Concept Review
Modeling

Why Modeling?

System modeling is a crucial engineering practice that provides a mathematical framework for understanding and predicting system behavior before physical implementation. It allows us to predict performance characteristics and system limitations accurately and it enables efficient control system design by providing a clear understanding of how the system will respond to different inputs and operating conditions and it allows us to monitor the behavior against it's digital twin in simulation to help with identifying faults. Modeling a system effectively allows us to understand complex systems and make informed decisions that can enhance system reliability and efficiency.

In the case of a DC motor, doing motor modeling builds fundamental understanding of electromagnetic and mechanical principles.

Contents

This document explores system modeling through step response modeling as well as frequency response modeling to help with analyzing, designing, and controlling systems effectively. Some of the examples are based around DC motors like the one found in the Qube-Servo, therefore, this document also outlines the basic DC motor concepts to understand the key motor parameters and equations.

1. Fundamental DC Motor Concepts

A DC motor converts electrical energy into mechanical energy through electromagnetic interaction. Consider the electromechanical system of a motor as shown in Figure 1. An RL series circuit can be used to model the internal resistance and inductance of the motor.

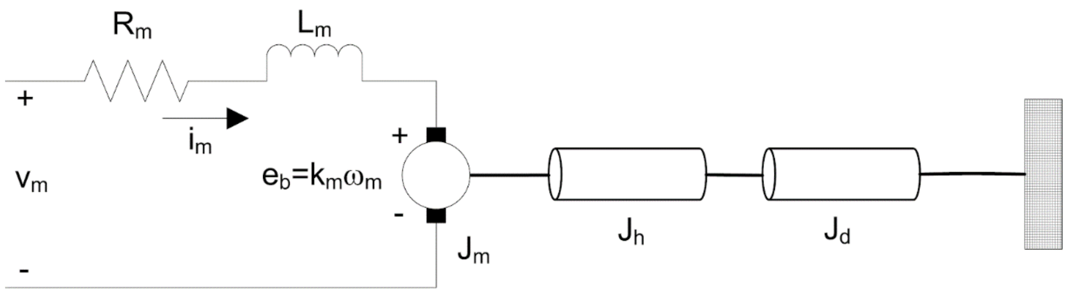

Figure 1. Typical Electromechanical Motor Configuration

a. Key Motor Parameters

i. Electrical Parameters

- **Voltage (v_m):** Applied electrical potential $[V]$
- **Current (i_m):** Current through motor windings $[A]$
- **Resistance (R_m):** Motor winding resistance $[\Omega]$
- **Inductance (L_m):** Motor winding inductance $[H]$
- **Back-EMF constant (k_m):** Voltage generated per unit speed $[V/rad/s]$

ii. Mechanical Parameters

- **Torque (τ)**: Motor output torque $[N \cdot m]$
- **Angular velocity (ω)**: Shaft rotational speed $[rad/s]$
- **Inertia (J)**: Combined motor and load inertia $[kg \cdot m^2]$
- **Viscous friction (B)**: Damping coefficient $[N \cdot m \cdot s/rad]$
- **Torque constant (k_t)**: Torque produced per unit current $[N \cdot m/A]$

b. Basic Motor Equations

The equations from this section form the foundation for all subsequent modeling approaches discussed in this document.

i. Electrical Equations

In the circuit diagram in Figure 1, Kirchoff's laws can dictate the relationship between the command voltage applied to the motor v_m as well as the voltage drops across the resistor v_r, inductor v_l and motor armature e_b. The last term reflects the back-emf generated by the spinning motor shaft and core.

$$v_m = v_r + v_l + e_b \tag{1.1}$$

The voltage drop across a resistor v_r with resistance R_m relates to the current flow i_m,

$$v_r = i_m R_m \tag{1.2}$$

The voltage drop across an inductor v_l with inductance L_m relates to the rate of change of current,

$$v_l = \frac{di}{dt} L_m \tag{1.3}$$

The back-emf (electromotive) voltage e_b is proportional to the speed of the motor shaft, ω_m, and the back-emf constant of the motor, k_m. It opposes the current flow. The back emf voltage is given by:

$$\boldsymbol{e_b = k_m \omega_m} \tag{1.4}$$

Where k_m is the motor speed constant in $V/rad/s$. Substituting equations 1.2 to 1.4 back into 1.1 yields the motor's electrical equation,

$$v_m = R_m i_m + \frac{di}{dt} L_m + k_m \omega_m \tag{1.5}$$

The inductance term is often dropped due to small effects on the overall model.

Note that some quantities are functions of time. The relevant **electrical equation for a DC motor** is:

$$\boldsymbol{v_m(t) = R_m i_m(t) + k_m\ \omega_m(t)} \quad \text{or} \quad \boldsymbol{v_m(t) = R_m i_m(t) + k_m\ \dot{\theta}_m(t)} \tag{1.6}$$

Where:

- $\boldsymbol{v_m(t)}$ is the applied voltage (control input).
- $\boldsymbol{R_m}$ is the motor's resistance.
- $\boldsymbol{i_m(t)}$ is the motor current.
- $\boldsymbol{k_m}$ is the back-EMF constant.
- $\boldsymbol{\omega_m(t)}$ or $\boldsymbol{\dot{\theta}_m(t)}$ are the angular velocity. ($\theta_m(t)$ is the angular position of the motor shaft)

Solving for the motor current $i_m(t)$, in equation 1.6, the equation becomes

$$\boldsymbol{i_m(t) = \frac{v_m(t) - k_m\ \omega_m(t)}{R_m}} \qquad 1.7$$

ii. Mechanical Equations

Next, we can consider the dynamic equations related to the mechanical motion of the motor shaft.

We start with Newton's law applied to rotational systems, The net motor torque τ_{net} is,

$$\tau_{net} = J_{eq}\dot{\omega} \qquad 1.8$$

Where J_{eq} is the total moment of inertia of the motor shaft, hub and associated load in kgm^2 and $\dot{\omega}$ is the angular acceleration of the motor.

The equivalent moment of inertia can be calculated as the sum of the individual components,

$$J_{eq} = J_m + J_h + J_d \qquad 1.9$$

J_m relates to the mass moment of inertia of motor core, shaft coils etc. The term J_h is an optional term representing a hub or attachment that holds the load to the motor shaft. The final term J_d represents the moment of inertia of the attached load.

The moment of inertia of a disk about its pivot, with mass m and radius r, is

$$J_d = \frac{1}{2}\, m\, r^2 \qquad 1.10$$

The net torque on the motor is given by,

$$\tau_{net} = \tau_{app} - b\omega \qquad 1.11$$

Where τ_{app} is the applied motor torque, ω is the motor speed, and b represents motor damping. Combining 1.11 with 1.9 yields **the relevant applied torque equation**

$$\boldsymbol{\tau_{app} = J_{eq}\dot{\omega} + b\omega} \qquad 1.12$$

Where $\dot{\omega}$ is the angular acceleration of the motor. If we use $\theta_m(t)$ as the angular position, and $\dot{\theta}_m(t)$ as the angular velocity, the above equation becomes:

$$\boldsymbol{\tau_{app} = J_{eq}\ddot{\theta} + b\omega} \qquad 1.13$$

Finally, we note that the applied torque τ_{app} is directly proportional to the motor current i_m generated as a result of the applied voltage command v_m. Which means the **torque equation can also be represented as follows**:

$$\boldsymbol{\tau_{app} = k_t i_m} \qquad 1.14$$

Where k_t is the motor torque constant in Nm/A.

iii. Transfer Functions

Transfer functions are like a "black box" that describes how a system will process and modify any input signal. A transfer function mathematically represents the relationship between the input and output of a linear time-invariant (LTI) system in the complex frequency domain (s-domain or Laplace domain).

Transfer functions transform complex differential equations from the time domain into simpler algebraic equations in the frequency domain, which simplifies the analysis and design of control systems. The frequency domain approach enables rapid system analysis through simple multiplication and division operations, eliminating the need for complex time-domain convolution calculations when solving differential equations.

The voltage $V_m(s)$ to speed $\Omega_m(s)$ transfer function of a DC motor is:

$$\frac{\Omega_m(s)}{V_m(s)} = \frac{K}{\tau s + 1} \tag{1.15}$$

The voltage $V_m(s)$ to position $\Theta_m(s)$ transfer function is

$$\frac{\Theta_m(s)}{V_m(s)} = \frac{K}{s\,(\tau s + 1)} \tag{1.16}$$

Where:

- K is the model steady–state gain

- τ is the model time constant

- $\Omega_m(s) = L\,[\omega_m(t)]$ is the Laplace transform of the motor/disc speed

- $\Theta_m(s) = L\,[\theta_m(t)]$ is the Laplace transform of the motor/disc position, and

- $V_m(s) = L\,[v_m(t)]$ is the Laplace transform of the applied motor voltage.

Note how the voltage to position transfer function is the same as equation 1.15 with an integrator in series. It multiplied by $1/s$ to integrate it in time to find the position of the motor.

2. Step Response

A step response test is a fundamental method for analyzing a system's dynamic behavior by observing how it responds to a sudden change in input. For DC motors, this typically involves instantly applying a fixed voltage and measuring how the motor's speed or position changes over time. This testing is crucial because it reveals essential system characteristics like response speed, stability, and accuracy, which are vital for control system design and optimization.

a. First Order Step Response

Step response modeling, also known as the *bump test*, is a simple test based on the step response of a stable system. A step input is given to the system and its response is recorded.

Consider a standard form of the transfer function of a first order system:

$$\frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1} \tag{2.1}$$

The step response shown in Figure 2 is generated using this transfer function with $K = 5$ rad/V.s and $\tau = 0.05$ s.

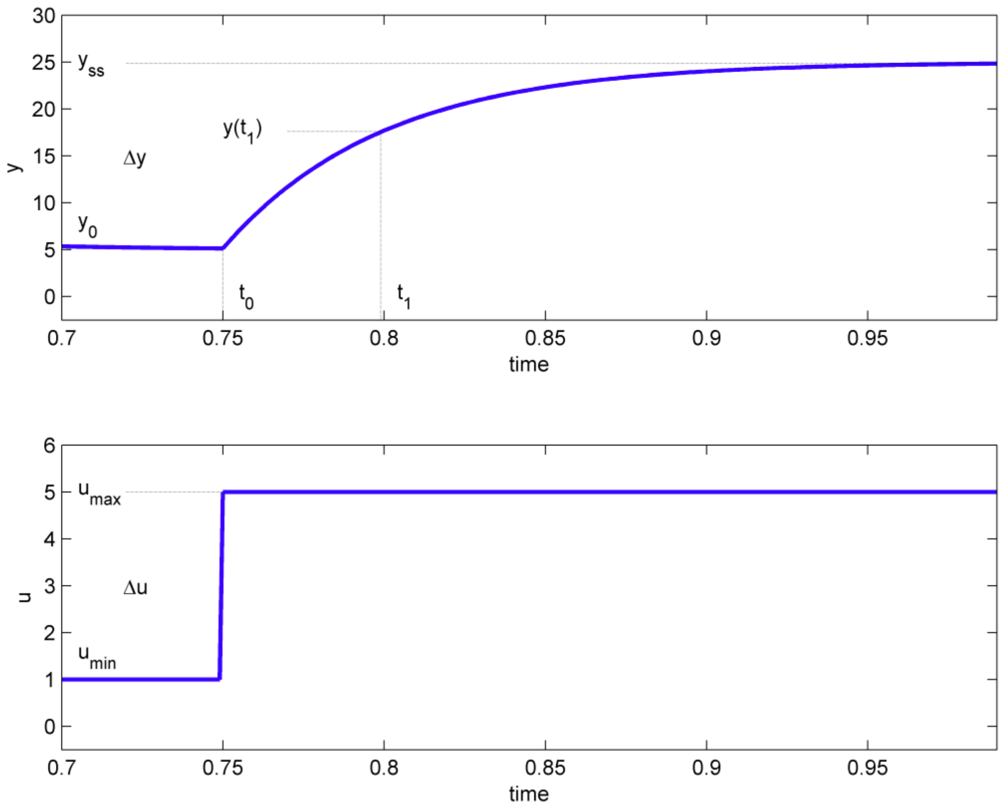

Figure 2. Step response of a first-order system

The step input begins at time t_0 at which the output signal is at y_0. The input signal has a minimum value of u_{min} and a maximum value of u_{max}. Once the step is applied, the output tries to follow it and eventually settles at its steady–state value y_{ss}. From the output and input signals, **the steady–state gain is**

$$K = \frac{\Delta y}{\Delta u} = \frac{y_{ss} - y_0}{u_{max} - u_{min}} \qquad 2.2$$

The **time constant of a system $\boldsymbol{\tau}$** is defined as the time it takes the system to respond to the application of a step input and reach $1 - {}^{1}\!/_{e} \approx 63.2\%$ of its way towards the steady–state value. In Figure 2, the system reaches this point at time t_1 and an output signal of $y(t_1)$:

$$y(t_1) = 0.632\,\Delta y + y_0 \qquad 2.3$$

Using the response in Figure 2, the time t_1 that corresponds to $y(t_1)$ can be identified. From this, the model time constant can be found as:

$$\tau = t_1 - t_0 \qquad 2.4$$

i. For Qube-Servo

The s–domain representation of a step input voltage with a time delay t_0 is given by

$$V_m(s) = \frac{A_v e^{(-st_0)}}{\tau s + 1} \qquad 2.5$$

Where A_v is the amplitude of the step and t_0 is the step time (i.e. the delay).

The voltage $V_m(s)$ to speed $\Omega_m(s)$ transfer function is:

$$\frac{\boldsymbol{\Omega_m(s)}}{\boldsymbol{V_m(s)}} = \frac{\boldsymbol{K}}{\boldsymbol{\tau s + 1}} \qquad 2.6$$

Where:

- K is the model steady–state gain

- τ is the model time constant

- $\Omega_m(s) = L\,[\omega_m(t)]$ is the Laplace transform of the motor/disc speed, and

- $V_m(s) = L\,[v_m(t)]$ is the Laplace transform of the applied motor voltage.

If we substitute the s domain equation (2.5) into the voltage to speed transfer function (2.6), we get:

$$\Omega_m(s) = \frac{KA_v e^{(-st_0)}}{(\tau s + 1)s} \tag{2.7}$$

We can then find the Qube–Servo motor speed step response in the time domain $\omega_m(t)$ by taking inverse Laplace of this equation

$$\omega_m(t) = KA_v \left(1 - e^{(-\frac{t-t_0}{\tau})}\right) + \omega_m(t_0) \tag{2.8}$$

noting the initial conditions $\omega_m(0^-) = \omega_m(t_0)$.

b. Second Order Step Response

The standard second-order transfer function has the form:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\,\zeta\,\omega_n\,s + \omega_n^2} \tag{2.9}$$

The properties of its response depend on the parameters ω_n and ζ, where:

- ω_n is the natural frequency of the system

- ζ is the damping ratio of the system

Consider a second order system as shown in equation 2.9 subjected to a step input given by

$$R(s) = \frac{R_0}{s} \tag{2.10}$$

with a step amplitude of $R_0 = 1.5$. The system response to this input is shown in Figure 3, where the red trace is the output response $y(t)$ and the blue trace is the step input $r(t)$.

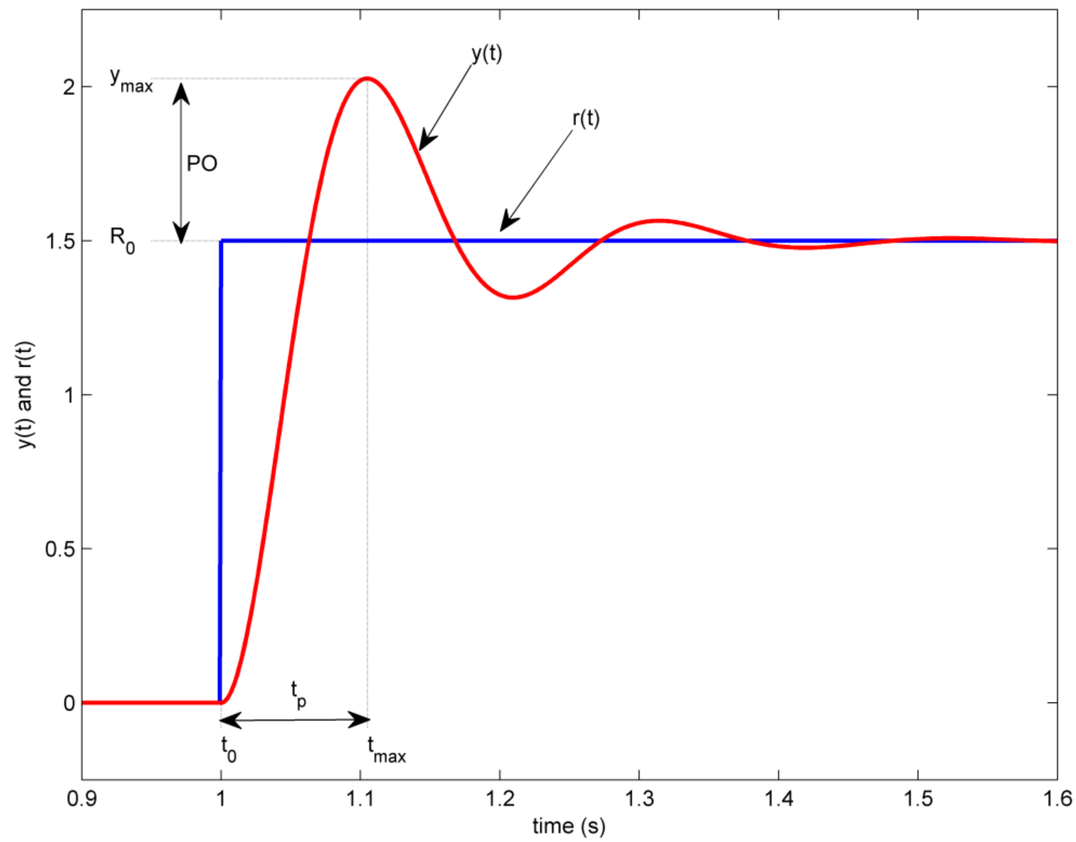

Figure 3. Standard second-order step response

i. Peak Time and Overshoot

The maximum value of the response is denoted by the variable y_{max} and it occurs at a time t_{max}. The final value of the system is denoted as R_0. For a response similar to Figure 3, the **percent overshoot** is found using. For situations where the starting value of R_0 are not 0, the denominator becomes $R_{max} - R_{min}$

$$PO = \frac{100(y_{max} - R_0)}{R_0} \tag{2.11}$$

The **peak time** of the system, which is the time from the initial step t_0, to the time it takes for the response to reach its maximum value is:

$$t_p = t_{max} - t_0 \tag{2.12}$$

In a second- order system, the **percentage overshoot** depends solely on the damping ratio parameter (ζ), and it can be calculated using the equation

$$PO = 100\, e^{\left(-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}\right)} \tag{2.13}$$

The **peak time** depends on both the damping ratio and natural frequency of the system, and it can be derived as:

$$t_p = \frac{\pi}{\omega_m \sqrt{1 - \zeta^2}} \tag{2.14}$$

Generally speaking, the damping ratio affects the shape of the response while the natural frequency affects the speed of the response. Once equations 2.11 and 2.12 are used to determine overshoot and peak time, equations 2.13 and 2.14 could be used to determine the damping ratio of the system (ζ) and the natural frequency of the system (ω_n). In real life scenarios, peak time and percentage overshoot are specified based on the needs of the system and then the controller gets modified to match those values.

c. Unity Feedback Loop

Unity feedback is fundamental in control systems because it enables direct comparison between the desired input and actual output, creating an error signal that drives system correction. This standard form makes it easier to analyze stability, calculate performance parameters (like steady-state error and settling time), and apply classical control techniques. A unity feedback loop looks like the diagram in Figure 4, where r is the reference value, e is the error between the output y and the reference value. $P(s)$ is the plant of the system and $C(s)$ is the controller that will be applied to the system.

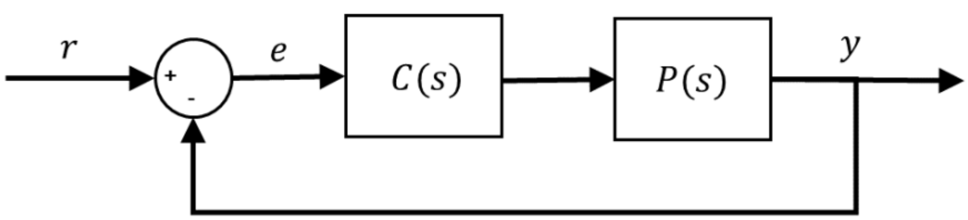

Figure 4. Unity feedback loop

The closed loop system transfer function for unity feedback as shown in Figure 4 will always be given by,

$$\frac{\boldsymbol{Y(s)}}{\boldsymbol{R(s)}} = \frac{\boldsymbol{C(s) \cdot P(s)}}{\boldsymbol{1 + C(s) \cdot P(s)}} \tag{2.15}$$

i. <u>For Qube – Servo</u>

The first order step response allowed for a transfer function to describe the speed of the system. To instead control the position of a Qube-Servo, a unity feedback loop is used.

The Qube-Servo voltage to position transfer function is

$$\frac{\Theta_m(s)}{V_m(s)} = \frac{K}{s\,(\tau s + 1)} \tag{2.16}$$

Where:

- K is the model steady–state gain

- τ is the model time constant

- $\Theta_m(s) = L\,[\theta_m(t)]$ is the Laplace transform of the motor/disc position, and

- $V_m(s) = L\,[v_m(t)]$ is the Laplace transform of the applied motor voltage.

Note how this is the voltage to speed transfer function from equation 2.6 multiplied by $1/s$ to integrate it in time to find the position of the motor.

If no modeling lab has been done, for Qube-Servo 3, $K = 24$ and $\tau = 0.1$ are good defaults if this equation needs to be used.

The controller in Figure 4 is denoted by $C(s)$. For this unity feedback loop, if a unity controller is used: $C(s) = 1$, equation 2.15 becomes:

$$\frac{Y(s)}{R(s)} = \frac{P(s)}{1 + P(s)} \tag{2.17}$$

We can use equation 2.17, and equation 2.16 as the plant to find the closed-loop transfer function of the Qube-Servo position control using unity feedback and a unity controller as shown in Figure 4. The equation for the reference input (desired position) $R(s) = \Theta_d(s)$, to the output (measured position) $Y(s) = \Theta_m(s)$ is:

$$\frac{\Theta_m(s)}{\Theta_d(s)} = \frac{\frac{K}{\tau}}{s^2 + \frac{1}{\tau}s + \frac{K}{\tau}} \tag{2.18}$$

3. Frequency Response Modeling

Frequency response modeling is a fundamental tool in control systems engineering that characterizes how a system (like a DC motor) responds to sinusoidal inputs at varying frequencies. For DC motors, this modeling approach provides crucial insights into the motor's dynamic behavior and limitations. It helps understand how well a motor can follow commands that change at different speeds - from very slow changes to rapid oscillations.

An input sinusoid $V_m(t)$ can be characterized by its amplitude and frequency. When applying an input sine wave to a DC motor, the resulting output of the DC motor will be a *scaled* and *delayed* sinusoid of the same frequency. For example, in Figure 5, t_1 denotes the length of a period of the sinusoid and t_2 shows the time delay between an input voltage signal, V_m, and a scaled output speed signal, Ω_m.

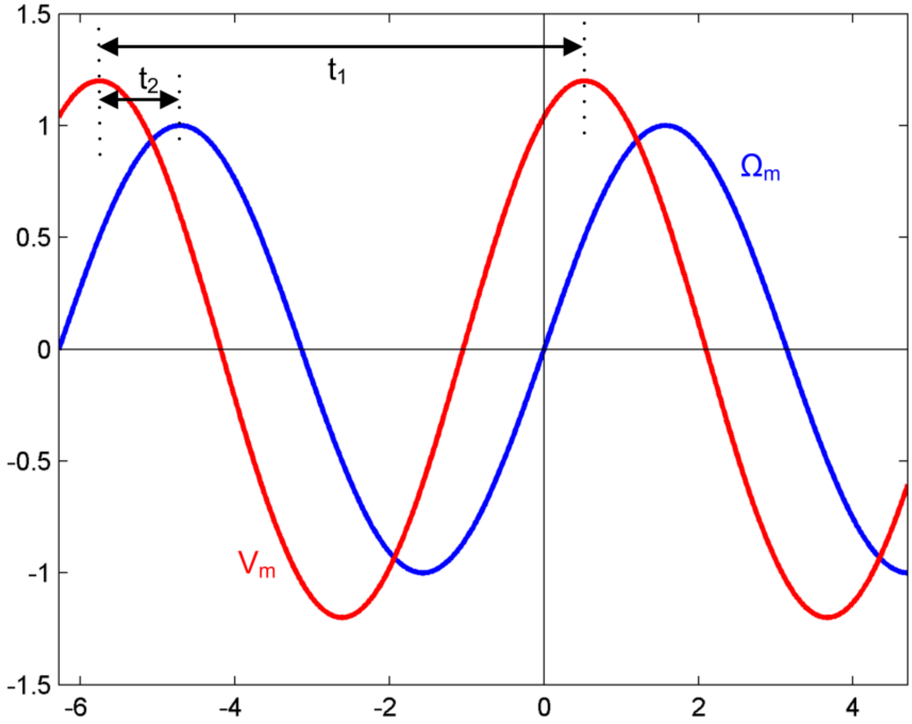

Figure 5. Period and phase delay of sinusoidal signals

a. Magnitude Response Analysis

The magnitude response of the resulting output changes with respect to the frequency of the applied sinusoid and can be used to find the time constant τ in a transfer function of a DC motor. The transfer function of a DC motor is shown in equation 2.6.

This magnitude can be represented using a Bode Magnitude plot, which allows to visualize how the output amplitude varies with frequency (in decibels (dB)). A Bode plot of the system's magnitude response similar to Figure 6 can be created by varying the frequency of the applied sinusoid.

i. Cutoff Frequency

The cutoff frequency (ω_c) is a key feature we look for in a Bode plot. The cutoff frequency ω_c is defined as the specific frequency point where a circuit starts to reduce or "attenuate" the intensity of the signal. It occurs at the point where the system's response has decreased by 3 dB compared to its maximum steady-state value. This -3 dB point is significant because it marks where the system's power has dropped to half of its maximum - that's why we sometimes call it the half-power point.

When working with power measurements, we use $10\ \log_{10}(0.5)\ \approx\ -3.01\ dB$, where 0.5 represents half of the original power and the power at the cutoff frequency (in dB) is -3.01 dB.

However, when dealing with voltage or current measurements, at the cutoff frequency, the output amplitude is approximately 70.7% (or 1/√2) of the input amplitude, we use $20\log_{10}(\frac{1}{\sqrt{2}}) \approx -3.01\ dB$, where $\frac{1}{\sqrt{2}} \approx 0.707$ represents the voltage or current ratio.

These equations give us the same result because power is proportional to voltage squared ($P \propto V^2$), from the power equation $P = V^2/R$. So when voltage drops to 1/√2 (about 0.707) of its original value $G(\omega_c) = K/\sqrt{2}$, it is describing the same cutoff point where the power drops to $(1/\sqrt{2})^2 = 1/2$ of its original value.

We often refer to this cutoff frequency as the system's "bandwidth" because it tells us something very practical: how quickly the system can respond to changes in input. It shows us the fastest rate at which our system can effectively react to signals we give it.

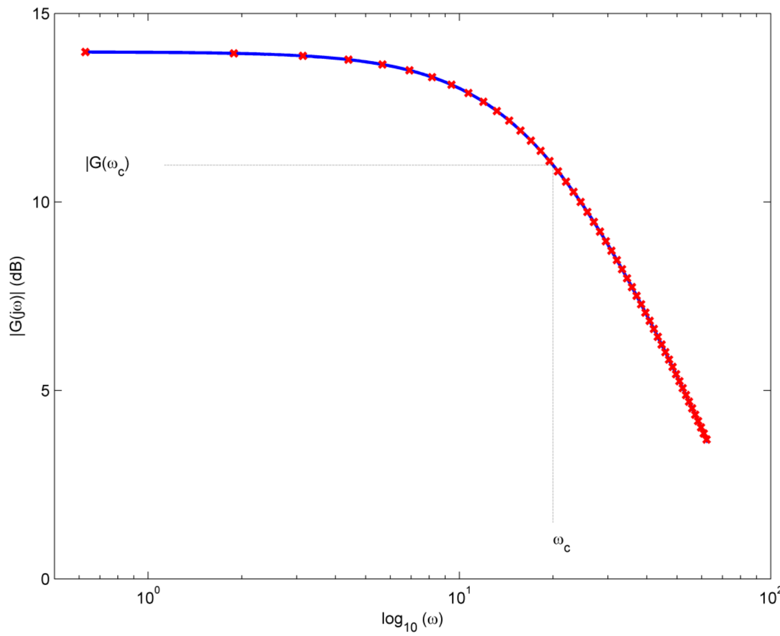

Figure 6. Magnitude Bode plot

ii. Magnitude Response of a DC motor

Recall the first order linear time invariant (LTI) model of the input voltage-to-speed DC motor transfer function shown in equation 2.6. The time constant τ can be obtained by applying sinusoidal inputs and observing the system response.

The frequency response is obtained by substituting $s = j\omega$ into the transfer function (equation 2.6). the magnitude of the frequency response at a frequency ω of the motor input voltage is then defined as

$$|G(j\omega)| = \left|\frac{\Omega_m(j\omega)}{V_m(j\omega)}\right| = \frac{K}{\tau j\omega + 1} \qquad 3.1$$

The magnitude or absolute value of a complex number $z = x + jy$ is defined as:

$$|z| = |x + jy| = \sqrt{\Re z^{\,2} + \Im z^2} = \sqrt{x^2 + y^2} \qquad 3.2$$

Therefore, determining the absolute value of the right side of equation 3.1 subsequently leads to the following expression for the **system's magnitude response** with respect to the input frequency:

$$|\boldsymbol{G}(\boldsymbol{\omega})| = \frac{\boldsymbol{K}}{\sqrt{\boldsymbol{1} + \boldsymbol{\tau}^2\boldsymbol{\omega}^2}} \qquad 3.3$$

The system's steady-state (or low frequency) gain can then be obtained by setting $\omega = 0$, i.e. applying a constant signal:

$$K = |G(0)| \qquad 3.4$$

Consider a DC motor with $K = 10\,rad/s/V$ and $\tau = 0.1\ seconds$ and using equation 3.3:

1. At very low frequency ($\omega \approx 0$): $|G(0)| = K = 10\,rad/s/V$ The motor follows inputs perfectly

2. At the cutoff frequency ($\omega = 1/\tau = 10\,rad/s$): $|G(10)| = \frac{10}{\sqrt{1+1}} = 7.07\,rad/s/V$ The response has dropped to about 70.7% of its original amplitude

3. At high frequency ($\omega = 100\,rad/s$): $|G(100)| = \frac{10}{\sqrt{1+100}} = 0.995\,rad/s/V$ The motor can barely follow the input

This shows how the motor's ability to follow input commands decreases as frequency increases.

b. Phase Delay Analysis

Another way to determine τ is to perform a phase delay analysis, i.e. to investigate by how much the system's response lags the system's sinusoidal input. The phase delay (or phase angle) of a complex number $z = x + jy$ is defined as

$$\angle z = \tan^{-1}\left(\frac{\Im\{z\}}{\Re\{z\}}\right) = \tan^{-1}\left(\frac{y}{x}\right) \qquad 3.5$$

The phase angle, or phase delay of a transfer function is:

$$\Phi_d = \Phi_{num} - \Phi_{den} \qquad 3.6$$

where Φ_{num} and Φ_{den} represent the phase angle/delay of the numerator and denominator of the transfer function, respectively.

The effect of this phase delay can also be observed in the delay of input/output graphs of Figure 6. Here, the phase shift can be expressed as

$$\Phi_d = \frac{t_2}{t_1} \times 360° \qquad 3.7$$

Where t_1 is the period T of the signal and t_2 is the time delay between input and output.

i. Phase Delay for a DC motor

Substituting $s = j\omega$ into the DC motor transfer function (equation 2.6), we get both magnitude and phase information. This equation is shown in equation 3.1.

To find the phase angle, the equation needs to be converted into a form that can separate real and imaginary parts:

$$G(j\omega) = \frac{K}{\tau j\omega + 1} \times \frac{\tau(-j)\omega + 1}{\tau(-j)\omega + 1} = \frac{K(1 - \tau j\omega)}{1 + (\tau\omega)^2} = \frac{K}{1 + (\tau\omega)^2} - j\frac{K\tau\omega}{1 + (\tau\omega)^2} \quad 3.8$$

Since K is real, $\angle(K) = 0$, the phase angle can be found using the arctangent of the ratio of imaginary to real parts:

$$\angle G(j\omega) = \angle(K) - \angle(\tau j\omega + 1) = \tan^{-1}\left(-\frac{K\tau\omega}{K}\right) = -\tan^{-1}(\tau\omega) \quad 3.9$$

The negative sign appears because the imaginary part is negative, indicating that the output lags behind the input.

Consider the same DC motor with $K = 10\ rad/s/V$ and $\tau = 0.1\ seconds$:

1. At very low frequency ($\omega \approx 0$): $\angle G(0) = -\tan^{-1}(0) = 0°$. The output almost perfectly aligns with the input (no delay).

2. At the cutoff frequency $\left(\omega = \frac{1}{\tau} = 10 rad/s\right)$: $\angle G(10) = -\tan^{-1}(1) = 45°$. The output lags the input by 45 degrees, showing that once the cutoff frequency is reached, there's a significant phase delay. The time delay here is:

$$t_2 = (-45° \times T) / 360° = (-45° \times 2\pi/10) / 360° \approx 0.079\ seconds$$

3. At high frequency $\left(\omega = 100\frac{rad}{s}\right)$: $\angle G(100) = -\tan^{-1}(10) = -84.3°$. The output severely lags the input which combined with the reduced magnitude, makes the motor unable to effectively follow high-frequency commands

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