IOP PUBLISHING EUROPEAN JOURNAL OF PHYSICS

Eur. J. Phys. **28** (2007) 551–561 doi:10.1088/0143-0807/28/3/017

A simple approach to Fourier aliasing

James Foadi

Department of Physics, University of York, York YO10 5DD, UK

E-mail: jf117@york.ac.uk

Received 13 February 2007, in final form 14 March 2007
Published 24 April 2007
Online at stacks.iop.org/EJP/28/551

Abstract
In the context of discrete Fourier transforms the idea of aliasing as due to approximation errors in the integral defining Fourier coefficients is introduced and explained. This has the positive pedagogical effect of getting to the heart of sampling and the discrete Fourier transform without having to delve into effective, but otherwise long and structured, introductions to the topic, commonly met in advanced, specialized books.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

A huge mass of books, reviews and articles has been published on the Fourier transform and concepts related to it. The discrete Fourier transform, too, has an overwhelming presence in the scientific literature (see for instance the excellent book by Briggs and Henson [1]). Most of the jargon used in this topic, though, has developed within electronics and information theory contexts. To undergraduate students terms like aliasing, or band-limited signals do not generally appear to have an immediate connection with the mathematical framework of Fourier theory. For the first- or second-year student, 'Fourier' is simply a trigonometric sum, while the Fourier transform always gems out of the expression for the Fourier coefficients in a trigonometric series. No more than algebra, trigonometry and some basic skills with integration techniques is required to grasp the meaning of Fourier theory. Why bother, then, to go through unfamiliar subjects and obscure jargon if the theory is as simple as that? Such a complication is not needed in an undergraduate physics course. Indeed, two examples are widely used to introduce the concept of aliasing in an easy way. One is the apparent slower or backward rotatory motion of turning objects filmed by cameras with slow shutter mechanisms; the other is given by plots of pure sines or cosines sampled with an insufficient number of points. Although such examples are certainly valid and effective, still they are too elementary to connect the problem with the technicalities of a discrete Fourier transform. There is a need, in the present literature, to provide elementary, but exhaustive, explanations of the subject. One can stay within a purely introductory mathematics context and still explain effectively terms like sampling, band-limited, and aliasing. The goal of this short article is to show how to deal with some of the discrete Fourier transform technicalities, without trespassing in fields different from algebra, trigonometry or calculus.

2. Fourier coefficients and discrete Fourier transform

One of the fundamental topics in calculus concerns the approximation of functions by finite or infinite summations. In particular, students find it quite natural to learn that a periodic function can be expressed as a finite or infinite summation of sines and cosines. This is why the best way to introduce the idea of Fourier transform is through the Fourier series. Furthermore, students are normally required to compute transforms of functions with real, rather than complex, values. Let us suppose, then, that $f(x)$ is a real, periodic function of period 1 (generalizations to other finite periods are straightforward). If the function behaves properly (see for instance reference [2], p 484), it can be expanded as a Fourier series in the following way:

$$f(x) = \sum_{n=-\infty}^{+\infty} C_n \exp(2\pi i n x). \tag{1}$$

The Fourier coefficients C_n are generally complex quantities computed with integrals over one period, typically $[0, 1]$ (but other, equal-length intervals can be used):

$$C_n = \int_0^1 f(x) \exp(-2\pi i n x) \mathrm{d}x. \tag{2}$$

In this article we will indicate with $|C_n|$ the modulus of C_n, while φ_n will be its argument (or phase) in radians. The above integral defines a Fourier coefficient, but a trivial extension of its integration interval to $(-\infty, +\infty)$ turns it into a Fourier transform. The constraint given to $f(x)$ of being a real function produces an interesting property for the C_n we have to take into account during the elaboration of the fast Fourier transform, notably,

$$(C_n)^* = C_{-n} \quad \Leftrightarrow \quad |C_{-n}| = |C_n|, \qquad \text{and} \qquad \varphi_{-n} = -\varphi_n. \tag{3}$$

The reader can easily prove property (3) simply by applying definition (2). Very often the function $f(x)$ to be analysed has an unknown analytical form; only a discrete sample of its values is available for computation. Integral (2) needs, in such a case, to be replaced by a discrete summation. If the range from 0 to 1 is sampled regularly at N points, integral (2) is approximated as follows:

$$C_n \approx \tilde{C}_n \equiv \frac{1}{N} \sum_{m=0}^{N-1} f\left(\frac{m}{N}\right) \exp\left(-2\pi i n \frac{m}{N}\right). \tag{4}$$

The set of all $\tilde{C}_n$ computed through (4) forms the *discrete Fourier transform* of the N sampled values $\{f(m/N), m = 0, 1, \ldots, N-1\}$. As said before, the discrete Fourier transform comes out quite naturally as an approximation to Fourier coefficients.

Formula (4) gives the impression that any Fourier coefficient C_n for series (1) can be approximated by the related $\tilde{C}_n$. This is actually not true. Only N frequencies can be created with formula (4); any other frequency will be a repetition of the first N frequencies, because $\tilde{C}_n$ has period N, i.e.:

$$\tilde{C}_{n+N} = \tilde{C}_n. \tag{5}$$

This fundamental result springs directly out of definition (4). We have:

$$\begin{aligned}\tilde{C}_{n+N} &= \frac{1}{N}\sum_{m=0}^{N-1} f\left(\frac{m}{N}\right)\exp\left[-2\pi\mathrm{i}(n+N)\frac{m}{N}\right]\\ &= \frac{1}{N}\sum_{m=0}^{N-1} f\left(\frac{m}{N}\right)\exp\left(-2\pi\mathrm{i}n\frac{m}{N}\right)\underbrace{\exp\left(-2\pi\mathrm{i}N\frac{m}{N}\right)}_{=1}\\ &= \frac{1}{N}\sum_{m=0}^{N-1} f\left(\frac{m}{N}\right)\exp\left(-2\pi\mathrm{i}n\frac{m}{N}\right) \equiv \tilde{C}_n.\end{aligned}$$

Let us make a numeric example and select $N = 65$ (we will assume N is odd for the remainder of this paper; a short comment for those cases with N being even will be added towards the end of the article). Then, the following quantities can be originated by formula (4):

$$\tilde{C}_0, \tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_{63}, \tilde{C}_{64}.$$

Any other $\tilde{C}_n$ with $n < 0$ or $n > 64$ assumes one of the 65 values previously displayed. Equivalently, we could have chosen to apply formula (4) to determine the following set:

$$\tilde{C}_{-32}, \tilde{C}_{31}, \tilde{C}_{-30}, \ldots, \tilde{C}_0, \ldots, \tilde{C}_{31}, \tilde{C}_{32}$$

which, again, consists of 65 values. In this case any other $\tilde{C}_n$ with $n < -32$ or $n > 32$ is already contained in the previous set. In general, at least for the kind of real functions we are considering, the discrete Fourier transform will generate coefficients with frequencies ranging from 0 to $n^* \equiv (N-1)/2$. All negative frequencies will be related to positive frequencies through property (3).

At this point a legitimate question comes to mind; if only a limited set of frequencies can be obtained through the discrete Fourier transform, how is it possible to have an infinite number of coefficients to determine the original function through formula (1)? The answer is: it is simply not possible. The availability of only a discrete sample of a continuous function makes it, in general, impossible to have a complete knowledge of the function itself (but see later for an important exception to this). The approximation of the Fourier coefficients C_n with the discrete Fourier transform $\tilde{C}_n$ is mirrored by the approximation of function $f(x)$ with a function $\tilde{f}(x)$ defined by:

$$\tilde{f}(x) \equiv \sum_{n=-n^*}^{n^*} \tilde{C}_n \exp(2\pi\mathrm{i}nx), \tag{6}$$

where $n^* = (N-1)/2$, as said before.

A couple of examples will help to clarify the argument. Consider, first, a trigonometric polynomial:

$$f(x) = 2\cos(\pi/2 - 2\pi x) + 6\cos(\pi - 6\pi x) + 4\cos(8\pi x) + 2\cos(3\pi/2 - 10\pi x). \tag{7}$$

Now, suppose only $N = 9$ values of $f(x)$ are known. The discrete Fourier transform carried out over these 9 points will produce 9 $\tilde{C}_n$, which form an approximation to the 9 C_n. The sampling introduces errors in the evaluation of the Fourier coefficients. It is quite natural to expect that $\tilde{f}(x)$ will be different from $f(x)$. This difference does indeed occur, and it is shown in the plot in figure 1. The dissimilarity of the two functions is immediately evident, and it is clear that $\tilde{f}(x)$ tries to interpolate $f(x)$ at those points forming the initial available sample.

Another interesting example is given by a mixture of VonMises functions. A VonMises function is the equivalent of the Gauss function for a periodic range (see reference [3] for further

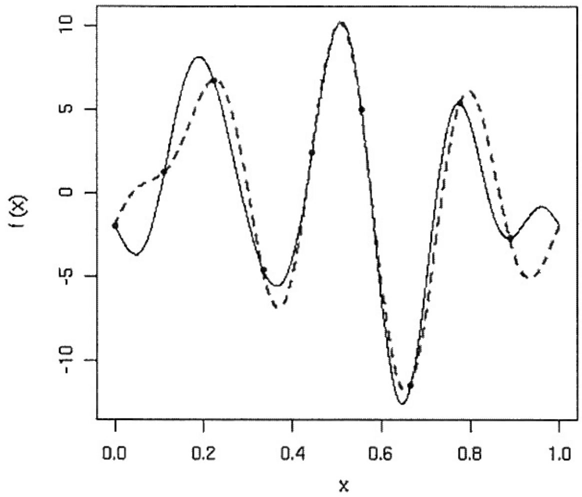

Figure 1. A trigonometric polynomial of degree 5 (continuous line), approximated by a Fourier summation (red, broken line). Only 9 sample points are available in this case (thick blue points).

details). The function width is characterized by a 'shape factor' G. Its analytic expression over one period is:

$$f(x) = \frac{1}{2\pi I_0(G)} \exp[G\cos(2\pi x)],$$

where I_0 is the modified Bessel function of order zero. This is an important and interesting example of a periodic function, not immediately related to trigonometric polynomials, and not having discontinuities in the first or higher derivatives; we can use it to illustrate aliasing, without having to worry about other effects, like the Gibbs phenomenon, typical of functions with discontinuities. In short, the partial sum of the Fourier series of such functions near each discontinuity exhibits a considerable overshoot. Increasing the number of terms forming a partial sum does not eliminate the overshooting, which never drops below 9% of half the discontinuity jump. This effect, first explained by the American physicist Josiah Willard Gibbs, makes introductory explanations on the convergence of the Fourier series unproductively complicated. The example we would like to introduce is, actually, a mixture of three VonMises functions:

$$\begin{aligned} f(x) = {} & \frac{2}{2\pi I_0(10)} \exp\{10\cos[2\pi(x-0.2)]\} \\ & + \frac{3}{2\pi I_0(20)} \exp\{20\cos[2\pi(x-0.6)]\} \\ & + \frac{1}{2\pi I_0(5)} \exp\{5\cos[2\pi(x-0.9)]\}. \end{aligned}$$

Let us suppose we have a sample of only 21 points available to compute the Fourier coefficients. Function $\tilde{f}(x)$ approximating $f(x)$ in this case is plotted against $f(x)$ in figure 2. There is still a dissimilarity between the two functions, although small on this occasion. Once more, we can observe a willingness of $\tilde{f}(x)$ to 'mimic' $f(x)$, at least at the sampled points. This 'mimicking' is known as *aliasing* in the context of discrete Fourier transforms. It is generally said that $\tilde{f}(x)$ is an 'alias' of function $f(x)$ at the sampled points. Whichever the starting sample, the approximated function built with coefficients coming out of a discrete Fourier

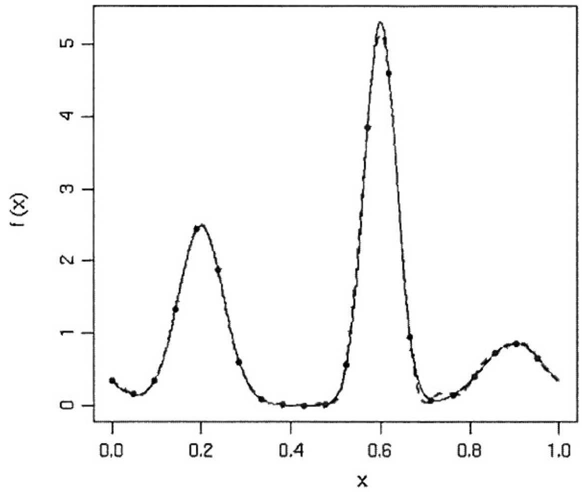

Figure 2. A mixture of VonMises functions between 0 and 1 (continuous line) is approximated by a Fourier summation (red, broken line). The thick, blue points form the initial available sample.

transform will always pass through those selected points. We will expand on this concept in the next two sections.

3. The approximating character of the discrete Fourier transform

After having shown qualitatively that Fourier coefficients computed through a discrete Fourier transform can raise approximation issues, it is time to delve quantitatively into such issues. Let us consider, then, definition (4):

$$\tilde{C}_n = \frac{1}{N}\sum_{m=0}^{N-1} f\left(\frac{m}{N}\right)\exp\left(-2\pi i n\frac{m}{N}\right)$$

and replace in it $f(m/N)$ according to series (1):

$$\begin{aligned}\tilde{C}_n &= \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n'=-\infty}^{+\infty} C_{n'}\exp\left(2\pi i n'\frac{m}{N}\right)\exp\left(-2\pi i n\frac{m}{N}\right)\\ &= \frac{1}{N}\sum_{n'=-\infty}^{+\infty} C_{n'}\left[\sum_{m=0}^{N-1}\exp\left(2\pi i m\frac{n'-n}{N}\right)\right].\end{aligned}$$

The term within square brackets sums the N complex roots of 1, of order N. In the complex plane this quantity is always zero, unless all complex vectors are collinear. This can only happen if $n' - n = pN$ (i.e. $n' = n + pN$), where p is an integer number. Therefore, the summation within square brackets equals N, and we are left with the following result:

$$\tilde{C}_n = \sum_{p=-\infty}^{+\infty} C_{n+pN}. \tag{8}$$

The above equation is an exact expression. It is a synthetic way to show that $\tilde{C}_n$ does not coincide with C_n. Rather, it equals C_n plus other terms:

$$\tilde{C}_n = \cdots + C_{n-2N} + C_{n-N} + C_n + C_{n+N} + C_{n+2N} + \cdots.$$

Equation (8) brings with it a somewhat depressing feeling; one becomes aware that by approximating C_n with $\tilde{C}_n$, an infinite number of terms are going to pollute the correct value of C_n. Luckily, in very many cases, nature provides functions having a limited frequency content. By this we mean that many functions can be generated through a finite Fourier summation. In this case there exists a maximum frequency, $n_{\max}$, such that $C_n = 0$ for all $n < -n_{\max}$ and $n > n_{\max}$. All those functions with the property just stated are known as *band-limited* functions. There is a wider class of functions which are not exactly band-limited, but whose Fourier coefficients beyond a given frequency are very close to zero. For all practical purposes these functions can be considered as band-limited. Let us assume that we are dealing with a band-limited function, whose highest frequency is $n_{\max}$. If the maximum frequency introduced by sampling, $n^* = (N-1)/2$ (or $N/2$ if N is even), is greater or equal than $n_{\max}$, we should expect the discrete Fourier transform to reproduce exactly all correct Fourier coefficients. It is, in fact, possible to show that, if $N > 2n_{\max}$, there is going to be no overlapping risk through formula (8). First of all, $n^* \geqslant n_{\max}$ is equivalent to $(N-1)/2 \geqslant n_{\max}$. This automatically implies $N > 2n_{\max}$, as an odd N which is greater than $2n_{\max}$ (remember we are at the moment assuming odd N's) is, by definition, greater than or equal to $2n_{\max} + 1$. Let us, next, consider any frequency n between $-n^*$ and n^*. From formula (8) we know that,

$$\tilde{C}_n = \cdots + C_{n-N} + C_n + C_{n+N} + \cdots.$$

If frequency n resides between $-n^*$ and n^*, then $-(N-1)/2 \leqslant n \leqslant (N-1)/2$; adding or subtracting N to this inequality produces other two inequalities:

$$\begin{cases} -(N-1)/2 - N \leqslant n - N \leqslant (N-1)/2 - N \\ -(N-1)/2 + N \leqslant n + N \leqslant (N-1)/2 + N \end{cases}$$

i.e.,

$$\begin{cases} (-3N+1)/2 \leqslant n - N \leqslant (-N-1)/2 \\ (N+1)/2 \leqslant n + N \leqslant (3N-1)/2 \end{cases}.$$

Given that $N > 2n_{\max}$, from the previous inequalities we deduce that $n - N < -n_{\max}$ and $n + N > n_{\max}$. As the sampled function is band-limited, $C_{n-N} = 0$ and $C_{n+N} = 0$; the same is, obviously, true for any C_{n-pN} and C_{n+pN}, with $p = 2, 3, \ldots$. In conclusion, $\tilde{C}_n$ is always equal to just one Fourier coefficient, C_n, the remaining infinite coefficients of sum (8) being zero.

In a nutshell, we have just shown that a periodic, band-limited function $f(x)$ can be correctly calculated starting from a sample of its values only if the size N of this sample obeys the following inequality,

$$N > 2n_{\max}, \tag{9}$$

where $n_{\max}$ is the highest frequency contained in the function. The 'important exception' introduced in section 2, when discussing the problem of recovering a function starting from a sample of it, is, thus, explained. If the number of sampling points satisfies condition (9), the whole function can be recovered starting from its discrete Fourier transform. Quantity $2n_{\max}$ is normally known as the *Nyquist sampling frequency* and the property just described goes under the name of the *Shannon sampling theorem*.

4. Aliasing

Now that the approximating character of the discrete Fourier transform has been explored in detail, we are in a position to explain why summation (6) did not reproduce exactly those

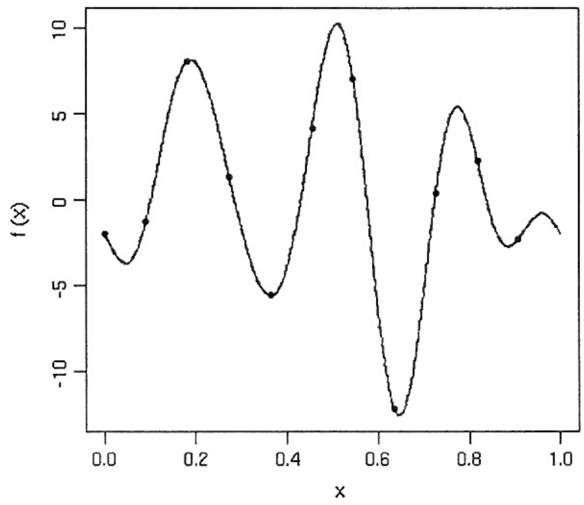

Figure 3. When the sampling size is at least 11, the discrete Fourier transform exactly reproduces (red, broken line) the trigonometric polynomial (continuous line) for all values between 0 and 1, not only at the sampled points (blue, thick points). There is no aliasing in this case.

Table 1. The complex coefficients for the example trigonometric polynomial used in section 2. They are zero for all frequencies higher than 5.

n	0	1	2	3	4	5	6	7	8	...
$\|C_n\|$	0	1	0	3	2	1	0	0	0	...
φ_n	0	$\pi/2$	0	π	0	$3\pi/2$	0	0	0	...

example functions of section 2. Let us consider the trigonometric polynomial (7), first. By repeatedly exploiting Euler's well known identity,

$$\cos\theta = \frac{\exp(i\theta) + \exp(-i\theta)}{2},$$

we can re-write the polynomial as a Fourier series,

$$f(x) = \sum_{n=-\infty}^{+\infty} C_n \exp(2\pi i n x)$$

where the coefficients are listed in table 1. All coefficients with $n < -5$ and $n > 5$ have amplitude zero. This is because a trigonometric polynomial is a band-limited function, i.e. it contains a finite number of frequencies. In this case $n_{\max} = 5$. According to condition (9), the minimum sampling required to reproduce this function in its entirety, using a Fourier summation, is $N > 10$ i.e., considering we are at the moment working with odd N's, $N \geqslant 11$. The sampling carried out in the example only contained 9 points, rather than 11. This is why the Fourier summation with 9 frequencies could not replicate exactly the function. A sampling of 11 points should be enough for its correct representation through the discrete Fourier transform. In figure 3, $\tilde{f}(x)$ and $f(x)$ are compared for this case. This time the matching is perfect; 11 sampled points are sufficient to replicate the whole function without errors.

Let us consider, next, the VonMises mixture. In figure 4 all amplitudes for the first 25 frequencies are shown. The coefficients' length decreases for higher frequencies and,

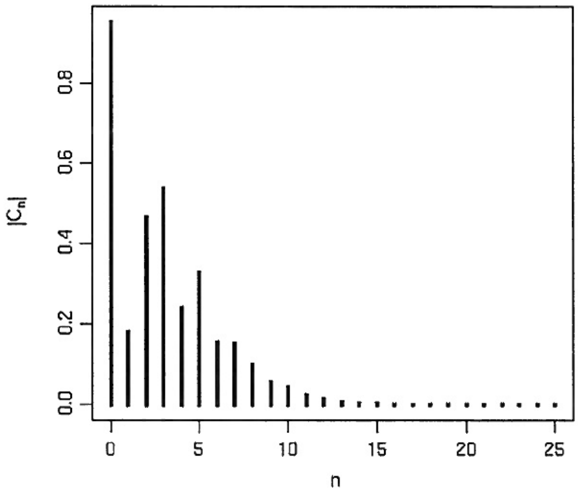

Figure 4. Amplitudes of Fourier coefficients for the mixture of VonMises functions.

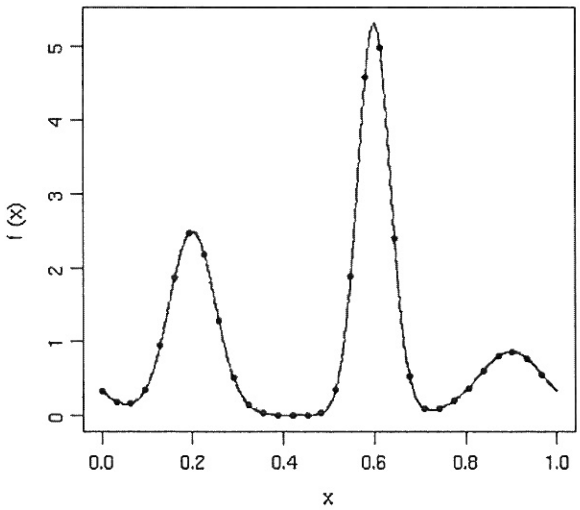

Figure 5. When the sampling size is at least 31, the discrete Fourier transform reproduces (red, broken line) the VonMises mixture to a high degree of precision (continuous line) for all values between 0 and 1, not only at the sampled points (blue, thick points). There is virtually no aliasing in this case.

therefore, contributions from high-frequency components alter the Fourier summation less and less. For all practical purposes we might say that for $n > 15$ all Fourier coefficients are zero; although these coefficients are never identically zero, still we can consider this function as a band-limited one, with $n_{\max} = 15$ playing the role of highest contained frequency. This being the case, sampling with $N = 31$ points should be enough to avoid aliasing (see figure 5).

It is an interesting feature of the discrete Fourier transform that the approximating function exactly coincides with the original one at all sampled points, even when the sampling is insufficient to avoid aliasing. The 'alias' function $\tilde{f}(x)$ tries to fit $f(x)$ in the best possible way. In so doing it passes exactly through all sampled points. In mathematical notation this means:

$$\tilde{f}\left(\frac{m}{N}\right) = f\left(\frac{m}{N}\right), \qquad m = 0, 1, \ldots, N-1. \tag{10}$$

To show that property (10) is verified for any value of N (and not only for $N > 2n_{\max}$), let us re-write expression (6) for any value N, at any of the points $0/N, 1/N, 2/N, \ldots, (N-1)/N$:

$$\tilde{f}\left(\frac{m}{N}\right) = \sum_{n=-(N-1)/2}^{(N-1)/2} \tilde{C}_n \exp\left(2\pi i n \frac{m}{N}\right).$$

In this expression let us replace $\tilde{C}_n$ with definition (4). We are left with the following quantity:

$$\begin{aligned}\tilde{f}\left(\frac{m}{N}\right) &= \sum_{n=-(N-1)/2}^{(N-1)/2} \frac{1}{N} \sum_{m'=0}^{N-1} f\left(\frac{m'}{N}\right) \exp\left(-2\pi i n \frac{m'}{N}\right) \exp\left(2\pi i n \frac{m}{N}\right) \\ &= \frac{1}{N} \sum_{m'=0}^{N-1} f\left(\frac{m'}{N}\right) \left[\sum_{n=-(N-1)/2}^{(N-1)/2} \exp\left(2\pi i n \frac{m-m'}{N}\right) \right].\end{aligned}$$

The quantity in square brackets, as previously explained, sums the N roots of 1, of order N; it is always zero, unless $m - m' = 0$, in which case it yields N. Thus,

$$\tilde{f}\left(\frac{m}{N}\right) = \frac{1}{N} \sum_{m'=0}^{N-1} f\left(\frac{m'}{N}\right) N\delta_{m'm} = f\left(\frac{m}{N}\right),$$

where $\delta_{m'm}$, the Kronecker symbol, is always zero, unless $m = m'$, when it is 1. With the above expression we have just proved property (10). Even for those cases where condition (9) is not respected, i.e. when the number of sampling points is not sufficiently high, we will observe that function $\tilde{f}(x)$ is an alias of function $f(x)$ at all sampled points. If condition (9) is applied, then $\tilde{f}(x)$ will match $f(x)$ throughout all intervals $[0, 1]$, not just at the sampled points.

5. Discrete sampling with an even number of points

Throughout this paper the number of sampled points, N, has been considered as an odd number. But, mainly due to the success of the fast Fourier transform, which put a strong emphasis on N as a power of 2, it is with even N's that discrete Fourier transforms are normally computed. It is important, therefore, to add a few comments to what has already been said, this time using an even N.

The main reason that forced us to work with an odd number of sampled points is related to the connection that was made between the coefficients of a Fourier series and their approximation through the discrete Fourier transform. To keep the level of difficulty to a minimum, we decided to handle only real, periodic functions. For them the Fourier series has a symmetrical expression (equation (1)), with the DC frequency ($n = 0$) being the central term of the expansion. Thus, for a band-limited function, the best way to connect a real, periodic function to its Fourier expansion is through the following relation:

$$f(x) = \sum_{n=-n_{\max}}^{n_{\max}} C_n \exp(2\pi i n x).$$

In fact, using Euler's trigonometric identity several times, the above real function can be written as a sum of pure real trigonometric functions:

$$f(x) = C_0 + 2\sum_{n=1}^{n_{\max}} |C_n| \cos(\varphi_n + 2\pi n x).$$

Function $\tilde{f}(x)$, approximating $f(x)$, can be built in a similar way:

$$\tilde{f}(x) = \tilde{C}_0 + 2\sum_{n=1}^{n^*} |\tilde{C}_n| \cos(\tilde{\varphi}_n + 2\pi n x) \tag{11}$$

where, as seen before, $n^* = (N-1)/2$ for odd N. There are, apparently, no conceptual problems in adopting, straightforwardly, the same expression for even N, where $n^* = N/2$. On the contrary, the periodicity of the discrete Fourier transform poses an important constraint in this case. Coefficient $\tilde{C}_{-N/2}$ is equal to $\tilde{C}_{N/2}$, because $-N/2 + N = N/2$. But, due to property (8), $\tilde{C}_{-N/2} = C_{-N/2} + C_{N/2} = 2|C_{N/2}| \cos\varphi_n$; this means that $\tilde{C}_{N/2}$ is always a real quantity. The amplitude and phase information of the related Fourier coefficient, $C_{N/2}$, are lost when $\tilde{C}_{N/2}$ is computed. Now, if $N/2 > n_{\max}$ this is not a big problem, as $|\tilde{C}_{N/2}| = |C_{N/2}| = 0$; things are different if $N = 2n_{\max}$. To see this let us re-write formula (11) for $n^* = N/2$:

$$\tilde{f}(x) = \tilde{C}_0 + 2\sum_{n=1}^{N/2-1} |\tilde{C}_n| \cos(\tilde{\varphi}_n + 2\pi n x) + \tilde{C}_{-N/2}\exp(-\mathrm{i}\pi N x) + \tilde{C}_{N/2}\exp(\mathrm{i}\pi N x).$$

Given that $\tilde{C}_{-N/2} = \tilde{C}_{N/2} = 2|C_{N/2}| \cos(\varphi_{N/2})$, the previous expression becomes:

$$\tilde{f}(x) = \tilde{C}_0 + 2\sum_{n=1}^{N/2-1} |\tilde{C}_n| \cos(\tilde{\varphi}_n + 2\pi n x) + 4|C_{N/2}| \cos\varphi_{N/2} \cos(\pi N x).$$

For even N $f(x)$ is:

$$f(x) = C_0 + 2\sum_{n=1}^{N/2-1} |C_n| \cos(\varphi_n + 2\pi n x) + 2|C_{N/2}| \cos(\varphi_{N/2} + \pi N x).$$

Even assuming $\tilde{C}_n = C_n$ for $0 \leqslant n \leqslant N/2 - 1$, $\tilde{f}(x)$ is different from $f(x)$, due to differences in the highest frequency. We were, indeed, expecting this, as $N/2$ is the first aliased frequency. What forces us to reconsider a different expansion for $\tilde{f}(x)$ is the fact that it is not equal to $f(x)$ even at the sampled points. In fact, when $m = 0, 1, \ldots, N-1$,

$$\tilde{f}\left(\frac{m}{N}\right) - f\left(\frac{m}{N}\right) = 2|C_{N/2}|[2\cos\varphi_{N/2}\cos(m\pi) - \cos(\varphi_{N/2} + m\pi)],$$

or, using $\cos(\varphi_{N/2} + m\pi) = \cos\varphi_{N/2}\cos(m\pi)$ and $\cos(m\pi) = (-1)^m$,

$$\tilde{f}\left(\frac{m}{N}\right) - f\left(\frac{m}{N}\right) = 2(-1)^m|C_{N/2}|\cos\varphi_{N/2}.$$

This is, of course, due to the amplitude and phase information lost when $\tilde{C}_{N/2}$ is computed, an operation which as a matter of fact considers twice the same contribution to $\tilde{f}(x)$. The interpolation property of the discrete Fourier transform for even N can be re-introduced in the formalism by adopting the following definition for the approximating function:

$$\tilde{f}(x) = \tilde{C}_0 + 2\sum_{n=1}^{N/2-1} |\tilde{C}_n| \cos(\tilde{\varphi}_n + 2\pi n x) + \tilde{C}_{N/2}\cos(\pi N x). \tag{12}$$

Aliasing will still occur for $N \leqslant 2n_{\max}$; the first even integer available to have a complete agreement between $\tilde{f}(x)$ and $f(x)$ is $N = 2n_{\max} + 2$. This value, as it should do, obeys the sampling criterium (9).

Acknowledgments

The author wishes to thank one of the referees for useful comments which have improved the readability of the original manuscript.

References

[1] Briggs W L and Henson V E 1995 *The DFT* (Philadelphia: SIAM)
[2] Kreyszig E 2006 *Advanced Engineering Mathematics* (Singapore: Wiley)
[3] Fisher N I 1993 *Statistical Analysis of Circular Data* (Cambridge: Cambridge University Press)