Solutions of the Exercises of Lesson 5

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1. 1. Classify the stability of the following systems in: Liapunov stable, asymptotically stable, or none of the above.

   a)

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -4x \end{aligned} \tag{1}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{2}$$

$$\left.\begin{matrix} \Delta = 4 > 0 \\ \tau = 0 \end{matrix}\right\} centers \tag{3}$$

$$\begin{aligned} p(\lambda) &=^2 +4 = 0 \\ \lambda &= \pm 2i \\ x(t) &= c1e^{\lambda_1 t}\vec{v_1} + c2e^{\lambda_2 t}\vec{v_2} \\ \lambda_1 &= \alpha_1 + \beta_1 i \\ e^{\lambda_1 t} = e^{(\alpha_1+\beta_1 i)t} = e^{\alpha_1 t}e^{\beta_1 it} &= e^{\alpha_1 t}(cos(\beta_1 t) + isen(\beta_2 t)) \\ y &= \dot{x} \\ \dot{y} &= \ddot{x} \\ \dot{y} &= -4x \\ \ddot{x} &= -4x \\ \ddot{x} + 4x &= 0 \\ \text{homogeneous EDO,second order} \\ x(t) &= c1cos(2t) + c2sen(2t) \\ y(t) &= -c1sent(2t) + 2c2cos(2t) \\ (x_0, y_0), t &= 0 \\ x_0 &= c1 \\ y_0 = 2c2, c2 &= \frac{y_0}{2} \\ x(t) &= x_0cos(2t) + \frac{y_0}{2}sen(2t) \\ y(t) &= -x_0sen(2t) + y_0cos(2t) \\ \text{if t} &\to \infty \\ x(t), y(t) &\leq 1 \\ Because &: \\ |sent(2t)| &\leq 1 \\ |cos(2t)| &\leq 1 \\ \textbf{Liapunov stable} \end{aligned} \tag{4}$$

   b)

$$\begin{aligned} \dot{x} &= 2y \\ \dot{y} &= x \end{aligned} \tag{5}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(6)

$$\left.\begin{array}{c}\Delta = 0\\ \tau = 1\end{array}\right\}\text{line of unstable fixed points} \tag{7}$$

$$\begin{aligned}p(\lambda) &= \lambda^2 - \lambda = 0\\ \lambda_1 &= 0\\ \lambda_2 &= 1\\ x(t) &= c1 + c2e^t\vec{v2}\\ Eigenvectors &:\\ |\Delta -_2|v &= 0\\ \lambda &= 0\\ \begin{bmatrix}0 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}v1\\ v2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} 2v2 + v2 &= 0\\ 3v2 &= 0\\ v(1,0)\lambda = 1\begin{bmatrix}-1 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}v1\\ v2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} - v1 + v2 &= 0\\ v2 &= v1\\ v2(1,1)\end{aligned} \tag{8}$$

$c)$

$$\begin{aligned}x &= x_0\\ \dot{y} &= x_0\\ y &= x_0t + y\\ c\text{if t} &\to \infty\\ \text{uniform motion}\\ \textbf{trajectories tend to infinite, unstable}\end{aligned} \tag{9}$$

$d)$

$$\begin{aligned}\dot{x} &= 0\\ \dot{y} &= -y\end{aligned} \tag{10}$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \tag{11}$$

$$\left.\begin{array}{c}\Delta = 0\\ \tau = -1\end{array}\right\}\text{line of stable fixed points} \tag{12}$$

$$p(\lambda) = \lambda^2 + \lambda = 0$$
$$\lambda_1 = 0$$
$$\lambda_2 = -1$$
$$x(t) = c1v\vec{1} + c2e^{-t}v\vec{2}$$
$$Eigenvectors:$$
$$\lambda = 0$$
$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} v(1,0)$$
$$\lambda = -1$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} v(0,1) \tag{13}$$
$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} c1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} c2$$
$$x_0 = c1$$
$$y_0 = c2$$
$$x(t) = x_0$$
$$y(t) = y_0 e^{-t}$$
$$\text{if t} \to \infty$$
$$y(t) \to 0$$
$$\textbf{Liapunov stable}$$

$e)$

$$\left.\begin{array}{r} \dot{x} = -x \\ \dot{y} = -5y \end{array}\right\} \text{decoupled system}$$
$$\text{separable variables}$$
$$\frac{dx}{dt} = -x$$
$$\frac{-dx}{x} = dt$$
$$-ln(x) = t + c$$
$$ln(x) = -t + c$$
$$x(t) = e^{-t}c$$
$$t = 0$$
$$x(0) = x_0$$
$$\boxed{x(t) = x_0 e^{-t}} \tag{14}$$
$$\dot{y} = -5y$$
$$\frac{dy}{y} = -5dt$$
$$ln(y) = -5t + c$$
$$t = 0$$
$$y(0) = y_0$$
$$y = e^{-5t}c$$
$$\boxed{y = e^{-5t}y_0}$$
$$\text{if t} \to \infty$$
$$(x_0, y_0) = (0,0)$$
$$\textbf{Asymptotically stable}$$

f)

$$\begin{aligned}
\left.\begin{aligned}\dot{x} &= x\\ \dot{y} &= y\end{aligned}\right\}\text{decoupled system}\\
\text{separable variables}\\
\frac{dx}{dt} &= x\\
ln(x) &= t + c\\
x(t) &= e^t c\\
t &= 0\\
x(0) &= x_0\\
x(t) &= x_0 e^t\\
\frac{dy}{y} &= y\\
Ln(y) &= t + c\\
y &= e^t c\\
t &= 0\\
y(0) &= y_0\\
y(t) &= y_0 e^t\\
\text{if t} &\rightarrow \infty\\
\text{trajectories go away from origin}\\
\textbf{unstable}
\end{aligned} \tag{15}$$

2. Consider the system

a) Write the system as x˙ = Ax.

$$\begin{aligned}
\dot{x} &= 4x - y\\
\dot{y} &= 2x + y
\end{aligned} \tag{16}$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}4 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \tag{17}$$

$$\begin{aligned}
\Delta &= 6\\
\tau &= 5 \;\; source\\
p(\lambda) &= \lambda^2 - 5\lambda + 6 = 0\\
\lambda_1 &= 3\\
\lambda_2 &= 2\\
Eigenvectors &:\\
\lambda_1 &= 3\\
\begin{bmatrix}1 & -1\\ 2-2 & \end{bmatrix}\begin{bmatrix}v1\\ v2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\\
v1 - v2 &= 0\\
v1 &= v2\\
v1(1,1)\\
\lambda_2 &= 2\\
\begin{bmatrix}2 & -1\\ 2-1 & \end{bmatrix}\begin{bmatrix}v1\\ v2\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\\
2v1 - v2 &= 0\\
2v1 &= v2\\
v2(1,2)
\end{aligned} \tag{18}$$

b) Find the general solution of the system.

$$\begin{aligned}
x(t) &= c1e^{3t}\vec{v1} + c2e^{2t}\vec{v2}\\
x(t) &= \begin{bmatrix}1\\ 1\end{bmatrix} c1e^{3t} + \begin{bmatrix}1\\ 2\end{bmatrix} c2e^{2t}
\end{aligned} \tag{19}$$

c) Classify the fixed point at the origin

$$\begin{aligned} \Delta &> 0 \\ \tau &> 0 \\ \Delta &< \frac{1}{4}\tau^2 \\ \text{source, unstable} \end{aligned} \tag{20}$$

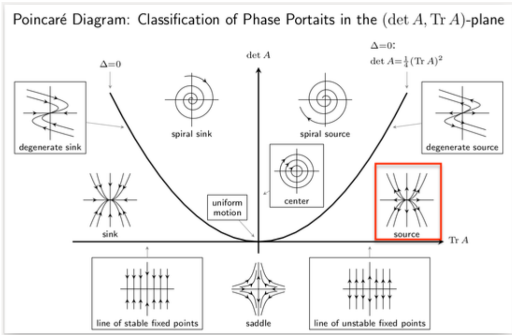


Figura 1: classification in axes trace and determinant

d) Solve the system subject to the initial condition (x0, y0) = (3, 4).

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} c1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} c2 \tag{21}$$

$$\begin{aligned} &\left. \begin{aligned} 3 &= c1 + c2 \\ 4 &= c1 + c2 \end{aligned} \right\} \\ c1 &= 2 \\ c2 &= 1 \\ &\left. \begin{aligned} x(t) &= 2e^{3t} + e^{2t} \\ y(t) &= \end{aligned} \right\} \end{aligned} \tag{22}$$

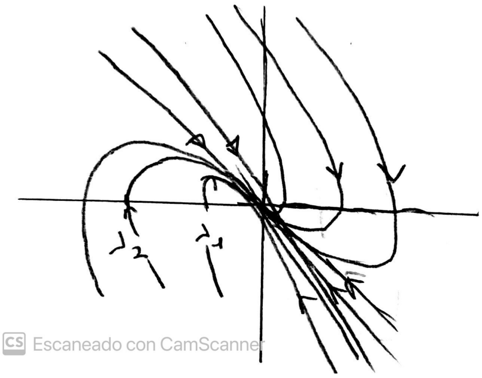
3. Plot the phase portrait and classify the fixed point of the following linear systems. If the eigenvectors are real, indicate them in your sketch.

   a)

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -2x - 3y \end{aligned} \tag{23}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{24}$$

$$
\begin{aligned}
p(\lambda) = \lambda^2 + 3\lambda + 2 = 0 \\
\lambda_1 = -1 \\
\lambda_2 = -2 \\
Eigenvectors: \\
\lambda = -1 \\
\begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
v2 = -v1 \\
v1(1, -1) \\
\lambda = -2 \\
\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
2v1 = -v2 \\
v2 = (2, -1) \\
\Delta > 0 \\
\tau < 0 \\
\Delta < \frac{1}{4}\tau^2 \\
\textbf{sink, stable}
\end{aligned} \tag{25}
$$

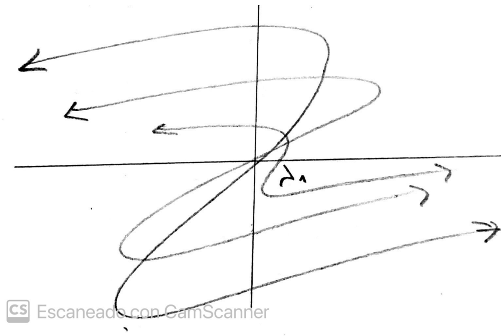


$b)$

$$\begin{aligned} \dot{x} &= 3x - 4y \\ \dot{y} &= x - y \end{aligned} \tag{26}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{27}$$

$$\begin{aligned} \Delta &= 1 \\ \tau &= 2 \\ p(\lambda) = \lambda^2 - 2\lambda + 1 &= 0 \\ Eigenvectors &: \\ \lambda &= 1 \\ \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 2v1 - 4v2 &= 0 \\ v1(2,4) & \\ \Delta &> 0 \\ \tau &> 0 \\ \Delta &= \frac{1}{4}\tau^2 \\ \text{one eigenvector} & \\ \textbf{Degenerate source} & \end{aligned} \tag{28}$$

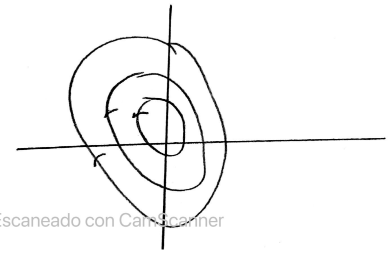


$c)$

$$\begin{aligned} \dot{x} &= 5x + 2y \\ \dot{y} &= -17x - 5y \end{aligned} \tag{29}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -17 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{30}$$

$$\begin{aligned} \Delta &= 9 \\ \tau &= 0 \\ p(\lambda) &= \lambda^2 + 9 \\ \lambda_1 &= 3i \\ \lambda_2 &= -3i \\ Eigenvectors &: \\ \lambda &= 3i \\ \begin{bmatrix} 5-3i & 2 \\ -17 & -5-3i \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ v1(5-3i, -2) \\ \lambda &= -3i \\ \begin{bmatrix} 5+3i & 2 \\ -17 & -5+3i \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ v2(5+3i, -2) \\ \Delta &> 0 \\ \tau &= 0 \\ \textbf{centers} \end{aligned} \tag{31}$$

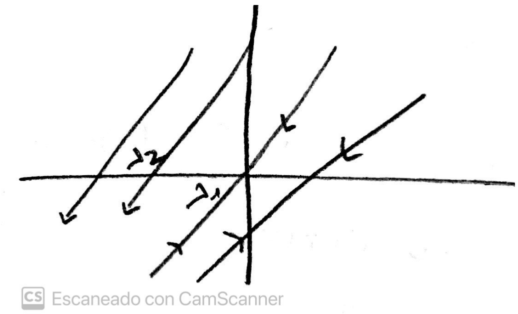


d)

$$\begin{aligned} \dot{x} &= 4x - 3y \\ \dot{y} &= 8x - 6y \end{aligned} \tag{32}$$

$$[\dot{x}\dot{y}] = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{33}$$

$$\begin{aligned} \Delta &= 0 \\ \tau &= -2 \\ p(\lambda) &= \lambda^2 + 2\lambda \\ \lambda_1 &= 0 \\ \lambda_2 &= -2 \\ Eigenvectors &: \\ \lambda &= 0 \\ \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ v1(4,3) \\ \lambda &= -2 \\ \begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ v2(2,1) \\ \Delta &= 0 \\ \tau &< 0 \\ \textbf{line of stable fixed points} \end{aligned} \tag{34}$$

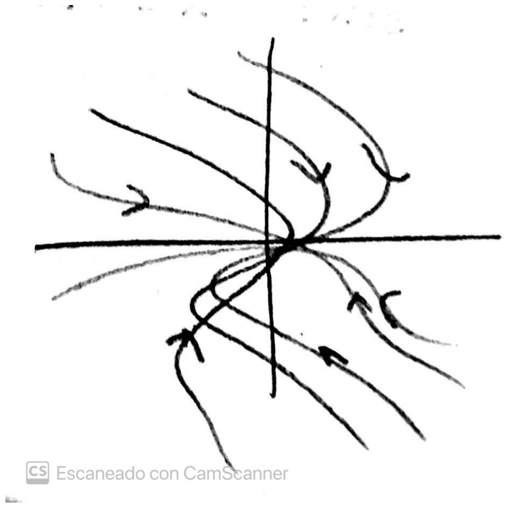


4. . Show that any matrix of the form A = 0 has only a one-dimensional eigenspace corresponding to the eigenvalue . Then solve the system x˙ = Ax and sketch the phase portrait.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{35}$$

c=0

$$\begin{aligned}
\dot{x} &= \lambda x + by \\
\dot{y} &= \lambda y \\
\frac{dy}{dt} &= \lambda y \\
ln(y) &= \lambda t + c \\
y(t) &= ce^{\lambda t} \\
\dot{x} &= \lambda x + by \\
\dot{x} &= \lambda x + bce^{\lambda t} \\
\dot{x} - \lambda x &= bce^{\lambda t} \\
e^{-\lambda t}(\dot{x} - \lambda x) &= bc \\
e^{-\lambda t}\dot{x} - e^{-\lambda t}\lambda x &= bc \\
d(xe^{-\lambda t}) &= e^{-\lambda t}\dot{x} - e^{-\lambda t}\lambda x \\
d(xe^{-\lambda t}) &= bc \\
xe^{-\lambda t} &= \int bcdt \\
x &= e^{\lambda t}(bct + k) \\
y &= ce^{\lambda t} Eigenvectors: \\
\begin{bmatrix} \lambda - \delta & b \\ 0 & \lambda - \delta \end{bmatrix} &= 0 \\
(\lambda - \delta)^2 &= 0 \\
\lambda &= \delta(double) \\
\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v1 \\ v2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
bv2 &= 0 \\
v(1,0) & \\
\textbf{one dimensional eigenspace vector} &
\end{aligned} \tag{36}$$



5. . (Damped harmonic oscillator) The motion of a damped harmonic oscillator is described by mx¨+bx˙ +kx 0, where b ¿0 is the damping constant.

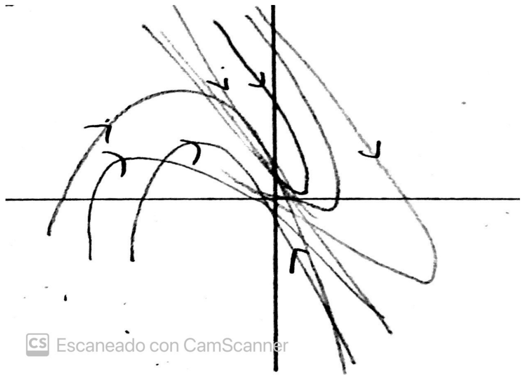
   a) Rewrite the equation as a two-dimensional linear system.Rewrite the equation as a two-dimensional linear system.

$$
\begin{aligned}
\ddot{x} &= \frac{-(b\dot{x}+kx)}{m} \\
\dot{y} = \ddot{x} &= \frac{-(b\dot{x}+kx)}{m} \\
\dot{x} &= y \\
\frac{-(b\dot{x}+kx)}{m} &= \frac{-by}{m} - \frac{kx}{m} \\
\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} &= \begin{bmatrix} o & y \\ \frac{-k}{m} & \frac{-b}{m}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}
\end{aligned} \tag{37}
$$

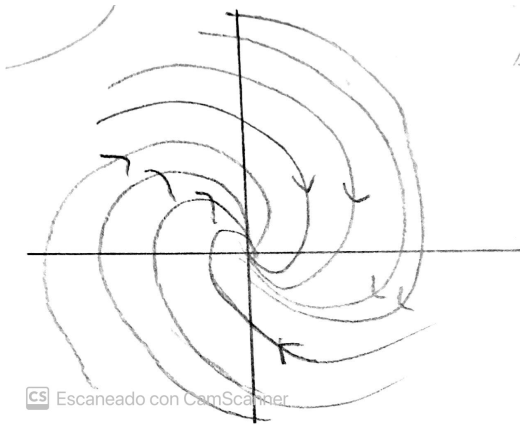
   b) Classify the fixed point at the origin and sketch the phase portrait. Be sure to show all the different cases that can occur, depending on the relative sizes of the parameters.

$$
\begin{aligned}
\Delta &= \frac{k}{m}y \\
\tau &= \frac{-b}{m} \\
p(\lambda) &= \lambda^2 + \frac{b}{m}\lambda + \frac{k}{m}y = 0 \\
\lambda &= \frac{\frac{-b}{m} \pm \sqrt{(\frac{b}{m})^2 - 4\frac{k}{m}y}}{2} \\
\lambda &= \frac{-b \pm \sqrt{b^2 - 4ky}}{2m}
\end{aligned} \tag{38}
$$

$$
\begin{gathered}
if \;\; b^2 - 4ky > 0 \\
\textbf{stable node, source}
\end{gathered} \tag{39}
$$

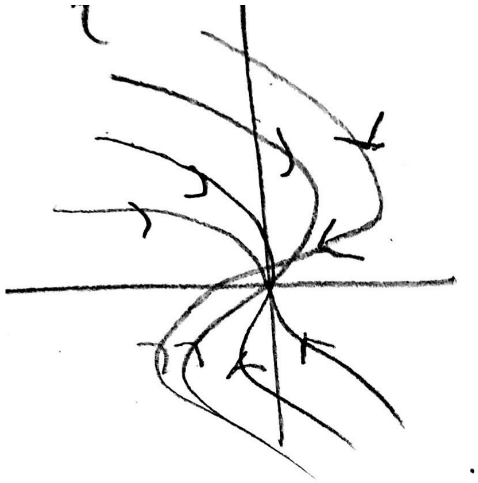


$$if \;\; b^2 - 4ky < 0 \tag{40}$$
spiral source



$$if \;\; b^2 - 4ky = 0 \tag{41}$$

degenerate source



c) How do your results relate to the standard notions of overdamped, critically damped, and underdamped vibrations?
source $\rightarrow overdamped$
$degeneratesource \rightarrow criticallydamped$
$spiralsource \rightarrow underdamped$

6. (Out of touch with their own feelings) Suppose Romeo and Juliet react to each other, but not to themselves: R˙ = aJ, J˙ = bR. What happens? Analyze the cualitative behaviour depending on the parameters.

$$\begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix} \tag{42}$$

$$\begin{aligned} \Delta &= -ba \\ \tau &= 0 \\ p(\lambda) &= \lambda^2 - ba = 0 \\ \lambda &= \pm\sqrt{ba} \\ if \;\; ba < 0 &\rightarrow center \\ if \;\; ba > 0 &\rightarrow \text{saddle node} \end{aligned} \tag{43}$$