Forward Kinematics

1 TRANSFORMATION MATRICES

The DH table for the QArm manipulator is presented in Table 1 (See 1. Frame Assignments for more information).

i	a_i	α_i	d_i	$ heta_i$
1	0	$-\pi/2$	λ_1	$ heta_1$
2	λ_2	0	0	$ heta_2$
3	0	$-\pi/2$	0	$ heta_3$
4	0	0	λ_3	$ heta_4$

Table 1. DH Table

Begin with the general form of the transformation matrix,

$${}^{i-1}T_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i\\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i\\ 0 & \sin\alpha_i & \cos\alpha_i & d_i\\ 0 & 0 & 0 & 1 \end{bmatrix}$$
(1)

Using Equation 1 and Table 1, we can find all the transformation matrices. For brevity, we can abbreviate cosines as c and sines as s. In addition, the revolute joint variables θ will be dropped. Thus, the term $\cos\theta_1$ can be abbreviated as c_1 . Similarly, a term such as $\sin(\theta_1 + \theta_2)$ can be abbreviated as s_{12} .

$${}^{0}T_{1} = \begin{bmatrix} c_{1} & 0 & -s_{1} & 0 \\ s_{1} & 0 & c_{1} & 0 \\ 0 & -1 & 0 & \lambda_{1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{1}T_{2} = \begin{bmatrix} c_{2} & -s_{2} & 0 & \lambda_{2} c_{2} \\ s_{2} & c_{2} & 0 & \lambda_{2} s_{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{2}T_{3} = \begin{bmatrix} c_{3} & 0 & -s_{3} & 0 \\ s_{3} & 0 & c_{3} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{3}T_{4} = \begin{bmatrix} c_{4} & -s_{4} & 0 & 0 \\ s_{4} & c_{4} & 0 & 0 \\ 0 & 0 & 1 & \lambda_{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(2)$$

2 FORWARD KINEMATICS FORMULATION

The joint vector $\vec{\theta}$ represents the joint states of the manipulator,

$$\vec{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} \tag{3}$$

and the position \overrightarrow{p} of the end-effector $\{4\}$ (expressed in base frame $\{0\}$)

$$\overrightarrow{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = {}^{0}p_4 \tag{4}$$

The goal of the forward kinematics formulation is to provide a mapping from the joint space to the position space

$$\overrightarrow{p} = f_{FPK}(\overrightarrow{\theta}) \tag{5}$$

Using the matrices in equation 2, we can proceed to derive the matrices ${}^{0}T_{2}$, ${}^{0}T_{3}$ and ${}^{0}T_{4}$.

$${}^{0}T_{2} = {}^{0}T_{1}{}^{1}T_{2} = \begin{bmatrix} c_{1}c_{2} & -c_{1}s_{2} & -s_{1} & \lambda_{2}c_{1}c_{2} \\ s_{1}c_{2} & -s_{1}s_{2} & c_{1} & \lambda_{2}s_{1}c_{2} \\ -s_{2} & -c_{2} & 0 & \lambda_{1} - \lambda_{2}s_{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
 (6)

$${}^{0}T_{3} = {}^{0}T_{2}{}^{2}T_{3} = \begin{bmatrix} c_{1}c_{23} & s_{1} & -c_{1}s_{23} & \lambda_{2}c_{1}c_{2} \\ s_{1}c_{23} & -c_{1} & -s_{1}s_{23} & \lambda_{2}s_{1}c_{2} \\ -s_{23} & 0 & -c_{23} & \lambda_{1} - \lambda_{2}s_{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
 (7)

$${}^{0}T_{2} = {}^{0}T_{1}{}^{1}T_{2} = \begin{bmatrix} c_{1}c_{2} & -c_{1}s_{2} & -s_{1} & \lambda_{2}c_{1}c_{2} \\ s_{1}c_{2} & -s_{1}s_{2} & c_{1} & \lambda_{2}s_{1}c_{2} \\ -s_{2} & -c_{2} & 0 & \lambda_{1} - \lambda_{2}s_{2} \end{bmatrix}$$

$${}^{0}T_{3} = {}^{0}T_{2}{}^{2}T_{3} = \begin{bmatrix} c_{1}c_{23} & s_{1} & -c_{1}s_{23} & \lambda_{2}c_{1}c_{2} \\ s_{1}c_{23} & -c_{1} & -s_{1}s_{23} & \lambda_{2}s_{1}c_{2} \\ -s_{23} & 0 & -c_{23} & \lambda_{1} - \lambda_{2}s_{2} \end{bmatrix}$$

$${}^{0}T_{4} = {}^{0}T_{3}{}^{3}T_{4} = \begin{bmatrix} c_{1}c_{23}c_{4} + s_{1}s_{4} & -c_{1}c_{23}s_{4} + s_{1}c_{4} & -c_{1}s_{23} & \lambda_{2}c_{1}c_{2} - \lambda_{3}c_{1}s_{23} \\ s_{1}c_{23}c_{4} - c_{1}s_{4} & -s_{1}c_{23}s_{4} - c_{1}c_{4} & -s_{1}s_{23} & \lambda_{2}s_{1}c_{2} - \lambda_{3}s_{1}s_{23} \\ -s_{23}c_{4} & s_{23}s_{4} & -c_{23} & \lambda_{1} - \lambda_{2}s_{2} - \lambda_{3}c_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(8)$$

Consider the vector representing the position of the end-effector frame $\{4\}$ \overrightarrow{p} . From the matrix ${}^{0}T_{4}$ in equation 8, we can extract this vector expressed in frame $\{0\}$ as 0p_4 , and can also extract the orientation (via rotation matrix) ${}^{0}R_{4}$ of the end-effector frame {4} with respect to the base frame {0}.

$$p_{x} = \lambda_{2}c_{1}c_{2} - \lambda_{3}c_{1}s_{23}$$

$$p_{y} = \lambda_{2}s_{1}c_{2} - \lambda_{3}s_{1}s_{23}$$

$$p_{z} = \lambda_{1} - \lambda_{2}s_{2} - \lambda_{3}c_{23}$$
(9)

Equation 9 represents the forward kinematics formulation f_{FPK} . Note that the wrist rotation angle $heta_4$ does not play a part in the position of the end-effector, and correspondingly does not appear in f_{FPK} . However, it does play a part in the end-effector orientation,

$${}^{0}R_{4} = \begin{bmatrix} c_{1}c_{23}c_{4} + s_{1}s_{4} & -c_{1}c_{23}s_{4} + s_{1}c_{4} & -c_{1}s_{23} \\ s_{1}c_{23}c_{4} - c_{1}s_{4} & -s_{1}c_{23}s_{4} - c_{1}c_{4} & -s_{1}s_{23} \\ -s_{23}c_{4} & s_{23}s_{4} & -c_{23} \end{bmatrix}$$
(10)