We need to find a matrix B rather than A:

$$A(x_{k+1} - x_k) = b - Ax_k$$

$$\longrightarrow$$

$$\boxed{B(x_{k+1} - x_k) = b - Ax_k}$$

so that we still have convergence,

$$\| x_{k+1} - x_k \| \to 0$$

and which is <u>simple to invert</u>.

Then, we have

$$\boxed{x_{k+1} = B^{-1}b + B^{-1}(B-A)x_k}$$

Convergence is secured if all eigenvalues of $B^{-1}(B-A)$ lie within the unit circle.

How to choose B?

Splitting $A = L + D + R$ ← upper right triangle.

left lower triangle

diagonal

Jacobi iteration

Set $B = D$.

$$\boxed{x_{k+1} = D^{-1}b + \underbrace{D^{-1}(D-A)}_{}x_k}$$

$J = D^{-1}(D-A)$

"Jacobi" matrix.

Convergence

Largest eigenvalue in J: λ_J.

$$x_{k+1} = D^{-1}b + J x_k = c + \lambda_J x_k$$

$$x_{k+1} = c + \lambda_J (c + \lambda_J x_{k-1}))$$

$$= c + \lambda_J c + \lambda_J^2 c + \cdots + \lambda_J^k c$$

$$= \frac{1 - \lambda_J^k}{1 - \lambda_J} c$$

Convergence rate:

$$\boxed{\lambda_J = \frac{\| x_{k+1} - x_k \|}{\| x_k - x \|}}$$

Solution.

$$\| x_{k+1} - x_k \| = | \lambda_J^k | \, \| c \|$$

$$\| x_k - x \| = \left| \frac{1 - \lambda_J^k - 1}{1 - \lambda_J} \right| \| c \| = \left| \frac{\lambda_J^k}{1 - \lambda_J} \right| \| c \|$$

Hence,

$$\boxed{\Lambda_J = |1 - \lambda_J|}$$

Convergence of Jacobi method is usually low since

$$|\lambda_J| \simeq 1.$$

Gauss-Seidel relaxation

$$(D+L)(x_{k+1} - x_k) = b - A x_k$$

$$A = L + D + R$$

$$\underbrace{(D+L)}\, x_{k+1} = b - R x_k$$

Invert $D+L$ through forwards substitution.

Gauss-Seidel convergence rate:

$$r_{GS} = \frac{\| x_{k+1} - x_k \|}{\| x_k - x \|} = |1 - \lambda_J^2|$$

This means that the convergence rate is significantly better than that of the Jacobi method, $r_J = |1 - \lambda_J|$.

SOR - Successive OverRelaxation.

This is an efficient algorithm that should be considered.

The idea is to place the new x_{k+1} somewhere between x_k and x_{k+1}^{GS} – the result of a Gauss-Seidel relaxation step:

$$x_{k+1} = \omega x_{k+1}^{GS} + (1-\omega) x_k$$

relaxation parameter.

Convergence when $0 < \omega < 2$.

$$(D+L)\, x_{k+1}^{GS} = b - R x_k$$

$$\downarrow$$

$$x_{k+1} = \omega x_{k+1}^{GS} + (1-\omega) x_k$$

$$\downarrow$$

$$\boxed{(D+L)\, x_{k+1} = \omega b - (R - (1-\omega) A)\, x_k}$$

Iteration matrix

$$S = -(D+L)^{-1}(R-(1-\omega)A)$$

Optimal value for ω:

$$\omega = \frac{2}{1+\sqrt{1-\lambda_J^2}}$$

This value leads to a convergence rate

$$\lambda_S = |1-\lambda_s|$$

where

$$\lambda_s = \left(\frac{\lambda_J}{1+\sqrt{1-\lambda_J^2}}\right)^2$$

Concrete example:

$$\lambda_J = 0.5$$

Ohm law:

$$i_{12} = \frac{1}{r_{12}} (V_2 - V_1)$$

$$i_{23} = \frac{1}{r_{23}} (V_3 - V_2)$$

$$i_{34} = \frac{1}{r_{34}} (V_4 - V_3)$$

$$i_{45} = \frac{1}{r_{45}} (V_5 - V_4)$$

Kirchhoff law:

$$i_{12} = i_{23}$$

$$i_{23} = i_{34}$$

$$i_{34} = i_{45}$$

Combining:

$$r_{23} (V_2 - V_1) = r_{12} (V_3 - V_2)$$

$$r_{34} (V_3 - V_2) = r_{23} (V_4 - V_3)$$

$$r_{45} (V_4 - V_3) = r_{34} (V_5 - V_4)$$

In the following, we will assume that A is <u>always positive definite</u>.

(One can do without this assumption, but this renders the algorithms more complex – we have to "square" A.)

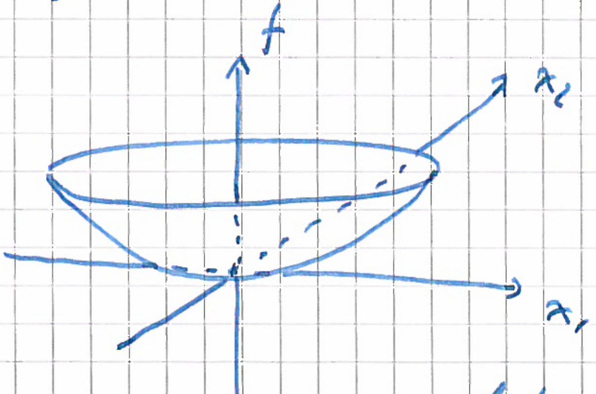
$Ax = b \quad \Longrightarrow$

$$\boxed{f(x) = \frac{1}{2} xAx - bx}$$

a function of x.

A positive definite $\Longrightarrow$

$f(x)$ has the shape



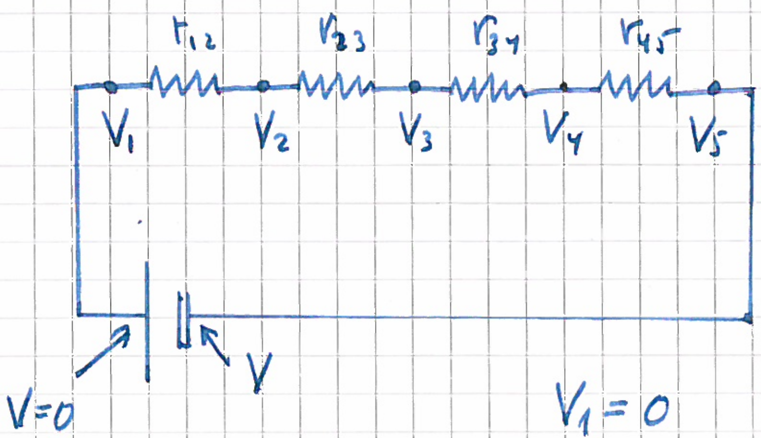
deformed cup.

$\lambda_J = 0.5$	$r_J = 0.5$
$\lambda_{GS} = 0.25$	$r_{GS} = 0.75$
$\lambda_S = 0.07$	$r_S = 0.93$

Different philosophy:

Steepest descent method.

Example of 4 resistors in series:



$V_1 = 0$

$V_5 = V$

Collecting terms:

$$(r_{12} + r_{23}) V_2 - r_{12} V_3 = r_{23} V_1 = 0$$

$$-r_{34} V_2 + (r_{23} + r_{34}) V_3 - r_{23} V_4 = 0$$

$$-r_{45} V_3 + (r_{34} + r_{45}) V_4 = r_{34} V_5 = r_{34} V$$

In matrix form:

$$\underbrace{\begin{pmatrix} (r_{12} + r_{23}) & -r_{12} & 0 \\ -r_{34} & (r_{23} + r_{34}) & -r_{23} \\ 0 & -r_{45} & (r_{34} + r_{45}) \end{pmatrix}} \begin{pmatrix} V_2 \\ V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r_{34} V \end{pmatrix}$$

This is matrix A.

It is not symmetric, but it is <u>positive definite.</u>

All eigenvalues are positive.

In our electrical example from page 48:

$$f(\vec{V}) = \frac{1}{2} \Big[(r_{12} + r_{23}) V_2^2 - (r_{12} + r_{34}) V_2 V_3 + (r_{23} + r_{34}) V_3^2 - (r_{23} + r_{45}) V_3 V_4 + (r_{34} + r_{45}) V_4^2 \Big] - r_{34} V V_4 .$$

Since $\nabla f(x) = Ax - b \underset{\uparrow}{=} 0$

when x is the solution to the equation $Ax = 0$.

Hence, finding the minimum of f is equivalent to solving $Ax = b$.

$$Ax = b \quad \Longleftrightarrow \quad \boxed{\left\{ x \;\middle|\; \min_{x'} f(x') \right\}}$$

Cauchy steepest descent method:

* Choose an initial x_0.
* Move in the direction of $-\nabla f(x_0)$
* Stop at $\min_x f(x)$ along this direction

  $\Rightarrow x_1$.

* repeat.

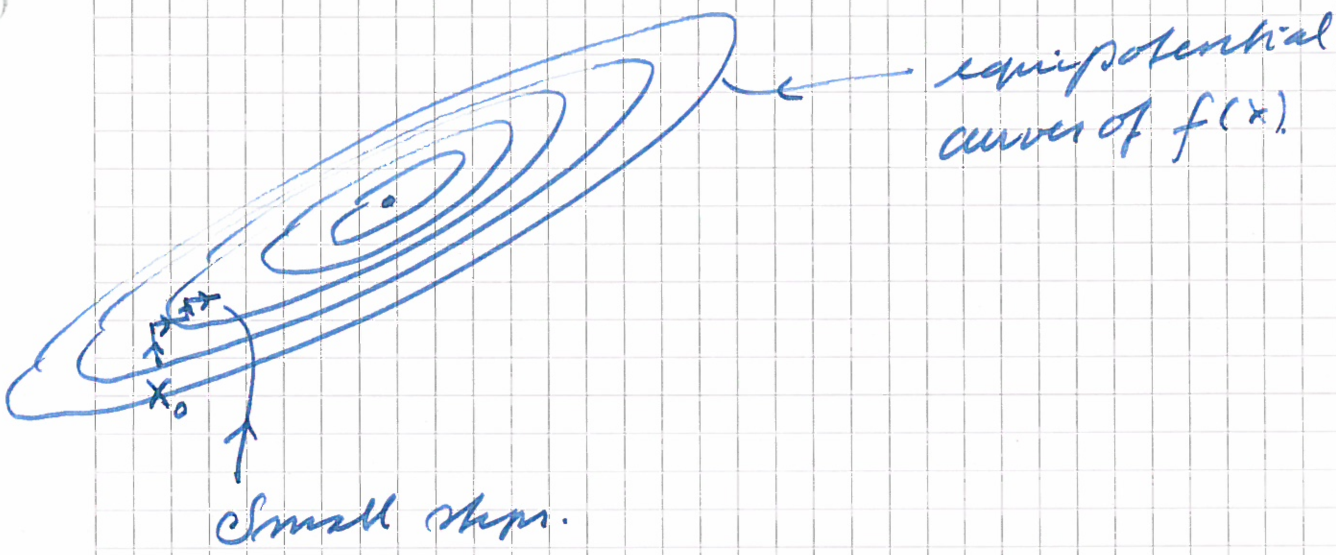
find $\frac{d}{d\lambda}\{f(x_0+\lambda\nabla f(x_0))\}=0$

There is a problem:

> Changes of direction are all orthogonal to each other.

- otherwise, $x_i = x_{i-1} + \lambda \nabla f(x_{i-1})$ would not be a minimum in the gradient direction.

Result: Small zig-zag steps.



A, b contains $N(N+1)$ parameters:

Number of minimizations $\sim N^2$,

Whereas dimensionality of x-space is N.