

Notes on Event Structures

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1 Prime Event Structures

In the imperative setting, the evaluation of a program is commonly accompanied by a memory that changes accordingly the execution of said program, where each step performed by the computation is not labeled. On the other side we have a process algebra approach, in which states are dropped and each step of the computation is labeled by the action that caused the occurrence of the computation. Although we intend to model an imperative language, our approach is similar to the latter. This decision comes from the use of event structures. By dropping the state we can use the usual definitions of event structures [Win84, Win82]. Since we want to model an imperative language, we need to have the notion of state. Well, since we label the transitions we perform, we can create a trace of the actions that were performed. By doing this, we can apply each instruction in the trace to a given state.

After summing up the main idea, let us delve into definitions. By now, the intention is just to write down the definitions. Later on, intuitions and examples shall be develop to help reader's comprehension.

Informally, an event structure [Win88] is composed of a set of events, together with a notion of causality given by a partial order on events: if $e \leq e'$ then e' can only occur if e already occurred, and a notion of conflict between events: if $e \# e'$ then either e occurs or e' occurs, which is a behavior similar to a non-deterministic choice.

Definition 1.1 (Prime Event Structures). Define an *event structure* to be a structure $E = (E, \leq, \#)$ consisting of a set E of events, which are partially ordered by \leq , the *causal dependency relation*, and a binary, symmetric, irreflexive relation $\# \subseteq E \times E$, the *conflict relation*, satisfying:

- $\{e' \mid e' \leq e\}$ is finite
- $e \# e' \leq e'' \Rightarrow e \# e''$

for all $e, e', e'' \in E$.

Definition 1.2 (Concurrent Event). Two events, e, e' are said *concurrent* iff $\neg(e \leq e') \wedge \neg(e \# e')$. In other words, two events are concurrent when they are not causally dependent and are not in conflict.

Definition 1.3 (Configuration). A configuration is a subset of the set of events, $x \subseteq E$, that are

$$\begin{aligned} \text{conflict-free: } & \forall e, e' \in x. \neg(e \# e') \\ \text{down-closed: } & \forall e, e'. e' \leq e \wedge e \in x \Rightarrow e' \in x \end{aligned}$$

We then denote by $\mathcal{C}^\infty(E)$ the set of all configurations and by $\mathcal{C}(E)$ the set of finite configurations.

Definition 1.4 (Covering chain). Let $E = (E, \leq, \#)$ be a prime event structure, $e \in E$, and $x \in \mathcal{C}(E)$. Denote by $x \xrightarrow{e} x \cup \{e\}$ if $e \notin x$ and $(x \cup \{e\}) \in \mathcal{C}(E)$. A *covering chain* on a configuration $x \in \mathcal{C}(E)$ is a finite sequence of events $e_0 e_1 \dots e_n$ such that

$$x_0 \xrightarrow{e_0} x_1 \xrightarrow{e_1} x_2 \xrightarrow{e_2} \dots \xrightarrow{e_n} x_{n+1} = x$$

Definition 1.5 (Cover). Let E be a prime event structure and $x, y \in \mathcal{C}(E)$. Say that y covers x , pictured as $x \multimap y$, if $x \subset y$ with nothing in between ($\nexists z. x \subset z \subset y$).

Later on we shall find useful to say $y \xrightarrow{e_1 \dots e_n} x_1, \dots, x_n$ when $y \multimap x_1, \dots, x_n$.

Definition 1.6 (Maximal configuration). Let E be a prime event structure and $x \in \mathcal{C}(E)$. Say that x is a maximal configuration iff $\nexists y \in \mathcal{C}(E)$ such that $x \multimap y$. Denote by $\mathcal{C}_{\max}(E)$ the set of maximal configurations.

Later on we shall find useful to simplify how covering chains are represented. We then let $\omega = e_0 e_1 \dots e_n$ and denote $x_0 \xrightarrow{e_0} x_1 \xrightarrow{e_1} x_2 \xrightarrow{e_2} \dots \xrightarrow{e_n} x_{n+1} = x$ simply by $x \xrightarrow{\omega} x \cup \{w\}$.

Since the causal relation is a partial order we know that it is transitive. Furthermore, the conflict relation is hereditary over events. Hence if we want to draw an event structure using these two relations we would have to add a lot of redundant information, which would make the event structure hard to understand. To ease such task, we find it useful to use the notions of *immediate causality*, pictured by \rightarrow , and *minimal conflict*, represented by \sim . Let E be an event structure such that $e_1, e_2 \in E$. We say $e_1 \rightarrow e_2$ iff $e_1 < e_2$ and $\nexists e_3 \in E. e_1 < e_3 < e_2$. We say $e_1 \sim e_2$ iff $e_1 \# e_2$ and whenever $e'_1 \leq e_1$, $e'_1 \# e'_2$, and $e'_2 \leq e_2$ we have $e_1 = e'_1$ and $e_2 = e'_2$. Note that it is possible to deduce the causal and conflict relations from the immediate causality and minimal conflict relations.

Example 1.7 aims to get the reader familiarized with event structures.

Example 1.7. Figure 1 shows an event structure with four events, a, b, c , and d , where: b causally depends on a , c and d are concurrent events which are in conflict with a , and consequently also with b . Furthermore, note that a is in minimal conflict with c and d . The set of configurations, *i.e.* the set of possible computations, is $\{\emptyset, \{a\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}\}$. Furthermore note that the configuration $\{c, d\}$, which is composed of two concurrent events, has two possible covering chains: $\emptyset \xrightarrow{d} \{d\} \xrightarrow{c} \{c, d\}$ and $\emptyset \xrightarrow{c} \{c\} \xrightarrow{d} \{c, d\}$.

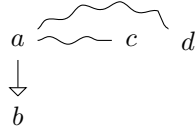


Figure 1: Example of an event structure

Definition 1.8 (Map event structures). Let $E = (E, \leq, \#)$, $E' = (E', \leq', \#')$ be event structures. A partial/total map f from E to E' is a partial/total function $f : E \rightarrow E'$ such that:

(Configuration Preserving) $\forall x \in \mathcal{C}(E) \Rightarrow f(x) \in \mathcal{C}(E')$

(Locally injective) $\forall (a \neq b) \in x \in \mathcal{C}(E)$, if f is defined in both then $f(a) \neq f(b)$

where $f(x) = \{f(e) \mid e \in x, f(e) \text{ defined}\}$

Example 1.9. In Figure 2 we have a map of event structures f that maps a to itself and the conflicting events c, b to d . We note that f is total and, consequently, it preserves the size of covering chains. Consider the covering chain $\emptyset \xrightarrow{c} \{c\} \xrightarrow{a} \{a, c\}$. The respective covering chain after applying f is $\emptyset \xrightarrow{d} \{d\} \xrightarrow{a} \{a, d\}$. This is only possible because of the local injective condition, in which different events of the same configuration must have different images.

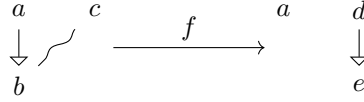


Figure 2: Map event structures

With these definitions presented, we are now prepared to advance to the next stage, where we present the language that we intend to model with prime event structures, *i.e.* its syntax and respective operational semantics in terms of a small-step and n-step. After presenting the language, we present the constructions made on prime event structures to capture the behavior of the language operator's. Then we show how to interpret commands of the language using prime event structures and, at last, we show that both semantics are sound and adequate.

1.1 Language

To present the language we consider a set of atomic actions Act ranged over by a (examples of atomic actions are assignments, or unitary application, etc...). Furthermore, we extend the set of atomic actions with sk and called the resultant set the set of labels, *i.e.* $L = Act \cup \{sk\}$.

The set of commands allowed by the language are given by the following grammar:

$$C ::= \text{skip} \mid a \in Act \mid C ; C \mid C + C \mid C \parallel C$$

To define the operational semantics, we add a new command to the language, denoted by \checkmark , that indicates the end of a computation.

We denote by $L = Act \cup \{sk\}$ the set of labels, which is ranged by l , and we consider a *terminal command*, denoted by \checkmark , representing the end of a computation. The small-step semantics is then defined as the smallest relation $\xrightarrow{l} \subseteq C \times L \times (C \cup \{\checkmark\})$ obeying the rules in Figure 3.

Define a word to be a sequence of labels:

$$\omega ::= l \mid l : \omega$$

where $l : \omega$ appends l to the beginning of ω . A word can also be seen as an element of L^+ , *i.e.* a possibly infinite sequence of labels without the empty sequence. Despite L^+ allows the possibility of having infinite words, by now we focus only on the finite words.

Define the n -step transition, $\xrightarrow{\omega} \subseteq C \times L^+ \times (C \cup \{\checkmark\})$, where n is the length of the words, as follows:

Example 1.10. The initial program is $a ; b + c \parallel d$, from which we have three possible transitions: by a , c or d . If we transit by a , we reach the command b , which we execute to finish the computation. Otherwise, we could either transit via c and then execute d , or transit via d and then execute c , in order to finish the computation.

With the support of Figure 6 together with the above explanation, we can straightforwardly deduce the words that can be formed by the n-step semantics: a , c , d , ab , cd , and dc .

$$\begin{array}{c}
\text{skip} \xrightarrow{sk} \checkmark \qquad a \xrightarrow{a} \checkmark \qquad \frac{C_1 \xrightarrow{l} \checkmark}{C_1 ; C_2 \xrightarrow{l} C_2} \qquad \frac{C_1 \xrightarrow{l} C'_1}{C_1 ; C_2 \xrightarrow{l} C'_1 ; C_2} \\
\\
\frac{C_1 \xrightarrow{l} \checkmark}{C_1 + C_2 \xrightarrow{l} \checkmark} \qquad \frac{C_1 \xrightarrow{l} C'_1}{C_1 + C_2 \xrightarrow{l} C'_1} \qquad \frac{C_2 \xrightarrow{l} \checkmark}{C_1 + C_2 \xrightarrow{l} \checkmark} \qquad \frac{C_2 \xrightarrow{l} C'_2}{C_1 + C_2 \xrightarrow{l} C'_2} \\
\\
\frac{C_1 \xrightarrow{l} \checkmark}{C_1 \parallel C_2 \xrightarrow{l} C_2} \qquad \frac{C_1 \xrightarrow{l} C'_1}{C_1 \parallel C_2 \xrightarrow{l} C'_1 \parallel C_2} \qquad \frac{C_2 \xrightarrow{l} \checkmark}{C_1 \parallel C_2 \xrightarrow{l} C_1} \qquad \frac{C_2 \xrightarrow{l} C'_2}{C_1 \parallel C_2 \xrightarrow{l} C_1 \parallel C'_2}
\end{array}$$

Figure 3: Rules of the small-step operational semantics

$$\frac{C \xrightarrow{l} C'}{C \xrightarrow{l} C'} \qquad \frac{C \xrightarrow{l} C'' \quad C'' \xrightarrow{\omega'} C'}{C \xrightarrow{l:\omega'} C'}$$

Figure 4: Rules of the n-step operational semantics

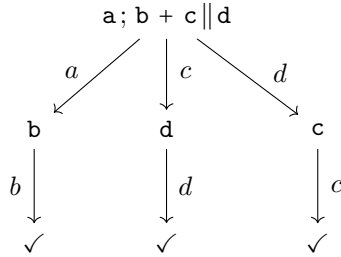


Figure 5: Labeled transition system of $a; b + c \parallel d$

Example 1.11. The initial program is $(a; b) \parallel c$, from which we have two possible transitions: either by a or by c . If we transit by c we go to the command $a; b$, where we execute a followed by b to complete the computation. On the other hand, case we transition by a , we reach the command $b \parallel c$, which allows two possible transitions: first b and then c or first c and then b .

With the support of Figure 6 together with the above explanation, we can straightforwardly deduce the words that can be formed by the n-step semantics: $a, c, ab, ac, ca, abc, acb$, and cab .

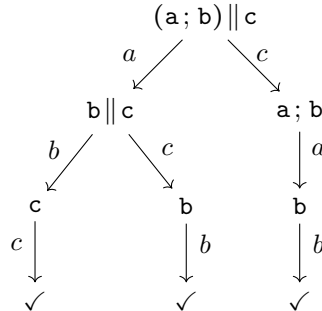


Figure 6: Labeled transition system of $(a; b) \parallel c$

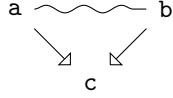
1.2 Constructions on Prime Event Structures

Having defined the language, *i.e.* its syntax and operational semantics, we now focus on event structures. We need to define the constructions on event structures that captures the effects of sequential composition,

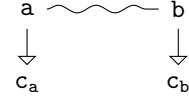
non-deterministic choice, and parallel composition.

To capture the behavior of the language's operators, we need to define them in terms of event structures.

Let us begin with sequential composition. Consider $\mathbf{a}; \mathbf{b}$ to be the sequential combination of two actions, a and b . According to the rules in Figure 3 we execute \mathbf{b} after \mathbf{a} has been executed, which with an event structure view means that b causally depends on a . As a first attempt to define sequential composition of two event structures, $E_1; E_2$, one might try to connect every event of E_1 with every event of E_2 . However, this approach fails to interpret programs like $(\mathbf{a} + \mathbf{b}); \mathbf{c}$, as show in Figure 7a. This failure arises because there are two ways to reach event a , which come from conflicting events. According to the definition of event structures, the conflict relation is hereditary, and an event is not in conflict with itself. Thus, we would end up with an invalid event structure. To address this issue, we introduce a 'copy' for each event of E_2 regarding the different ways it can be reached. For example, in the aforementioned program, we create two copies of a : one indicating it was reached by executing event b , and another indicating it was reached by executing event c , as can be seen in Figure 7b.



(a) Unwanted sequential composition



(b) Good sequential composition

To capture the intended behavior in prime event structures, we make use of maximal configurations, as shown in the next definition.

Definition 1.12 (Product between events and configurations). Let E be a set of events and $\mathcal{C}(E)$ a set of finite configurations of a event structure E . Then

$$E \times \mathcal{C}(E) = \{(e, x) \mid e \in E, x \in \mathcal{C}(E)\}$$

Now we define how to sequentially compose two event structures.

Definition 1.13 (PES sequential). Let $E_1 = (E_1, \leq_1, \#_1)$ and $E_2 = (E_2, \leq_2, \#_2)$ be event structures. Define $E_1; E_2 = (E, \leq, \#)$ as:

$$\begin{aligned} E &= E_1 \uplus (E_2 \times \mathcal{C}_{\max}(E_1)) \\ \leq &= \{e_1 \leq e'_1 \mid e_1 \leq_1 e'_1\} \cup \{(e_2, x) \leq (e'_2, x) \mid e_2 \leq_2 e'_2\} \cup \{e_1 \leq (e_2, x) \mid e_1 \in x\} \\ \# &= \{e \# e' \mid \exists (a \leq e, a' \leq e') . a \#_1 a' \vee a \#_{(2,x)} a'\} \end{aligned}$$

where we use the notation $a \#_{(2,x)} a'$ iff $(a, x) \#_{(2,x)} (a', x)$ iff $a \#_2 a'$.

Note that we multiplied E_2 with the maximal configurations of E_1 . That is due maximal configurations representing finished computations. In other words, the set of maximal configurations gives all the possible ways to reach the first event of E_2 .

Lemma 1.14. Let E_1 and E_2 be event structures. $E_1; E_2$ is an event structure.

Proof. Let $E_1 = (E_1, \leq_1, \#_1)$, $E_2 = (E_2, \leq_2, \#_2)$, and $E_1; E_2 = (E, \leq, \#)$.

We need to show that $\forall e, e', e'' \in E$:

1. $\{e' \mid e' \leq e\}$ is finite
 - (a) Case $e \in E_1$ we are done.
 - (b) Case $e \in E_2 \times \mathcal{C}_{\max}(E_1)$. Then $e = (e_2, x_1)$ with $e_2 \in E_2$ and $x_1 \in \mathcal{C}_{\max}(E_1)$. We know that $\{e' \mid e' \leq (e_2, x_1)\} = \{(e'_2, x_1) \mid (e'_2, x_1) \leq (e_2, x_1)\} \cup \{e_1 \mid e_1 \leq (e_2, x_1), e_1 \in x_1\}$. Both sets are finite because E_2 is an event structure and x_1 is finite, respectively. Since both sets are finite and the union of finite sets is finite, then $\{e' \mid e' \leq (e_2, x_1)\}$ is finite.
2. $e \# e' \leq e'' \Rightarrow e \# e''$
 - (a) Case $e, e', e'' \in E_1$ or $a, a', e, e', e'' \in E_2 \times \mathcal{C}_{\max}(E_1)$ we are done.
 - (b) Case $e, e' \in E_1$ and $e'' \in E_2 \times \mathcal{C}_{\max}(E_1)$. We want to show that $e \# e''$. Hence we have to show that $\exists (a \leq e, a'' \leq e'') . a \#_1 a'' \vee a \#_{(2,x)} a''$. Since $e, e' \in E_1$ then $a, a'' \in E_1$. Let $a = e$ and $e' = a''$. Hence we have $a \leq e \Leftrightarrow e \leq e \Leftrightarrow e \leq_1 e$. By the initial assumption, $e' \leq e'' \Leftrightarrow a'' \leq e''$. It lacks to show that $a \#_1 a''$. That follows directly from the initial assumption $e \# e' \Leftrightarrow e \#_1 e' \Leftrightarrow a \#_1 a''$.

□

At last we have the non-deterministic composition of event structures. Let us use the rules in Figure 3 to give the intuition behind the definition. Consider the command $\mathbf{a} + \mathbf{b}$. According to the operational semantics, if we execute a then we cannot execute b and if we execute b we cannot execute a . If we abstract ourselves and instead of $\mathbf{a} + \mathbf{b}$ we consider $\mathbf{C}_1 + \mathbf{C}_2$, we notice that if we execute an action from \mathbf{C}_1 , then it is no longer possible to execute any action of \mathbf{C}_2 and vice-versa. To capture this behavior in event structures, we need to put all the events corresponding to \mathbf{C}_1 in conflict with all the events corresponding to \mathbf{C}_2 . Formally:

Definition 1.15 (PES non-deterministic). Let $E_1 = (E_1, \leq_1, \#_1)$ and $E_2 = (E_2, \leq_2, \#_2)$ be event structures. Define $E_1 + E_2 = (E, \leq, \#)$ as:

$$\begin{aligned} E &= E_1 \uplus E_2 \\ \leq &= \leq_1 \cup \leq_2 \\ \# &= \#_1 \cup \#_2 \cup \{e_1 \# e_2 \mid e_1 \in E_1, e_2 \in E_2\} \end{aligned}$$

Equivalently, we can define the partial order in Definition 1.15 as follows:

$$e \leq e' = \begin{cases} e \leq_1 e' & \text{if } e, e' \in E_1 \\ e \leq_2 e' & \text{if } e, e' \in E_2 \end{cases}$$

Lemma 1.16. Let E_1 and E_2 be event structures. $E_1 + E_2$ is an event structure.

Proof. Let $E_1 = (E_1, \leq_1, \#_1)$, $E_2 = (E_2, \leq_2, \#_2)$, and $E_1 + E_2 = (E, \leq, \#)$.

We need to show that $\forall e, e', e'' \in E$:

1. $\{e' \mid e' \leq e\}$ is finite
 - (a) Case $e \in E_1$ then $\{e' \mid e' \leq e\} = e' \mid e' \leq_1 e$, which is finite since E_1 is an event structure.
 - (b) Case $e \in E_2$ then $\{e' \mid e' \leq e\} = e' \mid e' \leq_2 e$, which is finite since E_2 is an event structure.

2. $e \# e' \leq e'' \Rightarrow e \# e''$

We have two cases: $e, e', e'' \in E_1$ or $e, e', e'' \in E_2$. In both cases this holds because E_1 and E_2 are prime event structures.

□

The absence of communication in the language considered simplifies the definition of parallel composition in event structures, when compared to [Win88], since we do not need a mechanism of synchronization. In our case, we simply place ‘side-by-side’ the two event structures.

Definition 1.17 (PES parallel). Let $E_1 = (E_1, \leq_1, \#_1)$ and $E_2 = (E_2, \leq_2, \#_2)$ be event structures. Define $E_1 \parallel E_2 = (E, \leq, \#)$ as:

$$\begin{aligned} E &= E_1 \uplus E_2 \\ \leq &= \leq_1 \cup \leq_2 \\ \# &= \#_1 \cup \#_2 \end{aligned}$$

Lemma 1.18. Let E_1 and E_2 be event structures. $E_1 \parallel E_2$ is an event structure.

Proof. Let $E_1 = (E_1, \leq_1, \#_1)$, $E_2 = (E_2, \leq_2, \#_2)$, and $E_1 \parallel E_2 = (E, \leq, \#)$.

We need to show that $\forall e, e', e'' \in E$:

1. $\{e' \mid e' \leq e\}$ is finite
 - (a) Case $e \in E_1$ then $\{e' \mid e' \leq e\} = e' \mid e' \leq_1 e$, which is finite since E_1 is a prime event structure.
 - (b) Case $e \in E_2$ then $\{e' \mid e' \leq e\} = e' \mid e' \leq_2 e$, which is finite since E_2 is a prime event structure.

2. $e \# e' \leq e'' \Rightarrow e \# e''$

We have two cases: $e, e', e'' \in E_1$ or $e, e', e'' \in E_2$. In both cases this holds because E_1 and E_2 are prime event structures.

□

Definition 1.19. We interpret commands as prime event structures as follows ($\llbracket - \rrbracket : \mathcal{C} \rightarrow \mathcal{E}$):

$$\begin{aligned}\llbracket \text{skip} \rrbracket &= (\{sk\}, \{sk \leq sk\}, \emptyset) \\ \llbracket a \rrbracket &= (\{a\}, \{a \leq a\}, \emptyset) \\ \llbracket \mathcal{C}_1 ; \mathcal{C}_2 \rrbracket &= \llbracket \mathcal{C}_1 \rrbracket ; \llbracket \mathcal{C}_2 \rrbracket \\ \llbracket \mathcal{C}_1 + \mathcal{C}_2 \rrbracket &= \llbracket \mathcal{C}_1 \rrbracket + \llbracket \mathcal{C}_2 \rrbracket \\ \llbracket \mathcal{C}_1 \parallel \mathcal{C}_2 \rrbracket &= \llbracket \mathcal{C}_1 \rrbracket \parallel \llbracket \mathcal{C}_2 \rrbracket\end{aligned}$$

When the goal is to show the equivalence between the operational and the denotational semantics, Definition 1.38 is not suitable, since sequential composition is not left-monotone. This happens because the inclusion on the set of events is too restrict, *i.e.* the copies made by Definition 1.13 are distinct. Hence, we loose the inclusion on the set of events and obtain the following ordering:

Definition 1.20 (sub-PES). Let $E_1 = (E_1, \leq_1, \#_1)$ and $E_2 = (E_2, \leq_2, \#_2)$ be event structures. Say $E_1 \sqsubseteq E_2$ if:

$$\begin{aligned}\underline{E_1} &\sqsubseteq \underline{E_2} \text{ s.t. } \underline{E_i} = \{e \mid (e \vee (e, x)) \in E_i\} \\ \forall e, e' . e \leq_1 e' &\Leftrightarrow e, e' \in E_1 \wedge e \leq_2 e' \\ \forall e, e' . e \#_1 e' &\Leftrightarrow e, e' \in E_1 \wedge e \#_2 e'\end{aligned}$$

We say that two event structures E_1, E_2 are equivalent, denoted $E_1 \equiv E_2$, iff $E_1 \sqsubseteq E_2$ and $E_2 \sqsubseteq E_1$.

Note that in Definition 1.20, when comparing the set of events, we ignore the ‘copies’ of events. This comes as a consequence of Definition 1.13 in which we make ‘copies’ of the same event to distinguish the different ways an event can be reached. However, the ‘copies’ denote the same event. Thus we want to forget the different ways they can be reached and just focus on the event itself. Case we have not done that, sequential composition would not be monotone, *i.e.* if $E_1 \sqsubseteq E'_1$ and $E_2 \sqsubseteq E'_2$ then $E_1 ; E_2 \not\sqsubseteq E'_1 ; E'_2$. That is easily seen when the number of maximal configurations of E_1 is greater than that of E'_1 .

Remark 1. When $E_1 \sqsubseteq E_2$ and $E_2 \sqsubseteq E_1$ we say that $E_1 \equiv E_2$, *i.e.* that E_1 and E_2 are equivalent. Furthermore, it is clear that if $E_1 = E_2$ then $E_1 \equiv E_2$.

To finish this section of definitions, we define the set of initial events and the removal of an initial event from a prime event structure.

Definition 1.21 (Set of initial events). Let $E = (E, \leq, \#)$ be a prime event structure. Define the set of initial events as follows:

$$\mathcal{I}(E) = \{e' \mid \nexists e \in E . e \leq e' \wedge e \neq e'\}$$

When removing an initial event from an event structure, not only the event itself but also all conflicting events are eliminated. This decision aims to mimic, within event structures, what happens in a transition using the small-step semantics. In small-step semantics, once an action triggers a transition, that same action cannot be executed again. Furthermore, if the transition occurs within a non-deterministic program, only the program associated with the triggering action continues, while the others are discarded.

Definition 1.22 (Remove initial event). Let $E = (E, \leq, \#)$ be a prime event structure and $a \in \mathcal{I}(E)$. Define $E \setminus a = (E', \leq', \#')$ as

$$\begin{aligned}E' &= \{e \in E \mid \neg(e \# a), e \neq a\} \\ \leq' &= \{e \leq e' \mid e, e' \in E'\} \\ \# &= \{e \# e' \mid e, e' \in E'\}\end{aligned}$$

Lemma 1.23. Let E be an event structure and $a \in \mathcal{I}(E)$. $E \setminus a$ is a event structure.

Proof. Let $E = (E, \leq, \#)$ and $E \setminus a = (E', \leq', \#')$.

We need to show that $\forall e, e', e'' \in E$:

1. $\{e' \mid e' \leq e\}$ is finite

Since $\{e' \mid e' \leq e\}$ is finite, then so it is $\{e' \mid e' \leq' e\} = \{e' \mid e' \leq e\} \setminus \{a\}$.

2. $e \# e' \leq e'' \Rightarrow e \# e''$

Let $e, e', e'' \in E'$. Then $e, e', e'' \neq a$ and $\neg(e, e', e'' \# a)$. By Definition 1.22, $e \# e'$ entails $e \# e''$ and $e, e' \in E'$ and $e' \leq' e''$ entails $e' \leq e''$ and $e', e'' \in E$. Since E is an event structure, we have $e \# e' \leq e'' \Rightarrow e \# e''$. Thus $e \# e' \leq' e'' \Rightarrow e \# e''$ and $e, e', e'' \in E$.

We have two cases: $e, e', e'' \in E_1$ or $e, e', e'' \in E_2$. In both cases this holds because E_1 and E_2 are prime event structures.

□

1.3 Results

Here we present the results obtained. Similarly to the previous subsection, we will just list what was proved. We postpone the addition of the proofs as well as the examples for some results for future versions of the document.

For this section, we interpret \checkmark as the empty prime event structure, *i.e.* $\llbracket \checkmark \rrbracket = (\emptyset, \emptyset, \emptyset) = \emptyset$.

Lemma 1.24. Let E_1, E'_1, E_2, E'_2 be event structures. If $E_1 \sqsubseteq E'_1$ and $E_2 \sqsubseteq E'_2$ then $E_1 ; E_2 \sqsubseteq E'_1 ; E'_2$.

Proof. Let $E_1 = (E_1, \leq_1, \#_1)$, $E'_1 = (E'_1, \leq'_1, \#'_1)$, $E_2 = (E_2, \leq_2, \#_2)$, $E'_2 = (E'_2, \leq'_2, \#'_2)$, $E_1 ; E_2 = (E, \leq, \#)$, and $E'_1 ; E'_2 = (E', \leq', \#')$, such that $E_1 \sqsubseteq E'_1$ and $E_2 \sqsubseteq E'_2$.

1. $E \sqsubseteq E'$

By Definition 1.13, we have that $E = E_1 \uplus (E_2 \times \mathcal{C}_{\max}(E_1))$ and $E' = E'_1 \uplus (E'_2 \times \mathcal{C}_{\max}(E'_1))$. Since $E_1 \sqsubseteq E'_1$ and $E_2 \sqsubseteq E'_2$ then $E_1 \subseteq E'_1$ and $E_2 \subseteq E'_2$. Furthermore, $(E_2 \times \mathcal{C}_{\max}(E_1)) \subseteq (E'_2 \times \mathcal{C}_{\max}(E'_1))$. Hence, it follows directly that $E = E_1 \uplus (E_2 \times \mathcal{C}_{\max}(E_1)) \subseteq E'_1 \uplus (E'_2 \times \mathcal{C}_{\max}(E'_1)) = E'$.

2. $\forall e_0, e_1. e_0 \leq e_1 \Leftrightarrow e_0, e_1 \in E$ and $e_0 \leq' e_1$

\Rightarrow Assume $e_0 \leq e_1$. By Definition 1.13 we have:

(a) $e_0 \leq e_1 \in \leq_1$

Hence $e_0, e_1 \in E_1 \subseteq E$. Furthermore, since $E_1 \sqsubseteq E'_1$, then $e_0 \leq'_1 e_1$, which by Definition 1.13 gives $e_0 \leq' e_1$.

(b) $e_0 \leq e_1 \in \leq_{(2,x)}$

Hence $e_0, e_1 \in (E_2 \times \mathcal{C}_{\max}(E_1)) \subseteq E$. Furthermore, since $E_2 \sqsubseteq E'_2$, then $e_0 \leq'_{(2,x)} e_1$, which by Definition 1.13 gives $e_0 \leq' e_1$.

(c) $e_0 \leq (e_1, x)$ such that $e_0 \in x$.

By Definition 1.13, $e_0 \in E_1$ and $(e_1, x) \in (E_2 \times \mathcal{C}_{\max}(E_1))$, which entails $e_0, (e_1, x) \in E$. Furthermore, since $E_1 \sqsubseteq E'_1$ and $E_2 \sqsubseteq E'_2$, then $e_0 \in E'_1$ and $(e_1, x) \in (E'_2 \times \mathcal{C}_{\max}(E'_1))$, such that $e_0 \in x$. By Definition 1.13, $e_0 \leq' (e_1, x)$.

\Leftarrow Assume $e_0, e_1 \in E$ and $e_0 \leq' e_1$. The cases are distinguished by \leq' .

(a) $e_0 \leq' e_1 \in \leq'_1$

We have $E_1 \sqsubseteq E'_1$, which entails that $\forall e_0, e_1 \in E_1$ we have $e_0 \leq_1 e_1$. By Definition 1.13 we have $e_0 \leq e_1$.

(b) $e_0 \leq' e_1 \in \leq'_{(2,x)}$

We have $E_1 \sqsubseteq E'_1$ and $E_2 \sqsubseteq E'_2$, which gives us $(E_2 \times \mathcal{C}_{\max}(E_1)) \subseteq (E'_2 \times \mathcal{C}_{\max}(E'_1))$. It entails that $\forall e_0, e_1 \in (E_2 \times \mathcal{C}_{\max}(E_1))$ we have $e_0 \leq_{(2,x)} e_1$. By Definition 1.13 we have $e_0 \leq e_1$.

(c) $e_0 \leq' (e_1, x)$ such that $e_0 \in x$.

We know that $e_0, (e_1, x) \in E = E_1 \uplus (E_2 \times \mathcal{C}_{\max}(E_1))$, which entails that $e_0 \in E_1$ and $(e_1, x) \in (E_2 \times \mathcal{C}_{\max}(E_1))$, such that $e_0 \in x$. It follows directly from Definition 1.13 that $\forall e_0, (e_1, x)$ we have $e_0 \leq (e_1, x)$.

3. $\forall e_0, e_1. e_0 \# e_1 \Leftrightarrow e_0, e_1 \in E$ and $e_0 \#' e_1$

\Rightarrow Assume $e_0 \# e_1$.

From Definition 1.13 we have that $\exists (a_0 \leq e_0, a_1 \leq e_1)$ such that $a_0 \#_1 a_1$ or $a_0 \#_{(2,x)} a_1$. Since $a_0 \#_1 a_1$ or $a_0 \#_{(2,x)} a_1$ then $e_0, e_1 \in E$. Furthermore, since $E_1 \sqsubseteq E'_1$ and $E_2 \sqsubseteq E'_2$ then $a_0 \#'_1 a_1$ or $a_0 \#'_{(2,x)} a_1$, which entails $a_0 \leq e_0, a_1 \leq e_1$. Thus $e_0 \#' e_1$.

\Leftarrow Assume $e_0, e_1 \in E$ and $e_0 \#' e_1$.

It exists $a_0 \leq' e_0$ and $a_1 \leq' e_1$ such that $a_0 \leq'_1 a_1$ or $a_0 \leq'_{(2,x)} a_1$. Since $e_0, e_1 \in E \subseteq E'$ then $a_0, a_1 \in E$. Hence it follows directly that $e_0 \# e_1$. □

Lemma 1.25. Let E_1, E'_1, E_2, E'_2 be event structures. If $E_1 \sqsubseteq E'_1$ and $E_2 \sqsubseteq E'_2$ then $E_1 + E_2 \sqsubseteq E'_1 + E'_2$.

Proof. It follows directly from Definition 1.15. □

Lemma 1.26. Let E_1, E'_1, E_2, E'_2 be event structures. If $E_1 \sqsubseteq E'_1$ and $E_2 \sqsubseteq E'_2$ then $E_1 \parallel E_2 \sqsubseteq E'_1 \parallel E'_2$.

Proof. It follows directly from Definition 1.17. □

Lemma 1.27. Let $op \in \{;, \parallel, +\}$ and E_1, E_2, E'_1 and E'_2 be event structures. If $E_1 \sqsubseteq E'_1$ and $E_2 \sqsubseteq E'_2$ then $E_1 op E_2 \sqsubseteq E'_1 op E'_2$.

Proof.

$$\begin{aligned}
& E_1 \equiv E'_1 \text{ and } E_2 \equiv E'_2 \\
& \Rightarrow \{\text{Remark 1}\} \\
& E_1 \subseteq E'_1 \text{ and } E'_1 \subseteq E_1, E_2 \subseteq E'_2 \text{ and } E'_2 \subseteq E_2 \\
& \Rightarrow \{\text{op monotone}\} \\
& E_1 \text{ op } E_2 \subseteq E'_1 \text{ op } E'_2 \text{ and } E'_1 \text{ op } E'_2 \subseteq E_1 \text{ op } E_2 \\
& \Rightarrow \{\text{Remark 1}\} \\
& E_1 \text{ op } E_2 \equiv E'_1 \text{ op } E'_2
\end{aligned}$$

□

Lemma 1.28. Let E_1, E_2 be event structures. Consider $E_1; E_2$ such that $l \in \mathcal{I}(E_1; E_2)$. Then $(E_1; E_2) \setminus l \equiv (E_1 \setminus l); E_2$.

Proof. To prove $(E_1; E_2) \setminus l \equiv (E_1 \setminus l); E_2$, we need to verify that $(E_1; E_2) \setminus l \subseteq (E_1 \setminus l); E_2$ and $(E_1 \setminus l); E_2 \subseteq (E_1; E_2) \setminus l$.

Let $E_1 = (E_1, \leq_1, \#_1)$, $E_2 = (E_2, \leq_2, \#_2)$, $E_1; E_2 = (E_{1;2}, \leq_{1;2}, \#_{1;2})$, $(E_1; E_2) \setminus l = (E, \leq, \#)$, $E_1 \setminus l = (E_1^l, \leq_1^l, \#_1^l)$, $(E_1 \setminus l); E_2 = (E', \leq', \#')$, and $l \in \mathcal{I}(E_1; E_2)$.

- $(E_1; E_2) \setminus l \subseteq (E_1 \setminus l); E_2$

1. $E \subseteq E'$

Since $l \in \mathcal{I}(E_1; E_2)$, it is straightforward to see that $l \in \mathcal{I}(E_1)$. Consequently, $l \in E_1$. Now let $e \in E$. By Definition 1.22 and Definition 1.13 we have $e \in E_1 \uplus (E_2 \times \mathcal{C}_{\max}(E_1))$ such that $\neg(e \# l)$ and $e \neq l$. We then have two cases:

- (a) $e \in E_1$

We know that $e, l \in E_1$, $\neg(e \# l)$, and $e \neq l$. By Definition 1.22, $\neg(e \# l)$ entails $\neg(e \#_{1;2} l)$, which by Definition 1.13 entails $\forall (a \leq e, a' \leq l) . \neg(a \#_1 a') \vee \neg(a \#_{(2,x)} a')$. By letting $a = e$ and $e' = l$ we have $\neg(e \#_1 l)$. Thus $e \in E_1$. It then follows directly from Definition 1.22 that $e \in E_1^l$, which by Definition 1.13 entails $e \in E'$, since $E_1^l \subseteq E'$.

- (b) $e \in E_2 \times \mathcal{C}_{\max}(E_1)$

By Definition 1.20 we ignore the copies created by multiplying each maximal configuration of esE_1 with each event of E_2 to verify if a set of events is contained on another set of events. Informally, that means that by Definition 1.20 $E_2 \times \mathcal{C}_{\max}(E_1)$ and $E_2 \times \mathcal{C}_{\max}(E_1 \setminus l)$ are 'equal'. Hence, it follows directly that $e \in E'$.

2. $\forall e, e' . e \leq e' \Leftrightarrow e, e' \in E \wedge e \leq' e'$

\Rightarrow Assume $\forall e, e' . e \leq e'$.

By Definition 1.22, $e, e' \in E$ and $e \leq_{1;2} e'$. We have three cases:

- (i) $e \leq_{1;2} e'$ is of the form $e \leq_1 e'$.

By Definition 1.13. Consequently $e, e' \in E_1$ and $e, e' \neq l$, since $e, e' \in E$. Thus $e \leq_1^l e'$ that by Definition 1.22 gives $e \leq' e'$.

- (ii) $e \leq_{1;2} e'$ is of the form $e \leq_{(2,x)} e'$.

It entails $e \leq_2 e'$. It then follows directly that $e \leq' e'$ since $l \notin E_2$.

- (iii) $e \leq_{1;2} e'$ is of the form $e \leq_{1;2} (e', x)$.

It entails $e \in x$ with $x \in \mathcal{C}_{\max}(E_1)$. By removing l from x we obtain a configuration $x' \in \mathcal{C}_{\max}(E_1^l)$ (this is easy to see because we are removing the initial element of an already maximal configuration). Hence $e \in x'$, which by Definition 1.13 gives $e \leq' (e, x')$.

\Leftarrow Assume $e, e' \in E \wedge e \leq' e'$.

We have three cases:

- (i) $e \leq' e'$ is of the form $e \leq_1^l e'$.

By Definition 1.22 we have $e \leq_1 e'$ such that $e, e' \neq l$ and $\neg(e \#_1 l), \neg(e' \#_1 l)$. Hence $e, e' \in E_1$, and consequently $e, e' \in E$, since $e, e' \neq l$. It then follows from Definition 1.13 and Definition 1.22 that $e \leq e'$.

- (ii) $e \leq' e'$ is of the form $(e, x') \leq_{(2,x')} (e', x')$.

$(e, x') \leq_{(2,x')} (e', x')$ entails $e \leq_2 e'$. Since $l \notin E_2$, it follows directly that $e \leq e'$.

- (iii) $e \leq' e'$ is of the form $e \leq' (e', x')$.

$e \leq' (e', x')$ entails $e \in x'$. Since $x' \in \mathcal{C}_{\max}(E_1^l)$, by Definition 1.22 $\exists x \in \mathcal{C}_{\max}(E_1)$ such that $x = x' \cup \{l\}$; Hence $e \in x$. By Definition 1.13 and Definition 1.22 it follows that $e \leq (e', x)$.

$$3. \forall e, e'. e \# e' \Leftrightarrow e, e' \in E \wedge e \# e'$$

\Rightarrow Assume $\forall e, e'. e \# e'$.

By Definition 1.22 it entails $e \#_{1;2} e'$ and $e, e' \in E$. By Definition 1.13 it entails $\exists(a \leq_{1;2} e, a' \leq_{1;2} e'). a \#_1 a' \vee a \#_{(2,x)} a'$. We have two cases:

(i) $a \#_1 a'$.

It entails that $a, a' \in E_1$. If $a = l$ or $a' = l$ then e or e' are not in E (this would entail that $e \#_1 l$ or $e' \#_1 l$), due to Definition 1.22, which contradicts our initial assumption $e, e' \in E$. Hence $a \neq l \neq a'$ and $\neg(a \# l), \neg(a' \# l')$. In that case, by Definition 1.22 we have $a \#_1^l a'$, which by Definition 1.13 gives $e \# e'$.

(ii) $a \#_{(2,x)} a'$.

It follows directly that $a \#_{(2,x')} a'$ since $l \notin E_2$ and $x' = x \setminus l$.

\Leftarrow Assume $e, e' \in E \wedge e \# e'$.

$e \# e'$ entails $\exists(a \leq' e, a' \leq' e'). a \#_1^l a' \vee a \#_{(2,x')} a'$. We have two cases:

(i) $a \#_1^l a'$.

By Definition 1.22 it entails $a \#_1 a', a, a' \neq l$, and $\neg(a \# l), \neg(a' \# l)$, which gives us that $e, e' \neq l$ and $\neg(e \#_1 l), \neg(e' \#_1 l)$, by conflict inheritance. Hence we have $e, e' \in E_1$ such that $e, e' \neq l$, which by Definition 1.13 gives $e, e' \in E$ and $e \# e'$.

(ii) $a \#_{(2,x')} a'$.

Since $l \notin E_2$ we have that $a, a' \neq l$ and $\neg(a \# l), \neg(a' \# l)$, which gives us $e, e' \neq l$ and $\neg(e \#_1 l), \neg(e' \#_1 l)$. By Definition 1.22, $\exists x \in \mathcal{C}_{\max}(E_1)$ such that $x = x' \cup \{l\}$. By Definition 1.13 we have $a \#_{(2,x)} a'$ and consequently $e \# e'$.

- $(E_1 \setminus l); E_2 \sqsubseteq (E_1; E_2) \setminus l$

Here we only show that $E' \subseteq E$, since the remaining cases are similarly proved.

Let $e \in E'$. By Definition 1.13 and Definition 1.22, $E' = \{e \in E_1 \mid \neg(e \#_1 l), e \neq l\} \uplus (E_2 \times \mathcal{C}_{\max}(E_1 \setminus l))$. We have two cases:

1. $e \in \{e \in E_1 \mid \neg(e \#_1 l), e \neq l\}$

Since $\neg(e \#_1 l)$, then $\nexists (a \leq_1 e, a' \leq_1 l)$ such that $a \#_1 a'$, because of conflict inheritance. By Definition 1.13 $\neg(e \#_1; 2)$ and consequently $\neg(e \# l)$. Thus $e \in E$.

2. $e \in (E_2 \times \mathcal{C}_{\max}(E_1 \setminus l))$

It follows directly $e \in E$ because when comparing sets of events, we discard the copies of events from E_2 made through the multiplication with $\mathcal{C}_{\max}(E_1 \setminus l)$.

□

Lemma 1.29. Let E_1, E_2 be event structures. Consider $E_1 + E_2$ such that $l \in \mathcal{I}(E_1 + E_2)$. Then

$$(E_1 + E_2) \setminus l \equiv \begin{cases} E_1 \setminus l & \text{if } l \in \mathcal{I}(E_1) \\ E_2 \setminus l & \text{if } l \in \mathcal{I}(E_2) \end{cases}$$

Proof. We need to prove $(E_1 + E_2) \setminus l \sqsubseteq E_1 \setminus l$, $E_1 \setminus l \sqsubseteq (E_1 + E_2) \setminus l$ when $l \in \mathcal{I}(E_1)$ and $(E_1 + E_2) \setminus l \sqsubseteq E_2 \setminus l$, $E_2 \setminus l \sqsubseteq (E_1 + E_2) \setminus l$ when $l \in \mathcal{I}(E_2)$. Since both cases are identical, we focus solely when $l \in \mathcal{I}(E_1)$.

Let $E_1 = (E_1, \leq_1, \#_1)$, $E_2 = (E_2, \leq_2, \#_2)$, $E_1 + E_2 = (E, \leq, \#)$, $(E_1 + E_2) \setminus l = (E', \leq', \#')$, $E_1 \setminus l = (E_1^l, \leq_1^l, \#_1^l)$, and $l \in \mathcal{I}(E_1 + E_2)$.

Consider $l \in \mathcal{I}(E_1)$.

- $(E_1 + E_2) \setminus l \sqsubseteq E_1 \setminus l$

1. $E' \subseteq E_1^l$

Let $e \in E'$. By Definition 1.22, $E' = \{e \in E_1 \uplus E_2 \mid \neg(e \# l), e \neq l\}$. We have two cases:

(i) $e \in E_1$

Since $\neg(e \# l)$ and $e \neq l$, then by Definition 1.15 $\neg(e \#_1 l)$. By Definition 1.22 we have $e \in E_1^l$, since $e \neq l$.

(ii) $e \in E_2$

Then $e \notin E'$, since by Definition 1.15 we have $e \# l$.

2. $\forall e, e'. e \leq' e' \Leftrightarrow e, e' \in E' \wedge e \leq_1^l e'$

\Rightarrow Assume $\forall e, e'. e \leq' e'$.

By Definition 1.22, $e \leq' e'$ entails $e, e' \in E'$. From the proof of $E' \subseteq E_1^l$ we know that $e, e' \notin E_2$. Hence $\neg(e \leq_2 e')$. Thus it only remains that $e \leq_1 e'$ with $e, e' \in E_1$. Again, from the proof of $E' \subseteq E_1^l$ we know that $e, e' \in E_1^l$ since $e, e' \neq l$ and $\neg(e \# l), \neg(e' \# l)$. Thus, by Definition 1.22, $e \leq_1^l e'$.

\Leftarrow Assume $e, e' \in E' \wedge e \leq_1^l e'$.

By Definition 1.22, $e \leq_1^l e'$ entails $e \leq_1 e'$ with $e, e' \in E_1^l$. Hence $e, e' \neq l$, $\neg(e \#_1 l), \neg(e' \#_1 l)$, and $e, e' \in E_1$. By Definition 1.15 we have $e, e' \in E$ and $e \leq e'$. Furthermore, we also have $\neg(e \# l), \neg(e' \# l)$ and consequently $e, e' \in E'$. Thus $e \leq' e'$.

3. $\forall e, e'. e \# e' \Leftrightarrow e, e' \in E' \wedge e \#_1^l e'$

\Rightarrow Assume $\forall e, e'. e \# e'$.

By Definition 1.22 we have $e \# e'$ and $e, e' \in E'$ such that $e, e' \neq l$ and $\neg(e \# l), \neg(e' \# l)$. By Definition 1.15, $e \# e'$ entails $e \#_1 e'$ or $e \#_2 e'$ or $\{e \# e' \mid e \in E_1, e' \in E_2\}$. Since $l \in \mathcal{I}(E_1)$, then $l \in E_1$ and consequently $e, e' \notin E_2$. Thus, by exclusion of hypothesis we have $e \#_1 e'$ and consequently $\neg(e \#_1 l), \neg(e' \#_1 l)$. Thus by Definition 1.22 we have $e \#_1^l e'$.

\Leftarrow Assume $e, e' \in E' \wedge e \#_1^l e'$.

By Definition 1.22 we have $e \#_1 e'$ and $e, e' \in E_1^l$, such that $e, e' \neq l$ and $\neg(e \#_1 l), \neg(e' \#_1 l)$. Furthermore $e, e' \in E_1$. By Definition 1.15, $e, e' \in E$ and $\neg(e \# l), \neg(e' \# l)$. By Definition 1.22 we have $e \# e'$.

• $E_1 \setminus l \sqsubseteq (E_1 + E_2) \setminus l$

Here we only show that $E' \subseteq E$, since the remaining cases are similarly proved.

Let $e \in E_1^l$. By Definition 1.22, $e \in E_1$, $\neg(e \#_1 l)$, and $e \neq l$. By Definition 1.15 $e \in E$ and $\neg(e \# l)$. Thus, by Definition 1.22, $e \in E'$.

□

Lemma 1.30. Let E_1, E_2 be event structures. Consider $E_1 \parallel E_2$ such that $l \in \mathcal{I}(E_1 \parallel E_2)$. Then $(E_1 \parallel E_2) \setminus l \equiv (E_1 \setminus l) \parallel (E_2 \setminus l)$.

Proof. We need to prove $(E_1 \parallel E_2) \setminus l \sqsubseteq (E_1 \setminus l) \parallel (E_2 \setminus l)$ and $(E_1 \setminus l) \parallel (E_2 \setminus l) \sqsubseteq (E_1 \parallel E_2) \setminus l$.

Let $E_1 = (E_1, \leq_1, \#_1)$, $E_2 = (E_2, \leq_2, \#_2)$, $E_1 \parallel E_2 = (E, \leq, \#)$, $(E_1 \parallel E_2) \setminus l = (E', \leq', \#')$, $E_1 \setminus l = (E_1^l, \leq_1^l, \#_1^l)$, $E_2 \setminus l = (E_2^l, \leq_2^l, \#_2^l)$, $(E_1 \setminus l) \parallel (E_2 \setminus l) = (E^l, \leq^l, \#^l)$, and $l \in \mathcal{I}(E_1 \parallel E_2)$.

It is straightforward to see that if $l \in \mathcal{I}(E_1 \parallel E_2)$ then either $l \in \mathcal{I}(E_1)$ or $l \in \mathcal{I}(E_2)$, by Definition 1.17. Furthermore, consider that $l \in \mathcal{I}(E_1)$, then it follows that $E_2 \setminus l = E_2$. A similar behavior occurs in the other way around. Hence we focus when $l \in \mathcal{I}(E_1)$.

• $(E_1 \parallel E_2) \setminus l \sqsubseteq (E_1 \setminus l) \parallel (E_2 \setminus l)$

1. $E' \subseteq E^l$

Let $e \in E'$. By Definition 1.22 $e \in E_1 \uplus E_2$, $\neg(e \# l)$, and $e \neq l$. We have two cases:

(i) $e \in E_1$

Since $e, l \in E_1$, then by Definition 1.17 we have that $\neg(e \#_1 l)$. By Definition 1.22, $e \in E_1^l$. By Definition 1.17, $e \in E^l$.

(ii) $e \in E_2$

It follows directly that $e \in E^l$, since $l \in E_1$ and $E_2 \setminus l = E_2$.

2. $\forall e, e'. e \leq' e' \Leftrightarrow e, e' \in E' \wedge e \leq^l e'$

\Rightarrow Assume $\forall e, e'. e \leq' e'$

By Definition 1.22, $e \leq e'$ and $e, e' \in E'$. We have two cases:

(i) $e \leq e'$ is of the form $e \leq_1 e'$

Then $e, e' \in E_1$. Since $e, e' \in E'$, then $e, e' \neq l$ and $\neg(e \# l), \neg(e' \# l)$. By Definition 1.17 $\neg(e \#_1 l), \neg(e' \#_1 l)$. By Definition 1.22, $e, e' \in E_1^l$ and $e \leq_1^l e'$. By Definition 1.17, $e \leq^l e'$.

(ii) $e \leq e'$ is of the form $e \leq_2 e'$

It follows directly that $e \leq^l e'$, since $l \in E_1$ and $E_2 \setminus l = E_2$.

\Leftarrow Assume $e, e' \in E' \wedge e \leq^l e'$

By Definition 1.22 and by Definition 1.17 we have two cases:

(i) $e, e' \in E_1^l$ and $e \leq^l e'$ is of the form $e \leq_1^l e'$

Since $e, e' \in E_1^l$, by Definition 1.22 we have $e, e' \in E_1$, $e, e' \neq l$ and $\neg(e \#_1 l), \neg(e' \#_1 l)$. Hence, by Definition 1.17 and Definition 1.22 we have $e, e' \in E'$. From $e \leq_1^l e'$, we know that from Definition 1.22 we have $e \leq_1 e'$ and $e, e' \in E_1^l$. It then follows by Definition 1.17 and Definition 1.22 that $e \leq' e'$.

(ii) $e, e' \in E_2^l$ and $e \leq^l e'$ is of the form $e \leq_2^l e'$

It follows directly that $e \leq' e'$, since $l \in E_1$ and $E_2 \setminus l = E_2$.

3. $\forall e, e'. e \#^l e' \Leftrightarrow e, e' \in E' \wedge e \#^l e'$

The reasoning for this case is similar to the previous one, since the definitions are identical.

- $(E_1 \setminus l) \parallel (E_2 \setminus l) \sqsubseteq (E_1 \parallel E_2) \setminus l$

We only prove that $E^l \subseteq E'$, since the remaining cases are similarly proved.

Let $e \in E^l$. By Definition 1.17 we have two cases:

(i) $e \in E_1^l$

We know that $e \neq l$ and $\neg(e \#_1 l)$. By Definition 1.17 we have $\neg(e \# l)$ and consequently $e \in E'$.

(ii) $e \in E_2^l$

It follows directly that $e \in E'$, since $l \in E_1$ and $E_2 \setminus l = E_2$.

□

Lemma 1.31. Let E_1, E_2 be event structures. Then $E_1 \parallel E_2 = E_2 \parallel E_1$.

Proof. It follows directly from Definition 1.17.

□

Recall that $\llbracket \checkmark \rrbracket = (\emptyset, \emptyset, \emptyset)$.

Lemma 1.32 (Soundness I). If $C \xrightarrow{l} C'$ then $\llbracket C' \rrbracket \equiv \llbracket C \rrbracket \setminus l$.

Proof. Induction over rules in Figure 3.

- $\text{skip} \xrightarrow{sk} \checkmark$

It follows directly that $\llbracket \checkmark \rrbracket \equiv \llbracket \text{skip} \rrbracket \setminus sk \equiv \emptyset$.

- $a \xrightarrow{a} \checkmark$

It follows directly that $\llbracket \checkmark \rrbracket \equiv \llbracket a \rrbracket \setminus a \equiv \emptyset$.

- $C_1 ; C_2 \xrightarrow{l} C_2$

$$\begin{aligned}
& C_1 ; C_2 \xrightarrow{l} C_2 \\
& \Rightarrow \{\text{Figure 3 entails}\} \\
& C_1 \xrightarrow{l} \checkmark \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket \checkmark \rrbracket \equiv \llbracket C_1 \rrbracket \setminus l \\
& \Rightarrow \{\text{Lemma 1.24}\} \\
& \llbracket \checkmark \rrbracket ; \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) ; \llbracket C_2 \rrbracket \\
& \Rightarrow \{\llbracket \checkmark \rrbracket ; \llbracket C_2 \rrbracket = \llbracket C_2 \rrbracket, \text{ Lemma 1.28}\} \\
& \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket ; \llbracket C_2 \rrbracket) \setminus l \\
& \Rightarrow \{\text{Definition 1.19}\} \\
& \llbracket C_2 \rrbracket \equiv \llbracket C_1 ; C_2 \rrbracket \setminus l
\end{aligned}$$

- $C_1 ; C_2 \xrightarrow{l} C'_1 ; C_2$

$$\begin{aligned}
& C_1 ; C_2 \xrightarrow{l} C'_1 ; C_2 \\
& \Rightarrow \{\text{Figure 3 entails}\} \\
& C_1 \xrightarrow{l} C'_1 \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket C'_1 \rrbracket \equiv \llbracket C_1 \rrbracket \backslash l \\
& \Rightarrow \{\text{Lemma 1.24}\} \\
& \llbracket C'_1 \rrbracket ; \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \backslash l) ; \llbracket C_2 \rrbracket \\
& \Rightarrow \{\text{Lemma 1.28}\} \\
& \llbracket C'_1 \rrbracket ; \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket ; \llbracket C_2 \rrbracket) \backslash l \\
& \Rightarrow \{\text{Definition 1.19}\} \\
& \llbracket C'_1 ; C_2 \rrbracket \equiv \llbracket C_1 ; C_2 \rrbracket \backslash l
\end{aligned}$$

- $C_1 + C_2 \xrightarrow{l} \checkmark$

$$\begin{aligned}
& C_1 + C_2 \xrightarrow{l} \checkmark \\
& \Rightarrow \{\text{Figure 3 entails}\} \\
& C_1 \xrightarrow{l} \checkmark \text{ or } C_2 \xrightarrow{l} \checkmark \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket \checkmark \rrbracket \equiv \llbracket C_1 \rrbracket \backslash l \text{ or } \llbracket \checkmark \rrbracket \equiv \llbracket C_2 \rrbracket \backslash l \\
& \Rightarrow \{\text{Lemma 1.29 for both cases, Definition 1.19}\} \\
& \llbracket \checkmark \rrbracket \equiv \llbracket C_1 + C_2 \rrbracket \backslash l
\end{aligned}$$

- $C_1 + C_2 \xrightarrow{l} C'$

$$\begin{aligned}
& C_1 + C_2 \xrightarrow{l} C' \\
& \Rightarrow \{\text{Figure 3 entails}\} \\
& C_1 \xrightarrow{l} C'_1 \text{ or } C_2 \xrightarrow{l} C'_2 \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket C'_1 \rrbracket \equiv \llbracket C_1 \rrbracket \backslash l \text{ or } \llbracket C'_2 \rrbracket \equiv \llbracket C_2 \rrbracket \backslash l \\
& \Rightarrow \{\text{Lemma 1.29 for both cases, Definition 1.19}\} \\
& \llbracket C'_1 \rrbracket \equiv (\llbracket C_1 + C_2 \rrbracket) \backslash l \text{ or } \llbracket C'_2 \rrbracket \equiv \llbracket C_1 + C_2 \rrbracket \backslash l
\end{aligned}$$

- $C_1 \parallel C_2 \xrightarrow{l} C_2$

$$\begin{aligned}
& C_1 \parallel C_2 \xrightarrow{l} C_2 \\
& \Rightarrow \{\text{Figure 3 entails}\} \\
& C_1 \xrightarrow{l} \checkmark \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket \checkmark \rrbracket \equiv \llbracket C_1 \rrbracket \backslash l \\
& \Rightarrow \{\text{Lemma 1.26}\} \\
& \llbracket \checkmark \rrbracket \parallel \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \backslash l) \parallel C_2 \\
& \Rightarrow \{\llbracket \checkmark \rrbracket \parallel \llbracket C_2 \rrbracket = \llbracket C_2 \rrbracket\} \\
& \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \backslash l) \parallel C_2 \\
& \Rightarrow \{\llbracket C_2 \rrbracket = \llbracket C_2 \rrbracket \backslash l \text{ since } l \notin \mathcal{I}(\llbracket C_2 \rrbracket)\} \\
& \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \backslash l) \parallel (\llbracket C_2 \rrbracket \backslash l) \\
& \Rightarrow \{\text{Lemma 1.30, Definition 1.19}\} \\
& \llbracket C_2 \rrbracket \equiv \llbracket C_1 \parallel C_2 \rrbracket \backslash l
\end{aligned}$$

- $C_1 \parallel C_2 \xrightarrow{l} C'_1 \parallel C_2$

$$\begin{aligned}
& C_1 \parallel C_2 \xrightarrow{l} C'_1 \parallel C_2 \\
& \Rightarrow \{\text{Figure 3 entails}\} \\
& C_1 \xrightarrow{l} C'_1 \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket C'_1 \rrbracket \equiv \llbracket C_1 \rrbracket \setminus l \\
& \Rightarrow \{\text{Lemma 1.26}\} \\
& \llbracket C'_1 \rrbracket \parallel \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel C_2 \\
& \Rightarrow \{\llbracket C_2 \rrbracket = \llbracket C_2 \rrbracket \setminus l \text{ since } l \notin \mathcal{I}(\llbracket C_2 \rrbracket)\} \\
& \llbracket C'_1 \rrbracket \parallel \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel (\llbracket C_2 \rrbracket \setminus l) \\
& \Rightarrow \{\text{Lemma 1.30, Definition 1.19}\} \\
& \llbracket C'_1 \parallel C_2 \rrbracket \equiv \llbracket C_1 \parallel C_2 \rrbracket \setminus l
\end{aligned}$$

- $C_1 \parallel C_2 \xrightarrow{l} C_1$

$$\begin{aligned}
& C_1 \parallel C_2 \xrightarrow{l} C_1 \\
& \Rightarrow \{\text{Figure 3 entails}\} \\
& C_2 \xrightarrow{l} \checkmark \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket \checkmark \rrbracket \equiv \llbracket C_2 \rrbracket \setminus l \\
& \Rightarrow \{\text{Lemma 1.26}\} \\
& \llbracket C_1 \rrbracket \parallel \llbracket \checkmark \rrbracket \equiv \llbracket C_1 \rrbracket \parallel (\llbracket C_2 \rrbracket \setminus l) \\
& \Rightarrow \{\llbracket C_1 \rrbracket \parallel \llbracket \checkmark \rrbracket = \llbracket C_1 \rrbracket\} \\
& \llbracket C_1 \rrbracket \equiv \llbracket C_1 \rrbracket \parallel (\llbracket C_2 \rrbracket \setminus l) \\
& \Rightarrow \{\llbracket C_1 \rrbracket = \llbracket C_1 \rrbracket \setminus l \text{ since } l \notin \mathcal{I}(\llbracket C_1 \rrbracket)\} \\
& \llbracket C_1 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel (\llbracket C_2 \rrbracket \setminus l) \\
& \Rightarrow \{\text{Lemma 1.30, Definition 1.19}\} \\
& \llbracket C_1 \rrbracket \equiv \llbracket C_1 \parallel C_2 \rrbracket \setminus l
\end{aligned}$$

- $C_1 \parallel C_2 \xrightarrow{l} C_1 \parallel C'_2$

$$\begin{aligned}
& C_1 \parallel C_2 \xrightarrow{l} C_1 \parallel C'_2 \\
& \Rightarrow \{\text{Figure 3 entails}\} \\
& C_2 \xrightarrow{l} C'_2 \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket C'_2 \rrbracket \equiv \llbracket C_2 \rrbracket \setminus l \\
& \Rightarrow \{\text{Lemma 1.26}\} \\
& \llbracket C_1 \rrbracket \parallel \llbracket C'_2 \rrbracket \equiv C_1 \parallel (\llbracket C_2 \rrbracket \setminus l) \\
& \Rightarrow \{\llbracket C_1 \rrbracket = \llbracket C_1 \rrbracket \setminus l \text{ since } l \notin \mathcal{I}(\llbracket C_2 \rrbracket)\} \\
& \llbracket C_1 \rrbracket \parallel \llbracket C'_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel (\llbracket C_2 \rrbracket \setminus l) \\
& \Rightarrow \{\text{Lemma 1.30, Definition 1.19}\} \\
& \llbracket C_1 \parallel C'_2 \rrbracket \equiv \llbracket C_1 \parallel C_2 \rrbracket \setminus l
\end{aligned}$$

□

Lemma 1.33 (Soundness II). If $C \xrightarrow{\omega} C'$ then $\exists x \in \mathcal{C}(\llbracket C \rrbracket)$ such that $\emptyset \xrightarrow{\omega} Cx$.

Proof. Induction over the length of ω , which is denoted by $|\omega|$.

- $|\omega| = 1$

It follows directly that $\exists \{l\} \in \mathcal{C}(\llbracket \mathbf{C} \rrbracket) \cdot \emptyset \xrightarrow{l} \mathbf{C}$

- $|\omega| > 1$

$$\begin{aligned}
& \mathbf{C} \xrightarrow{\omega} \mathbf{C}' \\
& \Rightarrow \{\text{Definition 4}\} \\
& \mathbf{C} \xrightarrow{l} \mathbf{C}'' \quad \mathbf{C}'' \xrightarrow{\omega'} \mathbf{C}' \\
& \Rightarrow \{\text{Lemma 1.32, i.h.}\} \\
& \llbracket \mathbf{C}'' \rrbracket \equiv \llbracket \mathbf{C} \rrbracket \setminus l \quad \exists y \in \mathcal{C}(\llbracket \mathbf{C}'' \rrbracket) \cdot \emptyset \xrightarrow{\omega'} \mathbf{C} y \\
& \Rightarrow \{\text{Definition 1.22}\} \\
& \{l\} \cup y \in \mathcal{C}(\llbracket \mathbf{C} \rrbracket) \cdot \emptyset \xrightarrow{l} \mathbf{C} \{l\} \xrightarrow{\omega'} \mathbf{C} \{l\} \cup y = x
\end{aligned}$$

□

Lemma 1.34 (Adequacy I). Let $l \in \mathcal{I}(\llbracket \mathbf{C} \rrbracket)$. Then $\exists \mathbf{C}' \in (\mathbf{C} \cup \{\checkmark\})$ s.t $\mathbf{C} \xrightarrow{l} \mathbf{C}'$ and $\llbracket \mathbf{C} \rrbracket \setminus l \equiv \llbracket \mathbf{C}' \rrbracket$.

Proof. Induction over the interpretation of commands.

- $sk \in \mathcal{I}(\llbracket \text{skip} \rrbracket)$

Let $\mathbf{C}' = \checkmark$. It follows directly that $\text{skip} \xrightarrow{sk} \checkmark$ and that $\llbracket \text{skip} \rrbracket \setminus sk \equiv \llbracket \checkmark \rrbracket$.

- $a \in \mathcal{I}(\llbracket \mathbf{a} \rrbracket)$

Let $\mathbf{C}' = \checkmark$. It follows directly that $\mathbf{a} \xrightarrow{a} \checkmark$ and that $\llbracket \mathbf{a} \rrbracket \setminus a \equiv \llbracket \checkmark \rrbracket$.

- $l \in \mathcal{I}(\llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket)$

By Definition 1.13 we have that $l \in \mathcal{I}(\llbracket \mathbf{C}_1 \rrbracket)$. By i.h., $\exists \mathbf{C}'$ such that $\mathbf{C}_1 \xrightarrow{l} \mathbf{C}'$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l \equiv \llbracket \mathbf{C}' \rrbracket$. We have two cases:

1. $\mathbf{C}' = \checkmark$

We have $\mathbf{C}_1 \xrightarrow{l} \checkmark$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l \equiv \llbracket \checkmark \rrbracket$. By the rules in Figure 3, $\mathbf{C}_1 ; \mathbf{C}_2 \xrightarrow{l} \mathbf{C}_2$. By Definition 1.13, $(\llbracket \mathbf{C}_1 \rrbracket \setminus l) ; \llbracket \mathbf{C}_2 \rrbracket \equiv \llbracket \checkmark \rrbracket ; \llbracket \mathbf{C}_2 \rrbracket \equiv \llbracket \mathbf{C}_2 \rrbracket$.

2. $\mathbf{C}' = \mathbf{C}'_1$

We have $\mathbf{C}_1 \xrightarrow{l} \mathbf{C}'_1$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l \equiv \llbracket \mathbf{C}'_1 \rrbracket$. By the rules in Figure 3, $\mathbf{C}_1 ; \mathbf{C}_2 \xrightarrow{l} \mathbf{C}'_1 ; \mathbf{C}_2$. By Definition 1.13, $(\llbracket \mathbf{C}_1 \rrbracket \setminus l) ; \llbracket \mathbf{C}_2 \rrbracket \equiv \llbracket \mathbf{C}'_1 \rrbracket ; \llbracket \mathbf{C}_2 \rrbracket$. By Definition 1.19, $\llbracket \mathbf{C}'_1 ; \mathbf{C}_2 \rrbracket$.

- $l \in \mathcal{I}(\llbracket \mathbf{C}_1 + \mathbf{C}_2 \rrbracket)$

By Definition 1.15 we have two cases:

1. $l \in \mathcal{I}(\llbracket \mathbf{C}_1 \rrbracket)$

By i.h. $\exists \mathbf{C}' \cdot \mathbf{C}_1 \xrightarrow{l} \mathbf{C}'$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l \equiv \llbracket \mathbf{C}' \rrbracket$. By the rules in Figure 3 we have two cases:

- (a) $\mathbf{C}' = \checkmark$

We have $\mathbf{C}_1 \xrightarrow{l} \checkmark$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l \equiv \llbracket \checkmark \rrbracket$. By the rules in Figure 3 we have $\mathbf{C}_1 + \mathbf{C}_2 \xrightarrow{l} \checkmark$. By Lemma 1.29 we have $\llbracket \checkmark \rrbracket \equiv \llbracket \mathbf{C}_1 \rrbracket \setminus l \equiv (\llbracket \mathbf{C}_1 \rrbracket + \llbracket \mathbf{C}_2 \rrbracket) \setminus l$. By Definition 1.19, $\llbracket \mathbf{C}_1 + \mathbf{C}_2 \rrbracket \setminus l$.

- (b) $\mathbf{C}' = \mathbf{C}'_1$

We have $\mathbf{C}_1 \xrightarrow{l} \mathbf{C}'_1$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l \equiv \llbracket \mathbf{C}'_1 \rrbracket$. By the rules in Figure 3 we have $\mathbf{C}_1 + \mathbf{C}_2 \xrightarrow{l} \mathbf{C}'_1$. By Lemma 1.29 we have $\llbracket \mathbf{C}'_1 \rrbracket \equiv \llbracket \mathbf{C}_1 \rrbracket \setminus l \equiv (\llbracket \mathbf{C}_1 \rrbracket + \llbracket \mathbf{C}_2 \rrbracket) \setminus l$. By Definition 1.19, $\llbracket \mathbf{C}_1 + \mathbf{C}_2 \rrbracket \setminus l$.

2. $l \in \mathcal{I}(\llbracket \mathbf{C}_2 \rrbracket)$

By i.h. $\exists \mathbf{C}' \cdot \mathbf{C}_2 \xrightarrow{l} \mathbf{C}'$ and $\llbracket \mathbf{C}_2 \rrbracket \setminus l \equiv \llbracket \mathbf{C}' \rrbracket$. By the rules in Figure 3 we have two cases:

- (a) $\mathbf{C}' = \checkmark$

We have $\mathbf{C}_2 \xrightarrow{l} \checkmark$ and $\llbracket \mathbf{C}_2 \rrbracket \setminus l \equiv \llbracket \checkmark \rrbracket$. By the rules in Figure 3 we have $\mathbf{C}_1 + \mathbf{C}_2 \xrightarrow{l} \checkmark$. By Lemma 1.29 we have $\llbracket \checkmark \rrbracket \equiv \llbracket \mathbf{C}_2 \rrbracket \setminus l \equiv (\llbracket \mathbf{C}_1 \rrbracket + \llbracket \mathbf{C}_2 \rrbracket) \setminus l$. By Definition 1.19, $\llbracket \mathbf{C}_1 + \mathbf{C}_2 \rrbracket \setminus l$.

(b) $C' = C'_2$

We have $C_2 \xrightarrow{l} C'_2$ and $\llbracket C_2 \rrbracket \setminus l \equiv \llbracket C'_2 \rrbracket$. By the rules in Figure 3 we have $C_1 + C_2 \xrightarrow{l} C'_2$. By Lemma 1.29 we have $\llbracket C'_2 \rrbracket \equiv \llbracket C_2 \rrbracket \setminus l \equiv (\llbracket C_1 \rrbracket + \llbracket C_2 \rrbracket) \setminus l$. By Definition 1.19, $\llbracket C_1 + C_2 \rrbracket \setminus l$.

• $l \in \mathcal{I}(\llbracket C_1 \rrbracket \parallel \llbracket C_2 \rrbracket)$

By Definition 1.17 we have two cases:

1. $l \in \mathcal{I}(\llbracket C_1 \rrbracket)$

By i.h. $\exists C'. C_1 \xrightarrow{l} C'$ and $\llbracket C_1 \rrbracket \setminus l \equiv \llbracket C' \rrbracket$. By the rules in Figure 3 we have two cases:

(a) $C' = \checkmark$

We have $C_1 \xrightarrow{l} \checkmark$ and $\llbracket C_1 \rrbracket \setminus l \equiv \llbracket \checkmark \rrbracket$. By the rules in Figure 3 we have $C_1 \parallel C_2 \xrightarrow{l} C_2$. By Definition 1.17, $(\llbracket C_1 \rrbracket \setminus l) \parallel \llbracket C_2 \rrbracket$. Since $l \in \mathcal{I}(\llbracket C_1 \rrbracket)$, then $\llbracket C_2 \rrbracket = \llbracket C_2 \rrbracket \setminus l$. Hence, we have $(\llbracket C_1 \rrbracket \setminus l) \parallel \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel (\llbracket C_2 \rrbracket \setminus l)$. By Lemma 1.30 we have $(\llbracket C_1 \rrbracket \parallel \llbracket C_2 \rrbracket) \setminus l$. By Definition 1.19, $\llbracket C_1 \parallel C_2 \rrbracket \setminus l$.

(b) $C' = C'_1$

We have $C_1 \xrightarrow{l} C'_1$ and $\llbracket C_1 \rrbracket \setminus l \equiv \llbracket C'_1 \rrbracket$. By the rules in Figure 3 we have $C_1 \parallel C_2 \xrightarrow{l} C'_1 \parallel C_2$. By Definition 1.17, $(\llbracket C_1 \rrbracket \setminus l) \parallel \llbracket C_2 \rrbracket$. Since $l \in \mathcal{I}(\llbracket C_1 \rrbracket)$, then $\llbracket C_2 \rrbracket = \llbracket C_2 \rrbracket \setminus l$. Hence, we have $(\llbracket C_1 \rrbracket \setminus l) \parallel \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel (\llbracket C_2 \rrbracket \setminus l)$. By Lemma 1.30 we have $(\llbracket C_1 \rrbracket \parallel \llbracket C_2 \rrbracket) \setminus l$. By Definition 1.19, $\llbracket C_1 \parallel C_2 \rrbracket \setminus l$.

2. $l \in \mathcal{I}(\llbracket C_2 \rrbracket)$

By i.h. $\exists C'. C_2 \xrightarrow{l} C'$ and $\llbracket C_2 \rrbracket \setminus l \equiv \llbracket C' \rrbracket$. By the rules in Figure 3 we have two cases:

(a) $C' = \checkmark$

We have $C_2 \xrightarrow{l} \checkmark$ and $\llbracket C_2 \rrbracket \setminus l \equiv \llbracket \checkmark \rrbracket$. By the rules in Figure 3 we have $C_1 \parallel C_2 \xrightarrow{l} C_1$. By Definition 1.17, $\llbracket C_1 \rrbracket \parallel (\llbracket C_2 \rrbracket \setminus l)$. Since $l \in \mathcal{I}(\llbracket C_2 \rrbracket)$, then $\llbracket C_1 \rrbracket = \llbracket C_1 \rrbracket \setminus l$. Hence, we have $\llbracket C_1 \rrbracket \parallel (\llbracket C_2 \rrbracket \setminus l) \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel (\llbracket C_2 \rrbracket \setminus l)$. By Lemma 1.30 we have $(\llbracket C_1 \rrbracket \parallel \llbracket C_2 \rrbracket) \setminus l$. By Definition 1.19, $\llbracket C_1 \parallel C_2 \rrbracket \setminus l$.

(b) $C' = C'_2$

We have $C_2 \xrightarrow{l} C'_2$ and $\llbracket C_2 \rrbracket \setminus l \equiv \llbracket C'_2 \rrbracket$. By the rules in Figure 3 we have $C_1 \parallel C_2 \xrightarrow{l} C_1 \parallel C'_2$. By Definition 1.17, $\llbracket C_1 \rrbracket \parallel (\llbracket C_2 \rrbracket \setminus l)$. Since $l \in \mathcal{I}(\llbracket C_2 \rrbracket)$, then $\llbracket C_1 \rrbracket = \llbracket C_1 \rrbracket \setminus l$. Hence, we have $\llbracket C_1 \rrbracket \parallel (\llbracket C_2 \rrbracket \setminus l) \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel (\llbracket C_2 \rrbracket \setminus l)$. By Lemma 1.30 we have $(\llbracket C_1 \rrbracket \parallel \llbracket C_2 \rrbracket) \setminus l$. By Definition 1.19, $\llbracket C_1 \parallel C_2 \rrbracket \setminus l$.

□

Lemma 1.35 (Adequacy II). If $(x \neq \emptyset) \in \mathcal{C}(\llbracket C \rrbracket)$ s.t. $\emptyset \xrightarrow{\omega} c x$ then $\exists C'$ s.t. $C \xrightarrow{\omega} C'$.

Proof. Induction over the length of ω .

• $|\omega| = 1$

We have $\{l\} \in \mathcal{C}(C)$ such that $\emptyset \xrightarrow{l} c \{l\}$. Furthermore $l \in \mathcal{I}(\llbracket C \rrbracket)$. By Lemma 1.34, $C \xrightarrow{l} C'$ and $\llbracket C' \rrbracket \equiv \llbracket C \rrbracket \setminus l$. By the rules in Figure 4, $C \xrightarrow{l} C'$.

• $|\omega| > 1$

We have $x \in \mathcal{C}(\llbracket C \rrbracket)$ such that $\emptyset \xrightarrow{\omega} c x$. Since $\omega = l_0 l_1 \dots l_n$, then $\emptyset \xrightarrow{l_0} c \{l_0\} \xrightarrow{\omega'} c x$. Hence $l_0 \in \mathcal{I}(\llbracket C \rrbracket)$. By Lemma 1.34, $C \xrightarrow{l_0} C'$ and $\llbracket C' \rrbracket \equiv \llbracket C \rrbracket \setminus l_0$. By Definition 1.22, $\exists y \in \mathcal{C}(\llbracket C' \rrbracket)$ such that $\emptyset \xrightarrow{\omega'} c y$. By i.h. $\exists C''$ such that $C' \xrightarrow{\omega'} C''$. By the rules in Figure 4, $C \xrightarrow{\omega} C''$, where $\omega = l_0 : \omega'$.

□

1.4 Introducing cyclic behavior

We now introduce cyclic behavior to the language in Section 1.1. In order to avoid the introduction of the notion of state in the language, the cyclic behavior will be given by recursion. In that way, we do not need to associate the notion of state to a command in the operational semantics. We can just keep recording the actions that are being made by the program.

Another thing to have in mind is that with cyclic behavior we open the door to infinite computations. However, covering chains are only defined in finite sequence of words and infinite configurations are odd (according to Marc), because we would need to define precisely what it means to be an infinite configuration. Hence, the

words that we formed with the n -step will be always finite, despite the possibility of them being infinite. We can justify this by saying that we are only concerned on the ‘interesting words’, *i.e.* those who are finite.

To introduce recursion we need to add some restrictions when forming programs, since we do not want to allow commands like: $\mu X.X; a$ and $\mu X.a; X; b$.

Let $X \subseteq Var$, with Var a set of variables. The syntax is now given by:

$$C ::= \text{skip} \mid a \in Act \mid C; C \mid C + C \mid C \parallel C \mid \mu X.C \mid X$$

where **skip** is a command that does nothing; a is an atomic action from a pre-determined set of atomic actions, denoted as Act ; $C; C$ is the usual sequential composition of programs; $C \parallel C$ is the parallel composition of commands; $C + C$ represents the non-deterministic choice; $\mu X.C$ is the recursive command; and $X \in Var$ with Var a set of variables. Furthermore, we only consider closed commands, *i.e.* commands in which every variable X is bound by a recursion μX and in sequential composition we only allow recursion to occur at right.

We define the set of free-variables and bound-variables as follows:

$$\begin{array}{l|l} FV(\text{skip}) = \emptyset & BV(\text{skip}) = \emptyset \\ FV(a) = \emptyset & BV(a) = \emptyset \\ FV(C_1; C_2) = FV(C_1) \cup FV(C_2) & BV(C_1; C_2) = BV(C_1) \cup BV(C_2) \\ FV(C_1 \parallel C_2) = FV(C_1) \cup FV(C_2) & BV(C_1 \parallel C_2) = BV(C_1) \cup BV(C_2) \\ FV(C_1 + C_2) = FV(C_1) \cup FV(C_2) & BV(C_1 + C_2) = BV(C_1) \cup BV(C_2) \\ FV(X) = \{X\} & BV(X) = \emptyset \\ FV(\mu X.C) = FV(C) \setminus \{X\} & BV(\mu X.C) = \{X\} \cup BV(C) \end{array}$$

We restrict the sequential composition to those whose free-variables and bound-variables on the left are empty, *i.e.* $C_1; C_2$ if $FV(C_1) = \emptyset = BV(C_1)$. With this restriction we forbid program like $\mu X.X; a$, $\mu X.a; X; b$ (with the condition $FV(C_1) = \emptyset$) and $(\mu X.a; X); b$ (with the condition $BV(C_1) = \emptyset$). We want to forbid these kind of programs in sequential composition, because if C_1 never terminates then the sequential composition never terminates. This is also a restriction that comes from the fact that covering chains are only defined in finite sequences and that infinite configurations are odd in event structures. Note however that we allow programs like $\mu X.X \parallel a$ and $\mu X.X + a$, since they do not block the computation.

We add to Figure 3 the following rule for the recursion command:

$$\frac{C \xrightarrow{l} C'}{\mu X.C \xrightarrow{l} C'[X \leftarrow \mu X.C]}$$

We define substitution as follows [HS08, inspired by]:

Definition 1.36. Let $X \in Var$ and C, C' be commands. Define $C[X \leftarrow C']$, where we substitute every free occurrence of X in C by C' (while changing bound variables to avoid clashes) by induction on C as follows:

$$\begin{aligned} \text{skip}[X \leftarrow C'] &= \text{skip} \\ a[X \leftarrow C'] &= a \\ (C_1; C_2)[X \leftarrow C'] &= C_1; (C_2[X \leftarrow C']) \\ (C_1 \parallel C_2)[X \leftarrow C'] &= C_1[X \leftarrow C'] \parallel C_2[X \leftarrow C'] \\ (C_1 + C_2)[X \leftarrow C'] &= C_1[X \leftarrow C'] + C_2[X \leftarrow C'] \\ (\mu Y.C)[X \leftarrow C'] &= \mu Y.C[X \leftarrow C'] \end{aligned}$$

Example 1.37. Figure 8 illustrates the behavior of a non-deterministic toss coin, which produces a possibly empty sequence of a ’s that finishes with sk . To understand this we observe that the initial program has two possible transitions: (1) we execute sk that terminates the computation; (2) we execute a , and we transit to a command equal to the initial one in which we have two possible transitions again.

On the event structure side, we want to use the Knaster-Tarski Theorem to build the least-fix point. To define it, we will use an order that does not ignore copies, differently from what happens with Definition 1.20.

Definition 1.38. Let $E_1 = (E_1, \leq_1, \#_1)$ and $E_2 = (E_2, \leq_2, \#_2)$ be event structures. Say $E_1 \trianglelefteq E_2$ if:

$$\begin{aligned} E_1 &\subseteq E_2 \\ \forall e, e' . e \leq_1 e' &\Leftrightarrow e, e' \in E_1 \wedge e \leq_2 e' \\ \forall e, e' . e \#_1 e' &\Leftrightarrow e, e' \in E_1 \wedge e \#_2 e' \end{aligned}$$

Lemma 1.39. \trianglelefteq is a partial order.

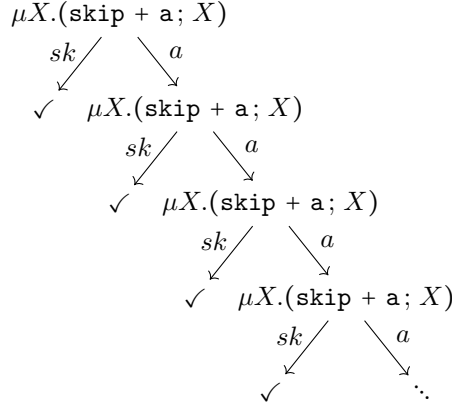


Figure 8: Unrolling the execution of $\mu X.(\text{skip} + a; X)$

Proof. Let E_1, E_2 , and E_3 be event structures.

- Reflexivity: $E_1 \trianglelefteq E_1$

It follows directly from Definition 1.38.

- Transitivity: $E_1 \trianglelefteq E_2, E_2 \trianglelefteq E_3 \Rightarrow E_1 \trianglelefteq E_3$

1. $E_1 \subseteq E_3$

Let $e \in E_1$. Since $E_1 \trianglelefteq E_2$ then $e \in E_2$. Since $E_2 \trianglelefteq E_3$ then $e \in E_3$. Hence $E_1 \subseteq E_3$.

2. $e \leq_1 e' \Leftrightarrow e, e' \in E_1, e \leq_3 e'$

\Rightarrow Let $e \leq_1 e'$.

Clearly $e, e' \in E_1$. Since $E_1 \trianglelefteq E_2$ then $e \leq_2 e'$. Furthermore $e, e' \in E_2$. Since $E_2 \trianglelefteq E_3$ then $e \leq_3 e'$.

\Leftarrow Let $e, e' \in E_1, e \leq_3 e'$.

Since $E_1 \trianglelefteq E_2$ then $e, e' \in E_2$. Since $E_2 \trianglelefteq E_3$ then $e \leq_2 e'$. Since $E_1 \trianglelefteq E_2$ $e \leq_1 e'$.

3. $e \#_1 e' \Leftrightarrow e, e' \in E_1, e \#_3 e'$

Similar to \leq , *i.e.* previous point.

- Antisymmetry: $E_1 \trianglelefteq E_2, E_2 \trianglelefteq E_1 \Rightarrow E_1 = E_2$

1. $E_1 = E_2$

Let $e \in E_1$. Since $E_1 \trianglelefteq E_2$ then $e \in E_2$. Let $e \in E_2$. Since $E_2 \trianglelefteq E_1$ then $e \in E_1$. Hence $E_1 = E_2$.

2. $(e \leq_1 e' \Leftrightarrow e, e' \in E_1, e \leq_2 e') = (e \leq_2 e' \Leftrightarrow e, e' \in E_2, e \leq_1 e')$

\Rightarrow Let $e \leq_1 e'$. Clearly $e, e' \in E_1$. From $E_1 \trianglelefteq E_2$, $e \leq_2 e'$.

Let $e \leq_2 e'$. Clearly $e, e' \in E_2$. From $E_2 \trianglelefteq E_1$, $e \leq_1 e'$.

\Leftarrow Let $e, e' \in E_1, e \leq_2 e'$. Since $E_1 \trianglelefteq E_2$ then $e \leq_1 e'$. Let $e, e' \in E_2, e \leq_1 e'$. Since $E_2 \trianglelefteq E_1$ then $e \leq_2 e'$.

Hence $\leq_1 = \leq_2$

- $(e \#_1 e' \Leftrightarrow e, e' \in E_1, e \#_2 e') = (e \#_2 e' \Leftrightarrow e, e' \in E_2, e \#_1 e')$

Similar reasoning as previous point.

□

Thus \trianglelefteq is a partial order.

Lemma 1.40. Define $\perp = (\emptyset, \emptyset, \emptyset)$. \perp is the least element of \trianglelefteq .

Proof. • \perp is an event structure

It follows directly that all conditions in Definition 1.1 are trivially satisfied because \perp has no events.

- For any event structure $E = (E, \leq, \#)$ we want to show $\perp \trianglelefteq E$.

1. $\emptyset \subseteq E$

Trivially holds.

2. $e \leq_{\perp} e' \Leftrightarrow e, e' \in \emptyset, e \leq e'$

Since \perp has no events and the causal relation is empty, it follows that $e \leq_{bot} e'$ and $e, e' \in \emptyset$ are false. Hence the condition trivially holds.

3. $e \#_{\perp} e' \Leftrightarrow e, e' \in \emptyset, e \# e'$

Similar to previous point.

□

Definition 1.41. Let $E_1 \trianglelefteq \dots \trianglelefteq E_n \trianglelefteq \dots$ be a ω -chain. Let $E^\omega = (E^\omega, \leq^\omega, \#^\omega)$ be its least upper bound where:

- $E^\omega = \cup_{n \in \omega} E_n$
- $\leq^\omega = \cup_{n \in \omega} \leq_n$
- $\#^\omega = \cup_{n \in \omega} \#_n$

Lemma 1.42. E^ω is an event structure.

Proof. • $\{e' \mid e' \leq^\omega e\}$ is finite

By Definition 1.41, $e' \leq^\omega e$ entails $\exists n \in \omega$ such that $e' \leq_n e$. Consequently, $e, e' \in E_n$. Furthermore, E_n is an event structure. Hence $\{e' \mid e' \leq_n e\}$ is finite. Thus $\{e' \mid e' \leq^\omega e\}$ is finite.

- $e \#^\omega e' \leq^\omega e'' \Rightarrow e \#^\omega e''$

By Definition 1.41, $e' \leq^\omega e$ entails $\exists n \in \omega$ such that $e \#_n e' \leq_n e''$, where E_n is an event structure. Hence $e \#_n e''$. Thus $e \#^\omega e''$.

□

Lemma 1.43. Let $E_1 \trianglelefteq \dots \trianglelefteq E_n \trianglelefteq \dots$ be a ω -chain. Then E^ω is its least upper bound.

Proof. • E^ω is an upper bound

$\forall n \in \omega$ we need to have $E_n \trianglelefteq E^\omega$. It follows directly from Definition 1.38 that $\forall n \in \omega E_n \trianglelefteq E^\omega$ since E^ω is by definition the union of all E_n .

- E^ω is the least upper bound

Let $E = (E, \leq, \#)$ be an upper bound of the chain. We need to show that if $E_n \trianglelefteq E^\omega$ and $E_n \trianglelefteq E$ then $E^\omega \trianglelefteq E$.

1. $E^\omega \subseteq E$

Let $e \in E^\omega$. By Definition 1.41, $\exists n \in \omega$ such that $e \in E_n$. By $E_n \trianglelefteq E$ we have $e \in E$.

2. $e \leq^\omega e' \Leftrightarrow e, e' \in E^\omega$ and $e \leq e'$

\Rightarrow Let $e \leq^\omega e'$.

By Definition 1.41, $\exists n \in \omega$ such that $e \leq_n e'$. It is then clear that $e, e' \in E_n \subseteq E^\omega$. Since $E_n \trianglelefteq E$ we have $e \leq e'$.

\Leftarrow $e, e' \in E^\omega$ and $e \leq e'$

By Definition 1.41, $\exists n \in \omega$ such that $e, e' \in E_n$. Since $E_n \trianglelefteq E$, $e \leq_n e'$. By Definition 1.41, $e \leq^\omega e'$.

3. $e \#^\omega e' \Leftrightarrow e, e' \in E^\omega$ and $e \# e'$

Similar to previous point.

□

Now we show that the operators of the language are monotone w.r.t to Definition 1.38. We highlight that the sequential composition is only right monotone because of the restriction imposed in the syntax, in which the free-variables and bounded-variables of the first command must be empty.

Lemma 1.44. Let E, E_1, E_2 be event structures. If $E_1 \trianglelefteq E_2$ then $E; E_1 \trianglelefteq E; E_2$.

Proof. Let $E = (E, \leq, \#)$, $E_1 = (E_1, \leq_1, \#_1)$, $E_2 = (E_2, \leq_2, \#_2)$, $E; E_1 = (E^1, \leq^1, \#^1)$, and $E; E_2 = (E^2, \leq^2, \#^2)$, such that $E_1 \trianglelefteq E_2$.

1. $E^1 \subseteq E^2 \Leftrightarrow E \uplus (E_1 \times \mathcal{C}_{\max}(E)) \subseteq E \uplus (E_2 \times \mathcal{C}_{\max}(E))$

By Definition 1.13 we have two cases:

- (a) $e \in E$

Then we are done.

(b) $e \in E_1 \times \mathcal{C}_{\max}(E)$

We know that e is of the form (e_1, x) where $e_1 \in E_1$ and $x \in \mathcal{C}_{\max}(E)$. Since $E_1 \trianglelefteq E_2$ then $e_1 \in E_2$ and consequently $e \in \mathcal{C}_{\max}(E)$.

2. $\forall e, e' . e \leq^1 e' \Leftrightarrow e, e' \in E^1$ and $e \leq 2e'$

\Rightarrow Assume $e \leq^1 e'$. Clearly $e, e' \in E^1$. By Definition 1.13 we have three cases:

(a) $e \leq^1 e'$ is of the form $e \leq e'$

Hence $e, e' \in E$. By Definition 1.13 $e \leq^2 e'$.

(b) $e \leq e'$ is of the form $e \leq_{(1,x)} e'$

We know that e, e' are of the form $(e, x), (e', x) \in E_1 \times \mathcal{C}_{\max}(E)$, which entails $e, e' \in E_1$ and $x \in \mathcal{C}_{\max}(E)$. Since $E_1 \trianglelefteq E_2$, $e, e' \in E_2$ and $e \leq_2 e'$. By Definition 1.13 we have $(e, x) \leq^2 (e', x)$.

(c) $e \leq^1 e'$ is of the form $e \leq^1 (e', x)$

We know that $e \in E$, $e \in x \in \mathcal{C}_{\max}(E)$, and $(e', x) \in E_1 \times \mathcal{C}_{\max}(esE)$, with the last entailing $e' \in E_1$. Since $E_1 \trianglelefteq E_2$, $e' \in E_2$ and consequently $(e', x) \in E_2 \times \mathcal{C}_{\max}(E)$. By Definition 1.13 we have $e \leq^2 (e', x)$.

\Leftarrow Assume $e, e' \in E^1$ and $e \leq^2 e'$. The cases are distinguished by \leq^2 .

(a) $e \leq^2 e'$ is of the form $e \leq e'$

Hence $e, e' \in E$. By Definition 1.13, $e \leq^1 e'$.

(b) $e \leq^2 e$ is of the form $e \leq_{(2,x)} e'$

We know that e, e' are of the form $(e, x), (e', x) \in E_2 \times \mathcal{C}_{\max}(E)$, which entails $e, e' \in E_2$ and $x \in \mathcal{C}_{\max}(E)$. Since $E_1 \trianglelefteq E_2$ and $e, e' \in E^1$, which entails for this case that $e, e' \in E_1$, then $e \leq_1 e'$. By Definition 1.13 we have $(e, x) \leq^1 (e', x)$.

(c) $e \leq^2 e'$ is of the form $e \leq^2 (e', x)$

We know that $e \in E$, $e \in x \in \mathcal{C}_{\max}(E)$, and $(e', x) \in E_2 \times \mathcal{C}_{\max}(esE)$, with the last entailing $e' \in E_2$. Since $E_1 \trianglelefteq E_2$ and $e' \in E^1$, which entails for this case that $e' \in E_1$, then $(e', x) \in E_1 \times \mathcal{C}_{\max}(E)$. By Definition 1.13 we have $e \leq^1 (e', x)$.

3. $\forall e, e' . e \#^1 e' \Leftrightarrow e, e' \in E$ and $e \#^2 e'$

\Rightarrow Assume $e \#^1 e'$.

Clearly $e, e' \in E^1$. From Definition 1.13 we have that $\exists (a \leq^1 e, a' \leq^1 e')$ such that $a \# a'$ or $a \#_{(1,x)} a'$. For the former we are done. For the latter, since $E_1 \subseteq E_2$ then $a \#_{(2,x)} a'$, such that $a \leq^2 e$ or $a' \leq^2 e'$. Thus $e \#^2 e'$.

\Leftarrow Assume $e, e' \in E^1$ and $e \#^2 e'$.

From Definition 1.13 it exists $a \leq^2 e$ and $a' \leq^2 e'$ such that $a \# a'$ or $a \#_{(2,x)} a'$. Since $e, e' \in E^1$ and $E_1 \trianglelefteq E_2$ then it follows directly that $e \#^1 e'$.

□

Lemma 1.45. Let E_1, E'_1, E_2, E'_2 be event structures. If $E_1 \trianglelefteq E'_1$ and $E_2 \trianglelefteq E'_2$ then $E_1 \parallel E_2 \trianglelefteq E'_1 \parallel E'_2$.

Proof. It follows directly from Definition 1.17. □

Lemma 1.46. Let E_1, E'_1, E_2, E'_2 be event structures. If $E_1 \trianglelefteq E'_1$ and $E_2 \trianglelefteq E'_2$ then $E_1 + E_2 \trianglelefteq E'_1 + E'_2$.

Proof. It follows directly from Definition 1.15. □

Definition 1.47. Let op be an n -ary operation on the class of event structures. Say op is *monotonic* iff when for event structures we have

$$E_1 \trianglelefteq E'_1, \dots, E_n \trianglelefteq E'_n \text{ then } op(E_1, \dots, E_n) \trianglelefteq op(E'_1, \dots, E'_n)$$

Say op is continuous iff for all countable chains

$$\begin{aligned} E_{11} \trianglelefteq E_{12} \trianglelefteq \dots \trianglelefteq E_{1i} \trianglelefteq \dots \\ \vdots \\ E_{n1} \trianglelefteq E_{n2} \trianglelefteq \dots \trianglelefteq E_{ni} \trianglelefteq \dots \end{aligned}$$

we have

$$op\left(\bigsqcup_i E_{1i}, \dots, \bigsqcup_i E_{ni}\right) = \bigsqcup_i op(E_{1i}, \dots, E_{ni})$$

where \bigsqcup denotes the least upper bound w.r.t \trianglelefteq .

The next lemma will be very useful when proving the continuity of operators.

Lemma 1.48. Let op be a unary operation on event structures. Then op is continuous iff

1. op is monotonic
2. if $E_1 \sqsubseteq \dots \sqsubseteq E_n \sqsubseteq \dots$ is a ω -chain then each event of $op(\sqcup_n E_n)$ is an event of $\sqcup_n op(E_n)$.

Proof. • \Rightarrow : Assume op is continuous.

We have $op(\sqcup_n E_n) = \sqcup_n op(E_n)$. Let $E_1 \sqsubseteq \dots \sqsubseteq E_n \sqsubseteq \dots$ and $E'_1 \sqsubseteq \dots \sqsubseteq E'_n \sqsubseteq \dots$ be two ω -chains such that $E_1 \sqsubseteq E'_1, \dots, E_n \sqsubseteq E'_n$. We want to show that $E_1 \sqsubseteq E'_1 \Rightarrow op E_1 \sqsubseteq op E'_1, \dots, E_n \sqsubseteq E'_n \Rightarrow op E_n \sqsubseteq op E'_n, \dots$. For that we can make use of the least upper bound, i.e. $\sqcup_n E_n \sqsubseteq \sqcup_n E'_n \Rightarrow op(\sqcup_n E_n) \sqsubseteq op(\sqcup_n E'_n)$. Since op is continuous, $op \sqcup_n E_n \sqsubseteq \sqcup_n op E'_n$. Hence op is monotonic. Now it lacks to show that each event of $op(\sqcup_n E_n)$ is an event of $\sqcup_n op(E'_n)$. But that comes freely since $op(\sqcup_n E_n) = \sqcup_n op(E_n)$.

• \Leftarrow : Assume 1. and 2. above.

We want to show $op(\sqcup_n E_n) = \sqcup_n op(E_n)$. Let $E_1 \sqsubseteq \dots \sqsubseteq E_n \sqsubseteq \dots$ be a ω -chain. By 1. we know that op is monotonic, hence $E_n \sqsubseteq \sqcup_n E_n$ entails $op(E_n) \sqsubseteq op(\sqcup_n E_n)$ that leads to $op(\sqcup_n E_n) \sqsubseteq \sqcup_n op(E_n)$. By 2., each event of $op(\sqcup_n E_n)$ is an event of $\sqcup_n op(E_n)$. Hence by Definition 1.38, $op(\sqcup_n E_n) = \sqcup_n op(E_n)$. \square

Lemma 1.49. $\sqcup_m(E; E_m) = E; \sqcup_m E_m$.

Proof. By Lemma 1.44 we know that sequential composition is monotone w.r.t \sqsubseteq at right. It lacks to show that each event of $E; \sqcup_m E_m$ is an event of $\sqcup_m(E; E_m)$. Let $E_1 \sqsubseteq \dots \sqsubseteq E_m \sqsubseteq \dots$ be an ω -chain such that $\sqcup_m E_m$ is its least upper bound and $E; E_1 \sqsubseteq \dots \sqsubseteq E; E_m \sqsubseteq \dots$ be another ω -chain with $\sqcup_m(E; E_m)$ as its least upper bound. Let e be an event of $E; \sqcup_m E_m$. By Definition 1.13 we have two cases:

1. e is an event of E

Then we are done, since $\forall m, e$ is an event of $E; E_m$. Hence it is an event of $\sqcup_m(E; E_m)$.

2. e is an event of $(\cup_{m \in \omega} E_m) \times \mathcal{C}_{\max}(E)$

We know that e is of the form (e_m, x) with e_m an event of $\sqcup_m E_m$ and $x \in \mathcal{C}_{\max}(E)$. The former entails $\exists m$ such that e_m is an event of E_m . By Definition 1.13 we have (e_m, x) as an event of $E; E_m$. Consequently (e_m, x) is an event of $\sqcup_m(E; E_m)$.

By Lemma 1.48 we are done. \square

Lemma 1.50. $\sqcup_{n,m}(E_n \parallel E_m) = \sqcup_n E_n \parallel \sqcup_m E_m$.

Proof. By Lemma 1.45 we know that parallel composition is monotone w.r.t \sqsubseteq . It lacks to show that each event of $\sqcup_n E_n \parallel \sqcup_m E_m$ is an event of $\sqcup_{n,m}(E_n \parallel E_m)$.

Let $E_1 \sqsubseteq \dots \sqsubseteq E_n \sqsubseteq \dots$ and $E'_1 \sqsubseteq \dots \sqsubseteq E'_m \sqsubseteq \dots$ be ω -chains with least upper bound $\sqcup_n E_n$ and $\sqcup_m E_m$, respectively. Let e be an event of $\sqcup_n E_n \parallel \sqcup_m E_m$. By Definition 1.17 we have two cases:

1. e is an event of $\sqcup_n E_n$

Then $\exists n \in \omega$ such that e is an event of E_n . By Definition 1.17, e is an event of $E_n \parallel E_m$ and consequently is an event of $\sqcup_{n,m}(E_n \parallel E_m)$.

2. e is an event of $\sqcup_m E_m$ Similar to previous point.

By Lemma 1.48 we are done. \square

Lemma 1.51. $\sqcup_{n,m}(E_n + E_m) = \sqcup_n E_n + \sqcup_m E_m$.

Proof. By Lemma 1.46 we know that non-deterministic composition is monotone w.r.t \sqsubseteq . It lacks to show that each event of $\sqcup_n E_n + \sqcup_m E_m$ is an event of $\sqcup_{n,m}(E_n + E_m)$.

Let $E_1 \sqsubseteq \dots \sqsubseteq E_n \sqsubseteq \dots$ and $E'_1 \sqsubseteq \dots \sqsubseteq E'_m \sqsubseteq \dots$ be ω -chains with least upper bound $\sqcup_n E_n$ and $\sqcup_m E_m$, respectively. Let e be an event of $\sqcup_n E_n + \sqcup_m E_m$. By Definition 1.15 we have two cases:

1. e is an event of $\sqcup_n E_n$

Then $\exists n \in \omega$ such that e is an event of E_n . By Definition 1.15, e is an event of $E_n + E_m$ and consequently is an event of $\sqcup_{n,m}(E_n + E_m)$.

2. e is an event of $\sqcup_m E_m$ Similar to previous point.

By Lemma 1.48 we are done. \square

Lemma 1.52. Let Γ be a continuous operation on event structures. Let $\perp = (\emptyset, \emptyset, \emptyset) \in E$. Define $fix(\Gamma)$ to be the least upper bound of the chain $\perp \trianglelefteq \Gamma(\perp) \trianglelefteq \dots \trianglelefteq \Gamma^n(\perp) \trianglelefteq \dots$. Then $\Gamma(fix(\Gamma)) = fix(\Gamma)$.

Proof. $\Gamma(fix(\Gamma)) = fix(\Gamma) \Leftrightarrow \Gamma(\bigsqcup_n \Gamma^n(\perp)) = \bigsqcup_n (\Gamma^n(\perp))$. Since Γ is continuous, $\Gamma(\bigsqcup_n \Gamma^n(\perp)) = \bigsqcup_n \Gamma \Gamma^n(\perp) = \bigsqcup_n \Gamma^{n+1}(\perp)$. We note that: $\perp \sqcup \bigsqcup_n \Gamma^n(\perp) = \bigsqcup_n \Gamma^n(\perp)$. Since \perp is the ‘identity of the least upper bound’ we have: $\perp \sqcup \bigsqcup_n \Gamma^n(\perp) = \bigsqcup_n \Gamma^n(\perp) \Leftrightarrow \bigsqcup_n \Gamma \Gamma^n(\perp) = \bigsqcup_n \Gamma^n(\perp) \Leftrightarrow \Gamma(\bigsqcup_n \Gamma^n(\perp)) = \bigsqcup_n \Gamma^n(\perp)$.

Now we need to show that $fix(\Gamma)$ is the least fixpoint. Let E be an event structure, $\Gamma(E) \trianglelefteq E$, and $\perp \trianglelefteq E$. By the monotonic property $\Gamma(\perp) \trianglelefteq \Gamma(E)$. Since $\Gamma(E) \trianglelefteq E$ then $\Gamma(\perp) \trianglelefteq E$. By induction $\Gamma^n(E) \trianglelefteq E$. Thus $fix(\Gamma) = \bigsqcup_n \Gamma^n(\perp) \trianglelefteq E$. Hence $fix(\Gamma)$ is the least fixpoint. \square

Definition 1.53. Define an environment to be a function $\gamma : Var \rightarrow E$ from variables to event structures. For a command C and an environment γ define $\llbracket C \rrbracket_\gamma$ as follows:

$$\begin{aligned} \llbracket \text{skip} \rrbracket_\gamma &= (\{sk\}, \{sk \leq sk\}, \emptyset) \\ \llbracket a \rrbracket_\gamma &= (\{a\}, \{a \leq a\}, \emptyset) \\ \llbracket C_1 ; C_2 \rrbracket_\gamma &= \llbracket C_1 \rrbracket_\gamma ; \llbracket C_2 \rrbracket_\gamma \\ \llbracket C_1 + C_2 \rrbracket_\gamma &= \llbracket C_1 \rrbracket_\gamma + \llbracket C_2 \rrbracket_\gamma \\ \llbracket C_1 \parallel C_2 \rrbracket_\gamma &= \llbracket C_1 \rrbracket_\gamma \parallel \llbracket C_2 \rrbracket_\gamma \\ \llbracket X \rrbracket_\gamma &= \gamma(X) \\ \llbracket \mu X. C \rrbracket_\gamma &= fix(\Gamma^{C, \gamma}) \end{aligned}$$

where $\Gamma^{C, \gamma} : E \rightarrow E$ is given by $\Gamma^{C, \gamma}(E) = \llbracket C \rrbracket_{\gamma(X \leftarrow E)}$.

Remark 2. ‘Another way to see’ $\Gamma^{C, \gamma}$ is

$$\Gamma^{C, \gamma} := E \mapsto \Gamma^C(\gamma(X_1), \gamma(X_2), \dots, \gamma(X_n), E)$$

where we make a connection with $FV(C) = \{X_1, X_2, \dots, X_n, X\}$.

We now show that $\Gamma^{C, \gamma}$ is continuous. For that it is useful to know that curry and fix are continuous [AJ94].

Lemma 1.54. $\Gamma^{C, \gamma}$ is continuous.

Proof. •

$$\begin{aligned} &\Gamma^{C_1 ; C_2, \gamma}(\bigsqcup_n E_n) \\ &= \{\text{Definition 1.53}\} \\ &\quad \llbracket C_1 ; C_2 \rrbracket_{\gamma(X \leftarrow \bigsqcup_n E_n)} \\ &= \{\text{Definition 1.53}\} \\ &\quad \llbracket C_1 \rrbracket ; \llbracket C_2 \rrbracket_{\gamma(X \leftarrow \bigsqcup_n E_n)} \\ &= \{\text{Definition 1.53}\} \\ &\quad \llbracket C_1 \rrbracket ; \bigsqcup_n \Gamma^{C_2, \gamma}(E_n) \\ &= \{\text{Lemma 1.49}\} \\ &\quad = \bigsqcup_n (\llbracket C_1 \rrbracket ; \Gamma^{C_2, \gamma}(E_n)) \\ &= \{\text{Definition 1.53}\} \\ &\quad = \bigsqcup_n (\llbracket C_1 \rrbracket ; \llbracket C_2 \rrbracket_{\gamma(X \leftarrow E_n)}) \\ &= \{\text{Definition 1.53}\} \\ &\quad \bigsqcup_n \llbracket C_1 ; C_2 \rrbracket_{\gamma(X \leftarrow E_n)} \end{aligned}$$

•

$$\begin{aligned}
& \Gamma^{\mathbf{C}_1 \parallel \mathbf{C}_2, \gamma}(\bigsqcup_n E_n) \\
&= \{\text{Definition 1.53}\} \\
& \llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket_{\gamma(X \leftarrow \bigsqcup_n E_n)} \\
&= \{\text{Definition 1.53}\} \\
& \llbracket \mathbf{C}_1 \rrbracket_{\gamma(X \leftarrow \bigsqcup_n E_n)} \parallel \llbracket \mathbf{C}_2 \rrbracket_{\gamma(X \leftarrow \bigsqcup_n E_n)} \\
&= \{\text{Definition 1.53}\} \\
& \bigsqcup_n \Gamma^{\mathbf{C}_1, \gamma}(E_n) \parallel \bigsqcup_n \Gamma^{\mathbf{C}_2, \gamma}(E_n) \\
&= \{\text{Lemma 1.50}\} \\
&= \bigsqcup_n (\Gamma^{\mathbf{C}_1, \gamma}(E_n) \parallel \Gamma^{\mathbf{C}_2, \gamma}(E_n)) \\
&= \{\text{Definition 1.53}\} \\
&= \bigsqcup_n (\llbracket \mathbf{C}_1 \rrbracket_{\gamma(X \leftarrow E_n)} \parallel \llbracket \mathbf{C}_2 \rrbracket_{\gamma(X \leftarrow E_n)}) \\
&= \{\text{Definition 1.53}\} \\
& \bigsqcup_n \llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket_{\gamma(X \leftarrow E_n)}
\end{aligned}$$

•

$$\begin{aligned}
& \Gamma^{\mathbf{C}_1 + \mathbf{C}_2, \gamma}(\bigsqcup_n E_n) \\
&= \{\text{Definition 1.53}\} \\
& \llbracket \mathbf{C}_1 + \mathbf{C}_2 \rrbracket_{\gamma(X \leftarrow \bigsqcup_n E_n)} \\
&= \{\text{Definition 1.53}\} \\
& \llbracket \mathbf{C}_1 \rrbracket_{\gamma(X \leftarrow \bigsqcup_n E_n)} + \llbracket \mathbf{C}_2 \rrbracket_{\gamma(X \leftarrow \bigsqcup_n E_n)} \\
&= \{\text{Definition 1.53}\} \\
& \bigsqcup_n \Gamma^{\mathbf{C}_1, \gamma}(E_n) + \bigsqcup_n \Gamma^{\mathbf{C}_2, \gamma}(E_n) \\
&= \{\text{Lemma 1.51}\} \\
&= \bigsqcup_n (\Gamma^{\mathbf{C}_1, \gamma}(E_n) + \Gamma^{\mathbf{C}_2, \gamma}(E_n)) \\
&= \{\text{Definition 1.53}\} \\
&= \bigsqcup_n (\llbracket \mathbf{C}_1 \rrbracket_{\gamma(X \leftarrow E_n)} + \llbracket \mathbf{C}_2 \rrbracket_{\gamma(X \leftarrow E_n)}) \\
&= \{\text{Definition 1.53}\} \\
& \bigsqcup_n \llbracket \mathbf{C}_1 + \mathbf{C}_2 \rrbracket_{\gamma(X \leftarrow E_n)}
\end{aligned}$$

•

$$\begin{aligned}
& \Gamma^{\mu X.\mathbf{C},\gamma}(\bigsqcup_n E_n) \\
&= \{\text{Definition 1.53}\} \\
& \llbracket \mu X.\mathbf{C} \rrbracket_{\gamma(X \leftarrow \bigsqcup_n E_n)} \\
&= \{\text{Definition 1.53}\} \\
& fix(\Gamma^{\mathbf{C},\gamma(X \leftarrow \bigsqcup_n E_n)}) = fix(E \mapsto \Gamma^{\mathbf{C},\gamma(X \leftarrow \bigsqcup_n E_n)}(E)) \\
&= \{\text{Definition 1.53}\} \\
& fix(E \mapsto \llbracket \mathbf{C} \rrbracket_{\gamma(X \leftarrow \bigsqcup_n E_n, Y \leftarrow E)}) \\
&= \{\text{i.h.}\} \\
& fix(E \mapsto \bigsqcup_n \llbracket \mathbf{C} \rrbracket_{\gamma(X \leftarrow E_n, Y \leftarrow E)}) \\
&= \{\text{curry continuous ; doubts in this step}\} \\
& fix(\bigsqcup_n (E \mapsto \llbracket \mathbf{C} \rrbracket_{\gamma(X \leftarrow E_n, Y \leftarrow E)})) \\
&= \{fix \text{ continuous}\} \\
& \bigsqcup_n fix(E \mapsto \llbracket \mathbf{C} \rrbracket_{\gamma(X \leftarrow E_n, Y \leftarrow E)}) \\
&= \{\text{Definition 1.53}\} \\
& \bigsqcup_n fix(E \mapsto \Gamma^{\mathbf{C},\gamma(X \leftarrow E_n)}(E)) = \bigsqcup_n fix(\Gamma^{\mathbf{C},\gamma(X \leftarrow E_n)}) \\
&= \{\text{Definition 1.53}\} \\
& \bigsqcup_n \llbracket \mu X.\mathbf{C} \rrbracket_{\gamma(X \leftarrow E_n)} \\
&= \{\text{Definition 1.53}\} \\
& \bigsqcup_n \Gamma^{\mu X.\mathbf{C},\gamma}(E_n)
\end{aligned}$$

□

Lemma 1.55. $\llbracket \mathbf{C}'[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_{\gamma}] \rrbracket_{\gamma} = \llbracket \mathbf{C}' \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_{\gamma})}$

Proof. • $\llbracket \text{skip}[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_{\gamma}] \rrbracket_{\gamma}$

It follows directly that $\llbracket \text{skip} \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_{\gamma})}$.

• $\llbracket \mathbf{a}[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_{\gamma}] \rrbracket_{\gamma}$

It follows directly that $\llbracket \mathbf{a} \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_{\gamma})}$.

•

$$\begin{aligned}
& \llbracket (\mathbf{C}_1 ; \mathbf{C}_2)[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_{\gamma}] \rrbracket_{\gamma} \\
&= \{\text{Definition 1.36}\} \\
& \llbracket \mathbf{C}_1 ; (\mathbf{C}_2[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_{\gamma}]) \rrbracket_{\gamma} \\
&= \{\text{Definition 1.53}\} \\
& \llbracket \mathbf{C}_1 \rrbracket ; (\llbracket \mathbf{C}_2 \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_{\gamma})}) \\
&= \{\text{i.h.}\} \\
& \llbracket \mathbf{C}_1 \rrbracket ; \llbracket \mathbf{C}_2 \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_{\gamma})} \\
&= \{\text{Definition 1.53}\} \\
& \llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_{\gamma})}
\end{aligned}$$

•

$$\begin{aligned}
& \llbracket (\mathbf{C}_1 \parallel \mathbf{C}_2)[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma] \rrbracket_\gamma \\
& = \{\text{Definition 1.36}\} \\
& \llbracket \mathbf{C}_1[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma] \parallel \mathbf{C}_2[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma] \rrbracket_\gamma \\
& = \{\text{Definition 1.53}\} \\
& \llbracket \mathbf{C}_1 \rrbracket_\gamma[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma] \parallel \llbracket \mathbf{C}_2 \rrbracket_\gamma[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma] \\
& = \{\text{i.h.}\} \\
& \llbracket \mathbf{C}_1 \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma)} \parallel \llbracket \mathbf{C}_2 \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma)} \\
& = \{\text{Definition 1.53}\} \\
& \llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma)}
\end{aligned}$$

•

$$\begin{aligned}
& \llbracket (\mathbf{C}_1 + \mathbf{C}_2)[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma] \rrbracket_\gamma \\
& = \{\text{Definition 1.36}\} \\
& \llbracket \mathbf{C}_1[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma] + \mathbf{C}_2[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma] \rrbracket_\gamma \\
& = \{\text{Definition 1.53}\} \\
& \llbracket \mathbf{C}_1 \rrbracket_\gamma[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma] + \llbracket \mathbf{C}_2 \rrbracket_\gamma[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma] \\
& = \{\text{i.h.}\} \\
& \llbracket \mathbf{C}_1 \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma)} + \llbracket \mathbf{C}_2 \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma)} \\
& = \{\text{Definition 1.53}\} \\
& \llbracket \mathbf{C}_1 + \mathbf{C}_2 \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma)}
\end{aligned}$$

•

$$\begin{aligned}
& \llbracket (\mu Y.\mathbf{C}')[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma] \rrbracket_\gamma \\
& = \{\text{Definition 1.36}\} \\
& \llbracket \mu Y.(\mathbf{C}'[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma]) \rrbracket_\gamma \\
& = \{\text{Definition 1.53}\} \\
& \text{fix}(\Gamma^{\mathbf{C}'[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma], \gamma}) = \text{fix}(\mathbf{E} \mapsto \Gamma^{\mathbf{C}'[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma], \gamma}(\mathbf{E})) \\
& = \{\text{Definition 1.53}\} \\
& \text{fix}(\mathbf{E} \mapsto \llbracket \mathbf{C}'[X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma] \rrbracket_{\gamma(Y \leftarrow \mathbf{E})}) \\
& = \{\text{i.h.}\} \\
& \text{fix}(\mathbf{E} \mapsto \llbracket \mathbf{C}' \rrbracket_{\gamma(Y \leftarrow \mathbf{E}, X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma)}) \\
& = \{\text{Definition 1.53}\} \\
& \text{fix}(\mathbf{E} \mapsto \Gamma^{\mathbf{C}', \gamma(Y \leftarrow \mathbf{E}, X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma)}(\mathbf{E})) = \text{fix}(\Gamma^{\mathbf{C}', \gamma(X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma)}(\mathbf{E})) \\
& = \text{Definition 1.53;} \\
& \llbracket \mu Y.\mathbf{C}' \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma)}
\end{aligned}$$

□

Lemma 1.56. $\llbracket \mu X.\mathbf{C} \rrbracket_\gamma = \llbracket \mathbf{C} \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma)}$

Proof.

$$\begin{aligned}
& \llbracket \mu X.\mathbf{C} \rrbracket_\gamma \\
& = \text{Definition 1.53} \\
& \text{fix}(\Gamma^{\mathbf{C}, \gamma}) \\
& = \{\text{Lemma 1.52}\} \\
& \Gamma^{\mathbf{C}, \gamma}(\text{fix}(\Gamma^{\mathbf{C}, \gamma})) \\
& = \{\text{Definition 1.53}\} \\
& \Gamma^{\mathbf{C}, \gamma}(\llbracket \mu X.\mathbf{C} \rrbracket_\gamma) \\
& = \{\text{Definition 1.53}\} \\
& \llbracket \mathbf{C} \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.\mathbf{C} \rrbracket_\gamma)}
\end{aligned}$$

□

To show the equivalence between the operational and the denotational semantics, we reuse what was done in Section 1.3. The only lemmas in which we need to add the proof for the recursion case are the following:

Lemma 1.57 (Soundness I). If $C \xrightarrow{l} C'$ then $\forall \gamma, \llbracket C' \rrbracket_\gamma \equiv \llbracket C \rrbracket_\gamma \backslash l$.

Proof.

$$\begin{aligned}
& \mu X.C \\
& \Rightarrow \{\text{Figure 3 entails}\} \\
& C \xrightarrow{l} C' \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket C' \rrbracket_\gamma \equiv \llbracket C \rrbracket_\gamma \backslash l \\
& \Rightarrow \{\text{setting } \gamma = \gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)\} \\
& \llbracket C' \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)} \equiv \llbracket C \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)} \backslash l \\
& \Rightarrow \{\text{Lemma 1.55, Lemma 1.56}\} \\
& \llbracket C' [X \leftarrow \llbracket \mu X.C \rrbracket_\gamma] \rrbracket_\gamma \equiv \llbracket \mu X.C \rrbracket_\gamma \backslash l
\end{aligned}$$

□

Lemma 1.58 (Adequacy I). Let $l \in \mathcal{I}(\llbracket C \rrbracket)$. Then $\exists C' \in (C \cup \{\checkmark\})$ s.t $C \xrightarrow{l} C'$ and $\llbracket C \rrbracket \backslash l \equiv \llbracket C' \rrbracket$.

Proof. • $l \in \mathcal{I}(\llbracket \mu X.C \rrbracket_\gamma)$

By Definition 1.53 and Definition 1.41, $l \in \mathcal{I}(\llbracket C \rrbracket_\gamma)$. By i.h., $\exists C'$ such that $C \xrightarrow{l} C'$ and $\llbracket C' \rrbracket_\gamma \equiv \llbracket C \rrbracket_\gamma \backslash l$. By the rules in Figure 3 and by setting $\gamma = \gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)$, $\mu X.C \xrightarrow{l} C' [X \leftarrow \mu X.C]$ and $\llbracket C' \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)} \equiv \llbracket C \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)} \backslash l$

□

For Lemma 1.33 and Lemma 1.35 we only need to adapt $\llbracket - \rrbracket$ to $\llbracket - \rrbracket_\gamma$.

Example 1.59. The event structure in Example 1.7 corresponds to the command in Example 1.10.

To see how the semantics relate, recall the configurations in Example 1.7 and the words in Example 1.10.

Let us select the words cd and dc . It is straightforward to see that each word corresponds to a covering chain, $\emptyset \xrightarrow{d} c \{d\} \xrightarrow{c} c \{d, c\}$ and $\emptyset \xrightarrow{c} c \{c\} \xrightarrow{d} c \{d, c\}$, respectively. Both covering chains correspond to the configuration $\{d, c\}$.

Conversely, the configuration $\{d, c\}$ is obtained by two covering chains: $\emptyset \xrightarrow{d} c \{d\} \xrightarrow{c} c \{d, c\}$ and $\emptyset \xrightarrow{c} c \{c\} \xrightarrow{d} c \{d, c\}$. It is straightforward to see that each covering chain corresponds to the words dc and cd , respectively.

Example 1.60. Figure 9 shows the event structure corresponding to the interpretation of $\llbracket (a; b) \parallel c \rrbracket$. The set of configurations is $\{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$, where we note that in the presence of concurrent events, a configuration has more than one possible covering chain.

To see the equivalence between both semantics through an example, recall the words that can be formed by the n-step in Example 1.10: $a, c, ab, ac, ca, abc, acb$, and cab .

Each word corresponds to a covering chain, which represents a configuration. For example the words ac and ca correspond to the covering chains $\emptyset \xrightarrow{a} c \{a\} \xrightarrow{c} c \{a, c\}$ and $\emptyset \xrightarrow{c} c \{c\} \xrightarrow{a} c \{a, c\}$, respectively. These covering chains correspond to the configuration $\{a, c\}$. Conversely, for each covering chain, there exists a corresponding word.

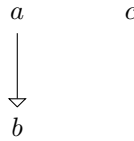


Figure 9: Event structure of $\llbracket (a; b) \parallel c \rrbracket$

2 Probabilistic Event Structures

Probabilistic event structures [Win14] are event structures together with a valuation on configurations $v : \mathcal{C}(E) \rightarrow [0, 1]$, which are seen as the probability of reaching at least this configuration, such that $v(\emptyset) = 1$ and a condition that assures the non-existence of negative probabilities. The definition of probabilistic event structures in [Win14] makes use of a *drop condition* function, which is intuitively seen as the probability of reaching at least a configuration y without reaching any of the x_1, \dots, x_n with $y \subseteq x_1, \dots, x_n$. In order to abstract the reader from that definition, we make use of [Win14, Proposition 1] in Definition 2.1, which says that the drop condition can be described in terms of a sum.

Definition 2.1 (Probabilistic event structure). Let $E = (E, \leq, \#)$ be an event structure. A configuration-valuation on E is a function $v : \mathcal{C}(E) \rightarrow [0, 1]$ such that $v(\emptyset) = 1$ and $\forall y, x_1, \dots, x_n \in \mathcal{C}(E)$ such that $y \subseteq x_1, \dots, x_n$

$$v(y) - \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|+1} v\left(\bigcup_{i \in I} x_i\right) \geq 0 \quad (1)$$

where $v(x) = 0$ whenever $x \notin \mathcal{C}(E)$.

A probabilistic event structure, $P = (E, v)$, comprises an event structure $E = (E, \leq, \#)$ together with a configuration-valuation $v : \mathcal{C}(E) \rightarrow [0, 1]$.

From Equation 1 we can conclude that the valuation on configurations is decreasing, *i.e.* $x \subseteq y \Rightarrow v(x) \geq v(y)$. This captures what happens with the execution of a probabilistic program. To understand this behavior note that we can represent the execution of a program by a tree, where nodes represent commands and edges denote transitions between commands. Furthermore, the root of the tree corresponds to the initial command. As we traverse the tree, the probability either remains the same or it decreases. Essentially, commands near to the root have higher probabilities compared to those farther away. It lacks to establish a connection between configurations and commands in the tree structure. The root is the initial command and the corresponding configuration is the empty one. Hence, it follows straightforwardly that the probability of the empty configuration should be the same as the probability of the initial command, which is 1. As we move away from the root, more actions from the program are performed, leading to the growth of configurations. Consequently, if a command C_1 is closer to the root than a command C_2 , we can deduce that the probability of the latter is either lower or the same as the probability of the former. In terms of configurations, this corresponds exactly with the decreasing feature of the valuation, as the configuration associated with command C_1 is either included or the same as the configuration associated with command C_2 .

Example 2.2 intends to introduce the reader to probabilistic event structures.

Example 2.2. Figure 10 shows a probabilistic event structure very similar to the event structure in Figure 1, the only difference being the addition of a new event τ , for which the events a , c , and d are causally dependent. The event τ is used to indicate that the events that are causally immediate to it, *i.e.* $\tau \rightarrow a$, $\tau \rightarrow c$, and $\tau \rightarrow d$ arose from a probabilistic choice and consequently they have probabilities associated, as can be seen by the configuration-valuation.

The set of configurations is composed of $\{\emptyset, \{\tau\}, \{\tau, a\}, \{\tau, c\}, \{\tau, d\}, \{\tau, a, b\}, \{\tau, c, d\}\}$, where $\{\tau, a, b\}$ and $\{\tau, c, d\}$ are maximal configurations with probability p and $1 - p$, respectively.

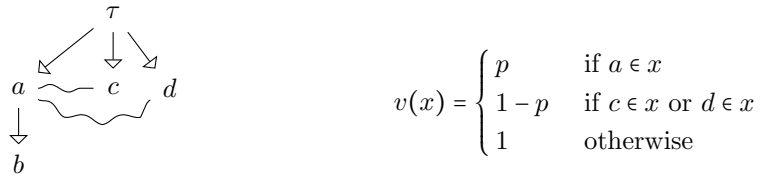


Figure 10: Example of a probabilistic event structure

2.1 Language

To present the language we consider a set of atomic actions Act ranged over by a (examples of atomic actions are assignments, or unitary application, etc...). Furthermore, we extend the set of atomic actions with sk and called the resultant set the set of labels, *i.e.* $L = Act \cup \{sk\}$. Moreover, we add an "invisible" label τ to L , $L' = L \cup \{\tau\}$, to represent the execution of a probabilistic choice.

The set of commands allowed by the language are given by the following grammar (where $p \in]0, 1[$):

$$C ::= \text{skip} \mid a \in Act \mid C; C \mid C +_p C \mid C \parallel C$$

In the design of this language we made two choices: the first was to substitute the non-deterministic operator by the probabilistic operator and the second concerns the intervals for which p ranges. The justification for the former is related with the chosen probabilistic event structure. In sum, Winskel probabilistic event structures are not suitable to model a language that posses both non-deterministic and probabilistic operators, as explained in [dV19]. Regarding the latter, the intervals chosen are influenced by Definition 2.13, since it is no reasonable to remove an initial event when its probability is zero.

We extend the set of labels with a new label τ , *i.e.* $L' = L \cup \{\tau\}$ and let it be ranged by l' . Similarly to process algebra, τ will be used to denote an invisible transition.

We fix $D(X) = \{\psi : X \rightarrow [0, 1] \mid \text{sup}(X) \text{ finite}, \sum_{x \in X} \psi(x) = 1\}$ as being the probabilistic finite support functor and we define the small-step transition step (labeled Segala automaton), $\rightarrow \subseteq \mathbb{C} \times D(L' \times (\mathbb{C} \cup \{\checkmark\}))$, as the smallest relation obeying the following rules:

$$\begin{array}{c}
\text{skip} \rightarrow 1 \cdot (sk, \checkmark) \qquad a \rightarrow 1 \cdot (a, \checkmark) \qquad \mathbb{C}_1 +_p \mathbb{C}_2 \rightarrow p \cdot (\tau, \mathbb{C}_1) + (1-p) \cdot (\tau, \mathbb{C}_2) \\
\\
\frac{\mathbb{C}_1 \rightarrow 1 \cdot (l, \checkmark)}{\mathbb{C}_1 ; \mathbb{C}_2 \rightarrow 1 \cdot (l, \mathbb{C}_2)} \qquad \frac{\mathbb{C}_1 \rightarrow 1 \cdot (l, \mathbb{C}'_1)}{\mathbb{C}_1 ; \mathbb{C}_2 \rightarrow 1 \cdot (l, \mathbb{C}'_1 ; \mathbb{C}_2)} \qquad \frac{\mathbb{C}_1 \rightarrow \sum_i p_i \cdot (\tau, \mathbb{C}_i)}{\mathbb{C}_1 ; \mathbb{C}_2 \rightarrow \sum_i p_i \cdot (\tau, \mathbb{C}_i ; \mathbb{C}_2)} \\
\\
\frac{\mathbb{C}_1 \rightarrow 1 \cdot (l, \checkmark)}{\mathbb{C}_1 \parallel \mathbb{C}_2 \rightarrow 1 \cdot (l, \mathbb{C}_2)} \qquad \frac{\mathbb{C}_1 \rightarrow 1 \cdot (l, \mathbb{C}'_1)}{\mathbb{C}_1 \parallel \mathbb{C}_2 \rightarrow 1 \cdot (l, \mathbb{C}'_1 \parallel \mathbb{C}_2)} \qquad \frac{\mathbb{C}_1 \rightarrow \sum_i p_i \cdot (\tau, \mathbb{C}_i)}{\mathbb{C}_1 \parallel \mathbb{C}_2 \rightarrow \sum_i p_i \cdot (\tau, \mathbb{C}_i \parallel \mathbb{C}_2)} \\
\\
\frac{\mathbb{C}_2 \rightarrow 1 \cdot (l, \checkmark)}{\mathbb{C}_1 \parallel \mathbb{C}_2 \rightarrow 1 \cdot (l, \mathbb{C}_1)} \qquad \frac{\mathbb{C}_2 \rightarrow 1 \cdot (l, \mathbb{C}'_2)}{\mathbb{C}_1 \parallel \mathbb{C}_2 \rightarrow 1 \cdot (l, \mathbb{C}_1 \parallel \mathbb{C}'_2)} \qquad \frac{\mathbb{C}_2 \rightarrow \sum_j p_j \cdot (\tau, \mathbb{C}_j)}{\mathbb{C}_1 \parallel \mathbb{C}_2 \rightarrow \sum_j p_j \cdot (\tau, \mathbb{C}_1 \parallel \mathbb{C}_j)}
\end{array}$$

Figure 11: Rules of the probabilistic small-step operational semantics

Define a word to be a sequence of labels:

$$\omega ::= l' \mid l' : \omega$$

where $l' : \omega$ appends l' to the beginning of ω . A word can also be seen as an element of $(L')^+$, *i.e.* a possibly infinite sequence of labels without the empty sequence. Despite $(L')^+$ allows the possibility of having infinite words, by now we focus only on the finite words.

Define the n -step transition, $\xrightarrow{\omega_p} \subseteq \mathbb{C} \times D((L')^+ \times (\mathbb{C} \cup \{\checkmark\}))$, where n is the length of the words, as follows:

$$\frac{\mathbb{C} \rightarrow \sum_i p_i (l', \mathbb{C}_i)}{\mathbb{C} \twoheadrightarrow \sum_i p_i (l', \mathbb{C}_i)} \qquad \frac{\mathbb{C} \rightarrow \sum_i p_i (l', \mathbb{C}_i) \quad \forall i \mathbb{C}_i \twoheadrightarrow \sum_j p_j \cdot (\omega_{ij}, \mathbb{C}_{ij})}{\mathbb{C} \twoheadrightarrow \sum_i p_i (\sum_j p_j \cdot (l' : \omega_{ij}, \mathbb{C}_{ij}))}$$

Figure 12: Rules of the n -step operational semantics

The left rule represents the execution of a single step in a computation, while the right rule represents multiple steps of the computation. The latter rule can be understood as follows: if \mathbb{C} transits to $\sum_i p_i \cdot (l', \mathbb{C}_i)$ and for each \mathbb{C}_i we transit to $\sum_j p_j \cdot (\omega_{ij}, \mathbb{C}_{ij})$, then by appending l' to each ω_{ij} , we can transit from \mathbb{C} to $\sum_i p_i (\sum_j p_j \cdot (l' : \omega_{ij}, \mathbb{C}_{ij}))$. In this transition, for each i , we multiply the probabilities obtained from the small-step transition with the probabilities obtained from the n -step transition.

Example 2.3. In Figure 14 we use straight arrows to denote a transition from a command to a distribution, which we denote by \bullet , labeled by the triggering action and wiggly arrows to represent a transition from a distribution to a command labeled by the associated probability.

From $(a ; b) +_p (c \parallel d)$ we transit with τ to the distribution $p \cdot a ; b + (1-p) \cdot c \parallel d$, which transits with probability p to $a ; b$ and with probability $1-p$ to $c \parallel d$. For the former, by executing first a and then b we reach the end of the computation. For the latter, since it is a concurrent program, to finish the computation we can either execute first c and then d or we can execute first d and then c .

Based on Figure 14 and following the rules in Figure 12, we can deduce that with probability p the word τab leads to a final computation and the same behavior is captured with probability $1-p$ with the words τcd and τdc .

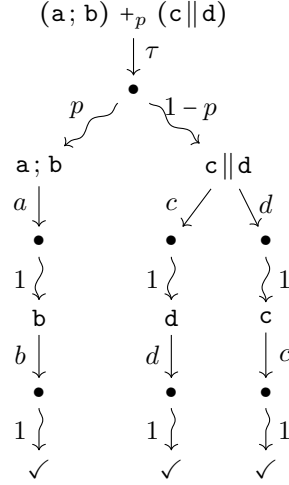


Figure 13: Segala Automaton of $(a||b) +_p c$

Example 2.4. In Figure 14 we use straight arrows to denote a transition from a configuration to a distribution, which we denote by \bullet , labeled by the triggering action and wiggly arrows to represent a transition from a distribution to a configuration labeled by the associated probability.

From $(a||b) +_p c$ we transit with τ to the distribution $p \cdot a||b + (1-p) \cdot c$, which transits with probability p to $a||b$ and with probability $1-p$ to c . By executing c the computation finishes. On the other side we have two possible transitions: either we transit with a , leading to the distribution $1 \cdot b$, which terminates after executing b , or we transit with b which goes to the distribution $1 \cdot a$ that after executing a terminates the computation. The words that lead $(a||b) +_p c$ to \checkmark are $\tau ab, \tau ba$ with probability p and τc with probability $1-p$.

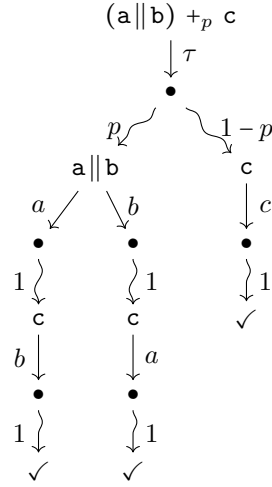


Figure 14: Segala Automaton of $(a||b) +_p c$

2.2 Constructions on Probabilistic Event Structures

The constructions on probabilistic event structures are an extension of the ones defined previously. Hence, the explanation of the sequential and parallel composition will be focused on the valuation and we detail more the probabilistic choice.

Let P_1 and P_2 be two probabilistic event structures. For the valuation of the sequential composition we note the following: either the configuration belongs to $\mathcal{C}(P_1)$ and in that case the valuation of the sequential composition equals the valuation of P_1 , or the configuration has elements of both probabilistic event structures. In that case, we multiply valuation of a maximal configuration in P_1 with the valuation of a configuration in P_2 whose events are reached by the maximal configuration of P_1 .

Definition 2.5 (Prob PES sequential). Let $P_1 = (E_1, \leq_1, \#_1, v_1)$ and $P_2 = (E_2, \leq_2, \#_2, v_2)$ be probabilistic

event structures. Define $P_1; P_2 = (E, \leq, \#, v)$ as:

$$\begin{aligned} E &= E_1 \uplus (E_2 \times \mathcal{C}_{\max}(P_1)) \\ \leq &= \{e_1 \leq e'_1 \mid e \leq_1 e'\} \cup \{(e_2, x) \leq (e'_2, x) \mid e_2 \leq_2 e'_2\} \cup \{e_1 \leq (e_2, x) \mid e_1 \in x\} \\ \# &= \{e \# e' \mid \exists (a \leq e, a' \leq e') \cdot a \#_1 a' \vee a \#_{(2,x)} a'\} \\ \forall x \in \mathcal{C}(P_1; P_2) \cdot v(x) &= \begin{cases} v_1(x_1) & \text{if } x \in \mathcal{C}(P_1) \\ v_1(x_1) \cdot v_{(2,x_1)}(x_2) & \text{if } \exists x_1 \in \mathcal{C}_{\max}(P_1), x_2 \in \mathcal{C}(P_{(2,x_1)}) \text{ s.t. } x = x_1 \cup x_2 \end{cases} \end{aligned}$$

where we use the notation $a \#_{(2,x)} a'$ iff $(a, x) \#_{(2,x)} (a', x)$ iff $a \#_2 a'$, and $v_{(2,x_1)}(x_2) = v_2\{e_2 \mid (e_2, x_1) \in x_2\}$.

Lemma 2.6. Let P_1 and P_2 be probabilistic event structures. $P_1; P_2$ is a probabilistic event structure.

Proof. Let $P_1 = (E_1, v_1)$, $P_2 = (E_2, v_2)$, and $P_1; P_2 = (E, v)$.

By Lemma 1.14 we know that E is an event structure. Hence we focus solely on the valuation part.

1. $v(\emptyset) = 1$

Since $\emptyset \in \mathcal{C}(P_1)$ then $v(\emptyset) = v_1(\emptyset) = 1$.

2. $d_v^{(n)}[y; x_1, \dots, x_n] \geq 0$ for all $n \geq 1$ and $y, x_1, \dots, x_n \in \mathcal{C}(P_1; P_2)$ with $y \subseteq x_1, \dots, x_n$

We have two cases based on n :

(a) $n = 0$

We have $d_v^{(0)}[y;] = v(y)$. By Definition 2.5 we have two cases:

i. $y \in \mathcal{C}(P_1; P_2)$ such that $y \in \mathcal{C}(P_1)$

It follows directly that $v(y) = v_1(y) \geq 0$.

ii. $y \in \mathcal{C}(P_1; P_2)$ such that $\exists y_1 \in \mathcal{C}_{\max}(P_1), y_2 \in \mathcal{C}(P_{(2,y_1)})$ with $y_1 \cup y_2 = y$.

It follows directly that $v(y) = v_1(y) \cdot v_{(2,y_1)}(y_2) = v_1(y) \cdot v_2(y_2) \geq 0$, since v_1, v_2 are valuations.

(b) $n > 0$

By [Win14, Proposition 5] we only need to check the condition for $y \multimap x_1, \dots, x_n$. We have three cases:

i. $y \in \mathcal{C}(P_1)$ but $y \notin \mathcal{C}_{\max}(P_1)$

By [Win14, Proposition 5] we know that $x_1, \dots, x_n \in \mathcal{C}(P_1)$, since \multimap is a 'single-step' relation.

We have $y, x_1, \dots, x_n \in \mathcal{C}(P_1)$. It then follows directly that $d_v^{(n)}[y; x_1, \dots, x_n] = d_{v_1}^{(n)}[y; x_1, \dots, x_n] \geq 0$.

ii. $y \in \mathcal{C}_{\max}(P_1)$

By [Win14, Proposition 5] we know that $x_1, \dots, x_n \in \mathcal{C}(P_1; P_2)$ such that $x_1, \dots, x_n \notin \mathcal{C}(P_1)$.

Hence $\exists x'_1, \dots, x'_n \in \mathcal{C}(P_{(2,y)})$ such that $x_1 = y \cup x'_1, \dots, x_n = y \cup x'_n$. Furthermore $\bigcup_{i \in I} x_i = y \cup \bigcup_{i \in I} x'_i$ and let $I \subseteq \{1, \dots, n\}$.

$$\begin{aligned} d_v^{(n)}[y; x_1, \dots, x_n] &= \sum_I (-1)^{|I|} v\left(y \cup \bigcup_{i \in I} x_i\right) \\ &= \sum_I (-1)^{|I|} v\left(y \cup \bigcup_{i \in I} (y \cup x'_i)\right) \\ &= \sum_I (-1)^{|I|} v\left(y \cup \bigcup_{i \in I} x'_i\right) \\ &= \sum_I (-1)^{|I|} v_1(y) \cdot v_2\left(\bigcup_{i \in I} x'_i\right) \\ &= v_1(y) \cdot \sum_I (-1)^{|I|} v_2\left(\bigcup_{i \in I} x'_i\right) \\ &= v_1(y) \cdot d_{v_2}^{(n)}[\emptyset; x'_1, \dots, x'_n] \end{aligned}$$

Since $v_1(y) \geq 0$ and $d_{v_2}^{(n)}[\emptyset; x'_1, \dots, x'_n] \geq 0$, then $v_1(y) \cdot d_{v_2}^{(n)}[\emptyset; x'_1, \dots, x'_n] \geq 0$.

iii. $y \in \mathcal{C}(P_1; P_2)$ but $y \notin \mathcal{C}(P_1)$.

By [Win14, Proposition 5] we know that $\exists y_1 \in \mathcal{C}_{\max}(P_1)$ and $y', x'_1, \dots, x'_n \in \mathcal{C}(P_{(e,y_1)})$ such that $y = y_1 \cup y', x_1 = y_1 \cup x'_1, \dots, x_n = y_1 \cup x'_n$. Furthermore $\bigcup_{i \in I} x_i = \bigcup_{i \in I} (y_1 \cup x'_i) = y_1 \cup \bigcup_{i \in I} x'_i$ and let

$$I \subseteq \{1, \dots, n\}.$$

$$\begin{aligned}
d_v^{(n)}[y; x_1, \dots, x_n] &= \sum_I (-1)^{|I|} v \left(y \cup \bigcup_{i \in I} x_i \right) \\
&= \sum_I (-1)^{|I|} v \left((y_1 \cup y') \cup \bigcup_{i \in I} (y_1 \cup x'_i) \right) \\
&= \sum_I (-1)^{|I|} v \left((y_1 \cup y') \cup \bigcup_{i \in I} x'_i \right) \\
&= \sum_I (-1)^{|I|} v_1(y_1) \cdot v_2(y') \cdot v_2 \left(\bigcup_{i \in I} x'_i \right) \\
&= v_1(y) \cdot \sum_I (-1)^{|I|} v_2(y') \cdot v_2 \left(\bigcup_{i \in I} x'_i \right) \\
&= v_1(y) \cdot d_{v_2}^{(n)}[y'; x'_1, \dots, x'_n]
\end{aligned}$$

Since $v_1(y) \geq 0$ and $d_{v_2}^{(n)}[y'; x'_1, \dots, x'_n] \geq 0$, then $v_1(y) \cdot d_{v_2}^{(n)}[y'; x'_1, \dots, x'_n] \geq 0$.

□

For the probabilistic choice, $E_1 +_p E_2$ we note that the invisible action τ should be the initial event, to be in accordance with the operational semantics. Furthermore, the behavior of the probabilistic choice is very similar to that of the non-deterministic choice, in which if we choose a side we cannot execute the other. In terms of event structures, this means that the events of both sides should be in conflict. Regarding the valuations, if the configuration obtained by removing τ belongs to $\mathcal{C}(E_1)$, then we multiply by p the valuation in P_1 , otherwise we multiply by $(1 - p)$.

Definition 2.7 (PES probabilistic choice). Let $P_1 = (E_1, \leq_1, \#_1, v_1)$ and $P_2 = (E_2, \leq_2, \#_2, v_2)$ be probabilistic event structures. Define $P_1 +_p P_2 = (E, \leq, \#)$ as:

$$\begin{aligned}
E &= \{\tau\} \uplus (E_1 \uplus E_2) \\
\leq &= \{\tau \leq e \mid e \in E\} \cup \leq_1 \cup \leq_2 \\
\# &= \#_1 \cup \#_2 \cup \{e_1 \# e_2 \mid e_1 \in E_1, e_2 \in E_2\} \\
\forall x \in \mathcal{C}(P_1 +_p P_2). v(x) &= \begin{cases} p \cdot v_1(x \setminus \tau) & \text{if } x \setminus \tau \in \mathcal{C}(P_1) \\ (1 - p) \cdot v_2(x \setminus \tau) & \text{if } x \setminus \tau \in \mathcal{C}(P_2) \\ 1 & \text{if } x = \{\tau\} \vee x = \emptyset \end{cases}
\end{aligned}$$

Lemma 2.8. Let P_1 and P_2 be probabilistic event structures. $P_1 +_p P_2$ is a probabilistic event structure.

Proof. Let $P_1 = (E_1, \leq_1, \#_1, v_1)$, $P_2 = (E_2, \leq_2, \#_2, v_2)$, and $P_1 +_p P_2 = (E, \leq, \#, v)$. Let $e, e', e'' \in E$. We have four conditions to check:

1. $\{e' \mid e' \leq e\}$ is finite

We have three cases:

- (a) $e = \tau$

It follows directly that $\{e' \mid e' \leq \tau\} = \{\tau\}$ since $\tau \in \mathcal{I}(P_1 +_p P_2)$.

- (b) $e \in E_1$

We have that $\{e' \mid e' \leq e\} = \{\tau\} \cup \{e' \mid e' \leq_1 e\}$. Since P_1 is a probabilistic event structure, then we know that $\{e' \mid e' \leq_1 e\}$ is finite. Hence $\{\tau\} \cup \{e' \mid e' \leq_1 e\}$ is finite.

- (c) $e \in E_2$

We have that $\{e' \mid e' \leq e\} = \{\tau\} \cup \{e' \mid e' \leq_2 e\}$. Since P_2 is a probabilistic event structure, then we know that $\{e' \mid e' \leq_2 e\}$ is finite. Hence $\{\tau\} \cup \{e' \mid e' \leq_2 e\}$ is finite.

2. $e \# e' \leq e'' \Rightarrow e \# e''$

Since τ is not in conflict with any event, this condition trivially holds because we either have $e, e', e'' \in E_1$ or $e, e', e'' \in E_2$ and P_1, P_2 are probabilistic event structures.

3. $v(\emptyset) = 1$

It follows directly from the definition.

4. $d_v^{(n)}[y; x_1, \dots, x_n] \geq 0$ for all $n \geq 1$ and $y, x_1, \dots, x_n \in \mathcal{C}(P_1 +_p P_2)$ with $y \subseteq x_1, \dots, x_n$

By [Win14, Proposition 5] we only need to check the condition for $y \sqsubset x_1, \dots, x_n$, i.e. $y \xrightarrow{e_1, \dots, e_n} x_1, \dots, x_n$. We have three cases:

- (a) $y = \emptyset$

We then have $\emptyset \xrightarrow{\tau} \{\tau\}$.

It follows that $d_v^{(1)}[\emptyset; \{\tau\}] = v(\emptyset) - v(\{\tau\}) = 1 - 1 = 0$

- (b) $y \setminus \tau \in \mathcal{C}(P_1)$ We have three cases (let $1 \leq i \leq n$):

- i. $\forall e_i \in E_1$

We have $x_1 \setminus \tau, \dots, x_n \setminus \tau \in \mathcal{C}(P_1)$. Let $I \subseteq \{1, \dots, n\}$.

$$\begin{aligned} d_v^{(n)}[y; x_1, \dots, x_n] &= \sum_I (-1)^{|I|} v(y \cup \bigcup_{i \in I} x_i) \\ &= p \cdot \sum_I (-1)^{|I|} v_1((y \setminus \tau) \cup \bigcup_{i \in I} (x_i \setminus \tau)) \\ &= p \cdot d_{v_1}^{(n)}[y; x_1, \dots, x_n] \end{aligned}$$

Since $p \in]0, 1[$ and $d_{v_1}^{(n)}[y; x_1, \dots, x_n]$, because P_1 is a probabilistic event structure, then $p \cdot d_{v_1}^{(n)}[y; x_1, \dots, x_n] \geq 0$.

- ii. $\forall e_i \in E_2$

Since $y \setminus \tau \in \mathcal{C}(P_1)$, $x_i = \{e_i\} \cup y$, and for all $1 \leq i \leq n$ we have $e_i \# e \in y \setminus \tau \in \mathcal{C}(P_1)$, then $v(x_i) = 0$. Hence

$$\begin{aligned} d_v^{(n)}[y; x_1, \dots, x_n] &= \sum_I (-1)^{|I|} v(y \cup \bigcup_{i \in I} x_i) \\ &= v(y) \\ &= p \cdot v_1(y \setminus \tau) \end{aligned}$$

Since $p \in]0, 1[$ and $v_1(y \setminus \tau) \geq 0$, because P_1 is a probabilistic event structure, we have $p \cdot v_1(y \setminus \tau) \geq 0$.

- iii. $\exists e_i \in E_2$

Since $y \setminus \tau \in \mathcal{C}(P_1)$, $x_i = \{e_i\} \cup y$, and for all $1 \leq i \leq n$ such that $e_i \# e \in y \setminus \tau \in \mathcal{C}(P_1)$, we have $v(x_i) = 0$. Hence

$$\begin{aligned} d_v^{(n)}[y; x_1, \dots, x_n] &= \sum_I (-1)^{|I|} v(y \cup \bigcup_{i \in I} x_i) \\ &= \sum_{I'} (-1)^{|I'|} v(y \cup \bigcup_{i \in I'} x_i) \\ &= p \cdot \sum_{I'} (-1)^{|I'|} v_1((y \setminus \tau) \cup \bigcup_{i \in I'} (x_i \setminus \tau)) \\ &= p \cdot d_{v_1}^{(m)}[y; x_1, \dots, x_m] \end{aligned}$$

where $I' = \{1, \dots, m\}$, i.e. I' is I without those $e_i \in E_2$.

Since $p \in]0, 1[$ and $d_{v_1}^{(m)}[y; x_1, \dots, x_m]$, because P_1 is a probabilistic event structure, then $p \cdot d_{v_1}^{(m)}[y; x_1, \dots, x_m] \geq 0$.

- (c) $y \setminus \tau \in \mathcal{C}(P_2)$

Similar to previous case.

□

Remark 3. Another way of representing $P_1 +_p P_2$ is by putting the probabilities explicit on both sides, i.e. $p \cdot P_1 + (1 - p) \cdot P_2$. That leaves us with $P_1 +_p P_2 = p \cdot P_1 + (1 - p) \cdot P_2$

Remark 4. When showing equivalence between the operational and denotational semantics, it will be useful to have a general definition of Definition 2.7. Consider that we have a finite number n of probabilistic event

structures P_n . Let $1 \leq i \leq n$ and define $\sum_i p_i \cdot P_i$, with $\sum_i p_i = 1$ as follows:

$$\begin{aligned} E &= \{\tau\} \uplus \biguplus_i E_i \\ \leq &= \{\tau \leq e \mid e \in E\} \cup \bigcup_i \leq_i \\ \# &= \bigcup_i \#_i \cup \{e_i \# e_j \mid e_i \in E_i, e_j \in E_j, 1 \leq i \neq j \leq n\} \\ \forall x \in \mathcal{C}\left(\sum_i p_i \cdot P_i\right) \cdot v(x) &= \begin{cases} p_i \cdot v_i(x \setminus \tau) & \text{if } x \setminus \tau \in \mathcal{C}(P_i) \\ 1 & \text{if } x = \{\tau\} \vee x = \emptyset \end{cases} \end{aligned}$$

Showing that this definition is in fact a probabilistic event structure is very similar to what was done to $P_1 +_p P_2$.

For the parallel composition, and by taken advantage of the intuition that the parallel composition is just putting "side-by-side" the two event structures, the valuation is the multiplication of the valuations resulting from projecting the configuration in $P_1 \parallel P_2$ into the respective configuration of P_1 and P_2 .

Definition 2.9 (PES parallel). Let $P_1 = (E_1, \leq_1, \#_1, v_1)$ and $P_2 = (E_2, \leq_2, \#_2, v_2)$ be probabilistic event structures. Define $P_1 \parallel P_2 = (E, \leq, \#, v)$ as:

$$\begin{aligned} E &= E_1 \uplus E_2 \\ \leq &= \leq_1 \cup \leq_2 \\ \# &= \#_1 \cup \#_2 \\ \forall x \in \mathcal{C}(P_1 \parallel P_2) \cdot v(x) &= v_1(x \cap E_1) \cdot v_2(x \cap E_2) \end{aligned}$$

Lemma 2.10. Let P_1 and P_2 be probabilistic event structures. $P_1 \parallel P_2$ is a probabilistic event structure.

Proof. Let $P_1 = (E_1, v_1)$, $P_2 = (E_2, v_2)$, and $P_1 \parallel P_2 = (E, v)$.

By Lemma 1.18 we know that E is an event structure. Hence we focus solely on the valuation part.

1. $v(\emptyset) = 1$

$$v(\emptyset) = v_1(\emptyset \cap E_1) \cdot v_2(\emptyset \cap E_2) = v_1(\emptyset) \cdot v_2(\emptyset) = 1 \cdot 1 = 1$$

2. $d_v^{(n)}[y; x_1, \dots, x_n] \geq 0$ for all $n \geq 1$ and $y, x_1, \dots, x_n \in \mathcal{C}(P_1 +_p P_2)$ with $y \subseteq x_1, \dots, x_n$

By [Win14, Proposition 5] we only need to check the condition for $y \sqsubset x_1, \dots, x_n$, i.e. $y \stackrel{e_1, \dots, e_n}{\sqsubset} x_1, \dots, x_n$.

We want to show that $d_v^{(n)}[y; x_1, \dots, x_n] = d_{v_1}^{(n_1)}[y \cap E_1; x_1 \cap E_1, \dots, x_n \cap E_1] \cdot d_{v_2}^{(n_2)}[y \cap E_2; x_1 \cap E_2, \dots, x_n \cap E_2]$

Let $I_1 \subseteq \{1, \dots, n_1\}$, $I_2 \subseteq \{1, \dots, n_2\}$, and $I = I_1 \uplus I_2$.

$$\begin{aligned} & d_{v_1}^{(n_1)}[y \cap E_1; x_1 \cap E_1, \dots, x_n \cap E_1] \cdot d_{v_2}^{(n_2)}[y \cap E_2; x_1 \cap E_2, \dots, x_n \cap E_2] \\ &= \sum_{I_1} (-1)^{|I_1|} v_1 \left((y \cap E_1) \cup \left(\bigcup_{i \in I_1} x_i \cap E_1 \right) \right) \cdot \sum_{I_2} (-1)^{|I_2|} v_2 \left((y \cap E_2) \cup \left(\bigcup_{j \in I_2} x_j \cap E_2 \right) \right) \\ &= \sum_{I_1} (-1)^{|I_1|} v_1 \left(\left(y \cup \bigcup_{i \in I_1} x_i \right) \cap E_1 \right) \cdot \sum_{I_2} (-1)^{|I_2|} v_2 \left(\left(y \cup \bigcup_{j \in I_2} x_j \right) \cap E_2 \right) \\ &= \sum_{I_1} \sum_{I_2} (-1)^{|I_1|+|I_2|} v_1 \left(\left(y \cup \bigcup_{i \in I_1} x_i \right) \cap E_1 \right) v_2 \left(\left(y \cup \bigcup_{j \in I_2} x_j \right) \cap E_2 \right) \\ &= \sum_{I_1, I_2} (-1)^{|I_1|+|I_2|} v_1 \left(\left(y \cup \bigcup_{i \in I_1} x_i \right) \cap E_1 \right) v_2 \left(\left(y \cup \bigcup_{j \in I_2} x_j \right) \cap E_2 \right) \\ &= \sum_{I_1, I_2} (-1)^{|I_1|+|I_2|} v \left(y \cup \bigcup_{i \in (I_1 \uplus I_2)} x_i \right) \\ &= \sum_I (-1)^{|I|} v \left(y \cup \bigcup_{i \in I} x_i \right) \\ &= d_v^{(n)}[y; x_1, \dots, x_n] \geq 0 \end{aligned}$$

□

Definition 2.11. We interpret commands as probabilistic event structures as follows ($\llbracket - \rrbracket : \mathcal{C} \rightarrow \mathcal{P}$):

$$\begin{aligned}\llbracket \text{skip} \rrbracket &= (\{sk\}, \{sk \leq sk\}, \emptyset, v(\{sk\}) = 1) \\ \llbracket a \rrbracket &= (\{a\}, \{a \leq a\}, \emptyset, v(\{a\}) = 1) \\ \llbracket C_1 +_p C_2 \rrbracket &= \llbracket C_1 \rrbracket +_p \llbracket C_2 \rrbracket \\ \llbracket C_1 ; C_2 \rrbracket &= \llbracket C_1 \rrbracket ; \llbracket C_2 \rrbracket \\ \llbracket C_1 \parallel C_2 \rrbracket &= \llbracket C_1 \rrbracket \parallel \llbracket C_2 \rrbracket\end{aligned}$$

Recall that Definition 2.31 is not suitable when we want to relate the operational and the denotational semantics. Hence we extend Definition 1.20 to the probabilistic setting. For that we introduce some notation. Let E a set of events and $y \in \mathcal{C}(E)$ a configuration. We denote $\underline{E} = \{e \mid (e \vee(e, x)) \in E\}$ and $\underline{y} = \{e \mid (e \vee(e, x)) \in y\}$, in order to ignore the copies.

Definition 2.12 (sub-PES). Let $P_1 = (E_1, \leq_1, \#_1, v_1)$ and $P_2 = (E_2, \leq_2, \#_2, v_2)$ be probabilistic event structures. Say $P_1 \sqsubseteq P_2$ iff:

$$\begin{aligned}\underline{E_1} &\subseteq \underline{E_2} \\ \forall e, e' . e \leq_1 e' &\Leftrightarrow e, e' \in E_1 \wedge e \leq_2 e' \\ \forall e, e' . e \#_1 e' &\Leftrightarrow e, e' \in E_1 \wedge e \#_2 e' \\ \forall x \in \mathcal{C}(P_1), y \in \mathcal{C}(P_2) . \underline{x} \subseteq \underline{y} &\Rightarrow v_1(x) \geq v_2(y)\end{aligned}$$

We say that two event structures P_1, P_2 are equivalent, denoted $P_1 \equiv P_2$, iff $P_1 \sqsubseteq P_2$ and $P_2 \sqsubseteq P_1$.

Remark 5. When $P_1 \sqsubseteq P_2$ and $P_2 \sqsubseteq P_1$ we say that $P_1 \equiv P_2$, i.e. that P_1 and P_2 are equivalent. Furthermore, it is clear that if $P_1 = P_2$ then $P_1 \equiv P_2$.

To define \sqsubseteq in the probabilistic setting, we based ourselves on the fact that given two configurations x, y such that $x \subseteq y$ the probability of x must be greater or equal to the probability of y , i.e. $v(x) \geq v(y)$.

To remove the initial event of a probabilistic event structure, we need to guarantee that the probability of said event is not zero. Because if the event had probability zero, removing it would lead to a division by zero. Furthermore, this is the reason why $p \in]0, 1[$ in the probabilistic operator.

Definition 2.13 (Remove initial event). Let $P = (E, \leq, \#, v)$ be a prime event structure and $a \in \mathcal{I}(E)$, s.t. $v(\{a\}) \neq 0$. Define $P \setminus a = (E', \leq', \#', v')$ as

$$\begin{aligned}E' &= \{e \in E \mid \neg(e \# a), e \neq a\} \\ \leq' &= \{e \leq e' \mid e, e' \in E'\} \\ \# &= \{e \# e' \mid e, e' \in E'\} \\ \forall x \in \mathcal{C}(P \setminus a) . v'(x) &= \frac{v(x \cup \{a\})}{v(\{a\})}\end{aligned}$$

Lemma 2.14. Let P be a probabilistic event structure. $P \setminus a$ is a probabilistic event structure.

Proof. Let $P = (E, v)$ and $P \setminus a = (E', v')$. By Lemma 3.14 we know that E' is an event structure. Hence we focus solely on the valuation part.

1. $v'(\emptyset) = 1$

$$v'(\emptyset) = \frac{v_1(\emptyset \cup \{a\})}{v_1(\{a\})} = \frac{v_1(\{a\})}{v_1(\{a\})} = 1$$

2. $d_v^{(n)}[y; x_1, \dots, x_n] \geq 0$ for all $n \geq 1$ and $y, x_1, \dots, x_n \in \mathcal{C}(P_1 +_p P_2)$ with $y \subseteq x_1, \dots, x_n$

By [Win14, Proposition 5] we only need to check the condition for $y \multimap x_1, \dots, x_n$, i.e. $y \xrightarrow{e_1, \dots, e_n} x_1, \dots, x_n$.

Let $I \subseteq \{1, \dots, n\}$.

$$\begin{aligned}d_v^{(n)}[y; x_1, \dots, x_n] &\geq 0 \Leftrightarrow \sum_I (-1)^{|I|} v \left(y \cup \bigcup_{i \in I} x_i \right) \geq 0 \\ &\Leftrightarrow \sum_I (-1)^{|I|} \frac{v_1((y \cup \bigcup_{i \in I} x_i) \cup \{a\})}{v_1(\{a\})} \geq 0 \\ &\Leftrightarrow \sum_I (-1)^{|I|} v_1 \left(\left(y \cup \bigcup_{i \in I} x_i \right) \cup \{a\} \right) \geq 0 \\ &\Leftrightarrow d_{v_1}^{(n)}[y \cup \{a\}; x_1 \cup \{a\}, \dots, x_n \cup \{a\}] \geq 0\end{aligned}$$

□

2.3 Results

Lemma 2.15. Let P_1, P'_1, P_2, P'_2 be event structures. If $P_1 \sqsubseteq P'_1$ and $P_2 \sqsubseteq P'_2$ then $P_1 ; P_2 \sqsubseteq P'_1 ; P'_2$.

Proof. Let $P_1 = (E_1, v_1), P'_1 = (E'_1, v'_1), P_2 = (E_2, v_2), P'_2 = (E'_2, v'_2), P_1 ; P_2 = (E, v), P'_1 ; P'_2 = (E', v')$. Due to Lemma 1.24, we only need to show $\forall x \in \mathcal{C}(P_1 ; P_2), y \in \mathcal{C}(P'_1 ; P'_2). \underline{x} \subseteq \underline{y} \Rightarrow v(x) \geq v'(y)$.

Let $x \in \mathcal{C}(P_1 ; P_2)$ and $y \in \mathcal{C}(P'_1 ; P'_2)$ such that $\underline{x} \subseteq \underline{y}$. We have three cases:

1. $x \in \mathcal{C}(P_1 ; P_2)$ such that $x \in \mathcal{C}(P_1)$ and $y \in \mathcal{C}(P'_1 ; P'_2)$ such that $y \in \mathcal{C}(P'_1)$

It follows directly that $v(x) \geq v'(y) \Leftrightarrow v_1(x) \geq v'_1(y)$, since $v(x) = v'_1(x)$, $v'(y) = v'_1(y)$ and $P_1 \sqsubseteq P'_1$.

2. $x \in \mathcal{C}(P_1 ; P_2)$ such that $x \in \mathcal{C}(P_1)$ and $y \in \mathcal{C}(P'_1 ; P'_2)$ such that $\exists y_1 \in \mathcal{C}_{\max}(P'_1), y_2 \in \mathcal{C}(P'_2)$ such that $y = y_1 \cup y_2$

We know that $v(x) = v_1(x)$ and that $v'(y) = v'_1(y_1) \cdot v'_2(y_2)$. Since $P_1 \sqsubseteq P'_1$ then $\underline{x} \subseteq \underline{y_1}$ and $v_1(x) \geq v'_1(y_1)$. It then follows directly that $v(x) = v_1(x) \geq v'_1(y_1) \geq v'_1(y_1) \cdot v'_2(y_2) = v'(y)$.

3. $x \in \mathcal{C}(P_1 ; P_2)$ such that $\exists x_1 \in \mathcal{C}(P_1), x_2 \in \mathcal{C}(P_2)$ such that $x = x_1 \cup x_2$ and $y \in \mathcal{C}(P'_1 ; P'_2)$ such that $\exists y_1 \in \mathcal{C}_{\max}(P'_1), y_2 \in \mathcal{C}(P'_2)$ such that $y = y_1 \cup y_2$

We know that $v(x) = v_1(x_1) \cdot v_2(x_2)$ and $v'(y) = v'_1(y_1) \cdot v'_2(y_2)$. Since $P_1 \sqsubseteq P'_1$ then $\underline{x_1} \subseteq \underline{y_1}$ and $v_1(x_1) \geq v'_1(y_1)$, and $P_2 \sqsubseteq P'_2$ then $\underline{x_2} \subseteq \underline{y_2}$ and $v_2(x_2) \geq v'_2(y_2)$.

Furthermore,

$$\begin{aligned} v_1(x_1) \geq v'_1(y_1) &\Leftrightarrow v'_1(y_1) \leq v_1(x_1) \\ &\Leftrightarrow \frac{v'_1(y_1)}{v_1(x_1)} \leq 1 \\ &\Leftrightarrow \left(\frac{v'_1(y_1)}{v_1(x_1)} = 1 \right) \text{ or } \left(\frac{v'_1(y_1)}{v_1(x_1)} < 1 \right) \end{aligned}$$

Now we show that $v(x) \geq v'(y)$.

$$\begin{aligned} v(x) \geq v'(y) &\Leftrightarrow v_1(x_1) \cdot v_2(x_2) \geq v'_1(y_1) \cdot v'_2(y_2) \\ &\Leftrightarrow v_2(x_2) \geq \frac{v'_1(y_1)}{v_1(x_1)} \cdot v'_2(y_2) \end{aligned}$$

We have two cases:

1. $\frac{v'_1(y_1)}{v_1(x_1)} = 1$

$$v_2(x_2) \geq \frac{v'_1(y_1)}{v_1(x_1)} \cdot v'_2(y_2) \Leftrightarrow v_2(x_2) \geq v'_2(y_2) \text{ and we are done.}$$

2. $\frac{v'_1(y_1)}{v_1(x_1)} < 1$

$$\text{Since } v_2(x_2) \geq v'_2(y_2) \text{ and } v'_2(y_2) \geq \frac{v'_1(y_1)}{v_1(x_1)} \cdot v'_2(y_2) \text{ it follows that } v_2(x_2) \geq v'_2(y_2) \geq \frac{v'_1(y_1)}{v_1(x_1)} \cdot v'_2(y_2)$$

□

Lemma 2.16. Let P_1, P'_1, P_2, P'_2 be event structures. If $P_1 \sqsubseteq P'_1$ and $P_2 \sqsubseteq P'_2$ then $P_1 +_p P_2 \sqsubseteq P'_1 +_p P'_2$.

Proof. Let $P_1 = (E_1, v_1), P'_1 = (E'_1, v'_1), P_2 = (E_2, v_2), P'_2 = (E'_2, v'_2), P_1 +_p P_2 = (E, v), P'_1 +_p P'_2 = (E', v')$.

The conditions to check are:

1. $\underline{E} \subseteq \underline{E'}$
2. $\forall e, e'. e \leq e' \Leftrightarrow e, e' \in E \wedge e \leq' e'$
3. $\forall e, e'. e \# e' \Leftrightarrow e, e' \in E \wedge e \# e'$
4. $\forall x \in \mathcal{C}(P_1 +_p P_2), y \in \mathcal{C}(P'_1 +_p P'_2). \underline{x} \subseteq \underline{y} \Rightarrow v(x) \geq v'(y)$

The first three conditions follow directly from Definition 2.7. Hence we focus on the last one.

Let $x \in \mathcal{C}(P_1 +_p P_2)$ and $y \in \mathcal{C}(P'_1 +_p P'_2)$ such that $\underline{x} \subseteq \underline{y}$. We have two cases:

1. $x \setminus \tau \in \mathcal{C}(P_1)$ and $y \setminus \tau \in \mathcal{C}(P'_1)$

It follows directly that $v(x) \geq v'(y)$, since $P_1 \sqsubseteq P'_1$ and $v(x) = p \cdot v_1(x \setminus \tau) \geq p \cdot v'_1(y \setminus \tau) = v'(y)$.

2. $x \setminus \tau \in \mathcal{C}(P_2)$ and $y \setminus \tau \in \mathcal{C}(P'_2)$

It follows directly that $v(x) \geq v'(y)$, since $P_2 \sqsubseteq P'_2$ and $v(x) = (1-p) \cdot v_2(x \setminus \tau) \geq (1-p) \cdot v'_2(y \setminus \tau) = v'(y)$. □

Lemma 2.17. Let P_1, P'_1, P_2, P'_2 be event structures. If $P_1 \sqsubseteq P'_1$ and $P_2 \sqsubseteq P'_2$ then $P_1 \parallel P_2 \sqsubseteq P'_1 \parallel P'_2$.

Proof. Let $P_1 = (E_1, v_1), P'_1 = (E'_1, v'_1), P_2 = (E_2, v_2), P'_2 = (E'_2, v'_2), P_1 \parallel P_2 = (E, v), P'_1 \parallel P'_2 = (E', v')$. Due to Lemma 1.26, we only need to show $\forall x \in \mathcal{C}(P_1 \parallel P_2), y \in \mathcal{C}(P'_1 \parallel P'_2). \underline{x} \subseteq \underline{y} \Rightarrow v(x) \geq v'(y)$.

Let $x \in \mathcal{C}(P_1 \parallel P_2)$ and $y \in \mathcal{C}(P'_1 \parallel P'_2)$ such that $\underline{x} \subseteq \underline{y}$. Since $P_1 \sqsubseteq P'_1$ then $\forall x_1 \in \mathcal{C}(P_1), y_1 \in \mathcal{C}(P'_1)$ such that $\underline{x_1} \subseteq \underline{y_1}$ we have $v_1(x_1) \geq v'_1(y_1)$ and that $P_2 \sqsubseteq P'_2$ entails $\forall x_2 \in \mathcal{C}(P_2), y_2 \in \mathcal{C}(P'_2)$ such that $\underline{x_2} \subseteq \underline{y_2}$ we have $v_2(x_2) \geq v'_2(y_2)$. By Definition 2.9, $v(x) = v_1(x_1) \cdot v_2(x_2)$ and $v'(y) = v'_1(y_1) \cdot v'_2(y_2)$, where $x_1 = x \cap E_1, x_2 = x \cap E_2, y_1 = y \cap E'_1$, and $y_2 = y \cap E'_2$. We then have:

$$v(x) = v_1(x_1) \cdot v_2(x_2) \geq v'_1(y_1) \cdot v'_2(y_2) = v'(y)$$
□

Lemma 2.18. Let P_1 and P_2 be probabilistic event structures. Consider $P_1 ; P_2$ such that $l \in \mathcal{I}(P_1 ; P_2)$. Then $(P_1 ; P_2) \setminus l \equiv (P_1 \setminus l) ; P_2$.

Proof. Let $P_1 = (E_1, v_1), P_2 = (E_2, v_2), P_1 ; P_2 = (E_{1;2}, v_{1;2}), (P_1 ; P_2) \setminus l = (E, v), P_1 \setminus l = (E_1^l, v_1^l), (P_1 \setminus l) ; P_2 = (E', v')$, and $l \in \mathcal{I}(P_1 ; P_2)$.

Due to Lemma 1.28 we only need to show

1. $\forall x \in \mathcal{C}((P_1 ; P_2) \setminus l), y \in \mathcal{C}((P_1 \setminus l) ; P_2). \underline{x} \subseteq \underline{y} \Rightarrow v(x) \geq v'(y)$

Let $x \in \mathcal{C}((P_1 ; P_2) \setminus l)$ and $y \in \mathcal{C}((P_1 \setminus l) ; P_2)$ such that $\underline{x} \subseteq \underline{y}$. We have two cases:

- (a) $\{l\} \cup x \in \mathcal{C}(P_1 ; P_2)$ such that $\{l\} \cup x \in \mathcal{C}(P_1)$

$$v(x) = \frac{v_{1;2}(\{l\} \cup x)}{v_{1;2}(\{l\})} = \frac{v_1(\{l\} \cup x)}{v_1(\{l\})} = v_1^l(x) = v'(x)$$

Since $(P_1 \setminus l) ; P_2$ is a probabilistic event structure, then for $x, y \in \mathcal{C}((P_1 \setminus l) ; P_2)$ such that $\underline{x} \subseteq \underline{y}$, we have $v'(x) \geq v'(y)$, since $d_{v'}^{(1)}[x; y] \geq 0 \Leftrightarrow v'(x) - v'(y) \geq 0 \Leftrightarrow v'(x) \geq v'(y)$.

Hence $v(x) \geq v'(y)$.

- (b) $\{l\} \cup x \in \mathcal{C}(P_1 ; P_2)$ such that $\exists (\{l\} \cup x_1) \in \mathcal{C}_{\max}(P_1), x_2 \in \mathcal{C}(P_2)$ where $\{l\} \cup x = \{l\} \cup x_1 \cup x_2$

$$v(x) = \frac{v_{1;2}(\{l\} \cup x)}{v_{1;2}(\{l\})} = \frac{v_1(\{l\} \cup x_1) \cdot v_2(x_2)}{v_1(\{l\})} = v_1^l(x_1) \cdot v_2(x_2) = v'(x)$$

Since $(P_1 \setminus l) ; P_2$ is a probabilistic event structure, we obtain $v(x) \geq v'(y)$.

2. $\forall x \in \mathcal{C}((P_1 \setminus l) ; P_2), y \in \mathcal{C}((P_1 ; P_2) \setminus l). \underline{x} \subseteq \underline{y} \Rightarrow v'(x) \geq v(y)$

Let $x \in \mathcal{C}((P_1 \setminus l) ; P_2)$ and $y \in \mathcal{C}((P_1 ; P_2) \setminus l)$ such that $\underline{x} \subseteq \underline{y}$. We have two cases:

- (a) $x \in \mathcal{C}(P_1 \setminus l ; P_2)$ such that $x \in \mathcal{C}(P_1 \setminus l)$

$$v'(x) = v_1^l(x) = \frac{v_1(\{l\} \cup x)}{v_1(\{l\})} = \frac{v_{1;2}(\{l\} \cup x)}{v_{1;2}(\{l\})} = v(x)$$

Since $(P_1 ; P_2) \setminus l$ is a probabilistic event structure, we obtain $v'(x) \geq v(y)$.

- (b) $x \in \mathcal{C}(P_1 \setminus l ; P_2)$ such that $\exists x_1 \in \mathcal{C}_{\max}(P_1 \setminus l), x_2 \in \mathcal{C}(P_2)$ such that $x = x_1 \cup x_2$

$$v'(x) = v_1^l(x_1) \cdot v_2(x_2) = \frac{v_1(\{l\} \cup x_1) \cdot v_2(x_2)}{v_1(\{l\})} = \frac{v_{1;2}(\{l\} \cup x)}{v_{1;2}(\{l\})} = v(x)$$

Since $(P_1 ; P_2) \setminus l$ is a probabilistic event structure, we obtain $v'(x) \geq v(y)$. □

Lemma 2.19. Let P_1 and P_2 be probabilistic event structures. Consider $P_1 \parallel P_2$ such that $l \in \mathcal{I}(P_1 \parallel P_2)$. Then

$$(P_1 \parallel P_2) \setminus l \equiv \begin{cases} (P_1 \setminus l) \parallel P_2 & \text{if } l \in \mathcal{I}(P_1) \\ P_1 \parallel (P_2 \setminus l) & \text{if } l \in \mathcal{I}(P_2) \end{cases}$$

Proof. Let $P_1 = (E_1, v_1)$, $P_2 = (E_2, v_2)$, $P_1 \parallel P_2 = (E, v)$, $(P_1 \parallel P_2) \setminus l = (E', v')$, $P_1 \setminus l = (E_1^l, v_1^l)$, $P_2 \setminus l = (E_2^l, v_2^l)$, $(P_1 \setminus l) \parallel P_2 = (E^l, v^l)$, and $l \in \mathcal{I}(P_1 \parallel P_2)$.

Due to Lemma 1.28, and similarly to it, we focus when $l \in \mathcal{I}(E_1)$ and only show

1. $\forall x \in \mathcal{C}((P_1 \parallel P_2) \setminus l), y \in \mathcal{C}((P_1 \setminus l) \parallel P_2). \underline{x} \subseteq \underline{y} \Rightarrow v'(x) \geq v^l(y)$

$$\begin{aligned} v'(x) &= \frac{v(\{l\} \cup x)}{v(\{l\})} = \frac{v_1((\{l\} \cup x) \cap E_1) \cdot v_2((\{l\} \cup x) \cap E_2)}{v_1(\{l\})} \\ &= \frac{v_1(\{l\} \cup (x \cap E_1)) \cdot v_2(x \cap E_2)}{v_1(\{l\})} = v_1^l(x \cap E_1) \cdot v_2(x \cap E_2) = v^l(x) \end{aligned}$$

Since $(P_1 \setminus l) \parallel P_2$ is a probabilistic event structure, we obtain $v'(x) \geq v^l(y)$.

2. $\forall x \in \mathcal{C}((P_1 \setminus l) \parallel P_2), y \in \mathcal{C}((P_1 \parallel P_2) \setminus l). \underline{x} \subseteq \underline{y} \Rightarrow v^l(x) \geq v'(y)$

$$\begin{aligned} v^l(x) &= v_1^l(x \cap E_1) \cdot v_2(x \cap E_2) = \frac{v_1(\{l\} \cup (x \cap E_1)) \cdot v_2(x \cap E_2)}{v_1(\{l\})} \\ &= \frac{v_1((\{l\} \cup x) \cap E_1) \cdot v_2((\{l\} \cup x) \cap E_2)}{v_1(\{l\})} = \frac{v(\{l\} \cup x)}{v(\{l\})} = v'(x) \end{aligned}$$

Since $(P_1 \parallel P_2) \setminus l$ is a probabilistic event structure, we obtain $v^l(x) \geq v'(y)$. □

Lemma 2.20. Let P_1, P_2 be probabilistic event structures. Then $P_1 \parallel P_2 = P_2 \parallel P_1$. □

Proof. It follows directly from Definition 2.9. □

Lemma 2.21. Let C be a command and $l \in \mathcal{I}(\llbracket C \rrbracket)$. Then $v(\{l\}) = 1$.

Proof. • $sk \in \mathcal{I}(\llbracket \text{skip} \rrbracket)$

It follows directly that $v(\{sk\}) = 1$.

- $a \in \mathcal{I}(\llbracket a \rrbracket)$

It follows directly that $v(\{a\}) = 1$.

- $\tau \in \mathcal{I}(\llbracket C_1 +_p C_2 \rrbracket)$

It follows directly that $v(\{\tau\}) = 1$.

- $l' \in \mathcal{I}(\llbracket C_1 ; C_2 \rrbracket)$

By Definition 2.5 we have $l' \in \mathcal{I}(C_1)$. By i.h., $v(\{l'\}) = 1$ and since $l' \in \mathcal{I}(C_1 ; C_2)$ we are done.

- $l' \in \mathcal{I}(\llbracket C_1 \parallel C_2 \rrbracket)$

By Definition 2.9 we have $l' \in \mathcal{I}(C_1)$ or $l' \in \mathcal{I}(C_2)$. By i.h., $v(\{l'\}) = 1$ for both cases. Since $l' \in \mathcal{I}(C_1 \parallel C_2)$ we are done. □

Lemma 2.22. Let $P, \sum_i p_i \cdot P_i$ be probabilistic event structures. Then $(\sum_i p_i \cdot P_i) ; P = \sum_i p_i \cdot (P_i ; P)$

Proof. Follows directly from the respective definitions. □

Lemma 2.23. Let $P, \sum_i p_i \cdot P_i$ be probabilistic event structures.

1. $(\sum_i p_i \cdot P_i) \parallel P \subseteq \sum_i p_i \cdot (P_i \parallel P)$
2. $x \in \mathcal{C}_{\max}((\sum_i p_i \cdot P_i) \parallel P)$ iff $x \in \mathcal{C}_{\max}(\sum_i p_i \cdot (P_i \parallel P))$

Proof. Let $P = (E, \leq, \#, v)$, $\sum_i p_i \cdot P_i = (E_i, \leq_i, \#_i, v_i)$, $(\sum_i p_i \cdot P_i) \parallel P = (E', \leq'_i, \#'_i, v'_i)$, and $\sum_i p_i \cdot (P_i \parallel P) = (E''_i, \leq''_i, \#''_i, v''_i)$.

1. $(\sum_i p_i \cdot P_i) \parallel P \sqsubseteq \sum_i p_i \cdot (P_i \parallel P)$

By Remark 4, we know that $E_i'' = \{\tau\} \uplus \uplus_i (E_i \uplus E)$. Let $e \in E$ and for all i pick an initial element $l_i \in \mathcal{I}(P_i)$. Rename the events of E in $E_i \uplus E$ as $(e, \{\tau, l_i\})$. Intuitively, we are making a copy of an event in E for each i .

We need to verify the following conditions:

- (a) $E_i' \subseteq E_i''$
- (b) $e \leq_i' e' \Leftrightarrow e, e' \in E_i' \wedge e \leq_i'' e'$
- (c) $e \#_i' e' \Leftrightarrow e, e' \in E_i' \wedge e \#_i'' e'$
- (d) $\forall x \in \mathcal{C}((\sum_i p_i \cdot P_i) \parallel P), y \in \mathcal{C}(\sum_i p_i \cdot (P_i \parallel P)). \underline{x} \subseteq \underline{y} \Rightarrow v_i'(x) \geq v_i''(y)$

The first three conditions follow directly from Remark 4 and Definition 2.9. Hence we focus on the last condition, which we prove by contradiction.

Let $x \in \mathcal{C}((\sum_i p_i \cdot P_i) \parallel P)$, $y \in \mathcal{C}(\sum_i p_i \cdot (P_i \parallel P))$, such that $\underline{x} \subseteq \underline{y}$.

We want to show that $\exists x \in \mathcal{C}((\sum_i p_i \cdot P_i) \parallel P), y \in \mathcal{C}(\sum_i p_i \cdot (P_i \parallel P)). \underline{x} \subseteq \underline{y} \Rightarrow v_i'(x) \leq v_i''(y)$.

Let $l \in \mathcal{I}((\sum_i p_i \cdot P_i) \parallel P)$ such that $\{l\} \in \mathcal{C}(P)$. By Definition 2.9, $v_i'(\{l\}) = v(\{l\}) = 1$. On the other side, we have $\{\tau, l_x\} \in \mathcal{C}(\sum_i p_i \cdot (P_i \parallel P))$. Hence $v_i''(\{\tau, l_x\}) = p_i \cdot v(\{l\})$. Since $\{\tau\} \subseteq \{\tau, l_x\}$ and $v(\{\tau\}) \geq p_i \cdot v(\{l\})$, we are done since the assumption was contradicted.

2. $x \in \mathcal{C}_{\max}((\sum_i p_i \cdot P_i) \parallel P)$ iff $x \in \mathcal{C}_{\max}(\sum_i p_i \cdot (P_i \parallel P))$

For both cases, it is relevant to notice the following: let P_1, P_2 be two probabilistic event structures such that $x_1 \in \mathcal{C}(P_1)$ and $x_2 \in \mathcal{C}(P_2)$. Then $x = x_1 \cup x_2 \in \mathcal{C}(P_1 \parallel P_2)$, since by Definition 2.9 there is no conflict between events of P_1 and events of P_2 .

\Leftarrow If $x \in \mathcal{C}_{\max}((\sum_i p_i \cdot P_i) \parallel P)$ then $x \in \mathcal{C}_{\max}(\sum_i p_i \cdot (P_i \parallel P))$

Let $x \in \mathcal{C}_{\max}((\sum_i p_i \cdot P_i) \parallel P)$. We can represent x as follows: $x = x \cap (E \uplus E_i) = (x \cap E) \uplus (x \cap E_i)$, for E_i in $\uplus_i E_i$ and where $x \cap E \in \mathcal{C}_{\max}(P)$ and $x \cap E_i \in \mathcal{C}_{\max}(\sum_i p_i \cdot P_i)$. Hence, it follows directly that $(x \cap E) \uplus (x \cap E_i) = x \cap (E \uplus E_i) = x \in \mathcal{C}_{\max}(\sum_i p_i \cdot (P_i \parallel P))$.

\Rightarrow If $x \in \mathcal{C}_{\max}(\sum_i p_i \cdot (P_i \parallel P))$ then $x \in \mathcal{C}_{\max}((\sum_i p_i \cdot P_i) \parallel P)$

Let $x \in \mathcal{C}_{\max}(\sum_i p_i \cdot (P_i \parallel P))$. Hence $\exists x_i \in \mathcal{C}_{\max}(\sum_i p_i \cdot P_i), y \in \mathcal{C}_{\max}(P)$ such that $x = x_i \cup y$. Since $x_i \in \mathcal{C}_{\max}(\sum_i p_i \cdot P_i)$, then $\exists i. x_i \in \mathcal{C}_{\max}(P_i)$. We then have $x_i \cup y = x \in \mathcal{C}_{\max}(P_i \parallel P)$ and consequently $x \in \mathcal{C}_{\max}((\sum_i p_i \cdot P_i) \parallel P)$.

□

Lemma 2.24. Let $\mathcal{C} = \mathcal{C}_1 ; \mathcal{C}_2$ or $\mathcal{C} = \mathcal{C}_1 \parallel \mathcal{C}_2$. If $\mathcal{C} \rightarrow \sum_i p_i \cdot (\tau, \mathcal{C}_i)$ then $x \in \mathcal{C}_{\max}(\llbracket \mathcal{C} \rrbracket)$ and $x \in \mathcal{C}_{\max}(\sum_i p_i \cdot \llbracket \mathcal{C}_i \rrbracket)$ such that $\exists \llbracket \mathcal{C}_i \rrbracket. v(x) = v_i(x)$.

Proof. 1. $\mathcal{C} \equiv \mathcal{C}_1 ; \mathcal{C}_2$

We know that $\mathcal{C}_1 ; \mathcal{C}_2 \rightarrow \sum_i p_i \cdot (\tau, \mathcal{C}_i ; \mathcal{C}_2)$. By the rules in Figure 11 we have $\mathcal{C}_1 \rightarrow \sum_i p_i \cdot (\tau, \mathcal{C}_i)$. By i.h. we have $x_1 \in \mathcal{C}_{\max}(\llbracket \mathcal{C}_1 \rrbracket)$ and $x_1 \in \mathcal{C}_{\max}(\sum_i p_i \cdot \llbracket \mathcal{C}_i \rrbracket)$ such that $\exists \llbracket \mathcal{C}_i \rrbracket. v_1(x_1) = v_i(x_1)$. Let $x_2 \in \mathcal{C}_{\max}(\llbracket \mathcal{C}_2 \rrbracket)$. By Definition 2.5 we have $x_1 \cup x_2 \in \mathcal{C}_{\max}(\llbracket \mathcal{C}_1 ; \mathcal{C}_2 \rrbracket)$ and $x_1 \cup x_2 \in \mathcal{C}_{\max}((\sum_i p_i \cdot \llbracket \mathcal{C}_i \rrbracket) ; \mathcal{C}_2)$, and $v(x) = v_1(x_1) \cdot v_2(x_2) = v_i(x_1) \cdot v_2(x_2)$. By Lemma 2.22 we have $x_1 \cup x_2 \in \mathcal{C}_{\max}(\sum_i p_i \cdot \llbracket \mathcal{C}_i ; \mathcal{C}_2 \rrbracket)$.

2. $\mathcal{C} \equiv \mathcal{C}_1 \parallel \mathcal{C}_2$

We know that $\mathcal{C}_1 \parallel \mathcal{C}_2 \rightarrow \sum_i p_i \cdot (\tau, \mathcal{C}_i \parallel \mathcal{C}_2)$. By the rules in Figure 11 we have $\mathcal{C}_1 \rightarrow \sum_i p_i \cdot (\tau, \mathcal{C}_i)$. By i.h. we have $x_1 \in \mathcal{C}_{\max}(\llbracket \mathcal{C}_1 \rrbracket)$ and $x_1 \in \mathcal{C}_{\max}(\sum_i p_i \cdot \llbracket \mathcal{C}_i \rrbracket)$ such that $\exists \llbracket \mathcal{C}_i \rrbracket. v_1(x_1) = v_i(x_1)$. Let $x_2 \in \mathcal{C}_{\max}(\llbracket \mathcal{C}_2 \rrbracket)$. By Definition 2.9 we have $x_1 \cup x_2 \in \mathcal{C}_{\max}(\llbracket \mathcal{C}_1 \rrbracket \parallel \llbracket \mathcal{C}_2 \rrbracket)$ and $x_1 \cup x_2 \in \mathcal{C}_{\max}((\sum_i p_i \cdot \llbracket \mathcal{C}_i \rrbracket) \parallel \mathcal{C}_2)$, and $v(x) = v_1(x_1) \cdot v_2(x_2) = v_i(x_1) \cdot v_2(x_2)$. By Lemma 2.23 we have $x_1 \cup x_2 \in \mathcal{C}_{\max}(\sum_i p_i \cdot \llbracket \mathcal{C}_i \rrbracket \parallel \mathcal{C}_2)$.

A similar reasoning is applied when $\mathcal{C}_1 \parallel \mathcal{C}_2 \rightarrow \sum_j p_j \cdot (\tau, \mathcal{C}_2 \parallel \mathcal{C}_j)$.

□

Lemma 2.25 (Soundness I). • If $\mathcal{C} \rightarrow 1 \cdot (l, \mathcal{C}')$ then $\llbracket \mathcal{C}' \rrbracket = \llbracket \mathcal{C} \rrbracket \setminus l$

- If $\mathcal{C} \rightarrow \sum_i p_i (\tau, \mathcal{C}_i)$ then $\llbracket \mathcal{C} \rrbracket \sqsubseteq \sum_i p_i \llbracket \mathcal{C}_i \rrbracket$

Proof. Induction over rules in Figure 11.

- **skip** $\rightarrow 1 \cdot (sk, \checkmark)$

It follows directly that $\llbracket \checkmark \rrbracket \equiv \llbracket \text{skip} \rrbracket \setminus sk \equiv \emptyset$.

- $\mathbf{a} \rightarrow 1 \cdot (a, \checkmark)$

It follows directly that $\llbracket \checkmark \rrbracket \equiv \llbracket \mathbf{a} \rrbracket \setminus a \equiv \emptyset$.

- $\mathbf{C}_1 +_p \mathbf{C}_2 \rightarrow p \cdot (\tau, \mathbf{C}_1) + (1-p) \cdot (\tau, \mathbf{C}_2)$

It follows directly that $p \cdot \llbracket \mathbf{C}_1 \rrbracket + (1-p) \cdot \llbracket \mathbf{C}_2 \rrbracket = \llbracket \mathbf{C}_1 +_p \mathbf{C}_2 \rrbracket$.

- $\mathbf{C}_1 ; \mathbf{C}_2 \rightarrow 1 \cdot (l, \mathbf{C}_2)$

$$\begin{aligned}
& \mathbf{C}_1 ; \mathbf{C}_2 \xrightarrow{l} \mathbf{C}_2 \\
& \Rightarrow \{\text{Figure 11 entails}\} \\
& \mathbf{C}_1 \xrightarrow{l} \checkmark \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket \checkmark \rrbracket \equiv \llbracket \mathbf{C}_1 \rrbracket \setminus l \\
& \Rightarrow \{\text{Lemma 2.15}\} \\
& \llbracket \checkmark \rrbracket ; \llbracket \mathbf{C}_2 \rrbracket \equiv (\llbracket \mathbf{C}_1 \rrbracket \setminus l) ; \llbracket \mathbf{C}_2 \rrbracket \\
& \Rightarrow \{\llbracket \checkmark \rrbracket ; \llbracket \mathbf{C}_2 \rrbracket = \llbracket \mathbf{C}_2 \rrbracket, \text{ Lemma 2.18}\} \\
& \llbracket \mathbf{C}_2 \rrbracket \equiv (\llbracket \mathbf{C}_1 \rrbracket ; \llbracket \mathbf{C}_2 \rrbracket) \setminus l \\
& \Rightarrow \{\text{Definition 2.11}\} \\
& \llbracket \mathbf{C}_2 \rrbracket \equiv \llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket \setminus l
\end{aligned}$$

- $\mathbf{C}_1 ; \mathbf{C}_2 \rightarrow 1 \cdot (l, \mathbf{C}'_1 ; \mathbf{C}_2)$

$$\begin{aligned}
& \mathbf{C}_1 ; \mathbf{C}_2 \xrightarrow{l} \mathbf{C}'_1 ; \mathbf{C}_2 \\
& \Rightarrow \{\text{Figure 11 entails}\} \\
& \mathbf{C}_1 \xrightarrow{l} \mathbf{C}'_1 \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket \mathbf{C}'_1 \rrbracket \equiv \llbracket \mathbf{C}_1 \rrbracket \setminus l \\
& \Rightarrow \{\text{Lemma 2.15}\} \\
& \llbracket \mathbf{C}'_1 \rrbracket ; \llbracket \mathbf{C}_2 \rrbracket \equiv (\llbracket \mathbf{C}_1 \rrbracket \setminus l) ; \llbracket \mathbf{C}_2 \rrbracket \\
& \Rightarrow \{\text{Lemma 2.18}\} \\
& \llbracket \mathbf{C}'_1 \rrbracket ; \llbracket \mathbf{C}_2 \rrbracket \equiv (\llbracket \mathbf{C}_1 \rrbracket ; \llbracket \mathbf{C}_2 \rrbracket) \setminus l \\
& \Rightarrow \{\text{Definition 2.11}\} \\
& \llbracket \mathbf{C}'_1 ; \mathbf{C}_2 \rrbracket \equiv \llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket \setminus l
\end{aligned}$$

- $\mathbf{C}_1 ; \mathbf{C}_2 \rightarrow \sum_i p_i \cdot (\tau, \mathbf{C}_i ; \mathbf{C}_2)$

$$\begin{aligned}
& \mathbf{C}_1 ; \mathbf{C}_2 \rightarrow \sum_i p_i \cdot (\tau, \mathbf{C}_i ; \mathbf{C}_2) \\
& \Rightarrow \{\text{Figure 11 entails}\} \\
& \mathbf{C}_1 \rightarrow \sum_i p_i \cdot (\tau, \mathbf{C}_i) \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket \mathbf{C}_1 \rrbracket \equiv \sum_i p_i \cdot \llbracket \mathbf{C}_i \rrbracket \\
& \Rightarrow \{\text{Lemma 2.15}\} \\
& \llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket \equiv (\sum_i p_i \cdot \llbracket \mathbf{C}_i \rrbracket) ; \llbracket \mathbf{C}_2 \rrbracket \\
& \Rightarrow \{\text{Lemma 2.22}\} \\
& \llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket \equiv \sum_i p_i \cdot \llbracket \mathbf{C}_i ; \mathbf{C}_2 \rrbracket
\end{aligned}$$

- $C_1 \parallel C_2 \rightarrow 1 \cdot (l, C_2)$

$$\begin{aligned}
& C_1 \parallel C_2 \xrightarrow{l} C_2 \\
& \Rightarrow \{\text{Figure 11 entails}\} \\
& C_1 \xrightarrow{l} \checkmark \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket \checkmark \rrbracket \equiv \llbracket C_1 \rrbracket \setminus l \\
& \Rightarrow \{\text{Lemma 2.17}\} \\
& \llbracket \checkmark \rrbracket \parallel \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel \llbracket C_2 \rrbracket \\
& \Rightarrow \{\llbracket \checkmark \rrbracket \parallel \llbracket C_2 \rrbracket = \llbracket C_2 \rrbracket\} \\
& \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel \llbracket C_2 \rrbracket \\
& \Rightarrow \{\text{Lemma 2.19, Definition 2.11}\} \\
& \llbracket C_2 \rrbracket \equiv \llbracket C_1 \parallel C_2 \rrbracket \setminus l
\end{aligned}$$

- $C_1 \parallel C_2 \rightarrow 1 \cdot (l, C'_1 \parallel C_2)$

$$\begin{aligned}
& C_1 \parallel C_2 \xrightarrow{l} C'_1 \parallel C_2 \\
& \Rightarrow \{\text{Figure 11 entails}\} \\
& C_1 \xrightarrow{l} C'_1 \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket C'_1 \rrbracket \equiv \llbracket C_1 \rrbracket \setminus l \\
& \Rightarrow \{\text{Lemma 2.17}\} \\
& \llbracket C'_1 \rrbracket \parallel \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel \llbracket C_2 \rrbracket \\
& \Rightarrow \{\text{Lemma 2.19, Definition 2.11}\} \\
& \llbracket C'_1 \parallel C_2 \rrbracket \equiv \llbracket C_1 \parallel C_2 \rrbracket \setminus l
\end{aligned}$$

- $C_1 \parallel C_2 \rightarrow \sum_i p_i \cdot (\tau, C_i \parallel C_2)$

$$\begin{aligned}
& C_1 \parallel C_2 \rightarrow \sum_i p_i \cdot (\tau, C_i \parallel C_2) \\
& \Rightarrow \{\text{Figure 11 entails}\} \\
& C_1 \rightarrow \sum_i p_i \cdot (\tau, C_i) \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket C_1 \rrbracket \equiv \sum_i p_i \cdot \llbracket C_i \rrbracket \\
& \Rightarrow \{\text{Lemma 2.17}\} \\
& \llbracket C_1 \parallel C_2 \rrbracket \equiv (\sum_i p_i \cdot \llbracket C_i \rrbracket) \parallel \llbracket C_2 \rrbracket \\
& \Rightarrow \{\text{Lemma 2.23, Definition 2.11}\} \\
& \llbracket C_1 \parallel C_2 \rrbracket \equiv \sum_i p_i \cdot \llbracket C_i \parallel C_2 \rrbracket
\end{aligned}$$

- $C_1 \parallel C_2 \xrightarrow{l} C_1$

$$\begin{aligned}
& C_1 \parallel C_2 \xrightarrow{l} C_1 \\
& \Rightarrow \{\text{Figure 11 entails}\} \\
& C_2 \xrightarrow{l} \checkmark \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket \checkmark \rrbracket \equiv \llbracket C_2 \rrbracket \setminus l \\
& \Rightarrow \{\text{Lemma 2.17}\} \\
& \llbracket C_1 \rrbracket \parallel \llbracket \checkmark \rrbracket \equiv \llbracket C_1 \rrbracket \parallel (\llbracket C_2 \rrbracket \setminus l) \\
& \Rightarrow \{\llbracket C_1 \rrbracket \parallel \llbracket \checkmark \rrbracket = \llbracket C_1 \rrbracket\} \\
& \llbracket C_1 \rrbracket \equiv \llbracket C_1 \rrbracket \parallel (\llbracket C_2 \rrbracket \setminus l) \\
& \Rightarrow \{\text{Lemma 2.19, Definition 2.11}\} \\
& \llbracket C_1 \rrbracket \equiv \llbracket C_1 \parallel C_2 \rrbracket \setminus l
\end{aligned}$$

- $C_1 \parallel C_2 \xrightarrow{l} C_1 \parallel C'_2$

$$\begin{aligned}
& C_1 \parallel C_2 \xrightarrow{l} C_1 \parallel C'_2 \\
& \Rightarrow \{\text{Figure 11 entails}\} \\
& C_2 \xrightarrow{l} C'_2 \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket C'_2 \rrbracket \equiv \llbracket C_2 \rrbracket \setminus l \\
& \Rightarrow \{\text{Lemma 2.17}\} \\
& \llbracket C_1 \rrbracket \parallel \llbracket C'_2 \rrbracket \equiv \llbracket C_1 \rrbracket \parallel (\llbracket C_2 \rrbracket \setminus l) \\
& \Rightarrow \{\text{Lemma 2.19, Definition 2.11}\} \\
& \llbracket C_1 \parallel C'_2 \rrbracket \equiv \llbracket C_1 \parallel C_2 \rrbracket \setminus l
\end{aligned}$$

- $C_1 \parallel C_2 \rightarrow \sum_j p_j \cdot (\tau, C_1 \parallel C_j)$

$$\begin{aligned}
& C_1 \parallel C_2 \rightarrow \sum_j p_j \cdot (\tau, C_1 \parallel C_j) \\
& \Rightarrow \{\text{Figure 11 entails}\} \\
& C_2 \rightarrow \sum_j p_j \cdot (\tau, C_j) \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket C_2 \rrbracket \sqsubseteq \sum_j p_j \cdot \llbracket C_j \rrbracket \\
& \Rightarrow \{\text{Lemma 2.17}\} \\
& \llbracket C_1 \parallel C_2 \rrbracket \sqsubseteq \llbracket C_1 \rrbracket \parallel (\sum_j p_j \cdot \llbracket C_j \rrbracket) \\
& \Rightarrow \{\text{Lemma 2.23, Definition 2.11}\} \\
& \llbracket C_1 \parallel C_2 \rrbracket \sqsubseteq \sum_j p_j \cdot \llbracket C_1 \parallel C_j \rrbracket
\end{aligned}$$

□

Lemma 2.26 (Soundness II). If $C \twoheadrightarrow \sum_i p_i (\sum_j p_j (\omega_{ij}, \checkmark))$ then $\forall_{i,j}, \exists x_{ij} \in \mathcal{C}_{\max}(\llbracket C \rrbracket) \cdot \emptyset \xrightarrow{\omega_{ij}}_{\mathcal{C}} x_{ij}$ and $p_{ij} = v(x_{ij})$.

Proof. Induction over rules in Figure 12.

- $\text{skip} \twoheadrightarrow 1 \cdot (sk, \checkmark)$

It follows directly that $\{sk\} \in \mathcal{C}_{\max}(\llbracket \text{skip} \rrbracket) \cdot \emptyset \xrightarrow{sk}_{\mathcal{C}} \{sk\}$ and $v(\{sk\}) = 1$.

- $\mathbf{a} \rightarrow 1 \cdot (a, \checkmark)$

It follows directly that $\{a\} \in \mathcal{C}_{\max}(\llbracket \mathbf{a} \rrbracket) \cdot \emptyset \xrightarrow{a} \{a\}$ and $v(\{a\}) = 1$.

- $\mathbf{C}_1 +_p \mathbf{C}_2 \rightarrow \sum_i p_i \left(\sum_j p_j (\omega_{ij}, \checkmark) \right)$

$$\begin{aligned}
& \mathbf{C}_1 +_p \mathbf{C}_2 \rightarrow \sum_i p_i \left(\sum_j p_j (\omega_{ij}, \checkmark) \right) \\
& \Rightarrow \{\text{Figure 12 entails}\} \\
& \mathbf{C}_1 +_p \mathbf{C}_2 \rightarrow p \cdot (\tau, \mathbf{C}_1) + (1-p) \cdot (\tau, \mathbf{C}_2) \\
& \mathbf{C}_1 \rightarrow \sum_l p_l \cdot (\omega_l, \checkmark) \\
& \mathbf{C}_2 \rightarrow \sum_k p_k \cdot (\omega_k, \checkmark) \\
& \Rightarrow \{\text{Lemma 2.25, i.h.}\} \\
& \llbracket \mathbf{C}_1 +_p \mathbf{C}_2 \rrbracket \subseteq p \cdot \llbracket \mathbf{C}_1 \rrbracket + (1-p) \cdot \llbracket \mathbf{C}_2 \rrbracket \\
& \forall l, \exists x_l \in \mathcal{C}_{\max}(\llbracket \mathbf{C}_1 \rrbracket) \cdot \emptyset \xrightarrow{\omega_l} x_l, \quad v_1(x_l) = p_l \\
& \forall k, \exists x_k \in \mathcal{C}_{\max}(\llbracket \mathbf{C}_2 \rrbracket) \cdot \emptyset \xrightarrow{\omega_k} x_k, \quad v_2(x_k) = p_k \\
& \Rightarrow \{\text{Definition 3.7}\} \\
& \forall i, \exists x_i = \{\tau\} \cup x_l \in \mathcal{C}_{\max}(\llbracket \mathbf{C}_1 +_p \mathbf{C}_2 \rrbracket) \cdot \emptyset \xrightarrow{\tau} \{\tau\} \xrightarrow{\omega_l} \{\tau\} \cup x_l, \quad v(x_i) = p \cdot p_l \\
& \text{or} \\
& \exists x_i = \{\tau\} \cup x_k \in \mathcal{C}_{\max}(\llbracket \mathbf{C}_1 +_p \mathbf{C}_2 \rrbracket) \cdot \emptyset \xrightarrow{\tau} \{\tau\} \xrightarrow{\omega_k} \{\tau\} \cup x_k, \quad v(x_i) = (1-p) \cdot p_k
\end{aligned}$$

- $\mathbf{C}_1 ; \mathbf{C}_2 \rightarrow \sum_i p_i \left(\sum_j p_j (\omega_{ij}, \checkmark) \right)$

$$\begin{aligned}
& \mathbf{C}_1 ; \mathbf{C}_2 \rightarrow \sum_i p_i \left(\sum_j p_j (\omega_{ij}, \checkmark) \right) \\
& \Rightarrow \{\text{Figure 12 entails}\} \\
& \mathbf{C}_1 ; \mathbf{C}_2 \rightarrow \sum_i p_i \cdot (l', \mathbf{C}_i) \\
& \forall i \quad \mathbf{C}_i \rightarrow \sum_j p_j (\omega_{ij}, \checkmark)
\end{aligned}$$

According to $\mathbf{C}_1 ; \mathbf{C}_2 \rightarrow \sum_i p_i \cdot (l', \mathbf{C}_i)$ we have three cases:

1.

$$\begin{aligned}
& \mathbf{C}_1 ; \mathbf{C}_2 \rightarrow 1 \cdot (l, \mathbf{C}_2) \\
& \mathbf{C}_2 \rightarrow \sum_j p_j (\omega_j, \checkmark) \\
& \Rightarrow \{\text{Lemma 2.25, i.h.}\} \\
& \llbracket \mathbf{C}_2 \rrbracket \equiv \llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket \setminus l \\
& \forall j, \exists x_j \in \mathcal{C}_{\max}(\llbracket \mathbf{C}_2 \rrbracket) \cdot \emptyset \xrightarrow{\omega_j} x_j, \quad v_2(x_j) = p_j \\
& \Rightarrow \{\text{Definition 2.13}\} \\
& \forall j, \{l\} \cup x_j \in \mathcal{C}_{\max}(\llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket) \cdot \emptyset \xrightarrow{l} \{l\} \xrightarrow{\omega_j} \{l\} \cup x_j, \quad v(\{l\} \cup x_j) = p_j
\end{aligned}$$

2.

$$\begin{aligned}
& \mathbf{C}_1 ; \mathbf{C}_2 \rightarrow 1 \cdot (l, \mathbf{C}'_1 ; \mathbf{C}_2) \\
& \mathbf{C}'_1 ; \mathbf{C}_2 \twoheadrightarrow \sum_j p_j (\omega_j, \checkmark) \\
& \Rightarrow \{\text{Lemma 2.25, i.h.}\} \\
& \llbracket \mathbf{C}'_1 ; \mathbf{C}_2 \rrbracket \equiv \llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket \setminus l \\
& \forall j, \exists x_j \in \mathcal{C}_{\max}(\llbracket \mathbf{C}'_1 ; \mathbf{C}_2 \rrbracket) . \emptyset \xrightarrow{\omega_j} \subset x_j, \quad v'(x_j) = p_j \\
& \Rightarrow \{\text{Definition 2.13}\} \\
& \forall j, \{l\} \cup x_j \in \mathcal{C}_{\max}(\llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket) . \emptyset \xrightarrow{l} \subset \{l\} \xrightarrow{\omega_j} \subset \{l\} \cup x_j, \quad v(\{l\} \cup x_j) = p_j
\end{aligned}$$

3.

$$\begin{aligned}
& \mathbf{C}_1 ; \mathbf{C}_2 \rightarrow \sum_i p_i \cdot (\tau, \mathbf{C}_i ; \mathbf{C}_2) \\
& \forall i \ \mathbf{C}_i ; \mathbf{C}_2 \twoheadrightarrow \sum_j p_j (\omega_{ij}, \checkmark) \\
& \Rightarrow \{\text{Lemma 2.25, i.h.}\} \\
& \llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket \equiv \sum_i p_i \cdot \llbracket \mathbf{C}_i ; \mathbf{C}_2 \rrbracket \\
& \forall i, j, \exists x_j \in \mathcal{C}_{\max}(\llbracket \mathbf{C}_i ; \mathbf{C}_2 \rrbracket) . \emptyset \xrightarrow{\omega_j} \subset x_j, \quad v_i(x_j) = p_j \\
& \Rightarrow \{\text{Remark 4, Lemma 2.24}\} \\
& \forall i, j, \{\tau\} \cup x_j \in \mathcal{C}_{\max}(\llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket) . \emptyset \xrightarrow{\tau} \subset \{\tau\} \xrightarrow{\omega_j} \subset \{\tau\} \cup x_j, \\
& \quad v(\{\tau\} \cup x_j) = p_i v_i(x_j) = p_i p_j
\end{aligned}$$

- $\mathbf{C}_1 \parallel \mathbf{C}_2 \twoheadrightarrow \sum_i p_i (\sum_j p_j (\omega_{ij}, \checkmark))$

$$\begin{aligned}
& \mathbf{C}_1 \parallel \mathbf{C}_2 \twoheadrightarrow \sum_i p_i \left(\sum_j p_j (\omega_{ij}, \checkmark) \right) \\
& \Rightarrow \{\text{Figure 12 entails}\} \\
& \mathbf{C}_1 \parallel \mathbf{C}_2 \rightarrow \sum_i p_i \cdot (l', \mathbf{C}_i) \\
& \forall i \ \mathbf{C}_i \twoheadrightarrow \sum_j p_j (\omega_{ij}, \checkmark)
\end{aligned}$$

According to $\mathbf{C}_1 \parallel \mathbf{C}_2 \rightarrow \sum_i p_i \cdot (l', \mathbf{C}_i)$ we have three cases (we only consider the cases where \mathbf{C}_1 is the command that performs a computation, since for the other way around the reasoning is similar):

1.

$$\begin{aligned}
& \mathbf{C}_1 \parallel \mathbf{C}_2 \rightarrow 1 \cdot (l, \mathbf{C}_2) \\
& \mathbf{C}_2 \twoheadrightarrow \sum_j p_j (\omega_j, \checkmark) \\
& \Rightarrow \{\text{Lemma 2.25, i.h.}\} \\
& \llbracket \mathbf{C}_2 \rrbracket \equiv \llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket \setminus l \\
& \forall j, \exists x_j \in \mathcal{C}_{\max}(\llbracket \mathbf{C}_2 \rrbracket) . \emptyset \xrightarrow{\omega_j} \subset x_j, \quad v_2(x_j) = p_j \\
& \Rightarrow \{\text{Definition 2.13}\} \\
& \forall j, \{l\} \cup x_j \in \mathcal{C}_{\max}(\llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket) . \emptyset \xrightarrow{l} \subset \{l\} \xrightarrow{\omega_j} \subset \{l\} \cup x_j, \quad v(\{l\} \cup x_j) = p_j
\end{aligned}$$

2.

$$\begin{aligned}
& \mathbf{C}_1 \parallel \mathbf{C}_2 \rightarrow 1 \cdot (l, \mathbf{C}'_1 \parallel \mathbf{C}_2) \\
& \mathbf{C}'_1 \parallel \mathbf{C}_2 \twoheadrightarrow \sum_j p_j (\omega_j, \checkmark) \\
& \Rightarrow \{\text{Lemma 2.25, i.h.}\} \\
& \llbracket \mathbf{C}'_1 \parallel \mathbf{C}_2 \rrbracket \equiv \llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket \setminus l \\
& \forall j, \exists x_j \in \mathcal{C}_{\max}(\llbracket \mathbf{C}'_1 \parallel \mathbf{C}_2 \rrbracket) \cdot \emptyset \xrightarrow{\omega_j} \subset x_j, \quad v'(x_j) = p_j \\
& \Rightarrow \{\text{Definition 2.13}\} \\
& \forall j, \{l\} \cup x_j \in \mathcal{C}_{\max}(\llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket) \cdot \emptyset \xrightarrow{l} \subset \{l\} \xrightarrow{\omega_j} \subset \{l\} \cup x_j, \quad v(\{l\} \cup x_j) = p_j
\end{aligned}$$

3.

$$\begin{aligned}
& \mathbf{C}_1 \parallel \mathbf{C}_2 \rightarrow \sum_i p_i \cdot (\tau, \mathbf{C}_i \parallel \mathbf{C}_2) \\
& \forall i \mathbf{C}_i \parallel \mathbf{C}_2 \twoheadrightarrow \sum_j p_j (\omega_{ij}, \checkmark) \\
& \Rightarrow \{\text{Lemma 2.25, i.h.}\} \\
& \llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket \equiv \sum_i p_i \cdot \llbracket \mathbf{C}_i \parallel \mathbf{C}_2 \rrbracket \\
& \forall i, j, \exists x_j \in \mathcal{C}_{\max}(\llbracket \mathbf{C}_i \parallel \mathbf{C}_2 \rrbracket) \cdot \emptyset \xrightarrow{\omega_j} \subset x_j, \quad v_i(x_j) = p_j \\
& \Rightarrow \{\text{Remark 4, Lemma 2.24}\} \\
& \forall i, j, \{\tau\} \cup x_j \in \mathcal{C}_{\max}(\llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket) \cdot \emptyset \xrightarrow{\tau} \subset \{\tau\} \xrightarrow{\omega_j} \subset \{\tau\} \cup x_j, \\
& v(\{\tau\} \cup x_j) = p_i v_i(x_j) = p_i p_j
\end{aligned}$$

□

Lemma 2.27 (Adequacy I). Let $l' \in \mathcal{I}(\llbracket \mathbf{C} \rrbracket)$.

1. If $l' \neq \tau$ then $\exists \mathbf{C}' \in (\mathbf{C} \cup \{\checkmark\}) \cdot \mathbf{C} \rightarrow 1 \cdot (l', \mathbf{C}')$ and $\llbracket \mathbf{C} \rrbracket \setminus l' \equiv \llbracket \mathbf{C}' \rrbracket$.
2. If $l' = \tau$ then $\exists \mathbf{C}', \mathbf{C}'', \exists e' \in \mathcal{I}(\llbracket \mathbf{C}' \rrbracket) \cdot \mathbf{C} \rightarrow p \cdot \mathbf{C}' + (1-p) \cdot \mathbf{C}''$ and $\llbracket \mathbf{C} \rrbracket \equiv p \cdot \llbracket \mathbf{C}' \rrbracket + (1-p) \cdot \llbracket \mathbf{C}'' \rrbracket$, with $p = v(\{\tau, e'\})$.

Proof. • $sk \in \mathcal{I}(\llbracket \text{skip} \rrbracket)$

Let $\mathbf{C}' = \checkmark$. It follows directly that $\text{skip} \rightarrow 1 \cdot (sk, \checkmark)$ and that $\llbracket \text{skip} \rrbracket \setminus sk \equiv \llbracket \checkmark \rrbracket$.

- $a \in \mathcal{I}(\llbracket \mathbf{a} \rrbracket)$

Let $\mathbf{C}' = \checkmark$. It follows directly that $\mathbf{a} \rightarrow 1 \cdot (a, \checkmark)$ and that $\llbracket \mathbf{a} \rrbracket \setminus a \equiv \llbracket \checkmark \rrbracket$.

- $\tau \in \mathcal{I}(\llbracket \mathbf{C}_1 +_p \mathbf{C}_2 \rrbracket)$

By Definition 2.7 we have that $\llbracket \mathbf{C}_1 +_p \mathbf{C}_2 \rrbracket = p \cdot \llbracket \mathbf{C}_1 \rrbracket + (1-p) \cdot \llbracket \mathbf{C}_2 \rrbracket$, hence $\tau \in \mathcal{I}(p \cdot \llbracket \mathbf{C}_1 \rrbracket + (1-p) \cdot \llbracket \mathbf{C}_2 \rrbracket)$. Let $l \in \mathcal{I}(\llbracket \mathbf{C}_1 \rrbracket)$. By Definition 2.7 and Lemma 2.21 we have that $v(\{\tau, l\}) = p \cdot v_1(\{l\}) = p$. It then follows directly that $\mathbf{C}_1 +_p \mathbf{C}_2 \rightarrow p \cdot (\tau, \mathbf{C}_1) + (1-p) \cdot (\tau, \mathbf{C}_2)$ and $\llbracket \mathbf{C}_1 +_p \mathbf{C}_2 \rrbracket = p \cdot \llbracket \mathbf{C}_1 \rrbracket + (1-p) \cdot \llbracket \mathbf{C}_2 \rrbracket$. Similarly we do the same when $l \in \mathcal{I}(\llbracket \mathbf{C}_2 \rrbracket)$.

- $l' \in \mathcal{I}(\llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket)$

We have two cases:

1. $l' \neq \tau$

By Definition 2.5 we have that $l \in \mathcal{I}(\llbracket \mathbf{C}_1 \rrbracket)$. By i.h., $\exists \mathbf{C}'$ such that $\mathbf{C}_1 \rightarrow 1 \cdot (l, \mathbf{C}')$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l \equiv \llbracket \mathbf{C}' \rrbracket$. We have two cases:

- (a) $\mathbf{C}' = \checkmark$

We have $\mathbf{C}_1 \rightarrow 1 \cdot (l, \checkmark)$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l \equiv \llbracket \checkmark \rrbracket$. By the rules in Figure 11, $\mathbf{C}_1 ; \mathbf{C}_2 \rightarrow 1 \cdot (l, \mathbf{C}_2)$. By Definition 2.5, $(\llbracket \mathbf{C}_1 \rrbracket \setminus l) ; \llbracket \mathbf{C}_2 \rrbracket \equiv \llbracket \checkmark \rrbracket ; \llbracket \mathbf{C}_2 \rrbracket = \llbracket \mathbf{C}_2 \rrbracket$.

- (b) $\mathbf{C}' = \mathbf{C}'_1$

We have $\mathbf{C}_1 \xrightarrow{l} \mathbf{C}'_1$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l \equiv \llbracket \mathbf{C}'_1 \rrbracket$. By the rules in Figure 11, $\mathbf{C}_1 ; \mathbf{C}_2 \rightarrow 1 \cdot (l, \mathbf{C}'_1 ; \mathbf{C}_2)$. By Definition 2.5, $(\llbracket \mathbf{C}_1 \rrbracket \setminus l) ; \llbracket \mathbf{C}_2 \rrbracket \equiv \llbracket \mathbf{C}'_1 \rrbracket ; \llbracket \mathbf{C}_2 \rrbracket$. By Definition 2.11, $\llbracket \mathbf{C}'_1 ; \mathbf{C}_2 \rrbracket$.

2. $l' = \tau$

We have $\tau \in \mathcal{I}(\llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket)$, which by Definition 2.5 gives us that $\tau \in \mathcal{I}(\llbracket \mathbf{C}_1 \rrbracket)$. By i.h., $\exists \mathbf{C}', \mathbf{C}''$ such that $\mathbf{C}_1 \rightarrow p \cdot (\tau, \mathbf{C}') + (1-p) \cdot (\tau, \mathbf{C}'')$ with $p = v(\{\tau, e'\})$ and $e' \in \mathcal{I}(\mathbf{C}')$, and $\llbracket \mathbf{C}_1 \rrbracket \sqsubseteq p \cdot \llbracket \mathbf{C}' \rrbracket + (1-p) \cdot \llbracket \mathbf{C}'' \rrbracket$. By the rules in Figure 11, we have $\mathbf{C}_1 ; \mathbf{C}_2 \rightarrow p \cdot (\tau, \mathbf{C}' ; \mathbf{C}_2) + (1-p) \cdot (\tau, \mathbf{C}'' ; \mathbf{C}_2)$. By Lemma 2.15, $\llbracket \mathbf{C}_1 \rrbracket ; \llbracket \mathbf{C}_2 \rrbracket \sqsubseteq (p \cdot \llbracket \mathbf{C}' \rrbracket + (1-p) \cdot \llbracket \mathbf{C}'' \rrbracket) ; \llbracket \mathbf{C}_2 \rrbracket$. By Lemma 2.22 and Definition 2.11, $\llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket \sqsubseteq p \cdot \llbracket \mathbf{C}' ; \mathbf{C}_2 \rrbracket + (1-p) \cdot \llbracket \mathbf{C}'' ; \mathbf{C}_2 \rrbracket$.

- $l \in \mathcal{I}(\llbracket \mathbf{C}_1 \rrbracket \parallel \llbracket \mathbf{C}_2 \rrbracket)$

We have two cases:

1. $l' \neq \tau$

By Definition 2.9 we have two cases:

(a) $l \in \mathcal{I}(\llbracket \mathbf{C}_1 \rrbracket)$

By i.h. $\exists \mathbf{C}' . \mathbf{C}_1 \rightarrow 1 \cdot (l, \mathbf{C}')$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l \equiv \llbracket \mathbf{C}' \rrbracket$. By the rules in Figure 11 we have two cases:

i. $\mathbf{C}' = \checkmark$

We have $\mathbf{C}_1 \rightarrow 1 \cdot (l, \checkmark)$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l \equiv \llbracket \checkmark \rrbracket$. By the rules in Figure 3 we have $\mathbf{C}_1 \parallel \mathbf{C}_2 \rightarrow 1 \cdot (l, \mathbf{C}_2)$. By Definition 2.9, $(\llbracket \mathbf{C}_1 \rrbracket \setminus l) \parallel \llbracket \mathbf{C}_2 \rrbracket$. By Lemma 2.19 we have $(\llbracket \mathbf{C}_1 \rrbracket \parallel \llbracket \mathbf{C}_2 \rrbracket) \setminus l$. By Definition 2.11, $\llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket \setminus l$.

ii. $\mathbf{C}' = \mathbf{C}'_1$

We have $\mathbf{C}_1 \rightarrow 1 \cdot (l, \mathbf{C}'_1)$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l \equiv \llbracket \mathbf{C}'_1 \rrbracket$. By the rules in Figure 11 we have $\mathbf{C}_1 \parallel \mathbf{C}_2 \rightarrow 1 \cdot (l, \mathbf{C}'_1 \parallel \mathbf{C}_2)$. By Definition 2.9, $(\llbracket \mathbf{C}_1 \rrbracket \setminus l) \parallel \llbracket \mathbf{C}_2 \rrbracket$. By Lemma 2.19 we have $(\llbracket \mathbf{C}_1 \rrbracket \parallel \llbracket \mathbf{C}_2 \rrbracket) \setminus l$. By Definition 2.11, $\llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket \setminus l$.

(b) $l \in \mathcal{I}(\llbracket \mathbf{C}_2 \rrbracket)$

By i.h. $\exists \mathbf{C}' . \mathbf{C}_2 \rightarrow 1 \cdot (l, \mathbf{C}')$ and $\llbracket \mathbf{C}_2 \rrbracket \setminus l \equiv \llbracket \mathbf{C}' \rrbracket$. By the rules in Figure 11 we have two cases:

i. $\mathbf{C}' = \checkmark$

We have $\mathbf{C}_2 \rightarrow 1 \cdot (l, \checkmark)$ and $\llbracket \mathbf{C}_2 \rrbracket \setminus l \equiv \llbracket \checkmark \rrbracket$. By the rules in Figure 11 we have $\mathbf{C}_1 \parallel \mathbf{C}_2 \rightarrow 1 \cdot (l, \mathbf{C}_1)$. By Definition 2.9, $\llbracket \mathbf{C}_1 \rrbracket \parallel (\llbracket \mathbf{C}_2 \rrbracket \setminus l)$. By Lemma 2.19 we have $(\llbracket \mathbf{C}_1 \rrbracket \parallel \llbracket \mathbf{C}_2 \rrbracket) \setminus l$. By Definition 2.11, $\llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket \setminus l$.

ii. $\mathbf{C}' = \mathbf{C}'_2$

We have $\mathbf{C}_2 \rightarrow 1 \cdot (l, \mathbf{C}'_2)$ and $\llbracket \mathbf{C}_2 \rrbracket \setminus l \equiv \llbracket \mathbf{C}'_2 \rrbracket$. By the rules in Figure 11 we have $\mathbf{C}_1 \parallel \mathbf{C}_2 \rightarrow 1 \cdot (l, \mathbf{C}_1 \parallel \mathbf{C}'_2)$. By Definition 2.9, $\llbracket \mathbf{C}_1 \rrbracket \parallel (\llbracket \mathbf{C}_2 \rrbracket \setminus l)$. By Lemma 2.19 we have $(\llbracket \mathbf{C}_1 \rrbracket \parallel \llbracket \mathbf{C}_2 \rrbracket) \setminus l$. By Definition 2.11, $\llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket \setminus l$.

2. $l' = \tau$

We have $\tau \in \mathcal{I}(\llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket)$, which by Definition 2.9 entails $\tau \in \mathcal{I}(\mathbf{C}_1)$ or $\tau \in \mathcal{I}(\mathbf{C}_2)$. We have two cases:

(a) $\tau \in \mathcal{I}(\mathbf{C}_1)$

By i.h., $\exists \mathbf{C}', \mathbf{C}''$ such that $\mathbf{C}_1 \rightarrow p \cdot (\tau, \mathbf{C}') + (1-p) \cdot (\tau, \mathbf{C}'')$ with $p = v(\{\tau, e'\})$ and $e' \in \mathcal{I}(\mathbf{C}')$, and $\llbracket \mathbf{C}_1 \rrbracket \sqsubseteq p \cdot \llbracket \mathbf{C}' \rrbracket + (1-p) \cdot \llbracket \mathbf{C}'' \rrbracket$. By the rules in Figure 11, we have $\mathbf{C}_1 \parallel \mathbf{C}_2 \rightarrow p \cdot (\tau, \mathbf{C}' \parallel \mathbf{C}_2) + (1-p) \cdot (\tau, \mathbf{C}'' \parallel \mathbf{C}_2)$. By Lemma 2.17, $\llbracket \mathbf{C}_1 \rrbracket \parallel \llbracket \mathbf{C}_2 \rrbracket \sqsubseteq (p \cdot \llbracket \mathbf{C}' \rrbracket + (1-p) \cdot \llbracket \mathbf{C}'' \rrbracket) \parallel \llbracket \mathbf{C}_2 \rrbracket$. By Lemma 2.23 and Definition 2.11, $\llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket \sqsubseteq p \cdot \llbracket \mathbf{C}' \parallel \mathbf{C}_2 \rrbracket + (1-p) \cdot \llbracket \mathbf{C}'' \parallel \mathbf{C}_2 \rrbracket$.

(b) $\tau \in \mathcal{I}(\mathbf{C}_2)$

By i.h., $\exists \mathbf{C}', \mathbf{C}''$ such that $\mathbf{C}_2 \rightarrow p \cdot (\tau, \mathbf{C}') + (1-p) \cdot (\tau, \mathbf{C}'')$ with $p = v(\{\tau, e'\})$ and $e' \in \mathcal{I}(\mathbf{C}')$, and $\llbracket \mathbf{C}_2 \rrbracket \sqsubseteq p \cdot \llbracket \mathbf{C}' \rrbracket + (1-p) \cdot \llbracket \mathbf{C}'' \rrbracket$. By the rules in Figure 11, we have $\mathbf{C}_1 \parallel \mathbf{C}_2 \rightarrow p \cdot (\tau, \mathbf{C}_1 \parallel \mathbf{C}') + (1-p) \cdot (\tau, \mathbf{C}_1 \parallel \mathbf{C}'')$. By Lemma 2.17, $\llbracket \mathbf{C}_1 \rrbracket \parallel \llbracket \mathbf{C}_2 \rrbracket \sqsubseteq \mathbf{C}_1 \parallel (p \cdot \llbracket \mathbf{C}' \rrbracket + (1-p) \cdot \llbracket \mathbf{C}'' \rrbracket)$. By Lemma 2.23 and Definition 2.11, $\llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket \sqsubseteq p \cdot \llbracket \mathbf{C}_1 \parallel \mathbf{C}' \rrbracket + (1-p) \cdot \llbracket \mathbf{C}_1 \parallel \mathbf{C}'' \rrbracket$.

□

Lemma 2.28 (Adequacy II). $\forall x \in \mathcal{C}_{\max}(\llbracket \mathbf{C} \rrbracket) . \emptyset \xrightarrow{\omega_x} x$ then $\mathbf{C} \rightarrow \sum_x v(x)(\omega_x, \checkmark)$.

Proof. • $\{sk\} \in \mathcal{C}_{\max}(\llbracket \mathbf{skip} \rrbracket) . \emptyset \xrightarrow{sk} \{sk\}$

We know that $v(\{sk\}) = 1$. It then follows directly that $\mathbf{skip} \rightarrow 1 \cdot (sk, \checkmark)$.

- $\{a\} \in \mathcal{C}_{\max}(\llbracket \mathbf{a} \rrbracket) . \emptyset \xrightarrow{a} \{a\}$

We know that $v(\{a\}) = 1$. It then follows directly that $\mathbf{a} \rightarrow 1 \cdot (a, \checkmark)$.

- $x \in \mathcal{C}_{\max}(\llbracket \mathbf{C}_1 +_p \mathbf{C}_2 \rrbracket) . \emptyset \xrightarrow{\omega_x} x$

By Definition 2.7 we have $\emptyset \xrightarrow{\tau} \{\tau\} \xrightarrow{\omega'_x} x$ since $\tau \in \mathcal{I}(\llbracket \mathbf{C}_1 +_p \mathbf{C}_2 \rrbracket)$. By Lemma 2.27, $\mathbf{C}_1 +_p \mathbf{C}_2$ with $p = v(\{\tau, e_1\})$ where $e_i \in \mathcal{I}(\llbracket \mathbf{C}_i \rrbracket)$ and $\llbracket \mathbf{C}_1 +_p \mathbf{C}_2 \rrbracket = p \cdot \llbracket \mathbf{C}_1 \rrbracket + (1-p) \cdot \llbracket \mathbf{C}_2 \rrbracket$ (the equality comes from Definition 2.11). We then have $x \in \mathcal{C}_{\max}(p \cdot \llbracket \mathbf{C}_1 \rrbracket + (1-p) \cdot \llbracket \mathbf{C}_2 \rrbracket)$. By Definition 2.7 we have $x \setminus \tau \in \mathcal{C}_{\max}(\llbracket \mathbf{C}_1 \rrbracket)$ or

$x \setminus \tau \in \mathcal{C}_{\max}(\llbracket \mathcal{C}_2 \rrbracket)$. Let $x_1 = x \setminus \tau$ for the former and $x_2 = x \setminus \tau$ for the latter. By i.h. $\mathcal{C}_1 \rightarrow \sum_{x_1} v(x_1)(\omega_{x_1}, \checkmark)$ and $\mathcal{C}_2 \rightarrow \sum_{x_2} v(x_2)(\omega_{x_2}, \checkmark)$. By the rules in Figure 12, we obtain

$$\mathcal{C}_1 +_p \mathcal{C}_2 \rightarrow p \cdot \sum_{x_1} v(x_1)(\tau : \omega_{x_1}, \checkmark) + (1-p) \cdot \sum_{x_2} v(x_2)(\tau : \omega_{x_2}, \checkmark)$$

- $x \in \mathcal{C}_{\max}(\llbracket \mathcal{C}_1 ; \mathcal{C}_2 \rrbracket) \cdot \emptyset \xrightarrow{\omega_x} x$

We know that $\omega_x = l_0 l_1 \dots l_n$. Hence $\emptyset \xrightarrow{l_0} \emptyset \cdot \{l_0\} \xrightarrow{\omega'_x} x$. We then have $l_0 \in \mathcal{I}(\llbracket \mathcal{C}_1 ; \mathcal{C}_2 \rrbracket)$. By Lemma 2.27 we have two cases:

1. $l_0 \neq \tau$

Hence $\mathcal{C}_1 ; \mathcal{C}_2 \rightarrow 1 \cdot (l_0, \mathcal{C}')$ and $\llbracket \mathcal{C}_1 ; \mathcal{C}_2 \rrbracket \setminus l_0 \equiv \llbracket \mathcal{C}' \rrbracket$, such that $\mathcal{C}' = \mathcal{C}_2$ or $\mathcal{C}' = \mathcal{C}'_1 ; \mathcal{C}_2$. By Definition 2.13 we have $x \setminus l_0 \in \mathcal{C}_{\max}(\llbracket \mathcal{C}' \rrbracket)$ such that $\emptyset \xrightarrow{\omega'_x} x \setminus l_0$. By i.h. $\mathcal{C}' \rightarrow \sum_{x \setminus l_0} v(x \setminus l_0)(\omega'_x, \checkmark)$. By the rules in Figure 12

$$\mathcal{C}_1 ; \mathcal{C}_2 \rightarrow \sum_{x \setminus l_0} v(x \setminus l_0)(\omega_x, \checkmark)$$

2. $l_0 = \tau$

Hence $\mathcal{C}_1 ; \mathcal{C}_2 \rightarrow p \cdot (\tau, \mathcal{C}') + (1-p) \cdot (\tau, \mathcal{C}'')$ with $p = v(\{\tau, e'\})$ where $e' \in \mathcal{I}(\llbracket \mathcal{C}' \rrbracket)$ and $\llbracket \mathcal{C}_1 ; \mathcal{C}_2 \rrbracket \sqsubseteq p \cdot \llbracket \mathcal{C}' ; \mathcal{C}_2 \rrbracket + (1-p) \cdot \llbracket \mathcal{C}'' ; \mathcal{C}_2 \rrbracket$. By Lemma 2.24 we have $x \in \mathcal{C}_{\max}(p \cdot \llbracket \mathcal{C}' ; \mathcal{C}_2 \rrbracket + (1-p) \cdot \llbracket \mathcal{C}'' ; \mathcal{C}_2 \rrbracket)$. By Definition 2.7 we know that $x \setminus \tau \in \mathcal{C}_{\max}(\mathcal{C}' ; \mathcal{C}_2)$ or $x \setminus \tau \in \mathcal{C}_{\max}(\mathcal{C}'' ; \mathcal{C}_2)$. Let $x' = x \setminus \tau$ for the former and $x'' = x \setminus \tau$ for the latter. By i.h., $\mathcal{C}' ; \mathcal{C}_2 \rightarrow \sum_{x'} v(x')(\omega_{x'}, \checkmark)$ and $\mathcal{C}'' ; \mathcal{C}_2 \rightarrow \sum_{x''} v(x'')(\omega_{x''}, \checkmark)$. By the rules in Figure 12

$$\mathcal{C}_1 ; \mathcal{C}_2 \rightarrow p \cdot \sum_{x'} v(x')(\tau : \omega'_{x'}, \checkmark) + (1-p) \cdot \sum_{x''} v(x'')(\tau : \omega'_{x''}, \checkmark)$$

- $x \in \mathcal{C}_{\max}(\llbracket \mathcal{C}_1 \parallel \mathcal{C}_2 \rrbracket) \cdot \emptyset \xrightarrow{\omega_x} x$

We know that $\omega_x = l_0 l_1 \dots l_n$. Hence $\emptyset \xrightarrow{l_0} \emptyset \cdot \{l_0\} \xrightarrow{\omega'_x} x$. We then have $l_0 \in \mathcal{I}(\llbracket \mathcal{C}_1 \parallel \mathcal{C}_2 \rrbracket)$. By Lemma 2.27 we have two cases (where we focus on the cases where $l_0 \in \mathcal{I}(\mathcal{C}_1)$):

1. $l_0 \neq \tau$

Hence $\mathcal{C}_1 \parallel \mathcal{C}_2 \rightarrow 1 \cdot (l_0, \mathcal{C}')$ and $\llbracket \mathcal{C}_1 \parallel \mathcal{C}_2 \rrbracket \setminus l_0 \equiv \llbracket \mathcal{C}' \rrbracket$, such that $\mathcal{C}' = \mathcal{C}_2$ or $\mathcal{C}' = \mathcal{C}'_1 \parallel \mathcal{C}_2$. By Definition 2.13 we have $x \setminus l_0 \in \mathcal{C}_{\max}(\llbracket \mathcal{C}' \rrbracket)$ such that $\emptyset \xrightarrow{\omega'_x} x \setminus l_0$. By i.h. $\mathcal{C}' \rightarrow \sum_{x \setminus l_0} v(x \setminus l_0)(\omega'_x, \checkmark)$. By the rules in Figure 12

$$\mathcal{C}_1 \parallel \mathcal{C}_2 \rightarrow \sum_{x \setminus l_0} v(x \setminus l_0)(\omega_x, \checkmark)$$

2. $l_0 = \tau$

Hence $\mathcal{C}_1 \parallel \mathcal{C}_2 \rightarrow p \cdot (\tau, \mathcal{C}' \parallel \mathcal{C}_2) + (1-p) \cdot (\tau, \mathcal{C}'' \parallel \mathcal{C}_2)$ with $p = v(\{\tau, e'\})$ where $e' \in \mathcal{I}(\llbracket \mathcal{C}' \rrbracket)$ and $\llbracket \mathcal{C}_1 \parallel \mathcal{C}_2 \rrbracket \sqsubseteq p \cdot \llbracket \mathcal{C}' \parallel \mathcal{C}_2 \rrbracket + (1-p) \cdot \llbracket \mathcal{C}'' \parallel \mathcal{C}_2 \rrbracket$. By Lemma 2.24 we have $x \in \mathcal{C}_{\max}(p \cdot \llbracket \mathcal{C}' \parallel \mathcal{C}_2 \rrbracket + (1-p) \cdot \llbracket \mathcal{C}'' \parallel \mathcal{C}_2 \rrbracket)$. By Definition 2.7 we know that $x \setminus \tau \in \mathcal{C}_{\max}(\mathcal{C}' \parallel \mathcal{C}_2)$ or $x \setminus \tau \in \mathcal{C}_{\max}(\mathcal{C}'' \parallel \mathcal{C}_2)$. Let $x' = x \setminus \tau$ for the former and $x'' = x \setminus \tau$ for the latter. By i.h., $\mathcal{C}' \parallel \mathcal{C}_2 \rightarrow \sum_{x'} v(x')(\omega_{x'}, \checkmark)$ and $\mathcal{C}'' \parallel \mathcal{C}_2 \rightarrow \sum_{x''} v(x'')(\omega_{x''}, \checkmark)$. By the rules in Figure 12

$$\mathcal{C}_1 \parallel \mathcal{C}_2 \rightarrow p \cdot \sum_{x'} v(x')(\tau : \omega'_{x'}, \checkmark) + (1-p) \cdot \sum_{x''} v(x'')(\tau : \omega'_{x''}, \checkmark)$$

□

In Lemma 2.25 and Lemma 2.27 we see the usefulness of introducing the label τ . It helps us identifying the situations where a transition occurred due to the probabilistic command and when it did not.

Proposition 2.26 assures us that for every word originated by the n-step semantics we have a maximal configuration who matches the word and the respective probability. Proposition 2.28 tells us that for every maximal configuration of a command \mathcal{C} , the n-step reaches a distribution that whose probabilities are given by the valuations of the maximal configurations and the words are the respective covering chains.

2.4 Introducing cyclic behavior

We now introduce cyclic behavior to the language in Section 2.1. In order to avoid the introduction of the notion of state in the language, the cyclic behavior will be given by recursion. In that way, we do not need to associate the notion of state to a command in the operational semantics. We can just keep recording the actions that are being made by the program.

Another thing to have in mind is that with cyclic behavior we open the door to infinite computations. However, covering chains are only defined in finite sequence of words and infinite configurations are odd (according to Marc), because we would need to define precisely what it means to be an infinite configuration. Hence, the words that we formed with the n -step will be always finite, despite the possibility of them being infinite. We can justify this by saying that we are only concerned on the ‘interesting words’, *i.e.* those who are finite.

To introduce recursion we need to add some restrictions when forming programs, since we do not want to allow commands like: $\mu X.X; a$ and $\mu X.a; X; b$.

Let $X \subseteq Var$, with Var a set of variables. The syntax is now given by:

$$C ::= \text{skip} \mid a \in Act \mid C; C \mid C +_p C \mid C \parallel C \mid \mu X.C \mid X$$

We define the set of free-variables and bound-variables as follows:

$ \begin{aligned} FV(\text{skip}) &= \emptyset \\ FV(a) &= \emptyset \\ FV(C_1; C_2) &= FV(C_1) \cup FV(C_2) \\ FV(C_1 \parallel C_2) &= FV(C_1) \cup FV(C_2) \\ FV(C_1 +_p C_2) &= FV(C_1) \cup FV(C_2) \\ FV(X) &= \{X\} \\ FV(\mu X.C) &= FV(C) \setminus \{X\} \end{aligned} $	$ \begin{aligned} BV(\text{skip}) &= \emptyset \\ BV(a) &= \emptyset \\ BV(C_1; C_2) &= BV(C_1) \cup BV(C_2) \\ BV(C_1 \parallel C_2) &= BV(C_1) \cup BV(C_2) \\ BV(C_1 +_p C_2) &= BV(C_1) \cup BV(C_2) \\ BV(X) &= \emptyset \\ BV(\mu X.C) &= \{X\} \cup BV(C) \end{aligned} $
--	---

We restrict the sequential composition to those whose free-variables and bound-variables on the left are empty, *i.e.* $C_1; C_2$ if $FV(C_1) = \emptyset = BV(C_1)$. With this restriction we forbid program like $\mu X.X; a$, $\mu X.a; X; b$ (with the condition $FV(C_1) = \emptyset$) and $(\mu X.a; X); b$ (with the condition $BV(C_1) = \emptyset$). We want to forbid these kind of programs in sequential composition, because if C_1 never terminates then the sequential composition never terminates. This is also a restriction that comes from the fact that covering chains are only defined in finite sequences and that infinite configurations are odd in event structures. Note however that we allow programs like $\mu X.X \parallel a$ and $\mu X.X + a$, since they do not block the computation.

We add to Figure 11 the following rules for the recursion command:

$$\frac{C \rightarrow 1 \cdot (l, C')}{\mu X.C \rightarrow 1 \cdot (l, C'[X \leftarrow \mu X.C])} \quad \frac{C \rightarrow \sum_i p_i \cdot (\tau, C_i)}{\mu X.C \rightarrow \sum_i p_i \cdot (\tau, C_i[X \leftarrow \mu X.C])}$$

We define substitution as follows [HS08, inspired by]:

Definition 2.29. Let $X \in Var$ and C, C' be commands. Define $C[X \leftarrow C']$, where we substitute every free occurrence of X in C by C' (while changing bound variables to avoid clashes) by induction on C as follows:

$$\begin{aligned}
 \text{skip}[X \leftarrow C'] &= \text{skip} \\
 a[X \leftarrow C'] &= a \\
 (C_1; C_2)[X \leftarrow C'] &= C_1; (C_2[X \leftarrow C']) \\
 (C_1 \parallel C_2)[X \leftarrow C'] &= C_1[X \leftarrow C'] \parallel C_2[X \leftarrow C'] \\
 (C_1 +_p C_2)[X \leftarrow C'] &= C_1[X \leftarrow C'] +_p C_2[X \leftarrow C'] \\
 (\mu Y.C)[X \leftarrow C'] &= \mu Y.C[X \leftarrow C']
 \end{aligned}$$

Example 2.30. Figure 15 illustrates a probabilistic coin toss scenario where each time we toss the coin, it executes with probability p the command **skip** or continues the tossing with probability $1 - p$. To understand this behavior, focus on the initial command. From there, we transit to a distribution formed by the commands **skip** and $\mu X.(X +_p \text{skip})$, which is the same as the initial command. From this distribution we transit to **skip** with probability p or to $\mu X.(X +_p \text{skip})$ with probability $1 - p$, enabling us to repeat the process.

On the event structure side, we want to use the Knaster-Tarski Theorem to build the least-fix point. To define it, we will use an order that does not ignore copies, differently from what happens with Definition 2.12.

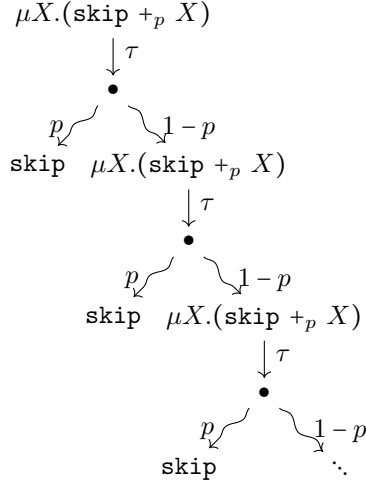


Figure 15: Fragment of the execution of $\mu X.(\text{skip} +_p X)$

Definition 2.31. Let $P_1 = (E_1, \leq_1, \#_1, v_1)$ and $P_2 = (E_2, \leq_2, \#_2, v_2)$ be event structures. Say $P_1 \trianglelefteq P_2$ if:

$$\begin{aligned}
 &E_1 \subseteq E_2 \\
 &\forall e, e' . e \leq_1 e' \Leftrightarrow e, e' \in E_1 \wedge e \leq_2 e' \\
 &\forall e, e' . e \#_1 e' \Leftrightarrow e, e' \in E_1 \wedge e \#_2 e' \\
 &\forall x \in \mathcal{C}(P)_1 . v_1(x) = v_2(x)
 \end{aligned}$$

Lemma 2.32. \trianglelefteq is a partial order.

Proof. Due to Lemma 1.39 we only need to check the condition of the valuations. Consider $P_1 = (E_1, v_1)$, $P_2 = (E_2, v_2)$, and $P_3 = (E_3, v_3)$ to be probabilistic event structures.

- Reflexivity: $P_1 = P_1$
We want to show that $\forall x \in \mathcal{C}(P_1) . v_1(x) = v_1(x)$. It holds straightforwardly.
- Transitivity: $P_1 \trianglelefteq P_2, P_2 \trianglelefteq P_3 \Rightarrow P_1 \trianglelefteq P_3$
We want to show $\forall x \in \mathcal{C}(P_1) . v_1(x) = v_3(x)$. From $P_1 \trianglelefteq P_2$, $\forall x \in \mathcal{C}(P_1) . v_1(x) = v_2(x)$. From $P_2 \trianglelefteq P_3$, $\forall x \in \mathcal{C}(P_2) . v_2(x) = v_3(x)$. Hence, $\forall x \in \mathcal{C}(P_1) . v_1(x) = v_3(x)$.
- Antisymmetry: $P_1 \trianglelefteq P_2, P_2 \trianglelefteq P_1 \Rightarrow P_1 = P_2$
We want to show $\forall x \in \mathcal{C}(P_1), \mathcal{C}(P_2) . v_1(x) = v_2(x)$. From $P_1 \trianglelefteq P_2$, $\forall x \in \mathcal{C}(P_1) . v_1(x) = v_2(x)$. From $P_2 \trianglelefteq P_1$, $\forall x \in \mathcal{C}(P_2) . v_2(x) = v_1(x)$. Hence, $\forall x \in \mathcal{C}(P_1), \mathcal{C}(P_2) . v_1(x) = v_2(x)$.

□

Lemma 2.33. Define $\perp = (\emptyset, \emptyset, \emptyset, v_\perp(\emptyset) = 1)$. \perp is the least element of \trianglelefteq .

Proof. We first show that \perp is a probabilistic event structure. From Lemma 1.40 $\perp = \emptyset\emptyset\emptyset$ is an event structure. It lacks to see the conditions on the valuations. It follows directly the definition that $v(\emptyset) = 1$. Furthermore the only configuration in $\mathcal{C}(\perp)$ is \emptyset . Hence we trivially have that $v(\emptyset) \geq 0$.

To show that \perp is the least element, consider any probabilistic event structure P . We need to show that $\perp \trianglelefteq P$. Due to Lemma 1.40 we focus solely on the valuations. Since the empty configuration is the only one in $\mathcal{C}(\emptyset)$ and since P is a probabilistic event structure it holds that $v_\perp(\emptyset) = 1 = v(\emptyset)$. □

Definition 2.34. Let $P_1 \trianglelefteq \dots \trianglelefteq P_n \trianglelefteq \dots$ be a ω -chain. Let $P^\omega = (E^\omega, \leq^\omega, \#^\omega, v^\omega)$ be its least upper bound where:

- $E^\omega = \cup_{n \in \omega} E_n$
- $\leq^\omega = \cup_{n \in \omega} \leq_n$
- $\#^\omega = \cup_{n \in \omega} \#_n$
- $\forall x \in \mathcal{C}(P^\omega), \exists n \in \omega . x \in \mathcal{C}(P_n) . v^\omega(x) = v_n(x)$

Lemma 2.35. P^ω is an event structure.

Proof. Due to Lemma 1.42 we focus only on the valuation part, where we have two conditions to verify:

- $v^\omega(\emptyset) = 1$

From Definition 2.34 we know that $\exists n \in \omega . v^\omega(\emptyset) = v_n(\emptyset) = 1$.

- $\forall y, x_1, \dots, x_m \in \mathcal{C}(P^\omega)$ such that $y \sqsubseteq x_1, \dots, x_m$, $v^\omega(y) - \sum_{\emptyset \neq I \subseteq \{1, \dots, m\}} (-1)^{|I|+1} v^\omega(\cup_{i \in I} x_i) \geq 0$

Following [Win14, Propostion 5] we only need to focus on $y \sqsubset x_1, \dots, x_m$. From Definition 2.34 we know it $\exists n \in \omega . v^\omega(y) = v_n(y)$. We then have three cases, depending if the events are in E_n , in E_{n+1} , or in both.

1. the events are in E_n

We know that $\sum_{\emptyset \neq I \subseteq \{1, \dots, m\}} (-1)^{|I|+1} v^\omega(\cup_{i \in I} x_i) = \sum_{\emptyset \neq I \subseteq \{1, \dots, m\}} (-1)^{|I|+1} v_n(\cup_{i \in I} x_i)$ and consequently $v_n(y) - \sum_{\emptyset \neq I \subseteq \{1, \dots, m\}} (-1)^{|I|+1} v_n(\cup_{i \in I} x_i) \geq 0$, since P_n is a probabilistic event structure

2. the events are in E_{n+1}

We know that $v_n(y) = v_{n+1}(y)$ since $P_n \trianglelefteq P_{n+1}$. Furthermore $\sum_{\emptyset \neq I \subseteq \{1, \dots, m\}} (-1)^{|I|+1} v^\omega(\cup_{i \in I} x_i) = \sum_{\emptyset \neq I \subseteq \{1, \dots, m\}} (-1)^{|I|+1} v_{n+1}(\cup_{i \in I} x_i)$ and consequently $v_{n+1}(y) - \sum_{\emptyset \neq I \subseteq \{1, \dots, m\}} (-1)^{|I|+1} v_{n+1}(\cup_{i \in I} x_i) \geq 0$, since P_{n+1} is a probabilistic event structure

3. the events are in both

Since $P_n \trianglelefteq P_{n+1}$ we know that $\forall x \in \mathcal{C}(P_n) . v_n(x) = v_{n+1}(x)$, which leads us to the previous case.

□

Lemma 2.36. Let $P_1 \trianglelefteq \dots \trianglelefteq P_n \trianglelefteq \dots$ be a ω -chain. Then P^ω is its least upper bound.

Proof. Due to Lemma 1.43 we focus only on the valuations.

- P^ω is an upper bound

$\forall n \in \omega$ we need to have $P_n \trianglelefteq P^\omega$. It follows directly from Definition 2.31 that $\forall n \in \omega$ we have $P_n \trianglelefteq P^\omega$ since by Definition 2.34, $\forall x \in \mathcal{C}(P^\omega)$, $\exists n \in \omega . v^\omega(x) = v_n(x)$.

- P^ω is the least upper bound

Let $P = (E, v)$ be an upper bound of the chain. We need to show that if $P_n \trianglelefteq P^\omega$ and $P_n \trianglelefteq P$ then $P^\omega \trianglelefteq P$. From $P_n \trianglelefteq P^\omega$, $\forall x \in \mathcal{C}(P_n) . v_n(x) = v^\omega(x)$. From Definition 2.34, $\forall x \in \mathcal{C}(P^\omega)$, $\exists n \in \omega . v^\omega(x) = v_n(x)$. From $P_n \trianglelefteq P$, $\forall x \in \mathcal{C}(P_n) . v_n(x) = v(x)$. Thus $\forall x \in \mathcal{C}(P^\omega)$, $\exists n \in \omega . v^\omega(x) = v_n(x) = v(x)$.

□

Lemma 2.37. Let P, P_1, P_2 be probabilistic event structures. If $P_1 \trianglelefteq P_2$ then $P ; P_1 \trianglelefteq P ; P_2$.

Proof. Due to Lemma 1.44 we only focus on the valuations. Let $P = (E, v), P_1 = (E_1, v_1), P_2 = (E_2, v_2), P ; P_1 = (E^1, v^1), P ; P_2 = (E^2, v^2)$. We want to show $\forall x \in \mathcal{C}(P ; P_1) . v^1(x) = v^2(x)$. According to Definition 2.5 we have two cases:

1. $x \in \mathcal{C}(P ; P_1)$ such that $x \in \mathcal{C}(P)$

Then we are done because $v^1(x) = v(x) = v^2(x)$.

2. $x \in \mathcal{C}(P ; P_1)$ such that $\exists y \in \mathcal{C}_{\max}(P), y' \in \mathcal{C}(P_1) . x = y \cup y'$

Then we have $v^1(x) = v(y) \cdot v_1(y')$. Since $P_1 \trianglelefteq P_2$, $\forall y' \in \mathcal{C}(P_1) . v_1(y') = v_2(y')$. Then $v(y) \cdot v_1(y') = v(y) \cdot v_2(y') = v^2(x)$. Hence $v^1(x) = v^2(x)$.

□

Lemma 2.38. Let P_1, P'_1, P_2, P'_2 be probabilistic event structures. If $P_1 \trianglelefteq P'_1$ and $P_2 \trianglelefteq P'_2$ then $P_1 \parallel P_2 \trianglelefteq P'_1 \parallel P'_2$.

Proof. Due to Lemma 1.45 we only focus on the valuations. Let $P_1 = (E_1, v_1), P_2 = (E_2, v_2), P'_1 = (E'_1, v'_1), P'_2 = (E'_2, v'_2), P_1 \parallel P_2 = (E, v), P'_1 \parallel P'_2 = (E', v')$. We want to show $\forall x \in \mathcal{C}(P_1 \parallel P_2) . v(x) = v'(x)$.

Let $x \in \mathcal{C}(P_1 \parallel P_2) . v(x) = v_1(x \cap E_1) \cdot v_2(x \cap E_2)$, such that $x_1 = x \cap E_1$ and $x_2 = x \cap E_2$. Since $P_1 \trianglelefteq P'_1$ and $P_2 \trianglelefteq P'_2$ then $\forall x_1 \in \mathcal{C}(P_1) . v_1(x_1) = v'_1(x_1)$ and $\forall x_2 \in \mathcal{C}(P_2) . v_2(x_2) = v'_2(x_2)$, respectively. Hence $v(x) = v_1(x_1) \cdot v_2(x_2) = v'_1(x_1) \cdot v'_2(x_2) = v'(x)$.

□

Lemma 2.39. Let P_1, P'_1, P_2, P'_2 be probabilistic event structures. If $P_1 \trianglelefteq P'_1$ and $P_2 \trianglelefteq P'_2$ then $P_1 +_p P_2 \trianglelefteq P'_1 +_p P'_2$.

Proof. Let $P_1 = (E_1, v_1), P'_1 = (E'_1, v'_1), P_2 = (E_2, v_2), P'_2 = (E'_2, v'_2), P_1 +_p P_2 = (E, v), P'_1 +_p P'_2 = (E', v')$.

The conditions to check are:

1. $E \subseteq E'$
2. $\forall e, e'. e \leq e' \Leftrightarrow e, e' \in E \wedge e \leq' e'$
3. $\forall e, e'. e \# e' \Leftrightarrow e, e' \in E \wedge e \# e'$
4. $\forall x \in \mathcal{C}(P_1 +_p P_2). v(x) \geq v'(x)$

The first three conditions follow directly from Definition 2.7. Hence we focus on the last one.

Let $x \in \mathcal{C}(P_1 +_p P_2)$. We have two cases:

1. $x \setminus \tau \in \mathcal{C}(P_1)$

It follows directly that $v(x) = v'(x)$, since $P_1 \trianglelefteq P'_1$ and $v(x) = p \cdot v_1(x \setminus \tau) = p \cdot v'_1(x \setminus \tau) = v'(x)$.

2. $x \setminus \tau \in \mathcal{C}(P_2)$

It follows directly that $v(x) = v'(x)$, since $P_2 \trianglelefteq P'_2$ and $v(x) = (1-p) \cdot v_2(x \setminus \tau) = (1-p) \cdot v'_2(x \setminus \tau) = v'(x)$. \square

Definition 1.47 and Lemma 1.48 are the same as in Section 1.4.

Lemma 2.40. $\sqcup_m(P; P_m) = P; \sqcup_m P_m$.

Proof. By Lemma 2.37, the sequential composition is monotone at right. Furthermore, showing that each event of $P; \sqcup_m P_m$ is an event of $\sqcup_m(P; P_m)$ is already done in Lemma 1.49. By Lemma 1.48 we are done. \square

Lemma 2.41. $\sqcup_{n,m}(P_n \parallel P_m) = \sqcup_n P_n \parallel \sqcup_m P_m$.

Proof. By Lemma 2.38, the parallel composition is monotone at right. Furthermore, showing that each event of $\sqcup_n P_n \parallel \sqcup_m P_m$ is an event of $\sqcup_{n,m}(P_n \parallel P_m)$ is already done in Lemma 1.50. By Lemma 1.48 we are done. \square

Lemma 2.42. $\sqcup_{n,m}(P_n +_p P_m) = \sqcup_n P_n +_p \sqcup_m P_m$.

Proof. By Lemma 2.39 we know that the probabilistic choice is monotone. It lacks to show that each event of $\sqcup_n P_n +_p \sqcup_m P_m$ is an event of $\sqcup_{n,m}(P_n +_p P_m)$.

Let $P_1 \trianglelefteq \dots \trianglelefteq P_n \trianglelefteq \dots$ and $P'_1 \trianglelefteq \dots \trianglelefteq P'_m \trianglelefteq \dots$ be ω -chains with least upper bound $\sqcup_n P_n$ and $\sqcup_m P_m$, respectively. Let e be an event of $\sqcup_n P_n +_p \sqcup_m P_m$. By Definition 2.7 we have three cases:

1. $e = \tau$

It follows directly from Definition 2.7 that τ is an event of $P_n +_p P_m$. Consequently it is an event of $\sqcup_{n,m}(P_n +_p P_m)$.

2. e is an event of $\sqcup_n P_n$

By Definition 2.34, $\exists n \in \omega. e$ is an event of P_n . By Definition 2.7, e is an event of $P_n +_p P_m$ and consequently it is an event of $\sqcup_{n,m}(P_n +_p P_m)$.

3. e is an event of $\sqcup_m P_m$

Similar to the previous point.

By Lemma 1.48 we are done. \square

Lemma 1.52 does not change.

Definition 2.43. Define an environment to be a function $\gamma : Var \rightarrow P$ from variables to probabilistic event structures. For a command C and an environment γ define $\llbracket C \rrbracket_\gamma$ as follows:

$$\begin{aligned}
\llbracket \text{skip} \rrbracket_\gamma &= (\{sk\}, \{sk \leq sk\}, \emptyset, v(\{sk\} = 1)) \\
\llbracket a \rrbracket_\gamma &= (\{a\}, \{a \leq a\}, \emptyset, v(\{a\} = 1)) \\
\llbracket C_1; C_2 \rrbracket_\gamma &= \llbracket C_1 \rrbracket_\gamma; \llbracket C_2 \rrbracket_\gamma \\
\llbracket C_1 +_p C_2 \rrbracket_\gamma &= \llbracket C_1 \rrbracket_\gamma +_p \llbracket C_2 \rrbracket_\gamma \\
\llbracket C_1 \parallel C_2 \rrbracket_\gamma &= \llbracket C_1 \rrbracket_\gamma \parallel \llbracket C_2 \rrbracket_\gamma \\
\llbracket X \rrbracket_\gamma &= \gamma(X) \\
\llbracket \mu X. C \rrbracket_\gamma &= fix(\Gamma^{C, \gamma})
\end{aligned}$$

where $\Gamma^{C, \gamma} : P \rightarrow P$ is given by $\Gamma^{C, \gamma}(P) = \llbracket C \rrbracket_{\gamma(X \leftarrow P)}$.

Remark 6. ‘Another way to see’ $\Gamma^{\mathbb{C},\gamma}$ is

$$\Gamma^{\mathbb{C},\gamma} := P \mapsto \Gamma^{\mathbb{C}}(\gamma(X_1), \gamma(X_2), \dots, \gamma(X_n), P)$$

where we make a connection with $FV(\mathbb{C}) = \{X_1, X_2, \dots, X_n, X\}$.

We now show that $\Gamma^{\mathbb{C},\gamma}$ is continuous. For that it is useful to know that *curry* and *fix* are continuous [AJ94].

Lemma 2.44. $\Gamma^{\mathbb{C},\gamma}$ is continuous.

Proof. We only do for the probabilistic choice, since for the remaining cases the prove is the same as in Lemma 1.54.

$$\begin{aligned} & \Gamma^{\mathbb{C}_1 +_p \mathbb{C}_2, \gamma}(\bigsqcup_n P_n) \\ &= \{\text{Definition 2.43}\} \\ & \llbracket \mathbb{C}_1 +_p \mathbb{C}_2 \rrbracket_{\gamma(X \leftarrow \bigsqcup_n P_n)} \\ &= \{\text{Definition 2.43}\} \\ & \llbracket \mathbb{C}_1 \rrbracket_{\gamma(X \leftarrow \bigsqcup_n P_n)} +_p \llbracket \mathbb{C}_2 \rrbracket_{\gamma(X \leftarrow \bigsqcup_n P_n)} \\ &= \{\text{Definition 2.43}\} \\ & \bigsqcup_n \Gamma^{\mathbb{C}_1, \gamma}(P_n) +_p \bigsqcup_n \Gamma^{\mathbb{C}_2, \gamma}(P_n) \\ &= \{\text{Lemma 2.42}\} \\ &= \bigsqcup_n (\Gamma^{\mathbb{C}_1, \gamma}(P_n) +_p \Gamma^{\mathbb{C}_2, \gamma}(P_n)) \\ &= \{\text{Definition 2.43}\} \\ &= \bigsqcup_n (\llbracket \mathbb{C}_1 \rrbracket_{\gamma(X \leftarrow P_n)} +_p \llbracket \mathbb{C}_2 \rrbracket_{\gamma(X \leftarrow P_n)}) \\ &= \{\text{Definition 2.43}\} \\ & \bigsqcup_n \llbracket \mathbb{C}_1 +_p \mathbb{C}_2 \rrbracket_{\gamma(X \leftarrow P_n)} \end{aligned}$$

□

Lemma 2.45. $\llbracket \mathbb{C}'[X \leftarrow \llbracket \mu X. \mathbb{C} \rrbracket_{\gamma}] \rrbracket_{\gamma} = \llbracket \mathbb{C}' \rrbracket_{\gamma(X \leftarrow \llbracket \mu X. \mathbb{C} \rrbracket_{\gamma})}$

Proof. We only show the probabilistic choice, since the proof for the other cases is in Lemma 1.55.

$$\begin{aligned} & \llbracket (\mathbb{C}_1 +_p \mathbb{C}_2)[X \leftarrow \llbracket \mu X. \mathbb{C} \rrbracket_{\gamma}] \rrbracket_{\gamma} \\ &= \{\text{Definition 2.29}\} \\ & \llbracket \mathbb{C}_1[X \leftarrow \llbracket \mu X. \mathbb{C} \rrbracket_{\gamma}] +_p \mathbb{C}_2[X \leftarrow \llbracket \mu X. \mathbb{C} \rrbracket_{\gamma}] \rrbracket_{\gamma} \\ &= \{\text{Definition 2.43}\} \\ & \llbracket \mathbb{C}_1 \rrbracket_{\gamma(X \leftarrow \llbracket \mu X. \mathbb{C} \rrbracket_{\gamma})} +_p \llbracket \mathbb{C}_2 \rrbracket_{\gamma(X \leftarrow \llbracket \mu X. \mathbb{C} \rrbracket_{\gamma})} \\ &= \{\text{i.h.}\} \\ & \llbracket \mathbb{C}_1 \rrbracket_{\gamma(X \leftarrow \llbracket \mu X. \mathbb{C} \rrbracket_{\gamma})} +_p \llbracket \mathbb{C}_2 \rrbracket_{\gamma(X \leftarrow \llbracket \mu X. \mathbb{C} \rrbracket_{\gamma})} \\ &= \{\text{Definition 2.43}\} \\ & \llbracket \mathbb{C}_1 +_p \mathbb{C}_2 \rrbracket_{\gamma(X \leftarrow \llbracket \mu X. \mathbb{C} \rrbracket_{\gamma})} \end{aligned}$$

□

Lemma 1.56 is the same.

Lemma 2.46. If $\mu X. \mathbb{C} \rightarrow \sum_i p_i \cdot (\tau, \mathbb{C}_i[X \leftarrow \mu X. \mathbb{C}])$ then $x \in \mathcal{C}_{\max}(\llbracket \mu X. \mathbb{C} \rrbracket_{\gamma})$ and $x \in \mathcal{C}_{\max}(\sum_i p_i \cdot \llbracket \mathbb{C}_i[X \leftarrow \mu X. \mathbb{C}] \rrbracket_{\gamma})$ such that $\exists \llbracket \mathbb{C}_i \rrbracket_{\gamma} \cdot v(x) = v_i(x)$.

Proof.

$$\begin{aligned}
& \mu X.C \rightarrow \sum_i p_i \cdot (\tau, \mathsf{C}_i[X \leftarrow \mu X.C]) \\
& \Rightarrow \{\text{rules in Figure 11}\} \\
& C \rightarrow \sum_i p_i(\tau, \mathsf{C}_i) \\
& \Rightarrow \{\text{i.h.}\} \\
& x \in \mathcal{C}_{\max}(\llbracket C \rrbracket_\gamma) \text{ and } x \in \mathcal{C}_{\max}(\sum_i p_i \cdot \llbracket \mathsf{C}_i \rrbracket_\gamma) \text{ s.t. } \exists \llbracket \mathsf{C}_i \rrbracket_\gamma \cdot v_i(x) = v(x) \\
& \Rightarrow \{\gamma = \gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)\} \\
& x \in \mathcal{C}_{\max}(\llbracket C \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)}) \text{ and } x \in \mathcal{C}_{\max}(\sum_i p_i \cdot \llbracket \mathsf{C}_i \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)}) \\
& \Rightarrow \{\text{Lemma 1.56 and Lemma 1.55}\} \\
& x \in \mathcal{C}_{\max}(\mu X.C)_\gamma \text{ and } x \in \mathcal{C}_{\max}(\sum_i p_i \llbracket \mathsf{C}_i[X \leftarrow \llbracket \mu X.C \rrbracket_\gamma] \rrbracket_\gamma)
\end{aligned}$$

□

To show the equivalence between the operational and the denotational semantics, we reuse what was done in Section 1.3. Furthermore, we only show the proof for the recursion case, since the remaining cases are very similar.

Lemma 2.47 (Soundness I). • If $C \rightarrow 1 \cdot (l, C')$ then $\llbracket C' \rrbracket_\gamma = \llbracket C \rrbracket_\gamma \setminus l$

• If $C \rightarrow \sum_i p_i(\tau, \mathsf{C}_i)$ then $\llbracket C \rrbracket_\gamma \sqsubseteq \sum_i p_i \llbracket \mathsf{C}_i \rrbracket_\gamma$

Proof. •

$$\begin{aligned}
& \mu X.C \rightarrow 1 \cdot (l, C'[x \leftarrow \mu X.C]) \\
& \Rightarrow \{\text{Figure 11}\} \\
& C \rightarrow 1 \cdot (l, C') \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket C \rrbracket_\gamma \setminus l \equiv \llbracket C' \rrbracket_\gamma \\
& \Rightarrow \{\gamma = \gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)\} \\
& \llbracket C \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)} \setminus l \equiv \llbracket C' \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)} \\
& \Rightarrow \{\text{Lemma 1.56, Lemma 2.45}\} \\
& \llbracket \mu X.C \rrbracket_\gamma \setminus l \equiv \llbracket C'[X \leftarrow \llbracket \mu X.C \rrbracket_\gamma] \rrbracket_\gamma
\end{aligned}$$

•

$$\begin{aligned}
& \mu X.C \rightarrow \sum_i p_i \cdot (\tau, \mathsf{C}_i[x \leftarrow \mu X.C]) \\
& \Rightarrow \{\text{Figure 11}\} \\
& C \rightarrow \sum_i p_i \cdot (\tau, \mathsf{C}_i) \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket C \rrbracket_\gamma \sqsubseteq \sum_i p_i \llbracket \mathsf{C}_i \rrbracket_\gamma \\
& \Rightarrow \{\gamma = \gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)\} \\
& \llbracket C \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)} \sqsubseteq \sum_i p_i \cdot \llbracket \mathsf{C}_i \rrbracket_{\gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)} \\
& \Rightarrow \{\text{Lemma 1.56, Lemma 2.45}\} \\
& \llbracket \mu X.C \rrbracket_\gamma \sqsubseteq \sum_i p_i \llbracket \mathsf{C}_i[X \leftarrow \llbracket \mu X.C \rrbracket_\gamma] \rrbracket_\gamma
\end{aligned}$$

□

Lemma 2.48 (Soundness II). If $C \twoheadrightarrow \sum_i p_i (\sum_j p_j (\omega_{ij}, \checkmark))$ then $\forall_{i,j}, \exists x_{ij} \in \mathcal{C}_{\max}(\llbracket C \rrbracket_\gamma) \cdot \emptyset \xrightarrow{\omega_{ij}} x_{ij}$ and $p_{ij} = v(x_{ij})$.

Proof. •

$$\begin{aligned}
& \mu X.C \rightarrow \sum_j p_j(\omega_j, \checkmark) \\
& \Rightarrow \{\text{Figure 12}\} \\
& \mu X.C \rightarrow 1 \cdot (l, C'[X \leftarrow \mu X.C]) \\
& C'[X \leftarrow \mu X.C] \rightarrow \sum_j p_j(\omega_j, \checkmark) \\
& \Rightarrow \{\text{Lemma 2.47, i.h.}\} \\
& \llbracket \mu X.C \rrbracket_\gamma \setminus l \equiv \llbracket C'[X \leftarrow \mu X.C] \rrbracket_\gamma \\
& \forall j \exists x_j \in \mathcal{C}_{\max}(\llbracket C'[X \leftarrow \mu X.C] \rrbracket_\gamma) \cdot \emptyset \xrightarrow{\omega_j} c x_j \text{ and } p_j = v'(x_j) \\
& \Rightarrow \{\text{Definition 2.13}\} \\
& \forall j \exists \{l\} \cup x_j \in \mathcal{C}_{\max}(\llbracket \mu X.C \rrbracket_\gamma) \cdot \emptyset \xrightarrow{l} c \{l\} \xrightarrow{\omega_j} c \{l\} \cup x_j \text{ and } p_j = v(\{l\} \cup x_j)
\end{aligned}$$

•

$$\begin{aligned}
& \mu X.C \rightarrow \sum_i p_i(\sum_j p_j(\omega_{ij}, \checkmark)) \\
& \Rightarrow \{\text{Figure 12}\} \\
& \mu X.C \rightarrow \sum_i p_i \cdot (\tau, C_i[X \leftarrow \mu X.C]) \\
& \forall i \ C_i[X \leftarrow \mu X.C] \rightarrow \sum_j p_j(\omega_{ij}, \checkmark) \\
& \Rightarrow \{\text{Lemma 2.47, i.h.}\} \\
& \llbracket \mu X.C \rrbracket_\gamma \sqsubseteq \sum_i p_i \llbracket C_i[X \leftarrow \mu X.C] \rrbracket_\gamma \\
& \forall i, j \exists x_{ij} \in \mathcal{C}_{\max}(\llbracket C_i[X \leftarrow \mu X.C] \rrbracket_\gamma) \cdot \emptyset \xrightarrow{\omega_{ij}} c x_{ij} \text{ and } p_{ij} = v(x_{ij}) \\
& \Rightarrow \{\text{Lemma 2.46}\} \\
& \forall i, j \exists x_{ij} \in \mathcal{C}_{\max}(\llbracket \mu X.C \rrbracket_\gamma) \cdot \emptyset \xrightarrow{\omega_{ij}} c \{l\} \cup x_{ij} \text{ and } p_{ij} = v(x_{ij})
\end{aligned}$$

□

Lemma 2.49 (Adequacy I). Let $l' \in \mathcal{I}(\llbracket C \rrbracket_\gamma)$.

1. If $l' \neq \tau$ then $\exists C' \in (C \cup \{\checkmark\}) \cdot C \rightarrow 1 \cdot (l', C')$ and $\llbracket C \rrbracket_\gamma \setminus l' \equiv \llbracket C' \rrbracket_\gamma$.
2. If $l' = \tau$ then $\exists C', C'', \exists e' \in \mathcal{I}(\llbracket C' \rrbracket_\gamma) \cdot C \rightarrow p \cdot C' + (1-p) \cdot C''$ and $\llbracket C \rrbracket_\gamma \sqsubseteq p \cdot \llbracket C' \rrbracket_\gamma + (1-p) \cdot \llbracket C'' \rrbracket_\gamma$, with $p = v(\{\tau, e'\})$.

Proof. • $l' \neq \tau \in \mathcal{I}(\llbracket \mu X.C \rrbracket_\gamma)$

By Definition 2.43 and Definition 2.34, $l' \in \mathcal{I}(\llbracket C \rrbracket_\gamma)$. By i.h., $\exists C'$ such that $C \rightarrow 1 \cdot (l', C')$ and $\llbracket C \rrbracket_\gamma \setminus l' \equiv \llbracket C' \rrbracket_\gamma$. By Figure 11 and by letting $\gamma = \gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)$ and Lemma 1.56 and Lemma 2.45, $\mu X.C \rightarrow 1 \cdot (l', C'[X \leftarrow \mu X.C])$ and $\llbracket \mu X.C \rrbracket_\gamma \setminus l' \equiv \llbracket C'[X \leftarrow \mu X.C] \rrbracket_\gamma$.

- $l' = \tau \in \mathcal{I}(\llbracket \mu X.C \rrbracket_\gamma)$

By Definition 2.43 and Definition 2.34, $l' \in \mathcal{I}(\llbracket C \rrbracket_\gamma)$. By i.h. $\exists C', C'', \exists e \in \mathcal{I}(\llbracket C' \rrbracket_\gamma)$ such that $C \rightarrow p \cdot (\tau, C') + (1-p) \cdot (\tau, C'')$ with $p = v(\{\tau, e\})$ and $\llbracket C \rrbracket_\gamma \sqsubseteq p \cdot \llbracket C' \rrbracket_\gamma + (1-p) \cdot \llbracket C'' \rrbracket_\gamma$. By Figure 11 and by letting $\gamma = \gamma(X \leftarrow \llbracket \mu X.C \rrbracket_\gamma)$ and Lemma 1.56 and Lemma 2.45, $\mu X.C \rightarrow p \cdot (\tau, C'[X \leftarrow \mu X.C]) + (1-p) \cdot (\tau, C''[X \leftarrow \mu X.C])$ and $\llbracket \mu X.C \rrbracket_\gamma \sqsubseteq p \cdot \llbracket C'[X \leftarrow \mu X.C] \rrbracket_\gamma + (1-p) \cdot \llbracket C''[X \leftarrow \mu X.C] \rrbracket_\gamma$.

□

Lemma 2.50 (Adequacy II). $\forall x \in \mathcal{C}_{\max}(\llbracket C \rrbracket_\gamma) \cdot \emptyset \xrightarrow{\omega_x} c x$ then $C \rightarrow \sum_x v(x)(\omega_x, \checkmark)$.

Proof. Let $x \in \mathcal{C}_{\max}(\llbracket \mu X.C \rrbracket_\gamma) \cdot \emptyset \xrightarrow{\omega_x} c x$. We know that $\emptyset \xrightarrow{l_0} c \{l_0\} \xrightarrow{\omega_{x'}} c x$ such that $l_0 \in \mathcal{I}(\llbracket \mu X.C \rrbracket_\gamma)$ and $x' = x \setminus l_0$. By Lemma 2.49 we have two cases:

1. $l_0 \neq \tau$

Hence $\mu X.C \rightarrow 1 \cdot (l_0, C'[X \leftarrow \mu X.C])$ and $\llbracket \mu X.C \rrbracket_\gamma \setminus l_0 \equiv \llbracket C'[X \leftarrow \mu X.C] \rrbracket_\gamma$. By Definition 2.13, $x' \in \mathcal{C}_{\max}(\llbracket \mu X.C \rrbracket_\gamma \setminus l_0)$, hence $x \in \mathcal{C}_{\max}(\llbracket C'[X \leftarrow \mu X.C] \rrbracket_\gamma)$. By i.h., $C'[X \leftarrow \mu X.C] \rightarrow \sum_{x'} v(x')(\omega_{x'}, \checkmark)$. By Figure 12, $\mu X.C \rightarrow \sum_{x'} v(x')(\omega_x, \checkmark)$.

2. $l_0 = \tau$

Hence $\mu X.C \rightarrow p \cdot (\tau, C'[X \leftarrow \mu X.C]) + (1-p) \cdot (\tau, C''[X \leftarrow \mu X.C])$ with $p = v(\{\tau, e\})$, where $e \in \mathcal{I}(\llbracket C \rrbracket_\gamma)$, and $\llbracket \mu X.C \rrbracket_\gamma \sqsubseteq p \cdot \llbracket C'[X \leftarrow \mu X.C] \rrbracket_\gamma + (1-p) \cdot \llbracket C''[X \leftarrow \mu X.C] \rrbracket_\gamma$. By Lemma 2.46, $x \in \mathcal{C}_{\max}(p \cdot \llbracket C'[X \leftarrow \mu X.C] \rrbracket_\gamma + (1-p) \cdot \llbracket C''[X \leftarrow \mu X.C] \rrbracket_\gamma)$. By Definition 2.7, $x \setminus \tau \in \mathcal{C}_{\max}(\llbracket C'[X \leftarrow \mu X.C] \rrbracket_\gamma)$ or $x \setminus \tau \in \mathcal{C}_{\max}(\llbracket C''[X \leftarrow \mu X.C] \rrbracket_\gamma)$. Let $x' = x \setminus \tau$ for the former and $x'' = x \setminus \tau$ for the latter. By i.h. $C'[X \leftarrow \mu X.C] \twoheadrightarrow \sum_{x'} v(x')(\omega_{x'}, \checkmark)$ and $C''[X \leftarrow \mu X.C] \twoheadrightarrow \sum_{x''} v(x'')(\omega_{x''}, \checkmark)$. By Figure 12, $\mu X.C \twoheadrightarrow p \cdot \sum_{x'} v(x')(\tau : \omega_{x'}, \checkmark) + (1-p) \cdot \sum_{x''} v(x'')(\tau : \omega_{x''}, \checkmark)$.

□

Example 2.51. The probabilistic event structure in Example 2.2 corresponds to the command in Example 2.3.

To see how both semantics relate with each other, recall the maximal configurations in Example 2.2 and the words that lead to the end of a computation in Example 2.3.

Similarly to what was shown in Example 1.59, it is straightforward to see that each word corresponds to a covering chain and vice-versa. What is left to verify is the probability. From Example 2.3 we know that the word τab has probability p , which is the same probability of the corresponding covering chain. Similarly, the words τcd and τdc have probability $1 - p$, which equals the probability of the respective covering chains.

Conversely, if we pick a covering chain of a maximal configuration, we quickly notice that its probability and the probability of the respective word is the same.

3 Unitary Event Structures

A quantum event structure [Win14] is an event structure together with a function that maps events to unitary operators or projections on a finite-dimensional Hilbert space \mathcal{H} , with a condition saying that operators of concurrent events must commute.

For reasons that shall be detailed in Section 3.4, we add two new conditions to Winskel's definition and we call the resultant structure *unitary event structures*. We impose the minimal conflict to be transitive and the sum of events in minimal conflict should be a unitary operator. The intuition behind the restrictions is to consider events in minimal conflicts as measurements and by allowing the sum of such events to be a unitary operator rather than the identity, we gain the flexibility to measure in any basis, rather than being restricted to the computational basis¹. To define unitary event structures we make use of the equivalence class of an event e , which is composed by itself or by the events in which e is in minimal conflict, *i.e.* $[e] = \{e' \mid e = e', e \sim e'\}$.

Definition 3.1 (Unitary Event Structure). A unitary event structure over a finite-dimensional Hilbert space \mathcal{H} , is a pair $U = (E, Q : E \rightarrow Op(\mathcal{H}))$ comprised of an event structure $E = (E, \leq, \#)$, where Q maps events $e \in E$ to projection/unitary operators on \mathcal{H} such that:

- $\forall e_1, e_2 \in E, e_1 \text{ co } e_2 \Rightarrow Q(e_1)Q(e_2) = Q(e_2)Q(e_1)$
- \sim is transitive
- $\forall e \in E, \sum_{e' \in [e]} Q(e')$ is unitary

Definition 3.2. Let $\emptyset \xrightarrow{e_1} x_1 \xrightarrow{e_2} x_2 \dots \xrightarrow{e_n} x_n = x \in \mathcal{C}(E)$ be a covering chain. Denote the covering chain by ω and define the operator $A_\omega = Q_{e_n} Q_{e_{n-1}} \dots Q_{e_2} Q_{e_1}$. Case $\omega = \emptyset$, then we define $A_\emptyset = Id$.

Lemma 3.3. Let $x \in \mathcal{C}(U)$ such that $\omega_1 \equiv \emptyset \xrightarrow{e_1} x_1 \xrightarrow{e_2} x_2 \dots \xrightarrow{e_n} x_n = x$ and $\omega_2 \equiv \emptyset \xrightarrow{e'_1} x_{\sigma(1)} \xrightarrow{e'_2} x_{\sigma(2)} \dots \xrightarrow{e'_n} x_{\sigma(n)} = x$ are two covering chains. Then $A_{\omega_1} = A_{\omega_2}$.

Proof. We know that:

- in a configuration there are no elements in conflict
- for each configuration exists at least one covering chain

If there is more than one covering chain for a configuration, then it means that the configuration has concurrent events, which by Definition 3.1 entails that their associated quantum operators commute. If they commute, then the order in which they are applied is not relevant, hence the operators associated to the covering chains are equal. \square

Despite knowing that measurements are the cause of probabilities, from Definition 3.1 we note that no probabilities are associated to unitary event structures, unlikely to what happens with probabilistic event structures. However, according to [Win14, Theorem 3] there is a way to transform quantum event structures without conflicting events, also known as an elementary quantum event structures, into a probabilistic event structure. In Section 3.4 we explore how this is done and how the additional restrictions allows us to remove the elementary condition of [Win14, Theorem 3].

Example 3.4 is designed for the reader to get used to unitary event structures.

Example 3.4. In Figure 16 we have depicted a quantum event structure composed of the events $H_1, \tau_0^1, \tau_1^1, X_1$, and Z_1 . H_1 is the initial event, followed by τ_0^1 , which leads to X_1 , and τ_1^1 , which leads to Z_1 . Note that τ_0^1 and τ_1^1 are in conflict (specifically, in minimal conflict). Furthermore, since the conflict relation is hereditary, X_1 and Z_1 are in conflict.

The set of configurations is $\{\emptyset, \{H_1\}, \{H_1, \tau_0^1\}, \{H_1, \tau_1^1\}, \{H_1, \tau_0^1, X_1\}, \{H_1, \tau_1^1, Z_1\}\}$. As said in Definition 3.1, from a configuration x we can define the operator A_x . For example, if we consider the maximal configurations $\{H_1, \tau_0^1, X_1\}$ and $\{H_1, \tau_1^1, Z_1\}$, the respective operators are $X(1)P_0^1H(1)$ and $X(1)P_1^1H(1)$. The former applies the Hadamard gate to qubit 1, projects it to $|0\rangle$, and then applies the X gate. The latter, after applying the Hadamard gate, projects the qubit to $|1\rangle$ and then applies the Z gate.

¹in a system with only one qubit, the computational basis is given by $|0\rangle$ and $|1\rangle$

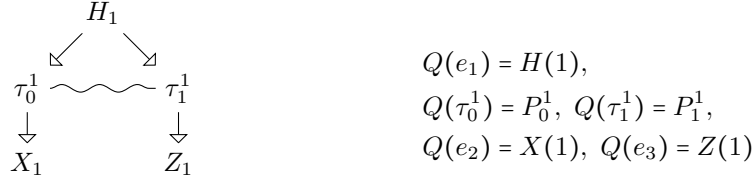


Figure 16: Example unitary event structure

3.1 Language

We adapt the language shown in Section 1.1 to the quantum setting. For that we need some preliminaries. We consider at our disposal a finite number of qubits N , whose associated space is \mathbb{C}^{2^N} . Each qubit is denoted by a natural number n and we let $\bar{n} \subseteq N$ be a subset of the set of qubits. We will need the notion of a partial density operator, which is a density operator whose trace is less or equal to one. We denote by \mathcal{H} its associated space and we denote by $\mathcal{D}_{\leq 1}(\mathcal{H})$ the set of partial density operators. We shall use ρ to represent a partial density operator. The set of actions is now composed by a set of unitary gates \mathcal{U} together with a set of projections $\{P_0^n, P_1^n\}$ in which P_0^n and P_1^n represent the projection of qubit n into $|0\rangle$ and $|1\rangle$, respectively. The set of labels is then $L' = L \cup \{P_0^n, P_1^n\}$, with $L = Act \cup \{sk\}$.

The set of commands allowed by the language are given by the following grammar:

$$\mathbf{C} ::= \text{skip} \mid U(\bar{n}) \mid \mathbf{C} ; \mathbf{C} \mid \mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2) \mid \mathbf{C} \parallel \mathbf{C}$$

where $U(\bar{n})$ applies the unitary gate U to the qubits presented in \bar{n} and $\mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2)$ represents the measurement of a qubit n and if the measurement was made by P_0^n then we execute \mathbf{C}_1 , else if the measurement was made by P_1^n then we execute \mathbf{C}_2 . Note that the behavior of $\mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2)$ is similar to that of a classical if clause.

The set of qubits being used in a command \mathbf{C} is defined as follows:

$$\begin{aligned} \text{qVar}(\text{skip}) &= \emptyset \\ \text{qVar}(U(\bar{n})) &= \bar{n} \\ \text{qVar}(\mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2)) &= \{n\} \cup \text{qVar}(\mathbf{C}_1) \cup \text{qVar}(\mathbf{C}_2) \\ \text{qVar}(\mathbf{C}_1 ; \mathbf{C}_2) &= \text{qVar}(\mathbf{C}_1) \cup \text{qVar}(\mathbf{C}_2) \\ \text{qVar}(\mathbf{C}_1 \parallel \mathbf{C}_2) &= \text{qVar}(\mathbf{C}_1) \cup \text{qVar}(\mathbf{C}_2) \end{aligned}$$

We restrict the parallel operator to only compose commands with disjoint variables, *i.e.* $\mathbf{C}_1 \parallel \mathbf{C}_2$ iff $\text{qVar}(\mathbf{C}_1) \cap \text{qVar}(\mathbf{C}_2) = \emptyset$.

To define the operational semantics, we add a new symbol, denoted by \checkmark , that indicates the end of a computation. We define the small-step transition step $\xrightarrow{a} \subseteq \mathbf{C} \times L' \times (\mathbf{C} \cup \{\checkmark\})$, as the smallest relation obeying the following rules:

$$\begin{array}{c} \text{skip} \xrightarrow{sk} \checkmark \qquad U(\bar{n}) \xrightarrow{U(\bar{n})} \checkmark \qquad \mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2) \xrightarrow{P_0^n} \mathbf{C}_1 \qquad \mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2) \xrightarrow{P_1^n} \mathbf{C}_2 \\[10pt] \frac{\mathbf{C}_1 \xrightarrow{l} \checkmark}{\mathbf{C}_1 ; \mathbf{C}_2 \xrightarrow{l} \mathbf{C}_2} \qquad \frac{\mathbf{C}_1 \xrightarrow{l'} \mathbf{C}'_1}{\mathbf{C}_1 ; \mathbf{C}_2 \xrightarrow{l'} \mathbf{C}'_1 ; \mathbf{C}_2} \\[10pt] \frac{\mathbf{C}_1 \xrightarrow{l} \checkmark}{\mathbf{C}_1 \parallel \mathbf{C}_2 \xrightarrow{l} \mathbf{C}_2} \qquad \frac{\mathbf{C}_1 \xrightarrow{l'} \mathbf{C}'_1}{\mathbf{C}_1 \parallel \mathbf{C}_2 \xrightarrow{l'} \mathbf{C}'_1 \parallel \mathbf{C}_2} \qquad \frac{\mathbf{C}_2 \xrightarrow{l} \checkmark}{\mathbf{C}_1 \parallel \mathbf{C}_2 \xrightarrow{l} \mathbf{C}_1} \qquad \frac{\mathbf{C}_2 \xrightarrow{l'} \mathbf{C}'_2}{\mathbf{C}_1 \parallel \mathbf{C}_2 \xrightarrow{l'} \mathbf{C}_1 \parallel \mathbf{C}'_2} \end{array}$$

Figure 17: Rules of the small-step operational semantics

Define a word to be a sequence of labels:

$$\omega ::= l' \mid l' : \omega$$

where $l' : \omega$ appends l' to the beginning of ω . A word can also be seen as an element of $(L')^+$, *i.e.* a possibly infinite sequence of labels without the empty sequence. Despite $(L')^+$ allows the possibility of having infinite words, by now we focus only on the finite words.

Define the n -step transition, $\xrightarrow{\omega} \subseteq \mathbb{C} \times (L')^+ \times (\mathbb{C} \cup \{\checkmark\})$, where n is the length of the words, as follows:

$$\frac{\mathbb{C} \xrightarrow{l} \mathbb{C}'}{\mathbb{C} \xrightarrow{l} \mathbb{C}'} \quad \frac{\mathbb{C} \xrightarrow{l} \mathbb{C}'' \quad \mathbb{C}'' \xrightarrow{\omega'} \mathbb{C}'}{\mathbb{C} \xrightarrow{l' : \omega'} \mathbb{C}'}$$

Figure 18: Rules of the n -step operational semantics

3.2 Constructions on Unitary Event Structures

To define the constructions on unitary event structures, we extend the definitions of sequential and parallel composition from Section 1.2 to include the corresponding mapping of events to unitary or projection operators. Additionally, we define the measurement composition by making slight adjustments to the definition of non-deterministic composition provided in Section 1.2.

Now we define how to sequentially compose two unitary event structures.

Definition 3.5 (qES sequential). Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$ and $U_2 = (E_2, \leq_2, \#_2, Q_2)$ be unitary event structures. Define $U_1 ; U_2 = (E, \leq, \#, Q)$ as:

$$\begin{aligned} E &= E_1 \uplus (E_2 \times \mathcal{C}_{\max}(U_1)) \\ \leq &= \{e_1 \leq e'_1 \mid e \leq_1 e'_1\} \cup \{(e_2, x) \leq (e'_2, x) \mid e_2 \leq_2 e'_2\} \cup \{e_1 \leq (e_2, x) \mid e_1 \in x\} \\ \# &= \{e \# e' \mid \exists (a \leq e, a' \leq e') . a \#_1 a' \vee a \#_{(2,x)} a'\} \\ Q(e) &= \begin{cases} Q_1(e) & \text{if } e \in E_1 \\ Q_{(2,x)}(e) & \text{if } e \in E_2 \times \mathcal{C}_{\max}(U_1) \end{cases} \end{aligned}$$

where we use the notation $a \#_{(2,x)} a'$ iff $(a, x) \#_{(2,x)} (a', x)$ iff $a \#_2 a'$.

Lemma 3.6. Let U_1 and U_2 be unitary event structures. $U_1 ; U_2$ is a unitary event structure.

Proof. Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$, $U_2 = (E_2, \leq_2, \#_2, Q_2)$, and $U_1 ; U_2 = (E, \leq, \#, Q)$.

Due to Lemma 1.14 we only need show the conditions added in the definition of unitary event structures.

1. $\forall e, e' \in E, e \text{ co } e' \Rightarrow [Q(e), Q(e')] = 0$

Since $e \text{ co } e'$ only if $e, e' \in E_1$ or $e, e' \in E_2 \times \mathcal{C}_{\max}(U_1)$ we are done.

2. \sim is transitive

It follows directly since \sim only occurs between events of the same set of events.

3. $\forall e \in E \sum_{e' \in [e]} Q(e')$ is unitary

We have two cases, since there is no minimal conflict between events in E_1 and $E_2 \times \mathcal{C}_{\max}(U_1)$:

- (a) $\forall e \in E_1$

Since E_1 is a unitary event structure, we are done.

- (b) $\forall e \in E_2 \times \mathcal{C}_{\max}(U_1)$

Since E_2 is a unitary event structure, we are done.

□

Similarly to the previous definition, we need to take into account the restriction in Definition 3.1 that requires that the sum of operators associated with events in minimal conflict must be the identity, which is a unitary.

Definition 3.7 (qES measurement). Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$ and $U_2 = (E_2, \leq_2, \#_2, Q_2)$ be unitary event structures. Define $M(n) \rightarrow (U_1, U_2) = (E, \leq, \#, Q)$ as:

$$\begin{aligned} E &= \{\tau_0^n, \tau_1^n\} \uplus E_1 \uplus E_2 \\ \leq &= \{\tau_0^n \leq e \mid e \in E_1\} \cup \{\tau_1^n \leq e \mid e \in E_2\} \cup \leq_1 \cup \leq_2 \\ \# &= \{e \# e' \mid (e = \tau_0^n \vee e \in E_1), (e' = \tau_1^n \vee e' \in E_2)\} \cup \#_1 \cup \#_2 \\ Q(e) &= \begin{cases} P_0^n & \text{if } e = \tau_0^n \\ P_1^n & \text{if } e = \tau_1^n \\ Q_1(e) & \text{if } e \in E_1 \\ Q_2(e) & \text{if } e \in E_2 \end{cases} \end{aligned}$$

such that $Q(\tau_0^n) + Q(\tau_1^n) = Id$.

Remark 7. We sometimes find it useful to write $M(n) \rightarrow (U_1, U_2)$ as $P_0^n; U_1 + P_1^n; U_2$, where

$$\begin{aligned} P_0^n &= (\{\tau_0^n\}, \{\tau_0^n \leq \tau_0^n\}, \emptyset, Q(\tau_0^n) = P_0^n) \\ P_1^n &= (\{\tau_1^n\}, \{\tau_1^n \leq \tau_1^n\}, \emptyset, Q(\tau_1^n) = P_1^n) \end{aligned}$$

Lemma 3.8. Let U_1 and U_2 be unitary event structures. $M(n) \rightarrow (U_1, U_2)$ is a unitary event structure.

Proof. Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$, $U_2 = (E_2, \leq_2, \#_2, Q_2)$, and $M(n) \rightarrow (U_1, U_2) = (E, \leq, \#, Q)$.

We need to prove:

1. $\{e' \mid e' \leq e\}$ is finite

We have four cases:

- (a) $e = \tau_0^n$

It follows directly that $\{e' \mid e' \leq \tau_0^n\} = \{\tau_0^n\}$ since $\tau_0^n \in \mathcal{I}(M(n) \rightarrow (U_1, U_2))$.

- (b) $e = \tau_1^n$

It follows directly that $\{e' \mid e' \leq \tau_1^n\} = \{\tau_1^n\}$ since $\tau_1^n \in \mathcal{I}(M(n) \rightarrow (U_1, U_2))$.

- (c) $e \in E_1$

We have that $\{e' \mid e' \leq e\} = \{\tau_0^n\} \cup \{e' \mid e' \leq_1 e\}$. Since U_1 is a unitary event structure, then we know that $\{e' \mid e' \leq_1 e\}$ is finite. Hence $\{\tau_0^n\} \cup \{e' \mid e' \leq_1 e\}$ is finite.

- (d) $e \in E_2$

We have that $\{e' \mid e' \leq e\} = \{\tau_1^n\} \cup \{e' \mid e' \leq_2 e\}$. Since U_2 is a unitary event structure, then we know that $\{e' \mid e' \leq_2 e\}$ is finite. Hence $\{\tau_1^n\} \cup \{e' \mid e' \leq_2 e\}$ is finite.

2. $e \# e' \leq e'' \Rightarrow e \# e''$ It follows directly by Definition 3.7 that $e \# e''$.

3. $e \text{ co } e' \Rightarrow [Q(e), Q(e')] = 0$

The concurrent events are either in U_1 or in U_2 , which are unitary event structures, hence the condition trivially holds.

4. \sim is transitive

It follows directly since the conflict relation is inherited from U_1, U_2 , which are unitary event structures, and from the fact that the new events, τ_0^n and τ_1^n , are in minimal conflict between them, *i.e.* $\tau_0^n \sim \tau_1^n$.

5. $\forall e \in E, \sum_{e' \in [e]} Q(e')$ is unitary

We have two cases (since if $e_1 \in E_1, e_2 \in E_2$ then $\neg(e_1 \sim e_2)$):

- (a) $e = \tau_0^n, e' = \tau_1^n$ or vice-versa

It follows directly from Definition 3.7 that $Q(\tau_0^n) + Q(\tau_1^n) = Id$, which is unitary.

- (b) $\forall e \in E_1$ or $\forall e \in E_2$

It follows directly from U_1 and U_2 being unitary events structures.

□

When defining the parallel composition we must consider the restriction in Definition 3.1 requiring that the operators associated with concurrent events must commute. In Definition 1.17 every event in U_1 is concurrent with every event of U_2 . It then follows that the associated operators must commute.

Definition 3.9 (qES parallel). Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$ and $U_2 = (E_2, \leq_2, \#_2, Q_2)$ be event structures. Define $U_1 \parallel U_2 = (E, \leq, \#, Q)$ as:

$$\begin{aligned} E &= E_1 \uplus E_2 \\ \leq &= \leq_1 \cup \leq_2 \\ \# &= \#_1 \cup \#_2 \\ Q(e) &= \begin{cases} Q_1(e) & \text{if } e \in E_1 \\ Q_2(e) & \text{if } e \in E_2 \end{cases} \end{aligned}$$

such that, $\forall e_1 \in E_1, e_2 \in E_2. [Q_1(e_1), Q_2(e_2)] = 0$.

Lemma 3.10. Let U_1 and U_2 be unitary event structures. $U_1 \parallel U_2$ is a unitary event structure.

Proof. Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$, $U_2 = (E_2, \leq_2, \#_2, Q_2)$, and $U_1 \parallel U_2 = (E, \leq, \#, Q)$.

Due to Lemma 1.18 we only need show the conditions added in the definition of unitary event structures.

1. $\forall e, e' \in E, e \text{ co } e' \Rightarrow [Q(e), Q(e')] = 0$

We have two cases:

- (a) $e, e' \in E_1$ or $e, e' \in E_2$

The condition trivially holds, since U_1 and U_2 are unitary event structures.

- (b) $e \in E_1$ and $e' \in E_2$

It follows directly from Definition 3.9.

2. \sim is transitive

It follows directly since the parallel composition does not create new conflicts and that the conflict relation is inherited from U_1 and U_2 which are unitary event structures.

3. $\forall e \in E, \sum_{e' \in [e]} Q(e')$ is unitary

Since there is no minimal conflict between events in E_1 and E_2 , it follows directly that if $\forall e \in E_1$ or $\forall e \in E_2$ the condition holds since U_1 and U_2 are unitary event structures. □

Definition 3.11. We interpret commands as unitary event structures as follows ($\llbracket - \rrbracket : \mathbb{C} \rightarrow \mathcal{U}$):

$$\begin{aligned} \llbracket \text{skip} \rrbracket &= (\{sk\}, \{sk \leq sk\}, \emptyset, Q(sk) = Id) \\ \llbracket U(\vec{n}) \rrbracket &= (\{U_{\vec{n}}\}, \{U_{\vec{n}} \leq U_{\vec{n}}\}, \emptyset, Q(U_{\vec{n}}) = U(\vec{n})) \\ \llbracket M(n) \rightarrow (C_1, C_2) \rrbracket &= P_0^n; \llbracket C_1 \rrbracket + P_1^n; \llbracket C_2 \rrbracket \\ \llbracket C_1; C_2 \rrbracket &= \llbracket C_1 \rrbracket; \llbracket C_2 \rrbracket \\ \llbracket C_1 \parallel C_2 \rrbracket &= \llbracket C_1 \rrbracket \parallel \llbracket C_2 \rrbracket \end{aligned}$$

For what comes, we will need the following definition on unitary event structures.

Definition 3.12 (sub-qES). Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$ and $U_2 = (E_2, \leq_2, \#_2, Q_2)$ be unitary event structures. Say $U_1 \sqsubseteq U_2$ if:

$$\begin{aligned} E_1 &\subseteq E_2 \text{ s.t. } E_i = \{e \mid (e \vee (e, x)) \in E_i\} \\ \forall e, e' . e \leq_1 e' &\Leftrightarrow e, e' \in E_1 \wedge e \leq_2 e' \\ \forall e, e' . e \#_1 e' &\Leftrightarrow e, e' \in E_1 \wedge e \#_2 e' \\ \forall e \in E_1 . Q_1(e) &= Q_2(e) \end{aligned}$$

Remark 8. When $U_1 \sqsubseteq U_2$ and $U_2 \sqsubseteq U_1$ we say that $U_1 \equiv U_2$, i.e. that U_1 and U_2 are equivalent. Furthermore, it is clear that if $U_1 = U_2$ then $U_1 \equiv U_2$.

Definition 3.13 (Remove initial event). Let $U = (E, \leq, \#, Q)$ be a unitary event structure and $a \in \mathcal{I}(U)$. Define $U \setminus a = (E', \leq', \#', Q')$ as

$$\begin{aligned} E' &= \{e \in E \mid \neg(e \# a), e \neq a\} \\ \leq' &= \{e \leq e' \mid e, e' \in E'\} \\ \# &= \{e \# e' \mid e, e' \in E'\} \\ Q' &= Q|_{E'} \end{aligned}$$

Lemma 3.14. Let U be a unitary event structure and $a \in \mathcal{I}(U)$. $U \setminus a$ is a unitary event structure.

Proof. Let $U = (E, \leq, \#, Q)$ and $U \setminus a = (E', \leq', \#', Q')$.

Due to Lemma 3.14 we only need to check the conditions added in the definition of unitary event structures.

1. $\forall e, e' \in E', e \text{ co } e' \Rightarrow [Q'(e), Q'(e')] = 0$

It follows directly from Definition 3.13 that $[Q'(e), Q'(e')] = [Q|_{E'}(e), Q|_{E'}(e')] = 0$

2. \sim is transitive

It follows directly since the conflict relation $\#'$ is the restriction of $\#$ to the events of E' .

3. $\forall e \in E, \sum_{e' \in [e]} Q(e')$ is unitary

It follows directly from Definition 3.13 that $\sum_{e' \in [e]} Q'(e') = \sum_{e' \in [e]} Q|_{E'}(e')$ which is unitary. □

3.3 Results

Here we present the results obtained. Similarly to the previous subsection, we will just list what was proved. We postpone the addition of the proofs as well as the examples for some results for future versions of the document.

For this section, we interpret \checkmark as the empty prime event structure, *i.e.* $\llbracket \checkmark \rrbracket = (\emptyset, \emptyset, \emptyset, Id)$.

Lemma 3.15. Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$ and $U_2 = (E_2, \leq_2, \#_2, Q_2)$ be unitary event structures. If $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$ then $U_1; U_2 \sqsubseteq U'_1; U'_2$.

Proof. Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$, $U'_1 = (E'_1, \leq'_1, \#'_1, Q'_1)$, $U_2 = (E_2, \leq_2, \#_2, Q_2)$, $U'_2 = (E'_2, \leq'_2, \#'_2, Q'_2)$, $U_1; U_2 = (E, \leq, \#, Q)$, and $U'_1; U'_2 = (E', \leq', \#', Q')$, such that $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$.

Due to Lemma 1.24 we only need to show $\forall e \in E. Q(e) = Q'(e)$.

Let $e \in E$. We have two cases:

1. $e \in E_1$

Since $U_1 \sqsubseteq U'_1$, it follows directly that $Q(e) = Q_1(e) = Q'_1(e) = Q'(e)$.

2. $e \in E_2 \times C_{\max}(U_1)$

By Definition 3.5, we know that $Q(e) = Q_{(2,x)}(e) = Q_2(e)$. Since $U_2 \sqsubseteq U'_2$, then $Q_2(e) = Q'_2(e)$. By Definition 3.5, we have that $Q'_2(e) = Q'_{(2,x)}(e) = Q'(e)$.

□

Lemma 3.16. Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$ and $U_2 = (E_2, \leq_2, \#_2, Q_2)$ be unitary event structures. If $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$ then $M(n) \rightarrow (U_1, U_2) \sqsubseteq M(n) \rightarrow (U'_1, U'_2)$.

Proof. Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$, $U'_1 = (E'_1, \leq'_1, \#'_1, Q'_1)$, $U_2 = (E_2, \leq_2, \#_2, Q_2)$, $U'_2 = (E'_2, \leq'_2, \#'_2, Q'_2)$, $M(n) \rightarrow (U_1, U_2) = (E, \leq, \#, Q)$, and $M(n) \rightarrow (U'_1, U'_2) = (E', \leq', \#', Q')$, such that $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$.

We have to show that:

1. $E \sqsubseteq E'$
2. $\forall e, e'. e \leq e' \Leftrightarrow e, e' \in E \wedge e \leq' e'$
3. $\forall e, e'. e \# e' \Leftrightarrow e, e' \in E \wedge e \# e'$
4. $\forall e \in E. Q(e) = Q'(e)$

The first three conditions follow directly from Definition 3.7. For the last condition we argue as follows: If $e = \tau_0^n$ or $e = \tau_1^n$ then we are done, since $\tau_0^n, \tau_1^n \in E'$ and $Q(e) = Q'(e)$. If $e \in E_1$ then it follows from $U_1 \sqsubseteq U'_1$ and Definition 3.7 that $Q(e) = Q'(e)$. Similarly when $e \in E_2$. □

Lemma 3.17. Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$ and $U_2 = (E_2, \leq_2, \#_2, Q_2)$ be unitary event structures. If $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$ then $U_1 \parallel U_2 \sqsubseteq U'_1 \parallel U'_2$.

Proof. Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$, $U'_1 = (E'_1, \leq'_1, \#'_1, Q'_1)$, $U_2 = (E_2, \leq_2, \#_2, Q_2)$, $U'_2 = (E'_2, \leq'_2, \#'_2, Q'_2)$, $U_1 \parallel U_2 = (E, \leq, \#, Q)$, and $U'_1 \parallel U'_2 = (E', \leq', \#', Q')$, such that $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$.

Due to Lemma 1.26 we only need to show $\forall e \in E. Q(e) = Q'(e)$.

Let $e \in E$. If $e \in E_1$ then by Definition 3.9 we have $Q(e) = Q_1(e)$, which by $U_1 \sqsubseteq U'_1$ gives $Q_1(e) = Q'_1(e)$ that by Definition 3.9 gives $Q'_1(e) = Q'(e)$. Similarly when $e \in E_2$. □

Lemma 3.18. Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$ and $U_2 = (E_2, \leq_2, \#_2, Q_2)$ be unitary event structures. Consider $U_1; U_2$ such that $l \in \mathcal{I}(U_1; U_2)$. Then $(U_1; U_2) \setminus l \equiv (U_1 \setminus l); U_2$.

Proof. Let

$$\begin{aligned} U_1 &= (E_1, \leq_1, \#_1, Q_1) \\ U_2 &= (E_2, \leq_2, \#_2, Q_2) \\ U_1; U_2 &= (E_{1;2}, \leq_{1;2}, \#_{1;2}, Q_{1;2}) \\ (U_1; U_2) \setminus l &= (E, \leq, \#, Q) \\ U_1 \setminus l &= (E_1^l, \leq_1^l, \#_1^l, Q_1^l) \\ (U_1 \setminus l); U_2 &= (E', \leq', \#', Q') \\ l &\in \mathcal{I}(U_1; U_2) \end{aligned}$$

Due to Lemma 1.24 we focus only on the quantum part.

- $(U_1; U_2) \setminus l \sqsubseteq (U_1 \setminus l); U_2$

We need to show $\forall e \in E. Q(e) = Q'(e)$. Let $e \in E$. By Definition 3.13, $Q(e) = Q_{1;2|E}(e)$. By Definition 3.5 we have two cases:

- $e \in E_1$

By Definition 3.5 and since $e \neq l$ and $\neg(e \# l)$, which gives $e \in E_1^l$, then $Q_{1;2|E}(e) = Q_{1|E_1^l}(e)$. By Definition 3.13, $Q_{1|E_1^l}(e) = Q_1^l(e)$. By Definition 3.5, $Q_1^l(e) = Q'(e)$.

- $e \in E_2 \times \mathcal{C}_{\max}(U_1)$

By Definition 3.5 and since $l \notin E_2$, $Q_{1;2|E}(e) = Q_{(2,x)}(e) = Q_2(e)$. Hence, again by Definition 3.5, $Q_2(e) = Q_{(2,x)}(e) = Q'(e)$.

- $(U_1 \setminus l); U_2 \sqsubseteq (U_1; U_2) \setminus l$

Similar reasoning to the previous bullet.

□

Lemma 3.19. Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$ and $U_2 = (E_2, \leq_2, \#_2, Q_2)$ be unitary event structures. Consider $\mathbb{M}(n) \rightarrow (U_1, U_2)$ such that $l \in \mathcal{I}(\mathbb{M}(n) \rightarrow (U_1, U_2))$. Then

$$(\mathbb{M}(n) \rightarrow (U_1, U_2)) \setminus l \equiv \begin{cases} U_1 & \text{if } l = \tau_0^n \\ U_2 & \text{if } l = \tau_1^n \end{cases}$$

Proof. Let

$$\begin{aligned} U_1 &= (E_1, \leq_1, \#_1, Q_1) \\ U_2 &= (E_2, \leq_2, \#_2, Q_2) \\ \mathbb{M}(n) \rightarrow (U_1, U_2) &= (E, \leq, \#, Q) \\ (\mathbb{M}(n) \rightarrow (U_1, U_2)) \setminus l &= (E^l, \leq^l, \#^l, Q^l) \end{aligned}$$

Let us focus on the case where $l = \tau_0^n$.

To prove $(\mathbb{M}(n) \rightarrow (U_1, U_2)) \setminus \tau_0^n \equiv U_1$, we only consider the condition on quantum operators, since the other cases are similar to the proof done in Lemma 1.29.

We then show for:

- $(\mathbb{M}(n) \rightarrow (U_1, U_2)) \setminus \tau_0^n \sqsubseteq U_1$

- $\forall e \in E^l. Q^l(e) = Q_1(e)$

Let $e \in E^l$. By Definition 3.13, $Q^l(e) = Q_{|E^l}(e)$. Since $\tau_0^n \leq e$ we know that $e \in E_1$, hence by Definition 3.7, $Q_{|E^l}(e) = Q_1(e)$.

- $U_1 \sqsubseteq (\mathbb{M}(n) \rightarrow (U_1, U_2)) \setminus \tau_0^n$

- $\forall e \in E_1. Q_1(e) = Q^l(e)$

Similar reasoning to the previous case.

The reasoning when $l = \tau_1^n$ is equal to the one shown here.

□

Lemma 3.20. Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$ and $U_2 = (E_2, \leq_2, \#_2, Q_2)$ be unitary event structures. Consider $U_1 \parallel U_2$ such that $l \in \mathcal{I}(U_1 \parallel U_2)$. Then $(U_1 \parallel U_2) \setminus l \equiv (U_1 \setminus l) \parallel (U_2 \setminus l)$.

Proof. Let

$$\begin{aligned} U_1 &= (E_1, \leq_1, \#_1, Q_1) \\ U_2 &= (E_2, \leq_2, \#_2, Q_2) \\ U_1 \parallel U_2 &= (E_{1 \parallel 2}, \leq_{1 \parallel 2}, \#_{1 \parallel 2}, Q_{1 \parallel 2}) \\ (U_1 \parallel U_2) \setminus l &= (E, \leq, \#, Q) \\ U_1 \setminus l &= (E_1^l, \leq_1^l, \#_1^l, Q_1^l) \\ U_2 \setminus l &= (E_2^l, \leq_2^l, \#_2^l, Q_2^l) \\ (U_1 \setminus l) \parallel (U_2 \setminus l) &= (E', \leq', \#', Q') \\ l &\in \mathcal{I}(U_1 \parallel U_2) \end{aligned}$$

Due to Lemma 1.26 we focus only on the quantum part. Furthermore, we consider that $l \in \mathcal{I}(U_1)$, which entails that $U_2 \setminus l = U_2$, and consequently $Q_2^l = Q_2$.

- $(U_1 \parallel U_2) \setminus l \sqsubseteq (U_1 \setminus l) \parallel (U_2 \setminus l)$

We need to show $\forall e \in E. Q(e) = Q'(e)$. Let $e \in E$. By Definition 3.13, $Q(e) = Q_{1 \parallel 2|E}(e)$. By Definition 3.9 we have two cases:

1. $e \in E_1$

Since $l \in \mathcal{I}(U_1)$ then $e \neq l$ and $\neg(e \# l)$. Thus $e \in E_1^l$. Consequently, $Q_{1 \parallel 2|E}(e) = Q|_{E_1^l}(e)$. By Definition 3.13, $Q|_{E_1^l}(e) = Q_1^l(e)$. By Definition 3.9, $Q_1^l(e) = Q'(e)$.

2. $e \in E_2$

Then $Q_{1 \parallel 2|E}(e) = Q_2(e)$. Since $l \in \mathcal{I}(U_1)$, then $Q_2(e) = Q_2^l(e)$. Definition 3.9, $Q_2^l(e) = Q'(e)$.

- $(U_1 \setminus l) \parallel (U_2 \setminus l) \sqsubseteq (U_1 \parallel U_2) \setminus l$

We need to show $\forall e \in E'. Q'(e) = Q(e)$. By Definition 3.9 we have two cases:

1. $e \in E_1^l$

Then $Q'(e) = Q_1^l(e)$. By Definition 3.13, $Q_1^l(e) = Q_1|_{E_1^l}(e)$. By Definition 3.9, $Q_1|_{E_1^l}(e) = Q_{1 \parallel 2|E_1^l \uplus E_2}(e)$. By Definition 3.13, $Q_{1 \parallel 2|E_1^l \uplus E_2}(e) = Q(e)$.

2. $e \in E_2^l$

Then $Q'(e) = Q_2^l(e)$. Since $l \in \mathcal{I}(U_1)$ then $Q_2^l(e) = Q_2(e)$. By Definition 3.9, $Q_2(e) = Q_{1 \parallel 2}(e)$. By Definition 3.13, $Q_{1 \parallel 2}(e) = Q(e)$.

□

Lemma 3.21. Let U_1, U_2 be unitary event structures. Then $U_1 \parallel U_2 = U_2 \parallel U_1$.

Proof. It follows directly from Definition 3.9. □

Lemma 3.22 (Soundness I). If $\mathcal{C} \xrightarrow{l} \mathcal{C}'$ then $\llbracket \mathcal{C}' \rrbracket \equiv \llbracket \mathcal{C} \rrbracket \setminus l$.

Proof. Induction over rules in Figure 17.

- $\text{skip} \xrightarrow{sk} \checkmark$

It follows directly that $\llbracket \checkmark \rrbracket \equiv \llbracket \text{skip} \rrbracket \setminus sk \equiv \emptyset$.

- $\mathbf{a} \xrightarrow{a} \checkmark$

It follows directly that $\llbracket \checkmark \rrbracket \equiv \llbracket \mathbf{a} \rrbracket \setminus a \equiv \emptyset$.

- $\mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2) \xrightarrow{\tau_0^n} \mathbf{C}_1$

It follows directly since $P_0^n; \llbracket \mathbf{C}_1 \rrbracket \sqsubseteq P_0^n; \llbracket \mathbf{C}_1 \rrbracket + P_1^n; \llbracket \mathbf{C}_2 \rrbracket = \llbracket \mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2) \rrbracket$.

- $\mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2) \xrightarrow{\tau_1^n} \mathbf{C}_2$

It follows directly since $P_1^n; \llbracket \mathbf{C}_0 \rrbracket \sqsubseteq P_0^n; \llbracket \mathbf{C}_1 \rrbracket + P_1^n; \llbracket \mathbf{C}_2 \rrbracket = \llbracket \mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2) \rrbracket$.

- $\mathbf{C}_1; \mathbf{C}_2 \xrightarrow{l} \mathbf{C}_2$

$$\begin{aligned}
& \mathbf{C}_1; \mathbf{C}_2 \xrightarrow{l} \mathbf{C}_2 \\
& \Rightarrow \{ \xrightarrow{l} \text{ entails} \} \\
& \mathbf{C}_1 \xrightarrow{l} \checkmark \\
& \Rightarrow \{ \text{i.h.} \} \\
& \llbracket \checkmark \rrbracket \equiv \llbracket \mathbf{C}_1 \rrbracket \setminus l \\
& \Rightarrow \{ \text{Lemma 3.15} \} \\
& \llbracket \checkmark \rrbracket; \llbracket \mathbf{C}_2 \rrbracket \equiv (\llbracket \mathbf{C}_1 \rrbracket \setminus l); \llbracket \mathbf{C}_2 \rrbracket \\
& \Rightarrow \{ \llbracket \checkmark \rrbracket; \llbracket \mathbf{C}_2 \rrbracket = \llbracket \mathbf{C}_2 \rrbracket, \text{ Lemma 3.18} \} \\
& \llbracket \mathbf{C}_2 \rrbracket \equiv (\llbracket \mathbf{C}_1 \rrbracket; \llbracket \mathbf{C}_2 \rrbracket) \setminus l \\
& \Rightarrow \{ \text{Definition 3.11} \} \\
& \llbracket \mathbf{C}_2 \rrbracket \equiv \llbracket \mathbf{C}_1; \mathbf{C}_2 \rrbracket \setminus l
\end{aligned}$$

- $C_1 ; C_2 \xrightarrow{l} C'_1 ; C_2$

$$\begin{aligned}
& C_1 ; C_2 \xrightarrow{l} C'_1 ; C_2 \\
& \Rightarrow \{ \xrightarrow{l} \text{ entails} \} \\
& C_1 \xrightarrow{l} C'_1 \\
& \Rightarrow \{ \text{i.h.} \} \\
& \llbracket C'_1 \rrbracket \equiv \llbracket C_1 \rrbracket \setminus l \\
& \Rightarrow \{ \text{Lemma 3.15} \} \\
& \llbracket C'_1 \rrbracket ; \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) ; \llbracket C_2 \rrbracket \\
& \Rightarrow \{ \text{Lemma 3.18} \} \\
& \llbracket C'_1 \rrbracket ; \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket ; \llbracket C_2 \rrbracket) \setminus l \\
& \Rightarrow \{ \text{Definition 3.11} \} \\
& \llbracket C'_1 ; C_2 \rrbracket \equiv \llbracket C_1 ; C_2 \rrbracket \setminus l
\end{aligned}$$

- $C_1 \parallel C_2 \xrightarrow{l} C_2$

$$\begin{aligned}
& C_1 \parallel C_2 \xrightarrow{l} C_2 \\
& \Rightarrow \{ \xrightarrow{l} \text{ entails} \} \\
& C_1 \xrightarrow{l} \checkmark \\
& \Rightarrow \{ \text{i.h.} \} \\
& \llbracket \checkmark \rrbracket \equiv \llbracket C_1 \rrbracket \setminus l \\
& \Rightarrow \{ \text{Lemma 3.17} \} \\
& \llbracket \checkmark \rrbracket \parallel \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel C_2 \\
& \Rightarrow \{ \llbracket \checkmark \rrbracket \parallel \llbracket C_2 \rrbracket = \llbracket C_2 \rrbracket \} \\
& \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel C_2 \\
& \Rightarrow \{ \llbracket C_2 \rrbracket = \llbracket C_2 \rrbracket \setminus l \text{ since } l \notin \mathcal{I}(\llbracket C_2 \rrbracket) \} \\
& \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel (\llbracket C_2 \rrbracket \setminus l) \\
& \Rightarrow \{ \text{Lemma 3.20, Definition 3.11} \} \\
& \llbracket C_2 \rrbracket \equiv \llbracket C_1 \parallel C_2 \rrbracket \setminus l
\end{aligned}$$

- $C_1 \parallel C_2 \xrightarrow{l} C'_1 \parallel C_2$

$$\begin{aligned}
& C_1 \parallel C_2 \xrightarrow{l} C'_1 \parallel C_2 \\
& \Rightarrow \{ \xrightarrow{l} \text{ entails} \} \\
& C_1 \xrightarrow{l} C'_1 \\
& \Rightarrow \{ \text{i.h.} \} \\
& \llbracket C'_1 \rrbracket \equiv \llbracket C_1 \rrbracket \setminus l \\
& \Rightarrow \{ \text{Lemma 3.17} \} \\
& \llbracket C'_1 \rrbracket \parallel \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel C_2 \\
& \Rightarrow \{ \llbracket C_2 \rrbracket = \llbracket C_2 \rrbracket \setminus l \text{ since } l \notin \mathcal{I}(\llbracket C_2 \rrbracket) \} \\
& \llbracket C'_1 \rrbracket \parallel \llbracket C_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel (\llbracket C_2 \rrbracket \setminus l) \\
& \Rightarrow \{ \text{Lemma 3.20, Definition 3.11} \} \\
& \llbracket C'_1 \parallel C_2 \rrbracket \equiv \llbracket C_1 \parallel C_2 \rrbracket \setminus l
\end{aligned}$$

- $C_1 \parallel C_2 \xrightarrow{l} C_1$

$$\begin{aligned}
& C_1 \parallel C_2 \xrightarrow{l} C_1 \\
& \Rightarrow \{ \xrightarrow{l} \text{ entails} \} \\
& C_2 \xrightarrow{l} \checkmark \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket \checkmark \rrbracket \equiv \llbracket C_2 \rrbracket \setminus l \\
& \Rightarrow \{\text{Lemma 3.17}\} \\
& \llbracket C_1 \rrbracket \parallel \llbracket \checkmark \rrbracket \equiv \llbracket C_1 \rrbracket \parallel (\llbracket C_2 \rrbracket \setminus l) \\
& \Rightarrow \{ \llbracket C_1 \rrbracket \parallel \llbracket \checkmark \rrbracket = \llbracket C_1 \rrbracket \} \\
& \llbracket C_1 \rrbracket \equiv \llbracket C_1 \rrbracket \parallel (\llbracket C_2 \rrbracket \setminus l) \\
& \Rightarrow \{ \llbracket C_1 \rrbracket = \llbracket C_1 \rrbracket \setminus l \text{ since } l \notin \mathcal{I}(\llbracket C_1 \rrbracket) \} \\
& \llbracket C_1 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel (\llbracket C_2 \rrbracket \setminus l) \\
& \Rightarrow \{\text{Lemma 3.20, Definition 3.11}\} \\
& \llbracket C_1 \rrbracket \equiv \llbracket C_1 \parallel C_2 \rrbracket \setminus l
\end{aligned}$$

- $C_1 \parallel C_2 \xrightarrow{l} C_1 \parallel C'_2$

$$\begin{aligned}
& C_1 \parallel C_2 \xrightarrow{l} C_1 \parallel C'_2 \\
& \Rightarrow \{ \xrightarrow{l} \text{ entails} \} \\
& C_2 \xrightarrow{l} C'_2 \\
& \Rightarrow \{\text{i.h.}\} \\
& \llbracket C'_2 \rrbracket \equiv \llbracket C_2 \rrbracket \setminus l \\
& \Rightarrow \{\text{Lemma 3.17}\} \\
& \llbracket C_1 \rrbracket \parallel \llbracket C'_2 \rrbracket \equiv C_1 \parallel (\llbracket C_2 \rrbracket \setminus l) \\
& \Rightarrow \{ \llbracket C_1 \rrbracket = \llbracket C_1 \rrbracket \setminus l \text{ since } l \notin \mathcal{I}(\llbracket C_2 \rrbracket) \} \\
& \llbracket C_1 \rrbracket \parallel \llbracket C'_2 \rrbracket \equiv (\llbracket C_1 \rrbracket \setminus l) \parallel (\llbracket C_2 \rrbracket \setminus l) \\
& \Rightarrow \{\text{Lemma 3.20, Definition 3.11}\} \\
& \llbracket C_1 \parallel C'_2 \rrbracket \equiv \llbracket C_1 \parallel C_2 \rrbracket \setminus l
\end{aligned}$$

□

Lemma 3.23 (Soundness II). If $C \xrightarrow{\omega} C'$ then $\exists x \in \mathcal{C}(\llbracket C \rrbracket)$ such that $\emptyset \xrightarrow{\omega} Cx$.

Proof. • $|\omega| = 1$

It follows directly that $\exists \{l\} \in \mathcal{C}(\llbracket C \rrbracket) . \emptyset \xrightarrow{l} C\{l\}$

- $|\omega| > 1$

$$\begin{aligned}
& C \xrightarrow{\omega} C' \\
& \Rightarrow \{\text{Definition 18}\} \\
& C \xrightarrow{l} C'' \quad C'' \xrightarrow{\omega'} C' \\
& \Rightarrow \{\text{Lemma 3.22, i.h.}\} \\
& \llbracket C'' \rrbracket \equiv \llbracket C \rrbracket \setminus l \quad \exists y \in \mathcal{C}(\llbracket C'' \rrbracket) . \emptyset \xrightarrow{\omega'} C'y \\
& \Rightarrow \{\text{Definition 3.13}\} \\
& \{l\} \cup y \in \mathcal{C}(\llbracket C \rrbracket) . \emptyset \xrightarrow{l} C\{l\} \xrightarrow{\omega'} C\{l\} \cup y = x
\end{aligned}$$

□

Lemma 3.24 (Adequacy I). Let $l \in \mathcal{I}(\llbracket \mathbf{C} \rrbracket)$. Then $\exists \mathbf{C}' \in (\mathbf{C} \cup \{\checkmark\})$ s.t. $\mathbf{C} \xrightarrow{l} \mathbf{C}'$ and $\llbracket \mathbf{C} \rrbracket \setminus l \equiv \llbracket \mathbf{C}' \rrbracket$.

Proof. Induction over the interpretation of commands.

- $sk \in \mathcal{I}(\text{skip})$

Let $\mathbf{C}' = \checkmark$. It follows directly that $\text{skip} \xrightarrow{sk} \checkmark$ and that $\llbracket \text{skip} \rrbracket \setminus sk \equiv \llbracket \checkmark \rrbracket$.

- $a \in \mathcal{I}(\mathbf{a})$

Let $\mathbf{C}' = \checkmark$. It follows directly that $\mathbf{a} \xrightarrow{a} \checkmark$ and that $\llbracket \mathbf{a} \rrbracket \setminus a \equiv \llbracket \checkmark \rrbracket$.

- $l \in \mathcal{I}(\mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2))$

By Definition 3.11, $\llbracket \mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2) \rrbracket = P_0^n; \llbracket \mathbf{C}_1 \rrbracket + P_1^n; \llbracket \mathbf{C}_2 \rrbracket$. We have two cases:

1. $l' = \tau_0^n$

By Lemma 3.19, $(P_0^n; \llbracket \mathbf{C}_1 \rrbracket + P_1^n; \llbracket \mathbf{C}_2 \rrbracket) \setminus \tau_0^n \equiv \llbracket \mathbf{C}_1 \rrbracket$. Furthermore $\mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2) \xrightarrow{\tau_0^n} \mathbf{C}_1$.

2. $l' = \tau_1^n$

By Lemma 3.19, $(P_0^n; \llbracket \mathbf{C}_1 \rrbracket + P_1^n; \llbracket \mathbf{C}_2 \rrbracket) \setminus \tau_1^n \equiv \llbracket \mathbf{C}_2 \rrbracket$. Furthermore $\mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2) \xrightarrow{\tau_1^n} \mathbf{C}_2$.

- $l' \in \mathcal{I}(\mathbf{C}_1; \mathbf{C}_2)$

By Definition 3.5 we have that $l' \in \mathcal{I}(\llbracket \mathbf{C}_1 \rrbracket)$. By i.h., $\exists \mathbf{C}'$ such that $\mathbf{C}_1 \xrightarrow{l'} \mathbf{C}'$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l' \equiv \llbracket \mathbf{C}' \rrbracket$. We have two cases:

1. $\mathbf{C}' = \checkmark$

We have $\mathbf{C}_1 \xrightarrow{l'} \checkmark$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l' \equiv \llbracket \checkmark \rrbracket$. By the rules in Figure 17, $\mathbf{C}_1; \mathbf{C}_2 \xrightarrow{l'} \mathbf{C}_2$. By Definition 3.5, $(\llbracket \mathbf{C}_1 \rrbracket \setminus l'); \llbracket \mathbf{C}_2 \rrbracket \equiv \llbracket \checkmark \rrbracket; \llbracket \mathbf{C}_2 \rrbracket = \llbracket \mathbf{C}_2 \rrbracket$.

2. $\mathbf{C}' = \mathbf{C}'_1$

We have $\mathbf{C}_1 \xrightarrow{l'} \mathbf{C}'_1$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l' \equiv \llbracket \mathbf{C}'_1 \rrbracket$. By the rules in Figure 17, $\mathbf{C}_1; \mathbf{C}_2 \xrightarrow{l'} \mathbf{C}'_1; \mathbf{C}_2$. By Definition 3.5, $(\llbracket \mathbf{C}_1 \rrbracket \setminus l'); \llbracket \mathbf{C}_2 \rrbracket \equiv \llbracket \mathbf{C}'_1 \rrbracket; \llbracket \mathbf{C}_2 \rrbracket$. By Definition 3.11, $\llbracket \mathbf{C}'_1; \mathbf{C}_2 \rrbracket$.

- $l' \in \mathcal{I}(\mathbf{C}_1 \parallel \mathbf{C}_2)$

By Definition 3.9 we have two cases:

1. $l' \in \mathcal{I}(\llbracket \mathbf{C}_1 \rrbracket)$

By i.h. $\exists \mathbf{C}' . \mathbf{C}_1 \xrightarrow{l'} \mathbf{C}'$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l' \equiv \llbracket \mathbf{C}' \rrbracket$. By the rules in Figure 17 we have two cases:

- (a) $\mathbf{C}' = \checkmark$

We have $\mathbf{C}_1 \xrightarrow{l'} \checkmark$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l' \equiv \llbracket \checkmark \rrbracket$. By the rules in Figure 17 we have $\mathbf{C}_1 \parallel \mathbf{C}_2 \xrightarrow{l'} \mathbf{C}_2$. By Definition 3.9, $(\llbracket \mathbf{C}_1 \rrbracket \setminus l') \parallel \llbracket \mathbf{C}_2 \rrbracket$. Since $l' \in \mathcal{I}(\llbracket \mathbf{C}_1 \rrbracket)$, then $\llbracket \mathbf{C}_2 \rrbracket = \llbracket \mathbf{C}_2 \rrbracket \setminus l'$. Hence, we have $(\llbracket \mathbf{C}_1 \rrbracket \setminus l') \parallel \llbracket \mathbf{C}_2 \rrbracket = (\llbracket \mathbf{C}_1 \rrbracket \setminus l') \parallel (\llbracket \mathbf{C}_2 \rrbracket \setminus l')$. By Lemma 3.20 we have $(\llbracket \mathbf{C}_1 \rrbracket \parallel \llbracket \mathbf{C}_2 \rrbracket) \setminus l'$. By Definition 3.11, $\llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket \setminus l'$.

- (b) $\mathbf{C}' = \mathbf{C}'_1$

We have $\mathbf{C}_1 \xrightarrow{l'} \mathbf{C}'_1$ and $\llbracket \mathbf{C}_1 \rrbracket \setminus l' \equiv \llbracket \mathbf{C}'_1 \rrbracket$. By the rules in Figure 17 we have $\mathbf{C}_1 \parallel \mathbf{C}_2 \xrightarrow{l'} \mathbf{C}'_1 \parallel \mathbf{C}_2$. By Definition 3.9, $(\llbracket \mathbf{C}_1 \rrbracket \setminus l') \parallel \llbracket \mathbf{C}_2 \rrbracket$. Since $l' \in \mathcal{I}(\llbracket \mathbf{C}_1 \rrbracket)$, then $\llbracket \mathbf{C}_2 \rrbracket = \llbracket \mathbf{C}_2 \rrbracket \setminus l'$. Hence, we have $(\llbracket \mathbf{C}_1 \rrbracket \setminus l') \parallel \llbracket \mathbf{C}_2 \rrbracket = (\llbracket \mathbf{C}_1 \rrbracket \setminus l') \parallel (\llbracket \mathbf{C}_2 \rrbracket \setminus l')$. By Lemma 3.20 we have $(\llbracket \mathbf{C}_1 \rrbracket \parallel \llbracket \mathbf{C}_2 \rrbracket) \setminus l'$. By Definition 3.11, $\llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket \setminus l'$.

2. $l' \in \mathcal{I}(\llbracket \mathbf{C}_2 \rrbracket)$

By i.h. $\exists \mathbf{C}' . \mathbf{C}_2 \xrightarrow{l'} \mathbf{C}'$ and $\llbracket \mathbf{C}_2 \rrbracket \setminus l' = \llbracket \mathbf{C}' \rrbracket$. By the rules in Figure 17 we have two cases:

- (a) $\mathbf{C}' = \checkmark$

We have $\mathbf{C}_2 \xrightarrow{l'} \checkmark$ and $\llbracket \mathbf{C}_2 \rrbracket \setminus l' \equiv \llbracket \checkmark \rrbracket$. By the rules in Figure 17 we have $\mathbf{C}_1 \parallel \mathbf{C}_2 \xrightarrow{l'} \mathbf{C}_1$. By Definition 3.9, $\llbracket \mathbf{C}_1 \rrbracket \parallel (\llbracket \mathbf{C}_2 \rrbracket \setminus l')$. Since $l' \in \mathcal{I}(\llbracket \mathbf{C}_2 \rrbracket)$, then $\llbracket \mathbf{C}_1 \rrbracket = \llbracket \mathbf{C}_1 \rrbracket \setminus l'$. Hence, we have $\llbracket \mathbf{C}_1 \rrbracket \parallel (\llbracket \mathbf{C}_2 \rrbracket \setminus l') = (\llbracket \mathbf{C}_1 \rrbracket \setminus l') \parallel (\llbracket \mathbf{C}_2 \rrbracket \setminus l')$. By Lemma 3.20 we have $(\llbracket \mathbf{C}_1 \rrbracket \parallel \llbracket \mathbf{C}_2 \rrbracket) \setminus l'$. By Definition 3.11, $\llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket \setminus l'$.

- (b) $\mathbf{C}' = \mathbf{C}'_2$

We have $\mathbf{C}_2 \xrightarrow{l'} \mathbf{C}'_2$ and $\llbracket \mathbf{C}_2 \rrbracket \setminus l' \equiv \llbracket \mathbf{C}'_2 \rrbracket$. By the rules in Figure 17 we have $\mathbf{C}_1 \parallel \mathbf{C}_2 \xrightarrow{l'} \mathbf{C}_1 \parallel \mathbf{C}'_2$. By Definition 3.9, $\llbracket \mathbf{C}_1 \rrbracket \parallel (\llbracket \mathbf{C}_2 \rrbracket \setminus l')$. Since $l' \in \mathcal{I}(\llbracket \mathbf{C}_2 \rrbracket)$, then $\llbracket \mathbf{C}_1 \rrbracket = \llbracket \mathbf{C}_1 \rrbracket \setminus l'$. Hence, we have $\llbracket \mathbf{C}_1 \rrbracket \parallel (\llbracket \mathbf{C}_2 \rrbracket \setminus l') = (\llbracket \mathbf{C}_1 \rrbracket \setminus l') \parallel (\llbracket \mathbf{C}_2 \rrbracket \setminus l')$. By Lemma 3.20 we have $(\llbracket \mathbf{C}_1 \rrbracket \parallel \llbracket \mathbf{C}_2 \rrbracket) \setminus l'$. By Definition 3.11, $\llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket \setminus l'$.

□

Lemma 3.25 (Adequacy II). If $(x \neq \emptyset) \in \mathcal{C}(\llbracket \mathbb{C} \rrbracket)$ s.t. $\emptyset \xrightarrow{\omega} \subset x$ then $\exists \mathbb{C}'$ s.t. $\mathbb{C} \xrightarrow{\omega} \mathbb{C}'$.

Proof. Induction over the length of ω .

- $|\omega| = 1$

We have $\{l'\} \in \mathcal{C}(\mathbb{C})$ such that $\emptyset \xrightarrow{l'} \subset \{l'\}$. Furthermore $l' \in \mathcal{I}(\llbracket \mathbb{C} \rrbracket)$. By Lemma 3.24, $\mathbb{C} \xrightarrow{l'} \mathbb{C}'$ and $\llbracket \mathbb{C}' \rrbracket \equiv \llbracket \mathbb{C} \rrbracket \setminus l'$. By the rules in Figure 18, $\mathbb{C} \xrightarrow{l'} \mathbb{C}'$.

- $|\omega| > 1$

We have $x \in \mathcal{C}(\llbracket \mathbb{C} \rrbracket)$ such that $\emptyset \xrightarrow{\omega} \subset x$. Since $\omega = l_0 l_1 \dots l_n$, then $\emptyset \xrightarrow{l_0} \subset \{l_0\} \xrightarrow{\omega'} \subset x$. Hence $l_0 \in \mathcal{I}(\llbracket \mathbb{C} \rrbracket)$. By Lemma 3.24, $\mathbb{C} \xrightarrow{l_0} \mathbb{C}'$ and $\llbracket \mathbb{C}' \rrbracket \equiv \llbracket \mathbb{C} \rrbracket \setminus l_0$. By Definition 3.13, $\exists y \in \mathcal{C}(\llbracket \mathbb{C}' \rrbracket)$ such that $\emptyset \xrightarrow{\omega'} \subset y$. By i.h. $\exists \mathbb{C}''$ such that $\mathbb{C}' \xrightarrow{\omega'} \mathbb{C}''$. By the rules in Figure 18, $\mathbb{C} \xrightarrow{\omega} \mathbb{C}''$, where $\omega = l_0 : \omega'$.

□

3.4 Unitary Event Structures with initial state

According to [Win14, Theorem 3], an elementary quantum event structure, *i.e.* an event structure without conflicting events, paired with an initial state ρ , along with a valuation function defined as $v(x) = \text{Tr}(A_x^\dagger A_x \rho)$, corresponds to a probabilistic event structure. In this section, we show that the conditions added to Winskel's definition of quantum event structures allow us to eliminate the elementary condition in [Win14, Theorem 3]. The reason to such restrictions lies in the fact that in a probabilistic event structure, the probability of conflicting events cannot be greater than one. For example, if we consider conflicting events a and b , each with probability one, the sum condition in Definition 2.1 fails when we take y as the empty set and x_1, x_2 as $\{a\}$ and $\{b\}$, respectively. Through some calculations, the sum simplifies to $v(\emptyset) - (v(\{a\}) + v(\{b\})) = 1 - (1 + 1) = -1$, which does not meet the criteria of being greater than or equal to 0. To avoid such scenario, we need to ensure that the sum of the probabilities of events in minimal conflict does not exceed one. This is achieved by the restrictions we introduced. Since a measurement is composed of orthogonal operations, the probability of sequentially applying two or more orthogonal operations to a given state is zero. Note that this corresponds to ill-configurations. Furthermore, the sum of the probabilities of events in minimal conflict is less or equal to one, because the sum of the operators of events in minimal conflict is a unitary, and unitary operators preserve the trace.

The difference between our definition and Winskel's quantum event structures is the restrictions that we add. Therefore, we use as basis the proof outlined in [Win14, Theorem 3], which allows us to only show the case in which all the events are mapped to projections such that either all events are in conflict or there are events in conflict. To show the former we have everything. On the other hand, showing the latter requires extra machinery, which we show here.

According to [Win14, Proposition 3], to show that a structure is a probabilistic event structure we only need to show that the condition in Definition 2.1 holds for $y \xrightarrow{e_1} \subset x_1, \dots, y \xrightarrow{e_n} \subset x_n$. We then build a unitary event structure formed by the events of $y \xrightarrow{e_1} \subset x_1, \dots, y \xrightarrow{e_n} \subset x_n$.

Definition 3.26. Let $U = (E, \leq, \#, Q)$ be a unitary event structure and $y \in \mathcal{C}(U)$. Define $\tilde{U}_y = (\tilde{E}, \tilde{\leq}, \tilde{\#}, \tilde{Q})$ as follows:

$$\begin{aligned} \tilde{E} &= \{e \mid y \xrightarrow{e} \subset y \cup \{e\}\} \\ \tilde{\leq} &= \{(e, e) \mid e \in \tilde{E}\} \\ \tilde{\#} &= \# \cap (\tilde{E} \times \tilde{E}) \\ \tilde{Q} &= Q|_{\tilde{E}} \end{aligned}$$

Lemma 3.27. $\tilde{U}_y = (\tilde{E}, \tilde{\leq}, \tilde{\#}, \tilde{Q})$ is a unitary event structure.

Proof. We show \tilde{U}_y obeys the conditions of a unitary event structure.

- $\{e' \mid e' \tilde{\leq} e\}$ is finite

Trivially holds because every $e \in \tilde{E}$ is only causally related to itself.

- $e \tilde{\#} e' \tilde{\leq} e'' \Rightarrow e \tilde{\#} e''$

Trivially holds because every $e \in \tilde{E}$ is only causally related to itself. Hence $e' \tilde{\leq} e''$ is by definition $e' \tilde{\leq} e'$. It then follows directly $e \tilde{\#} e' \tilde{\leq} e'' \Rightarrow e \tilde{\#} e'$.

- $e \text{ co } e' \Rightarrow [\tilde{Q}(e), \tilde{Q}(e')] = 0$

It follows directly since if $e \text{ co } e'$ in \tilde{U} , then $e \text{ co } e'$ in U , in which $[Q(e), Q(e')] = 0$. Hence $[\tilde{Q}(e), \tilde{Q}(e')] = [Q|_{\tilde{E}}(e), Q|_{\tilde{E}}(e')] = 0$

- \sim is transitive

It follows directly from the fact that $\#$ is inherited from U .

- $\forall e \in \tilde{E}, \sum_{e' \in [e]} Q(e')$ is unitary

Let $e \in \tilde{E}$. By definition we have $\tilde{Q} = Q|_{\tilde{E}}$. It then follows that $\sum_{e' \in [e]} \tilde{Q}([e]) = \sum_{e' \in [e]} Q|_{\tilde{E}}(e')$ is unitary.

□

Next, we merge the events, from the previous definition, that are in conflict. Note that this gives a conflict relation that is empty, hence an elementary unitary event structure.

Definition 3.28. Let \tilde{U}_y be a unitary event structure. Define $\hat{U} = (\hat{E}, \hat{\leq}, \hat{\#}, \hat{Q})$ as follows:

$$\begin{aligned}\hat{E} &= \{[e] \mid e \in \tilde{U}\} \\ \hat{\leq} &= \{([e], [e']) \mid [e] \in \hat{E}\} \\ \hat{\#} &= \emptyset \\ \hat{Q}([e]) &= \begin{cases} Q(e) & \text{if } |[e]| = 1 \\ U & \text{if } |[e]| > 1 \end{cases}\end{aligned}$$

Note that when $|[e]| > 1$ we have $\hat{Q}([e]) = U = \sum_{e' \in [e]} Q(e')$.

Lemma 3.29. $\hat{U} = (\hat{E}, \hat{\leq}, \hat{\#}, \hat{Q})$ is a unitary event structure.

Proof. We show \hat{U} obeys the conditions of a unitary event structure.

see this

- $\{[e'] \mid [e'] \hat{\leq} [e]\}$ is finite

Trivially holds because every $[e] \in \hat{E}$ is only causally related to itself.

- $[e] \hat{\#} [e'] \hat{\leq} [e''] \Rightarrow [e] \hat{\#} [e'']$

Trivially holds because the conflict relation is empty.

- $[e] \text{ co } [e'] \Rightarrow [\hat{Q}([e]), \hat{Q}([e'])] = 0$

We have three cases:

1. $|[e]| = |[e']| = 1$

It follows directly that $[\hat{Q}([e]), \hat{Q}([e'])] = [Q(e), Q(e')] = 0$

2. $|[e]| = 1$ and $|[e']| > 1$

We have $[\hat{Q}([e]), \hat{Q}([e'])] = [Q(e), U] = 0$, since the event e and all the events in $[e']$ are concurrent.

3. $|[e]| > 1$ and $|[e']| > 1$

We have $[\hat{Q}([e]), \hat{Q}([e'])] = [U_1, U_2] = 0$, since all the events in $[e]$ are concurrent with the events in $[e']$.

- \sim is transitive

It follows directly since $\hat{\#} = \emptyset$.

- $\forall [e] \in \hat{E}, \sum_{[e'] \in [[e]]} \hat{Q}([e'])$ is unitary

Since $\hat{\#} = \emptyset$ then $\sum_{[e'] \in [[e]]} \hat{Q}([e']) = \hat{Q}([e])$ since $[[e]] = 1$. By Definition 3.28 we have two cases:

1. $|[e]| = 1$

Then $\hat{Q}([e]) = Q(e')$ which is a unitary.

2. $|[e]| > 1$

Then $\hat{Q}([e]) = U$ which is a unitary.

□

We then define a map of event structures between the underlying event structures of \tilde{U} and \hat{U} , $proj_{\tilde{U}} : (\tilde{E}, \tilde{\leq}, \tilde{\#}) \rightarrow (\hat{E}, \hat{\leq}, \hat{\#})$, where $e \mapsto [e]$. A feature of $proj_{\tilde{U}}$ is that for a given configuration $x \in \mathcal{C}(\tilde{E}_y)$ we have $|x| = |proj_{\tilde{U}}(x)|$, since $proj_{\tilde{U}}$ is a total map of event structures (recall Example 1.9).

We now define the map of event structures $proj_{\tilde{E}} : (\tilde{E}, \tilde{\leq}, \tilde{\#}) \rightarrow (\hat{E}, \hat{\leq}, \hat{\#})$, where

$$proj_{\tilde{E}} : \tilde{E} \rightarrow \hat{E} \\ e \mapsto [e]$$

Lemma 3.30. $proj_{\tilde{E}} : (\tilde{E}, \tilde{\leq}, \tilde{\#}) \rightarrow (\hat{E}, \hat{\leq}, \hat{\#})$ is a map of events structures.

Proof. We show that $proj_{\tilde{E}}$ satisfies the conditions to be a map of event structures.

- $\forall x \in \mathcal{C}(\tilde{E}) \Rightarrow proj_{\tilde{E}}(x) \in \mathcal{C}(\hat{E})$

It follows straightforwardly since $\hat{\#}$ is empty.

- $\forall (e \neq e') \in x \in \mathcal{C}(\tilde{E})$, if $proj_{\tilde{E}}$ is defined in both then $proj_{\tilde{E}}(e) \neq proj_{\tilde{E}}(e')$

Let $(e \neq e') \in x \in \mathcal{C}(\tilde{E})$. Since $proj_{\tilde{E}}$ is total, $proj_{\tilde{E}}$ is defined in both and since $e, e' \in x$ then $\neg(e \tilde{\#} e')$. Hence it follows straightforwardly that $proj_{\tilde{E}}(e) \neq proj_{\tilde{E}}(e')$. □

Lemma 3.31. Consider \tilde{E}_y , \hat{E} , and $proj_{\tilde{E}}$. If $x \in \mathcal{C}(\tilde{E}_y)$ then $|proj_{\tilde{E}}(x)| = |x|$.

Proof. Let $x \in \mathcal{C}(\tilde{E}_y)$. We know that $proj_{\tilde{E}}$ is total, hence $proj_{\tilde{E}}(x) = \{proj_{\tilde{E}}(e) \mid e \in x\} = \{[e] \mid e \in x\}$. Furthermore, $proj_{\tilde{E}}$ is locally injective. Hence if $e_1, \dots, e_n \in x$ then $proj_{\tilde{E}}(e_1), \dots, proj_{\tilde{E}}(e_n) \in proj_{\tilde{E}}(x)$. Thus $|x| = |proj_{\tilde{E}}(x)|$. □

Lemma 3.32. Consider \tilde{E}_y , \hat{E} . Let $\hat{x} = \{[e_1], \dots, [e_n]\} \in \mathcal{C}(\hat{E})$ and $\tilde{x} \in \mathcal{C}(\tilde{E}_y)$. Then $\{\tilde{x} \mid proj_{\tilde{E}}(\tilde{x}) = \hat{x}\} = \{\{\tilde{e}_1, \dots, \tilde{e}_n\} \mid \forall i, \tilde{e}_i \in [e_i]\}$.

Proof. We have two cases:

- $\{\tilde{x} \mid proj_{\tilde{E}}(\tilde{x}) = \hat{x}\} \subseteq \{\{\tilde{e}_1, \dots, \tilde{e}_n\} \mid \forall i, \tilde{e}_i \in [e_i]\}$

Let $\tilde{x} = \{\tilde{e}_1, \dots, \tilde{e}_n\} \in \mathcal{C}(\tilde{E}_y)$. By Definition 1.8, $proj_{\tilde{E}}(\tilde{x}) = \hat{x} \in \mathcal{C}(\hat{E})$. Furthermore, $proj_{\tilde{E}}(\tilde{x}) = proj_{\tilde{E}}(e_1), \dots, proj_{\tilde{E}}(e_n) = \{[e_1], \dots, [e_n]\}$. By definition of $[e]$, we know that $\forall i, \tilde{e}_i \in \tilde{x}$ we have $\tilde{e}_i \in [e_i]$. Hence, we are done.

- $\{\{\tilde{e}_1, \dots, \tilde{e}_n\} \mid \forall i, \tilde{e}_i \in [e_i]\} \subseteq \{\tilde{x} \mid proj_{\tilde{E}}(\tilde{x}) = \hat{x}\}$

Let $\{\tilde{e}_1, \dots, \tilde{e}_n\} \in \{\{\tilde{e}_1, \dots, \tilde{e}_n\} \mid \forall i, \tilde{e}_i \in [e_i]\}$. We need to show that $\{\tilde{e}_1, \dots, \tilde{e}_n\} \in \mathcal{C}(\tilde{E}_y)$.

1. $\forall \tilde{e}, \tilde{e}' \in \{\tilde{e}_1, \dots, \tilde{e}_n\}. \neg(\tilde{e} \tilde{\#} \tilde{e}')$

Let $\tilde{e}, \tilde{e}' \in \{\tilde{e}_1, \dots, \tilde{e}_n\}$. Then we know that $\tilde{e} \in [e]$ and $\tilde{e}' \in [e']$. By definition of $[e]$, we have that $\neg(\tilde{e} \tilde{\#} \tilde{e}')$, which by Definition 3.26 means $\neg(\tilde{e} \tilde{\#} \tilde{e}')$.

2. $\forall \tilde{e}, \tilde{e}'. \tilde{e}' \tilde{\leq} \tilde{e} \wedge \tilde{e} \in \{\tilde{e}_1, \dots, \tilde{e}_n\} \Rightarrow \tilde{e}' \in \{\tilde{e}_1, \dots, \tilde{e}_n\}$

By Definition 3.26, the causal relation is the equality. Hence this condition trivially holds.

Since $\{\tilde{e}_1, \dots, \tilde{e}_n\} \in \mathcal{C}(\tilde{E}_y)$, it lacks to show that $proj_{\tilde{E}_y}(\{\tilde{e}_1, \dots, \tilde{e}_n\}) \in \mathcal{C}(\tilde{E}_y) = \hat{x}$. That comes directly from applying $proj_{\tilde{E}_y}$ to $\{\tilde{e}_1, \dots, \tilde{e}_n\} \in \mathcal{C}(\tilde{E}_y)$, as follows:

$$proj_{\tilde{E}_y}(\{\tilde{e}_1, \dots, \tilde{e}_n\}) = \{proj_{\tilde{E}_y}(\tilde{e}_1), \dots, proj_{\tilde{E}_y}(\tilde{e}_n)\} = \{[e_1], \dots, [e_n]\} = \hat{x}$$

□

To prove our claim we make use of the following auxiliary result.

Lemma 3.33. Consider \tilde{U}_y together with an initial state ρ_y , such that all the events of \tilde{U}_y are projections.

For any $\tilde{x} \in \mathcal{C}(\tilde{U}_y)$ let $\tilde{v}(\tilde{x}) = \frac{v(y \cup \tilde{x})}{v(y)}$. Then $d_{\tilde{v}}^{(n)}[\emptyset; \{e_1\}, \dots, \{e_n\}] \geq 0$.

Proof. To show $d_v^{(n)}[\emptyset; \{e_1\}, \dots, \{e_n\}] \geq 0$, it is helpful to consider \hat{E} and $proj_{\hat{E}}$.

Recall that \hat{E} does not have events in conflict. Hence, by [Win14, Corollary 3], we have that \hat{U} with ρ_y and $\hat{v}(\hat{x}) = Tr(A_{\hat{x}}^\dagger A_{\hat{x}} \rho_y)$ is a probabilistic event structure.

We need to show that $\hat{v}(\hat{x}) = Tr(A_{\hat{x}}^\dagger A_{\hat{x}} \rho_y) = \sum_{\substack{\tilde{y} \in \mathcal{C}(\hat{U}) \\ f(\tilde{y}) = \hat{x}}} \tilde{v}(\tilde{y})$.

We note that $\hat{Q}([e]) = U = \sum_{\tilde{e} \in [e]} \tilde{Q}(\tilde{e})$ when $|[e]| > 1$ and that for a configuration $\hat{x} \in \mathcal{C}(\hat{E})$, the operator $A_{\hat{x}} = \hat{Q}([e_n]) \cdots \hat{Q}([e_1]) = \prod_{[e] \in \hat{x}} \hat{Q}([e])$.

Let us expand $\prod_{[e] \in \hat{x}} \hat{Q}([e])$:

$$\begin{aligned} \prod_{[e] \in \hat{x}} \hat{Q}([e]) &= \prod_{[e] \in \hat{x}} \left(\sum_{\tilde{e} \in [e]} \tilde{Q}(\tilde{e}) \right) = \prod_{i=1}^n \left(\sum_{\tilde{e}_i \in [e_i]} \tilde{Q}(\tilde{e}_i) \right) = \sum_{\substack{\tilde{e}_1, \dots, \tilde{e}_n \in \tilde{E}_y \\ \forall i, \tilde{e}_i \in [e_i]}} \left(\prod_{i=1}^n \tilde{Q}(\tilde{e}_i) \right) = \sum_{\substack{\tilde{e}_1, \dots, \tilde{e}_n \in \tilde{E}_y \\ \forall i, \tilde{e}_i \in [e_i]}} \left(\prod_{\tilde{e} \in \{\tilde{e}_1, \dots, \tilde{e}_n\}} \tilde{Q}(\tilde{e}) \right) = \\ &\stackrel{\text{(Lemma 3.32)}}{=} \sum_{\substack{\tilde{x} \in \mathcal{C}(\tilde{U}_y) \\ proj_{\tilde{E}_y}(\tilde{x}) = \hat{x}}} \left(\prod_{\tilde{e} \in \tilde{x}} \tilde{Q}(\tilde{e}) \right) = \sum_{\substack{\tilde{x} \in \mathcal{C}(\tilde{U}_y) \\ proj_{\tilde{E}_y}(\tilde{x}) = \hat{x}}} A_{\tilde{x}} \end{aligned}$$

Then it follows directly that:

$$\begin{aligned} \hat{v}(\hat{x}) &= Tr(A_{\hat{x}}^\dagger A_{\hat{x}} \rho_y) = Tr \left(\left(\sum_{\substack{\tilde{y} \in \mathcal{C}(\tilde{U}_y) \\ proj_{\tilde{E}_y}(\tilde{y}) = \hat{x}}} A_{\tilde{y}}^\dagger \right) \left(\sum_{\substack{\tilde{y}' \in \mathcal{C}(\tilde{U}_y) \\ proj_{\tilde{E}_y}(\tilde{y}') = \hat{x}}} A_{\tilde{y}'} \right) \rho_y \right) = Tr \left(\sum_{\substack{\tilde{y} \in \mathcal{C}(\tilde{U}) \\ proj_{\tilde{E}_y}(\tilde{y}) = \hat{x}}} \sum_{\substack{\tilde{y}' \in \mathcal{C}(\tilde{U}) \\ proj_{\tilde{E}_y}(\tilde{y}') = \hat{x}}} A_{\tilde{y}}^\dagger A_{\tilde{y}'} \rho_y \right) \\ &\stackrel{(\star)}{=} Tr \left(\sum_{\substack{\tilde{E} \in \mathcal{C}(\tilde{U}) \\ proj_{\tilde{E}_y}(\tilde{y}) = \hat{x}}} A_{\tilde{y}}^\dagger A_{\tilde{y}} \rho_y \right) = \sum_{\substack{\tilde{E} \in \mathcal{C}(\tilde{U}) \\ proj_{\tilde{E}_y}(\tilde{y}) = \hat{x}}} (Tr(A_{\tilde{y}}^\dagger A_{\tilde{y}} \rho_y)) = \sum_{\substack{\tilde{y} \in \mathcal{C}(\tilde{U}) \\ proj_{\tilde{E}_y}(\tilde{y}) = \hat{x}}} \tilde{v}(\tilde{y}) \end{aligned}$$

Where in step (\star) we note that if $\tilde{y} \neq \tilde{y}'$ then $A_{\tilde{y}}^\dagger A_{\tilde{y}'} = 0$. That is straightforward to see because $proj_{\tilde{E}_y}(\tilde{y}) = \hat{x} = proj_{\tilde{E}_y}(\tilde{y}')$. Hence it exists $\tilde{e} \in \tilde{y}$ and $\tilde{e}' \in \tilde{y}'$ such that $\tilde{Q}(\tilde{e}) \cdot \tilde{Q}(\tilde{e}') = 0$. In other words, \tilde{e} and \tilde{e}' are in conflict.

Now we are ready to show that $d_v^{(n)}[\emptyset; \{e_1\}, \dots, \{e_n\}] \geq 0$.

By [Win14, Proposition 1],

$$\begin{aligned} d_v^{(n)}[\emptyset; \{e_1\}, \dots, \{e_n\}] &= v(\emptyset) - \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} (-1)^{|I|+1} v \left(\bigcup_{i \in I} \{e_i\} \right) \\ &= \sum_{I \subseteq \{1, \dots, n\}} (-1)^{|I|} v \left(\bigcup_{i \in I} \{e_i\} \right) \end{aligned}$$

Now we note that it exists events that are in conflict, and since the union of events that are in conflict do not form a configuration, we have that its valuation is zero. We can then remove those terms from the sum.

$$\begin{aligned} &\sum_{I \subseteq \{1, \dots, n\}} (-1)^{|I|} v \left(\bigcup_{i \in I} \{e_i\} \right) \\ &= \sum_{\substack{I \subseteq \{1, \dots, n\} \\ \forall i, j \in I. e_i \text{ co } e_j}} (-1)^{|I|} v \left(\bigcup_{i \in I} \{e_i\} \right) \\ &= \sum_{\substack{I \subseteq \{1, \dots, n\} \\ \bigcup_{i \in I} \{e_i\} \in \mathcal{C}(\tilde{U})}} (-1)^{|I|} v \left(\bigcup_{i \in I} \{e_i\} \right) \end{aligned}$$

On the other side, on \hat{U} , we have:

$$\begin{aligned}
d_{\hat{v}}^{(k)}[\emptyset; \hat{x}_1, \dots, \hat{x}_k] &= \hat{v}(\emptyset) - \sum_{\emptyset \neq J \subseteq \{1, \dots, k\}} (-1)^{|J|+1} \hat{v}(\bigcup_{j \in J} \hat{x}_j) \\
&= \sum_{J \subseteq \{1, \dots, k\}} (-1)^{|J|} \hat{v}(\bigcup_{j \in J} \hat{x}_j) \\
&= \sum_{J \subseteq \{1, \dots, k\}} (-1)^{|J|} \hat{v}(\bigcup_{j \in J} \{\hat{e}_j\}) \\
&= \sum_{\substack{J \subseteq \{1, \dots, k\} \\ \tilde{y} \in \mathcal{C}(\tilde{U}_y) \\ \text{proj}_{\tilde{E}_y}(\tilde{y}) = \bigcup_{j \in J} \{\hat{e}_j\}}} (-1)^{|J|} \tilde{v}(\tilde{y}) \\
&= \sum_{\substack{J \subseteq \{1, \dots, k\} \\ I \subseteq \{1, \dots, n\} \\ \bigcup_{i \in I} \{e_i\} \in \mathcal{C}(\tilde{U}_y) \\ \text{proj}_{\tilde{E}_y}(\bigcup_{i \in I} \{e_i\}) = \bigcup_{j \in J} \{\hat{e}_j\}}} (-1)^{|J|} \tilde{v}(\bigcup_{i \in I} \{e_i\})
\end{aligned}$$

By Lemma 3.31 we know that $|\text{proj}_{\tilde{E}_y}(\bigcup_{i \in I} \{e_i\})| = |\bigcup_{i \in I} \{e_i\}| = |I|$. Furthermore $|\bigcup_{j \in J} \{\hat{e}_j\}| = |J|$. Since $\text{proj}_{\tilde{E}_y}(\bigcup_{i \in I} \{e_i\}) = \bigcup_{j \in J} \{\hat{e}_j\}$, again by Lemma 3.31 we have $|\text{proj}_{\tilde{E}_y}(\bigcup_{i \in I} \{e_i\})| = |\bigcup_{j \in J} \{\hat{e}_j\}|$. Thus $|I| = |J|$. Hence

$$\begin{aligned}
&\sum_{\substack{J \subseteq \{1, \dots, k\} \\ I \subseteq \{1, \dots, n\} \\ \bigcup_{i \in I} \{e_i\} \in \mathcal{C}(\tilde{U}_y) \\ \text{proj}_{\tilde{U}_y}(\bigcup_{i \in I} \{e_i\}) = \bigcup_{j \in J} \{\hat{e}_j\}}} (-1)^{|J|} \tilde{v}(\bigcup_{i \in I} \{e_i\}) = \sum_{\substack{J \subseteq \{1, \dots, k\} \\ I \subseteq \{1, \dots, n\} \\ \bigcup_{i \in I} \{e_i\} \in \mathcal{C}(\tilde{U}_y) \\ \text{proj}_{\tilde{U}_y}(\bigcup_{i \in I} \{e_i\}) = \bigcup_{j \in J} \{\hat{e}_j\}}} (-1)^{|I|} \tilde{v}(\bigcup_{i \in I} \{e_i\}) \\
&= \sum_{\substack{I \subseteq \{1, \dots, n\} \\ \bigcup_{i \in I} \{e_i\} \in \mathcal{C}(\tilde{U}_y)}} \left(\sum_{\substack{J \subseteq \{1, \dots, k\} \\ \text{proj}_{\tilde{U}_y}(\bigcup_{i \in I} \{e_i\}) = \bigcup_{j \in J} \{\hat{e}_j\}}} (-1)^{|I|} \tilde{v}(\bigcup_{i \in I} \{e_i\}) \right) \stackrel{(*)}{=} \sum_{\substack{I \subseteq \{1, \dots, n\} \\ \bigcup_{i \in I} \{e_i\} \in \mathcal{C}(\tilde{U}_y)}} (-1)^{|I|} \tilde{v}(\bigcup_{i \in I} \{e_i\}) \\
&= d_{\tilde{v}}^{(n)}[\emptyset; \tilde{x}_1, \dots, \tilde{x}_n]
\end{aligned}$$

In step (*) the sum no longer depends on J , hence we drop it.

We shown that $d_{\tilde{v}}^{(n)}[\emptyset; \tilde{x}_1, \dots, \tilde{x}_n] = d_{\hat{v}}^{(k)}[\emptyset; \hat{x}_1, \dots, \hat{x}_k]$.

Hence $d_{\tilde{v}}^{(n)}[\emptyset; \tilde{x}_1, \dots, \tilde{x}_n] \geq 0$. □

The idea behind Lemma 3.33 is the following: if we show that $d_{\tilde{v}}^{(n)}[\emptyset; \{e_1\}, \dots, \{e_n\}] \geq 0$ knowing that $\tilde{v}(\tilde{x}) = \frac{v(y \cup \tilde{x})}{v(y)}$ then it follows directly that $d_v^{(n)}[y; y \cup \{e_1\}, \dots, y \cup \{e_n\}] \geq 0$. In other words, the condition in Definition 2.1 is satisfied.

Now we are ready to extend [Win14, Theorem 3].

Lemma 3.34. Let $U = (E, \leq, \#, Q)$ be a unitary event structure with initial state ρ . For each $x \in \mathcal{C}(U)$ let $v(x) = \text{Tr}(\rho_x) = \text{Tr}(A_x^\dagger A_x \rho)$. Then $U = (E, \leq, \#, v)$ is a probabilistic event structure.

Proof. From [Win14, Proposition 5] we need to show $d_v^{(n)}[y; x_1, \dots, x_n]$ when $y \stackrel{e_1, \dots, e_n}{\sim} x_1, \dots, x_n$. We then identify the following cases:

1. $v(\emptyset) = 1$
2. $\exists e_i \in e_1, \dots, e_n$ such that $Q(e_i)$ is a unitary
3. $\forall e_i \in e_1, \dots, e_n$ we have $Q(e_i)$ is a projection
 - (a) all the events are concurrent
 - (b) all events are in conflict
 - (c) there are events in conflict

The proof of 1, 2, and 3.(a) can be found in [Win14, Theorem 3]. Thus, we only show the proof of 3.(b) and 3.(c).

3.(b) Case every event is in conflict we know that the sum of the associated quantum operators is a unitary. Hence we are in case 2. and consequently $d_v^{(n)}[y; x_1, \dots, x_n] = 0$.

3.(c) Case there is events in conflict :

$$\begin{aligned}
d_v^{(n)}[y; x_1, \dots, x_n] &= v(y) - \sum_{\substack{\emptyset \neq I \subseteq \{1, \dots, n\} \\ |I| > 1}} (-1)^{|I|+1} v(\bigcup_{i \in I} x_i) \\
&= v(y) - \left(\sum_{\substack{\emptyset \neq I \subseteq \{1, \dots, n\} \\ |I| = 1}} (-1)^{|I|+1} v(\bigcup_{i \in I} x_i) + \sum_{\substack{\emptyset \neq I \subseteq \{1, \dots, n\} \\ |I| > 1}} (-1)^{|I|+1} v(\bigcup_{i \in I} x_i) \right) \\
&= v(y) - \sum_{i=1}^n v(x_i) - \sum_{\substack{\emptyset \neq I \subseteq \{1, \dots, n\} \\ |I| > 1}} (-1)^{|I|+1} v(\bigcup_{i \in I} x_i)
\end{aligned}$$

Focus on $\sum_{\substack{\emptyset \neq I \subseteq \{1, \dots, n\} \\ |I| > 1}} (-1)^{|I|+1} v(\bigcup_{i \in I} x_i)$. Since $x_i = \{e_i\} \cup y$ then $\sum_{\substack{\emptyset \neq I \subseteq \{1, \dots, n\} \\ |I| > 1}} (-1)^{|I|+1} v(\bigcup_{i \in I} x_i) = \sum_{\substack{\emptyset \neq I \subseteq \{1, \dots, n\} \\ |I| > 1}} (-1)^{|I|+1} v(\bigcup_{i \in I} \{e_i\} \cup y)$.

We know that with $|I| > 1$ we are not considering singletons. Hence we are either making the union of events that are concurrent or are in conflict. W.l.o.g consider $v(\{e_j, e_k\} \cup y)$ with $1 \leq j \neq k \leq n$. Case $e_j \# e_k$ then we know that $Q(e_j) \cdot Q(e_k) = 0 = Q(e_k) \cdot Q(e_j)$ and consequently we have $v(\{e_j, e_k\} \cup y) = \text{Tr}(A_y^\dagger \cdot (Q(e_j) \cdot Q(e_k))^\dagger \cdot Q(e_j) \cdot Q(e_k) \cdot A_y \rho) = 0$. On the other side, case e_j co e_k then we know that $Q(e_j) \cdot Q(e_k) = Q(e_k) \cdot Q(e_j)$ and consequently $v(\{e_j, e_k\} \cup y) \geq 0$.

When events are in conflict their contribution to the sum is null, hence we can discard them. As a consequence, the sum is composed of elements that are concurrent. Hence we have

$$\begin{aligned}
&v(y) - \sum_{i=1}^n v(x_i) - \sum_{\substack{\emptyset \neq I \subseteq \{1, \dots, n\} \\ |I| > 1}} (-1)^{|I|+1} v(\bigcup_{i \in I} x_i) \\
&= v(y) - \sum_{i=1}^n v(x_i) - \sum_{\substack{\emptyset \neq I \subseteq \{1, \dots, n\} \\ |I| > 1 \\ \bigcup_{i \in I} x_i \in \mathcal{C}(\mathcal{U})}} (-1)^{|I|+1} v(\bigcup_{i \in I} x_i) \\
&= v(y) - \sum_{\substack{\emptyset \neq I \subseteq \{1, \dots, n\} \\ |I| = 1}} (-1)^{|I|+1} v(\bigcup_{i \in I} x_i) - \sum_{\substack{\emptyset \neq I \subseteq \{1, \dots, n\} \\ |I| > 1 \\ \forall (i \neq j) \in I. e_i \text{ co } e_j}} (-1)^{|I|+1} v(\bigcup_{i \in I} x_i) \\
&= v(y) - \sum_{\substack{\emptyset \neq I \subseteq \{1, \dots, n\} \\ \forall (i \neq j) \in I. e_i \text{ co } e_j}} (-1)^{|I|+1} v(\bigcup_{i \in I} x_i)
\end{aligned}$$

Despite removing the valuations from ill-configurations, we are not in case 3.(a) since there are still events in conflict. We thus resort to Lemma 3.33. Concretely:

$$\begin{aligned}
&d_v^{(n)}[\emptyset; \tilde{x}_1, \dots, \tilde{x}_n] \geq 0 \\
&\Leftrightarrow \tilde{v}(\emptyset) - \sum_{\substack{I \subseteq \{1, \dots, n\} \\ \bigcup_{i \in I} \{e_i\} \in \mathcal{C}(\tilde{E}_y)}} (-1)^{|I|+1} \tilde{v}(\bigcup_{i \in I} \{e_i\}) \geq 0 \\
&\Leftrightarrow \sum_{\substack{I \subseteq \{1, \dots, n\} \\ \bigcup_{i \in I} \{e_i\} \in \mathcal{C}(\tilde{E}_y)}} (-1)^{|I|} \tilde{v}(\bigcup_{i \in I} \{e_i\}) \geq 0 \\
&\Leftrightarrow \sum_{\substack{I \subseteq \{1, \dots, n\} \\ \bigcup_{i \in I} \{e_i\} \in \mathcal{C}(\tilde{E}_y)}} (-1)^{|I|} \frac{v(y \cup \bigcup_{i \in I} \{e_i\})}{v(y)} \geq 0 \\
&\Leftrightarrow \sum_{\substack{I \subseteq \{1, \dots, n\} \\ \bigcup_{i \in I} \{e_i\} \in \mathcal{C}(\tilde{E}_y)}} (-1)^{|I|} v(y \cup \bigcup_{i \in I} \{e_i\}) \geq 0 \\
&\Leftrightarrow d_v^{(n)}[y; x_1, \dots, x_n] \geq 0
\end{aligned}$$

With all cases proved, we have that $\mathcal{U} = (\mathcal{U}, \rho, v)$ is a probabilistic event structure. \square

The intuition behind the n -step in Section 3 is: given a command \mathcal{C} and a list of instructions, which is a word, $a : \omega'$ we reach a command \mathcal{C}' in n -steps. If we give an initial state to \mathcal{C} , the evolution of the state will

correspond to the application of the word to the initial state. We define a state to be a partial density operator, *i.e.* a density operator whose trace is less or equal to one and denote the set of partial density operator as $\mathcal{D}_{\leq 1}(\mathbb{C}^2)$. We now define how a word is applied to a state:

Definition 3.35. Let ω be a word and ρ a partial density operator. Define $\omega(\rho)$ inductively as follows:

$$\omega(\rho) = \begin{cases} a\rho a^\dagger & \text{if } \omega = a \\ \omega'(a\rho a^\dagger) & \text{if } \omega = a : \omega' \end{cases}$$

Note that when applying a word ω to a state ρ , the first action to be applied on ρ is the head of ω .

Following Lemma 3.34 we can define a new denotational semantics for quantum event structures who takes into consideration initial states.

Definition 3.36. We define $(\llbracket - \rrbracket_- : \mathbb{C} \times \mathcal{D}_{\leq 1}(\mathbb{C}^2) \rightarrow (\mathbf{U}, \rho, v)$, where \mathbf{U} is a unitary event structure) as follows:

$$\begin{aligned} \llbracket \text{skip} \rrbracket_\rho &= (\llbracket \text{skip} \rrbracket, \rho, v(\{sk\}) = 1) \\ \llbracket \mathbf{U}(\vec{n}) \rrbracket_\rho &= (\llbracket \mathbf{U}(\vec{n}) \rrbracket, \rho, v(\{U_{\vec{n}}\}) = 1) \\ \llbracket \mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2) \rrbracket_\rho &= (\llbracket \mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2) \rrbracket, \rho, v) \\ \llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket_\rho &= (\llbracket \mathbf{C}_1 ; \mathbf{C}_2 \rrbracket, \rho, v) \\ \llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket_\rho &= (\llbracket \mathbf{C}_1 \parallel \mathbf{C}_2 \rrbracket, \rho, v) \end{aligned}$$

At this point we can establish an equivalence between the semantics with or without initial state. However we note that for the former we first need to show the equivalence without initial state.

Consider the case without initial state. We observe that both the operational and denotational semantics presented in this section closely resemble those developed in Section 1. Consequently, the results obtained in Section 1.3 can be straightforwardly adapted to the quantum setting. It is worth to emphasize that removing an initial element form $\mathbf{M}(n) \rightarrow (\mathbf{U}_1, \mathbf{U}_2)$ is equal to \mathbf{U}_1 or to \mathbf{U}_2 if the event removed is τ_0^n or τ_1^n , respectively.

To show the equivalence of the semantics with an initial state we state the following.

Lemma 3.37 (Soundness). Let ρ be an initial state. If $\mathbb{C} \xrightarrow{\omega} \mathbb{C}'$ then $\exists x \in \mathcal{C}(\llbracket \mathbb{C} \rrbracket_\rho)$ such that $\emptyset \xrightarrow{\omega} c x$ and $v(x) = \omega(\rho)$.

Lemma 3.38 (Adequacy). Let ρ be an initial state. If $(x \neq \emptyset) \in \mathcal{C}(\llbracket \mathbb{C} \rrbracket_\rho)$ s.t. $\emptyset \xrightarrow{\omega} c x$ then $\exists \mathbb{C}'$ s.t. $\mathbb{C} \xrightarrow{\omega} \mathbb{C}'$ and $v(x) = \omega(\rho)$.

To prove the above statements we make use of Lemma 3.23 and Lemma 3.25, respectively. What is left to show is that $v(x) = \omega(\rho)$. However that comes freely because the operations applied on $\xrightarrow{\omega}$ and on $\emptyset \xrightarrow{\omega} c x$ are the same.

Example 3.39. In Figure 19, we have the labeled transition system of $\mathbf{H}(n); \mathbf{M}(n) \rightarrow (\mathbf{X}(n), \mathbf{Z}(n))$. This program applies first the Hadamard gate to qubit n and then measures it. If the measurement was made by P_0^n then we apply the X gate to qubit n and we are done. On the other side, if the measurement was performed by P_1^n we apply the Z gate to qubit n finishing the computation. With the help of Figure 19, it is straightforward to see that the words that lead to a terminal command are: $\mathbf{H}(n)P_0^n\mathbf{X}(n)$ and $\mathbf{H}(n)P_1^n\mathbf{Z}(n)$. By applying each word to the state $\rho = |0\rangle\langle 0|$, we obtain the following possible final states: $(\mathbf{H}(n)P_0^n\mathbf{X}(n))(|0\rangle\langle 0|) = \frac{1}{2}|1\rangle\langle 1|$ and $(\mathbf{H}(n)P_1^n\mathbf{Z}(n))(|0\rangle\langle 0|) = \frac{1}{2}|1\rangle\langle 1|$.

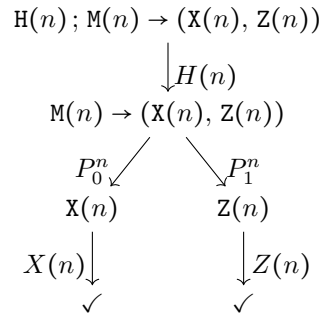


Figure 19: Labeled transition system of $\mathbf{H}(n); \mathbf{M}(n) \rightarrow (\mathbf{X}(n), \mathbf{Z}(n))$

3.5 Introducing cyclic behavior

Differently from what was done in Section 1.4 and Section 2.4, here the cyclic behavior will not be given by recursion. Instead it will be given by a while loop. By doing this we still manage to keep the intended philosophy in the operational semantics, because the while loop is defined in terms of a measurement. In other words, Section 3.1 already has all that we need to implement the while loop.

The set of commands allowed by the language are given by the following grammar:

$$\mathbf{C} ::= \text{skip} \mid U(\vec{n}) \mid \mathbf{C}; \mathbf{C} \mid \mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2) \mid \mathbf{C} \parallel \mathbf{C} \mid \text{while } \mathbf{M}(n) \rightarrow \mathbf{C}$$

where $U(\vec{n})$ applies the unitary gate U to the qubits presented in \vec{n} , the parallel composition is disjoint ² $\mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2)$ represents the measurement of a qubit n such that if the measurement is made by P_0^n then we execute \mathbf{C}_1 , else if the measurement is made by P_1^n then we execute \mathbf{C}_2 , and while $\mathbf{M}(n) \rightarrow \mathbf{C}$ is a while loop that stops the computation if the measurement is made by P_0^n . Note that the behavior of $\mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2)$ is similar to that of a classical if clause.

Remark 9. In this section, we used a while loop instead of a recursive command for cyclic behavior, unlike Sections 1.1 and 2.1. The reason to opt by a while loop in this section comes from the behavior of a measurement resembling an if-then-else command. Furthermore projections decide if the computation stops or continues, allowing us to implement the while loop without needing a notion of state. On the other hand, implementing the while loop in Sections 1.1 and 2.1 would require a notion of state associated with the command, which would obliged us to change the operational semantics we have designed without loops.

The set of qubits being used in a command \mathbf{C} is defined as follows:

$$\begin{aligned} \text{qVar}(\text{skip}) &= \emptyset \\ \text{qVar}(U(\vec{n})) &= \vec{n} \\ \text{qVar}(\mathbf{M}(n) \rightarrow (\mathbf{C}_1, \mathbf{C}_2)) &= \{n\} \cup \text{qVar}(\mathbf{C}_1) \cup \text{qVar}(\mathbf{C}_2) \\ \text{qVar}(\mathbf{C}_1; \mathbf{C}_2) &= \text{qVar}(\mathbf{C}_1) \cup \text{qVar}(\mathbf{C}_2) \\ \text{qVar}(\mathbf{C}_1 \parallel \mathbf{C}_2) &= \text{qVar}(\mathbf{C}_1) \cup \text{qVar}(\mathbf{C}_2) \\ \text{qVar}(\text{while } \mathbf{M}(n) \rightarrow \mathbf{C}) &= \{n\} \cup \text{qVar}(\mathbf{C}) \end{aligned}$$

We add the following rules to Figure 17.

$$\text{while } \mathbf{M}(n) \rightarrow \mathbf{C} \xrightarrow{P_0^n} \checkmark \quad \text{while } \mathbf{M}(n) \rightarrow \mathbf{C} \xrightarrow{P_1^n} \mathbf{C}; \text{while } \mathbf{M}(n) \rightarrow \mathbf{C}$$

Example 3.40. Figure 20 illustrates the behavior of a quantum toss coin, which, similarly to Example 1.37, produces a possibly empty sequence $H(n)P_1^n$ that finishes with P_0^n . To understand this we observe that the initial program has two possible transitions: (1) transits through P_0^n and the computation finishes; (2) transits through P_1^n to $H(n); \text{while } \mathbf{M}(n) \rightarrow H(n)$, which executes $H(n)$ to transit to $\text{while } \mathbf{M}(n) \rightarrow H(n)$, which is the same command as the initial one.

Definition 3.41. Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$ and $U_2 = (E_2, \leq_2, \#_2, Q_2)$ be unitary event structures. Say $U_1 \trianglelefteq U_2$ if:

$$\begin{aligned} E_1 &\subseteq E_2 \\ \forall e, e' . e \leq_1 e' &\Leftrightarrow e, e' \in E_1 \wedge e \leq_2 e' \\ \forall e, e' . e \#_1 e' &\Leftrightarrow e, e' \in E_1 \wedge e \#_2 e' \\ \forall e \in E_1 . Q_1(e) &= Q_2(e) \end{aligned}$$

Lemma 3.42. \trianglelefteq is a partial order.

Proof. Due to Lemma 1.39 we only focus on the condition of the quantum operators. Let $U_1 = (E_1, \leq_1, \#_1, Q_1)$, $U_2 = (E_2, \leq_2, \#_2, Q_2)$, and $U_3 = (E_3, \leq_3, \#_3, Q_3)$ be quantum event structures.

- Reflexivity: $U_1 \trianglelefteq U_1$

It follows directly that $\forall e \in E_1 . Q_1(e) = Q_1(e)$

- Transitivity : $U_1 \trianglelefteq U_2, U_2 \trianglelefteq U_3 \Rightarrow U_1 \trianglelefteq U_3$

From $U_1 \trianglelefteq U_2$, $\forall e \in E_1 . Q_1(e) = Q_2(e)$. From $U_2 \trianglelefteq U_3$, $\forall e \in E_2 . Q_2(e) = Q_3(e)$. Hence $\forall e \in E_1 . Q_1(e) = Q_2(e) = Q_3(e)$.

² $\mathbf{C}_1 \parallel \mathbf{C}_2$ being disjoint means that \mathbf{C}_1 and \mathbf{C}_2 do not share any qubit

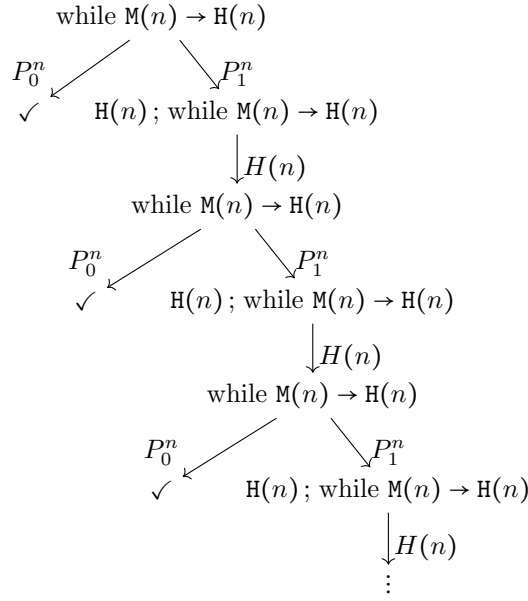


Figure 20: Fragment of the execution of $\text{while } M(n) \rightarrow H(n)$

- Antisymmetry: $U_1 \trianglelefteq U_2, U_2 \trianglelefteq U_1 \Rightarrow U_1 = U_2$

From Lemma 1.39 we know that $E_1 = E_2$. Hence it follows directly that $U_1 = U_2$.

□

Lemma 3.43. Define $\perp = (\emptyset, \emptyset, \emptyset, ! : \emptyset \rightarrow \text{Op}(\mathcal{H}))$. \perp is the least element of \trianglelefteq .

Proof. We begin by showing that \perp is a unitary event structure. We already know that $(\emptyset, \emptyset, \emptyset)$ is an event structure, hence it lacks to verify the conditions regarding the quantum operator. However, since there are no events, the conditions trivially holds.

To show that \perp is the least element, consider any unitary event structure U . We need to show $\perp \trianglelefteq U$. Due to Lemma 1.40 we only focus on the quantum operator. We need to show that for every event in \perp , its mapping through $!$ and Q is the same. We show it by contradiction. Thus, we need to find an event $e \in \emptyset$ such that its mapping through $!$ and Q is not the same. However, there are no events in \perp . Thus the condition holds. □

Definition 3.44. Let $U_1 \trianglelefteq \dots \trianglelefteq U_n \trianglelefteq \dots$ be a ω -chain. Let $U^\omega = (E^\omega, \leq^\omega, \#^\omega, Q^\omega)$ be its least upper bound where:

- $E^\omega = \cup_{n \in \omega} E_n$
- $\leq^\omega = \cup_{n \in \omega} \leq_n$
- $\#^\omega = \cup_{n \in \omega} \#_n$
- $Q^\omega(e) \Leftrightarrow \exists n \in \omega. e \in E_n \text{ and } Q_n(e) = Q^\omega(e)$

Lemma 3.45. U^ω is a unitary event structure.

Proof. Due to Lemma 1.42 we focus on the quantum operator condition.

- $\forall e, e' \in E^\omega, e \text{ co } e' \Rightarrow [Q^\omega(e), Q^\omega(e')] = 0$

Let $e, e' \in E^\omega$ such that $e \text{ co } e'$. We have two cases:

1. $e, e' \in E_n$

By Definition 3.44 we have $Q^\omega(e) = Q_n(e)$ and $Q^\omega(e') = Q_n(e')$ and since U_n is a unitary event structure we are done.

2. $e \in E_n$ and $e' \in E_m$ such that $U_n \trianglelefteq U_m$

By Definition 3.44 we have $Q^\omega(e) = Q_n(e)$ and $Q^\omega(e') = Q_m(e')$. From $U_n \trianglelefteq U_m$, we have that $Q_n(e) = Q_m(e)$ and since U_m is a unitary event structure, then we are done.

some difficulties in understanding how to prove the quantum condition with the empty function

- \sim is transitive

We want to show that for $e, e', e'' \in E^\omega$ if $e \sim e'$ and $e' \sim e''$ then $e \sim e''$. According to Definition 3.44 we have three cases:

1. $e, e', e'' \in E_n$

Then we are done, since U_n is a unitary event structure.

2. $e \in E_n$ and $e', e'' \in E_m$ such that $U_n \trianglelefteq U_m$

From $U_n \trianglelefteq U_m$ we know that $E_n \subseteq E_m$, hence $e \in E_m$. Since U_m is a unitary event structure we are done.

3. $e \in E_n$, $e' \in E_m$, and $e'' \in E_k$ such that $U_n \trianglelefteq U_m \trianglelefteq U_k$

From $U_n \trianglelefteq U_m \trianglelefteq U_k$ we have that $E_n \subseteq E_m \subseteq E_k$. Hence, $e, e' \in E_k$. Since U_k is a unitary event structure we are done.

- $\forall e \in E^\omega, |[e]| > 1 \Rightarrow \sum_{e' \in [e]} Q(e') = U$

Let $e \in E^\omega, |[e]| > 1$. By Definition 3.44, $\exists n \in \omega$ such that $e \in E_n$ and $Q^\omega(e) = Q_n(e)$. Since U_n is a unitary event structure we have $\sum_{e' \in [e]} Q^\omega(e') = \sum_{e' \in [e]} Q_n(e') = U$.

□

Lemma 3.46. Let $U_1 \trianglelefteq \dots \trianglelefteq U_n \trianglelefteq \dots$ be a ω -chain. Then U^ω is its least upper bound.

Proof. Due to Lemma 1.43 we only need to focus on the quantum condition.

- U^ω is an upper bound

$\forall n \in \omega$ we need to have $U_n \trianglelefteq U^\omega$. We need to check that $\forall e \in E_n, Q_n(e) = Q^\omega(e)$. It follows directly from Definition 3.44 that $\exists n \in \omega, e \in E_n$ and $Q^\omega(e) = Q_n(e)$.

- U^ω is a least upper bound

Let $U = (E, \leq, \#, Q)$ be an upper bound of the chain. We need to show that if $U_n \trianglelefteq U^\omega$ and $U_n \trianglelefteq U$ then $U^\omega \trianglelefteq U$. From $U_n \trianglelefteq U^\omega$, $\forall e \in E_n, Q_n(e) = Q^\omega(e)$. By Definition 3.44, $\exists n \in \omega, e \in E_n$ and $Q^\omega(e) = Q_n(e)$. From $U_n \trianglelefteq U$, $\forall e \in E_n, Q_n(e) = Q(e)$. Thus $\forall e \in U^\omega, \exists n \in \omega, e \in E_n$ and $Q^\omega(e) = Q_n(e) = Q(e)$.

□

Lemma 3.47. Let U, U_1, U_2 be event structures. If $U_1 \trianglelefteq U_2$ then $U; U_1 \trianglelefteq U; U_2$.

Proof. Due to Lemma 1.44 we focus solely on the quantum condition. Let $U = (E, \leq, \#, Q)$, $U_1 = (E_1, \leq_1, \#_1, Q_1)$, $U_2 = (E_2, \leq_2, \#_2, Q_2)$, $U; U_1 = (E^1, \leq^1, \#^1, Q^1)$, and $U; U_2 = (E^2, \leq^2, \#^2, Q^2)$, such that $U_1 \trianglelefteq U_2$.

We want to show $\forall e \in E^1, Q^1(e) = Q^2(e)$. Let $e \in E^1$. By Definition 3.5 we have two cases:

1. $e \in E$

It follows directly that $Q^1(e) = Q(e) = Q^2(e)$.

2. $e \in E_1 \times C_{\max}(U)$

We have $Q^1(e) = Q_{(1,x)}(e) = Q_1(e)$. From $U_1 \trianglelefteq U_2$, $Q_1(e) = Q_2(e)$. By Definition 3.5, $Q_2(e) = Q_{(2,x)}(e) = Q^2(e)$. Thus $Q^1(e) = Q^2(e)$.

□

Lemma 3.48. Let U_1, U'_1, U_2, U'_2 be event structures. If $U_1 \trianglelefteq U'_1$ and $U_2 \trianglelefteq U'_2$ then $U_1 \parallel U_2 \trianglelefteq U'_1 \parallel U'_2$.

Proof. Due to Lemma 1.45 we focus solely on the quantum condition. $U_1 = (E_1, \leq_1, \#_1, Q_1)$, $U_2 = (E_2, \leq_2, \#_2, Q_2)$, $U'_1 = (E'_1, \leq'_1, \#'_1, Q'_1)$, $U'_2 = (E'_2, \leq'_2, \#'_2, Q'_2)$, $U_1 \parallel U_2 = (E, \leq, \#, Q)$, and $U'_1 \parallel U'_2 = (E', \leq', \#', Q')$, such that $U_1 \trianglelefteq U'_1$ and $U_2 \trianglelefteq U'_2$.

We want to show that $\forall e \in E, Q(e) = Q'(e)$. Let $e \in E$. By Definition 3.9 we have two cases:

1. $e \in E_1$

We know that $Q(e) = Q_1(e)$. Since $U_1 \trianglelefteq U'_1$, $Q_1(e) = Q'_1(e)$. By Definition 3.9, $Q'_1(e) = Q'(e)$. Thus $Q(e) = Q'(e)$.

2. $e \in E_2$

Similar to the previous.

□

Lemma 3.49. Let U_1, U'_1, U_2, U'_2 be event structures. If $U_1 \trianglelefteq U'_1$ and $U_2 \trianglelefteq U'_2$ then $M(n) \rightarrow (U_1, U_2) \trianglelefteq M(n) \rightarrow (U'_1, U'_2)$.

Proof. $U_1 = (E_1, \leq_1, \#_1, Q_1)$, $U_2 = (E_2, \leq_2, \#_2, Q_2)$, $U'_1 = (E'_1, \leq'_1, \#'_1, Q'_1)$, $U'_2 = (E'_2, \leq'_2, \#'_2, Q'_2)$, $M(n) \rightarrow (U_1, U_2) = (E, \leq, \#, Q)$, and $M(n) \rightarrow (U'_1, U'_2) = (E', \leq', \#', Q')$, such that $U_1 \trianglelefteq U'_1$ and $U_2 \trianglelefteq U'_2$.

The conditions to check are:

1. $E \subseteq E'$
2. $\forall e, e'. e \leq e' \Leftrightarrow e, e' \in E \wedge e \leq' e'$
3. $\forall e, e'. e \# e' \Leftrightarrow e, e' \in E \wedge e \# e'$
4. $\forall e \in E. Q(e) = Q'(e)$

The first three conditions follow directly from Definition 3.7. Hence we focus on the last one.

Let $e \in E$. We have the four cases:

1. $e = \tau_0^n$
By Definition 3.7 we are done since, $Q(\tau_0^n) = Q'(\tau_0^n)$.
2. $e = \tau_1^n$
By Definition 3.7 we are done since, $Q(\tau_1^n) = Q'(\tau_1^n)$.
3. $e \in E_1$
We know that $Q(e) = Q_1(e)$. From $U_1 \trianglelefteq U'_1$, $Q_1(e) = Q'_1(e)$. By Definition 3.7, $Q'_1(e) = Q'(e)$. Thus $Q(e) = Q'(e)$.
4. $e \in E_2$
Similar to the previous point.

□

Definition 1.47 and Lemma 1.48 are similar.

Lemma 3.50. $\sqcup_m(U; U_m) = U; \sqcup_m U_m$.

Proof. Similar to Lemma 1.49.

□

Lemma 3.51. $\sqcup_{n,m}(U_n \parallel U_m) = \sqcup_n U_n \parallel \sqcup_m U_m$.

Proof. Similar to Lemma 1.50.

□

Lemma 3.52. $\sqcup_{n,m}(M(q) \rightarrow (U_n, U_m)) = M(q) \rightarrow (\sqcup_n U_n, \sqcup_m U_m)$.

Proof. By Lemma 3.49, the measurement is monotone. It lacks to show that each event of $M(q) \rightarrow (\sqcup_n U_n, \sqcup_m U_m)$ is an event of $\sqcup_{n,m}(M(q) \rightarrow (U_n, U_m))$. Let e be an event of $M(q) \rightarrow (\sqcup_n U_n, \sqcup_m U_m)$. We have four cases:

1. $e = \tau_0^n$
It follows directly from Definition 3.7.
2. $e = \tau_1^n$
It follows directly from Definition 3.7.
3. e is an event of $\sqcup_n U_n$
By Definition 3.44, $\exists n \in \omega$ such that e is an event of U_n . By Definition 3.7, e is an event of $M(q) \rightarrow (U_n, U_m)$. By Definition 3.44, e is an event of $\sqcup_{n,m}(M(q) \rightarrow (U_n, U_m))$.
4. e is an event of $\sqcup_m U_m$
Similar to the previous point.

□

Lemma 1.52 is similar.

Definition 3.53. We interpret commands as unitary event structures as follows ($\llbracket - \rrbracket : \mathcal{C} \rightarrow \mathcal{U}$):

$$\begin{aligned}\llbracket \text{skip} \rrbracket &= (\{sk\}, \{sk \leq sk\}, \emptyset, Q(sk) = Id) \\ \llbracket U_{\vec{n}} \rrbracket &= (\{U_{\vec{n}}\}, \{U_{\vec{n}} \leq U_{\vec{n}}\}, \emptyset, Q(U_{\vec{n}}) = U(\vec{n})) \\ \llbracket M(n) \rightarrow (C_1, C_2) \rrbracket &= P_0^n; \llbracket C_1 \rrbracket + P_1^n; \llbracket C_2 \rrbracket \\ \llbracket C_1; C_2 \rrbracket &= \llbracket C_1 \rrbracket; \llbracket C_2 \rrbracket \\ \llbracket C_1 \parallel C_2 \rrbracket &= \llbracket C_1 \rrbracket \parallel \llbracket C_2 \rrbracket \\ \llbracket \text{while } M(n) \rightarrow C \rrbracket &= fix(\Gamma^n)\end{aligned}$$

where $\Gamma^n : \mathcal{U} \rightarrow \mathcal{U}$ is given by $\Gamma^n(U) = P_0^n + P_1^n; U$.

Furthermore, note that $\llbracket \text{while } M(n) \rightarrow C \rrbracket = M(n) \rightarrow (\checkmark, \llbracket C; \text{while } M(n) \rightarrow C \rrbracket) = P_0^n + P_1^n; \llbracket C; \text{while } M(n) \rightarrow C \rrbracket$ and $\perp = \llbracket \checkmark \rrbracket$. These facts will be useful when showing the equivalence between the semantics.

We note that Γ^n is continuous because it is composed of continuous functions.

To show the equivalence between the operational and the denotational semantics, we reuse what was done in Section 3.3. The only lemmas in which we need to add the proof for the recursion case are the following:

Lemma 3.54 (Soundness I). If $C \xrightarrow{l} C'$ then $\llbracket C' \rrbracket \equiv \llbracket C \rrbracket \setminus l$.

Proof. • while $M(n) \rightarrow C \xrightarrow{\tau_0^n} \checkmark$

We know that $\llbracket \text{while } M(n) \rightarrow C \rrbracket = \llbracket M(n) \rightarrow (\checkmark, C; \text{while } M(n) \rightarrow C) \rrbracket$. Hence $\llbracket \text{while } M(n) \rightarrow C \rrbracket \setminus \tau_0^n = \llbracket M(n) \rightarrow (\checkmark, C; \text{while } M(n) \rightarrow C) \rrbracket \setminus \tau_0^n$, which by Lemma 3.19 gives $\llbracket \checkmark \rrbracket$. Hence $\llbracket \text{while } M(n) \rightarrow C \rrbracket \setminus \tau_0^n = \llbracket \checkmark \rrbracket$.

• while $M(n) \rightarrow C \xrightarrow{\tau_1^n} C; \text{while } M(n) \rightarrow C$

We know that $\llbracket \text{while } M(n) \rightarrow C \rrbracket = \llbracket M(n) \rightarrow (\checkmark, C; \text{while } M(n) \rightarrow C) \rrbracket$. Hence $\llbracket \text{while } M(n) \rightarrow C \rrbracket \setminus \tau_1^n = \llbracket M(n) \rightarrow (\checkmark, C; \text{while } M(n) \rightarrow C) \rrbracket \setminus \tau_1^n$, which by Lemma 3.19 gives $\llbracket C; \text{while } M(n) \rightarrow C \rrbracket$. Hence $\llbracket \text{while } M(n) \rightarrow C \rrbracket \setminus \tau_1^n = \llbracket C; \text{while } M(n) \rightarrow C \rrbracket$. □

Lemma 3.55 (Adequacy I). Let $l \in \mathcal{I}(\llbracket C \rrbracket)$. Then $\exists C' \in (\mathcal{C} \cup \{\checkmark\})$ s.t. $C \xrightarrow{l} C'$ and $\llbracket C \rrbracket \setminus l \equiv \llbracket C' \rrbracket$.

Proof. • $l \in \mathcal{I}(\llbracket \text{while } M(n) \rightarrow C \rrbracket)$

Since $\llbracket \text{while } M(n) \rightarrow C \rrbracket = \llbracket M(n) \rightarrow (\checkmark, C; \text{while } M(n) \rightarrow C) \rrbracket$, we have that $l = \tau_0^n$ or $l = \tau_1^n$. We have two cases:

1. $l = \tau_0^n$

Let $C' = \checkmark$. We know that $\llbracket \text{while } M(n) \rightarrow C \rrbracket = \llbracket M(n) \rightarrow (\checkmark, C; \text{while } M(n) \rightarrow C) \rrbracket$. Hence $\llbracket M(n) \rightarrow (\checkmark, C; \text{while } M(n) \rightarrow C) \rrbracket \setminus \tau_0^n$, which by Lemma 3.19 gives $\llbracket \checkmark \rrbracket$. Thus it follows directly that while $M(n) \rightarrow C \xrightarrow{\tau_0^n} \checkmark$.

2. $l = \tau_1^n$

Let $C' = C; \text{while } M(n) \rightarrow C$. We know that $\llbracket \text{while } M(n) \rightarrow C \rrbracket = \llbracket M(n) \rightarrow (\checkmark, C; \text{while } M(n) \rightarrow C) \rrbracket$. Hence $\llbracket M(n) \rightarrow (\checkmark, C; \text{while } M(n) \rightarrow C) \rrbracket \setminus \tau_1^n$, which by Lemma 3.19 gives $\llbracket \checkmark \rrbracket$. Thus it follows directly that while $M(n) \rightarrow C \xrightarrow{\tau_1^n} C; \text{while } M(n) \rightarrow C$. □

Similarly to what was done in Section 3.3, we can consider the equivalence between semantics with an initial state. However, doing it is very similar to what we already have, hence we postpone it.

Example 3.56. The quantum event structure in Example 3.4 corresponds to the interpretation of the command in Example 3.39.

To see the equivalence between both semantics, recall the maximal configurations in Example 3.4 and the words used in Example 3.39. It is trivial to see that for each word we have a corresponding covering chain, and vice-versa.

It lacks to verify the probability when an initial state is given. Consider that the initial state is $\rho = |0\rangle\langle 0|$. Applying the word $H(n)P_0^nX(n)$ to ρ yields a probability of 0.5, which matches the probability of the respective covering chain. Similarly, when we apply the word $H(n)P_1^nZ(n)$ to ρ , we obtain a probability of 0.5, once again matching the probability of the respective covering chain.

Conversely, if we obtained the probability from the trace of $A_x\rho$, where x is a configuration from a covering chain, we observe that applying the respective word to ρ gives the same probability. Concretely, the covering

chain of $\{H_1, \tau_0^1, X_1\}$ is $\emptyset \xrightarrow{H_1} \{H_1\} \xrightarrow{\tau_0^1} \{H_1, \tau_0^1\} \xrightarrow{X_1} \{H_1, \tau_0^1, X_1\}$. The associated operator A_x is $X(1)P_0^1H(1)$. By applying A_x to ρ we obtain the state $|1\rangle\langle 1|$ with probability 0.5, which corresponds to the probability of applying the respective word to ρ .

Example 3.57. Figure 21 shows the event structure corresponding to the interpretation of $\llbracket H(n); M(n) \rightarrow (\text{skip}, X(n)) \rrbracket$. The set of configurations is $\{\emptyset, \{H_n\}, \{H_n, \tau_0^n\}, \{H_n, \tau_1^n\}, \{H_n, \tau_0^n, sk\}, \{H_n, \tau_1^n, X_n\}\}$.

To see the equivalence between both semantics through an example, we first derive the words that can be formed by the n -step in Example 3.39: $H(n)$, $H(n)P_0^n$, $H(n)P_1^n$, $H(n)P_0^n sk$ and $H(n)P_1^n X(n)$.

Each word corresponds to a covering chain, which represents a configuration. For example the words $H(n)P_0^n sk$ and $H(n)P_1^n X(n)$ correspond to the covering chains $\emptyset \xrightarrow{H_n} \{H_n\} \xrightarrow{\tau_0^n} \{H_n, \tau_0^n\} \xrightarrow{sk} \{H_n, \tau_0^n, sk\} = x_1$ and $\emptyset \xrightarrow{H_n} \{H_n\} \xrightarrow{\tau_1^n} \{H_n, \tau_1^n\} \xrightarrow{X_n} \{H_n, \tau_1^n, X_n\} = x_2$, respectively.

Furthermore, given as initial state $\rho = |0\rangle\langle 0|$, we have the following probabilities: $v(x_1) = 0.5$ and $v(x_2) = 0.5$, which correspond to the probabilities obtained by respectively applying the words $H(n)P_0^n sk$ and $H(n)P_1^n X(n)$ to the same state, as shown in Example 3.39.

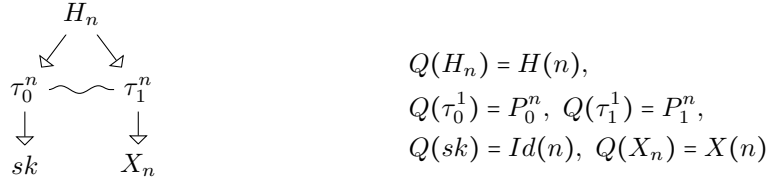


Figure 21: Event structure of $\llbracket H(n); M(n) \rightarrow (\text{skip}, X(n)) \rrbracket$

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