IQIC ENDSEM MARKING SCHEME

SET 1

1)
$$|\psi\rangle = \frac{|00\rangle + n|11\rangle}{\sqrt{1+|n|^2}}$$

$$\rho_{AB} = |\psi\rangle\langle\psi| = \frac{1}{1+|n|^2}\left(|00\rangle\langle 00| + n^*|00\rangle\langle 11| + n|11\rangle\langle 00| + |n|^2|11\rangle\langle 11|\right)$$

missing * or || also okay ↓ full marks

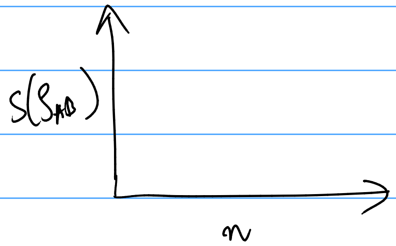
$$\rho_A = \rho_B = \frac{1}{1+|n|^2}\left(|0\rangle\langle 0| + |n|^2|1\rangle\langle 1|\right)$$

Only $\rho_{AB} \Rightarrow 0.5$

$\rho_A = \rho_B \Rightarrow 1$

$$S(\rho_{AB}) = 0$$

0.5



0.5

$$S(\rho_A) = S(\rho_B) =$$

$$-\left[\frac{1}{1+|n|^2}\log\left(\frac{1}{1+|n|^2}\right) + \frac{|n|^2}{1+|n|^2}\log\left(\frac{|n|^2}{1+|n|^2}\right)\right]$$

(OR)

$$-\frac{1}{1+|n|^2}\left[\log\left(\frac{1}{1+|n|^2}\right) + |n|^2\log\left(\frac{|n|^2}{1+|n|^2}\right)\right]$$

(OR)

$$= -\frac{1}{1+|n|^2}\left[\log\left(1+|n|^2\right)\left(-1-|n|^2\right) + |n|^2\log|n|^2\right]$$

(OR)

$$= \log\left(1+|n|^2\right) - \frac{|n|^2\log|n|^2}{1+|n|^2}$$

Any form correctly ⇒ 1

If only mod is missing : Full marks

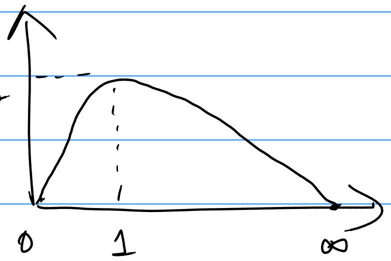
Any form partial / only for S_A or S_B = 0.5

Using the first formalism

Let $p = \frac{1}{1+|n|^2}$ $\qquad n=0 \rightarrow p=1$

$\qquad n=\infty \rightarrow p=0$

$S(\rho_B) = S(\rho_A) = 1$



Axes present, graph correct = 2

Anything missing = 1.

Graph wrong = 0

16)

Required state = $U|\phi\rangle$

where U is $\sigma_x \otimes I$, $\sigma_z \otimes I$, $i\sigma_y \otimes I$, $I \otimes I$

and $|\phi\rangle$ is $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$

Sent state = $U|\psi\rangle$

where $|\psi\rangle$ is $\frac{|00\rangle + n|11\rangle}{\sqrt{1+|n|^2}}$

Probability of success = $\| \langle\psi| U^{\dagger} U |\phi\rangle \|^2$

$= \| \langle\psi|\phi\rangle \|^2$

$= \left\| \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}} \right) \left(\frac{|00\rangle + n|11\rangle}{\sqrt{1+|n|^2}} \right) \right\|^2$

$= \frac{\| 1+n \|^2}{2(1+|n|^2)}$ $\implies$ Correct ans

= Full 5 marks.

$I \otimes I |\psi\rangle$

$\sigma_z \otimes I |\psi\rangle = \dfrac{|00\rangle - n|11\rangle}{\sqrt{1+|n|^2}}$

$\sigma_x \otimes I |\psi\rangle = \dfrac{|10\rangle + n|01\rangle}{\sqrt{1+|n|^2}}$

$\sigma_x . \sigma_z \otimes I |\psi\rangle = \dfrac{|10\rangle - n|01\rangle}{\sqrt{1+|n|^2}}$

$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_x . \sigma_z = i\sigma_y$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\longrightarrow$ Any of these 0.5 marks each.

(ii) Assumption, $|\psi\rangle, |\phi\rangle \neq 0$

Hermitian

$|\phi\rangle\langle\psi| = |\psi\rangle\langle\phi|$ (0.5)

$|\phi\rangle = \alpha|\psi\rangle$ (0.5)

$\alpha = \alpha^*$ (0.5)

Projector

$P^2 = P$ (0.5)

$|\psi\rangle\langle\phi|\psi\rangle\langle\phi| = |\psi\rangle\langle\phi|$

$\langle\phi|\psi\rangle = 1$ (0.5)

$K^2 = |\psi\rangle\langle\phi|\psi\rangle\langle\phi|$ (0.5)

Claim: There always exists a vector $|t\rangle$ such that $\langle\psi|t\rangle \neq 0$ and $\langle\phi|t\rangle \neq 0$

Proof: Consider 2 exhaustive cases:

Case 1: $\langle\psi|\phi\rangle \neq 0$

Then $|t\rangle = |\psi\rangle$ satisfies the condition

Case 2: Let $\langle\psi|\phi\rangle = 0$

Then $|t\rangle = |\psi\rangle + |\phi\rangle$ satisfies the condition.

Let $P_1 = \dfrac{|\psi\rangle\langle t|}{\sqrt{\langle t|\psi\rangle}}$ $\qquad P_1^2 = P_1$

$P_2 = \dfrac{|t\rangle\langle\phi|}{\sqrt{\langle\phi|t\rangle}}$ $\qquad P_2^2 = P_2$.

$\alpha = \dfrac{\sqrt{\langle t|\psi\rangle\langle\phi|t\rangle}}{\langle t|t\rangle}$

We have $\alpha P_1 P_2 = \dfrac{\cancel{\sqrt{\langle t|\psi\rangle\langle\phi|t\rangle}}}{\cancel{\langle t|t\rangle}} \dfrac{|\psi\rangle\cancel{\langle t|t\rangle}\langle\phi|}{\cancel{\sqrt{\langle t|\psi\rangle\langle\phi|t\rangle}}}$

$= |\psi\rangle\langle\phi|$

$= K$ //

Full answer = 2

Missing out $\langle\phi|\psi\rangle = 0$ but rest present 1.5

Else 0.

(1b) Before $= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 1

After $= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 1+1

computation

final ans

Ans: Both density matrices are same. If it were different this violates no-signalling principle. 2

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

If Alice gets $|+\rangle$, Bob gets

State on d.m (0.5)

$$|-\rangle = \frac{1}{2}\left(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|\right)$$

If Alice gets $|-\rangle$, Bob gets $|+\rangle$

$$\frac{1}{2}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right)$$

state on dm (0.5)

SET 3 Defns — 1 mark

Q3) Any proof using strong subadditivity
$S(A,B,C) + S(B) \leq S(A,B) + S(B,C)$ — Full 5 marks

OR

$I(X;Y|Z) = H(X|Z) + H(Y|Z) - H(X,Y|Z)$
$= H(X|Z) - H(X|Y,Z)$

$I(A;C|B) = H(A|B) - H(A|B,C)$

Full 5 marks

Q8) a) Express as matrix — 1/2 $\quad \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
eigenvalues — 1 without coeff: $\pm\sqrt{2}$, with $= \pm \frac{1}{\sqrt{2}}$
eigenvectors — 1

$|e_1\rangle = \dfrac{|0\rangle + (\sqrt{2}-1)|1\rangle}{\sqrt{1+(\sqrt{2}-1)^2}} \qquad e_2 = \dfrac{|0\rangle - (\sqrt{2}+1)|1\rangle}{\sqrt{1+(\sqrt{2}+1)^2}}$

Direct + minimal 3

Final $\frac{1}{\sqrt{2}}|e_1\rangle\langle e_1| + \frac{1}{\sqrt{2}}|e_2\rangle\langle e_2|$ 1/2

Direct 2.5

$\frac{1}{\sqrt{2}}\left(|0\rangle\langle +| + |1\rangle\langle -|\right)$

b) Matrix (1/2 if nothing else)
eigenvectors — 1 $\quad |+\rangle$
eigenvalues — 1 $\quad 1, 0$

If direct answer, full marks 2

13) Any helpful info / formulas = 1
Avg value of σ_x in $|0\rangle = 0$ 1
σ_y in $|0\rangle = 0$ 1
S.D. σ_x in $|0\rangle = 1$ 1
σ_y in $|0\rangle = 1$ 1

$$\begin{bmatrix} |0\rangle & |1\rangle \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \langle +| \\ \langle -| \end{bmatrix}$$

$$\begin{pmatrix} |0\rangle & |1\rangle \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \langle +| \\ \frac{1}{\sqrt{2}} \langle -| \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} |0\rangle\langle +| + \frac{1}{\sqrt{2}} |1\rangle\langle -|$$

- Writing CC^T or $CC^\dagger$ = 0.5 //

SET 4

4) $S(USU^\dagger)$

$= -\mathrm{Tr}\left(USU^\dagger \log USU^\dagger\right)$ → 1 //

$= -\mathrm{Tr}\left(USU^\dagger U \log(S) U^\dagger\right)$ → 2.

$= -\mathrm{Tr}\left(US \log S\, U^\dagger\right)$ → 2

$= -\mathrm{Tr}\left(U^\dagger U S \log S\right)$

$= -\mathrm{Tr}\left(S \log S\right)$

$= S(S)$ //

5 marks.

Another proof
$S(S) = -\sum_x \lambda_x \log \lambda_x$
eigenvalues don't change under unitary transform.

ρ is a diagonalizable matrix.
let $\rho = U \rho_D U^\dagger$ } 0.5

Defn $\log \rho = U \log(\rho_D) U^\dagger$. — 0.5

let $\rho' = V \rho V^\dagger$
$= V U \rho_D U^\dagger V^\dagger$

$$\log \rho' = \log (V U \rho_D U^\dagger V^\dagger)$$
$$= V U \log \rho_D U^\dagger V^\dagger$$
$$= V \log (U \rho_D U^\dagger) V^\dagger$$
$$= V \log(\rho) V^\dagger$$

1

Another valid proof:

let $\rho = U \rho_D U^\dagger$
$\therefore \rho_D = U^\dagger \rho U$ — 1

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho) \quad \text{— 1}$$
$$= -\mathrm{Tr}(\rho \log (U \rho_D U^\dagger)) \quad \text{— 1}$$
$$= -\mathrm{Tr}(\rho\, U \log \rho_D\, U^\dagger)$$
$$= -\mathrm{Tr}(U^\dagger \rho U \log \rho_D) \quad \text{1}$$
$$= -\mathrm{Tr}(\rho_D \log \rho_D)$$

→ Unitary doesn't change eigenvalues.
→ ρ_D is invariant under unitary. — 1

(9) $Q \otimes S = \sigma_z \otimes -\dfrac{\sigma_x - \sigma_z}{\sqrt{2}}$

Each matrix = 0.5 marks.

$$= \left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Each expectation value = 0.5 marks

Final add = 1 mark.

$R \otimes S = \sigma_x \otimes -\dfrac{\sigma_z - \sigma_x}{\sqrt{2}}$

No computation only answer = 0

$$= \left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix} 0 & 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \end{bmatrix}$$

$R \otimes T = \sigma_x \otimes \dfrac{\sigma_z - \sigma_x}{\sqrt{2}}$

$$= \left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix} 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \end{bmatrix}$$

$Q \otimes T = \sigma_z \otimes \dfrac{\sigma_z - \sigma_x}{\sqrt{2}}$

$$\left(-\frac{1}{\sqrt{2}}\right) = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

14) a) Let P be a projector.
We have $P^2 = P$ — 0.5

Let $|x\rangle$ be a non-trivial eigenvector of P

We have, $P|x\rangle = \lambda|x\rangle$

$$\langle x|P|x\rangle = \lambda\langle x|x\rangle$$

Any wrong defn of projector ↓ lose 0.5 mk.

$$P^2|x\rangle = P(P|x\rangle)$$
$$= P(\lambda|x\rangle)$$
$$= \lambda^2|x\rangle$$
$$\langle x|P^2|x\rangle = \lambda^2\langle x|x\rangle. \quad 0.5$$

Since $P = P^2$

$$\langle x|P|x\rangle = \langle x|P^2|x\rangle$$
$$\lambda\cancel{\langle x|x\rangle} = \lambda^2\cancel{\langle x|x\rangle}$$
$$\lambda = \lambda^2$$

0.5 $\therefore \lambda \in \{0,1\}$ — (1.5) —

b) $\forall\, |x\rangle$, we have

$$\langle x|A^+A|x\rangle$$
$$= \||A|x\rangle\|^2$$
$$\geq 0$$

Hence A^+A is positive. — 1 —

Also $\langle Ax, Ax\rangle$

c). Let $|x\rangle$ be a non-trivial eigenvector of H.

$$H|x\rangle = \lambda|x\rangle$$

$$\langle x|H|x\rangle = \lambda\langle x|x\rangle$$

$$\Rightarrow \langle Hx|x\rangle = \lambda\langle x|x\rangle$$

Proved = 1 mk
Assumed = 0.5 mk

$$\lambda^*\langle x|x\rangle = \lambda\langle x|x\rangle$$

$$(\lambda^* - \lambda)(\langle x|x\rangle) = 0$$

Since $\langle x|x\rangle \neq 0$

$$\underline{\underline{\lambda^* = \lambda}}$$

$\therefore$ All eigenvalues of Hermitian operators are real.

Let

$$H|x\rangle = \lambda_1|x\rangle$$

$$H|y\rangle = \lambda_2|y\rangle$$

where $\lambda_1 \neq \lambda_2$.

We have

$$\langle y|H|x\rangle = \lambda_1\langle y|x\rangle$$

$$\Rightarrow \langle Hy|x\rangle = \lambda_1\langle y|x\rangle$$

$$\Rightarrow \lambda_2^*\langle y|x\rangle = \lambda_1\langle y|x\rangle$$

(Since $\lambda_2 = \lambda_2^*$) $\Rightarrow (\lambda_2 - \lambda_1)(\langle y|x\rangle) = 0$ (1.5)

Since $\lambda_2 \neq \lambda_1$

$$\langle y|x\rangle = 0.$$

SET 5

since $\langle GHZ|W\rangle = 0$
$|GHZ\rangle$ and $|W\rangle$ are eigenvectors
e. values $= 1/2, 1/2$

5) $S(\rho_{abc}) = 1$

Compute matrix -0.5

$|GHZ\rangle\langle GHZ|$
$$= \frac{1}{2}\left(|000\rangle\langle 000| + |000\rangle\langle 111| + |111\rangle\langle 000| + |111\rangle\langle 111| \right)$$

$$GHZ_A = GHZ_B = GHZ_C = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1| \right)$$

$$GHZ_{AB} = GHZ_{BC} = GHZ_{AC} = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11| \right)$$

$|W\rangle\langle W|$
$$= \frac{1}{3}\begin{pmatrix} |100\rangle\langle 100| + |100\rangle\langle 010| + |100\rangle\langle 001| \\ + |010\rangle\langle 100| + |010\rangle\langle 010| + |010\rangle\langle 001| \\ + |001\rangle\langle 100| + |001\rangle\langle 010| + |001\rangle\langle 001| \end{pmatrix}$$

$$W_A = W_B = W_C = \frac{1}{3}|1\rangle\langle 1| + \frac{2}{3}|0\rangle\langle 0|$$

$$W_{AB} = W_{BC} = W_{AC} = \frac{1}{3}\left(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 10| + |10\rangle\langle 01| \right)$$

$$\therefore \rho_A = \rho_B = \rho_C = \frac{1}{2}\left(GHZ_A + W_A \right)$$
$$= \frac{1}{2}\left(\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} + \frac{2|0\rangle\langle 0| + 1|1\rangle\langle 1|}{3} \right)$$
$$= \frac{1}{2}\left(\frac{7|0\rangle\langle 0| + 5|1\rangle\langle 1|}{6} \right)$$
$$= \frac{7}{12}|0\rangle\langle 0| + \frac{5}{12}|1\rangle\langle 1|$$

Any 2 out of 3 states $= 1$ mk.
Only 1 $= 0.5$

$$\rho_{AB} = \rho_{BC} = \rho_{CA}$$

$$= \frac{5}{12}|00\rangle\langle 00| + \frac{2}{12}|01\rangle\langle 01| + \frac{2}{12}|10\rangle\langle 10|$$
$$+ \frac{3}{12}|11\rangle\langle 11| + \frac{2}{12}|01\rangle\langle 10| + \frac{2}{12}|10\rangle\langle 01|$$

2 out of 3 = 1

1 out of 3 = 0.5

Now, we have

(a) $S(\rho_{ABC}) = 1$ $\Rightarrow 0.5$

Entropy, CVE formula = 0.5

$$S(\rho_{AB}) = S(\rho_{AC}) = S(\rho_{BC})$$

$$= -\sum_{x \in eig(\rho_{AB})} x \log x$$

$$- \left(\frac{5}{12}\log\frac{5}{12} + \frac{1}{3}\log\frac{1}{3} + \frac{1}{4}\log\frac{1}{4}\right)$$

$= 1.40$ $= 0.5$

$$S(\rho_A) = S(\rho_B) = S(\rho_C)$$

$$= -\left(\frac{5}{12}\log\frac{5}{12} + \frac{7}{12}\log\frac{7}{12}\right)$$

$= 0.98$ //

(b) $S(B|AC) = S(A|BC) = S(C|AB)$

$= 1 - 1.40$

$= -0.40$ $= 0.5$

(c) $S(A|B) = S(A|C) = S(C|B)$

$= 1.40 - 0.98$ $= 0.5$

$= 0.42$

(10) Alice and Bob share a state $|\phi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

Bob's state before the protocol

$$= \left(|\phi\rangle\langle\phi|\right)_B = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} \quad 1$$

Alice then performs the protocol:

$$\alpha|0\rangle + \beta|1\rangle \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

1

$$= \frac{\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle}{\sqrt{2}}$$

$$= |\phi^+\rangle \underbrace{\frac{\alpha|0\rangle + \beta|1\rangle}{2}}_{(a)} + |\phi^-\rangle \underbrace{\frac{\alpha|0\rangle - \beta|1\rangle}{2}}_{(b)}$$

$$+ |\psi^+\rangle \underbrace{\frac{\alpha|1\rangle + \beta|0\rangle}{2}}_{(c)} + |\psi^-\rangle \underbrace{\frac{\alpha|1\rangle - \beta|1\rangle}{2}}_{(d)}$$

Alice's state	Probability	Bob's density matrix
$\vert\phi^+\rangle$	$1/4$	$(\alpha\vert 0\rangle + \beta\vert 1\rangle)(\alpha^*\langle 0\vert + \beta^*\langle 1\vert) = \begin{bmatrix} \vert\alpha\vert^2 & \alpha\beta^* \\ \beta\alpha^* & \vert\beta\vert^2 \end{bmatrix}$
$\vert\phi^-\rangle$	$1/4$	$(\alpha\vert 0\rangle - \beta\vert 1\rangle)(\alpha^*\langle 0\vert - \beta^*\langle 1\vert) = \begin{bmatrix} \vert\alpha\vert^2 & -\alpha\beta^* \\ -\beta\alpha^* & \vert\beta\vert^2 \end{bmatrix}$

Density matrix expression = 1 mark.

$|\psi^+\rangle$ $\quad$ $1/4$ $\quad$ $(\alpha|1\rangle+\beta|0\rangle)(\alpha^*\langle 1| + \beta^*\langle 0|)$

$$= \begin{bmatrix} |\beta|^2 & \beta\alpha^* \\ \alpha\beta^* & |\alpha|^2 \end{bmatrix}$$

$|\psi^-\rangle$ $\quad$ $1/4$ $\quad$ $(\alpha|1\rangle-\beta|0\rangle)(\alpha^*\langle 1| - \beta^*\langle 0|)$

$$\begin{bmatrix} |\beta|^2 & -\beta\alpha^* \\ -\alpha\beta^* & |\alpha|^2 \end{bmatrix}$$

$\therefore$ Bob's ensemble after measurement but before communication is described as:

$$\frac{1}{4}\left(\begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{bmatrix} + \begin{bmatrix} |\alpha|^2 & -\alpha\beta^* \\ -\beta\alpha^* & |\beta|^2 \end{bmatrix} + \begin{bmatrix} |\beta|^2 & \beta\alpha^* \\ \alpha\beta^* & |\alpha|^2 \end{bmatrix} + \begin{bmatrix} |\beta|^2 & -\beta\alpha^* \\ -\alpha\beta^* & |\alpha|^2 \end{bmatrix} \right)$$

$$= \frac{1}{4}\begin{bmatrix} 2|\alpha|^2+2|\beta|^2 & 0 \\ 0 & 2|\alpha|^2+2|\beta|^2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$ Ans 1

1

As Bob has the same density matrix before and after the measurement, the no-signalling principle has not been violated.

15) a) Let A, B be 2 Hermitian operators.

We have:

$$A = A^{+}$$
$$B = B^{+}$$

0.5

If (c) is proved as well, + 0.5 more.

$$\therefore \quad (A \otimes B)^{+} = A^{+} \otimes B^{+} = A \otimes B$$

Hence $A \otimes B$ is Hermitian.

b) Let U, W be 2 unitary operators.

We have

$$\left.\begin{array}{l} UU^{+} = U^{+}U = I \\ WW^{+} = W^{+}W = I \end{array}\right\} \quad (1)$$

$$(U \otimes W)(U \otimes W)^{+} = (U \otimes W)(U^{+} \otimes W^{+}) = UU^{+} \otimes WW^{+} \quad \text{——} \quad (2)$$

From (1) and (2) we have:

$$UU^{+} \otimes WW^{+} = U^{+}U \otimes W^{+}W = (U^{+} \otimes W^{+})(U \otimes W) = (U \otimes W)^{+}(U \otimes W)$$

2 marks.

Fixed dimension proof = 1 mark

From (1) and (2), we also have:

$$UU^{+} \otimes WW^{+} = I \otimes I = I$$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} a_1^* & a_3^* \\ a_2^* & a_4^* \end{bmatrix} = \begin{bmatrix} a_1 a_1^* + a_2 a_2^* & a_1 a_3^* + a_2 a_4^* \\ a_3 a_1^* + a_4 a_2^* & a_3 a_3^* + a_4 a_4^* \end{bmatrix}$$

$\therefore$ We have

$$(U\otimes W)(U\otimes W)^{+} = (U\otimes W)^{+}(U\otimes W) = I$$

c) Let A be an $m\times n$ matrix
B be a $p\times q$ matrix

Let the i,jth element of A be $A_{i,j}$
Let the k,lth element of B be $B_{k,l}$

$A\otimes B$ has mp rows and pq columns.
Let an element of $i.m+k$th row, and $j.n+l$th col be indexed as $(A\otimes B)_{ik,jl}$

We have $\quad A\otimes B_{ik,jl} = (A_{i,j})(B_{k,l})$

$$(A\otimes B)^{T}_{ik,jl} = A\otimes B_{jl,ik}$$

$$= (A_{j,i})(B_{l,k})$$

Fixed dim proof only = 1 mark

$$= (A^{T})_{i,j}\cdot(B^{T})_{k,l}$$

Full

Block matrix mult allowed.

$$= (A^{T}\otimes B^{T})_{ik,jl}$$

$$\therefore (A\otimes B)^{T} = A^{T}\otimes B^{T}$$

0.5 if both (a) and (c) are proved ? as (c) is needed for (a).

SET 2

~~Q2~~ correct state mistaken state correct − mistaken

(2) $S(|\psi\rangle\langle\psi| \,\|\, \rho)$ Formula = 1

$= Tr(|\psi\rangle\langle\psi| \log |\psi\rangle\langle\psi|) - Tr(|\psi\rangle\langle\psi| \log \rho)$

$= 0 - Tr(|\psi\rangle\langle\psi| \log \rho)$ 1

$= - \langle\psi| \log \rho |\psi\rangle$

$\log \rho = \begin{pmatrix} \log 1/2 & & & \\ & \log 0 & & \\ & & \log 0 & \\ & & & \log 1/2 \end{pmatrix}$ =1

$= - \begin{pmatrix} 1 & & & \\ & \log 0 & & \\ & & \log 0 & \\ & & & 1 \end{pmatrix}$

$0 \cdot \log 0 = 0$ 1

$- \langle\psi| \log \rho |\psi\rangle$

$= \frac{\langle 00| + \langle 11|}{\sqrt{2}} \left(|00\rangle\langle 00| + |11\rangle\langle 11| + \log 0\, |01\rangle\langle 01| + \log 0\, |10\rangle\langle 10| \right) \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

$= \frac{1}{2}(1 + 1 + 0 + 0)$

$= 1$ //

OR

$\frac{1}{2}(\ln 2 + \ln 2)$

$= \underline{0.693}$ 1

Other way also full marks Formula = 1

$$S\left(\rho \,\|\, |\psi\rangle\langle\psi|\right)$$

$$\left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00|+\langle 11|}{\sqrt{2}}\right)\left(\frac{|00\rangle-|11\rangle}{\sqrt{2}}\right)$$

1.5

$$= \frac{|00\rangle+|11\rangle}{\sqrt{2}} \quad 0 \quad = 0$$

$$\frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)\left(\frac{|00\rangle-|11\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\left(\frac{|00\rangle}{\sqrt{2}} - \frac{|11\rangle}{\sqrt{2}}\right)$$

1.5

$$= \frac{|00\rangle-|11\rangle}{2\sqrt{2}}$$

as kernel contains above vector.

$\therefore$ kernel of $|\psi\rangle\langle\psi| \cap$ support of $\rho \neq$ null

$\therefore$ Relative entropy $= +\infty$. 1

Correct ans with formula but didn't take care of log

2

Only written any 1 tensor prod = 0.5 marks

Q7)

$$\rho_{12} = p\, I/4 + (1-p)\, |\psi\rangle\langle\psi|$$

$$= \frac{p}{4}\left(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11| \right)$$

$$+ \frac{(1-p)}{1+|n|^2}\left(|00\rangle\langle 00| + n^* |00\rangle\langle 11| + n|11\rangle\langle 00| + |n|^2\, |11\rangle\langle 11| \right)$$

$$= \left(\frac{p}{4} + \frac{1-p}{1+|n|^2}\right)|00\rangle\langle 00| + \frac{p}{4}|01\rangle\langle 01| + \frac{p}{4}|10\rangle\langle 10|$$

$$+ \frac{(1-p)}{1+|n|^2} n^* |00\rangle\langle 11| + \frac{(1-p)}{1+|n|^2} n\, |11\rangle\langle 00| + \left(\frac{p}{4} + \frac{(1-p)|n|^2}{1+|n|^2}\right)|11\rangle\langle 11|$$

let us rename

$$\alpha_p = \frac{p}{4} + \frac{1-p}{1+|n|^2}$$

$$\beta_p = \frac{p}{4}$$

$$\gamma_p = \frac{1-p}{1+|n|^2} n^*$$

$$\delta_p = \frac{1-p}{1+|n|^2} n$$

$$\Omega_p = \frac{p}{4} + \frac{(1-p)|n|^2}{1+|n|^2}$$

$$\therefore\ \rho_{12} = \alpha_p |00\rangle\langle 00| + \beta_p |01\rangle\langle 01| + \beta_p |10\rangle\langle 10| + \gamma_p |00\rangle\langle 11| + \delta_p |11\rangle\langle 00| + \Omega_p |11\rangle\langle 11|$$

$$\rho_{34} = \alpha_q |00\rangle\langle 00| + \beta_q |01\rangle\langle 01| + \beta_q |10\rangle\langle 10| + \gamma_q |00\rangle\langle 11| + \delta_q |11\rangle\langle 00| + \Omega_q |11\rangle\langle 11|$$

$$(\rho_{12} \otimes \rho_{34})_{\underline{\underline{23\ 14}}} = \alpha_p\alpha_q\, |00\rangle\langle 00| \otimes |00\rangle\langle 00| + \alpha_p\beta_q |00\rangle\langle 00| \otimes |01\rangle\langle 01|$$

$$+ \alpha_p\beta_q\, |01\rangle\langle 01| \otimes |00\rangle\langle 00| + \alpha_p\gamma_q \otimes |00\rangle\langle 01| \otimes |00\rangle\langle 01| +$$

$$\Downarrow$$

$$(\underline{\underline{36 \text{ terms}}})$$

$$|\psi\rangle \otimes |\phi\rangle = \frac{1}{\sqrt{(1+|n|^2)(1+|m|^2)}} |0000\rangle + m|0011\rangle + n|1100\rangle + nm|1111\rangle$$

1

$$|\phi^+_{23}\rangle \left(|00\rangle + nm|11\rangle\right)$$

$$|\phi^-\rangle_{23} \left(|00\rangle - nm|11\rangle\right)$$

$$|\psi^+\rangle_{23} \left(m|01\rangle + n|10\rangle\right)$$

$$|\psi^-\rangle_{23} \left(m|01\rangle - n|10\rangle\right)$$

Till here (4 marks)

Measure
→ 1 of the outcomes → Full 5 marks!

Any effort in right direction = 1 mark.

Replaced any 3 correctly with Bell state = 2 marks.

$$|00\rangle = \frac{|\phi^+\rangle + |\phi^-\rangle}{\sqrt{2}} \qquad |11\rangle = \frac{|\phi^+\rangle - |\phi^-\rangle}{\sqrt{2}}$$

$$|01\rangle = \frac{|\psi^+\rangle + |\psi^-\rangle}{\sqrt{2}} \qquad |10\rangle = \frac{|\psi^+\rangle - |\psi^-\rangle}{\sqrt{2}}$$

Any valid info on ent swap = 1 mark.

Q12) PROOF 1

$$e^{i\alpha\sigma_x} = I\cos\alpha + i\sigma_x \sin\alpha$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= U\, g\, U^+$$

$$= \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad 1$$

$$e^{i\alpha\sigma_x} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{i\alpha} & \\ & e^{-i\alpha} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{i\alpha} & e^{i\alpha} \\ e^{-i\alpha} & -e^{-i\alpha} \end{pmatrix} \quad 1$$

$$= \tfrac{1}{2}\begin{pmatrix} e^{i\alpha}+e^{-i\alpha} & e^{i\alpha}-e^{-i\alpha} \\ e^{i\alpha}-e^{-i\alpha} & e^{i\alpha}+e^{-i\alpha} \end{pmatrix}$$

$$= \tfrac{1}{2}\begin{pmatrix} 2\cos\alpha & 2i\sin\alpha \\ 2i\sin\alpha & 2\cos\alpha \end{pmatrix}$$

$$= \cos\alpha\, I + i\sin\alpha\,\sigma_x .$$

PROOF 2

0.5

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$$

$$1 = I + \frac{i\alpha\sigma_x}{1!} + \frac{(i\alpha\sigma_x)^2}{2!} + \frac{(i\alpha\sigma_x)^3}{3!} \dots$$

$$1 = I\left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} \dots\right) + i\sigma_x\left(\frac{\alpha}{1!} - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} \dots\right)$$

$$= I\cos\alpha + i\sigma_x \sin\alpha$$

0.5 b) ii)
$$e^{2i\sigma x} = I\cos(2) + i\sigma_x \sin(2)$$

$$= \begin{bmatrix} \cos 2 & i\sin 2 \\ i\sin 2 & \cos 2 \end{bmatrix}$$

(iii)
$$\sigma_x = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\left(e^{\sigma x}\right)^2 = e^{2\sigma x} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^2 & 0 \\ 0 & 1/e^2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^2 & e^2 \\ 1/e^2 & -1/e^2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^2+1/e^2 & e^2-1/e^2 \\ e^2-1/e^2 & e^2+1/e^2 \end{bmatrix}$$ — 0.5

(i) $$e^{\sigma_x} = \frac{1}{2}\begin{bmatrix} e+1/e & e-1/e \\ e-1/e & e+1/e \end{bmatrix}$$ — 0.5

$$\sigma_y = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

$$e^{\sigma_y} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & 1/e \end{bmatrix}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} e & -ie \\ 1/e & i/e \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e+1/e & -ie+i/e \\ ie-i/e & e+1/e \end{bmatrix}$$ — 0.5

$e^{\sigma_x}\cdot e^{\sigma_y}$ — 0.5

$$= \frac{1}{4}\begin{bmatrix} (e+1/e)^2 + i(e-1/e)^2 & i\left(-e+\frac{1}{e}\right)(e+1/e) + (e-1/e)(e+1/e) \\ (e+1/e)(e-1/e) + i(e+1/e)(e-1/e) & i(-e+1/e)(e-1/e) + (e+1/e)^2 \end{bmatrix}$$

Other allowed answer:

$$\frac{e+e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} \cdots$$

$$\frac{e-e^{-1}}{2} = 1 + \frac{1}{3!} + \frac{1}{5!} \cdots$$

$$e^{\sigma_x} = I\left(\frac{e+1/e}{2}\right) + \sigma_x\left(\frac{e-1/e}{2}\right) \longrightarrow 0.5$$

$$e^{\sigma_y} = I\left(\frac{e+1/e}{2}\right) + \sigma_y\left(\frac{e-1/e}{2}\right) \longrightarrow 0.5$$

Prod = 0.5

square = 0.5 //

Not fully simplified / complex exponents (partial marks)