Fast Fourier Transform - FFT

$$H_n = \sum_{k=0}^{N-1} h_k \, e^{2\pi i k n / N}$$

$$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n \, e^{-2\pi i k n / N}$$

N values

N values

N^2 operations!

This is too much to be practical.

FFT: $\boxed{N^2 \text{ operations} \rightarrow N \log N \text{ operations!}}$

$$W_N \equiv e^{2\pi i / N}$$

$$H_n^{(N)} = \sum_{k=0}^{N-1} W_N^{nk} h_k$$

Basic idea is to write Fourier transform over N points as two Fourier transforms over $\frac{N}{2}$ points.

$$H_n^{(N)} = \sum_{k=0}^{N-1} W_N^{nk} h_k$$

$$= \sum_{k=0}^{\frac{N}{2}-1} W_N^{n(2k)} h_{2k}$$

$$W_N = e^{2\pi i/N}$$

$$+ \sum_{k=0}^{\frac{N}{2}-1} W_N^{n(2k+1)} h_{2k+1}$$

$$= \sum_{k=0}^{\frac{N}{2}-1} W_{\frac{N}{2}}^{nk} h_{2k}$$

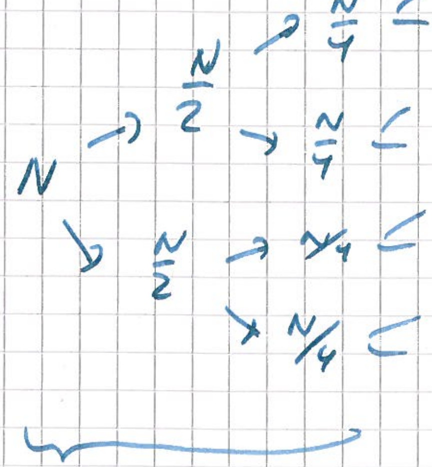
$$+ W_N^{n} \sum_{k=0}^{\frac{N}{2}-1} W_{\frac{N}{2}}^{nk} h_{2k+1}$$

$$= H_n^{(\frac{N}{2}),\,\mathrm{even}} + W_N^{n} H_n^{(\frac{N}{2}),\,\mathrm{odd}}$$

nth component of Fourier transform of length $\frac{N}{2}$, on <u>even</u> components of h_k.

nth component of Fourier transform of length $\frac{N}{2}$, on <u>odd</u> components of h_k.

This is done recursively:



log N levels.

Each level requires N operations $\Rightarrow$

N log N operations in total.

Most difficult problem in using FFT: Packing of components of H_m:

(Especially in higher dimensions)

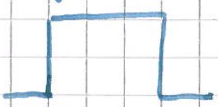
$N = 8$:

$\mathrm{Re}\, h_0$	$\mathrm{Re}\, H_0$	1
$\mathrm{Im}\, h_0$	$\mathrm{Im}\, H_0$	
$\mathrm{Re}\, h_1$	$\mathrm{Re}\, H_1$	2
$\mathrm{Im}\, h_1$	$\mathrm{Im}\, H_1$	
$\mathrm{Re}\, h_2$	$\mathrm{Re}\, H_2$	3
$\mathrm{Im}\, h_2$	$\mathrm{Im}\, H_2$	
$\mathrm{Re}\, h_3$	$\mathrm{Re}\, H_3$	4
$\mathrm{Im}\, h_3$	$\mathrm{Im}\, H_3$	
$\mathrm{Re}\, h_4$	$\mathrm{Re}\, H_4 = \mathrm{Re}\, H_{-4}$	Nyquist frequency
$\mathrm{Im}\, h_4$	$\mathrm{Im}\, H_4 = \mathrm{Im}\, H_{-4}$	5
$\mathrm{Re}\, h_5$	$\mathrm{Re}\, H_{-3}$	6
$\mathrm{Im}\, h_5$	$\mathrm{Im}\, H_{-3}$	
$\mathrm{Re}\, h_6$	$\mathrm{Re}\, H_{-2}$	7
$\mathrm{Im}\, h_6$	$\mathrm{Im}\, H_{-2}$	
$\mathrm{Re}\, h_7$	$\mathrm{Re}\, H_{-1}$	8
$\mathrm{Im}\, h_7$	$\mathrm{Im}\, H_{-1}$	

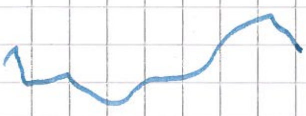
Convolution and Deconvolution

$$r * s(t) = \int_{-\infty}^{+\infty} d\tau \, r(\tau) \, s(t-\tau)$$

response



signal



Typical situation in experiments: Measure signal s with instrument that transforms it.

We discretize:

$$s(t) \to s_j$$

$$r(t) \to r_j$$

$$(r * s)_j = \sum_{k=-\frac{M}{2}+1}^{\frac{M}{2}} r_k \, s_{j-k}$$

$$r_k \neq 0 \text{ for } -\frac{M}{2} \le k \le \frac{M}{2} .$$

Interpretation of r_j:

Input signal s_j in channel j:

r_0 gives how much of s_j remains in this channel.

r_1 gives how much of s_j is sent into channel $j+1$.

And so on...

Identity response:

$$r_0 = 1, \quad r_k = 0, \quad k \neq 0.$$

$$s_j \to \begin{matrix} j-1 \\ j \\ j+1 \end{matrix} < \begin{matrix} r_{-1} \\ r_0 \\ r_1 \end{matrix} \to \begin{matrix} r_{-1}\, s_{j-1} \\ r_0\, s_j \\ r_{+1}\, s_{j+1} \end{matrix}$$

Discrete convolution theorem:

s_j is periodic with period N.

$s_0, s_1, \ldots, s_{N-1}$ determines s completely.

r_k has range $M \leq N$.

Define $r_k = 0$ for k that are outside the range: $-\frac{M}{2}+1 \leq k \leq \frac{M}{2}$.

This is called <u>padding</u>.

Then we have

$$\boxed{\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} r_k \, s_{j-k} \iff S_n R_n}$$

$n = 0, \ldots, N-1$

Fourier transformed.

What if s is not periodic with period N?

$j = 0$

$$(r * s)_0 = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} r_k \, s_{-k}$$

We fold the data from "higher" channels into the 0-channel:

$s_{-1} = s_{N-1}$

$s_{-2} = s_{N-2}$

$s_{-3} = s_{N-3}$

$$s_{-n} = \underbrace{s_{N-n}}$$

s is <u>made</u> periodic

This is fixed through <u>padding</u>.

$s_k = 0, \; k = N+1, \ldots, N+M$

$N \to N+M.$

Deconvolution:

$$(r * s)_j = \sum_{-\frac{N}{2}+1}^{\frac{N}{2}} r_k \, s_{j-k}$$

known

unknown.

This is an equation set that needs to be solved with respect to s_j.

Better to calculate $R_n S_n$ from $(r*s)_j$. Divide by R_n $(\neq 0)$ and FFT backwards.

In practice, deconvolution is difficult. Data are very sensitive to noise.

Wavelet

Fourier transform:

$$H(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt \; h(t) \, e^{2\pi i f t}$$

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} df \; H(f) \, \underbrace{e^{-2\pi i f t}}$$

Orthonormal set of functions.

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \; e^{2\pi i (f-g) t} = \delta(f-g)$$

But $\dfrac{e^{2\pi i f t}}{\sqrt{2\pi}}$ has very little structure. It is not possible to see in $H(f)$ where a given feature in $h(t)$ comes from.

Wavelets are an expansion of $h(t)$ in a different basis set than $e^{2\pi i f t}$, one that retains accessible knowledge of features of $h(t)$ in t-space:

$$\boxed{\psi_{a,b}(t) = \frac{1}{\sqrt{b}}\,\psi\left(\frac{t-a}{b}\right)}$$

position

scale

this is a compact "mother" function.

Compact: $\psi \equiv 0$ outside a finite interval.

$$h(t) = \sum_{a,b}{}' F_{a,b} \, \psi_{a,b}(t)$$

There are many different $\psi(t)$ that can be used as basis functions.

Wavelets are used for:

① Data filtering

② Data compression

③ Accelleration of matrix inversion

④ Fractal analysis

⑤ Data reconstruction ....

etc.