TFY4240 - Electromagnetic Theory
Assignment 1

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The problem is to dtermine the potential in an infinitely long, square, hollow tube, where the four walls have different potentials $V(x, y)$. The boundary conditions are as follows:

$$\begin{aligned} V(0, y) &= 0, \\ V(L, y) &= 0, \\ V(x, 0) &= 0, \\ V(x, L) &= V_0(x) \end{aligned} \tag{1}$$

The equation that need to be solved is Laplace's equation:

$$\nabla^2 V = 0 \tag{2}$$

Assuming the solution take the form:

$$V(x, y) = X(x) \cdot Y(y) \tag{3}$$

where X and Y have to be determined. This is done by substituting (3) into (2) which leads to the following differential equation to be solved:

$$\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = 0 \tag{4}$$

Given the three first boundary conditions in (1) the general solution to this differential equation is as follows:

$$V(x, y) = \sum_{n=1}^{\infty} C_n \cdot \sin(n\pi\frac{x}{L}) \cdot \sinh(n\pi\frac{y}{L}) \tag{5}$$

The C_n-s are determined by the fourth boundary condition by what Griffiths calls Fourier's trick [1]. By multiplying $V(x, L)$ with $\sin(n\prime\pi\frac{x}{L})$ and integrate x from 0 to L one obtains:

$$\sum_{n=1}^{\infty} C_n \cdot \sinh(n\pi) \underbrace{\int_0^L \sin(n\pi\frac{x}{L}) \sin(n\prime\pi\frac{x}{L}) \mathrm{d}x}_{L/2} = \int_0^L V_0(x) \sin(n\prime\pi\frac{x}{L}) \mathrm{d}x \tag{6}$$

Which in turn yields the following expression for C_n:

$$C_n = \frac{2}{L\sinh(n\pi)} \int_0^L V_0(x) \sin(n\pi\frac{x}{L}) \mathrm{d}x \tag{7}$$

To make it dimensionless substitute $\xi = x/L$ and $\eta = y/L$ to obtain:

$$V(\xi, \eta) = \sum_{n=1}^{\infty} \underbrace{\left[\frac{2}{\sinh(n\pi)} \int_0^1 V_0(\xi\prime) \sin(n\pi\xi\prime) \mathrm{d}\xi\prime\right]}_{C_n} \cdot \sinh(n\pi\eta) \sin(n\pi\xi) \tag{8}$$

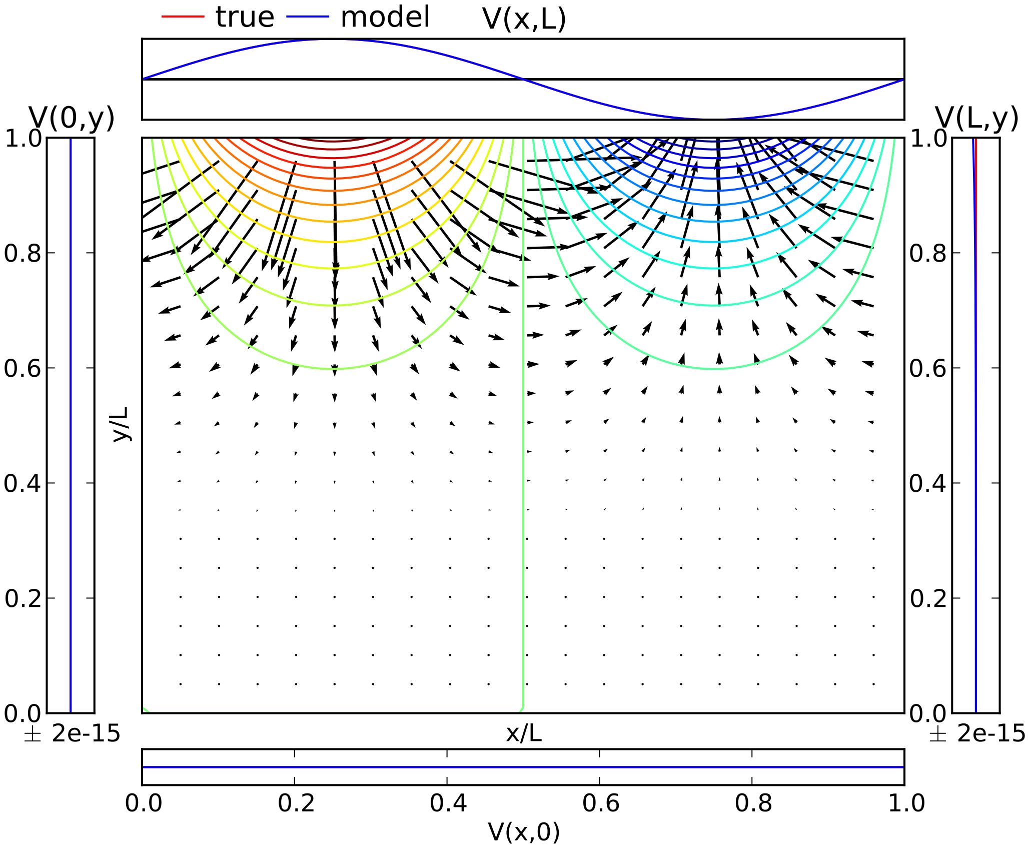

Figure 1: A simple sinusoidal potential

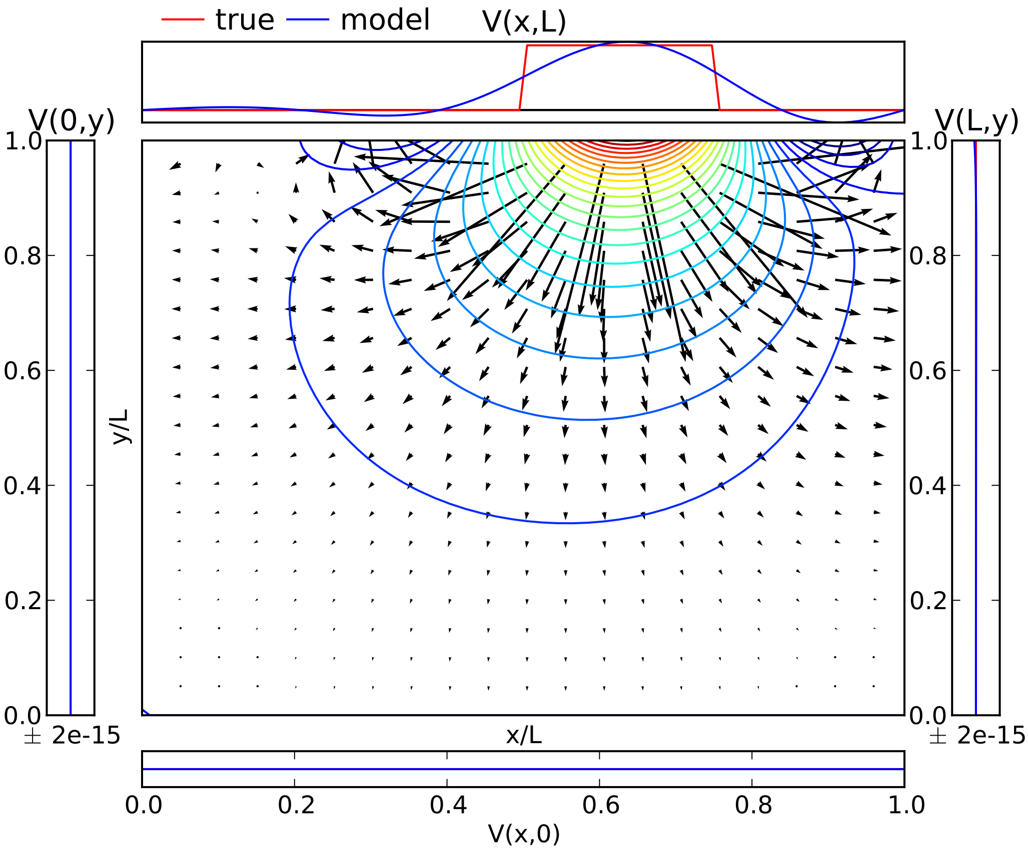

Figure 2: A discrete potential approximated by a 5th order Fourier-series.

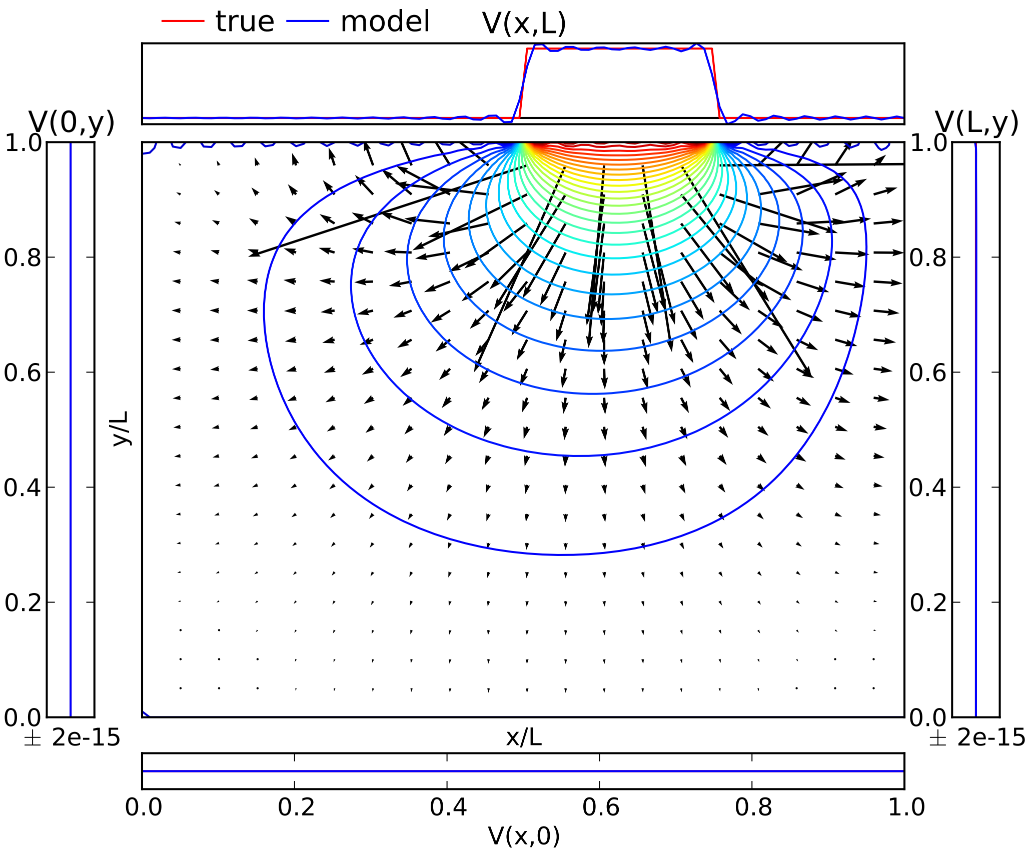

Figure 3: A discrete potential approximated by a 50th order Fourier-series.

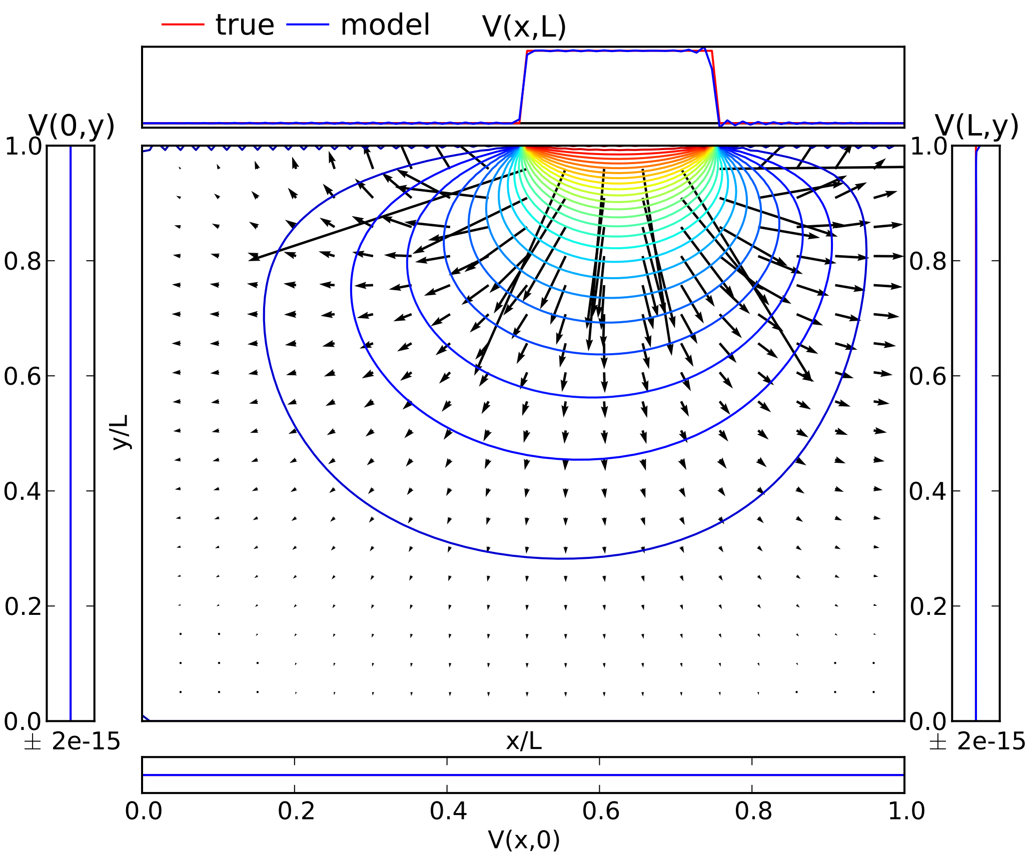

Figure 4: A discrete potential approximated by a 100th order Fourier-series.

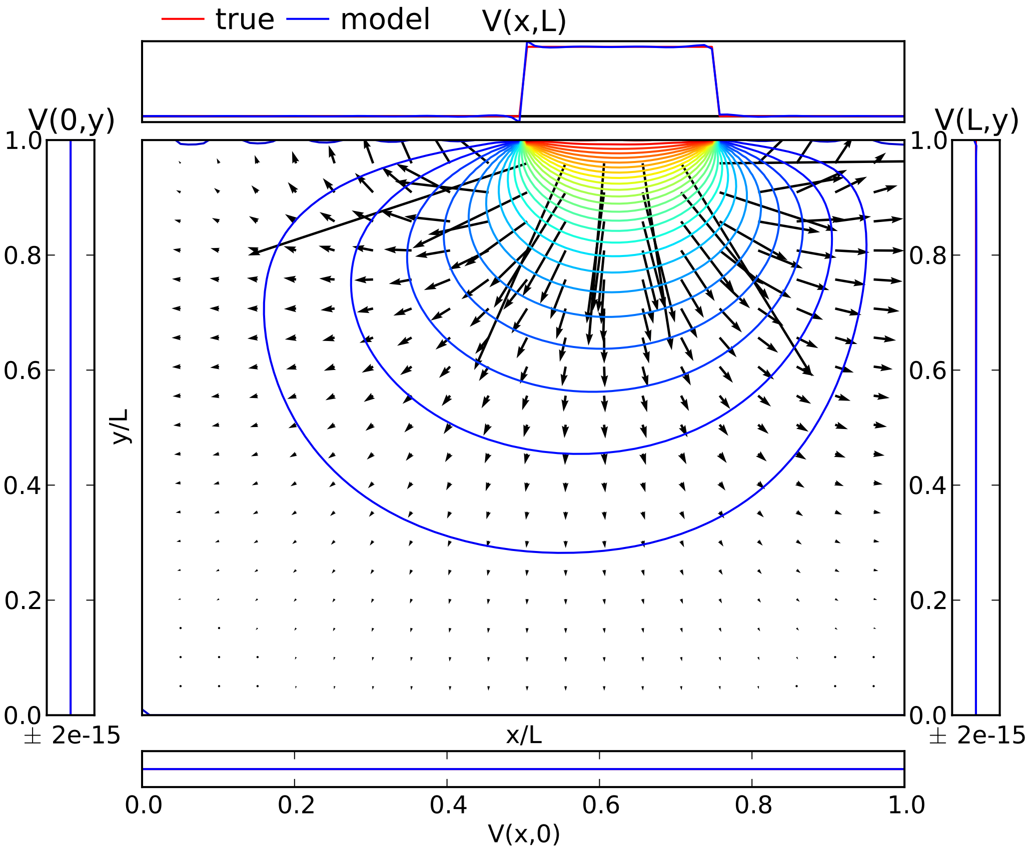

Figure 5: A discrete potential approximated by a 220th order Fourier-series.

which is easy to implement numerically. In *project.py* this is implemented and a graphical solution is presented. In the following figures the true potential at the boundaries is marked as a red line whereas the blue line is the numerically calculated potential. It should be noted that the range of the boundary potentials for $V(0, y)$, $V(x, 0)$ and $V(L, y)$ is $\pm 2 \cdot 10^{-15}$ and that the apparent divergence from 0 is due to rounding error. The arrows represent the eletrical field.

As can be seen from figures 2, 3, 4 and 5, adding more terms gives a better and better approximation of the true potential. By compairing figure 4 and 5, adding 120 more terms does little to better the approximation. Trying to solve for a somewhat more complicated potential also works reasonably well as is illustrated in figure 6. Though by studying figure 6 closer some concerns may be raised about convergence by looking closer at the edges of the potential at the 'top'. Since the edges of the true potential are not zero, the chosen Fourier-series of only sine functions have trouble approximating this. This becomes more apparent if a constant non-zero potential is tired approximated, as is shown in figure 7 and 8.

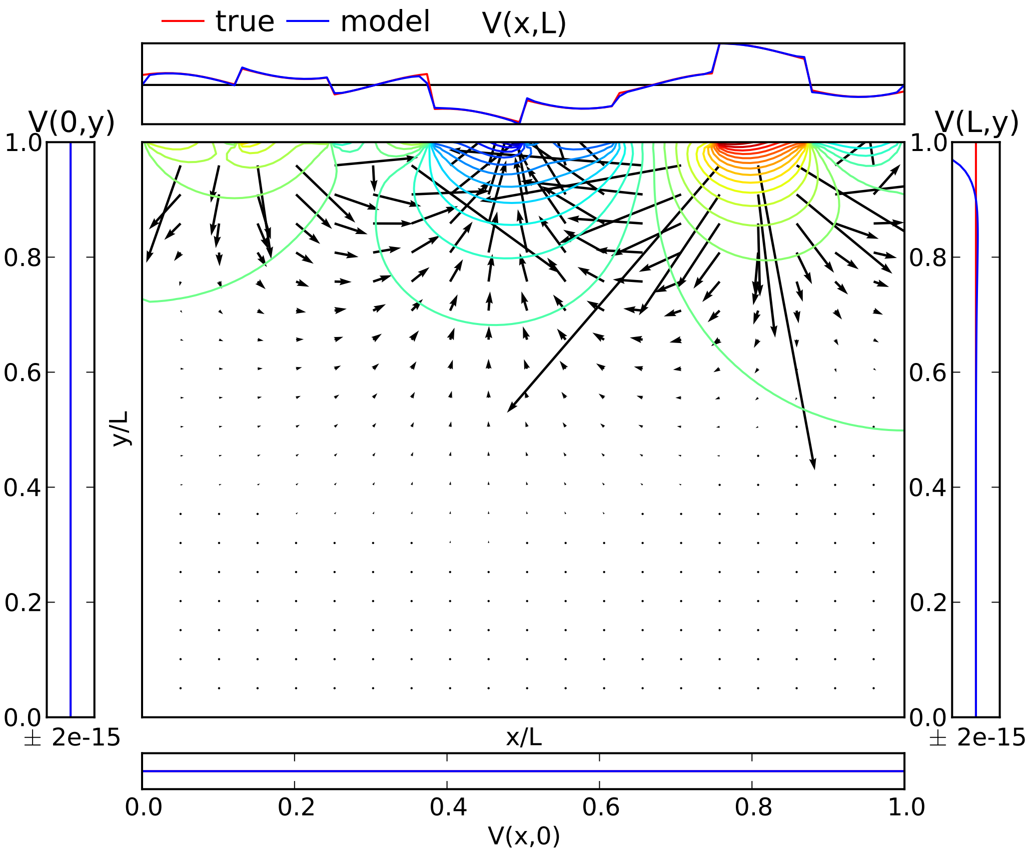

Figure 6: A somewhat complicated potential approximated by a 200th order Fourier-series.

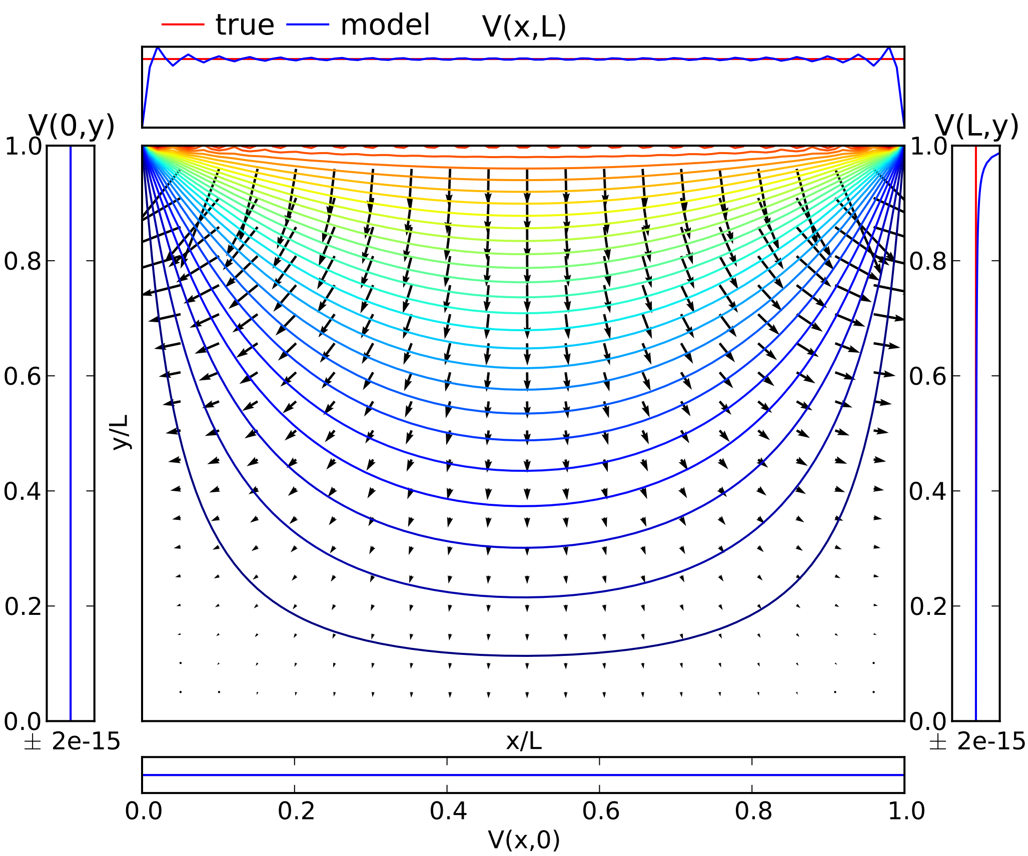

Figure 7: A constant non-zero potential approximated by a 50th order Fourier-series.

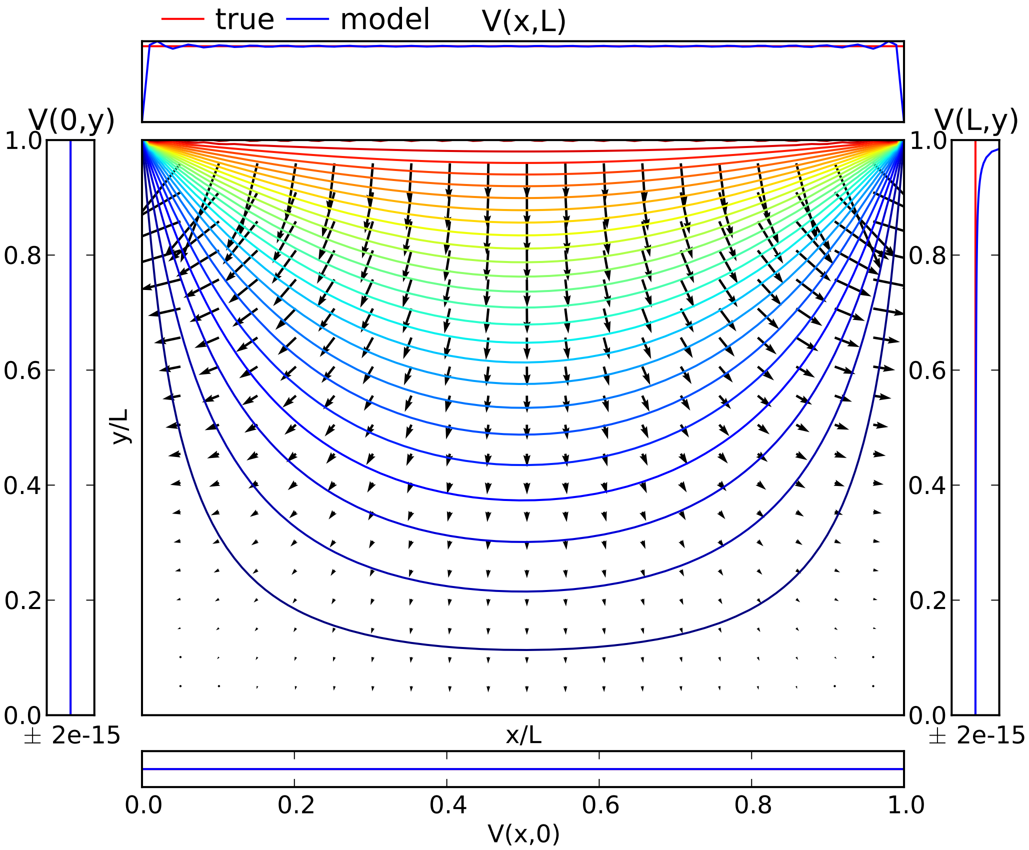

Figure 8: A constant non-zero potential approximated by a 150th order Fourier-series.

References

[1] David J. Griffiths. *Introduction to Electrodynamics*. Pearson Benjamin Cummings, 1301 Sansome St., San Fransosco, CA 94111, third edition, 2008.