Inverse Kinematics

1 Forward Kinematic equations

Recall the forward kinematics formulation f_{FPK}

$$\begin{gathered}\vec{p} = f_{FPK}(\vec{\theta}) \\ p_x = \lambda_2 c_1 c_2 - \lambda_3 c_1 s_{23} \\ p_y = \lambda_2 s_1 c_2 - \lambda_3 s_1 s_{23} \\ p_z = \lambda_1 - \lambda_2 s_2 - \lambda_3 c_{23}\end{gathered} \tag{1}$$

where $\vec{\theta}$ is the vector representing the joint states,

$$\vec{\theta} = \begin{bmatrix}\theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4\end{bmatrix} \tag{2}$$

and $\vec{p}$ is the position of the end-effector {4} (expressed in base frame {0})

$$\vec{p} = \begin{bmatrix}p_x \\ p_y \\ p_z\end{bmatrix} = {}^0p_4 \tag{3}$$

2 Inverse Kinematic Formulation

The goal of the inverse kinematics formulation f_{IPK} is to do the opposite. We will develop the formulation

$$\vec{\theta} = f_{IPK}(\vec{p}) \tag{4}$$

2.1 End-Effector Position

Begin with the equations describing p_x and p_y.

$$\begin{gathered}p_x = \lambda_2 c_1 c_2 - \lambda_3 c_1 s_{23} = c_1 (\lambda_2 c_2 - \lambda_3 s_{23}) \\ p_y = \lambda_2 s_1 c_2 - \lambda_3 s_1 s_{23} = s_1 (\lambda_2 c_2 - \lambda_3 s_{23})\end{gathered} \tag{5}$$

Next, we square and then add the equations describing p_x and p_y, giving,

$$\begin{gathered}p_x^2 + p_y^2 = c_1^2 (\lambda_2 c_2 - \lambda_3 s_{23})^2 + s_1^2 (\lambda_2 c_2 - \lambda_3 s_{23})^2 = (\lambda_2 c_2 - \lambda_3 s_{23})^2 \\ \sqrt{p_x^2 + p_y^2} = \lambda_2 c_2 - \lambda_3 s_{23}\end{gathered} \tag{6}$$

Equations 1 and 8 then provide the following set,

$$\begin{gathered}\sqrt{p_x^2 + p_y^2} = \lambda_2 c_2 - \lambda_3 s_{23} \\ \lambda_1 - p_z = \lambda_2 s_2 + \lambda_3 c_{23}\end{gathered} \tag{7}$$

This resembles Case (d) in Appendix A,

$$
\begin{aligned}
A\cos q_1 + B\cos(q_1+q_2) + C\sin(q_1+q_2) &= D \\
A\sin q_1 + B\sin(q_1+q_2) - C\cos(q_1+q_2) &= H
\end{aligned}
\tag{8}
$$

Where the following mapping applies,

$$
\begin{aligned}
A &= \lambda_2 \\
B &= 0 \\
C &= -\lambda_3 \\
D &= \pm\sqrt{p_x^2 + p_y^2} \\
H &= \lambda_1 - p_z \\
q_1 &= \theta_2 \\
q_2 &= \theta_3 \\
F &= \frac{D^2+H^2-A^2-C^2}{2A}
\end{aligned}
\tag{9}
$$

Note that there are two possible values for D. Regardless, the solution for q_2 involves squaring the D term which clears ambiguity (see appendix A). The two possible solutions for θ_3 are,

$$
\begin{aligned}
q_{2,1} = \theta_{3,1} = 2\tan^{-1}\left(\frac{C+\sqrt{C^2+F^2}}{F}\right) = 2\,atan2\left(C + \sqrt{C^2+F^2}\,,\ F\right) \\
q_{2,2} = \theta_{3,2} = 2\tan^{-1}\left(\frac{C-\sqrt{C^2+F^2}}{F}\right) = 2\,atan2\left(C - \sqrt{C^2+F^2}\,,\ F\right)
\end{aligned}
\tag{10}
$$

Note that the function $atan2$ takes the y and x components into account separately to provide a solution in the range $[-\pi, \pi]$ as supposed to the $atan$ function, which takes a single input p_y/p_x and only provides solutions in the range $[-\pi/2,\ \pi/2]$.

Before solving for θ_2, some additional mapping is required,

$$
\begin{aligned}
M &= A + C\sin q_2 \\
N &= -C\cos q_2 \\
\cos q_1 &= \frac{DM+HN}{M^2+N^2} \\
\sin q_1 &= \frac{H-N\cos q_1}{M}
\end{aligned}
\tag{11}
$$

The possible solutions for q_1 or θ_2 are calculated using the same equation,

$$
q_1 = \theta_2 = \tan^{-1}\left(\frac{\sin q_1}{\cos q_1}\right) = atan2\left(\sin q_1\ ,\ \cos q_1\right) \tag{12}
$$

The terms M and N have 2 possible values due to the two solutions for θ_3. For each of these, the $\cos q_1$ term has 2 possible values due to the D term. This results in a total of 4 possible solutions, all found using equation 12.

Now knowing θ_2 and θ_3, return to equation 5, and take the following ratio

$$\tan\theta_1 = \frac{{p_y}/{\lambda_2 c_2 - \lambda_3 s_{23}}}{{p_x}/{\lambda_2 c_2 - \lambda_3 s_{23}}} \tag{13}$$

Using the inverse tangent, we can solve for θ_1.

$$\theta_1 = \tan^{-1}\left(\frac{{p_y}/{\lambda_2 c_2 - \lambda_3 s_{23}}}{{p_x}/{\lambda_2 c_2 - \lambda_3 s_{23}}}\right) = \ atan2\left({p_y}/{\lambda_2 c_2 - \lambda_3 s_{23}}\ ,\ \ {p_x}/{\lambda_2 c_2 - \lambda_3 s_{23}}\right) \tag{14}$$

Although the term $\lambda_2 c_2 - \lambda_3 s_{23}$ is common in the numerator and denominator of the ratio in equation 14, note that it would be unwise to simply cancel it out. The sign of this term matters and will result in a unique θ_1 solution for each set of θ_2 and θ_3.

2.2 End-Effector Orientation

Although the wrist angle θ_4 is not calculated from the position above, an end-effector orientation may also be provided in the form of a rotation matrix. Knowing the possible values for θ_1, θ_2 and θ_3, one can solve the rotation matrix partially and match terms to derive θ_4.

A user cannot, however, specify an arbitrary position and orientation for the end-effector. There are 6 general degrees of freedom for position and orientation, however, there are only 4 degrees of freedom in the manipulator. Often, the orientation may simply be specified as a wrist angle γ, and the solution for the wrist angle is

$$\theta_4 = \gamma \tag{13}$$

See final equations on next page.

2.3 NET RESULT

The Inverse Kinematics formulation is then given as,

$$\theta_3 = 2\,atan2\left(-\lambda_3 \pm \sqrt{{\lambda_3}^2 + \left(\frac{p_x^2 + p_y^2 + (\lambda_1 - p_z)^2 - \lambda_2^2 - \lambda_3^2}{2\,\lambda_2}\right)^2}\;,\; \frac{p_x^2 + p_y^2 + (\lambda_1 - p_z)^2 - \lambda_2^2 - \lambda_3^2}{2\,\lambda_2}\right)$$

$$\text{(2 solutions possible)}$$

$$\theta_2 = atan2\left(\frac{\lambda_1 - p_z - \lambda_3\cos\theta_3\left(\frac{\pm(\lambda_2 - \lambda_3\sin\theta_3)\sqrt{p_x^2 + p_y^2} + (\lambda_3\cos\theta_3)(\lambda_1 - p_z)}{(\lambda_2 - \lambda_3\sin\phi_3)^2 + (\lambda_3\cos\theta_3)^2}\right)}{M}\;,\; \frac{\pm(\lambda_2 - \lambda_3\sin\theta_3)\sqrt{p_x^2 + p_y^2} + (\lambda_3\cos\theta_3)(\lambda_1 - p_z)}{(\lambda_2 - \lambda_3\sin\theta_3)^2 + (\lambda_3\cos\theta_3)^2}\right)$$

$$\text{(2 solutions possible for each } \phi_3\text{)} \tag{14}$$

$$\phi_1 = \tan^{-1}\left(\frac{{p_y}/{\lambda_2\cos\theta_2 - \lambda_3\sin(\theta_2 + \theta_3)}}{{p_x}/{\lambda_2\cos\theta_2 - \lambda_3\sin(\theta_2 + \theta_3)}}\right) = atan2\left({p_y}/{\lambda_2\cos\theta_2 - \lambda_3\sin(\theta_2 + \theta_3)}\;,\; {p_x}/{\lambda_2\cos\theta_2 - \lambda_3\sin(\theta_2 + \theta_3)}\right)$$

$$\text{(1 unique solution possible for each } \theta_2 \text{ and } \theta_3\text{)}$$

$$\theta_4 = \gamma$$

$$\text{(1 unique solution)}$$