On complete classes of valuated matroids

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Warm up:
Complete classes of matroids

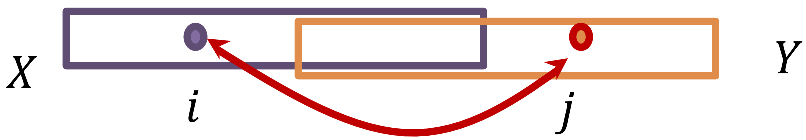
Matroids

$\mathcal{M} = (V, \mathcal{B})$, ground set V, bases $\emptyset \neq \mathcal{B} \subseteq 2^V$ such that

$$\forall X, Y \in \mathcal{B}, \forall i \in X \setminus Y \ \exists j \in Y \setminus X$$

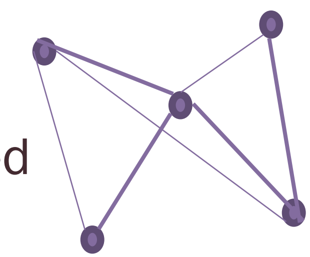
$$X - i + j, Y + i - j \in \mathcal{B}$$

symmetric basis exchange axiom

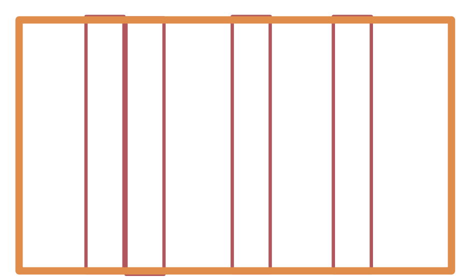


All bases have the same cardinality $d = \mathrm{rk}(\mathcal{M})$, the rank of $\mathcal{M}$

Basic examples



- Graphic matroid: bases are spanning forests of an undirected graph
- Linear matroid: Matrix $A \in \mathbb{F}^{m \times n}$, $\mathrm{rk}(A) = m$
  bases are index sets of nonsingular $m \times m$ submatrices

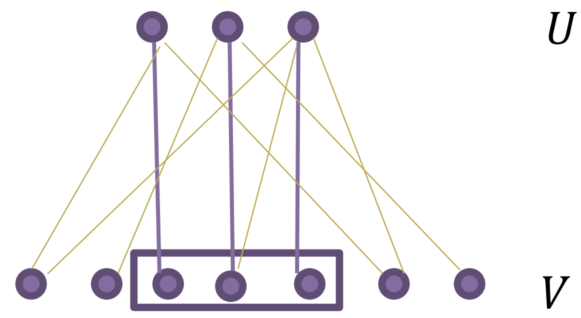


Transversal matroids and gammoids

Transversal matroid

Bipartite graph $G = (V, U; E), |U| = d$

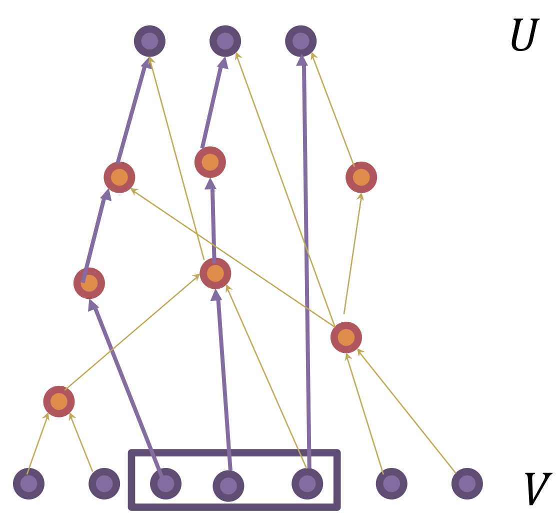
$X \subseteq V$ is a basis if there is a perfect matching between X and U



Gammoid

Directed graph $G = (N, A), U, V \subseteq N, |U| = d$

$X \subseteq V$ is a basis if there exist d node−disjoint paths between X and U



Operations on matroids

$$\mathcal{M} = (V, \mathcal{B})$$

- Deletion $Z \subseteq V$:

$$V' = B \setminus Z, \qquad \mathcal{B}' = \{X \in \mathcal{B}\colon X \cap Z = \emptyset\}$$

- Contraction $Z \subseteq V$:

$$V' = B \setminus Z, \qquad \mathcal{B}' = \{X \subseteq V \setminus Z\colon X \cup Z \in \mathcal{B}\}$$

- Duality

$$V' = V,\ \mathcal{B}' = \{V \setminus X\colon X \in \mathcal{B}\}$$

- Direct sum $\mathcal{M}_1 = (V_1, \mathcal{B}_1), \mathcal{M}_2 = (V_2, \mathcal{B}_2), V_1 \cap V_2 = \emptyset$

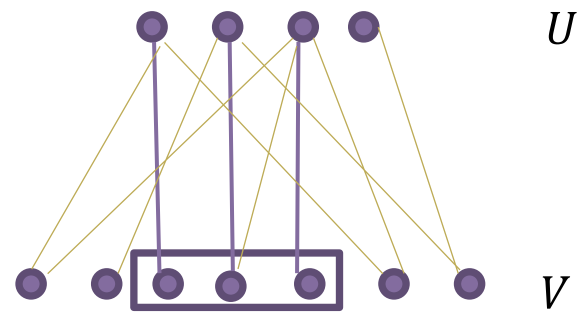
$$V' = V_1 \cup V_2, \qquad \mathcal{B}' = \{X_1 \cup X_2\colon X_1 \in \mathcal{B}_1, X_2 \in \mathcal{B}_2\}$$

- Truncation:

$$V' = V, \qquad \mathcal{B}' = \{X - v\colon X \in \mathcal{B}, v \in X\}$$

Operations on matroids

Induction by bipartite graph

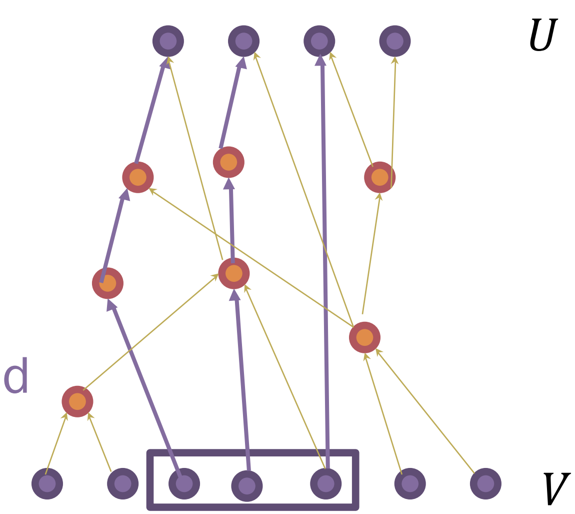


Bipartite graph $G = (V, U; E), \mathcal{M} = (U, \mathcal{B})$

$X \subseteq V$ is a basis if there is a perfect matching between X and a basis in $\mathcal{M}$

Transversal matroid = induction by bipartite graph from a free matroid

Induction by network



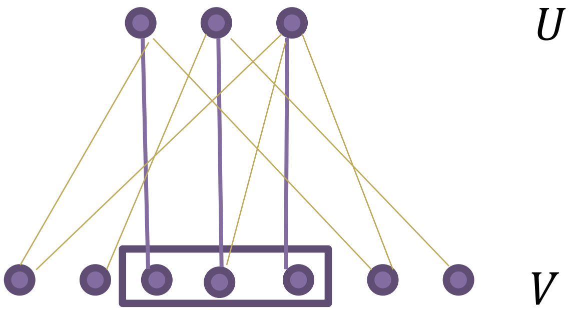
Directed graph $G = (N, A), U, V \subseteq N, \mathcal{M} = (U, \mathcal{B})$

$X \subseteq V$ is a basis if there exist *d* node−disjoint paths between X and a basis in $\mathcal{M}$

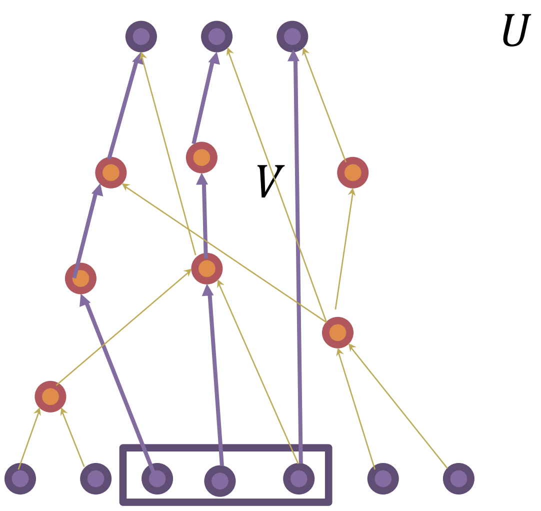
Gammoid = induction by network from a free matroid

Transversal matroids and gammoids

- Transversal matroid=
  induction by bipartite graph from a free matroid



- Gammoid=
  induction by network from a free matroid
- Induction by network=
  Induction by bipartite graph + contraction
- Gammoids=
  contractions of transversal matroids



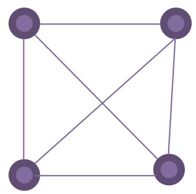
- Gammoids are closed under deletion, contraction, duality, direct sum, truncation, induction by network

Complete classes of matroids

- Ingleton 1977

> Let us call a class of matroids complete if it is closed under the operations of restriction, contraction, dualization, direct sum, truncation and induction. (The same concept could be defined by various different lists of permissible operations, depending on personal taste. This list certainly includes some logically redundant items.)

- Gammoids are the smallest complete class containing the free matroid on 1 element
- THEOREM (Brualdi 1971): Gammoids are strongly base orderable
- Graphic matroid of K_4: not strongly base orderable
- Vámos matroid: strongly base orderable but not gammoid
- Bonin & Savitsky 2016: hierarchy of complete classes



Valuated matroids

Valuated matroids

Dress & Wenzel 1990

$f\colon \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$ is a valuated matroid if

$$\forall X, Y \in \binom{V}{d}, \forall i \in X \setminus Y \;\; \exists j \in Y \setminus X$$
$$f(X) + f(Y) \le f(X - i + j) + f(Y + i - j)$$

Basic examples

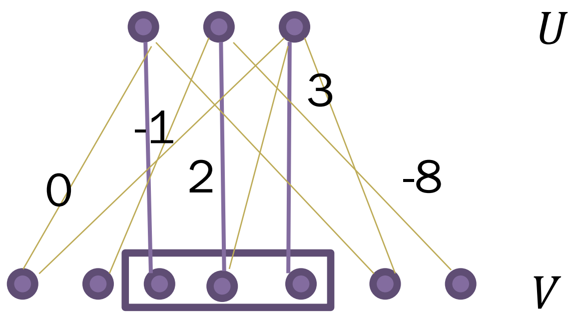
- Trivially valuated matroid: $\mathcal{M} = (V, \mathcal{B})$, $\mathrm{rk}(\mathcal{M}) = d$
$$f(X) = \begin{cases} 0 \;\; \text{if } X \in \mathcal{B} \\ -\infty \text{ if } X \notin \mathcal{B} \end{cases}$$

- Weighted matroids: $\mathcal{M} = (V, \mathcal{B})$, $\mathrm{rk}(\mathcal{M}) = d, w \in \mathbb{R}^V$
$$f(X) = \begin{cases} \sum_{i \in B} w_i \;\; \text{if } X \in \mathcal{B} \\ -\infty \text{ if } X \notin \mathcal{B} \end{cases}$$

Valuated transversal matroid

Bipartite graph $G = (V, U; E), |U| = d, w \in \mathbb{R}^E$

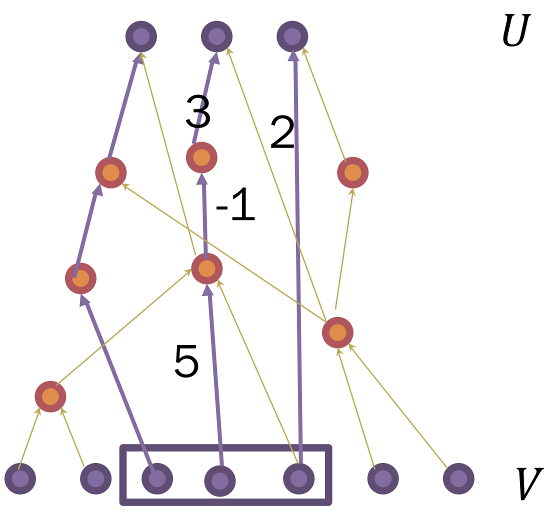
$f(X) =$max cost of a perfect matching between X and U



Valuated gammoid

Digraph $G = (N, A), U, V \subseteq N, |U| = d, w \in \mathbb{R}^A$

$f(X) =$ max cost of node-disjoint paths connecting X and U



Matrices with polynomial entries

A: $d \times n$ matrix over $\mathbb{R}[t]$

$f(X)$ =degree of the determinant of A_X for $X \in \binom{V}{d}$

$$\begin{array}{cccc} a & b & c & d \\ \multicolumn{4}{c}{\begin{pmatrix} t^2 & 1 & 1 & 1 \\ 1+t & 0 & 0 & 1 \end{pmatrix}} \end{array}$$

	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{b,c\}$	$\{b,d\}$	$\{c,d\}$
det	$-t-1$	$-t-1$	t^2-t-1	0	1	1
deg(t)	1	1	2	$-\infty$	0	0

Valuated matroids and matroids

$f: \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$ is a valuated matroid if

$$\forall X, Y \in \binom{V}{d}, \forall i \in X \setminus Y \;\; \exists j \in Y \setminus X$$
$$f(X) + f(Y) \le f(X - i + j) + f(Y + i - j)$$

- The support $\left\{X \in \binom{V}{d} : f(X) > -\infty\right\}$ forms the basis of a matroid
- For any cost function $p \in \mathbb{R}^V$, the set $\arg\max f(X) - p(X)$ forms the basis of a matroid
- Dress & Wenzel 1990: the greedy algorithm naturally extends to valuated matroids

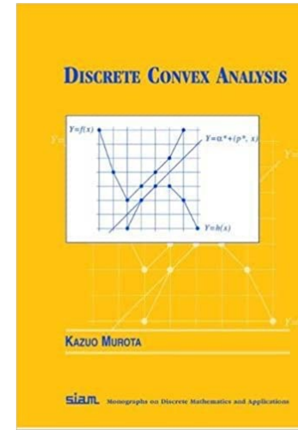
Valuated matroids everywhere

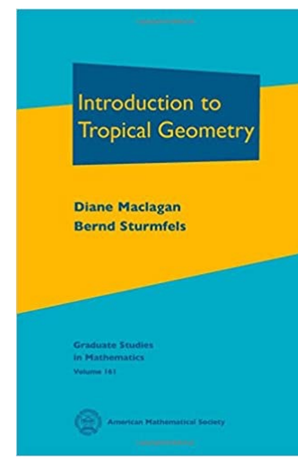
- Dress & Wenzel 1990: greedy algorithm naturally extends to valuated matroids

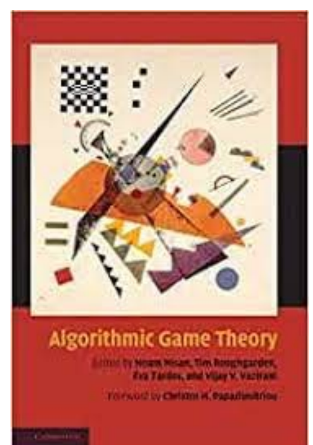
- Discrete convex analysis, Murota 1996-:
  Valuated matroids = M-concave functions on $\{0,1\}^n$

- Tropical geometry:
  Valuated matroids ~ Tropical linear spaces

- Game theory and mechanism design
  Generalized valuated matroids~
  Gross substitute valuations







Operations on valuated matroids

$$f\colon \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$$

- Deletion $Z \subseteq V$:

$$V' = B \setminus Z, d' = d, f'(X) = f(X)$$

- Contraction $Z \subseteq V$:

$$V' = B \setminus Z, d' = d - |Z|, f'(X) = f(X \cup Z)$$

- Duality

$$V' = V, \qquad d' = |V| - d, f'(X) = f(V - X)$$

- Direct sum $f_1\colon \binom{V_1}{d_1} \to \mathbb{R} \cup \{-\infty\}, f_2\colon \binom{V_2}{d_2} \to \mathbb{R} \cup \{-\infty\}$

$$V' = V_1 \cup V_2, \qquad d' = d_1 + d_2, \qquad f'(X_1 \cup X_2) = f_1(X_1) + f_2(X_2)$$

- Truncation:

$$V' = V, \qquad d' = d - 1, f'(X) = \max_v f(X + v)$$

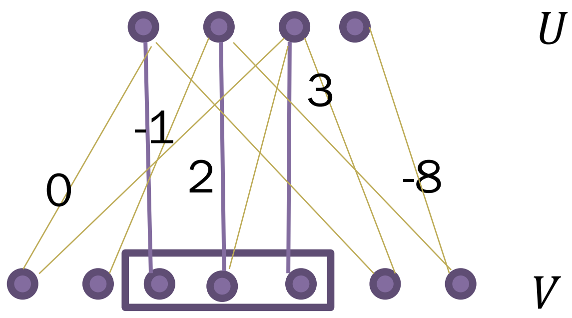
Induction by bipartite graph

Bipartite graph $G = (V, U; E)$,

g valuated matroid on U, $w \in \mathbb{R}^E$

$f(X) = \max\{w(M) + g(Y):$

M matching between X and $Y \subseteq U\}$



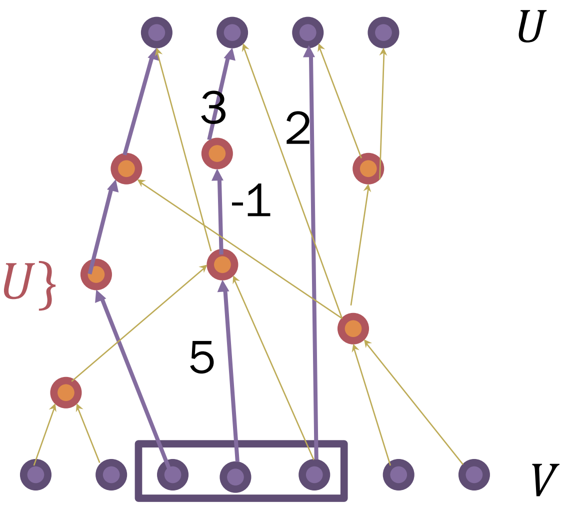
Induction by network

Directed graph $G = (N, A)$, $U, V \subseteq N$,

g valuated matroid on U, $w \in \mathbb{R}^A$

$f(X) = \max\{w(F) + g(Y):$

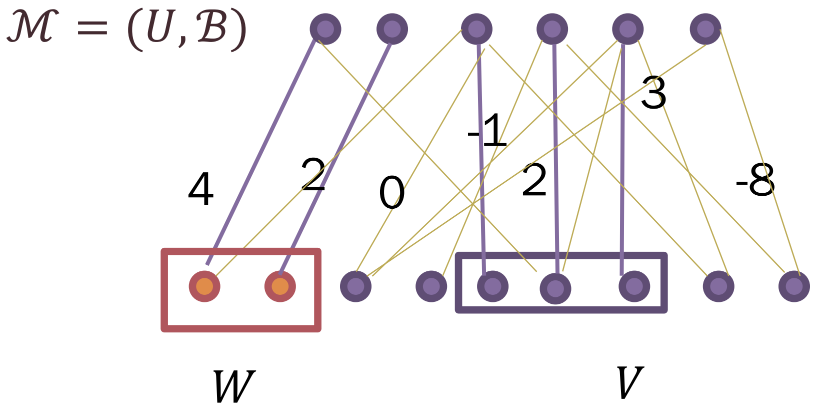
F node disjoint paths between X and $Y \subseteq U\}$



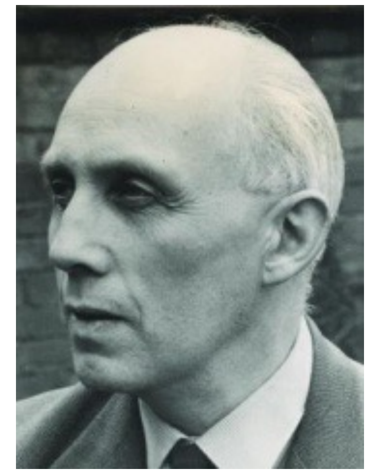
Complete classes of valuated matroids

- A class of valuated matroids is complete, if it is closed under contraction, deletion, duality, truncation, and induction by network.
- Smallest complete class: valuated gammoids
- *What is the smallest complete class containing all trivially valuated matroids?*

R-minor valuated matroids



- Bipartite graph $G = (W \cup V, U; E)$,
  $$\mathcal{M} = (U, \mathcal{B}), \mathrm{rk}(M) = d + |W|$$
- $f: \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$
  $$f(X) = \max\{w(M): M \text{ is a perfect matching between } X \cup W \text{ and some } Y \in \mathcal{B}\}$$
- R-induced valuated matroid: $W = \emptyset$



Richard Rado
(1906-1989)

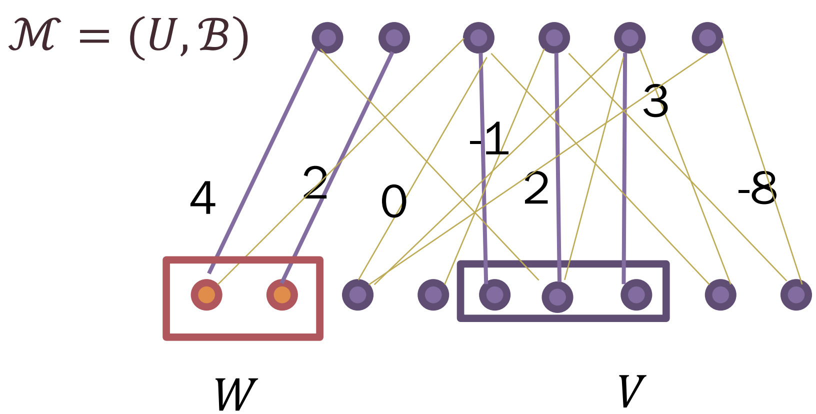
THEOREM (HLSV'21): R-minor valuated matroids are the smallest complete class containing all trivially valuated matroids

Is every valuated matroid R-minor?

THEOREM (HLSV'21): R-minor valuated matroids are the smallest complete class containing all trivially valuated matroids

- Variant of this question asked by Frank in 2003, popularized later by Murota
- Closely related questions on gross substitutes valuations: Hatfield & Milgrom 2005, Ostrovsky & Paes Leme 2015
- If the answer is yes…
  Valuated matroids = matroid + graphs + weights
- And the answer is…

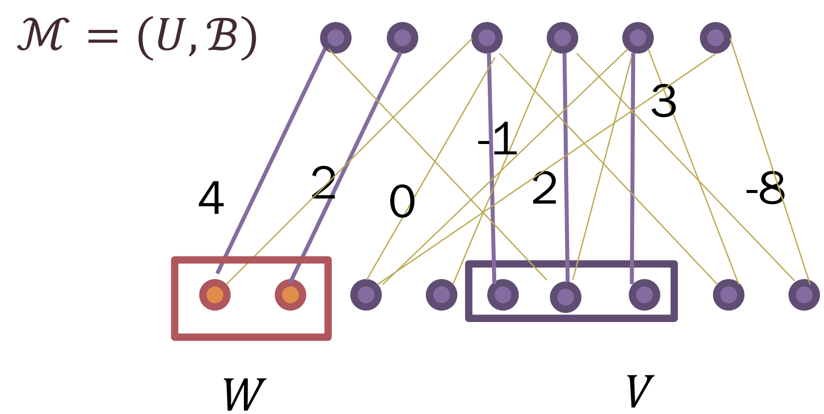
NO



THEOREM (HLSV'21): Not every valuated matroid is R-minor.

There are valuated matroids that do not arise from unvaluated matroids using induction by network and other simple operations.

THEOREM (HLSV'21): Not every valuated matroid is R-minor.

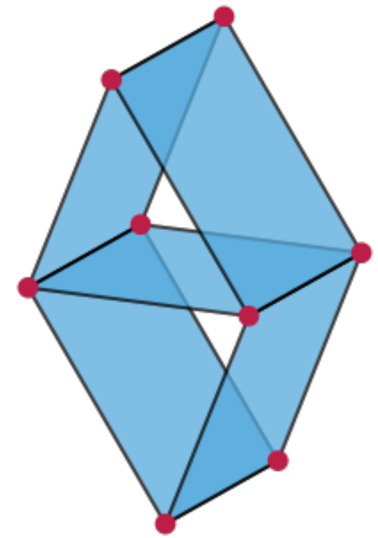


Difficulties

- Cannot argue with matroid invariants: every matroid rank function is included
- No (obvious) information theoretic arguments: the contracted set W can be arbitrarily large

Sparse paving matroids

- Rank d matroids where all circuits are of size d or $d+1$, and all size d circuits are hyperplanes
- Knuth 1974: construction of doubly exponentially many sparse paving matroids
- Conjecture (Mayhew, Newman, Welsh, Whittle 2011) asymptotically almost every matroid is sparse paving
- Underlying matroid of our construction



The Vámos matroid

- Ground set $V = \{1,2,\dots,32\} = P_1 \cup P_2 \cup \cdots \cup P_{16}$, $d = 4$

$$\mathcal{H} = \{P_i \cup P_j : ij \equiv 0 \ (mod\ 2)\}$$

$$\mathcal{B} = \binom{V}{4} \setminus \mathcal{H}$$

$$X^* = P_1 \cup P_2 = \{1,2,3,4\}$$

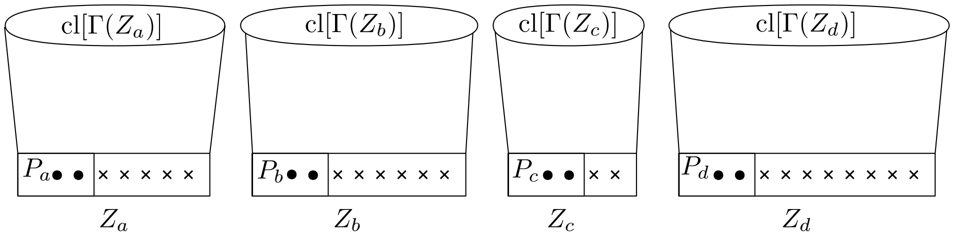
- $\mathcal{M} = (V, \mathcal{B})$ is a sparse paving matroid
- $X \in \binom{V}{4}$:

$$f(X) = \begin{cases} 0 \ \text{ if } X \in \mathcal{B} \\ -1 \text{ if } X = X^* \\ < -1 \text{ otherwise} \end{cases}$$

THEOREM (HLSV'21): Any such f is a valuated matroid that is not R-minor.

Proof ingredients

- Careful selection of smallest f and R-minor representation of a counterexample
- LP relaxation of R-minor representation, dual rephrased in terms of Lovász extension
- Exploiting that the matroid $\mathcal{M} = (V, \mathcal{B})$ of maximizers is not fully reducible: cannot be written as the sum of smaller matroids.
- Uncrossing, uncrossing & uncrossing:
  - Case $W = \emptyset$: the max weight independent matchings for $X \in \mathcal{B}$, along with the max weight independent matching for X^* can be recombined to give matching of weight ≥ -1 for some Y with $f(Y) < -1$
  - Case $W \neq \emptyset$: showing that nodes in W can be removed



Connections to mathematical economics

Gross substitutes

- V: set of n indivisible items
- $v: 2^V \to \mathbb{R}_+$ valuation function
- Given prices $p \in \mathbb{R}^V$, the agent wishes to buy a set of goods from
$$D(v,p) = \arg\max_S v(S) - p(S)$$
- Gross substitutes property: if the price of some items goes up, then the demand for all other goods may not decrease.
- Introduced by Kelso & Crawford in 1982 in the context of matching markets & auction algorithms
- Gül & Stacchetti 1999: Set of valuations for which Walrasian equilibrium exists

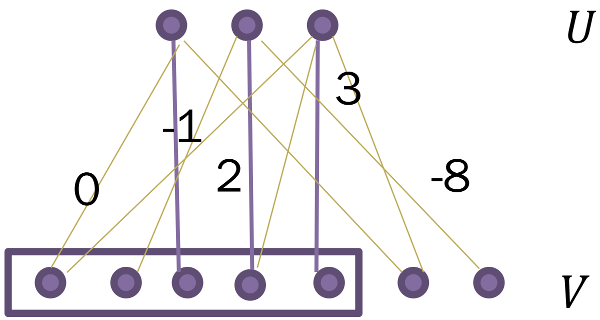
Gross substitutes

$f: 2^V \to \mathbb{R} \cup \{-\infty\}$ is a valuated generalized matroid if

- $\forall X, Y \subseteq V,\ \ |X| < |Y|\ \exists j \in Y \setminus X$
$$f(X) + f(Y) \leq f(X + j) + f(Y - j)$$
- $\forall X, Y \subseteq V,\ |X| = |Y|,\ \ \forall i \in X \setminus Y\ \ \exists j \in Y \setminus X$
$$f(X) + f(Y) \leq f(X - i + j) + f(Y + i - j)$$
- Valuated matroids = M-concave set functions ~ bases
  Valuated generalized matroids = M♮-concave set functions ~ independent sets
- Valuated generalized matroids are submodular functions

THEOREM (Fujishige & Yang 2003): Valuated generalized matroids = Gross substitutes valuation functions

Assignment valuations



Bipartite graph $G = (V, U; E), w \in \mathbb{R}^E$

$v(X) =$ max cost of a matching with endpoints in X

Characterizing gross substitutes valuations

- Hatfield & Milgrom 2005: does every GS valuation arise from assignment valuations using endowment (~contraction)?
- Ostrovsky & Paes Leme 2015: no, because not all matroid rank functions arise in this form

Matroid based valuation conjecture: every GS valuation arises from matroid rank functions using endowment (~contraction) and merge (~convolution)

- Tran 2020: merge only does not suffice

Matroid based valuation conjecture: every GS valuation arises from matroid rank functions using endowment (~contraction) and merge (~convolution)

- Stronger version: every GS valuation arises from matroid rank functions using endowment and induction by network
- This is a strengthening of the conjecture valuated matroids = R-minor
- Take a counterexample f from our construction scaled to value range $(-1{,}0]$

$$h(X) = \begin{cases} |X| \text{ if } |X| \leq 3 \\ 4 + f(X) \text{ if } |X| = 4 \\ 4 \text{ if } |X| \geq 5 \end{cases}$$

How did we get interested in this problem?

- Nash Social Welfare problem: given m indivisible items and n agents with valuation functions $v_i: 2^m \to \mathbb{R}_+$, find an allocation $[m] = S_1 \cup S_2 \cup \cdots \cup S_n$ that maximizes
$$\prod_{i=1}^{n} v_i(S_i)^{\frac{1}{n}}$$
- Cole & Gkatzelis 2015, AGSS 2017, BKV 2018: constant factor approximations for additive utilities
- Li & Vondrák 2021: estimation algorithm for conic combinations of Rado valuations
- Garg, Husić, V. 2021: constant factor approximation for Rado valuations
- Li & Vondrák 2021: constant factor approximation for general submodular valuations

Summary

- Valuated matroids ≠ R-minor: valuated matroids are (much?) more complex objects than matroids
- The matroid based valuation conjecture is false
- Implication on Lorentzian polynomials: not all Lorentzian polynomials can be obtained from generating functions of matroids using variable transformations by nonnegative matrix multiplication

Open questions

- Can we formulate a succinct property/inequality satisfied by R-minor but not all valuated matroids?
- Can we bound the size of the contracted set W in R-minor representations by poly($|V|$)?
  Kratsch & Wahlström 2020: bound for gammoids
- Understand larger complete classes of valuated matroids
- Can we 'well-approximate' valuated matroids using R-minor functions?
- How well can Rado/gross substitutes approximate nonnegative submodular functions?
  Dobzinski, Feige, Feldman 2020: lower bound $\Omega\left(\frac{\log n}{\log\log n}\right)$

Thank you!