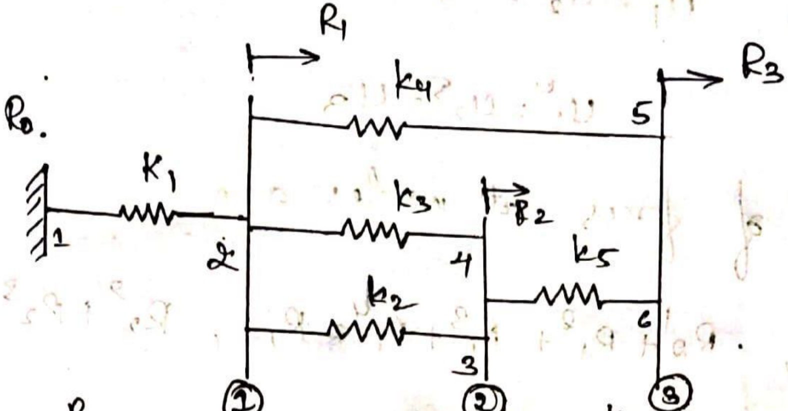
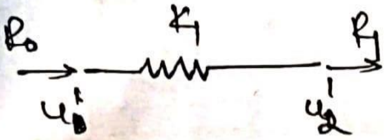
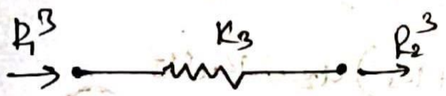
Q. 1)

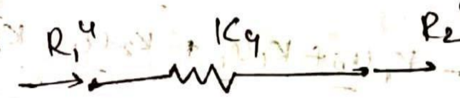




$$\begin{Bmatrix} R_1^1 \\ R_2^1 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{Bmatrix} u_1^1 \\ u_2^1 \end{Bmatrix}$$

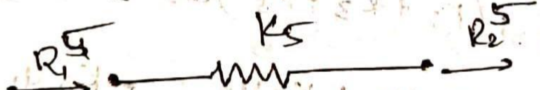
$$\begin{Bmatrix} R_2^1 \\ R_2^2 \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1^2 \\ u_2^2 \end{Bmatrix}$$





$$\begin{Bmatrix} R_1^3 \\ R_2^3 \end{Bmatrix} = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1^3 \\ u_2^3 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1^4 \\ R_2^4 \end{Bmatrix} = \begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} u_1^4 \\ u_2^4 \end{Bmatrix}$$



$$\begin{Bmatrix} R_1^5 \\ R_2^5 \end{Bmatrix} = \begin{bmatrix} k_5 & -k_5 \\ -k_5 & k_5 \end{bmatrix} \begin{Bmatrix} u_1^5 \\ u_2^5 \end{Bmatrix}$$

Above equations are elemental static equilibrium equation in terms of local displacements.

Compatibility ~~equation~~ condition to relate local and global displacements

$$u_1^1 = u_0, \quad u_1^2 = u_2^1 = u_1^3 = u_1^4 = u_1, \quad u_2^3 = u_2^2 = u_1^5 = u_2$$

$$u_2^4 = u_2^5 = u_3$$

Equilibrium of forces at nodes are

$$R_1^1 = R_0, \quad R_2^1 + R_1^2 + R_1^3 + R_1^4 = R_1, \quad R_2^2 + R_2^3 + R_1^5 = R_2$$

$$R_2^4 + R_2^5 = R_3$$

∴ Equilibrium forces can be written as

$$R_0 = K_1 u_0 - K_1 u_1 \longrightarrow (1)$$

$$R_1 = -K_1 u_0 + K_1 u_1 + K_2 u_1 - K_2 u_2 + K_3 u_1 - K_3 u_2 + K_4 u_1 - K_4 u_3$$

$$R_1 = -K_1 u_0 + (K_1 + K_2 + K_3 + K_4) u_1 - (K_2 + K_3) u_2 - K_4 u_3 \longrightarrow (2)$$

$$R_2 = -K_2 u_1 + K_2 u_2 - K_3 u_1 + K_3 u_2 + K_5 u_2 - K_5 u_3$$

$$R_2 = -(K_2 + K_3) u_1 + (K_2 + K_3 + K_5) u_2 - K_5 u_3 \longrightarrow (3)$$

$$R_3 = -K_4 u_1 + K_4 u_3 - K_5 u_2 + K_5 u_3 = -K_4 u_1 - K_5 u_2 + (K_4 + K_5) u_3 \longrightarrow (4)$$

∴ Equilibrium equation of system is (from eqn (1)–(4))

$$\begin{Bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 & 0 & 0 \\ -K_1 & (K_1+K_2+K_3+K_4) & -(K_2+K_3) & -K_4 \\ 0 & -(K_2+K_3) & K_2+K_3+K_5 & -K_5 \\ 0 & -K_4 & -K_5 & K_4+K_5 \end{bmatrix} \begin{Bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Considering element ①, ② & ③, which are movable, the equilibrium equation of system is written as

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{bmatrix} K_1+K_2+K_3+K_4 & -(K_2+K_3) & -K_4 \\ -(K_2+K_3) & K_2+K_3+K_5 & -K_5 \\ -K_4 & -K_5 & K_4+K_5 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\{F\} = [K]\{u\}$$

By energy method

WKT, Potential Energy = Strain Energy − Workdone.

∴ for each element

$$\pi_1 = \frac{1}{2} K_1 (u_2^1 - u_1^1) - R_1^1 u_1^1 - R_2^1 u_2^1$$

$$\pi_2 = \frac{1}{2} K_2 (u_2^2 - u_1^2) - R_1^2 u_1^2 - R_2^2 u_2^2$$

$$\pi_3 = \frac{1}{2} K_3 (u_2^3 - u_1^3) - R_1^3 u_1^3 - R_2^3 u_2^3$$

$$\pi_4 = \frac{1}{2} K_4 (u_2^4 - u_1^4) - R_1^4 u_1^4 - R_2^4 u_2^4$$

$$\pi_5 = \frac{1}{2} K_5 (u_2^5 - u_1^5) - R_1^5 u_1^5 - R_2^5 u_2^5$$

compatibility condition of local & global displacement relation is

$$u_1^1 = u_0 , \quad u_2^1 = u_1^2 = u_1^3 = u_1^4 = u_1 , \quad u_2^2 = u_2^3 = u_1^5 = u_2$$

$$u_2^4 = u_2^5 = u_3$$

Equilibrium forces,

$$R_1^1 = R_0 , \quad R_1 = R_2^1 + R_1^2 + R_1^3 + R_1^4 , \quad R_2 = R_2^2 + R_2^3 + R_2^5$$

$$R_3 = R_2^4 + R_2^5$$

$\therefore$ Above equation becomes,

$$\pi = \frac{1}{2}\Big[K_1(u_1-u_0)^2 + K_2(u_2-u_1)^2 + K_3(u_2-u_1)^2 + K_4(u_3-u_1)^2 + K_5(u_3-u_2)^2\Big]$$
$$+(-R_0u_0) - (R_2^1+R_1^2+R_1^3+R_1^4)u_1 - (R_2^2+R_2^3+R_1^5)u_2 - (R_2^4+R_2^5)u_3$$

$$\pi = \frac{1}{2}\Big[K_1(u_1-u_0)^2 + K_2(u_2-u_1)^2 + K_3(u_2-u_1)^2 + K_4(u_3-u_1)^2 + K_5(u_3-u_2)^2\Big]$$
$$-R_0u_0 - R_1u_1 - R_2u_2 - R_3u_3$$

$\therefore$ Differentiating (partial differentiation) of above equation

$$\frac{\partial\pi}{\partial u_0} = -(u_0-u_1)K_1 - R_0 = 0 \Rightarrow R_0 = K_1u_0 - K_1u_1 \longrightarrow (1)$$

$$\frac{\partial\pi}{\partial u_1} = (u_1-u_0)K_1 + (u_1-u_2)K_2 + K_3(u_1-u_2) + K_4(u_1-u_3) - R_1 = 0$$

$$R_1 = u_0K_1 + (K_1+K_2+K_3+K_4)u_1 - (K_2+K_3)u_2 - K_4u_3 \longrightarrow (2)$$

$$\frac{\partial\pi}{\partial u_2} = (u_2-u_1)K_2 + (u_2-u_1)K_3 + (u_2-u_3)K_5 - R_2 = 0$$

$$R_2 = -(K_2+K_3)u_1 + (K_2+K_3+K_5)u_2 - K_5u_3 \longrightarrow (3)$$

$$\frac{\partial\pi}{\partial u_3} = (u_3-u_1)K_4 + (u_3-u_2)K_5 - R_3 = 0$$

$$R_3 = -K_4u_1 - K_5u_2 + (K_4+K_5)u_3 \longrightarrow (4)$$

from above equation we can write equilibrium equation of system as

$$[F] = [K]\{u\}$$

$$\begin{Bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 & 0 & 0 \\ -K_1 & K_1+K_2+K_3+K_4 & -(K_2+K_3) & -K_4 \\ 0 & -(K_2+K_3) & K_2+K_3+K_5 & -K_5 \\ 0 & -K_4 & -K_5 & (K_4+K_5) \end{bmatrix} \begin{Bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

for 3 element

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{bmatrix} K_1+K_2+K_3+K_4 & -(K_2+K_3) & -K_4 \\ -(K_2+K_3) & K_2+K_3+K_5 & -K_5 \\ -K_4 & -K_5 & K_4+K_5 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\{F\} = [K]\{u\}$$

2. Matlab Code

```
%..............................................................
% MATLAB codes for Finite Element Analysis
% Discrete_Systems.m


% clear memory
clear all


% elementNodes: connections at elements


elementNodes=[1 2;2 3;2 3;2 4;3 4];


% numberElements: number of Elements


numberElements=size(elementNodes,1);


% By using the MATLAB function size, that returns the number
%of lines and columns of a rectangular matrix, we can detect  %the
number of elements by inspecting the number of lines of  %matrix
elementNodes.


% numberNodes: number of nodes


numberNodes=4;


% for structure:
% displacements: displacement vector
% force : force vector
% stiffness: stiffness matrix


% initializing the matrices


displacements=zeros(numberNodes,1);
force=zeros(numberNodes,1);
stiffness=zeros(numberNodes);


% applied load at node
2 force(2)=1;
force(3)=1; force(4)=1;
```

```
% computation of the system stiffness matrix
% We compute now the stiffness matrix for each element in turn and then assemble %
it in the global stiffness matrix.


for e=1:numberElements;
% elementDof: element degrees of freedom (Dof) elementDof=elementNodes(e,:)
;
stiffness(elementDof,elementDof)=stiffness(elementDof,elementDof)+[1 -1;-1 1] end


% boundary conditions and solution
% prescribed dofs
prescribedDof=1; %
free Dof : activeDof
activeDof=setdiff([1:numberNodes],[prescribedDof]);


% solution
displacements=stiffness(activeDof,activeDof)\force(activeDof);
% positioning all displacements
displacements1=zeros(numberNodes,1);
displacements1(activeDof)=displacements;
% output displacements/reactions
outputDisplacementsReactions(displacements1,stiffness,numberNodes,prescribedDof)

%............................................................. function
outputDisplacementsReactions...
(displacements,stiffness,GDof,prescribedDof)
% output of displacements and reactions in
% tabular form
% GDof: total number of degrees of freedom of
% the problem %
displacements
disp('Displacements')
%displacements=displacements1; jj=1:GDof;
format
[jj' displacements]


% reactions
F=stiffness*displacements;
reactions=F(jj); disp('reactions')
[jj' reactions]


end
%...................................................................
```

Output

Stiffness ans
= 4×4

1	-1	0	0
-1	4	-2	-1
0	-2	3	-1
0	-1	-1	2

Displacements

ans = 4×2

1.0000	0
2.0000	3.0000
3.0000	3.6000
4.0000	3.8000

Reactions

ans = 4×2

1.0000	-3.0000
2.0000	1.0000
3.0000	1.0000
4.0000	1.0000

Displacements:

Displacement of node 2 = 3

Displacement of node 3 = 3.60

Displacement of node 4 = 3.80

Reaction Forces:

Reaction force at node 1 = -3

2. (b) Given $K_1 = K_2 = K_3 = K_4 = K_5 = 1$.

$R_1 = R_2 = R_3 = 1$.

from equilibrium equation of system,

$$\begin{Bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 & 0 & 0 \\ -K_1 & K_1+K_2+K_3+K_4 & -(K_2+K_3) & -K_4 \\ 0 & -(K_2+K_3) & K_2+K_3+K_5 & -K_5 \\ 0 & -K_4 & -K_5 & K_4+K_5 \end{bmatrix} \begin{Bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} R_0 \\ 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -2 & -1 \\ 0 & -2 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$u_0 = 0$.
($\because$ fixed at 0)

$$\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{bmatrix} 4 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 4 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$R_0 = -u_1$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 3.6 \\ 3.8 \end{Bmatrix}$$

$u_1 = 3$

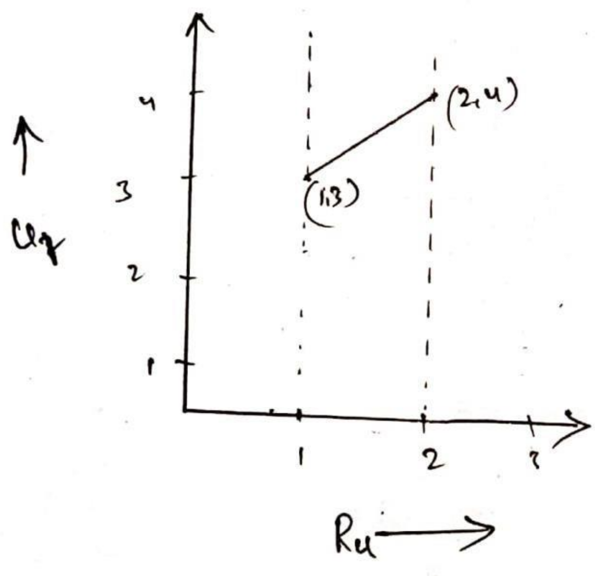
$\therefore \boxed{R_0 = -3}$

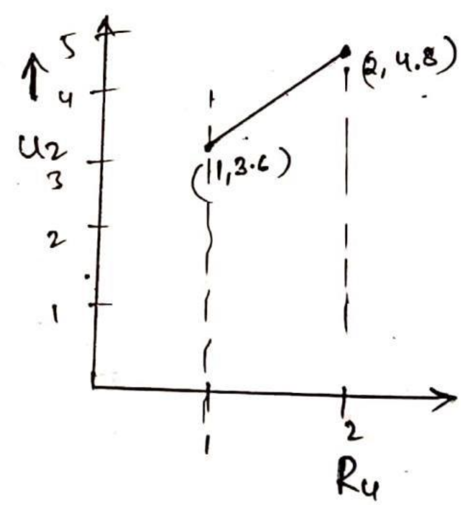
(c) Given $R_1 = R_2 = 1$ $\qquad R_3 \rightarrow 1 - 2$

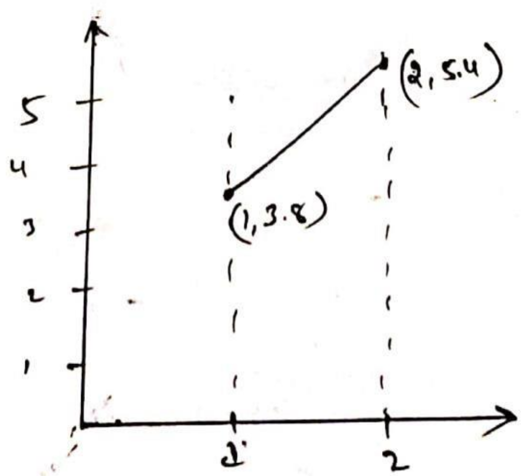
$$\begin{Bmatrix} 1 \\ 1 \\ R_3 \end{Bmatrix} = \begin{bmatrix} 4 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.4 & 1.2 \\ 1 & 1.2 & 1.6 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ R_3 \end{Bmatrix}$$

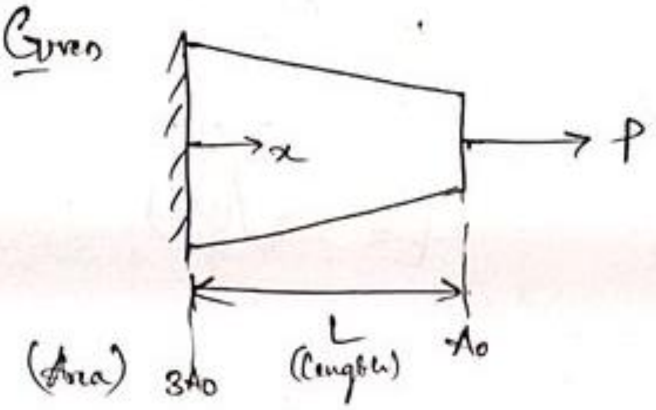
$u_1 = 2 + R_3$, $u_2 = 2.4 + 1.2 R_3$, $u_3 = 2.2 + 1.6 R_4$



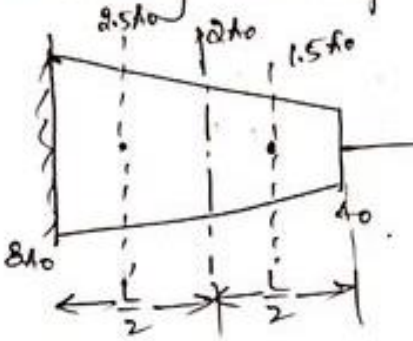


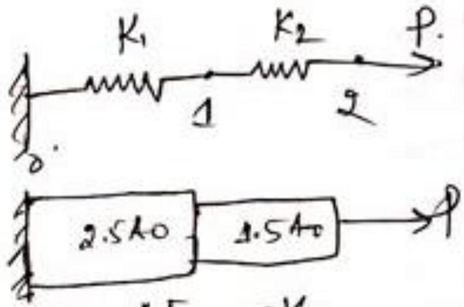


1. Given



(a) Considering two spring elements

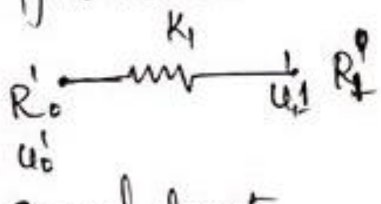




WKT, $K_0 = \frac{A_0E}{L}$, $K_1 = \frac{2.5A_0E}{L/2} = \frac{5}{2}\times 2\frac{A_0E}{L} = \underline{5K_0}$

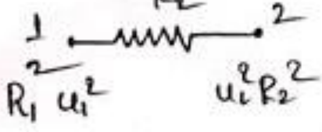
$$K_2 = \frac{1.5A_0E}{L/2} = \frac{3}{2}\times 2\frac{A_0E}{L} = \underline{3K_0}.$$

first element



$$\begin{Bmatrix} R_0^1 \\ R_1^1 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{Bmatrix} u_0^1 \\ u_1^1 \end{Bmatrix}$$

Second element



$$\begin{Bmatrix} R_1^2 \\ R_2^2 \end{Bmatrix} = \begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_1^2 \\ u_2^2 \end{Bmatrix}$$

Here, $u_0^1 = u_0$, $u_1^1 = u_1^2 = u_1$, $u_2^2 = u_2$

$$R_0^1 = R_0, \quad R_1^1 + R_1^2 = R_1, \quad R_2^2 = R_2$$

$$R_0 = K_1u_0 - K_1u_1 \longrightarrow (i).$$

$$R_1^1 + R_1^2 = R_1 = -K_1 u_0 + K_1 u_1 + K_2 u_1 - K_2 u_2$$

$$R_1 = -K_1 u_0 + (K_1 + K_2) u_1 - K_2 u_2 \longrightarrow (2)$$

$$R_2^2 = R_2 = -K_2 u_1 + K_2 u_2 \longrightarrow (3)$$

from (1), (2) & (3),

$$\begin{Bmatrix} R_0 \\ R_1 \\ R_2 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & (K_1+K_2) & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_0 \\ u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

DKI, $K_1 = 5K_0$, $K_2 = 3K_0$ & $R_1 = 0$

$$\begin{Bmatrix} 0 \\ P \end{Bmatrix} = \begin{bmatrix} 8K_0 & -3K_0 \\ -3K_0 & 3K_0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$8K_0 u_1 - 3K_0 u_2 = 0$$

$$u_1 = \frac{3}{8} u_2$$

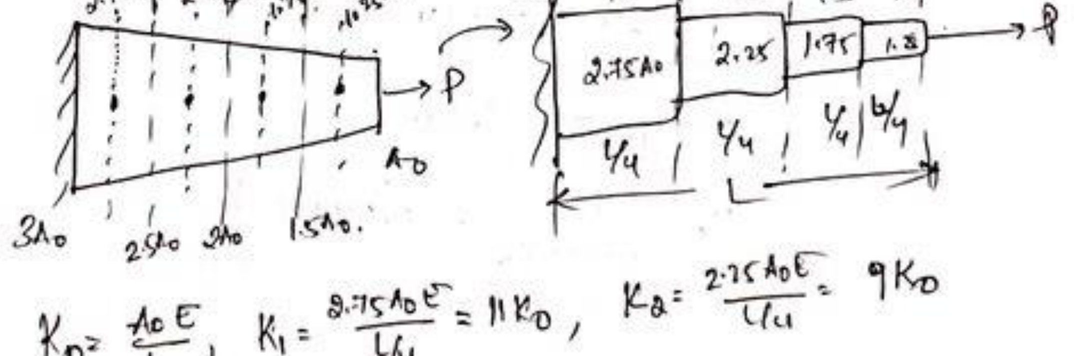
$$P = (-3u_1 + 3u_2) K_0$$

$$P = \left(-3 \times \frac{3}{8} + 3\right) u_2 K_0$$

$$P = \frac{15}{8} u_2 K_0$$

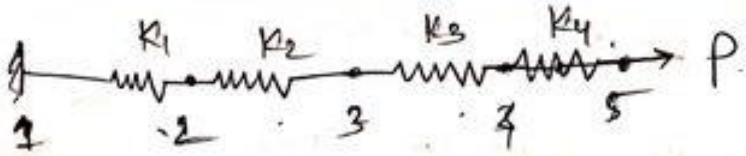
$$\boxed{u_2 = \frac{8P}{15K_0}} \quad u_2 = \cancel{\frac{P}{1.8K_0}} \quad u_2 = \frac{P}{1.875K_0} = 0.5333\frac{P}{K_0}$$

Considering four spring elements.



$$K_0 = \frac{A_0 E}{L}, \quad K_1 = \frac{2.75 A_0 E}{L/4} = 11K_0, \quad K_2 = \frac{2.25 A_0 E}{L/4} = 9K_0$$

$$K_3 = (1.75 \times 4) K_0 = 7K_0, \quad K_4 = 5K_0.$$



$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 & 0 & 0 & 0 \\ -K_1 & K_1+K_2 & -K_2 & 0 & 0 \\ 0 & -K_2 & K_2+K_3 & -K_3 & 0 \\ 0 & 0 & -K_3 & K_3+K_4 & -K_4 \\ 0 & 0 & 0 & -K_4 & K_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$$

$$\begin{Bmatrix} R_2 \\ R_3 \\ R_4 \\ R_5 \end{Bmatrix} = K_0 \begin{bmatrix} 20 & -9 & 0 & 0 \\ -9 & 16 & -7 & 0 \\ 0 & -7 & 12 & -5 \\ 0 & 0 & -5 & 5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$$

$R_2 = R_3 = R_4 = 0$; $R_5 = P$

$20u_2 = 9u_3$, $-9u_2 + 16u_3 - 7u_4 = 0 \rightarrow (1)$,

$-7u_3 + 12u_4 - 5u_5 = 0 \rightarrow (2)$, $P = (-5u_4 + 5u_5)K_0 \rightarrow (3)$

$u_2 = \frac{9}{20}u_3$, from (1), $-9\left(\frac{9}{20}\right)u_3 + 16u_3 = 7u_4$

$\frac{239}{20}u_3 = 7u_4$

from (2),

$(-7(0.585) + 12)u_4 = 5u_5$ $\quad u_3 = 0.585u_4$

$u_4 = 0.6325\,u_5$

from (3), $P = (-5(0.6325) + 5)u_5 K_0$

$P = 1.8375\,u_5 K_0$

$u_5 = \frac{P}{1.8375 K_0} = 0.54421 \frac{P}{K_0}$

b. Exact sol^n, $u = \frac{PL\ln(3)}{2EA_0} = 0.5493\ P/K_0$.

Percentage (%) error $= \frac{0.5493 - 0.54421}{0.5493} \times 100$.

$= 0.928\%$ error for $\underline{4 \text{ spring elements}}$

$= \frac{0.5493 - 0.5333}{0.5493} \times 100$

$= 2.912\%$ error for $\underline{2 \text{ spring}}$ elements.

C. Matlab code for determining the displacement at end of the bar for linear elements

```
clc;
clear all;
A1 = 30;
A2 = 10;
l = 100;
P = 100;
E = 200000;
n = input('Input array of number of elements')
disp_at_end = [];
for i = 1:length(n)
    node = zeros(n(i),2);
    for j = 1:n(i)
        node(j,:) = [j j+1];
    end
    le = l/n(i);
    n1 = max(node,[],'all');
    force = zeros(n1,1);
    stiffness = zeros(n1);
    displacement = zeros(n1,1);
    g = 1;
    h = setdiff(1:n1,g);
    r = n1;
    force(r,1) = P;
    reduced_force = force(h, :);
    for j = 1:n(i)
         t = 3 - (((2*j)-1)/n(i));
         T = (E*A2*t)/le;
         r = node(j,1); % left node no. of element j
```

```
        s = node(j,2); % right node no. of element j
        stiffness(r,r) = stiffness(r,r) + T;
        stiffness(s,s) = stiffness(s,s) + T;
        stiffness(r,s) = stiffness(r,s) - T;
        stiffness(s,r) = stiffness(s,r) - T;
    end
    disp_nodes = g;
    f_nodes = setdiff(1:n1,disp_nodes);
    f_stiffness = stiffness;
    f_stiffness = f_stiffness(:,f_nodes);
    f_stiffness = f_stiffness(f_nodes,:);
    unknown_disp = inv(f_stiffness)*reduced_force;
    for j = 1:length(f_nodes)
        displacement(f_nodes(j)) = unknown_disp(j);
    end
    disp_at_end(end+1) = displacement(n1);
    fprintf('\nLinear elements %d',n(i));
    fprintf('Displacement is %d',displacement(n1))
end
X = n;
Y = disp_at_end;
plot(X,Y);
grid on;
xlabel('Number of Elements')
ylabel('Dispacements at end')
```

Output

Input array of number of elements[2 4 6 8 10]

n =

  2 4 6 8 10

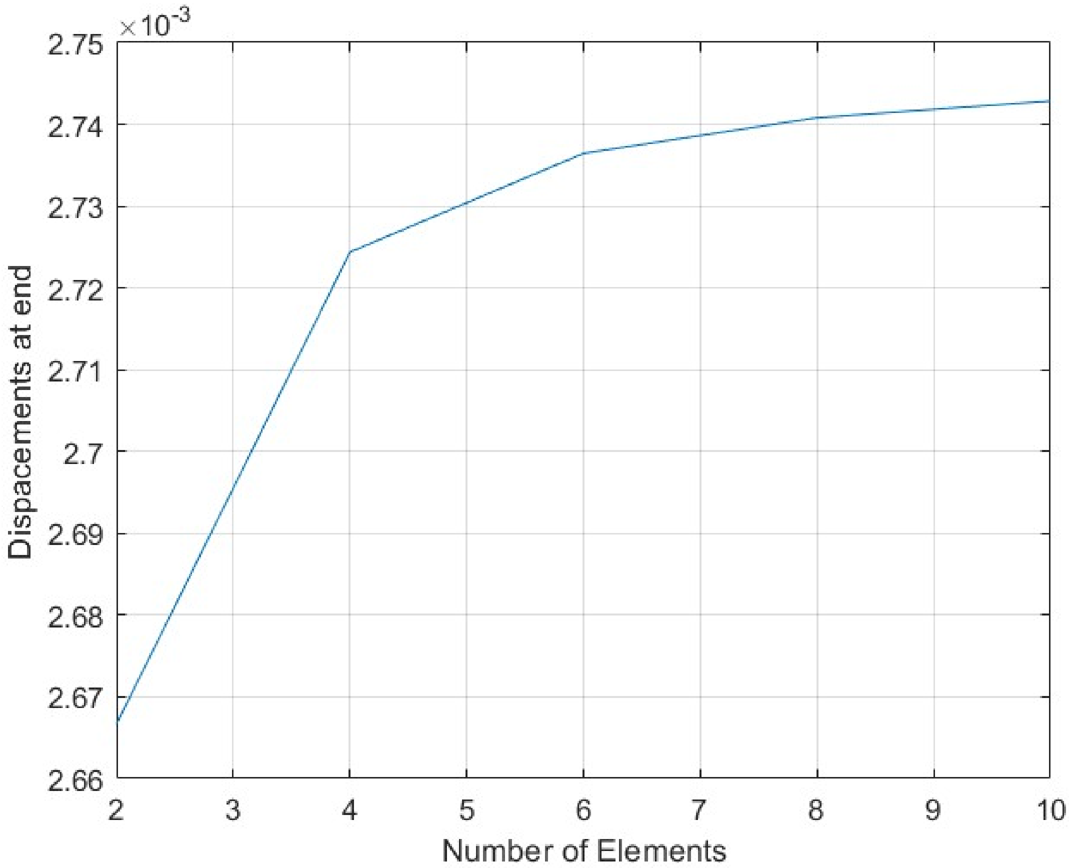
Linear elements 2   Displacement is 2.666667e-03

Linear elements 4   Displacement is 2.724387e-03

Linear elements 6   Displacement is 2.736453e-03

Linear elements 8   Displacement is 2.740812e-03

Linear elements 10   Displacement is 2.742855e-03



1) f.

As the number of elements increases, the accuracy tends to increase but at certain point for linear elements analysis the rate of convergence is decreasing as number of elements is increasing.