Dynamics - Torques

1 LAGRANGIAN

Beginning with the Lagrangian,

$$
\begin{aligned}
\mathcal{L} = \Big(&\frac{1}{2} I_{1A} + \frac{1}{2} m_2(\lambda_2 - \lambda_{c2})^2 c_2^2 + \frac{1}{2} I_{2A} s_2^2 + \frac{1}{2} I_{2L} c_2^2 + \frac{1}{2} m_3 \lambda_2^2 c_2^2 + \frac{1}{2} m_3 \lambda_{c3}^2 s_{23}^2 - m_3 \lambda_2 \lambda_{c3} c_2 s_{23} + \frac{1}{2} I_{3L} s_{23}^2 + \frac{1}{2} I_{3A} c_{23}^2 \\
&+ \frac{1}{2} m_4 (\lambda_3 - \lambda_{c4})^2 s_{23}^2 + \frac{1}{2} m_4 \lambda_2^2 c_2^2 - m_4(\lambda_3 - \lambda_{c4})\lambda_2 c_2 s_{23} + \frac{1}{2} I_{4L} s_{23}^2 + \frac{1}{2} I_{4A} c_{23}^2 + \frac{1}{2} m_L \lambda_3^2 s_{23}^2 \\
&+ \frac{1}{2} m_L \lambda_2^2 c_2^2 - m_L \lambda_2 \lambda_3 c_2 s_{23}\Big) \dot{\theta}_1^2 \\
&+ \Big(\frac{1}{2} m_2 (\lambda_2 - \lambda_{c2})^2 + \frac{1}{2} I_{2L} + \frac{1}{2} m_3 \lambda_2^2 + \frac{1}{2} m_3 \lambda_{c3}^2 - m_3 \lambda_2 \lambda_{c3} s_3 + \frac{1}{2} I_{3L} + \frac{1}{2} m_4 (\lambda_2 + \lambda_3 - \lambda_{c4})^2 \\
&+ \frac{1}{2} I_{4L} + \frac{1}{2} m_L \lambda_2^2 + \frac{1}{2} m_L \lambda_3^2 - m_L \lambda_2 \lambda_3 s_3\Big) \dot{\theta}_2^2 \\
&+ \Big(\frac{1}{2} m_3 \lambda_{c3}^2 + \frac{1}{2} I_{3L} + \frac{1}{2} m_4 (\lambda_{c4} - \lambda_3)^2 + \frac{1}{2} I_{4L} + \frac{1}{2} m_L \lambda_3^2\Big) \dot{\theta}_3^2 + \Big(\frac{1}{2} I_{4A}\Big) \dot{\theta}_4^2 - (I_{4A} c_{23})\, \dot{\theta}_1 \dot{\theta}_4 \\
&+ \Big(m_3 \lambda_{c3}^2 - m_3 \lambda_{c3} \lambda_2 s_3 + I_{3L} + m_4 (\lambda_3 - \lambda_{c4} - \lambda_2 s_3)\,(\lambda_3 - \lambda_{c4}) + I_{4L} - m_L\big(\lambda_3^2 - \lambda_2 \lambda_3 s_3\big)\Big) \dot{\theta}_2 \dot{\theta}_3 \\
&- m_1 g\, (\lambda_1 - \lambda_{c1}) - m_2 g\, (\lambda_1 - (\lambda_2 - \lambda_{c2}) s_2) - m_3 g\, (\lambda_1 - \lambda_2 s_2 - \lambda_{c3} c_{23}) \\
&- m_4 g\, (\lambda_1 - \lambda_2 s_2 - (\lambda_3 - \lambda_{c4}) c_{23}) - m_L g\, (\lambda_1 - \lambda_2 s_2 - \lambda_3 c_{23})
\end{aligned}
\tag{1}
$$

We can use the following equation to derive the joint torques,

$$\tau_i = \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}_i}\right) - \frac{d\mathcal{L}}{d\theta_i} \tag{2}$$

2 JOINT 4

The torque required at the fourth joint is then given by,

$$\tau_4 = \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}_4}\right) - \frac{d\mathcal{L}}{d\theta_4} \tag{3a}$$

$$\frac{d\mathcal{L}}{d\dot{\phi}_4} = I_{4A}\dot{\theta}_4 - I_{4A} c_{23} \dot{\theta}_1 \,, \quad \frac{d\mathcal{L}}{d\theta_4} = 0 \tag{3b}$$

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}_4}\right) = I_{4A}\ddot{\theta}_4 - I_{4A} c_{23} \ddot{\theta}_1 + I_{4A} s_{23} \dot{\theta}_1 \dot{\theta}_2 + I_{4A} s_{23} \dot{\theta}_1 \dot{\theta}_3 \tag{3c}$$

$$\tau_4 = -\{I_{4A} c_{23}\}\ddot{\theta}_1 + \{I_{4A}\}\ddot{\theta}_4 + \{I_{4A} s_{23}\}\dot{\theta}_1 \dot{\theta}_2 + \{I_{4A} s_{23}\}\dot{\theta}_1 \dot{\theta}_3 \tag{3d}$$

3 JOINT 3

The torque required at the fourth joint is then given by,

$$\tau_3 = \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}_3}\right) - \frac{d\mathcal{L}}{d\theta_3} \tag{4a}$$

$$\begin{aligned}\frac{d\mathcal{L}}{d\dot{\theta}_3} &= (m_3\lambda_{c3}^2 + I_{3L} + m_4(\lambda_{c4} - \lambda_3)^2 + I_{4L} + m_L\lambda_3^2)\dot{\theta}_3 \\ &+ \big(m_3\lambda_{c3}^2 - m_3\lambda_{c3}\lambda_2 s_3 + I_{3L} + m_4(\lambda_3 - \lambda_{c4} - \lambda_2 s_3)\,(\lambda_3 - \lambda_{c4}) + I_{4L} \\ &- m_L(\lambda_3^2 - \lambda_2\lambda_3 s_3)\big)\dot{\theta}_2\end{aligned} \tag{4b}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}_3}\right) &= (m_3\lambda_{c3}^2 + I_{3L} + m_4(\lambda_{c4} - \lambda_3)^2 + I_{4L} + m_L\lambda_3^2)\ddot{\theta}_3 \\ &+ \big(m_3\lambda_{c3}^2 - m_3\lambda_{c3}\lambda_2 s_3 + I_{3L} + m_4(\lambda_3 - \lambda_{c4} - \lambda_2 s_3)\,(\lambda_3 - \lambda_{c4}) + I_{4L} \\ &- m_L(\lambda_3^2 - \lambda_2\lambda_3 s_3)\big)\ddot{\phi}_2 + (m_L\lambda_2\lambda_3 c_3 - m_3\lambda_{c3}\lambda_2 c_3 - m_4\lambda_2(\lambda_3 - \lambda_{c4})c_3)\dot{\theta}_2\dot{\theta}_3\end{aligned} \tag{4c}$$

$$\begin{aligned}\frac{d\mathcal{L}}{d\theta_3} &= (m_3\lambda_{c3}^2 s_{23}c_{23} - m_3\lambda_2\lambda_{c3}c_2c_{23} + I_{3L}s_{23}c_{23} - I_{3A}s_{23}c_{23} + m_4(\lambda_3 - \lambda_{c4})^2 s_{23}c_{23} \\ &- m_4(\lambda_3 - \lambda_{c4})\lambda_2 c_2 c_{23} + I_{4L}s_{23}c_{23} - I_{4A}s_{23}c_{23} + m_L\lambda_3^2 s_{23}c_{23} - m_L\lambda_2\lambda_3 c_2 c_{23})\dot{\theta}_1^2 \\ &- (m_3\lambda_2\lambda_{c3}c_3 + m_L\lambda_2\lambda_3 c_3)\dot{\theta}_2^2 + (I_{4A}s_{23})\,\dot{\theta}_1\dot{\theta}_4 \\ &- \big(m_3\lambda_{c3}\lambda_2 c_3 + m_4\lambda_2 c_3(\lambda_3 - \lambda_{c4}) - m_L(\lambda_2\lambda_3 c_3)\big)\dot{\theta}_2\dot{\theta}_3 - m_3 g\,\lambda_{c3}s_{23} \\ &- m_4 g(\lambda_3 - \lambda_{c4})s_{23} - m_L g\lambda_3 s_{23}\end{aligned}$$

$$\begin{aligned}\tau_3 = &\{m_3\lambda_{c3}^2 - m_3\lambda_{c3}\lambda_2 s_3 + I_{3L} + m_4(\lambda_3 - \lambda_{c4} - \lambda_2 s_3)\,(\lambda_3 - \lambda_{c4}) + I_{4L} - m_L(\lambda_3^2 - \lambda_2\lambda_3 s_3)\}\,\ddot{\theta}_2 \\ &+ \{m_3\lambda_{c3}^2 + I_{3L} + m_4(\lambda_{c4} - \lambda_3)^2 + I_{4L} + m_L\lambda_3^2\}\,\ddot{\theta}_3 \\ &- \{m_3\lambda_{c3}^2 s_{23}c_{23} - m_3\lambda_2\lambda_{c3}c_2c_{23} + I_{3L}s_{23}c_{23} - I_{3A}s_{23}c_{23} + m_4(\lambda_3 - \lambda_{c4})^2 s_{23}c_{23} \\ &- m_4(\lambda_3 - \lambda_{c4})\lambda_2 c_2 c_{23} + I_{4L}s_{23}c_{23} - I_{4A}s_{23}c_{23} + m_L\lambda_3^2 s_{23}c_{23} - m_L\lambda_2\lambda_3 c_2 c_{23}\}\,\dot{\theta}_1^2 \\ &+ \{m_3\lambda_2\lambda_{c3}c_3 + m_L\lambda_2\lambda_3 c_3\}\dot{\theta}_2^2 - \{I_{4A}s_{23}\}\,\dot{\theta}_1\dot{\theta}_4 \\ &+ g\{m_3\,\lambda_{c3}s_{23} + m_4(\lambda_3 - \lambda_{c4})s_{23} + m_L\lambda_3 s_{23}\}\end{aligned} \tag{4d}$$

4 JOINT 2

The torque required at the fourth joint is then given by,

$$\tau_2 = \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}_2}\right) - \frac{d\mathcal{L}}{d\theta_2} \tag{5a}$$

$$\begin{aligned}\frac{d\mathcal{L}}{d\dot{\theta}_2} = {} & (m_2(\lambda_2 - \lambda_{c2})^2 + I_{2L} + m_3\lambda_2^2 + m_3\lambda_{c3}^2 - 2m_3\lambda_2\lambda_{c3}s_3 + I_{3L} + m_4(\lambda_2 + \lambda_3 - \lambda_{c4})^2 + I_{4L} + m_L\lambda_2^2 + m_L\lambda_3^2 \\ & - 2m_L\lambda_2\lambda_3 s_3)\dot{\theta}_2 \\ & + \left(m_3\lambda_{c3}^2 - m_3\lambda_{c3}\lambda_2 s_3 + I_{3L} + m_4(\lambda_3 - \lambda_{c4} - \lambda_2 s_3)(\lambda_3 - \lambda_{c4}) + I_{4L} - m_L(\lambda_3^2 - \lambda_2\lambda_3 s_3)\right)\dot{\theta}_3\end{aligned} \tag{5b}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}_2}\right) = {} & (m_2(\lambda_2 - \lambda_{c2})^2 + I_{2L} + m_3\lambda_2^2 + m_3\lambda_{c3}^2 - 2m_3\lambda_2\lambda_{c3}s_3 + I_{3L} + m_4(\lambda_2 + \lambda_3 - \lambda_{c4})^2 + I_{4L} + m_L\lambda_2^2 \\ & + m_L\lambda_3^2 - 2m_L\lambda_2\lambda_3 s_3)\ddot{\theta}_2 \\ & + \left(m_3\lambda_{c3}^2 - m_3\lambda_{c3}\lambda_2 s_3 + I_{3L} + m_4(\lambda_3 - \lambda_{c4} - \lambda_2 s_3)(\lambda_3 - \lambda_{c4}) + I_{4L} - m_L(\lambda_3^2 - \lambda_2\lambda_3 s_3)\right)\ddot{\theta}_3 \\ & - (2m_3\lambda_2\lambda_{c3}c_3 + 2m_L\lambda_2\lambda_3 c_3)\dot{\theta}_2\dot{\theta}_3 - \left(m_3\lambda_{c3}\lambda_2 c_3 + m_4\lambda_2 c_3(\lambda_3 - \lambda_{c4}) - m_L(\lambda_2\lambda_3 c_3)\right)\dot{\theta}_3^2\end{aligned} \tag{5c}$$

$$\begin{aligned}\frac{d\mathcal{L}}{d\theta_2} = {} & (-m_2(\lambda_2 - \lambda_{c2})^2 s_2 c_2 + I_{2A}s_2c_2 - I_{2L}s_2c_2 - m_3\lambda_2^2 s_2c_2 + m_3\lambda_{c3}^2 s_{23}c_{23} + m_3\lambda_2\lambda_{c3}s_2s_{23} - m_3\lambda_2\lambda_{c3}c_2c_{23} \\ & + I_{3L}s_{23}c_{23} - I_{3A}s_{23}c_{23} + m_4(\lambda_3 - \lambda_{c4})^2 s_{23}c_{23} - m_4\lambda_2^2 s_2c_2 + m_4(\lambda_3 - \lambda_{c4})\lambda_2 s_2 s_{23} \\ & - m_4(\lambda_3 - \lambda_{c4})\lambda_2 c_2 c_{23} + I_{4L}s_{23}c_{23} - I_{4A}s_{23}c_{23} + m_L\lambda_3^2 s_{23}c_{23} - m_L\lambda_2^2 s_2c_2 + m_L\lambda_2\lambda_3 s_2 s_{23} \\ & - m_L\lambda_2\lambda_3 c_2 c_{23})\dot{\theta}_1^2 + (I_{4A}s_{23})\,\dot{\theta}_1\dot{\theta}_4 + m_2 g\,(\lambda_2 - \lambda_{c2})c_2 + m_3 g\,(\lambda_2 c_2 - \lambda_{c3}s_{23}) \\ & + m_4 g\,(\lambda_2 c_2 - (\lambda_3 - \lambda_{c4})s_{23}) + m_L g\,(\lambda_2 c_2 - \lambda_3 s_{23})\end{aligned}$$

$$\begin{aligned}\tau_2 = {} & \{m_2(\lambda_2 - \lambda_{c2})^2 + I_{2L} + m_3\lambda_2^2 + m_3\lambda_{c3}^2 - 2m_3\lambda_2\lambda_{c3}s_3 + I_{3L} + m_4(\lambda_2 + \lambda_3 - \lambda_{c4})^2 + I_{4L} + m_L\lambda_2^2 + m_L\lambda_3^2 \\ & - 2m_L\lambda_2\lambda_3 s_3\}\,\ddot{\theta}_2 \\ & + \{m_3\lambda_{c3}^2 - m_3\lambda_{c3}\lambda_2 s_3 + I_{3L} + m_4(\lambda_3 - \lambda_{c4} - \lambda_2 s_3)(\lambda_3 - \lambda_{c4}) + I_{4L} - m_L(\lambda_3^2 - \lambda_2\lambda_3 s_3)\}\,\ddot{\theta}_3 \\ & - \{-m_2(\lambda_2 - \lambda_{c2})^2 s_2c_2 + I_{2A}s_2c_2 - I_{2L}s_2c_2 - m_3\lambda_2^2 s_2c_2 + m_3\lambda_{c3}^2 s_{23}c_{23} + m_3\lambda_2\lambda_{c3}s_2s_{23} \\ & - m_3\lambda_2\lambda_{c3}c_2c_{23} + I_{3L}s_{23}c_{23} - I_{3A}s_{23}c_{23} + m_4(\lambda_3 - \lambda_{c4})^2 s_{23}c_{23} - m_4\lambda_2^2 s_2c_2 \\ & + m_4(\lambda_3 - \lambda_{c4})\lambda_2 s_2 s_{23} - m_4(\lambda_3 - \lambda_{c4})\lambda_2 c_2 c_{23} + I_{4L}s_{23}c_{23} - I_{4A}s_{23}c_{23} + m_L\lambda_3^2 s_{23}c_{23} \\ & - m_L\lambda_2^2 s_2c_2 + m_L\lambda_2\lambda_3 s_2s_{23} - m_L\lambda_2\lambda_3 c_2c_{23}\}\,\dot{\theta}_1^2 \\ & - \{m_3\lambda_{c3}\lambda_2 c_3 + m_4\lambda_2 c_3(\lambda_3 - \lambda_{c4}) - m_L(\lambda_2\lambda_3 c_3)\}\dot{\theta}_3^2 - \{I_{4A}s_{23}\}\,\dot{\theta}_1\dot{\theta}_4 \\ & - \{2m_3\lambda_2\lambda_{c3}c_3 + 2m_L\lambda_2\lambda_3 c_3\}\dot{\theta}_2\dot{\theta}_3 \\ & - g\,\{m_2(\lambda_2 - \lambda_{c2})c_2 + m_3(\lambda_2 c_2 - \lambda_{c3}s_{23}) + m_4(\lambda_2 c_2 - (\lambda_3 - \lambda_{c4})s_{23}) + m_L(\lambda_2 c_2 - \lambda_3 s_{23})\}\end{aligned} \tag{5d}$$

5 JOINT 1

The torque required at the fourth joint is then given by,

$$\tau_1 = \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}_1}\right) - \frac{d\mathcal{L}}{d\theta_1} \tag{5a}$$

$$\begin{aligned}\frac{d\mathcal{L}}{d\dot{\theta}_1} = \big(& I_{1A} + m_2(\lambda_2 - \lambda_{c2})^2 c_2^2 + I_{2A}s_2^2 + I_{2L}c_2^2 + m_3\lambda_2^2 c_2^2 + m_3\lambda_{c3}^2 s_{23}^2 - 2m_3\lambda_2\lambda_{c3}c_2 s_{23} + I_{3L}s_{23}^2 + I_{3A}c_{23}^2 \\ & + m_4(\lambda_3 - \lambda_{c4})^2 s_{23}^2 + m_4\lambda_2^2 c_2^2 - 2m_4(\lambda_3 - \lambda_{c4})\lambda_2 c_2 s_{23} + I_{4L}s_{23}^2 + I_{4A}c_{23}^2 + m_L\lambda_3^2 s_{23}^2 + m_L\lambda_2^2 c_2^2 \\ & - 2m_L\lambda_2\lambda_3 c_2 s_{23}\big)\dot{\theta}_1 - (I_{4A}c_{23})\dot{\theta}_4\end{aligned} \tag{5b}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}_1}\right) = \big(& I_{1A} + m_2(\lambda_2 - \lambda_{c2})^2 c_2^2 + I_{2A}s_2^2 + I_{2L}c_2^2 + m_3\lambda_2^2 c_2^2 + m_3\lambda_{c3}^2 s_{23}^2 - 2m_3\lambda_2\lambda_{c3}c_2 s_{23} + I_{3L}s_{23}^2 + I_{3A}c_{23}^2 \\ & + m_4(\lambda_3 - \lambda_{c4})^2 s_{23}^2 + m_4\lambda_2^2 c_2^2 - 2m_4(\lambda_3 - \lambda_{c4})\lambda_2 c_2 s_{23} + I_{4L}s_{23}^2 + I_{4A}c_{23}^2 + m_L\lambda_3^2 s_{23}^2 + m_L\lambda_2^2 c_2^2 \\ & - 2m_L\lambda_2\lambda_3 c_2 s_{23}\big)\ddot{\theta}_1 - (I_{4A}c_{23})\ddot{\theta}_4 \\ & + \big(-2m_2(\lambda_2 - \lambda_{c2})^2 s_2 c_2 + 2I_{2A}s_2 c_2 - 2I_{2L}s_2 c_2 - 2m_3\lambda_2^2 s_2 c_2 + 2m_3\lambda_{c3}^2 s_{23}c_{23} + 2m_3\lambda_2\lambda_{c3}s_2 s_{23} \\ & - 2m_3\lambda_2\lambda_{c3}c_2 c_{23} + 2I_{3L}s_{23}c_{23} - 2I_{3A}s_{23}c_{23} + 2m_4(\lambda_3 - \lambda_{c4})^2 s_{23}c_{23} - 2m_4\lambda_2^2 s_2 c_2 \\ & + 2m_4(\lambda_3 - \lambda_{c4})\lambda_2 s_2 s_{23} - 2m_4(\lambda_3 - \lambda_{c4})\lambda_2 c_2 c_{23} + 2I_{4L}s_{23}c_{23} - 2I_{4A}s_{23}c_{23} + 2m_L\lambda_3^2 s_{23}c_{23} \\ & - 2m_L\lambda_2^2 s_2 c_2 + 2m_L\lambda_2\lambda_3 s_2 s_{23} - 2m_L\lambda_2\lambda_3 c_2 c_{23}\big)\dot{\theta}_1\dot{\theta}_2 \\ & + \big(2m_3\lambda_{c3}^2 s_{23}c_{23} - 2m_3\lambda_2\lambda_{c3}c_2 c_{23} + 2I_{3L}s_{23}c_{23} - 2I_{3A}s_{23}c_{23} + 2m_4(\lambda_3 - \lambda_{c4})^2 s_{23}c_{23} \\ & - 2m_4(\lambda_3 - \lambda_{c4})\lambda_2 c_2 c_{23} + 2I_{4L}s_{23}c_{23} - 2I_{4A}s_{23}c_{23} + 2m_L\lambda_3^2 s_{23}c_{23} - 2m_L\lambda_2\lambda_3 c_2 c_{23}\big)\dot{\theta}_1\dot{\theta}_3 \\ & + (I_{4A}s_{23})\dot{\theta}_2\dot{\theta}_4 + (I_{4A}s_{23})\dot{\theta}_3\dot{\theta}_4\end{aligned} \tag{5c}$$

$$\frac{d\mathcal{L}}{d\theta_1} = 0$$

$$\begin{aligned}\tau_1 = \{& I_{1A} + m_2(\lambda_2 - \lambda_{c2})^2 c_2^2 + I_{2A}s_2^2 + I_{2L}c_2^2 + m_3\lambda_2^2 c_2^2 + m_3\lambda_{c3}^2 s_{23}^2 - 2m_3\lambda_2\lambda_{c3}c_2 s_{23} + I_{3L}s_{23}^2 + I_{3A}c_{23}^2 + m_4(\lambda_3 - \lambda_{c4})^2 s_{23}^2 \\ & + m_4\lambda_2^2 c_2^2 - 2m_4(\lambda_3 - \lambda_{c4})\lambda_2 c_2 s_{23} + I_{4L}s_{23}^2 + I_{4A}c_{23}^2 + m_L\lambda_3^2 s_{23}^2 + m_L\lambda_2^2 c_2^2 - 2m_L\lambda_2\lambda_3 c_2 s_{23}\}\,\ddot{\theta}_1 \\ & - \{I_{4A}c_{23}\}\,\ddot{\theta}_4 \\ & + \{-2m_2(\lambda_2 - \lambda_{c2})^2 s_2 c_2 + 2I_{2A}s_2 c_2 - 2I_{2L}s_2 c_2 - 2m_3\lambda_2^2 s_2 c_2 + 2m_3\lambda_{c3}^2 s_{23}c_{23} + 2m_3\lambda_2\lambda_{c3}s_2 s_{23} \\ & - 2m_3\lambda_2\lambda_{c3}c_2 c_{23} + 2I_{3L}s_{23}c_{23} - 2I_{3A}s_{23}c_{23} + 2m_4(\lambda_3 - \lambda_{c4})^2 s_{23}c_{23} - 2m_4\lambda_2^2 s_2 c_2 \\ & + 2m_4(\lambda_3 - \lambda_{c4})\lambda_2 s_2 s_{23} - 2m_4(\lambda_3 - \lambda_{c4})\lambda_2 c_2 c_{23} + 2I_{4L}s_{23}c_{23} - 2I_{4A}s_{23}c_{23} + 2m_L\lambda_3^2 s_{23}c_{23} \\ & - 2m_L\lambda_2^2 s_2 c_2 + 2m_L\lambda_2\lambda_3 s_2 s_{23} - 2m_L\lambda_2\lambda_3 c_2 c_{23}\}\,\dot{\theta}_1\dot{\theta}_2 \\ & + \{2m_3\lambda_{c3}^2 s_{23}c_{23} - 2m_3\lambda_2\lambda_{c3}c_2 c_{23} + 2I_{3L}s_{23}c_{23} - 2I_{3A}s_{23}c_{23} + 2m_4(\lambda_3 - \lambda_{c4})^2 s_{23}c_{23} \\ & - 2m_4(\lambda_3 - \lambda_{c4})\lambda_2 c_2 c_{23} + 2I_{4L}s_{23}c_{23} - 2I_{4A}s_{23}c_{23} + 2m_L\lambda_3^2 s_{23}c_{23} - 2m_L\lambda_2\lambda_3 c_2 c_{23}\}\,\dot{\theta}_1\dot{\theta}_3 \\ & + \{I_{4A}s_{23}\}\,\dot{\theta}_2\dot{\theta}_4 + \{I_{4A}s_{23}\}\,\dot{\theta}_3\dot{\theta}_4\end{aligned} \tag{5d}$$

6 NET DYNAMIC EQUATIONS

The equations can be represented in a matrix form as,

$$\tau = \boldsymbol{M}(\Theta)\ddot{\Theta} + \boldsymbol{B}(\Theta)\dot{\Theta}_i\dot{\Theta}_j + \boldsymbol{C}(\Theta)\dot{\Theta}_k^2 + \boldsymbol{G}(\Theta)$$

$$i,j \in \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}, \qquad k \in \{1,2,3,4\} \tag{6}$$

where M is the **Inertia** matrix, $\boldsymbol{B}$ is the **Coriolis** matrix, $\boldsymbol{C}$ is the **Centrifugal** matrix and $\boldsymbol{G}$ is the **Gravity** matrix. In this format, also known as the Configuration-Space equation, the Coriolis and Centrifugal coefficient matrices are functions of the manipulator state Θ alone. Depending on the manipulator's configuration, some of the matrix coefficients are not applicable (or 0). For the Quanser Arm, these matrices are of the form,

$$\begin{bmatrix}\tau_1\\\tau_2\\\tau_3\\\tau_4\end{bmatrix} = \begin{bmatrix}M_{11} & 0 & 0 & M_{14}\\0 & M_{22} & M_{23} & 0\\0 & M_{32} & M_{33} & 0\\M_{41} & 0 & 0 & M_{44}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\\ddot{\theta}_2\\\ddot{\theta}_3\\\ddot{\theta}_4\end{bmatrix} + \begin{bmatrix}B_{11} & B_{12} & 0 & 0 & B_{15} & B_{16}\\0 & 0 & B_{23} & B_{24} & 0 & 0\\0 & 0 & B_{33} & 0 & 0 & 0\\B_{41} & B_{42} & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\dot{\theta}_2\\\dot{\theta}_1\dot{\theta}_3\\\dot{\theta}_1\dot{\theta}_4\\\dot{\theta}_2\dot{\theta}_3\\\dot{\theta}_2\dot{\theta}_4\\\dot{\theta}_3\dot{\theta}_4\end{bmatrix} + \begin{bmatrix}0 & 0 & 0 & 0\\C_{21} & 0 & C_{23} & 0\\C_{31} & C_{32} & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}_1^2\\\dot{\theta}_2^2\\\dot{\theta}_3^2\\\dot{\theta}_4^2\end{bmatrix} + g\begin{bmatrix}0\\G_2\\G_3\\0\end{bmatrix} \tag{7}$$

For all the terms, a coefficient with subscript mn corresponds to the coefficients relating the torque on the mth joint to the corresponding nth kinematic term. For example, B_{24} relates the torque on joint 2 τ_2 to the corresponding 4th kinematic term $\dot{\theta}_2\dot{\theta}_3$.