# Building a model for the Solar system


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**Abstract**

*In this project we will make a model to simulate the dynamics of the planets in the solar system. We will use 4th order Runge Kutta (RK4) to calculate the position and velocity of the objects from their initial conditions. The goal of this project is to write an object oriented program. By writing classes which can perform a general task we can use it multiple times throughout the project and obtain a logical flow in the way we write the project. Using our program we will investigate properties of the planets and their orbits, and specifically pay attention to the conservation of momentum and energy. The C++ sourcecode for this project has a lot of content so I have chosen to exclude a "Code Details" section and rather added more comments directly in the code than usual. Details can be found at www.github.com/vegardbs/something*


## I. Introduction

The only force acting between the objects in our model solar system is the Newtonian force of gravity. When the Earth orbits the Sun each object is attracted to each other by an equal and oppositely directed force, $F_g$, which is given by

$$F_G = \frac{GM_{\mathrm{S}}M_{\mathrm{E}}}{r^2}, \tag{1}$$

Here $G = 6.67384 \times 10^{-11} \frac{m^3}{kgs^2}$ is the universal gravitational constant, $M_S$ and $M_E$ is the mass of the Sun and the Earth respectively, and $r$ is the distance between the two objects. We can reduce this two-body problem to a single body rotating about the origin by realizing that the mass of the Sun is far greater than the mass of the Earth and the motion of the Sun can be neglected.

The Earth orbits the Sun in the orbital plane and we take this to be the xy-plane. Newtons second law says that the sum of all forces acting on a body is equal to the mass of the object times its acceleration, that is $F_{res} = M \cdot a$. Combining this with *Eq.1* we obtain the following equations for the movement in the xy-plane.

$$\frac{d^2x}{dt^2} = \frac{F_G^x}{M_{\mathrm{E}}}, \tag{2}$$

and

$$\frac{d^2y}{dt^2} = \frac{F_G^x}{M_{\mathrm{E}}}, \tag{3}$$

We know that acceleration is the derivative of the velocity and that velocity is the derivative of position, thus we can rewrite these second order differential equations to two coupled first order equations.

$$v_x = \frac{dx}{dt}$$
$$\frac{dv_x}{dt} = \frac{F_G^x}{M_{\mathrm{E}}}$$

and identically for the y-coordinate. The Earth has a nearly circular orbit around the Sun and Newtons second law for centripetal forces gives

$$F_G = \frac{M_{\mathrm{E}}v^2}{r} = \frac{GM_{\mathrm{S}}M_{\mathrm{E}}}{r^2}, \tag{4}$$

with the result

$$v^2 r = GM_{\mathrm{S}} = 4\pi^2 \mathrm{AU}^3/\mathrm{yr}^2. \tag{5}$$

Later we will modify our program to handle three-body problems by including the planet Jupiter to our model Solar system. The addition of Jupiter will cause the orbit of the Earth to deviate and the goal is to find this deviation. This is done by adding the force between the Earth and Jupiter, which is given by

$$F_{\mathrm{E-J}} = \frac{GM_{\mathrm{J}}M_{\mathrm{E}}}{r_{\mathrm{E-J}}^2}, \tag{6}$$

Another thing that is important to remember while running our program is to check that energy an momentum is always conserved. That is we need to check that the values

$$E = \sum \frac{p_i^2}{m_i} + \sum \frac{GM_i}{r_{ij}}$$

and

$$L = r_i \times p_i$$

is always constant or at least does not deviate too far from the initial value.

## II. Methods

In this project we will use 4th order Runge Kutta (RK4). To show how we can solve a differential equation by RK4 we begin by considering the following definitions

$$\frac{dy}{dt} = f(t,y)$$

$$y(t) = \int f(t,y)dt$$

which we can discretize by writing

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f(t,y)dt$$

Now we Taylor expand $f(t,y)$ about the center of the integration interval, at $t_i + h/2$ where h is the step length. Using the shorthand $y(t_i + h/2) = y_{i+1/2}$ we obtain

$$y_{i+1} \int_{t_i}^{t_i+1} f(t,y)dt \simeq hf(t_{i+1/2}, y_{i+1/2}) + O(h^3)$$

To find the value of $y_{i+1/2}$ we use Euler's method to approximate it by

$$y_{i+1/2} = y_i + \frac{h}{2}\frac{dy}{dt} = y(t_i) + \frac{h}{2}f(t_i, y_i)$$

This is the basis of 2nd order Runge Kutta, and we define

$$k_1 = hf(t_i, y_i)$$

$$k_2 = hf(t_{i+1/2}, y + k_1/2)$$

with the result

$$y_{i+1} = y_i + k_2 + O(h^3) \tag{7}$$

RK4 is an improvement of this method where we use Simpson's rule to approximate the integral at $t_i + h/2$. The algorithm is as follows

1. We first compute the slope at $t_i$

   $$k_1 = hf(t_i, y_i)$$

2. Then we compute the slope at the midpoint by using Euler's method

   $$k_2 = hf(t_i + h/2, y_i + k_1/2)$$

3. Next we improve our value of the midpoint slope by computing

   $$k_3 = hf(t_i + h/2, y_i + k_2/2)$$

4. The end of the interval is completed by computing

   $$k_4 = hf(t_i + h, y_i + k_3)$$

5. and the final algorithm for RK4 becomes

   $$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{8}$$

## III. Results

Our first task is to investigate how the Earth orbits the Sun depending on the initial velocity. The real orbital velocity of the Earth is 29.8 km/s, and with that value we obtain the trajectory shown in *Fig.1*.

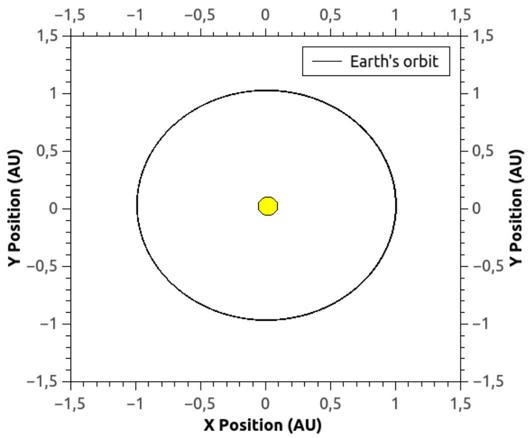


**Figure 1:** *The orbit of the Earth about the sun where $v_E = 29.8km/s$. The sun is fixed at the origin.*

We are interested in seeing how the initial velocity of the Earth affects its orbit, so we try with $v_E = 20km/s$.

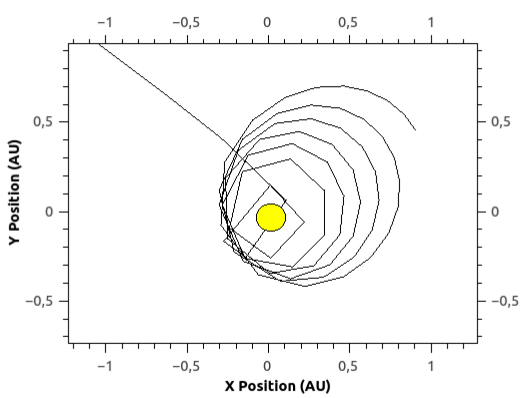


**Figure 2:** *The orbit of the Earth about the sun where $v_E = 20km/s$. The sun is fixed at the origin.*

As seen in *Fig.2*, this orbit is not stable and the Earth falls into the sun after only a few rotations. We see that the Earth is ejected far away from the Sun after they collide, but that is only because the Sun is a point particle in this program.

Now we try with $v_E = 40km/s$ and see what happens.

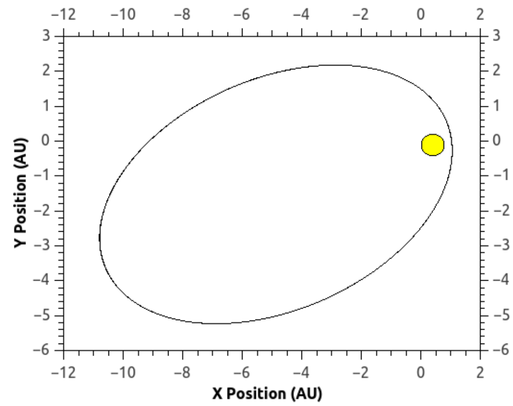


**Figure 3:** *The orbit of the Earth about the sun where $v_E = 40km/s$. The sun is fixed at the origin.*

The orbit is no longer circular, but follows an elliptical path. This orbit is stable, and the Earth neither crashes into the Sun nor is ejected away from it. We can calculate the escape velocity of the Earth from the Sun by

$$v_{escape} = \sqrt{\frac{2M_S G}{r_E}}$$

where $M_S$ is the mass of the Sun and $r_E$ is the distance between the Sun and the Earth. From this we obtain $v_{escape} = 42km/s$. Setting this as the initial velocity we obtain the orbit shown in *Fig.4*

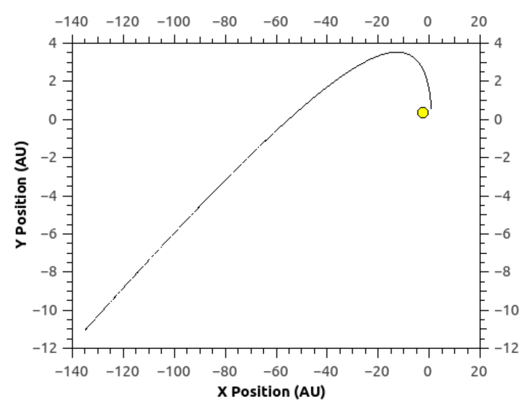


**Figure 4:** *The trajectory of the Earth with an initial velocity equal to the escape velocity. Here the Earth escapes the gravitational potential of the Sun and travels away indefinitely.*

Now we want to test the stability of our program as a function of the size of the time steps used.

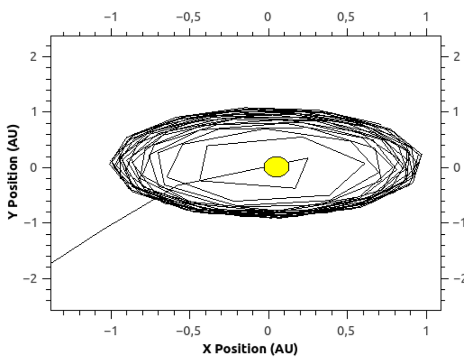


**Figure 5:** *The orbit of the Earth using a time step of 30 days.*

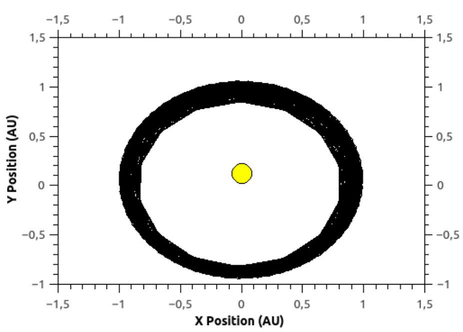


**Figure 6:** *The orbit of the Earth using a time step of 20 days.*

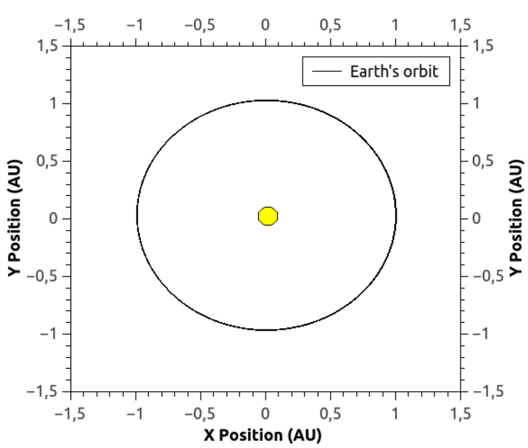


**Figure 7:** *The orbit of the Earth using a time step of 10 days.*

*Fig.5* through *Fig.7* shows the orbit of the Earth around the Sun fixed at the origin while using time steps of 30 days, 20 days and 10 days respectively. Using 30 days the orbit is not stable at all and the Earth eventually crashes into the Sun. With 20 days we see a clear improvement, however the orbit deviates quite a bit. Only by using a time step of 10 days or less do we obtain a satisfactory orbit with minimum deviations.

Before we add more planets to our solar system we want to makes sure that the momentum and energy of the system is conserved. Plotting the conserved quantities as functions of time, we obtain the plots in *Fig.8* and *Fig.9*. As expected both quantities are conserved. There are some oscillations in the plot for the angular momentum (*Fig.9*), but these oscillations exceeds the initial value as much as they deceed it, thus averaging out to zero.

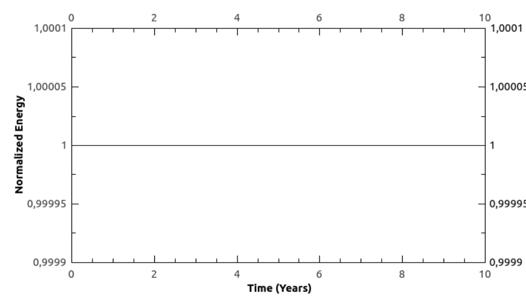


**Figure 8:** *The normalized energy of the system as a function of time*

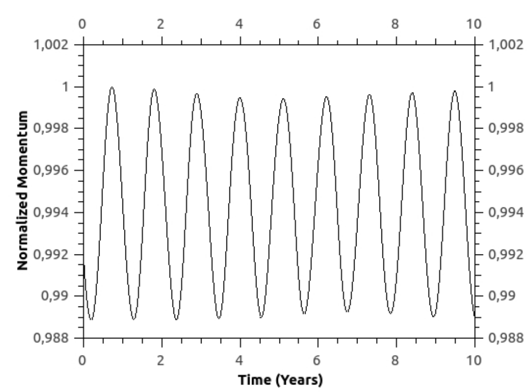


**Figure 9:** *The normalized angular momentum of the system as a function of time*

Now we want to add the planet Jupiter to our solar system and investigate its effect on the orbit of the Earth. In *Fig.10* and *Fig.11* we have plotted the orbit of the Earth with the orbit of Jupiter around a Sun fixed at the origin. In *Fig.10* we have used a 10 day timestep and in *Fig.11* we have used a 30 day timestep. Just as before the stability of the orbits is dependent on the length of the timesteps, and we in *Fig.11* that the orbit of the Earth becomes unstable and crashes into the Sun.

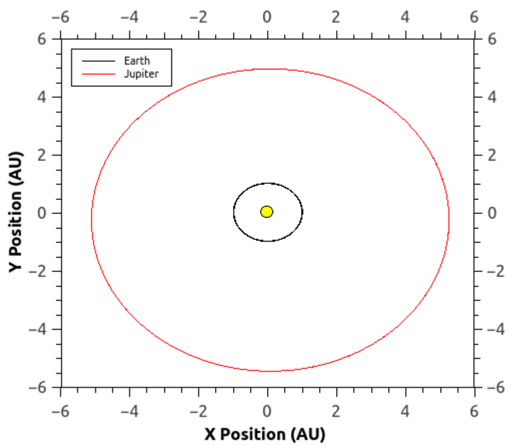


**Figure 10:** *The orbit of Earth and Jupiter using a 10 days timestep*

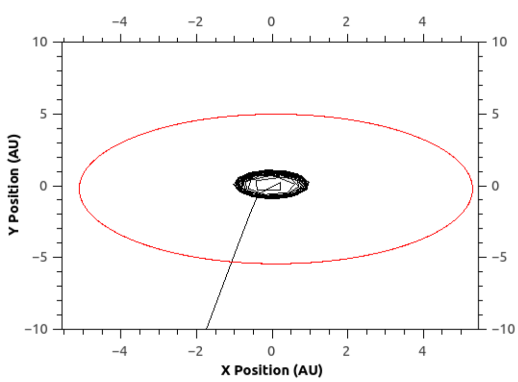


**Figure 11:** *The orbit of Earth and Jupiter using a 30 days timestep*

The orbit of the Earth is basically unchanged by the addition of Jupiter. The mass of the Sun is roughly 1000 times larger than the mass of Jupiter, thus the gravitational force between the Sun and Earth is much stronger than the gravitational force between Jupiter and Earth. We now increase the mass of Jupiter to see the effects on Earths orbit.

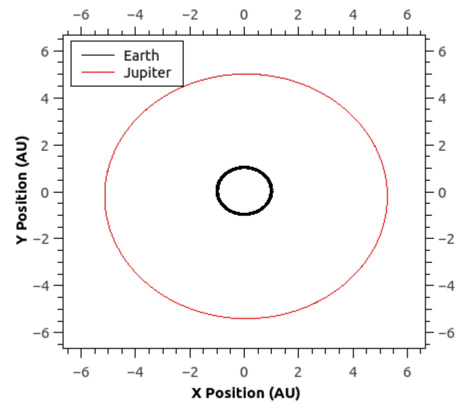


**Figure 12:** *The orbit of Earth and Jupiter using a 10 days timestep, with the mass of Jupiter 10 times as large as the real value.*

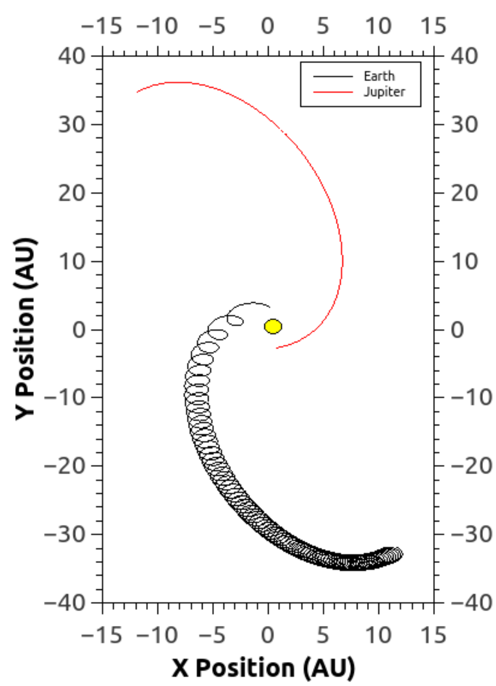


**Figure 13:** *The orbit of Earth and Jupiter using a 10 days timestep. with the mass of Jupiter 1000 times as large as the real value.*

As seen in *Fig.12*, increasing the mass of Jupiter by a factor of 10 makes the orbit of the Earth oscillate. The mass of Jupiter is now so large that the gravitational force between Jupiter and Earth pulls Earth out of its original orbit. In *Fig.13* the mass of Jupiter is comparable to the mass of the Sun (both are roughly $10^{30}$ kg). Since the Sun is still fixed both Jupiter and the Earth orbits the origin, but the orbits are now ellipses and the Earth obtains a secondary oscillating circular motion. When the Sun eclipses Jupiter from the Earths point of view, both objects pull the Earth towards it. As Jupiter moves out from behind the Sun the net force on the earth changes direction away from the origin. Then Jupiter moves behind the Sun again and the process is repeated.

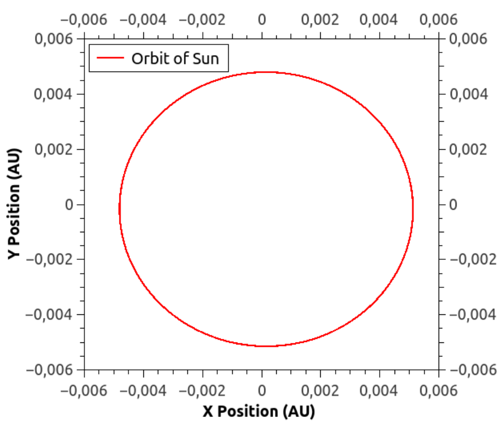


**Figure 14:** *The Sun orbits the center of mass at the origin for the solar system consisting of Earth and Jupiter*

Finally we release the Sun from its fixed position and observe what effect this has on our solar system. In *Fig.14* we plot the movement of the Sun in a solar system consisting of the Earth and Jupiter. We see that the Sun obtains a circular orbit of its own, but due to the high inertia of the Sun its orbital radius is far less than that of the planets. *Fig.15* shows the orbit of the Sun when we include all the planets of the solar system. There is a characteristic wobbling of the Sun about the centre of mass for the whole system. This wobbling is used by astrophysicists to determine if there are planets orbiting a star in a galaxy far far away.

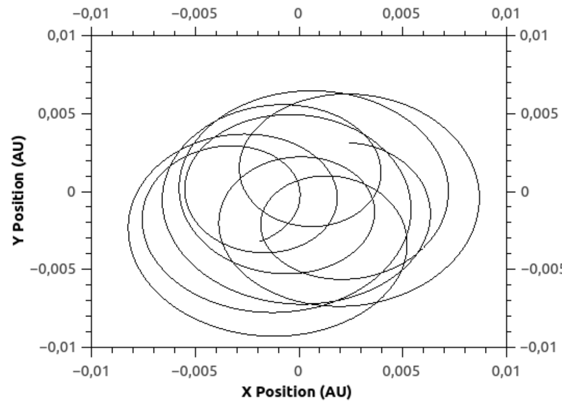


**Figure 15:** *The Sun orbits the center of mass at the origin for the solar system consisting of all the planets*

Lastly we plot the orbits of all the planets in the solar system in *Fig.16*. Here we see the different types of orbits ranging from the circular orbit of the inner planets to the elliptical orbits of the outer ones.

## IV. Conclusion

In this project we have constructed a solar system with many planetary objects. We created a program to solve the equations of motion for the different objects by rewriting the second order differential equation (*Eq.2* and *Eq.3*) to two coupled first order differential equations. By discretizing these equations we could solve them using RK4 as described in Section I. With our program we investigated the orbit of the planets under different conditions. We started by determining the escape velocity of the Earth orbiting a fixed Sun, and we tested the stability of its orbit as a function of the length of the time steps used in our computation. Always we made sure that the energy and angular moment of the total system was conserved. We then added Jupiter and observed increasing deviations in Earths orbit as a function of the mass of Jupiter. Releasing the Sun from its fixed position we found that also the Sun orbits a common centre of mass for the solar system.

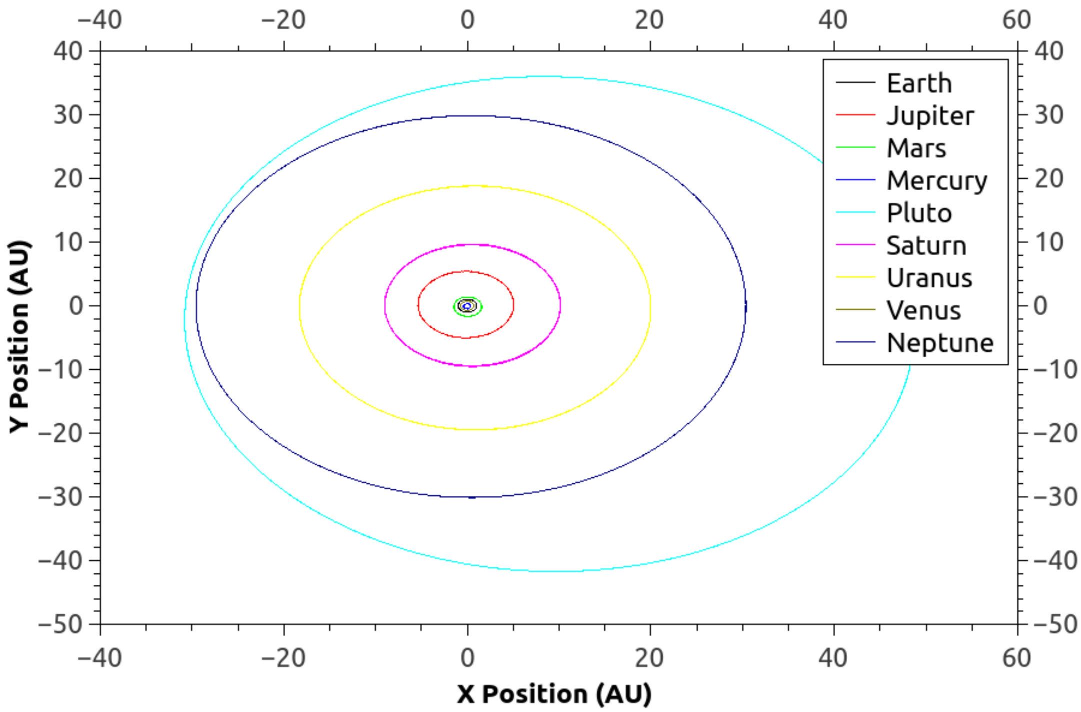


**Figure 16:** *The orbits of all the planets in the Solar System*