a}\sin at\right)$

$= y(t) = \frac{t}{4}\sin t + \frac{1}{16}\cos 3t + \frac{15}{16}\cos t$

prove that $L^{-1}\left\{\frac{p\times 1}{(p^2+a^2)^2}\right\} = \frac{t}{2a}\cdot\sin at$

$L^{-1}\{f(p)\cdot g(p)\} = F(t) * G(t)$

Let $f(p) = \frac{p}{p^2+a^2}$; $g(p) = \frac{1}{(p^2+a^2)}$

$= L^{-1}\{f(p)\} = \cos at$, $L^{-1}\{g(p)\} = \frac{1}{a}\sin at$

$F(t) = \cos at$, $G(t) = \frac{1}{a}\cdot\sin at$

$L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} = \cos at * \frac{1}{a}\sin at$

$= \int_0^t \cos au \cdot \frac{1}{a}\sin a(t-u)\cdot du$

$= \frac{1}{a}\int_0^t \cos au \cdot \sin(at-au)\, du$

$= \frac{1}{2a}\int_0^t 2\cdot\underset{A}{\cos au}\cdot\sin(\underset{B}{at-au})\, du$

$[\because 2\cos A\sin B = \sin(A+B) - \sin(A-B))$

$= \frac{1}{2a}\int_0^t [\sin[a\cancel{u}+at-a\cancel{u}) - \sin(au-at+au)]\, du$

$= \frac{1}{2a}\int_0^t (\sin at - \sin(2au-at))\, du$

$= \frac{1}{2a}\left[(\sin at)(u)_0^t - \left(-\frac{\cos(2au-at)}{2a}\right)_0^t\right]$

$= \frac{1}{2a}\left[t \cdot \sin at + \frac{1}{2a}\left[\cos at - \cos at\right]\right]$

$= \frac{1}{2a} \cdot t \cdot \sin at \Rightarrow \boxed{\frac{t}{2a} \sin at}$

* **Solution of integral equation by applying Laplace transforms:-**

Def:- (integral equation):- An equation of the form $F(t) = y(t) + \int_a^b k(u,t) \cdot F(u)\, du$ is called an integral equation.

Here, $k(u,t)$ is called kernel of integral equation.

Types of integral equation:-

① FREDHOLM Integral equation → (a, b are functions of t) (or) (a, b are constants)

② VOLTERRA integral equation → (lower limit $= 0$, upper limit $= t$)

In this chapter we shall discuss VOLTERRA integral equations only.

Ex:- ① Solve the integral equation $F(t) = a\sin t - 2\int_0^t F(u) \cos(t-u)\, du$

Given integral equation is $F(t) = a\sin t - 2\int_0^t F(u) \cdot \cos(t-u)\, du$

Convert this equation into convolution product,

$F(t) = a\sin t - 2(F(t) * \cos t)$

By applying Laplace transform on both sides,

$L\{F(t)\} = L\{a\sin t\} - L\{2 \cdot (F(t) * \cos t)\}$

$= L\{F(t)\} = a\left(\frac{1}{p^2+1}\right) - 2\left(L\{F(t)\} \cdot L\{\cos t\}\right)$

$= L\{F(t)\} = \frac{a}{p^2+1} - 2L\{F(t)\} \cdot \left(\frac{p}{p^2+1}\right)$

$= L\{F(t)\} + 2L\{F(t)\} \frac{p}{p^2+1} = \frac{a}{p^2+1}$

$= L\{F(t)\}\left(1 + \frac{2p}{p^2+1}\right) = \frac{a}{p^2+1}$

$= L\{F(t)\}\left(\frac{p^2+1+2p}{\cancel{p^2+1}}\right) = \frac{a}{\cancel{p^2+1}}$

$= L\{F(t)\}(p+1)^2 = a$

$= L\{F(t)\} = \frac{a}{(p+1)^2}$

$= F(t) = L^{-1}\left\{\frac{a}{(p+1)^2}\right\} = a.e^{-t}.L^{-1}\left\{\frac{1}{p^2}\right\}$

$= a.e^{-t}.\frac{t^1}{1!}$

$= F(t) = a.e^{-t}.\frac{t^1}{1!}$

$= F(t) = a.e^{-t}.t$ is the solution.

Ex:- ② Solve the integral equation $F(t) = e^{-t} - 2\int_0^t F(u).\cos(t-u)\,du$

Given integral equation $F(t) = e^{-t} - 2\int_0^t F(u).\cos(t-u)\,du$

convert this eq into convolution product

$F(t) = e^{-t} - 2\,F(u) * \cos t$

applying Laplace transform both side

$L\{F(t)\} = L\{e^{-t} - 2\,F(u) * \cos t\}$

$L\{F(t)\} = L\{e^{-t}\} - 2[L\{F(u)\}.L\{\cos t\}]$

$= \frac{1}{p+1} - 2\left[L\{F(u)\}.\frac{p}{p^2+1}\right]$

$L\{F(t)\} = \frac{1}{p+1} - 2\left[L\{F(u)\}.\frac{p}{p^2+1}\right]$

$L\{F(t)\} + 2L\{F(u)\}\frac{p}{p^2+1} = \frac{1}{p+1}$

$L\{F(t)\}\left[1+\frac{2p}{p^2+1}\right] = \frac{1}{p+1}$

$L\{F(t)\}\left[\frac{p^2+1+2p}{p^2+1}\right] = \frac{1}{p+1}$

$L\{F(t)\} = \frac{1}{p+1}.\frac{p^2+1}{p^2+1+2p}$

$L\{F(t)\} = \frac{p^2+1}{(p+1)(p+1)^2}$

$= \frac{p^2+1}{(p+1)^3}$

$L\{F(t)\} = \frac{p^2+1}{(p+1)^3}$

$F(t) = L^{-1}\left\{\frac{p^2+1}{(p+1)^3}\right\}$

$\because \{F.S.T\}$ use

$F(t) = e^{-t} L\left\{\frac{p^2+1}{p^3}\right\}$

$= e^{-t} L^{-1}\left\{\frac{p^2}{p^3}\right\} + L^{-1}\left\{\frac{1}{p^3}\right\}$

$= e^{-t} L^{-1}\left\{\frac{1}{p}\right\} + L^{-1}\left\{\frac{1}{p^3}\right\}$

$= e^{-t}.1 + \frac{t^2}{2!}$

$F(t) = e^{-t} + \frac{t^2}{2!}$

$\boxed{F(t) = \frac{2e^{-t}+t}{2}}$

Ex:-③ Solve the integral equation $F(t) = 1 + \int_0^t \sin(t-u) \cdot f(u)\,du$ and verify your solution.

Given integral equation is

$F(t) = 1 + \int_0^t \sin(t-u) \cdot F(u)\,du$

write this equation into Convolution product,

$F(t) = 1 + \sin t * F(t)$

Applying L.T on both sides,

$L\{F(t)\} = L\{1\} + L\{\sin t * F(t)\}$

$= L\{F(t)\} = \frac{1}{p} + L\{\sin t\} \cdot L\{F(t)\}$

$= L\{F(t)\} = \frac{1}{p} + \left(\frac{1}{p^2+1}\right) L\{F(t)\}$

$= L\{F(t)\} - \frac{1}{p^2+1} L\{F(t)\} = \frac{1}{p}$

$= L\{F(t)\}\left(1 - \frac{1}{p^2+1}\right) = \frac{1}{p}$

$= L\{F(t)\}\left(\frac{p^2+\cancel{1}-\cancel{1}}{p^2+1}\right) = \frac{1}{p}$

$= L\{F(t)\}\left(\frac{p^2}{p^2+1}\right) = \frac{1}{p}$

$= L\{F(t)\} = \frac{1}{p} \times \frac{p^2+1}{p^2}$

$= L\{F(t)\} = \frac{p^2+1}{p^3}$

$= L\{F(t)\} = \frac{1}{p} + \frac{1}{p^3}$

$= F(t) = L^{-1}\{\frac{1}{p}\} + L^{-1}\{\frac{1}{p^3}\}$

$= \boxed{F(t) = 1 + \frac{t^2}{2}}$ is the solution. $[\because L^{-1}\{\frac{1}{p^{n+1}}\} = \frac{t^n}{n!}$

Verification:

$L.H.S = F(t) = 1 + \frac{t^2}{2}$

$R.H.S = 1 + \int_0^t \sin(t-u) \cdot F(u) \cdot du$

$= 1 + \int_0^t \sin(t-u) \cdot (1 + \frac{u^2}{2}) du$

By using Leibnitz rule,

$= 1 + \left[(1+\frac{u^2}{2})\left(\frac{-\cos(t-u)}{-1}\right) - (u)\frac{\sin(t-u)}{-1} + 1 \cdot \frac{-\cos(t-u)}{1}\right]_0^t$

$= 1 + \left[\left((\cancel{1}+\frac{t^2}{2}) - \cancel{1}\right) - (1 \cdot \cancel{\cos t} - \cancel{\cos t})\right)$

$= 1 + \frac{t^2}{2}$

$\therefore L.H.S = R.H.S$ solution is the verified.

Ex: 4 Solve the integral equation, $F'(t) = t + \int_0^t F(t-u) \cdot \cos u \cdot du$ where $F(0) = 4$.

Given intergral equation is

$F'(t) = t + \int_0^t F(t-u) \cdot \cos u \cdot du$

write this equation into Convolution product,

$F'(t) = t + F(t) * \cos t$

Applying L.T on both sides,

$L\{F'(t)\} = L(t) + L\{F(t) * \cos t\}$

$= p \cdot L\{F(t)\} - F(0) = \frac{1}{p^2} + L\{F(t)\} \cdot L\{\cos t\}$

$= pL\{F(t)\} - 4 = \frac{1}{p^2} + L\{F(t)\}\left(\frac{p}{p^2+1}\right)$

$= pL\{F(t)\} - L\{F(t)\}\left(\frac{p}{p^2+1}\right) = \frac{1}{p^2} + 4$

$= L\{F(t)\}\left(p - \frac{p}{p^2+1}\right) = \frac{1+4p^2}{p^2}$

$= L\{F(t)\}\left(\frac{p^3+p-p}{p^2+1}\right) = \frac{1+4p^2}{p^2}$

$= L\{F(t)\} = \frac{1+4p^2}{p^2} \times \left(\frac{p^2+1}{p^3}\right)$

$= F(t) = L^{-1}\left\{\frac{(1+4p^2)(p^2+1)}{p^5}\right\}$

$= F(t) = L^{-1}\left\{\frac{4p^4+5p^2+1}{p^5}\right\}$

$= F(t) = L^{-1}\left\{\frac{4}{p}+\frac{5}{p^3}+\frac{1}{p^5}\right\}$

$= F(t) = 4(1) + 5 \cdot \frac{t^2}{2!} + \frac{t^4}{4!}$ $\quad \because L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{n!}$

$= \boxed{F(t) = 4 + \frac{5t^2}{2} + \frac{t^4}{24}}$

Ex:- 9 solve $\int_0^t F(u) \cdot F(t-u)\, du = 16 \sin 4t$

write the given integral equation into Convolution product

$F(t) * F(t) = 16 \sin 4t$

Applying L.t on both sides,

$L\{F(t) * F(t)\} = L\{16 \cdot \sin 4t\}$

$= (L\{F(t)\})^2 = 16 \cdot \left(\frac{4}{p^2+16}\right) = \frac{64}{p^2+16}$

$= L\{F(t)\} = \pm\sqrt{\frac{64}{p^2+16}}$

$= L\{F(t)\} = \pm \frac{8}{\sqrt{p^2+16}}$

$\left[\because L^{-1}\left\{\frac{1}{\sqrt{p^2+a^2}}\right\} = J_0(at),\right.$ $= F(t) = L^{-1}\left\{\pm \frac{8}{\sqrt{p^2+16}}\right\}$

when J_0 is called Bessel's function $\Big]$ $= \pm 8 J_0(4t)$

$= \boxed{F(t) = \pm 8 J_0(4t)}$

Ex:- Solve the integral equation $F'(t) = \sin t + \int_0^t F(t-u)\cos u \, du$; $F(0) = 0$.

Given integral equation $F'(t) = \sin t + \int_0^t F(t-u)\cos u \, du$

Converting this equation into Convolution theorm

$$F'(t) = \sin t + \cos t * F(t)$$

Both sides applying on L.T

$$L\{F'(t)\} = L\{\sin t\} + L\{\cos t * F(t)\}$$

$$p.L\{F(t)\} - F(0) = \frac{1}{p^2+1} + \frac{1}{p^2+1} \cdot L\{F(t)\}$$

$$p.L\{F(t)\} - \frac{p}{p^2+1} L\{F(t)\} = \frac{1}{p^2+1}$$

$$L\{F(t)\}\left[p - \frac{p}{p^2+1}\right] = \frac{1}{p^2+1}$$

$$L\{F(t)\}\left[\frac{p^3+p-p}{p^2+1}\right] = \frac{1}{p^2+1}$$

$$L\{F(t)\} = \frac{1}{\cancel{p^2+1}} \times \frac{\cancel{p^2+1}}{p^3}$$

$$L\{F(t)\} = \frac{1}{p^3}$$

$$F(t) = L^{-1}\left\{\frac{1}{p^3}\right\}$$

$$F(t) = \frac{t^2}{2!}$$

Ex:- Solve $2F(t) = 2 - t + \int_0^t F(t-u) \cdot F(u) \, du$.

write the given integral equation into Convolution theorm

$$2F(t) = 2 - t + F(t) * F(t)$$

Applying Laplace transform on both sides

$$L\{2F(t)\} = L\{2\} - L\{t\} + L\{F(t) * F(t)\}$$

$$= 2L\{F(t)\} = \frac{2}{p} - \frac{1}{p^2} + L\{F(t)\} \cdot L\{F(t)\}$$

$$= 2L\{F(t)\} = \frac{2}{p} - \frac{1}{p^2} + (L\{F(t)\})^2$$

$$= (L\{F(t)\})^2 - 2L\{F(t)\} = \frac{1}{p^2} - \frac{2}{p}$$

Adding "1" on both sides

$$= (L\{F(t)\})^2 - 2L\{F(t)\} = \frac{1}{p^2} - \frac{2}{p} + 1$$

$$= (L\{F(t)\}-1)^2 - \frac{1-2p+p^2}{p^2} = \frac{(p-1)^2}{p^2}$$

$$= L\{F(t)\}-1 = \sqrt{\frac{(p-1)^2}{p^2}} = \frac{p-1}{p}$$

$$= L\{F(t)\} = \frac{p-1}{p}+1 \Rightarrow L\{F(t)\} = \frac{p-1+p}{p}$$

$$= L\{F(t)\} = \frac{2p-1}{p} \Rightarrow L\{F(t)\} = \frac{2-1}{p}$$

$$= F(t) = aL^{-1}\left\{\frac{1}{(p+1)^2}\right\}$$

$$= a.e^{-t}.L^{-1}\left\{\frac{1}{p^2}\right\}$$

$$= a.e^{-t}.\frac{t^1}{1!} = ae^{-t}.t$$

Ex: Solve the integral equation $F(t) = t+2\int_0^t \cos(t-u)F(u)\,du$

Given integral equation is $F(t) = t+2\int_0^t \cos(t-u)F(u)\,du$

Converting this equation into convolution theorem

$$F(t) = t+2[F(t) * \cos t]$$

Applying L.t on both sides

$$L\{F(t)\} = L\{t\}+2[L\{F(t)\}.L\{\cos t\}]$$

$$L\{F(t)\} = \frac{1}{p^2}+2\left[L\{F(t)\}.\frac{p}{p^2+1}\right]$$

$$L\{F(t)\} - 2\left[L\{F(t)\}.\frac{p}{p^2+1}\right] = \frac{1}{p^2}$$

$$L\{F(t)\}\left[1-\frac{2p}{p^2+1}\right] = \frac{1}{p^2}$$

$$L\{F(t)\}\left[\frac{p^2+1-2p}{p^2+1}\right] = \frac{1}{p^2}$$

$$L\{F(t)\}\left[\frac{p^2+1-2p}{p^2+1}\right] = \frac{1}{p^2}$$

$$L\{F(t)\} = \frac{1}{p^2}.\frac{p^2+1}{p^2+1-2p}$$

$$L\{F(t)\} = \frac{p^2+1}{p^2(p-1)^2}$$

$$\frac{p^2+1}{p^2(p-1)^2} = \frac{A}{p}+\frac{B}{p^2}+\frac{C}{(p-1)}+\frac{D}{(p-1)^2}$$

$$=\frac{p^2+1}{p^2(p-1)^2} = \frac{Ap(p-1)^2+B(p-1)^2+Cp^2(p-1)+Dp^2}{p^2(p-1)^2}$$

$$p^2+1 = Ap(p-1)^2+B(p-1)^2+Cp^2(p-1)+Dp^2$$

If $p=1 \Rightarrow 2 = D(1)^2$

$D=2$

$$p^2+1 = Ap(p^2+1-2p)+B(p^2+1-2p)+Cp^3-Cp^2+Dp^2$$

$$p^2+1 = Ap^3+Ap-2Ap^2+Bp^2+B-2Bp+Cp^3-Cp^2+Dp^2$$

$$p^2+1 = p^3(A+C)+p^2(-2A+B-C+D)+p(A-2B)+B$$

$B=1$

Comparing "p" coefficients $A-2B=0$

$A-2(1)=0$

$A=2$

Comparing "p^2" coefficients $-2A+B-C+D=1$

$-2(2)+1-C+2=1 \Rightarrow -4+1-C+2=1$

$-1-C=1$

$-C=1+1$

$-C=2 \Rightarrow \boxed{C=-2}$

$$\frac{p^2+1}{p^2(p-1)^2} = \frac{A}{p}+\frac{B}{p^2}+\frac{C}{(p-1)}+\frac{D}{(p-1)^2}$$

$$L\{F(t)\} = \frac{p^2+1}{p^2(p-1)^2} \Rightarrow F(t) = L^{-1}\left\{\frac{p^2+1}{p^2(p-1)^2}\right\}$$

$$F(t) = L^{-1}\left\{\frac{A}{p}\right\}+L^{-1}\left\{\frac{B}{p^2}\right\}+L^{-1}\left\{\frac{C}{p-1}\right\}+L^{-1}\left\{\frac{P}{(p-1)^2}\right\}$$

$$F(t) = L^{-1}\left\{\frac{2}{p}\right\}+L^{-1}\left\{\frac{1}{p^2}\right\}+L^{-1}\left\{\frac{-2}{p-1}\right\}+L^{-1}\left\{\frac{2}{(p-1)^2}\right\}$$

$$F(t) = 2.(1)+\frac{t^1}{1!}-2.e^t+2.e^t L^{-1}\left\{\frac{1}{p^2}\right\}$$

$$F(t) = 2+t-2e^t+2e^t\frac{t^1}{1!}$$

$F(t) = 2te^t-2e^t+t+2$ is a solution

* Solve the integral function $y(t) = 1-e^{-t}+\int_0^t y(t-u)$ sinu du

Given integral equation $y(t) = 1-e^{-t}+\int_0^t y(t-u).\sin u \, du$

Converting this equation into Convolution theorem

$y(t) = 1 - e^{-t} + y(t) * \sin t$

Applying both sides Laplace transform

$= L\{y(t)\} = L\{1\} - L\{e^{-t}\} + L\{y(t) * \sin t\}$

$= L\{y(t)\} = \frac{1}{p} - \frac{1}{p+1} + L\{y(t)\} \cdot L\{\sin t\}$

$= L\{y(t)\} = \frac{1}{p} - \frac{1}{p+1} + L\{y(t)\} \cdot \frac{1}{p^2+1}$

$= L\{y(t)\} - \frac{1}{p^2+1} \cdot L\{y(t)\} = \frac{1}{p} - \frac{1}{p+1}$

$= L\{y(t)\} \left[1 - \frac{1}{p^2+1}\right] = \frac{1}{p} - \frac{1}{p+1}$

$= L\{y(t)\} \left[\frac{p^2+1-1}{p^2+1}\right] = \frac{p+1-p}{p(p+1)}$

$= L\{y(t)\} = \frac{1}{p(p+1)} \cdot \frac{p^2+1}{p^2}$

$= L\{y(t)\} = \frac{p^2+1}{p^3(p+1)}$

$$\frac{p^2+1}{p^3(p+1)} = \frac{A}{p+1} + \frac{B}{p} + \frac{C}{p^2} + \frac{D}{p^3}$$

$$\frac{p^2+1}{p^3(p+1)} = Ap^3 + Bp^2(p+1) + Cp(p+1) + D(p+1)$$

If $p = -1 \Rightarrow 2 = A(-1)^3$

$2 = -A$

$A = -2$

$p^2+1 = Ap^3 + Bp^3 + Bp^2 + Cp^2 + Cp + Dp + D$

$p^2+1 = p^3(A+B) + p^2(B+C) + p(D+p) + D$

$D = 1$

$B + C = 1$; $A + B = 0$

$2 + C = 1$; $-2 + B = 0$

$C = -1$; $B = 2$

$$L\{F(t)\} = \frac{p^2+1}{p^3(p+1)}$$

$$y(t) = L^{-1}\left\{\frac{p^2+1}{p^3(p+1)}\right\}$$

$$y(t) = L^{-1}\left\{\frac{-2}{p+1}\right\} + L^{-1}\left\{\frac{2}{p}\right\} + L^{-1}\left\{\frac{-1}{p^2}\right\} + L^{-1}\left\{\frac{1}{p^3}\right\}$$

$$y(t) = -2\cdot e^{-t} + 2(1) - 1\cdot\frac{t}{1!} + \frac{t^2}{2!}$$

$$y(t) = -2\cdot e^{-t} + 2 - t + \frac{t^2}{2}$$

Imp
Ex:- 10

Solve the integral equation $\int_0^t \frac{F(u)}{\sqrt{t-u}}\,du = 1+t+t^2$

Given integral equation is

$$\int_0^t \frac{F(u)}{\sqrt{t-u}}\,du = 1+t+t^2$$

$$= \int_0^t \frac{F(u)}{(t-u)^{1/2}}\,du = 1+t+t^2$$

$$= \int_0^t F(u)\cdot(t-u)^{-1/2}\cdot du = 1+t+t^2$$

$$= F(t) * t^{-\frac{1}{2}} = 1+t+t^2$$

Applying L.T on both sides

$$L\{F(t) * t^{-\frac{1}{2}}\} = L\{1\} + L\{t\} + L\{t^2\}$$

$$\left[\because L\{t^n\} = \frac{\Gamma(n+1)}{p^{n+1}} = \frac{n!}{p^{n+1}}\right]$$

$$= L\{F(t)\}\cdot L\{t^{-\frac{1}{2}}\} = \frac{1}{p} + \frac{1}{p^2} + \frac{2}{p^3}$$

$$= L\{F(t)\}\cdot\frac{\Gamma(-\frac{1}{2}+1)}{p^{-\frac{1}{2}+1}} = \frac{1}{p} + \frac{1}{p^2} + \frac{2}{p^3}$$

$$= L\{F(t)\}\left(\frac{\sqrt{\pi}}{\sqrt{p}}\right) = \frac{1}{p} + \frac{1}{p^2} + \frac{2}{p^3}$$

$$= L\{F(t)\} = \frac{\sqrt{p}}{\sqrt{\pi}}\left(\frac{1}{p} + \frac{1}{p^2} + \frac{2}{p^3}\right)$$

$$= L\{F(t)\} = \frac{1}{\sqrt{\pi}}\left[\frac{\sqrt{p}}{p} + \frac{\sqrt{p}}{p^2} + \frac{2\sqrt{p}}{p^3}\right]$$

$$= L\{F(t)\} = \frac{1}{\sqrt{\pi}}\left[\frac{1}{p^{1/2}} + \frac{1}{p^{3/2}} + \frac{2}{p^{5/2}}\right]$$

$$= F(t) = \frac{1}{\sqrt{\pi}}\left[L^{-1}\left\{\frac{1}{p^{\frac{1}{2}}}\right\} + L^{-1}\left\{\frac{1}{p^{3/2}}\right\} + 2L^{-1}\left\{\frac{1}{p^{5/2}}\right\}\right]$$

$$= F(t) = \frac{1}{\sqrt{\pi}}\left[\frac{t^{-\frac{1}{2}+1}}{\Gamma -\frac{1}{2}+1} + \frac{t^{\frac{1}{2}}}{\Gamma \frac{1}{2}+1} + \frac{2t^{3/2}}{\Gamma 3/2+1}\right]$$

$$\left[\because L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{n!} = \frac{t^n}{\Gamma n+1}\right]$$

$$= F(t) = \frac{1}{\sqrt{\pi}}\left(\frac{t^{-\frac{1}{2}}}{\sqrt{\pi}} + \frac{t^{\frac{1}{2}}}{\Gamma 3/2} + \frac{2t^{3/2}}{\Gamma 5/2}\right)$$

$$= F(t) = \frac{1}{\sqrt{\pi}}\left\{\frac{1}{\sqrt{\pi}\cdot\sqrt{t}} + \frac{\sqrt{t}}{(\frac{3}{2}-1)\Gamma\frac{3}{2}-1} + \frac{2t\sqrt{t}}{(\frac{5}{2}-1)\Gamma\frac{5}{2}-1}\right]$$

$$= F(t) = \frac{1}{\sqrt{\pi}}\left\{\frac{1}{\sqrt{\pi}\sqrt{t}} + \frac{\sqrt{t}}{\frac{1}{2}\sqrt{\pi}} + \frac{2t\sqrt{t}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\right]$$

$$\left[\because \Gamma n = (n-1)\Gamma n-1\right]$$

$$= F(t) = \frac{1}{\pi}\left[\frac{1}{\sqrt{t}} + 2\sqrt{t} + \frac{8t\sqrt{t}}{3}\right]$$

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Ex:- (11) Solve the integral equation $\int_0^t \frac{F(u)}{(t-u)^{1/3}}du = t(1+t)$

Given integral equation is

$$\int_0^t \frac{F(u)}{(t-u)^{1/3}}du = t(1+t) \Rightarrow t+t^2$$

$$= \int_0^t F(u)\cdot(t-u)^{-\frac{1}{3}}\cdot du = t+t^2$$

$$= F(t) * t^{-\frac{1}{3}} = t+t^2$$

Applying L.T on both sides

$$L\{F(t) * t^{-\frac{1}{3}}\} = L\{t\} + L\{t^2\}$$

$$= L\{F(t)\}\cdot L\{t^{-\frac{1}{3}}\} = \frac{1}{p^2} + \frac{2}{p^3}$$

$$= L\{F(t)\}\cdot\frac{\Gamma -\frac{1}{3}+1}{p^{-\frac{1}{3}+1}} = \frac{1}{p^2} + \frac{2}{p^3}$$

$$= L\{F(t)\}\cdot\frac{\Gamma\frac{2}{3}}{p^{2/3}} = \frac{1}{p^2} + \frac{2}{p^3}$$

$$= L\{F(t)\} = \frac{p^{2/3}}{\Gamma 2/3}\left[\frac{1}{p^2} + \frac{2}{p^3}\right]$$

$$= L\{F(t)\} = \frac{1}{\Gamma\frac{2}{3}}\left[\frac{1}{p^{2-\frac{2}{3}}} + \frac{2}{p^{3-\frac{2}{3}}}\right]$$

$$= L\{F(t)\} = \frac{1}{\Gamma\frac{2}{3}}\left[\frac{1}{p^{\frac{4}{3}}} + \frac{2}{p^{\frac{7}{3}}}\right]$$

$$= F(t) = \frac{1}{\Gamma\frac{2}{3}}\left\{L^{-1}\left\{\frac{1}{p^{4/3}}\right\} + 2L^{-1}\left\{\frac{1}{p^{7/3}}\right\}\right\}$$

$$= F(t) = \frac{1}{\Gamma\frac{2}{3}}\left[\frac{t^{\frac{1}{3}}}{\Gamma\frac{1}{3}+1} + \frac{2\cdot t^{\frac{4}{3}}}{\Gamma\frac{4}{3}+1}\right]$$

$$= F(t) = \frac{1}{\Gamma\frac{2}{3}}\left[\frac{t^{\frac{1}{3}}}{\Gamma\frac{4}{3}} + \frac{2t^{\frac{4}{3}}}{\Gamma\frac{7}{3}}\right]$$

$$= F(t) = \frac{1}{\Gamma\frac{2}{3}}\left\{\frac{t^{\frac{1}{3}}}{\left(\frac{4}{3}-1\right)\Gamma\frac{4}{3}-1} + \frac{2t^{\frac{4}{3}}}{\left(\frac{7}{3}-1\right)\Gamma\frac{7}{3}-1}\right\}$$

$$= F(t) = \frac{1}{\Gamma\frac{2}{3}}\left\{\frac{t^{\frac{1}{3}}}{\frac{1}{3}\Gamma\frac{1}{3}} + \frac{2t^{\frac{4}{3}}}{\frac{4}{3}\cdot\frac{1}{3}\cdot\Gamma\frac{1}{3}}\right\}$$

$$= F(t) = \frac{1}{\Gamma\frac{2}{3}\cdot\Gamma\frac{1}{3}}\left[3t^{\frac{1}{3}} + \frac{9}{2}t^{4/3}\right]$$

$$\because \Gamma n\,\Gamma 1-n = \frac{\pi}{\sin n\pi}$$

$$F(t) = \frac{\pi}{\sin\frac{1}{3}\pi} = \left[3t^{1/3} + \frac{9}{2}t^{4/3}\right]$$

$$= F(t) = \frac{2\pi}{\sqrt{3}}\left[3t^{\frac{1}{3}} + \frac{9}{2}t^{4/3}\right]$$

UNIT-5

FOURIER TRANSFORMS

Infinite and finite fourier transforms:-

Infinite fourier transforms:-

Let $f(x)$ be a function of x,

The fourier transform of $f(x)$ denoted by $F\{f(x)\}$ is defined as $F=\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \cdot f(x) \cdot dx = \bar{F}(p)$

* let $f(x)$ be a function of x, then, the inverse Fourier transform is defined as $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} \cdot \bar{F}(p) \cdot dp$

Ex:- ① find the FOURIER Transform of $f(x)$ defined by

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$ and hence evaluate

i) $\int_{-\infty}^{\infty} \frac{\sin ap \cdot \cos px}{p} dp$ ii) $\int_{0}^{\infty} \frac{\sin p}{p} dp$

By defination, $F\{F(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \cdot f(x) \cdot dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{ipx} \cdot 1 \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ipx}}{ip} \right]_{-a}^{a} \qquad \left[\because \int e^{ax} dx = \frac{e^{ax}}{a} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ipa}}{ip} - \frac{e^{-ipa}}{ip} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ipa} - e^{-ipa}}{ip} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2\cancel{i} \sin pa}{\cancel{i} p} \right] = \frac{1}{\sqrt{2\pi}} \cdot \frac{2 \cdot \sin pa}{p}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\sin pa}{p}$$

$$\boxed{F=\{f(x)\} = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin pa}{p}}$$

$\therefore$ By def. of inverse fourier transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} \cdot \bar{F}(p)\, dp$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} \cdot \sqrt{\frac{2}{\pi}} \frac{\sin pa}{p} \cdot dp$$

$$= f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-ipx} \cdot \frac{\sin pa}{p}\, dp = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

$$= \int_{-\infty}^{\infty} e^{-ipx} \cdot \frac{\sin pa}{p}\, dp = \begin{cases} \pi, & |x| < a \\ 0, & |x| > a \end{cases}$$

$$= \int_{-\infty}^{\infty} (\cos px - i \sin px) \frac{\sin pa}{p}\, dp = \begin{cases} \pi, & |x| < a \\ 0, & |x| > a \end{cases}$$

$$= \underbrace{\int_{-\infty}^{\infty} \cos px \cdot \frac{\sin pa}{p}\, dp}_{f(p)} - i \underbrace{\int_{-\infty}^{\infty} \sin px \cdot \frac{\sin pa}{p}\, dp}_{\text{odd function } (0)} = \begin{cases} \pi, & |x| < a \\ 0, & |x| > a \end{cases}$$

$$= \int_{-\infty}^{\infty} \cos px \cdot \frac{\sin ap}{p}\, dp = \begin{cases} \pi, & |x| < a \\ 0, & |x| > a \end{cases} \rightarrow \text{(i) ans.}$$

ii) $$= 2 \int_{0}^{\infty} \cos px \cdot \frac{\sin ap}{p}\, dp = \begin{cases} \pi, & |x| < a \\ 0, & |x| > a \end{cases}$$

$$= \int_{0}^{\infty} \cos px \cdot \frac{\sin ap}{p}\, dp = \begin{cases} \frac{\pi}{2}, & |x| < a \\ 0, & |x| > a \end{cases}$$

put $a=1$ and $x=0$

$$\int_{0}^{\infty} 1 \cdot \frac{\sin p}{p}\, dp = \frac{\pi}{2}$$

$$\Rightarrow \boxed{\int_{0}^{\infty} \frac{\sin p}{p}\, dp = \frac{\pi}{2}} \Rightarrow \text{ii) Ans}$$

2017 2020 10m Ex: ② Find the Fourier transform of $f(x)$ defined by

$$f(x) = \begin{cases} 1-x^2, & \text{if } |x|<1 \\ 0, & \text{if } |x|>1 \end{cases}$$

Hence Evaluate,

① $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cdot \cos\frac{x}{2}\, dx$ ② $\int_0^\infty \frac{x\cdot\cos x - \sin x}{x^3}\, dx$

By def. $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \cdot f(x)\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{ipx} \cdot (1-x^2)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[(1-x^2) \cdot \frac{e^{ipx}}{ip} - (-2x) \cdot \frac{e^{ipx}}{i^2p^2} + (-2) \cdot \frac{e^{ipx}}{i^3p^3} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[(1-x^2) \frac{e^{ipx}}{ip} + 2x \cdot \frac{e^{ipx}}{-p^2} - 2 \cdot \frac{e^{ipx}}{-ip^3} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[(1-x^2) \frac{e^{ipx}}{ip} - \frac{2x \cdot e^{ipx}}{p^2} + \frac{2e^{ipx}}{ip^3} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[(1-x^2) \frac{e^{ipx}}{ip} - \frac{2x e^{ipx}}{p^2} - \frac{2i e^{ipx}}{p^3} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{-2e^{ip}}{p^2} - \frac{2i e^{ip}}{p^3} \right) - \left(\frac{2 \cdot e^{-ip}}{p^2} - \frac{2i e^{-ip}}{p^3} \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{-2e^{ip}}{p^2} - \frac{2e^{-ip}}{p^2} \right) + \left(\frac{-2i e^{ip}}{p^3} + \frac{2i e^{-ip}}{p^3} \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-2}{p^2} (e^{ip} + e^{-ip}) - \frac{2i}{p^3} (e^{ip} - e^{-ip}) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-2}{p^2} (\cos p + \cancel{i\sin p} + \cos p - \cancel{i\sin p}) - \frac{2i}{p^3} (\cancel{\cos p} + i\sin p - \cancel{\cos p} + i\sin p) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-2}{p^2} (2\cos p) - \frac{2i}{p^3} (2i\sin p) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-4\cos p}{p^2} - \frac{4i^2 \sin p}{p^3} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\frac{4\cos p}{p^2} + \frac{4\sin p}{p^3} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-4p\cos p + 4\sin p}{p^3} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{4\sin p - 4p\cos p}{p^3} \right] //$$

By def of inverse fourier transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} \cdot F(p) \cdot dp$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} \cdot \frac{1}{\sqrt{2\pi}} \left\{ \frac{4\sin p - 4p\cos p}{p^3} \right\} dp = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} (\cos px - i\sin px) \left[\frac{\sin p - p\cos p}{p^3} \right] dp = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

$$= \int_{-\infty}^{\infty} \cos px \underbrace{\left[\frac{\sin p - p\cos p}{p^3} \right]}_{\text{even}} dp - i \int_{-\infty}^{\infty} \sin px \underbrace{\left[\frac{\sin p - p\cos p}{p^3} \right]}_{\text{odd}} dp = \begin{cases} \frac{\pi}{2}(1-x^2), & |x|<1 \\ 0, & |x|>1 \end{cases}$$

$$= 2\int_{0}^{\infty} \cos px \left[\frac{\sin p - p\cos p}{p^3} \right] dp = \begin{cases} \frac{\pi}{2}(1-x^2), & |x|<1 \\ 0, & |x|>1 \end{cases}$$

$$= \int_{0}^{\infty} \cos px \left[\frac{\sin p - p\cos p}{p^3} \right] dp = \begin{cases} \frac{\pi}{4}(1-x^2), & |x|<1 \\ 0, & |x|>1 \end{cases} \rightarrow \text{I}$$

put $x = \frac{1}{2}$ in eq I

$$\int_{0}^{\infty} \cos \frac{p}{2} \left\{ \frac{\sin p - p\cos p}{p^3} \right\} dp = \frac{\pi}{4} \left[1 - \left(\frac{1}{2}\right)^2 \right]$$

$$= \int_{0}^{\infty} \left(\frac{\sin p - p\cos p}{p^3} \right) \cos \frac{p}{2}\, dp = \frac{\pi}{4}\left(\frac{3}{4}\right) = \frac{3\pi}{16}$$

$$= \int_{0}^{\infty} \frac{p\cos p - \sin p}{p^3} \cdot \cos \frac{p}{2}\, dp = \frac{-3\pi}{16} \rightarrow \text{(i) ans}$$

put $x=0$ in eqn (I)

$$\int_0^\infty \cos p(0) \left(\frac{\sin p - p\cos p}{p^3}\right) dp = \frac{\pi}{4}(1-0^2)$$

$$= \frac{\pi}{4}$$

$$= \int_0^\infty \frac{p\cos p - \sin p}{p^3} dp = -\frac{\pi}{4} \quad \text{—(ii) Ans}$$

Ex: 3 Find the Fourier transform of $f(x) = e^{-\frac{x^2}{2}}$

By def. $F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{ipx} \cdot f(x)\, dx$

$$\therefore F\{e^{-\frac{x^2}{2}}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{ipx} \cdot e^{-\frac{x^2}{2}} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{ipx - \frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{\frac{2ipx - x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{\frac{(2ipx - x^2 + p^2) - p^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{\frac{2ipx - x^2 + p^2}{2}} \cdot e^{-\frac{p^2}{2}} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{p^2}{2}} \int_{-\infty}^\infty e^{\frac{2ipx - x^2 + p^2}{2}} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{p^2}{2}} \int_{-\infty}^\infty e^{-\left[\frac{x^2 - 2ipx - p^2}{2}\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{p^2}{2}} \cdot \int_{-\infty}^\infty e^{-\left[\frac{x^2 - 2ipx + (ip)^2}{2}\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{p^2}{2}} \int_{-\infty}^\infty e^{-\frac{(x-ip)^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{p^2}{2}} \cdot \int_{-\infty}^\infty e^{-\left(\frac{x-ip}{\sqrt{2}}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{p^2}{2}} \cdot \int_{-\infty}^{\infty} e^{-t^2} \cdot \sqrt{2}\, dt$$

$$= \frac{1}{\sqrt{\pi}} \cdot e^{-\frac{p^2}{2}} \int_{-\infty}^{\infty} \underbrace{e^{-t^2}}_{\text{even}} \cdot dt$$

$$= \frac{1}{\sqrt{\pi}} \cdot e^{-\frac{p^2}{2}} \cdot 2 \int_{0}^{\infty} e^{-t^2} dt$$

$$= \frac{1}{\sqrt{\pi}} \cdot e^{-\frac{p^2}{2}} \cdot \cancel{2} \int_{0}^{\infty} e^{-v} \cdot \frac{dv}{\cancel{2}\sqrt{v}}$$

$$= \frac{1}{\sqrt{\pi}} \cdot e^{-\frac{p^2}{2}} \int_{0}^{\infty} e^{-v} \cdot v^{-\frac{1}{2}} \cdot dv$$

$$= \frac{1}{\sqrt{\pi}} \cdot e^{-\frac{p^2}{2}} \int_{0}^{\infty} e^{-v} \cdot v^{\frac{1}{2}-1} \cdot dv$$

$$= \frac{1}{\sqrt{\pi}} \cdot e^{-\frac{p^2}{2}} \cdot \Gamma\tfrac{1}{2}$$

$$= \frac{1}{\cancel{\sqrt{\pi}}} \cdot e^{-\frac{p^2}{2}} \cdot \cancel{\sqrt{\pi}}$$

$$= e^{-\frac{p^2}{2}}$$

$$\boxed{\therefore F\left\{ e^{-\frac{x^2}{2}} \right\} = e^{-\frac{p^2}{2}}}$$

$\left[\text{put } \frac{x-ip}{\sqrt{2}} = t \right.$

$= \frac{x}{\sqrt{2}} - \frac{ip}{\sqrt{2}} = t$

$d\left(\frac{x}{\sqrt{2}} - \frac{ip}{\sqrt{2}}\right) = dt$

$= \frac{1}{\sqrt{2}} dx = dt$

$= dx = \sqrt{2}\, dt$

$x = -\infty \Rightarrow t = -\infty$

$\left. x = \infty \Rightarrow t = \infty \right]$

$\because \left[\text{put } t^2 = v \Rightarrow 2t\,dt = dv \right.$

$dt = \frac{dv}{2t}$

$dt = \frac{dv}{2\sqrt{v}}$

$t = 0 \Rightarrow v = 0$

$t = \infty \Rightarrow v = \infty$

$\left(\Gamma n = \int_{0}^{\infty} e^{-x} \cdot x^{n-1} \cdot dx \right)$

Def:- Convolution product in fourier transform:-

The Convolution product of two function $f(x)$ and $g(x)$ denot-ed by $f(x) * g(x)$ is defined as $f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \cdot g(x+u)\, du$

* Convolution in fourier transform:-

Statement:- if $F\{f(x)\}$, $F\{g(x)\}$ are the fourier transforms of $f(x)$, $g(x)$ respectively than $F\{f(x) * g(x)\} = F\{f(x)\} \cdot F\{g(x)\}$

Proof:- $L.H.S = F\{f(x) * g(x)\}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} [f(x) * g(x)]\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du \right] du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx}\, f(u) \cdot g(x-u)\, du \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx + ipu - ipu} \cdot F(u)\, g(x-u)\, dx \right] du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipu} \cdot e^{ip(x-u)} \cdot f(x) \cdot g(x-u) \cdot d(x-u) \Big] du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipu} f(u) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ip(x-u)} \cdot g(x-u)\, d(x-u) \right] du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipu} \cdot f(u) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipu} \cdot g(u)\, du \right] du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipu} \cdot f(u) \left(F(g(x)) \right) du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipu} f(u)\, du) \cdot F\{g(x)\}$$

Put $x-u=v$

$d(x-u)=dv$

$$= F\{f(x)\} \cdot F\{g(x)\}$$

$= R.H.S$

$L.H.S = R.H.S$

Properties of Fourier transforms:-

① Linearity property

② change of scale property

③ shifting property / shifting theorem

④ modulation property / modulation

① **Linearity property:-**

statement:-

$$F\{a \cdot f(x) + b \cdot g(x)\} = a \cdot F\{f(x)\} + b \cdot F\{g(x)\}$$

Proof: L.H.S $= F\{a \cdot f(x) + b \cdot g(x)\}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} [a f(x) + b g(x)] \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [e^{ipx} a \cdot f(x) + e^{ipx} \cdot b\, g(x)] \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} a\, f(x) \, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \cdot b\, g(x) \, dx$$

$$= a \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \cdot f(x) \, dx\right] + b \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} g(x) \, dx\right]$$

$$= a \cdot F\{f(x)\} + b \cdot F\{g(x)\} = R.H.S$$

L.H.S = R.H.S

② **change of Scale property:-**

statement:

$$F\{f(ax)\} = \frac{1}{a} F(P/a) \text{ where } F(P) = F\{f(x)\}$$

proof: L.H.S $= F\{f(ax)\}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(ax) \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipt/a} \cdot f(t) \frac{1}{a} \, dt$$

$$= \frac{1}{a} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(P/a)t} f(t) \, dt\right]$$

$$= \frac{1}{a} [F(P/a)] = R.H.S$$

③ **shifting property:**

statement: $F\{f(x-a)\} = e^{ipa}\,\bar{F}(p)$

proof: $L.H.S = F\{f(x-a)\}$

$$\left[\begin{array}{ll} x-a=t & \\ x=t+a & \\ dx=dt & \\ x=-\infty & t=-\infty \\ x=\infty & t=\infty \end{array}\right.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x-a)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ip(t+a)} f(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipt} \cdot e^{ipa} f(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipt} \cdot e^{ipa} f(t)\, dt$$

$$= e^{ipa} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipt} f(t)\, dt \right]$$

$$= e^{ipa}\,\bar{F}(p)$$

$$= L.H.S = R.H.S$$

④ **modulation property:**

statement: $F\{f(x)\cos ax\} = \frac{1}{2}\left[\bar{F}(p+a) + \bar{F}(p-a)\right]$

proof: $LHS = F\{f(x)\cos ax\}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x) \cos ax$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{ipx} f(x) \left[\frac{e^{iax} + e^{-iax}}{2} \right] dx \right.$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[e^{ipx} f(x)\, e^{iax} + e^{ipx} f(x)\, e^{-ipx} \right] dx \right]$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x) e^{iax}\, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x)\, e^{-iax} dx \right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx+iax} f(x)\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx-iax} f(x)\,dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(p+a)x} f(x)\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(p-a)x} f(x)\,dx\right]$$

$$= \frac{1}{2}\left[F(p+a) + F(p-a)\right]$$

$= L.H.S = R.H.S$

$t = -\infty$

$t = \infty$