Semester - IV

Mathematics syllabus

Real analysis

UNIT: 1 Real Numbers

(1) The Real numbers

The algebric and order properties of R Absolute value and real line.
completeness property of R. Applications of supremum property; intervals.

(2) Sequences

Sequences and Their Limits, Range and Boundedness of Sequences.
Limits of a sequences and convergent sequence.
The cauchy's criterion, property ~~of~~ Divergent Sequences, monotone.
Sequences, Necessary and sufficient condition for convergence of monotone sequences, limit point of sequences subsequences and and the Bolzano weierstrass Theorem - cauchy sequences cauchy's general principal of convergence Theorem

UNIT: 2 Infinite ~~series~~ Series

(3) Infinite Series

Introduction to Series, convergence of Series cauchy's general principle of convergence for Series, Test for of convergence of series of non Negative Terms

1. P-test

2. Cauchy's nth root test or Root test

3. D-Alembert's test or Ratio test

4. Alternating series - Leibnitz test

Absolute convergence and conditional convergence, Semi convergence.

Unit - III Infinite Continuity

1. Limits and continuity

Limits : Real valued functions Boundedness of a function, limits of functions, some extensions of the limit concept, Infinite limits, limits at infinity

Continuous function : continuous function, combinations of continuous functions, continuous functions on intervals, uniform continuity

Unit - IV

Differentiation and mean value theorems :-

5. Differentiation

The derivability of function on the interval at a point Derivability and continuity of a function, Graphical meaning of the Derivative mean value .... theorems, Role's theorem

Lagrange's theorem

Cauchy's mean value theorem

UNIT-V Integration

(6) Riemann integration:

Riemann intigral; Rie mann intigral functions, Darboun Theorem | Necessary And sufficient condition for R - integrability properties of integrable functions fundamental Theorem of integral calculus integral as the limit of a sum mean value Theorem

UNIT - I

Real numbers and Real Sequences

* **Sequence:** A function $S : Z^+ \to R$ is called a sequence of real number.

Eg: (i) The sequence $\{S_n\}$ is defined by $\frac{1}{\sqrt{n}}$ i.e., $\{S_n\} = \frac{1}{\sqrt{n}}$

$$= \frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \dots$$

So that $S_1 = 1, S_2 = \frac{1}{\sqrt{2}}$ $S_3 = \frac{1}{\sqrt{3}}$

(ii) $\{S_n\} = \begin{cases} \frac{1}{n} & \text{if } n \text{ is even} \\ \frac{-1}{n} & \text{if } n \text{ is odd} \end{cases}$

Constant Sequence: The sequence $\{S_n\}$ is defined by $\{S_n\} = c \in R$ is called a constant sequence.

Eg: $\{S_n\} = 0, 0 \dots 0$ where $S_1 = 0, S_2 = 0 \dots$

$\{S_n\} = 1, 1 \dots 1$ where $S_1 = 1, S_2 = 1 \dots$

Operations of Sequence:

(i) if $\{S_n\}, \{t_n\}$ are two sequences then their sum is defined by $\{S_n + t_n\}$ i.e..

$$\{S_n\} + \{t_n\} = \{S_n + t_n\}$$

Imp Theorem: Uniqueness of limits:

(5) Statement: A sequence can have atmost one limit (or)

$(l = l')$

A convergent sequence has unique limit

Proof: Let $\{S_n\}$ be a sequence and l, l' are two limit point of sequence $\{S_n\}$

Claim: ~~Let $\{S_n\}$ be a sequence and $l \neq l'$~~

$l = l'$ put $\varepsilon = \frac{1}{2}|l - l'| \Rightarrow 2\varepsilon = |l - l'|$

Suppose that $l \neq l'$

$\exists m_1 \in Z^+$ $\{S_n\}$ is converges to l / $|S_n - l| < \varepsilon$ $\forall n \geq m_1$

$\exists m_2 \in Z^+$, $\{S_n\}$ is converges to $l' = ||S_n - l'| < \varepsilon$ $\forall n \geq m_2$

Take $m = \max$ of $\{m_1, m_2\}$

By def, $|S_n - l| < \varepsilon, |S_n - l'| < \varepsilon, \forall n \geq m$

Consider, $|l - l'|$

$= |(S_n - l) - (S_n - l')|$

$\leq |(S_n - l) + (S_n - l')|$

$< 2\varepsilon = |l - l'|$

$|l - l'| < |l - l'|$

This is contradiction our supposition

supposition $l \neq l'$

$l = l'$ is True.

Hence every convergent sequence has unique limit.

Theorem: Every convergent sequence is bounded.

Proof: Given that $\{S_n\}$ is convergent sequence i.e. $\{S_n\} \to l$ [convergent to l]

$\Rightarrow \lim S_n = l$

put $\varepsilon = 1 > 0$ | $\exists$ +ve integer 'm'

By def, $|S_n - l| < \varepsilon = 1 \quad \forall n \geq m$

$|S_n - l| < 1$

$-1 < S_n - l < 1$

$l - 1 < S_n < l + 1$

Let $k_1 = \min\{S_1, S_2 \ldots S_n, l-1\}$

$k_2 = \max\{S_1, S_2 \ldots S_n, l+1\}$

$\Rightarrow k_1 < S_n < k_2 \quad \forall n \geq m$

$\therefore \{S_n\}$ is bounded.

Imp
(3)
*** (5) If $S_n = \sqrt{n+1} - \sqrt{n}$ prove that $\lim S_n = 0$

$S_n = \sqrt{n+1} - \sqrt{n}$

$= (\sqrt{n+1} - \sqrt{n}) \times \frac{(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$

$= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}}$

$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$

$< \frac{1}{\sqrt{n} + \sqrt{n}}$

$< \frac{1}{2\sqrt{n}}$

Let $\varepsilon > 0$

Consider $|S_n - l| < \varepsilon$

$< \left| \frac{1}{2\sqrt{n}} - 0 \right| < \varepsilon$

$\Rightarrow \frac{1}{2\sqrt{n}} < \varepsilon$

$\Rightarrow \sqrt{n} > \frac{1}{2\varepsilon}$

$\Rightarrow n > \frac{1}{4\varepsilon^2}$

$m = \left[\frac{1}{4\varepsilon^2}\right]$

for each $\varepsilon > 0$ $\exists m = \left[\frac{1}{4\varepsilon^2}\right] \in Z^+$

$\ni |S_n - 0| < \varepsilon \ \forall n \geq m.$

Note: $\boxed{\lim s_n = l \iff \lim (s_n - l) = 0}$

*** Imp

(4) Prove that $\lim \sqrt[n]{n} = 1$

(5) Sol

$t_n = \sqrt[n]{n} - 1$

$= n^{1/n} - 1$

$\therefore n^{1/n} = t_n + 1$

$n = (t_n + 1)^n$

$= 1 + n t_n + \frac{n(n-1)}{1 \cdot 2} t_n^2 + \cdots \cdots t_n^2$

$> \frac{n(n-1)}{2} t_n^2$

$n > \frac{n(n-1)}{2} t_n^2$

$t_n^2 < \frac{2}{n-1}$

$t_n < \sqrt{\frac{2}{n-1}}$

Let $\varepsilon > 0$

Consider $|t_n - l| < \varepsilon$

$\Leftarrow \left| \sqrt{\frac{2}{n-1}} - 0 \right| < \varepsilon$

$\Rightarrow \sqrt{\frac{2}{n-1}} < \varepsilon$

$\Rightarrow \frac{2}{n-1} < \varepsilon^2$

$n > \frac{2}{\varepsilon^2} + 1$

for each $\varepsilon > 0$ $\exists$ m $= \left[\frac{2}{\varepsilon^2} + 1\right] \in Z^+$

$\ni |t_n - 0| < \varepsilon \; \forall \; n \geq m.$

NOTE:

* $\lim S_n = 0 \Leftrightarrow \lim |S_n| = 0$

* if $\lim S_n = l$ Then $\lim |S_n| = |l|$

* if $\lim S_n = l$ and $\lim t_n = l'$ Then

$\lim (S_n + t_n) = l + l'$

* if $\lim S_n = l$ and $c \in R$ Then

$\lim (c \cdot S_n) = cl$.

* if $\lim S_n = 0$ and The sequence $\{t_n\}$ is bounded Then $\lim S_n t_n = 0$

* if $\lim S_n = l$ and $\lim t_n = l'$ Then

$\lim S_n \cdot t_n = l \cdot l'$

(5) Theorem: Sandwich (or) Sequence Theorem

Statement: If $\{S_n\}, \{t_n\}, \{U_n\}$ are Three Sequence Such That (i) $S_n \le U_n \le t_n$ for $n \ge K$ where K is some +ve integer

(ii) $\lim S_n = \lim t_n = l$. $\lim U_n = l$.

Proof: Given That $\{S_n\}, \{t_n\}$ and $\{U_n\}$ are Sequences, Let $\varepsilon > 0$

$\lim s_n = l$, $\exists\, m_1 \in Z^+$

$\ni |s_n - l| < \varepsilon \;\forall\, n \geq m_1$

$l - \varepsilon < s_n < l + \varepsilon \;\forall\, n \geq m_1$

$\lim t_n = l$, $\exists\, m_2 \in Z^+$

$\ni |t_n - l| < \varepsilon \;\forall\, n \geq m_2$

$l - \varepsilon < t_n < l + \varepsilon \;\forall\, n \geq m_2$

By hypothesis, we have

$s_n \leq u_n \leq t_n \quad \forall\, n \geq k$

we take $m = \max\{m_1, m_2, k\}$

$\Rightarrow l - \varepsilon \leq s_n \leq u_n \leq t_n \leq l + \varepsilon$

$\Rightarrow l - \varepsilon \leq u_n \leq l + \varepsilon$

$\Rightarrow |u_n - l| < \varepsilon$

$\Rightarrow \lim u_n = l$

Hence $\{u_n\}$ is converges to l.

** Imp

No 1 Show That $\lim \sqrt{\frac{n+1}{n}} = 1$

Sol We have $\sqrt{\frac{n+1}{n}} = \sqrt{1 + \frac{1}{n}}$

$= \left(1 + \frac{1}{n}\right)^{1/2}$

$\left[1 + \frac{1}{n} \cdot 1 + \frac{1}{2}\right]$ $1 + \frac{1}{2n} - \frac{1}{4n^2} + \cdots$

$< 1 + \frac{1}{2n}$

$$\sqrt{1+\frac{1}{n}} < 1+\frac{1}{2n} \quad — ①$$

We know that

$$1 < 1+\frac{1}{n}$$

$$\sqrt{1} < \sqrt{1+\frac{1}{n}}$$

from ① & ② $\quad 1 < \sqrt{1+\frac{1}{n}} \quad — ②$

$$1 < \sqrt{1+\frac{1}{n}} < 1+\frac{1}{2n}$$

— Apply limits

$$\lim 1 < \lim \sqrt{1+\frac{1}{n}} < \lim 1+\frac{1}{2n}$$

$$1 < \lim \sqrt{\frac{n+1}{n}} < 1$$

By Sandwich Theorem

$$\lim \sqrt{\frac{n+1}{n}} = 1$$

** IMP ② (S)

P.T $\lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right) = \frac{1}{2}$

Sol

Let $S_n = \dfrac{\sqrt{n^2+n} - n \times \sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}$

$$= \frac{\cancel{n^2}+n-\cancel{n^2}}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n}$$

$$= \frac{\cancel{n}}{\cancel{n}\left[\sqrt{1+\frac{1}{n}}+1\right]}$$

$$= \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

We have

$$1 < \sqrt{1+\frac{1}{n}+1} < 1+\frac{1}{2n}$$ [∴ adding Both Sides 1

$$2 < \sqrt{1+\frac{1}{n}}+1 < 2+\frac{1}{2n}$$

$$\frac{1}{2} > \frac{1}{\sqrt{1+\frac{1}{n}}+1} > \frac{2n}{4n+1}$$

$$\lim \frac{1}{2} > \lim\left(\frac{1}{\sqrt{1+\frac{1}{n}}+1}\right) > \lim \frac{2n}{4n+1}$$

Since $\lim \frac{1}{2} = \frac{1}{2}$ and

$$\lim \frac{2n}{4n+1} = \frac{1}{2}$$

$$\left[\because \lim \frac{2n}{4n+1}\right.$$

$$= \lim_{n\to\infty} \frac{2\cancel{n}}{\cancel{n}\left[4+\frac{1}{n}\right]} \qquad \left[\frac{1}{\infty}=0\right]$$

$$\left. = \frac{2}{4} = \frac{1}{2}\right]$$

By Sandwich Theorem

$$\lim_{n\to\infty} \sqrt{n^2+n} - n = \frac{1}{2}$$

3) P.T $\lim \left[\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots+\frac{1}{(n+n)^2}\right]=0$

Sol Given That

$$S_n = \left[\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots\frac{1}{(n+n)^2}\right]$$

if $n \geq m \geq 1$

$\Rightarrow n+n \geq m+n \geq 1+n$

$\Rightarrow (n+n)^2 \geq (m+n)^2 \geq (1+n)^2$

$\Rightarrow \frac{1}{(n+n)^2} \leq \frac{1}{(m+n)^2} \leq \frac{1}{(1+n)^2}$

$\frac{1}{(1+n)^2} \leq \frac{1}{(m+n)^2} \leq \frac{1}{(n+n)^2}$

we take $m = 1, 2, 3 \ldots n$ and add 'n' inequalities, we have

$\Rightarrow \frac{n}{(n+1)^2} \geq \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \frac{1}{(n+n)^2} \geq \frac{n}{(n+n)^2}$

$\Rightarrow \frac{n}{(n+1)^2} \geq S_n \geq \frac{n}{(n+n)^2}$

$\Rightarrow \frac{n}{4n^2} \leq S_n \leq \frac{n}{(n+1)^2} < \frac{n}{n^2}$

$\Rightarrow \frac{\cancel{n}}{4n^{\cancel{2}}} \leq S_n \leq \frac{\cancel{n}}{n^{\cancel{2}}}$

$\Rightarrow \frac{1}{4n} \leq S_n \leq \frac{1}{n}$

Apply lim on Both Sides

$\lim \frac{1}{4n} \leq \lim S_n \leq \lim \frac{1}{n}$

$0 \leq \lim S_n \leq 0$

By Sandwich Theorem, we have

$\therefore \lim S_n = 0.$

(iv) Strictly decresing:

A sequence $\{S_n\}$ is said to be strictly decresing if $S_1 > S_2 \cdots S_n > S_{n+1}$ $\forall n < n+1$ $\forall n \in Z^+$ (or) $S_n > S_m$ $\forall n < m$

Def: Monotone Sequence:

A sequence $\{S_n\}$, which is either increasing (or) decresing is called monotone sequence.

10m

Th A monotone seq convergent iff it is bounded (or)

a. $\{S_n\}$ is bounded increasing seq $\Leftrightarrow$ $\lim S_n = \sup\{S_n / n \in N\}$

b. $\{S_n\}$ is bounded decresing seq $\Leftrightarrow$ $\lim S_n = \inf\{S_n / n \in N\}$

Proof: If $\{S_n\}$ is convergent then $\{S_n\}$ is bounded

i) Let $\{S_n\}$ is bounded above.

Claim: $\{s_n\}$ is convergent

i.e $\lim s_n$ $\sup\{s_n | n \in N\}$

Since $\varepsilon > 0$ $\sup\{s_n\} = k$ (say)

$\rightarrow k - \varepsilon$ is not an upper bound

i.e $s_m > k - \varepsilon$ (i.e $m \in Z^+$)

Since $\{s_n\}$ is increasing $s_n \geq s_m > k - \varepsilon$

$\rightarrow s_n > k - \varepsilon$ — ①

$\sup\{s_n\} = k$

$\rightarrow s_n \leq k$.

$\rightarrow s_n \leq k + \varepsilon$ — ②

From ① & ②

$k - \varepsilon < s_n < k + \varepsilon$

$|s_n - k| < \varepsilon$

$\rightarrow \lim\{s_n\} = k$

$\therefore \lim s_n = \sup\{s_n / n \in N\}$

(ii) Let $\{s_n\}$ is bounded below.

i.e $\lim s_n = \inf\{s_n | n \in N\}$.

let $\varepsilon > 0$, $\inf\{s_n\} = u$ (say)

$\rightarrow u + \varepsilon$ is not an lower bound

$\exists\, m \in Z^+ / s_m < u + \varepsilon$

if $\{s_n\}$ is decreasing

$S_n < U + \varepsilon$ —①

and $\inf\{S_n\} = u$.

$\Rightarrow S_n \geq u$

$\Rightarrow S_n > U - \varepsilon$ —②

From ① & ② we have

$U - \varepsilon < S_n < U + \varepsilon$

$|S_n - u| < \varepsilon$

$\lim S_n = u$

$\therefore \lim S_n = \inf\{S_n \mid n \in N\}$

Hence monotone sequence is bounded, if and only if ~~Hence~~ it is convergent.

① Prove That Sequence $S_n = \frac{3n+4}{2n+1}$ is decresing and bounded below.

Sol Given That $S_n = \frac{3n+4}{2n+1}$ $\left[\because S_n \geq S_{n+1} \quad S_n - S_{n+1} \geq 0\right]$

$$S_{n+1} = \frac{3(n+1)+4}{2(n+1)+1}$$

$$= \frac{3n+7}{2n+3}$$

$$= \frac{3n+6-7}{n+2}$$

$$= \frac{3(n+2)-7}{n+2}$$

$$= \frac{3\cancel{(n+2)}}{\cancel{n+2}} - \frac{7}{n+2}$$

$$= 3 - \frac{7}{n+2} < 3$$

SM $\therefore \{S_n\}$ has bounded above

(S) (3) Prove That sequence $S_n = 2 - \frac{1}{2^{n-1}}$ is convergent

Sol

$$S_n = 2 - \frac{1}{2^{n-1}} \quad (1)$$

$$S_{n+1} = 2 - \frac{1}{2^{n+\cancel{1}-\cancel{1}}}$$

$$= 2 - \frac{1}{2^n}$$

We have $2^n > 2^{n-1}$

$$\frac{1}{2^n} < \frac{1}{2^{n-1}}$$

$$2 - \frac{1}{2^n} > 2 - \frac{1}{2^{n-1}}$$ "both sides adding ~~multiplying with~~ 2"

$S_{n+1} > S_n$.

$S_n < S_{n+1}$

Hence $\{S_n\}$ is increasing

$$S_n = 2 - \frac{1}{2^{n-1}} < 2$$

$\therefore S_n < 2$

Hence S_n is convergent.

(4) (8) If $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)}$ Then increasing and bounded above.

Sol

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)}$$ $\left[S_n < S_{n+1} \quad S_n - S_{n+1} < 0 \right]$

$$S_{n+1} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)}$$

$$S_n - S_{n+1} = \cancel{\frac{1}{1\cdot 2}} + \cancel{\frac{1}{2\cdot 3}} + \dots + \cancel{\frac{1}{n(n+1)}} - \cancel{\frac{1}{1\cdot 2}} - \cancel{\frac{1}{2\cdot 3}} \dots - \cancel{\frac{1}{n(n+1)}} - \frac{1}{(n+1)(n+2)}$$

$$= \frac{-1}{(n+1)(n+2)} < 0$$

$S_n - S_{n+1} < 0$

$S_n < S_{n+1}$

Also $S_n = \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \cdots + \frac{1}{n(n+1)}$

$= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots (\frac{1}{n} - \frac{1}{n+1})$

$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cdots + \cancel{\frac{1}{n}} - \frac{1}{n+1}$

$= 1 - \frac{1}{n+1}$

$= \frac{n}{n+1} < 1$

$S_n < 1$

$\therefore \{S_n\}$ is convergent.

5) P.T $S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ is convergent.

Sol $S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ $\quad [S_n < S_{n+1}$

$S_n - S_{n+1} < 0]$

$S_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n+1} + \frac{1}{2n+2}$

$S_n - S_{n+1} = \frac{1}{n+1} + \cancel{\frac{1}{n+2}} + \cdots + \cancel{\frac{1}{n+n}} - \cancel{\frac{1}{n+2}} - \cancel{\frac{1}{n+3}}$

$- \cancel{\frac{1}{n+n}} - \frac{1}{2n+1} - \frac{1}{2n+2}$

$$S_n - S_{n+1} = \frac{1}{n+1} - \frac{1}{2n+1} - \frac{1}{2n+2} < 0$$

(a) $S_n < S_{n+1}$ $\{S_n\}$ is incresing

Also $S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$

$$< \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = \cancel{n}\left(\frac{1}{\cancel{n}}\right) = 1$$

$S_n < 1$

$\therefore \{S_n\}$ is bounded above

Hence $\{S_n\}$ is convergent.

(6) If $\{S_n\}$ be a sequence defined by $a_1 = 1, a_{n+1} = \frac{2a_n + 3}{4}$ for $n \geq 1$. S.T $\{a_n\}$ is incresing and Evalute its limits.

Sol Given $a_1 = 1$

$$a_{n+1} = \frac{2a_n + 3}{4}$$

$$a_2 = \frac{2a_1 + 3}{4} = \frac{2+3}{4} = \frac{5}{4}$$

$$a_3 = \frac{2a_2 + 3}{4} = \frac{2\left(\frac{5}{4}\right) + 3}{4} = \frac{11}{8}$$

Consider $a_n < a_{n+1}$

$\Rightarrow 4x - 2x = 3$

$2x = 3$

$\lim_{x \to \infty} a_n = \frac{3}{2}$

(7) P.T the sequence $\{S_n\}$ is defined by $S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ is convergent

Sol

$$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

$S_n < S_{n+1}$

$S_n - S_{n+1} < 0$

$$S_{n+1} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!}$$

$$S_n - S_{n+1} = -\frac{1}{(n+1)!} < 0.$$

$$S_n - S_{n+1} < 0$$

$$S_n < S_{n+1}$$

$\therefore \{S_n\}$ is increasing

Also $S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$

$= 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \dots + \frac{1}{1\cdot 2\cdot 3 \cdots n}$

$< 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2\cdot 2} + \dots + \frac{1}{2\cdot 2 \cdots 2\,(n \text{ times})}$

$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots \frac{1}{2^n}$

$< 1 + \left[1 + \frac{1}{2} + \frac{1}{2^2} + \dots \frac{1}{2^n}\right]$

$< 1 + \left[\frac{\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}}\right]$ $\quad \left[\text{GP } \frac{a(1-r^n)}{1-r}\ r<1\right.$

$\frac{a(r^n - 1)}{r-1}$ if $r>1$

$< 1 + \left[2\left(1 - \frac{1}{2^n}\right)\right]$

$< 1 + 2 - \frac{1}{2^{n-1}}$

$< 3 - \frac{1}{2^{n-1}} < 3$

$\therefore \{S_n\}$ is bounded above

$\therefore \{S_n\}$ is convergent.

(6)

(8) Prove that seq $\{S_n\}$ is defined by $S_n = \left(1 + \frac{1}{n}\right)^n$
(9) is convergent

Sol $S_n = \left(1 + \frac{1}{n}\right)^n$

$= 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \dots +$

$\frac{n(n-1)\cdots 2\cdot 1}{n!}\cdot\frac{1}{n^n}$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{n!}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{n-1}{n}\right)\right]$$

$$S_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\cdots+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)$$

2 $$\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$$

We have $n < n+1$

$$\frac{1}{n} > \frac{1}{n+1}$$

$$\frac{-1}{n} < \frac{-1}{n+1}$$

$$1-\frac{1}{n} < 1-\frac{1}{n+1}$$

By $1-\frac{2}{n} < 1-\frac{2}{n+1}$

$$1-\frac{n-1}{n} < 1-\frac{n}{n+1}$$

$\therefore S_n < S_{n+1}$

$\{S_n\}$ is increasing

Also $S_n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\cdots\left(1-\frac{n-1}{n}\right)$

$$< 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$$

$$< 1+\left[1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right]$$

By above problem we have < 3

$\therefore S_n < 3$

(b) $\{S_n\}$ is bounded above

$\therefore$ Hence $\{S_n\}$ is Convergent.

(9) P.T the Seq $\{S_n\}$ is defined by $S_1 = \sqrt{c}$, $S_{n+1} = \sqrt{c+S_n}$ is convergent to the Positive root of $x^2 - x - c = 0$

Sol

$$S_1 = \sqrt{c}$$

and $S_{n+1} = \sqrt{c+S_n}$

Put $n=1$

$$S_2 = \sqrt{c+S_1}$$
$$= \sqrt{c+\sqrt{c}}$$

Put $n=2$

$$S_3 = \sqrt{c+\sqrt{c+\sqrt{c}}}$$

$S_n < S_{n+1}$

We have

$$S_n < S_{n+1}$$

$$c+S_n < c+S_{n+1}$$

$$\sqrt{c+S_n} < \sqrt{c+S_{n+1}}$$

$$S_{n+1} < S_{n+2}$$

$\therefore \{S_n\}$ is increasing

*Imp 2020

(1) State and Prove Bolzano - weierstrass Theorem

Statement : Every bounded sequence has a convergent subsequence.

Theorem: A sequence is convergent, iff it is bounded, and has only one - Limit Point

Proof; Let $\{S_n\}$ be a a convergent sequence if $\{S_n\}$ is convergent. then it is bounded

$\Rightarrow |S_n - l| < \varepsilon \ \forall \ n \geq m$

$\Rightarrow l - \varepsilon < S_n < l + \varepsilon \ \forall \ n \geq m$

$\Rightarrow S_n \in (l - \varepsilon, l + \varepsilon) \ \forall \ n \geq m$ — (1)

Let 'l'' be another limit of $\{S_n\}$

Let $\varepsilon > 0$ $(l' - \varepsilon, l' + \varepsilon) \cap (l - \varepsilon, l + \varepsilon) = \phi$

$S_n \in (l - \varepsilon, l + \varepsilon) \ \forall \ n \geq m$

$S_n \in (l' - \varepsilon, l' + \varepsilon)$ for almost $m-1$ of value m

$\therefore l'$ is not a Limit point.

hence $\{S_n\}$ has a unique point (Limit)

$\{S_n\}$ is bounded and it has unique Limit point.

Claim: $\{S_n\}$ is convergent.

Since $\{S_n\}$ is bounded and has one limit point

for each $\varepsilon > 0$ i.e. $S_n \in (l-\varepsilon, l+\varepsilon)$

Let ~~S_n, S_n~~ $S_{m_1}, S_{m_2} \dots$ are infinite terms of limit point.

take $n = \max\{m_1, m_2, m_3 \dots\}$

then $S_n \in (l-\varepsilon, l+\varepsilon) \ \forall \ n \geq m$

$$\Rightarrow l-\varepsilon < S_n < l+\varepsilon \ \forall \ n \geq m$$

$$|S_n - l| < \varepsilon \ \forall \ n \geq m$$

$\therefore$ for each $\varepsilon > 0$, $\exists m \in \mathbb{Z}^+ \ni |S_n - l| < \varepsilon \ \forall \ n \geq m$

Hence $\{S_n\}$ is convergent.

Main Proof:

By above theorem, we have every sequence has a unique limit point l (say).

So, $\exists$ any subsequence $\{S_{n_k}\}$ of $\{S_n\}$ is converges to l.

Hence every bounded sequence has convergent subsequence hence

$\Rightarrow k_1 < S_p < k_2$

$\Rightarrow k_1 < \{S_n\} < k_2$

$\therefore \{S_n\}$ is bounded

Hence every Cauchy sequence is bounded.

2020 SM

Theorem:

If The sequence $\{S_n\}$ is convergent. Then $\{S_n\}$ is Cauchy sequence.

(or)

Every convergent sequence is Cauchy sequence.

Proof: Let $\{S_n\}$ be a convergent sequence

Claim: $\{S_n\}$ is Cauchy sequence.

Since $\{S_n\}$ is convergent for each $\varepsilon = \frac{\varepsilon}{2} > 0$

$\exists\, m \in Z^+ \ni |n - l| < \varepsilon \;\forall\; n \geq m$

$\Rightarrow |S_n - l| < \frac{\varepsilon}{2} \;\forall\; n \geq m$

if $p, q \geq m$ then $|S_p - l| < \frac{\varepsilon}{2}$ and $|S_q - l| < \frac{\varepsilon}{2}$

consider $|S_p - S_q| = |S_p - l + l - S_q|$

$= |S_p - l| + |l - S_q|$

$= |S_p - l| + |S_q - l|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$< \frac{2\varepsilon}{2}$

$< \varepsilon$

$\therefore |S_p - S_q| < \varepsilon \ \forall \ p, q \geq m.$

Hence every convergent sequence is Cauchy sequence.

General principle

Theorem ② If the seq $\{S_n\}$ is Cauchy sequence then $\{S_n\}$ is convergent.

3/4

Proof: Let $\{S_n\}$ be a Cauchy sequence

Claim: $\{S_n\}$ convergent.

Since $\{S_n\}$ is Cauchy sequence, for each

$\varepsilon = \frac{\varepsilon}{\varepsilon} > 0 \ \exists \ m \in \mathbb{Z} \ \ni |S_p - S_q| < \varepsilon = \frac{\varepsilon}{\varepsilon} \ \forall \ p, q \geq$

$\Rightarrow |S_p - S_q| < \frac{\varepsilon}{\varepsilon} \ \forall \ p, q \geq m$

If $\{S_n\}$ is Cauchy seq then $\{S_n\}$ is bounded

$\Rightarrow \{S_n\}$ has one limit point l(say)

Let l' be another limit point

take $\varepsilon = |l-l'| > 0$ —(1)

let $p \geq m$ and $q \geq m$, Then

$|S_p - l| < \frac{\varepsilon}{\varepsilon}$ & $|S_q - l| < \frac{\varepsilon}{\varepsilon}$

Consider $|l-l'| = |l - S_p + S_p - S_q + S_q - l'|$

$= |l - S_p| + |S_p - S_q| + |S_q - l'|$

$= |S_p - l| + |S_p - S_q| + |S_q - l'|$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \frac{3\varepsilon}{3} < \varepsilon$

From (1)

$|l - l'| < |l - l'|$

This is impossible So That is contradiction

That l' is another limit point

$\therefore \{S_n\}$ has unique limit point.

By bolzano weierstrass theorem

$\{S_n\}$ is convergent.

IMP V.V.IMP I

Cauchy convergence criterion:

A seq is convergence iff it is Cauchy sequence.

(or)

Cauchy's general principle of convergence(or)

Cauchy's Theorem:

A sequence $\{S_n\}$ is convergence iff for each $\varepsilon > 0$ here exist $m \in \mathbb{Z}^+$ such that $|S_{n+p} - S_n| < \varepsilon$ for all $n \geq m$ and $p > 0$.

(or)

A sequence $\{S_n\}$ is convergence iff $\{S_n\}$ is a Cauchy sequence.

Proof of Theorem 1 & 2

UNIT - II

Infinite Series

Def of Series:

If seq $\{S_n\}$ is the sequence of real number and $u_1+u_2+u_3+\cdots+u_n+\cdots$, $n \in Z^+$ then the series summation $\sum_{n=1}^{\infty} u_n$ is called infinite series.

Convergence of Series:

Let $\sum_{n=1}^{\infty} u_n$ be the series of real number with partial sum $S_n = u_1+u_2+\cdots+u_n$ $n \in Z^+$ if the sequence $\{S_n\}$ converges to 's' we say that the series $\sum_{n=1}^{\infty} u_n$ converges to 's' the number 's' is called "sum of series" and it is also denoted by $\boxed{\sum_{n=1}^{\infty} u_n = s}$

* If the limit of sequence $\{S_n\}$ doesn't exist we say that the series $\sum_{n=1}^{\infty} u_n$ is diverges.

Hence By limit comparision test

Σv_n is divergent.

$\Sigma \frac{2n}{n^2+1}$ is divergent.

(3) Test the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}+(\beta/n)}$ (α, β are

imp (5) two real numbers)

Sol $\sum_{n=1}^{\infty} U_n = \cancel{\sum_{n=1}^{\infty} \frac{1}{n^{\alpha+\beta/n}}}$ $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha+\beta/n}}$

$$U_n = \frac{1}{n^{\alpha+\beta/n}}$$

For $n \geq 1$ $n^{\alpha+\beta/n} > n^{\alpha}$

$$\frac{1}{n^{\alpha}+(\beta/n)} < \frac{1}{n^{\alpha}}$$

If $V_n = \frac{1}{n^{\alpha}}$ then $\Sigma V_n = \Sigma \frac{1}{n^{\alpha}}$ $\begin{cases} \text{convergent if } \alpha > 1 \\ \text{divergent if } 0 < \alpha \leq 1 \\ \text{divergent if } \alpha \leq 0 \end{cases}$

$$\lim \frac{U_n}{V_n} = \lim \left(\frac{1}{n^{\alpha+\beta/n}} \cdot \frac{n^{\alpha}}{1} \right)$$

$$= \lim \left(\frac{1}{\cancel{n^{\alpha}} \cdot n^{\beta/n}} \cdot \frac{\cancel{n^{\alpha}}}{1} \right)$$

$$= \lim \cdot \frac{1}{n^{\beta/n}}$$

$$\lim \frac{u_n}{v_n} = \frac{1}{n^{\beta/n}} \neq 0$$

By limit comparision test.

Σu_n is $\begin{cases} \text{Convergent if } \alpha > 0 \\ \text{divergent if } 0 < \alpha \leq 1 \\ \text{divergent if } \alpha \leq 0 \end{cases}$

Hence $\Sigma \frac{1}{n^{\alpha+\beta/n}}$ is $\begin{cases} \text{Convergent if } \alpha > 0 \\ \text{divergent if } 0 < \alpha \leq 1 \\ \text{divergent if } \alpha \leq 0 \end{cases}$

(4) (5) Test the convergence of $\sum_{n=1}^{\infty} \frac{1}{2^n + 3^n}$

Let $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{2^n+3^n}$

i.e, $u_n = \frac{1}{2^n+3^n} = \frac{1}{3^n\left(1+\frac{2^n}{3^n}\right)}$

for $n \geq 1$

$$3^n\left(1+\frac{2^n}{3^n}\right) > 3^n$$

$$\frac{1}{3^n\left(1+\frac{2^n}{3^n}\right)} < \frac{1}{3^n}$$

If $v_n = \frac{1}{3^n}$, then $\Sigma v_n = \Sigma \frac{1}{3^n}$ (By Series

Σv_n is convergent. $\left[S_n = \sum_{n=0}^{\infty} \frac{1}{3^n} = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots \frac{1}{3^n} + \cdots \right.$

By Limit Comparision test

$$\lim \frac{u_n}{v_n} = \lim \left(\frac{1}{\cancel{3^n}\left(1+\left(\frac{2}{3}\right)^n\right)} \cdot \frac{\cancel{3^n}}{1} \right) = \frac{1\left(1-\frac{1}{3^n}\right)}{1-\frac{1}{3}}$$

$$= \lim \left(\frac{1}{1+\left(\frac{2}{3}\right)^n} \right) = \frac{3}{2}\left(1-\frac{1}{2^n}\right)$$

$$= \lim \frac{1}{1+0} = \lim 1 = 1 \neq 0$$

Σu_n is convergent.

$$= \frac{3}{2} - \frac{3}{2} \cdot \frac{1}{3^n}$$

$$= \frac{3}{2} - \frac{1}{2 \cdot 3^{n-1}}$$

(5) Test The convergence of $\Sigma \frac{1}{n} \sin \frac{1}{n}$

Sol Let $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$

i.e $u_n = \frac{1}{n} \sin \frac{1}{n} \left(\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \right.$

$$= \frac{1}{n}\left(\frac{1}{n} - \frac{1/n^3}{3!} + \frac{1/n^5}{5!} + \cdots \right)$$

$$u_n = \frac{1}{n^2}\left(1 - \frac{1/n^2}{3!} + \frac{1/n^4}{5!} + \cdots \right)$$

for $n \geq 1$

if $v_n = \frac{1}{n^2}$ then $\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent

Σv_n is convergent

Consider $\lim\left(\frac{U_n}{V_n}\right) = \lim\left(\frac{1}{n^2}\left(1 - \frac{1/n^2}{3!} + \frac{1/n^4}{5!} \dots\right)\right)$

$$= \lim\left(1 - \frac{1/n^2}{3!} + \frac{1/n^4}{5!} + \dots\right)$$

$$= 1 \neq 0$$

ΣU_n is convergent

Imp

(2) Test the convergence of $\sum_{n=1}^{\infty} \sqrt{n^2+1} - n$

Sol

$$\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \sqrt{n+1} - n$$

$$U_n = \frac{\sqrt{n^2+1} - n}{\sqrt{n^2+1} + n} \times \sqrt{n^2+1} + n$$

$$U_n = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$$

$$U_n = \frac{1}{\sqrt{n^2+1}+n}$$

$$U_n = \frac{1}{n\left(\sqrt{1+\frac{1}{n^2}}+1\right)}$$

for $n \geq 1$

$$n\left(\sqrt{1+\frac{1}{n^2}}+1\right) > n$$

If $V_n = \frac{1}{n}$ Then $\Sigma V_n = \Sigma \frac{1}{n}$ divergent

ΣV_n is divergent

$$\lim \frac{u_n}{v_n} = \lim \left(\frac{1}{\cancel{n}\sqrt{1+\frac{1}{n^2}}+1} \cdot \cancel{n} \right)$$

$$= \lim \frac{1}{\sqrt{1+\frac{1}{n^2}}+1}$$

$$= \frac{1}{2} \neq 0$$

Σu_n is divergent

Imp (5) Test The Convergence of $\sum_{n=1}^{\infty} \sqrt[3]{n^3+1} - n$

Sol Let $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \sqrt[3]{n^3+1} - n$

$$= \sum_{n=1}^{\infty} \underbrace{\sqrt[3]{n^3+1}}_{a} - \underbrace{\sqrt[3]{n^3}}_{b}$$

$$\text{i.e} = u_n = \frac{\left((n^3+1)^{1/3} - (n^3)^{1/3}\right)\left[\left((n^3+1)^{1/3}\right)^2 + (n^3+1)^{1/3} (n^3)^{1/3} + \left[(n^3)^{1/3}\right]^2\right]}{\left((n^3+1)^{1/2}\right)^2 + \left((n^3+1)^{1/3} \cdot (n^3)^{1/3} + (n^3)^{1/3}\right)^2}$$

$$= \frac{\left((n^3+1)^{1/3}\right)^3 - \left((n^3)^{1/3}\right)^{\cancel{3}}}{\left((n^3+1)^{1/3}\right)^2 + (n^3+1)^{1/3}(n^3)^{1/3} + \left((n^3)^{1/3}\right)^2}$$

$$= \frac{\cancel{n^3}+1-\cancel{n^3}}{(n^3+1)^{2/3} + (n^3+1)^{2/3} + (n^3)^{2/3}}$$

$$= \frac{1}{(n^3+1)^{2/3} + n\cdot(n^3+1)^{1/3} + n^2}$$

$$= \frac{1}{(n^3)^{2/3}\left(1+\frac{1}{n^3}\right)^{2/3} + n\cdot\left((n^3)^{1/3}\left(1+\frac{1}{n^3}\right)^{1/3}\right) + n^2}$$

$$= \frac{1}{n^2\left(1+\frac{1}{n^3}\right)^{2/3} + n^2\left(1+\frac{1}{n^3}\right)^{1/3} + n^2}$$

$$= \frac{1}{n^2\left[\left(1+\frac{1}{n^3}\right)^{2/3} + \left(1+\frac{1}{n^3}\right)^{1/3} + 1\right]}$$

for $n \geq 1$

$$\frac{1}{n^2\left[\left(1+\frac{1}{n^3}\right)^{2/3} + \left(1+\frac{1}{n^3}\right)^{1/3} + 1\right]} < \frac{1}{n^2}$$

If $v_n = \frac{1}{n^2}$; Then $\Sigma v_n = \frac{1}{n^2}$ is convergent

$\therefore \Sigma v_n$ is Convergent

Consider $\lim\left(\frac{u_n}{v_n}\right) = \lim\left(\frac{1}{n^2\left(1+\frac{1}{n^3}\right)^{2/3} + \left(1+\frac{1}{n^3}\right)^{1/3} + 1}\right)$

$$= \lim\left(\frac{1}{\left(1+\frac{1}{n^3}\right)^{2/3} + \left(1+\frac{1}{n^3}\right)^{1/3} + 1}\right)$$

$= \lim\left(\frac{1}{1+1\,1+1\,1}\right)$

$= \frac{1}{3} \neq 0$

$\therefore \Sigma U_n$ is convergent.

10M 1. Cauchy nth root test:-

Statement: If Summation U_n is a series of +ve terms. such That $\lim (U_n)^{1/n} = l$.

(a) ΣU_n is convergent if $l<1$ &

(b) ΣU_n is divergent if $l>1$

Proof: Given ΣU_n is series of +ve terms

i.e $U_n>0$.

and $\lim (U_n)^{1/n} = l \geq 0$

for each $\varepsilon>0$ $\exists$ $m \in \mathbb{Z}^+$ $| (U_n)^{1/n} - l | < \varepsilon$ $\forall$ $n \geq m$

$\Rightarrow l-\varepsilon < (U_n)^{1/n} < l+\varepsilon$ $\forall$ $n \geq m$

$\Rightarrow (l-\varepsilon)^n < U_n < (l+\varepsilon)^n$ $\forall$ $n \geq m$ —— (1)

(i) let $l<1$

Choose $\varepsilon>0$ such that $k = l+\varepsilon<1$

from (1) $U_n < (l+\varepsilon)^n$

$U_n < k^n$

$\left[0<l<1 \right.$

$l = 0.2$

$\varepsilon = 0.02$

$\left. l+\varepsilon = 0.22 \right]$

$\Sigma u_n < \Sigma k^n$ $\left[\because 0 < l < l+\varepsilon < 1 \right.$

$0 < l+\varepsilon < 1$

$\left. 0 < k < 1 \right]$

Σk^n is convergent $\left[\because k^n \text{ is Geometric Series because } 0<k<1 \right.$

By Comparision test

Σu_n is convergent.

(ii) Let $l > 1$

Chose $\varepsilon > 0$, $l - \varepsilon > 1$

$(l-\varepsilon)^n > 1^n$

$(l-\varepsilon)^n > 1$

from ① $(l-\varepsilon)^n < u_n$

$1 < (l-\varepsilon)^n < u_n$

$1 < u_n$

$\therefore \Sigma u_n$ is divergent.

అంటే ok?

$l = 2$ [illegible]

$\varepsilon = 0.2$

$l - \varepsilon = 1.8 > 1$

Eg: $k = \frac{1}{3}$

$k^n = \frac{1}{3^n}$

$\Sigma k^n = \Sigma \frac{1}{3^n}$

$= 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots$

$= \dfrac{1 - \frac{1}{3^n}}{1 - \frac{1}{3}}$

$\therefore \Sigma k^n$ is convergent

$\therefore \lim k^n = 1$

$\Sigma \left(1+\frac{1}{n}\right)^{-n^2}$ is convergent.

Imp **D. Alembert's Ratio test (or) Ratio test:**

Imp If Σu_n is a series of +ive Terms.

such that $\lim \frac{u_{n+1}}{u_n} = l$ (a) Σu_n is converges if $l<1$

(b) Σu_n diverges if $l>1$

Proof: Given Σu_n is a series of +ive terms

i.e $u_n > 0$ or $u_n \geq 1$

and $\lim \left(\frac{u_{n+1}}{u_n}\right) = l > 0$

for Each $\varepsilon > 0$ $\exists$ $m \in \mathbb{Z}^+$ $\left|\frac{u_{n+1}}{u_n} - l\right| < \varepsilon \; \forall n \geq m$

$= l - \varepsilon < \frac{u_{n+1}}{u_n} < l + \varepsilon \; \forall n \geq m$

Putting $n = m, m+1, \ldots n-1$

No. of terms $= n - \cancel{1} - m + \cancel{1}$

$= n - m$ terms

Put $n = m$

$l - \varepsilon < \frac{u_{m+1}}{u_m} < l + \varepsilon$

$n = m+1$

$l - \varepsilon < \frac{u_{m+2}}{u_{m+1}} < l + \varepsilon$

$$n = m+2$$

$$l-\varepsilon < \frac{U_{m+3}}{U_{m+2}} < l+\varepsilon \quad \Bigg\} \; (n-m) \text{ terms}$$

$\vdots$

$$n = n-1$$

$$l-\varepsilon < \frac{U_n}{U_{n-1}} < l+\varepsilon$$

multiply above $(n-m)$ inequalities.

$$(l-\varepsilon)^{n-m} < \frac{U_{m+1}}{U_m} \cdot \frac{U_{m+2}}{U_{m+1}} \cdots \frac{U_n}{U_{n-1}} < (l+\varepsilon)^{n-m}$$

$$(l-\varepsilon)^{n-m} < \frac{U_n}{U_m} < (l+\varepsilon)^{n-m}$$

$$U_m \cdot (l-\varepsilon)^{n-m} < U_n < (l+\varepsilon)^{n-m} \cdot U_m \quad \text{--- (1)}$$

(i) Let $l < 1$

Choose $\varepsilon > 0$ $k = l + \varepsilon < 1$

$$\left[\begin{aligned} l &= 0.2 \\ \varepsilon &= 0.02 \\ l+\varepsilon &= 0.22 \end{aligned}\right]$$

Now $0 \le l < 1$

$0 \le l < l+\varepsilon < 1$

$0 < l+\varepsilon < 1$

$0 < k < 1$

From (1) $U_n < (l+\varepsilon)^{n-m} U_m$

$$U_n < (k)^{n-m} \cdot U_m \Rightarrow U_n < \left(\frac{k^n}{k^m} \cdot U_m\right)$$

$$U_n < \left(\frac{U_m}{k^m}\right) k^n$$

$U_n < \alpha k^n$ where $\alpha = \dfrac{U_m}{k^m}$

$\therefore \Sigma \alpha k^n$ is convergent because $0 < k < 1$

By comparision test

ΣU_n is convergent

(ii) Let $l > 1$

Chose $\varepsilon > 0$ $k = l - \varepsilon > 1$

$k > 1$

$\left[\; l = 3 \quad \varepsilon = 0.02 \quad l - \varepsilon = 2.98 > 1 \;\right]$

From ①

$$(l - \varepsilon)^{n-m} \cdot U_m < U_n$$

$$(k)^{n-m} U_m < U_n \Rightarrow \frac{k^n}{k^m} U_m < U_n \Rightarrow \beta k^n < U_n$$

$\cancel{\frac{k^n}{k^m} \cdot U_m} \; \cancel{\left(\frac{U_m}{k^m}\right) k^n < U_n}$ Σk^n is divergent $(k > 1)$

~~$\beta k^n < U_n$ where~~ $\beta = \dfrac{U_m}{k^m}$

~~$\Sigma \beta k^n$ is divergent because $k > 1$.~~

By comparision test

ΣU_n is divergent.

Ques 1 Test of convergence of

(5) (i) $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots$

Sol

$$1+\sum_{n=1}^{\infty}\frac{1}{n!}$$

$$\sum_{n=1}^{\infty}\frac{1}{n!}$$

Let $\Sigma U_n = \Sigma \frac{1}{n}$

i.e $U_n = \frac{1}{n!}$

$$U_{n+1} = \frac{1}{(n+1)!}$$

$$= \frac{1}{n!\,(n+1)}$$

$$\lim\left(\frac{U_{n+1}}{U_n}\right) = \lim\left(\frac{1}{\cancel{n!}(n+1)}\cdot\frac{\cancel{n!}}{1}\right)$$

$$= \lim\left(\frac{1}{(n+1)}\right) < 1$$

ΣU_n is convergent

(5) (2) $\frac{1}{2}+\frac{1\cdot3}{2\cdot5}+\frac{1\cdot3\cdot5}{2\cdot5\cdot8}+\dots$

Sol

$$U_n = \frac{1\cdot3\cdot5\cdots\cdots(2n-1)}{2\cdot5\cdot8\cdots\cdots(3n-1)}$$

$$U_{n+1} = \frac{1\cdot3\cdot5\cdots(2n-1)(2n+2-1)}{2\cdot5\cdot8\cdots\cdots(3n-1)\,(3n+2)}$$

$$= \lim\left(\frac{2n+1}{3n+2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2+1/n}{3+2/n}\right) = \frac{2}{3} < 1$$

$\therefore$ ΣU_n is convergent.

(3) $\sum_{n=1}^{\infty}\frac{(n+1)!}{3n}$

Sol

$$U_n = \frac{(n+1)!}{3^n}$$

$$U_{n+1} = \frac{(n+1)!\,(n+2)}{3^n\cdot 3}$$

$$\lim\left(\frac{U_{n+1}}{U_n}\right) = \lim\left(\frac{\cancel{(n+1)!}\,(n+2)}{\cancel{3^n}\,3}\times\frac{\cancel{3^n}}{\cancel{(n+1)!}}\right)$$

$$= \lim\left(\frac{n+2}{3}\right)$$

$$= \lim\left(\frac{1+2/n}{3}\right)n$$

$$= \infty\left(\frac{1}{3}\right) < \infty > 1$$

ΣU_n is divergent

(4) $\sum_{n=1}^{\infty}\frac{n!}{n^n}$

$$U_n = \frac{n!}{n^n}$$

$$U_{n+1} = \left(\frac{\cancel{n!}\,(n+1)}{(\cancel{n^n})\,n+1}\times\frac{\cancel{n^n}}{\cancel{n!}}\right)$$

Imp
I

Leibnitz test:-

If $\{U_n\}$ is sequence of positive terms such that (a) $U_1 \geq U_2 \geq U_3 \dots \geq U_n \geq U_{n+1} \geq \dots$

and (b) $\lim U_n = 0$ such that

The alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} U_n$ is converges.

Proof: Given that $\{S_n\}$ is a sequence of positive terms.

i.e $U_n > 0$

and $U_1 \geq U_2 \geq U_3 \geq \dots U_n \geq U_{n+1} \geq$ (decreasing order)

$\lim U_n = 0$.

Let $S_n = U_1 - U_2 + U_3 - U_4 + \dots + (-1)^{n-1} U_n$ be the n^{th} partial sum.

take $n = 2n$

From ① $S_{2n} = U_1 - U_2 + U_3 - U_4 + \dots (-1)^{2n-1} U_{2n}$

$S_{2n} = U_1 - U_2 + U_3 - U_4 + \dots - U_{2n}$

$= U_1 - U_2 + U_3 - U_4 + \dots + U_{2n-1} - U_{2n}$ — ②

take $n = 2n+2$

$S_{2n+2} = U_1 - U_2 + U_3 - U_4 + \dots (-1)^{2n+2-1} \cdot U_{2n+2}$

$= u_1 - u_2 + u_3 - u_4 + \cdots - u_{2n+2}$

$= u_1 - u_2 + u_3 - u_4 + \cdots + u_{2n-1} - u_{2n} + u_{2n+1}$

$- u_{2n+2}$ — (3)

Substitute (2) in (3)

$S_{2n+2} = S_{2n} + u_{2n+1} - u_{2n+2}$

$S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2}$

$S_{2n+2} - S_{2n} \geq 0$ $\quad$ $[\because u_n \geq u_{n+1}$ Given

$S_{2n+2} \geq S_{2n}$ $\quad$ $n = 2n+1$

$\therefore \{S_{2n}\}$ is an increasing sequence $\quad$ $u_{2n+1} \geq u_{2n+2}$

from Eq^n (2)

$S_{2n} = u_1 - u_2 + u_3 - u_4 + \cdots - u_{2n}$

$= u_1 - (u_2 - u_3 + u_4 + \cdots + u_{2n-2} - u_{2n-1} + u_{2n})$

$= u_1 - ((u_2 - u_3) + (u_4 - u_5) + \cdots + (u_{2n-2} - u_{2n-1}) + u_{2n})$

[By cond $u_1 \geq u_2 \geq u_3 \cdots$]

$= u_1 - (\text{a positive terms}) \leq u_1$

$\therefore S_{2n} \leq u_1$

Hence $\{S_{2n}\}$ of $\{S_n\}$ is bounded

Pb: P.T $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$

Let $U_n=\frac{1}{n}$

$U_{n+1}=\frac{1}{n+1}$

Consider $U_n-U_{n+1}=\frac{1}{n}-\frac{1}{n+1}$

$=\frac{\cancel{n}+1-\cancel{n}}{n(n+1)}$

$=\frac{1}{n(n+1)}>0$

$U_n-U_{n+1}>0$

$U_n>0.$

$\therefore \{U_n\}$ is decreasing seq

$\lim U_n=\lim\left(\frac{1}{n}\right)$

$=0$

$\therefore \lim U_n=0$

By Leibnitz test

Hence $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{1}{n}$ is convergent

Hence $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(2n)}$ is convergent.

Absolutely Convergence and conditionally convergence.

(1) Prove that $\sum (-1)^{n-1} \frac{x^n}{n}$ is convergent for $-1 < x \leq 1$

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Sol: Let $x = 1$

Given $\sum (-1)^{n-1} \cdot \frac{x^n}{n} = \sum (-1)^{n-1} \cdot \frac{(1)^n}{n}$ $\left[\begin{matrix} -1<2 \\ x=1 \end{matrix} \right]$

$= \sum (-1)^{n-1} \cdot \frac{1}{n}$

Let $u_n = \frac{1}{n}$

$u_{n+1} = \frac{1}{n+1}$

Consider $u_n - u_{n+1} = \frac{1}{n} - \frac{1}{n+1}$

$= \frac{1}{n(n+1)} > 0$

$u_n - u_{n+1} > 0$

$u_n > u_{n+1}$

$\{S_n\}$ is decreasing

Consider $\lim u_n = \lim \frac{1}{n} = 0$

By Leibnitz test,

Hence $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{(2n-1)(2n)}$ is convergent.

Absolutely Convergence and conditionally convergence.

① Prove that $\sum (-1)^{n-1} \frac{x^n}{n}$ is convergent for $-1 < x \leq 1$

2020 I.

Sol: Let $x = 1$

Given $\sum (-1)^{n-1} \cdot \frac{x^n}{n} = \sum (-1)^{n-1} \cdot \frac{(1)^n}{n}$ $\left[\begin{array}{l} -1<2 \\ x=1 \end{array} \right]$

$= \sum (-1)^{n-1} \cdot \frac{1}{n}$

Let $u_n = \frac{1}{n}$

$u_{n+1} = \frac{1}{n+1}$

Consider $u_n - u_{n+1} = \frac{1}{n} - \frac{1}{n+1}$

$= \frac{1}{n(n+1)} > 0$

$u_n - u_{n+1} > 0$

$u_n > u_{n+1}$

$\{S_n\}$ is decreasing

Consider $\lim u_n = \lim \frac{1}{n} = 0$

By Leibnitz test,

$\Sigma(-1)^{n-1}\frac{x^n}{n}$ is convergent where $x=1$

Let $-1<x<1 \Rightarrow |x|<1$

Again Given $\Sigma(-1)^{n-1}\frac{x^n}{2}$

$$= \Sigma \frac{|x|^n}{n}$$

$$U_n = \frac{|x|^n}{n}$$

$$(U_n)^{1/n} = \frac{(|x|^n)^{1/n}}{x^{1/n}}$$

$$\lim (U_n)^{1/n} = \lim \frac{|x|}{n^{1/n}}$$

$$= |x|<1$$

By Cauchy's n^{th} root test

ΣU_n is convergent.

$\Rightarrow -1<x<1$ $\Sigma(-1)^{n-1}\frac{x^n}{n}$ is convergent

X Examine the convergence of $\sum_{n=1}^{\infty}(-1)^{n-1}(\sqrt{n+1}-\sqrt{n})$

Sol

$$U_n = \sqrt{n+1}-\sqrt{n} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}$$

Ex. Test the convergence of $\sum_{n=1}^{\infty} \sqrt{n^3+1} - \sqrt{n^3}$

Sol Let $\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \sqrt{n^3+1} - \sqrt{n^3}$

i.e $U_n = \sqrt{n^3+1} - \sqrt{n^3} \times \left(\frac{\sqrt{n^3+1} + \sqrt{n^3}}{\sqrt{n^3+1} + \sqrt{n^3}}\right)$

$$= \left(\frac{\cancel{n^3}+1-\cancel{n^3}}{\sqrt{n^3+1}+\sqrt{n^3}}\right)$$

$$= \frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}$$

$$= \frac{1}{n^{3/2}\left(\sqrt{1+\frac{1}{n^3}}+1\right)}$$

for $n \geq 1$, $n^{3/2}\left(\sqrt{1+\frac{1}{n^3}}+1\right) > n^{3/2}$

$$\frac{1}{n^{3/2}\left(\sqrt{1+\frac{1}{n^3}}+1\right)} < \frac{1}{n^{3/2}}$$

if $V_n = \frac{1}{n^{3/2}}$ then $\sum V_n = \sum \frac{1}{n^{3/2}}$ is convergent

Ex. $\sum \frac{(n)^{n^2}}{(n+1)^{n^2}}$

Sol $\sum U_n = \frac{\sum (n)^{n^2}}{(n+1)^{n^2}}$

i.e $U_n = \frac{(n)^{n^2}}{(n+1)^{n^2}}$

$$(U_n)^{1/n} = \frac{(n^{n^2})^{1/n}}{((n+1)^{n^2})^{1/n}} = \frac{n^n}{(n+1)^n}$$

$$= \frac{n^n}{n^n\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim (U_n)^{1/n} = \lim \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{e} < 1$$

$\sum U_n$ is convergent

$\sum \frac{n^{n^2}}{(n+1)^{n^2}}$ is convergent

Limits And Continuity

Real valued functions:-

A function $f: S \to R$ ($S \subseteq R$) is called Real valued function.

Constant functions: A function $f: R \to R$ is defined by $f(x) = k \in R$ $\forall x \in R$ is called constant functions.

Identity function:- A function $f: R \to R$ is defined by $f(x) = x$, $\forall x \in R$ is called Identity function.

Trigonometric function:-

A function $f: R \to R$ is defined by $f(x) = \sin x, \cos x, \tan x$ is called Trigonometric function.

Inverse trigonometric function:-

A function $f: R \to R$ is defined by $f(x) = \sin^{-1}x$, $\cos^{-1}x, \tan^{-1}x$ is called Inverse trigonometric function.

Power function:- A function $f: R \to R$ defined by $f(x) = x^n \in R$ $\forall x \in R$ $n \in R$ is called power function

Exponential function:- A function $f: R \to R$ is defined by $f(x) = e^x$, $\forall x \in R$ is called exponential function.

UNIT - III

Limits And continuity

1

Real valued functions :-

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Exponential function:- A function $f: R \to R$ is defined by $f(x) = e^x$, $\forall x \in R$ is called exponential function.

and limit exists

(ii) If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ Then $\lim_{x \to a} f(x)$ does not exists

NOTE :-

(i) $x \to a^- \Rightarrow x < a$.

(ii) $x \to a^+ \Rightarrow x > a$

(1) Prove That $\lim_{x \to 0} \frac{3x+|x|}{7x-5|x|}$ is doesnot exists.

Sol: Left Hand limit

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{3x+|x|}{7x-5|x|(-x)}$$

$$= \frac{3x-x}{7x+5(x)} = \frac{2x}{12x} = \frac{1}{6}$$

$$\lim_{x \to 0^-} f(x) = \frac{1}{6}$$

Right hand limit

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{3x+|x|}{7x-5|x|}$$

$$\frac{3x+x}{7x-5x} = \frac{4x}{2x} = 2$$

$$\therefore \lim_{x \to 0^+} f(x) = 2$$

(2) $\lim_{x \to 0} \frac{|x|}{x}$ does not Exists

Sol Left hand limit

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x} = \frac{-x}{x} = -1$$

$$\therefore \lim_{x \to 0^-} = -1$$

R.H.L

$$\lim f(x) = \lim_{x \to 0^+} \frac{|x|}{x}$$

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Sol

$f(x) = \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$ Examine $\lim_{x \to 0} f(x)$ 12

L.H.L

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} \right) \times \dfrac{e^{1/x}}{e^{1/x}}$

$= \dfrac{\lim_{x \to 0^-} e^{2/x} - e^{0}}{\lim_{x \to 0^-} e^{2/x} + e^{0}}$

$= \dfrac{0 - 1}{0 + 1} = -1$

R.H.L

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \dfrac{\left(e^{1/x} - e^{-1/x} \right)}{e^{1/x} + e^{-1/x}} \times \dfrac{e^{-1/x}}{e^{-1/x}}$

$\dfrac{\lim_{x \to 0^+} e^{0} - e^{-2/x}}{\lim_{x \to 0^+} e^{-2/x} + e^{0}}$

$\dfrac{1 - 0}{0 + 1} = 1.$

L.H.L $\neq$ R.H.L

$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

$\lim_{x \to 0} f(x)$ is dues not Exist.

(5) $f(x) = \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$ Examine $\lim_{x \to 0} f(x)$ 12

Sol

L.H.L

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} \right) \times \frac{e^{1/x}}{e^{1/x}}$$

$$= \frac{\lim_{x \to 0^-} e^{2/x} - e^{0}}{\lim_{x \to 0^-} e^{2/x} + e^{0}}$$

$$= \frac{0 - 1}{0 + 1} = -1$$

R.H.L

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\left(e^{1/x} - e^{-1/x}\right)}{e^{1/x} + e^{-1/x}} \times \frac{e^{-1/x}}{e^{-1/x}}$$

$$\frac{\lim_{x \to 0^+} e^{0} - e^{-2/x}}{\lim_{x \to 0^+} e^{-2/x} + e^{0}}$$

$$\frac{1 - 0}{0 + 1} = 1.$$

L.H.L $\neq$ R.H.L

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

$\lim_{x \to 0} f(x)$ is does not Exist

(5) Types of discontinuity :-

Imp

(1) Removable discontinuity

(2) Jump dc (3) Simple discontinuity.

(1) Removable discontinuity :-

If $\lim_{x \to a} f(x)$ Exists and $\lim_{x \to a} f(x) \neq f(a)$ (or)

$f(a)$ is not defined Then we say that f has removable discontinuity at a.

(2) Jump Discontinuity :-

If $\lim_{x \to a^-} f(x) = f(a)$ and $\lim_{x \to a^+} f(x) = f(a)$

both Exists are not Equal Then we say that f has Jump discontinuity at a.

(3) Simple discontinuity :-

A removable discontinuity (or) Jump discontinuity of a function is called simple discontinuity.

NOTE :-

* Let f is continuous on R If f is continuous at each point of R.

$|f|$ is also continuous at $a \in S$.

***

Imp Let $f: R \to R$ be such that

(1) $f(x) = \frac{\sin(a+1)x + \sin x}{x}$ for $x<0$,

10 M $f(x) = c$ for $x=0$ and $f(x) = \frac{(x+b)x}{bx}$ for $x>0$

Determine the value of a, b, c for which the function is continuous at $x=0$.

Sol Given that

$$f(x) = \begin{cases} \frac{\sin(a+1) + \sin x}{x} & \text{if } x<0 \\ \frac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & \text{if } x>0 \\ c & \text{if } x=0 \end{cases}$$

Now $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}$ $\left[x \to 0^+ \Rightarrow x>0\right]$

$$= \lim_{x\to 0^+} \frac{x^{1/2}\left[1+bx\right]^{1/2} - 1}{x^{1/2}\, bx}$$ $\left[x \cdot x^{1/2} = x^{1+1/2} = x^{3/2}\right]$

$$= \lim_{x\to 0^+} \frac{(1+bx)^{1/2} - 1}{bx}$$

* If $f: S \to \mathbb{R}$ is continuous

$|f|$ is also continuous at $a \in S$.

***

Imp ① Let $f: \mathbb{R} \to \mathbb{R}$ be such that $f(x) = \frac{\sin(a+1)x + \sin x}{x}$ for $x<0$,

10 M $f(x) = c$ for $x=0$ and $f(x) = \frac{(x+b)x}{bx}$ for $x>0$ which The

Determine The value of a, b, c for which the function is continuous at $x=0$.

Sol Given That

$$f(x) = \begin{cases} \frac{\sin(a+1)x + \sin x}{x} & \text{if } x<0 \\ \frac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & \text{if } x>0 \\ c & \text{if } x=0 \end{cases}$$

Now $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}$ $[x \to 0^+ \Rightarrow x>0]$

$$= \lim_{x\to 0^+} \frac{x^{1/2}\left[(1+bx)^{1/2} - 1\right]}{x^{1/2}\, bx}$$ $[x \cdot x^{1/2} = x^{1+1/2} = x^{3/2}]$

$$= \lim_{x\to 0^+} \frac{(1+bx)^{1/2} - 1}{bx}$$

Rationalizing) $= \lim_{x\to 0^+} \frac{(1+bx)^{1/2}-1}{bx} \times \frac{(1+bx)^{1/2}+1}{(1+bx)^{1/2}+1}$

$= \lim_{x\to 0^+} \frac{\left[(1+bx)^{1/2\cdot 2} - 1^2\right]}{bx\left[(1+bx)^{1/2}+1\right]}$ $\left[\because (a+b)(a-b) = a^2-b^2\right]$

$= \lim_{x\to 0^+} \frac{1+bx-1}{bx\left[(1+bx)^{1/2}+1\right]}$

$= \lim_{x\to 0^+} \frac{bx}{bx\left[(1+bx)^{1/2}+1\right]}$

$= \lim_{x\to 0^+} \frac{1}{(1+bx)^{1/2}+1}$ $\left[bx = 0 \quad b\neq 0, x=0\right]$

$= \frac{1}{(1+b\cdot 0)^{1/2}+1}$

$= \frac{1}{(1+0)^{1/2}+1}$

$= \frac{1}{(1)^{1/2}+1}$

$= \frac{1}{1+1} = \frac{1}{2}$

$\lim_{x\to 0^+} f(x) = 1/2$

Since $b\cdot x = 0$ and $x \to 0^+ \Rightarrow x$

$\boxed{\therefore b \neq 0}$

Ans

Now $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{\sin(a+1)x + \sin x}{x}$ $\left[\because x\to 0^- \quad x<0\right]$

$\left[\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right).\cos\left(\frac{C-D}{2}\right)\right]$

$$= \lim_{x\to 0^-} \frac{2\sin\left(\frac{ax+x+x}{2}\right)\cos\left(\frac{ax+x-x}{2}\right)}{x}$$

$$= \lim_{x\to 0^-} \frac{2\sin\frac{(a+2)x}{2}\cdot\cos\frac{ax}{2}}{x}$$

$$2\lim_{x\to 0^-} \frac{\sin\frac{(a+2)x}{2}}{x}\ (1)$$

$$2.\lim_{x\to 0^-} \frac{\sin\frac{(a+2)x}{2}}{x\left(\frac{a+2}{2}\right)}\left(\frac{a+2}{2}\right) \quad \left[\because \frac{\sin\theta}{\theta} = 1\right]$$

$$= 2\lim_{x\to 0^-} (1)\ \frac{a+2}{2}$$

$$= \cancel{2}.\frac{a+2}{\cancel{2}}$$

$$= a+2$$

$$\lim_{x\to 0^-} f(x) = a+2$$

and we have $f(0) = c$

Since f is continuous at $x=0$

we have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$$

$$\frac{1}{2} = a+2 = c$$

$$c = \frac{1}{2} = a+2$$

$\boxed{c=1/2}$ and $a+2 = 1/2$

Now, $a+2=\frac{1}{2} \Rightarrow a = \frac{1}{2}-2$

$$\Rightarrow a = \frac{1-4}{2} = \frac{-3}{2}$$

$$\boxed{\therefore a = \frac{-3}{2}}$$

$\therefore$ Hence $a = -3/2$; $b \neq 0$, $c = \frac{1}{2}$

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(2). Examine the continuity the function f defined by $f(x) = |x|+|x-1|$ at $x=0,1$

Sol Given That $f(x) = |x|+|x-1|$, $x=0$

$[|x| = x$ if $x<0]$

(i) $x \leq 0$, $|x| = -x$

and $|x-1| = -(x-1) = 1-x$

Now $f(x) = -x+1-x = 1-2x$

(ii) $x > 0$, $|x| = x$ $\quad [|x| = x$ if $x>0$

and $|x-1| = -(x-1)$ $\quad \Rightarrow x-1<0$

$1-x$ $\quad |x-1| = -(x-1)]$

Now $f(x) = x+1 - x = 1$

(iii) $x < 1$, $|x| = -x$

and $|x-1| = -(x-1)$ $\quad$ $\left[x < 1 \right.$

$x - 1 < 0$

$= 1 - x$ $\quad$ $\left. \Rightarrow |x-1| = -(x-1) \right]$

Now $f(x) = -x+1-x$

$= 1 - 2x$

(iv) $x > 1$, $|x| = x$ $\quad$ $\left[x > 1 \right.$

$x - 1 > 0$

and $|x-1| = x-1$ $\quad$ $\left. |x-1| = (x-1) \right]$

Now $f(x) = x + x - 1$

$= 2x - 1$

(a) at $x = 0$

Consider $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1-2x)$

$= 1 - 0$ $\quad$ $\left[x \to 0^- \right.$

$x < 0$

$= 1$ $\quad$ $\left. \Rightarrow f(x) = 1 - 2x \right]$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1)$

$= 1$

and $f(x) = 1 - 2x$

$f(0) = 1$.

Hence f is continuous at $x = 0$

(b) at $x = 1$.

Consider

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 1 - 2x$ $\quad$ $\left[x \to 1^- \right.$

$x < 1$

$\left. f(x) = 1 \right]$

$= 1-2(0)$

$= 1$

And $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x-1$ $\left[x \to 1^+ \Rightarrow x > 1 \quad f(x) = 2x \right]$

$= 2(1)-1$

$= 1$

And $f(x) = 2x-1 \Rightarrow f(x) = 2(0)-1$

$= 1$

Hence f is continuous at $x=1$

3) Examine the function for continuity at "0"

$f(x) = \frac{1}{x-a} \operatorname{cosec}(x-a)$ if $x \neq a$ and $f(x) = 0$ if $x = a$.

Sol we know that

$-1 < \sin x < 1$

$-1 < \sin(x-a) < 1$

$|\sin(x-a)| \leq 1$

and $\lim_{x \to a} (x-a) = 0$

Consider

$\lim_{x \to a} (x-a) \sin(x-a) = 0 \sin(x-a)$

$\Rightarrow \lim_{x \to a} (x-a) \sin(x-a) = 0$

$= \lim_{x \to a} \frac{1}{(x-a)\sin(x-a)} = \frac{1}{0}$

$= \lim_{x \to a} \frac{1}{x-a} \frac{1}{\sin(x-a)} = \frac{1}{0}.$

$\Rightarrow \lim_{x \to a} \frac{1}{x-a} \cdot \operatorname{cosec}(x-a) = \infty \quad \left[\frac{1}{\sin x} = \operatorname{cosec} x\right]$

$\lim_{x \to a} f(x) = \infty$ and $f(a) = 0$

$\lim_{x \to a} f(x)$ Exists and $\lim_{x \to a} f(x) \neq 0$

$\therefore f$ has Removeble discontinuity at $x = a$.

Imp

Q) Find the points discontinuity of $f(x) = \frac{1}{2^n}$,
for $\frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}$ where $n = 0, 1, 2 \ldots$ and $f(0) = 0$

Sol: Given $f(x) = \frac{1}{2^n}, \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}$

If $n = n-1$

$$f(x) = \frac{1}{2^{n-1}}, \frac{1}{2^n} < x \leq \frac{1}{2^{n-1}}$$

If $n = n+1$

$$f(x) = \frac{1}{2^{n+1}}, \frac{1}{2^{n+2}} < x \leq \frac{1}{2^{n+1}}$$

$\left[x \to \left(\frac{1}{2^n}\right)^- \quad x < \frac{1}{2^n} \right]$

Case (i):

(i) $\lim_{x \to \left(\frac{1}{2^n}\right)^-} f(x) = \lim_{x \to \left(\frac{1}{2^n}\right) - \left(\frac{1}{2^n}\right)} \left(\frac{1}{2^n}\right)$

$$= \frac{1}{2^n}$$

(ii) $\lim_{x \to \left(\frac{1}{2^n}\right)^+} f(x) = \lim_{x \to \left(\frac{1}{2^n}\right) +} \left(\frac{1}{2^{n-1}}\right)$

$$= \frac{1}{2^{n-1}} \qquad \left[x \to \left(\frac{1}{2^n}\right)^+ \quad x > \frac{1}{2^n} \right]$$

$$\therefore \lim_{x \to \left(\frac{1}{2^n}\right)^-} f(x) \neq \lim_{x \to \left(\frac{1}{2^n}\right)^+} f(x)$$

$\therefore$ Hence f has jump discontinuity at $x = \frac{1}{2^n}$

(b)

Case (ii):

(i) $$\lim_{x \to \left(\frac{1}{2^{n+1}}\right)^-} f(x) = \lim_{x \to \left(\frac{1}{2^{n+1}}\right)^-} \left(\frac{1}{2^{n+1}}\right)$$

$$= \frac{1}{2^{n+1}} \qquad \left[x \to \frac{1}{2^{n+1}} \quad x < \frac{1}{2^{n+1}} \right]$$

(ii) $$\lim_{x \to \left(\frac{1}{2^{n+1}}\right)^+} f(x) = \lim_{x \to \left(\frac{1}{2^{n+1}}\right)^+} \left(\frac{1}{2^{n-1}}\right)$$

$$= \frac{1}{2^{n-1}} \qquad \left[x \to \left(\frac{1}{2^{n+1}}\right)^+ \quad x > \left(\frac{1}{2^{n+1}}\right) \right]$$

$$\lim_{x \to \left(\frac{1}{2^{n+1}}\right)^-} f(x) \neq \lim_{x \to \left(\frac{1}{2^{n+1}}\right)^+} f(x)$$

Hence f has jump discontinuity at $x = \frac{1}{2^{n+1}}$

$< n+1+f$

more then $(n-1)$ points of the partition b/w a and x]

$< n$

$-n < f(x) - f(s) < n$

$f(s) - n < f(x) < n + f(s)$

$\therefore$ Therefore f is bounded

Imp 10M 2020 ~~Absolute~~ maximum – minimum Theorem

If $f: I = [a\ b] \to R$ is continuous on $[a\ b]$. f attains its bounds supremum & infimum.

proof: Using borel's Theorem:

Let $\varepsilon = 1 > 0$,

Since f is continuous on $[a\ b]$ then $\exists$

$\{a = t_0, t_1, t_2 \ldots t_{n-1}, t_n = b\}$ such that

$|f(x_1) - f(x_2)| < \varepsilon \ \forall \ x_1, x_2 \in [t_{r-1}, t_r]$

Let $x \in [a\ b]$, $t_1 t_2 \ldots t_r$ are in between a and x

consider $|f(x) - f(s)|$

$= f(x) - f(t_r) + f(t_r) - f(t_{r-1}) + \ldots f(t_1) - f(s)]$

$< 1 + 1 + \ldots (r+1 \text{ terms})$

$< r+1$

$|f(x) - f(a)| < \gamma + 1$

$< n - 1 + 1$

$< n$

$\Rightarrow -n < f(x) - f(s) < n$

$\Rightarrow f(s) - n < f(x) < f(s) + n$

Therefore f is bounded

Main proof:-

(i) Let $M = \{\sup f(x) \mid x \in [a\ b]\}$ and

$m = \{\inf f(x) \mid x \in [a\ b]\}$

then $m \leq f(x) \leq M \; \forall x \in [a\ b]$

Now we prove That f attains supremum and infimum.

Show That $c, d \in [a\ b]$ then $f(c) = M$ & $f(d) = m$

(i) Suppose that $f(x) < M \; \forall x \in [a\ b]$

$\Rightarrow M - f(x) > 0 \; \forall x \in [a\ b]$

where M is constant and

f is continuous on $[a\ b]$

$M - f(x)$ is continuous on $[a\ b]$

$\frac{1}{M - f(x)}$ is continuous on $[a\ b]$

by above Theorem

$\frac{1}{M - f(x)}$ is bounded on $[a\ b]$

Let $K > 0$, $\frac{1}{M - f(x)} \leq K \; \forall x \in [a\ b]$

$M - f(x) \geq \frac{1}{k} \; \forall x \in [a\, b]$

$f(x) \leq M - \frac{1}{k} \; \forall \; x \in [a\, b]$

$f(x) \leq M$

This is a contradiction. //

Hence $c \in [a\, b]$ Then $f(x) = M = \{ \sup (f(x) \mid x \in [a\, b] \}$

Suppose That $f(x) > m, \; \forall x \in [a\, b]$

$\Rightarrow f(x) - m > 0 \; \forall x \in [a\, b]$

where m is constant and

f is continuous on $[a\, b]$

$f(x) - m$ is continuous on $[a\, b]$

$\frac{1}{f(x) - m}$ is continuous on $[a\, b]$

By above Theorem

$\frac{1}{f(x) - m}$ is bounded on $[a\, b]$

Let $l > 0$, $\frac{1}{f(x) - m} \leq l \; \forall x \in [a\, b]$

$f(x) - m \geq \frac{1}{l}$

$f(x) \geq m + \frac{1}{l}$

$> m$

$f(x) > m$

This is contradiction

Hence $d \in [a\, b]$ then $f(d) = m$

$= \inf \{ f(x) \mid x \in [a\, b] \}$

Differentiation

Derivability of a function at a point:

* Let S be an aggregate [non-empty set] and $f: S \to R$ be a function let be a limit point of S.

1. If $\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$ where $x \neq 0$ exist then we say that f is derivable from left at 'c'. The limit is called the left derivable of f at c and it is denoted by $f'(x-c)$ or $[f'(c)]$ i.e,

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = f'(c)$$

2. If $\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$ where $x \neq c$ then we say that f is right derivable at c. The limit is called right derivable of f at c and it is denoted by $f'(x+c)$ or $Rf'(c)$. (Right derivable of f at c)

i.e $$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = f'(c)$$

3. If $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ where $x \neq c$ then we say that f is derivable at c. The limit point is called the derivable of f at c and it is denoted by $f'(c)$ i.e $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$

$$= \frac{x-a-(c-a)}{x-c}$$

$$= \frac{x-\cancel{a}-c+\cancel{a}}{x-c} = \frac{x-c}{x-c} = 1$$

$\therefore$ is not divisible at R.

3. Show that $f(x) = x \sin(1/x)$, $x \neq 0$ and $f(0)=0$ is continuous but not derivable at $x=0$

(Im) 2012 (S)

Sol: given that $f(x) = x\sin(1/x)$

(i) $\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin(1/x)$

$= 0 \cdot \sin(1/0) = 0$

and $f(0)=0$

$\therefore \lim_{x\to 0} f(x) = f(0)$

Hence f is continuous at '0'

(i) $\lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \left(\dfrac{\cancel{x} \sin(1/x) - 0}{x - 0} \right)$

$= \lim_{x \to 0} \left(\dfrac{\cancel{x} \sin(1/x)}{\cancel{x}} \right)$

$= \lim_{x \to 0} \sin(1/x)$

does not exist

$\therefore f$ is not derivable at '0'

Discuss the derivability of $f(x) = |x| + |x-1|$ on R where $x = 0, 1$

(i) $x \le 0$ $\quad |x| = -x$

$|x-1| = -(x-1)$

$= 1-x$

$f(x) = -x + 1 - x$

$= -2x + 1$

(ii) $0 < x < 1$, $|x| = x$

$|x-1| = -(x-1)$

$= 1-x$

$f(x) = \cancel{x} + 1 - \cancel{x} = 1$

$f'(x) = -\cos(1/x)\, x^{m-2} + m x^{m-1} \sin(1/x)$

$f'(0) = -\cos(1/0)\, 0^{m-2} + m \cdot 0^{m-1} \sin(1/0)$

$= 0$

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0)$$

Hence f is derivable where $m > 1$ or $m \geq 2$

* (5) Show that $f(x) = x^2 \cos(1/x), x \neq 0$, $f(x) = 0, x = 0$ if f is derivable everywhere but the derivable function is not continuous at $x = 0$.

Sol: Given that $f(x) = x^2 \cos(1/x), x \neq 0$

$f(x) = 0, x = 0$

(i) $$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \cos(1/x) - 0}{x - 0}$$

$$\lim_{x \to 0} x \cos(1/x)$$

$= 0$

$f(x) = x^2 \sin(1/x)(-1/x^2) + \cos(1/x)(2x)$

$= -\sin(1/x) + 2x\cos(1/x)$

$f'(0) = -\sin(1/0) + 2.\cos(1/0)$

$= -\sin(\infty) + 2\cos(\infty)$

$= 0$

Hence f is derivable at $x=0$

$f'(x) = -\sin(1/x) + 2x\cos(1/x)$

$\lim_{x\to 0} f'(x) = \lim_{x\to 0} -\sin(1/x) + 2x\cos(1/x)$

$f'(0) = -\sin(1/0) + 2(0)\cos(1/0)$

does not exist.

$\therefore f$ is not continuous at $x=0$

Hence prove that $f(x) = x^2\cos(1/x)$ is derivable and the derivable function is not continuous at $x=0$

* Show that $f(x) = x|x|$ $\forall x \in R$ is defferentiable at $x=0$ and the derived function $f(x)$ is not differentiable at $x=0$.

Sol Given that $f(x) - x|x|$ $\forall x \in R$

If $x<0 \Rightarrow |x| = -x$

* Show that $\log(1+x) - \frac{2x}{2+x}$ is increasing when $x > 0$

Sol

Given that

$$f(x) = \log(1+x) - \frac{2x}{2+x}$$

$$f'(x) = \frac{1}{1+x} - \frac{(2+x)(2) - (2x)(1)}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{4+\cancel{2x}-\cancel{2x}}{(2+x)^2}$$

$$= \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2}$$

$$= \frac{\cancel{4}+\cancel{4x}+x^2-\cancel{4}-\cancel{4x}}{(1+x)(2+x)^2}$$

$$= \frac{x^2}{(1+x)(2+x)^2} > 0$$

$f'(x) \geq 0$

$\therefore f$ is incoruing at $x > 0$

* Prove that $\tan x > x > \sin x \ \forall x \in (0, \pi/2)$ is incorueasing.

$f(x) = \tan x > x$ $\qquad$ $g(x) = x > \sin x$

$f(x) = \tan x - x > 0$ $\qquad$ $g(x) = x - \sin x > 0$

$f(x) > 0$ $\qquad$ $g(x) > 0$

Show that $f(x) = \frac{x}{\sin x}$ is increasing in $(0, \frac{\pi}{2})$

Sol

$$f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$$

$$f'(x) = \frac{\cancel{\sin x}(1 - x\cot x)}{\sin^{\cancel{2}} x}$$

$$f'(x) = \frac{1 - x\cot x}{\sin x} > 0$$

Hence f is increasing

Show that $f(x) = (x-1)e^x + 1 > 0 \ \forall x > 0$ [$e^x > 0$]

Sol

$$f'(x) = xe^x - e^x + 1$$

$$f'(x) = e^x + 1 + xe^x - e^x$$

$$f'(x) = xe^x > 0$$

Hence f is increasing

Imp Rolle's Theorem:

If a function $f : I : [a\ b] \to R$ is such that

10M (i) f is continuous on $[a\ b]$

(ii) f is derivable on $[a\ b]$ and

(iii) $f(a) = f(b)$ There exist $c \in (a\ b)$ such that $f'(c) = 0$

Ans we know that

f is continuous on $[a\ b]$

$\to f$ is bounded on $[a\ b]$ and attins Inf... and sup

Let $\alpha, \beta \in [a\ b] \Rightarrow f(\alpha) = M = \sup f$ and

$f(\beta) = m = \inf$

<u>Case (i)</u>: $M = m$

let $x \in [a\ b] \ni f(x) = m$

$f'(x) = 0$

$\exists\ c \in [a\ b]\ f'(c) = 0$

<u>Case (ii)</u> $M \neq m$

consider $f(a) = f(b)$ and $M \neq m$, we have

$M \neq f(a)$ then $M \neq f(b)$ and

$m \neq f(a)$ then $m \neq f(b)$

Let $M \neq f(m)$ then $M \neq f(b)$

$f(\alpha) = M \neq f(a)$ $\qquad$ $f(\alpha) = M \neq f(b)$

$f(\alpha) \neq f(a)$ $\qquad$ $f(\alpha) \neq f(b)$

$\alpha \neq a$ $\qquad$ $\alpha \neq b$

$\alpha \in (a\ b) \Rightarrow a < \alpha < b$

Since f is derivable on (a, b) and $\alpha \in (a\ b)$

$\Rightarrow f$ is derivable on α

Now we P.T $f'(\alpha) = 0$

if possible $f'(\alpha) < 0$

$\exists\ \delta \in (a\ b) \ni f(x) > f(\alpha)\ \forall\ x \in (-\delta, \alpha)$

$f(x) > f(\alpha) = M$

$f(x) > M$

f is contradiction

$f'(\alpha) = 0$

Similarly $f'(\alpha) > 0$

$\boxed{\therefore f'(\alpha) = 0}$

Verify Rolles Theorem in The [a b] for the function $f(x) = x(x-a)^{m}(x-b)^{n}$. m, n +ve integers.

Sol Clearly f is continuous on [a b]

f is derivable on (a b)

$f(a) = f(b)$

So, f satisfies Rolles Theorem

$\exists\, c \in (a,b) \to f'(c) = 0$

$\to f(x) = (x-a)^{m} \cdot (x-b)^{n}$

$f'(x) = (x-a)^{m}\left[n(x-b)^{n-1}\right] + (x-b)^{n}\left[m(x-a)^{m-1}\right]$

$= n(x-a)^{m}(x-b)^{n-1} + m(x-b)^{n}(x-a)^{m-1}$

$= n(x-a)^{m-1}(x-a)(x-b)^{n-1} + m(x-b)^{n-1}(x-b)(x-a)^{m-1}$

$= (x-a)^{m-1} \cdot (x-b)^{n-1}\left[n(x-a) + m(x-b)\right]$

$= (x-a)^{m-1}(x-b)^{n-1}\left[nx - na + mx - mb\right]$

$= (x-a)^{m-1}(x-b)^{n-1}\left[x(m+n) - (mb+na)\right]$

$f'(x) = (x-a)^{m-1}(x-b)^{n-1}(m+n)\left[x - \left(\frac{mb+na}{m+n}\right)\right]$

take $c = \frac{mb+na}{m+n} \Rightarrow c \in (a,b)$

$f'(x) = a(x-a)^{m-1}(x-b)^{n-1}(m+n)[x-c]$

$f'(c) = (c-a)^{m-1}(c-b)^{n-1}(m+a)[0]$

$f'(c) = 0$

2020 Imp 10m

Lagrange's Mean value Theorem or first mean Value Theorem

Theorem: If $f:[a\,b] \to R$ is a function such that i) f is continuous on $[a\,b]$ and

ii) f is derivable on $(a\,b)$ then

$\exists$ a Point $c \in (a\,b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$

or $f(b) - f(a) = f'(c)(b-a)$

Proof: Let $\phi : [a\,b] \to R$ is define

$\phi(x) = f(x) + k \cdot x$ ...

Consider $\phi(a) = \phi(b)$

$f(a) + k \cdot a = f(b) + k \cdot b$

$ka - kb = f(b) - f(a)$

$-k(b-a) = f(b) - f(a)$

$k = -\left[\frac{f(b)-f(a)}{b-a}\right]$

$x \in R$, x is continuous on $[a\,b] \to kx$ is continuous on $[a\,b]$ (consider)

Since

f is continuous on $[a\ b]$, kx is continuous on $[a\ b]$

$\Rightarrow \phi(x) = f(x) + kx$ is continuous on $[a\ b]$

f is derivable on $(a\ b)$, kx is derivable on $(a\ b)$ consider

$\phi(a) = \phi(b)$

By Rolles theorem

$\exists\ c \in (a\ b) \ni \phi'(c) = 0$

we have $\phi(x) = f(x) + kx$

$\phi'(x) = f'(x) + k$

$\phi'(c) = f'(c) + k$

$0 = f'(c) + k$

$k = -f'(c)$

$\rightarrow \cancel{\dfrac{[f(b) - f(a)]}{b-a} = f'(c)}$

$f'(c) = \dfrac{[f(b) - f(a)]}{b-a}$

* Discuss applicability of Lagranges mean value theorem for $f(x) = x(x-1)(x-2)$ on $[0\ \frac{1}{2}]$

Sol Given that $f(x) = x(x-1)(x-2)$

clearly f is continuous on $[0\ \frac{1}{2}]$

f is derivable on $(0\ \frac{1}{2})$

By Lagranges mean value Theorem

$\exists\, c \in (0, 1/2) \Rightarrow \dfrac{f(b) - f(a)}{b-a} = f'(c)$

$$\frac{f(1/2) - f(0)}{\frac{1}{2} - 0} = f'(c)$$

$$= \frac{\cancel{\frac{1}{2}}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right) - 0}{\cancel{1/2}} = f'(c)$$

$$= \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) = 3c^2 - 6c + 2$$

$$\frac{3}{4} = 3c^2 - 6c + 2$$

$$3c^2 - 6c + 2 - \frac{3}{4} = 0$$

$$3c^2 - 6c + \frac{5}{4} = 0$$

$$c = \frac{6 \pm \sqrt{36 - \cancel{4}(3)\left(\frac{5}{\cancel{4}}\right)}}{6}$$

$$c = \frac{6 \pm \sqrt{24}}{6}$$

Problem: Using Lagranges mean Value Theorem,

P.T $10.2 < \sqrt{105} < 10.25$

Sol Let $f : [a, b] \to R$ defined by $f(x) = \sqrt{x}$ $\forall x \in [100$

$$f'(x) = 1/2\sqrt{x}$$

Clearly

f is continuous on

I. Cauchy's mean value theorem:

Imp. If $f:[a\ b]\to R$ and $g:[a\ b]\to R$ such that (i) f, g are continuous on $[a\ b]$

(ii) f, g are derivable on $(a\ b)$

(iii) $g'(x)\neq 0\ \forall x\in(a\ b)$ $\exists\ c\in(a\ b)\ \ni \dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$

Sol. Hypothesis: -

$f:[a\ b]\to R$

$g:[a\ b]\to R$

(i) f, g are continuous on $[a\ b]$

(ii) f, g are derivable on $(a\ b)$

(iii) $g'(x)\neq 0\ \forall\ x\in(a\ b)$

$$\exists\ c\in(a\ b) \Rightarrow \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Let $\phi:[a\ b]\to R$ is defined by

$\phi(x)=f(x)+k\,g(x)$ and $\phi(a)=\phi(b)$

Consider $\phi(a) = \phi(b)$

$$f(a)+k\cdot g(a) = f(b)+k\cdot g(b)$$

$$k[g(a)-g(b)] = f(b)-f(a)$$

$$-k[g(b)-g(b)] = f(b)-f(a)$$

Now prove that $g(b)-g(a)\neq 0$

Suppose That $g(b) = g(a)$

g is continuous on $[a\ b]$

g is derivable on $(a\ b)$

then g Satisfies Rolle's Theorem

$\exists\ c \in (a\ b) \Rightarrow g'(c) = 0$

it is contradiction

By (iii) $g'(c) \neq 0$

Hence $g(b) \cdot g(a) \neq 0$

Hence $k = \dfrac{[f(b) - f(a)]}{g(b) - g(a)}$ —— ⊕

and f is continuous on $[a\ b]$ and g is continuous on $[a\ b]$ hence $\phi(x) = f(x) + kg(x)$ is continuous on $[a\ b]$

f is derivable on $(a\ b)$ and g is derivable on $(a\ b)$ and we know That

$\phi(a) = \phi(b)$

Hence ϕ Satisfies Rolle's Theorem

Hence $\exists\ c \in (a\ b) \mid \phi'(c) = 0$

$\phi'(x) = f'(x) + kg'(x)$

$\phi'(c) = f'(c) + kg'(c)$

$\phi'(c) = 0$

$$0 = f'(c) + kg'(c)$$

$$k = \frac{-f'(c)}{g'(c)}$$

from (*) $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$

Problem: Find 'c' of Cauchy mean value theorem

$f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x}$ $\forall x \in (a, b)$ $a, b > 0$

Sol. $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x}$

$f'(x) = \frac{-2}{x^3}$ and $g'(x) = \frac{-1}{x^2}$

Clearly f is continuous on $[a, b]$

f is derivable on (a, b)

By Cauchy's mean value theorem

$\exists\, c \in (a, b) \ni \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$

$$\Rightarrow \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}} = \frac{\frac{-2}{c^3}}{\frac{-1}{c^2}}$$

$$\Rightarrow \frac{\frac{a^2-b^2}{a^2b^2}}{\frac{a-b}{ab}} = \frac{2}{c}$$

$$\Rightarrow \frac{a^2-b^2}{ab(a-b)} = \frac{2}{c}$$

$$c = \frac{2ab(a-b)}{a^2-b^2}$$

$$c = \frac{2ab\cancel{(a-b)}}{\cancel{(a-b)}(a+b)} = \frac{2ab}{a+b}$$

~~Theorem~~

Problem: If f is continuous on $[a\ b]$ and f is derivable on $(a\ b)$ then prove that $\exists\, c \in (a\ b)$ such that $2c[f(b) - f(a)] = f'(c)(b^2 - a^2)$

Sol: Let $g : [a\ b] \to \mathbb{R}$ is defined by

$$g(x) = x^2 \ \forall\ x \in [a\ b]$$

$$g'(x) = 2x$$

clearly g is continuous on $[a\ b]$

g is derivable on $(a\ b)$

$g'(x) \neq 0$

by cauchy mean value Theorem

$$\exists\ c \in (a\ b) \Big/ \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(b) - f(a)}{b^2 - a^2} = \frac{f'(c)}{2c}$$

$$2c[f(b) - f(a)] = f'(c)(b^2 - a^2)$$

Hence proved

Since $x \in (a\,b)$

$a < c < b$

$a < \sqrt{ab} < b$

$c \in (a\,b)$ $\therefore$ $c = \sqrt{ab}$ $[f'(c) = 0]$

(cvm) 2020 $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in $[a\,b]$ find c?

Given $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\quad g'(x) = \frac{-1}{2x\sqrt{x}}$

$= \frac{1}{2\sqrt{x}}$

$(\sqrt{x}) = x^{1/2}$

$x^n = n \cdot x^{n-1}$

$= \frac{1}{2}x^{-1/2}$

clearly ~~de~~ f is continuous on $[a\,b]$

f is derivable on $(a\,b)$

cauchy's mean value theorem

$\exists\, c \in (a\,b) \ni \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$

$$\Rightarrow \frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}} = \frac{\frac{1}{\cancel{2\sqrt{c}}}}{\frac{-1}{\cancel{2}c\sqrt{c}}}$$

$$\Rightarrow \frac{\sqrt{b} - \sqrt{a}}{\frac{\sqrt{a} - \sqrt{b}}{\sqrt{a}\sqrt{b}}} = -c$$

$c = \sqrt{ab}$

Since $a \in (a,b)$

$a < c < b$

$a < \sqrt{ab} < b$

$c \in (a,b)$ " $c = \sqrt{ab}$ $[f'(x)=0]$

(VII) $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in $[a, b]$ find c?

Given $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\quad g'(x) = \frac{-1}{2x\sqrt{x}}$

$= \frac{1}{2\sqrt{x}}$

Clearly f is continuous on $[a, b]$

f is derivable on (a, b)

Cauchy's mean value theorem

$\exists\, c \in (a, b) \ni \dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$

$\Rightarrow \dfrac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}} = \dfrac{\frac{1}{\cancel{2\sqrt{c}}}}{\frac{-1}{\cancel{2}c\sqrt{c}}}$

$\Rightarrow \dfrac{\sqrt{b} - \sqrt{a}}{\frac{\sqrt{a} - \sqrt{b}}{\sqrt{a}\sqrt{b}}} = -c$

$\Rightarrow \quad \cancel{-\frac{(\sqrt{a}-\sqrt{b})}{\sqrt{a}\sqrt{b}} \times \sqrt{a}\sqrt{b}} \quad \cancel{\sqrt{b}-\sqrt{a}} \times \frac{\sqrt{a}\sqrt{b}}{-(\sqrt{b}-\sqrt{a})} = -c$

$\Rightarrow \quad \cancel{-\sqrt{ab}} \quad \cancel{-}\sqrt{ab} = \cancel{-}c$

$\boxed{\therefore c = \sqrt{ab}}$

\# $f(x) = x^2,\ g(x) = \cos x\ [0\ \frac{\pi}{2}]$ find c?

Given $f(x) = x^2 \quad g(x) = \cos x$

$f'(x) = 2x \quad g'(x) = -\sin x$

clearly f is continuous on $[0\ \frac{\pi}{2}]$

f is derivable on $(0\ \frac{\pi}{2})$

By Cauchy's mean value Theorem

$\exists\ c \in (0\ \frac{\pi}{2}) \Rightarrow \frac{f(\frac{\pi}{2}) - f(0)}{g(\frac{\pi}{2}) - g(0)} = \frac{f'(c)}{g'(c)}$

$\Rightarrow \frac{\frac{\pi^2}{4} - 0}{0-1} = \frac{2c}{-\sin c}$

$\Rightarrow \cancel{-}\frac{\pi^2}{4} = \frac{2c}{\cancel{-}\sin c}$

$\boxed{\pi^2 \sin c = 8c}$

\# $f(x) = x^2$ and $g(x) = x^3$ in $[1\,2]$ find c?

~~Given~~ $\cancel{g(x) = x^3}$

Given $f(x) = x^2,\ g(x) = x^3$

$f'(x) = 2x \quad g'(x) = 3x^2$

UNIT - V

Riemann Integration

Let $[a\ b]$ is a closed interval if a $a = x_0 < x_1 < x_2 \cdots < x_n = b$ Then The finite set $P = \{a = x_0, x_1, x_2 \cdots x_n = b\}$ is Called The Partition of $[a\ b]$

Upper and lower riemann sums:

1) Let $f : [a\ b] \to R$ be a bounded function. and $P = \{a = x_0, x_1 \cdots x_n = b\}$ be a Partition of $[a\ b]$. The sum $M_1\delta_1 + M_2\delta_2 + \cdots M_r\delta_r \cdots + M_n\delta_n = \sum_{r=1}^{n} M_r\delta_r$ is defined as The upper riemann sum (or) upper Darboux sum. of f corresponding to The Partition P and is denoted by $U(P, f)$ i.e

$$U(P, f) = \sum_{r=1}^{n} M_r\delta_r$$

2) Let $f : [a\ b] \to R$ be a bounded function and $P = \{a = x_0, x_1, x_2 \cdots x_n = b\}$ be a Partition of $[a\ b]$ The sum $m_1\delta_1 + m_2\delta_2 + \cdots + m_r\delta_r + \cdots m_n\delta_n = \sum_{r=1}^{n} m_r\delta_r$ is denoted as The lower riemann sum (or) lower ~~Dar~~ Darboux sum of f corresponding to The Partition P and is denoted $L(P, f)$ i.e

$$L(P, f) = \sum_{r=1}^{n} m_r\delta_r$$

*** Theorem: The necessary & sufficient condition for integrality

Imp 10M

Statement:- A bounded function $f:[a\ b] \to R$ is riemann integration $[a\ b]$ iff for each $\varepsilon > 0$ $\exists$ a partition of P of $[a\ b]$ such that $0 \leq U(P,f) - L(P,f) < \varepsilon$

Proof:-

Necessary condition:

Suppose f is a riemann integrable

i.e $\int_a^b f(x)\, dx = \overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx$ — ①

Let $\varepsilon > 0$

By Darboux Theorem, we have

$$U(P,f) < \overline{\int_a^b} f(x)\, dx + \frac{\varepsilon}{2} \quad — ②$$

$$L(P,f) > \underline{\int_a^b} f(x)\, dx - \frac{\varepsilon}{2} \quad — ③$$

from ① & ②

$$U(P,f) < \int_a^b f(x)\, dx + \frac{\varepsilon}{2} \quad — ④$$

from ① & ②

$$L(P,f) > \int_a^b f(x)\, dx + \frac{\varepsilon}{2}$$

$$\int_a^b f(x)\, dx < L(P,f) + \frac{\varepsilon}{2} \quad — ⑤$$

from ④ and ⑤

$U(P,f) < L(P,f) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$U(P,f) < L(P,f) + \varepsilon$

$U(P,f) - L(P,f) < \varepsilon$

Also we have

$U(P,f) - L(P,f) \geq 0$

Therefore

$0 \leq U(P,f) - L(P,f) < \varepsilon$

<u>Sufficient condition:</u>

Suppose for each $\varepsilon > 0$ $\exists$ partition $P \in \phi[a\,b]$

such that $0 \leq U(P,f) - L(P,f) < \varepsilon$

By def $\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) \mid P \in \phi[a\,b]\} \leq U(P,f)$

$\overline{\int_a^b} f(x)\,dx \leq U(P,f)$ — ①

By def $\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) \mid P \in \phi[a\,b]\} \geq L(P,f)$

$-\underline{\int_a^b} f(x)\,dx \leq -L(P,f)$ → ②

① + ②

$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < U(P,f) - L(P,f) < \varepsilon$ (from hypothesis)

$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \varepsilon$

and

By hypothesis $U(P,f) - L(P,f) \geq 0$

$$\Rightarrow \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \geq 0$$

$$\therefore 0 \leq \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \varepsilon$$

Since $\varepsilon > 0$, $-\varepsilon < 0$

$$\Rightarrow -\varepsilon < 0 \leq \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \varepsilon$$

$$\Rightarrow -\varepsilon \leq \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \varepsilon$$

$$\therefore \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx = 0$$

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

Hence f is riemann integral.

2010 Show That $f(x) = 3x+1$, is integrable on $[1\,2]$ and $\int_1^2 (3x+1)\,dx = \frac{11}{2}$

Sol Given That $f(x) = 3x+1$.

Let The Partition $P = \{1 = 1,\ 1+\frac{1}{n},\ 1+\frac{2}{n} \dots 1+\frac{r}{n} \dots 2 = 2$ on $[1\,2]$

Sub interval $I_r = [1+\frac{r-1}{n} \cdot 1+\frac{r}{n}]$ where $M_r = 1+\frac{r}{n}$

$m_r = 1+\frac{r-1}{n}$

$$\delta_r = M_r - m_r$$

$$\delta_r = \cancel{1}+\frac{r}{n} - \left(\cancel{1}+\frac{r-1}{n}\right) \Rightarrow \cancel{1}+\frac{r}{n} - \cancel{1} - \frac{r+1}{n}$$

$$= \frac{r}{n} - \frac{r+1}{n}$$

$$\frac{r-r+1}{n} = \frac{1}{n}$$

Since $f(x) = 3x+1$ is integrable on $[1,2]$,

$$M_r = \sup \text{ of } f \text{ in } I_r$$

and $m_r = \inf \text{ of } f \text{ in } I_r$

we have $U(P,f) = \sum_{r=1}^{n} M_r \delta_r$

$$= \sum_{r=1}^{n} (3x+1)\delta_r = \sum_{r=1}^{n} \left(3\left(1+\frac{r}{n}\right)+1\right)\frac{1}{n}$$

$$= \sum_{r=1}^{n} \left(3+\frac{3r}{n}+1\right)\frac{1}{n}$$

$$= \sum_{r=1}^{n} \left(4+\frac{3r}{n}\right)\frac{1}{n}$$

$$= \sum_{r=1}^{n} \left[\frac{4}{n}+\frac{3r}{n^2}\right]$$

$$= \sum_{r=1}^{n} \frac{4}{n} + \sum_{r=1}^{n} \frac{3(r)}{n^2}$$

$$= \left(\frac{4}{n}+\frac{4}{n}+\ldots n\text{ times}\right)+\left[\frac{3(1)}{n^2}+\frac{3(2)}{n^2}+\ldots\frac{3(n)}{n^2}\right]$$

$$= \frac{4}{n}(n) + \frac{3}{n^2}\left[1+2+3\ldots n\right]$$

$$\Rightarrow 4+\frac{3}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$\Rightarrow 4+\frac{3}{2n}(n+1)$$

$$U(P,f) = 4+\frac{3n}{2n}+\frac{3}{2n} = 4+\frac{3}{2}\left[1+\frac{1}{n}\right]$$

$$L(P,f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n}\left(3\left(1+\frac{r-1}{n}\right)+1\right)\frac{1}{n}$$

$$= \sum_{r=1}^{n}\left(3+\frac{3(r-1)}{n}+1\right)\frac{1}{n}$$

$$= \sum_{r=1}^{n}\left(4+\frac{3(r-1)}{n}\right)\frac{1}{n}$$

$$= \sum_{r=1}^{n} \frac{4}{n} + \frac{3(r-1)}{n^2} \qquad \because \left[\frac{n(n+1)}{2} - n\right] \Rightarrow \frac{n(n-1)}{2}$$

$$= \frac{4}{\cancel{n}}(\cancel{n}) + \frac{3(n-1)(n)}{2n^2}$$

$$= 4 + \frac{3(n-1)}{2n}$$

$$L(P,f) = 4 + \frac{3}{2}\left(\frac{n-1}{n}\right)$$

note: $\int_a^{\bar{b}} f(x)\,dx = \lim_{n \to \infty} f(x)$

$\int_{\underline{a}}^{b} f(x)\,dx = \lim_{x \to \infty} f(x)$

$$\int_a^{\bar{b}} f(x)\,dx = \lim_{x \to \infty} 4 + \frac{3}{2n}(n+1)$$

$$= 4 + \frac{3}{2}\left[1 + \frac{1}{\infty}\right]$$

$$4 + \frac{3}{2} = \frac{11}{2}$$

$$\int_{\underline{a}}^{b} f(x)\,dx = \lim_{x \to \infty} 4 + \frac{3}{2}\left(\frac{n-1}{n}\right)$$

$$= 4 + \frac{3}{2}\left(1 - \frac{1}{\infty}\right) = \frac{11}{2}$$

Hence $\int_a^{\bar{b}} f(x)\,dx = \int_{\underline{a}}^{b} f(x)\,dx$ — *

From riemann integrable def

$$\int_a^b f(x)\,dx = \frac{11}{2}.$$

$$= \int_1^2 (3x+1)\,dx = \frac{11}{2}$$

Hence prove[redacted]

* Q) 10M Prove That $f(x) = \sin x$ is integrable on $[0, \pi/2]$ and $\int_0^{\pi/2} \sin x \, dx = 1$

Sol: Given That $f(x) = \sin x$ and $[0, \pi/2]$

$$P = \left\{ 0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \frac{3\pi}{2n}, \dots \frac{n\pi}{2n} = \frac{\pi}{2} \right\}$$

$$I_r = \left[\underbrace{\frac{(r-1)\pi}{2n}}_{m}, \underbrace{\frac{r\pi}{2n}}_{M} \right] \quad r = 1, 2, \dots n$$

$$\delta_r = \frac{r\pi}{2n} - \left(\frac{r\pi - \pi}{2n} \right)$$

$$= \frac{\cancel{r\pi} - \cancel{r\pi} + \pi}{2n} = \frac{\pi}{2n}$$

$\because M_r = \sup g \cdot f$ in I_r
$= \sin x$
$= \sin \left(\frac{r\pi}{2n} \right)$

$m = \inf g \cdot f$ in I_r
$= \sin \left(\frac{(r-1)\pi}{2n} \right)$

$$U(P, f) = \sum_{r=1}^{n} M_r \delta_r$$

$$= \sum_{r=1}^{n} \sin \left(\frac{r\pi}{2n} \right) \frac{\pi}{2n}$$

$$= \frac{\pi}{2n} \sum_{r=1}^{n} \sin \left(\frac{r\pi}{2n} \right) = \frac{\pi}{2n} \left[\sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \dots \sin \frac{n\pi}{2n} \right]$$

Formula: $\sin a + \sin (a+d) + \dots + \sin (a + (n-1)d) = \dfrac{\sin \left(a + \left(\frac{n-1}{2} \right) d \right) \sin \frac{nd}{2}}{\sin \frac{d}{2}}$

$a = \pi/2n$

$d = \pi/2n$

$$= \frac{\pi}{2n} \left[\sin \left(\frac{\pi}{2n} + \left(\frac{n-1}{2} \right) \left(\frac{\pi}{2n} \right) \right) \sin \frac{n\pi}{4n} \right] \quad \because \left[\sin \frac{n\pi}{4n} \right.$$

$\left. \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \right]$

$$= \frac{\pi}{2n} \frac{1}{\sqrt{2}} \frac{\left[\sin \left(\pi/2n \right) + \frac{(n-1)\pi}{4n} \right]}{\sin \left(\pi/4n \right)}$$

$$\Rightarrow \frac{\pi}{2\sqrt{2}n}\left[\frac{\sin\left[\frac{2\pi - n\pi \cdot \pi}{4n}\right]}{\sin(\pi/4n)}\right]$$

$$\Rightarrow \frac{\pi}{2\sqrt{2}n}\left[\frac{\sin\left(\frac{\pi(n+1)}{4n}\right)}{\sin(\pi/4n)}\right]$$

$$\frac{\pi}{2\sqrt{2}n}\left[\frac{\sin\left(\frac{\pi(n+1)}{4n}\right)}{\sin \pi/4n}\right]$$

$$\frac{\pi}{2\sqrt{2}n}\left[\frac{\sin\left[\frac{\pi n}{4n} + \frac{\pi}{4n}\right]}{\sin(\pi/4n)}\right]$$

$$\Rightarrow \frac{\pi}{2\sqrt{2}n}\left[\frac{\sin\left(\frac{\pi n}{4n}\right)\cos\left(\frac{\pi}{4n}\right) + \cos\left(\frac{\pi n}{4n}\right)\cancel{\sin\left(\frac{\pi}{4n}\right)}}{\sin\left(\cancel{\pi/4n}\right)}\right]$$

$$\Rightarrow \frac{\pi}{2\sqrt{2}n}\left[\sin\left(\frac{\pi n}{4n}\right)\cdot\cot(\pi/4n) + \cos\left(\frac{\pi n}{4n}\right)\right] \qquad \left[\because \frac{\cos x}{\sin x} = \cot x\right.$$

$$\Rightarrow \frac{\pi}{2\sqrt{2}n}\left[\frac{1}{\sqrt{2}}\cot(\pi/4n) + \frac{1}{\sqrt{2}}\right]$$

$$U(P,f) \Rightarrow \frac{\pi}{4n}\left[\cot(\pi/4n) + 1\right]$$

$$\left[\int_a^b f(x)\,dx = \lim_{n\to\infty} U(P,f)\right]$$

$$\Rightarrow \lim_{n\to\infty}\left[\frac{\pi}{4n}\left(\cot(\pi/4n) + 1\right)\right]$$

$$\Rightarrow \lim_{n\to\infty}\frac{\pi}{4n}\cot\left(\frac{\pi}{4n}\right) + \lim_{n\to\infty}\frac{\pi}{4n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\pi/4n}{\tan \pi/4n} + 0 \quad \left[\because \lim_{x\to 0} \frac{\tan\theta}{\theta} = 1\right]$$

$$\Rightarrow 1 + 0 = 1$$

$$\Rightarrow \int_0^{\pi/2} \sin x \, dx = 1$$

Similarly we prove that $L(P,f)$ and $\int_0^{\frac{\pi}{2}} \sin x \, dx = 1$

$$\therefore \overline{\int_0^{\frac{\pi}{2}}} f(x)\, dx = \underline{\int_0^{\frac{\pi}{2}}} f(x)\, dx = 1 \longrightarrow (*)$$

By Riemann integrable def

$$* = \int_0^{\frac{\pi}{2}} f(x)\, dx = 1$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin x \, dx = 1$$

Hence proved.

* P.T $f(x) = x^2$ is integrable on $[0\ a]$ and $\int_0^a x^2 dx = \frac{a^3}{3}$

$$\therefore \left[P = \left\{0 = 0, \frac{a}{n}, \frac{2a}{n} \cdots \frac{na}{n} = a\right\}\right]$$

* P.T $f(x) = x^2$ is integrable on $[0\ 1]$ and
complete $U(P,f)$ and $L(P,f)$ $P = \left(0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \cdots 1 \geq 1\right]$

$$< \frac{\varepsilon}{f(b)-f(a)+1}\left[f(b)-f(a)\right] < \varepsilon$$

$\therefore U(P,f) - L(P,f) < \varepsilon$

we have

$U(P,f) - L(P,f) \geq 0$

$\therefore 0 \leq U(P,f) - L(P,f) < \varepsilon$

Hence f is integrable on $[a\ b]$

by f is decreasing on $[a\ b] \rightarrow f$ is riemann integrable on $[a\ b]$

* **Fundamental Theorem of integral Calculous**

(2) I 10M **Statement:-** If $f \in R[a\ b]$ & ϕ is primitive of f Then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

Given That ϕ is primitive on $[a\ b]$

i,e, $\phi'(x) = f(x)\ \forall\ x \in [a\ b] \longrightarrow (1)$

and $f \in R[a\ b]$ & $P = \{ a = x_0, x_1 \cdots x_n = b \}$

Let $\xi_r \in [x_{r-1}, x_r]$, $r = 1, 2 \cdots n$

i,e $x_{r-1} \leq (\xi_r)\,\delta_r = \int_a^b f(x)\,dx \longrightarrow (2)$

ϕ is derivable on $[a\ b] \Rightarrow \phi$ is continuous on $[a\ b]$ Then ϕ is continuous & derivable on $[x_{r-1}, x_r]$, $r = 1, 2 \cdots n$

By Lagrange's mean value Theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{\phi(x_r)-\phi(x_{r-1})}{x_r - x_{r-1}} = \phi'(\xi_r)$$

$$\sum_{r=1}^{n}[\phi(x_r)-\phi(x_{r-1})] = \sum_{r=1}^{n}(x_r - x_{r-1})\phi(\xi_r)$$

$$= \sum_{r=1}^{n}[\phi(x_r)-\phi(x_{r-1}) = \sum_{r=1}^{n}\phi'(\xi_r)\delta_r$$

$$= \sum_{r=1}^{n} f(\xi_r)\delta_r \text{ [From ①]}$$

$$\Rightarrow [\phi(x_1)-\phi(x_0)+\phi(x_2)+\dots+\phi(x_n)-\phi(x_{n-1})]$$

$$= \sum_{r=1}^{n} f(\xi_r)\delta_r$$

$$\Rightarrow \phi(x_n)-\phi(x_0) = \sum_{r=1}^{n} f(\xi_r)\delta_r$$

$$\Rightarrow \lim_{\|P\|\to 0} \phi(x_n)-\phi(x_0) = \lim_{\|P\|\to 0}\sum_{r=1}^{n} f(\xi_r)\delta_r.$$

(From ②)

$$\Rightarrow \phi(x_n)-\phi(x_0) = \int_a^b f(x)\,dx$$

$$\Rightarrow \phi(b)-\phi(a) = \int_a^b f(x)\,dx$$

* S.T $\int_0^1 x^4\,dx = \frac{1}{5}$

(5)

Given That $f(x) = x^4$

Since f is continuous on $[0\ 1]$

consider $\phi(x) = \frac{x^5}{5}$

By Lagrange's mean value Theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\frac{\phi(x_r)-\phi(x_{r-1})}{x_r - x_{r-1}} = \phi'(\xi_r)$$

$$\sum_{r=1}^{n}[\phi(x_r)-\phi(x_{r-1})] = \sum_{r=1}^{n}(x_r - x_{r-1})\phi(\xi_r)$$

$$= \sum_{r=1}^{n}[\phi(x_r)-\phi(x_{r-1}) = \sum_{r=1}^{n}\phi'(\xi_r)\delta_r$$

$$= \sum_{r=1}^{n} f(\xi_r)\delta_r \text{ [From ①]}$$

$$\Rightarrow [\phi(x_1)-\phi(x_0)+\phi(x_2)+\cdots+\phi(x_n)-\phi(x_{n-1})]$$

$$= \sum_{r=1}^{n} f(\xi_r)\delta_r$$

$$\Rightarrow \phi(x_n)-\phi(x_0) = \sum_{r=1}^{n} f(\xi_r)\delta_r$$

$$\Rightarrow \lim_{\|P\|\to 0} \phi(x_n)-\phi(x_0) = \lim_{\|P\|\to 0}\sum_{r=1}^{n} f(\xi_r)\delta_r.$$

(From ②)

$$\Rightarrow \phi(x_n)-\phi(x_0) = \int_a^b f(x)\,dx$$

$$\Rightarrow \phi(b)-\phi(a) = \int_a^b f(x)\,dx$$

* S.T $\int_0^1 x^4\,dx = \frac{1}{5}$

Sol

Given That $f(x) = x^4$

Since f is continuous on $[0\ 1]$

consider $\phi(x) = \frac{x^5}{5}$

$$\varphi'(x) = \frac{5x^4}{5}$$

$$\left[\because \varphi(x) = f(x)\right.$$

$$\frac{d}{dx}\varphi(x) = f(x)$$

$$\varphi'(x) = x^4 = f(x)$$

$$\left.\int \frac{d}{dx}\varphi(x) = \int f(x)\right]$$

Hence φ is premitive on $[0\ 1]$

By fundemental Theorem

$$\Rightarrow \int_a^b f(x)\,dx = \varphi(b) - \varphi(a)$$

$$\left[\because \varphi(x) = \int f(x) = \int x^4 = \frac{x^5}{5}\right]$$

$$= \int_0^1 x^4\,dx = \frac{1}{5} - \frac{0}{5} = \frac{1}{5}$$

$$\therefore \int_0^1 x^4\,dx = \frac{1}{5}$$

* Evaluate $\int_0^{\pi/4} \sec^4 x - \tan^4 x$

Hint $\sec^4 x - \tan^4 x = (\sec^2 x)^2 - (\tan^2 x)^2$

$$= (\sec^2 x - \tan^2 x)(\sec^2 x + \tan^2 x)$$

$$= (\sec^2 x + \tan^2 x)$$

$$= 2\sec^2 x - 1$$

Integral of the Limit of a sum:-

(5) * S.T $\lim_{x\to\infty} \sum_{r=1}^{n} \frac{n}{n^2+r^2} = \frac{\pi}{4}$

Sol

$$\lim_{x\to\infty} \sum_{r=1}^{n} \frac{n}{n^2+r^2}$$

$$\Rightarrow \lim_{x\to\infty} \sum_{r=1}^{n} \frac{n}{n^2\left(1+\left(\frac{r}{n}\right)^2\right)}$$

$$\Rightarrow \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n\left(1+\left(\frac{r}{n}\right)^2\right)} \qquad \left[\lim_{\|P\|\to 0} \sum_{r=1}^{n} f(\xi_r)\,\delta_r = \int_a^b f(x)\,dx\right]$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1+\left(\frac{r}{n}\right)^2}$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}\,dx \quad \text{term } \frac{r}{n} = t$$

By fundamental Theorem of Calculus

Let $f(x) = \frac{1}{1+x^2}$ $\forall\, x \in [0\ 1]$

Consider $\phi(x) = \tan^{-1} x$ $\qquad [f(x) = \phi'(x)]$

$$\phi'(x) = \frac{1}{1+x^2}$$

$$\phi'(x) = f(x)$$

$\because \phi$ is primitive

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}$$

* P.T $\lim_{n\to\infty} \left[\frac{1}{n} + \frac{1}{n+2} + \cdots \frac{1}{3n}\right] = \frac{1}{2}\log 3$]

$$\lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{n+2r}$$

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

$$\Rightarrow \int_0^{\pi/4} 2\sec^2 x - 1 = 2\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4} - [2\tan 0 - 0]$$

$$\Rightarrow \int_0^{\pi/4} 2\sec^2 x - 1 = 2(1) - \frac{\pi}{4} - 0$$

$$= 2 - \frac{\pi}{4} = \frac{8-\pi}{4}$$

(2) P.T $f(x) = x^2$ is intigrable on $[0\ a]$ and $\int_0^a x^2\,dx = \frac{a^3}{3}$

Given $f(x) = x^2$ and $[0\ a]$

$$P = \left\{ 0 = 0, \frac{a}{n}, \frac{2a}{n} \cdots \frac{na}{n} = a \right\}$$

$$I_r = \Big[\underbrace{\frac{(r-1)a}{n}}_{m}, \underbrace{\frac{ra}{n}}_{M} \Big]$$

$$\delta_r = \frac{ra}{n} - \left(\frac{ar - a}{n}\right)$$

$$= \frac{\cancel{ra} - \cancel{ar} + a}{n} = \frac{a}{n}$$

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r$$

$$= \sum_{r=1}^{n} \left(\frac{ar}{n}\right)^2 \frac{a}{n}$$

$$= \sum_{r=1}^{n} \frac{a^2 r^2}{n^2} \frac{a}{n}$$

$$= \sum_{r=1}^{n} \frac{a^3 r^2}{n^3}$$

$$= \frac{a^3}{n^3} \sum_{r=1}^{n} r^2$$

$$= \frac{a^3}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$= \frac{a^3}{\cancel{n^3}} \left[\frac{\cancel{n^3}(1+\frac{1}{n})(2+\frac{1}{n})}{6} \right]$$

$$\int_0^a U(P,f) \Rightarrow \frac{a^3}{6} \lim_{n\to\infty} \left[1+\frac{1}{n}\right]\left[2+\frac{1}{n}\right]$$

$$= \frac{a^3}{6}(1)(2)$$

$$= \frac{\cancel{2}a^3}{\cancel{6}\,3} = \frac{a^3}{3}$$

Similarly we can ~~R~~ prove $L(P,f)$ and $\int_a^b L(P,f) = \frac{a^3}{3}$

$$\int_a^a f(x)\,dx = \int_0^a f(x)\,dx = \frac{a^3}{3}$$

By riemann integral

$$\int_a^a f(x)\,dx = \frac{a^3}{3}$$

$$\therefore \int_0^a x^2\,dx = \frac{a^3}{3}$$

Hence proved.

$\phi'(x) = f(1+2x)$

$\phi'(x) = f(x)$ ϕ is primitive

$$\boxed{\int_a^b f(x)\,dx = \phi(b) - \phi(a)}$$

$$\int_0^1 \frac{1}{1+2x}\,dx = \frac{1}{2}\log(1+2) - \log(1)$$
$$= \frac{1}{2}\log 3 - 0$$
$$= \frac{1}{2}\log 3$$

first mean value Theorem:

If $f, g \in R[a\,b]$ & g keeps the same sign on $[a\,b]$ then $\exists\, \mu \in R$ lying between inf and sup of f $\ni$ $\int_a^b f(x)\cdot g(x)\,dx = \mu \int_a^b g(x)\,dx$

Let g be non negitive term.

i.e, $g(x) \geq 0 \;\forall\, x \in [a\,b]$

Since $f \in R[a\,b] \Rightarrow f$ is bounded on $[a\,b]$

i.e $m \leq f(x) \leq M \;\forall\, x \in [a\,b]$.

and $g(x) \geq 0 \Rightarrow m\,g(x) \leq f(x)\cdot g(x) \leq M g(x)$

$$\Rightarrow \int_a^b m\,g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq \int_a^b M g(x)\,dx$$

$$\Rightarrow m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx$$

$\exists\ \mu \in [m, M] \ni$

$$\int_a^b f(x)\cdot g(x)\,dx \leq \mu\int_a^b g(x)\,dx \quad \text{--- (1)}$$

Let g be non - Positive terms

$g(x) \leq 0$

$m\,g(x) \geq f(x)\,g(x) \geq M g(x)$

$$\Rightarrow \int_a^b m\,g(x) \geq \int_a^b f(x)g(x) \geq \int_a^b M g(x)$$

$$m\int_a^b g(x)\,dx \geq \int_a^b f(x)\,g(x)\,dx \geq M\int_a^b g(x)\,dx$$

$\exists\ \mu \in [m M]$

$$\int_a^b f(x)g(x)\,dx \geq \mu\int_a^b g(x)\,dx \quad \text{--- (2)}$$

from (1) & (2) $\therefore \int_a^b f(x)\,g(x)\,dx = \mu\int_a^b g(x)\,dx$

$$\int_0^{\pi/4} \sec^4 x - \tan^4 x$$

$f(x) = \sec^4 x - \tan^4 x$

$= 2\sec^2 x - 1$

Since f is continuous $[0 \frac{\pi}{4}]$

consider $\phi(x) = 2\tan x - x$

$$\phi'(x) = 2\sec^2 x - 1 = f(x)$$

hence ϕ is primitive on $[0 \frac{\pi}{4}]$

By fundamental theorem of calculus

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} 2\sec^2 x - 1 = 2\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4}.$$

$$- [2\tan 0 - 0]$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} 2\sec^2 x - 1 = 2(1) - \frac{\pi}{4} - 0$$

$$= 2 - \frac{\pi}{4} = \frac{8-\pi}{4}$$

~~P.T~~ $f(x) = x^2$ is intigrable on $[0\ a]$ and $\int_0^a x^2\,dx = \frac{a^3}{3}$

Given $f(x) = x^2$ and $[0\ a]$

$$P = \left\{0 = 0, \frac{a}{n}, \frac{2a}{n} \cdots \frac{na}{n} = a\right\}$$

$$I_r = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$$