TFY4235/FY8904 Computational Physics, Spring 2013

Solution Set 8

Problem 1.

We rewrite (1) as two first-order equations,

$$\begin{aligned} y' &= u \\ u' &= -\omega_0^2 y \ . \end{aligned} \tag{1}$$

Explicit Euler integration gives

$$\begin{aligned} y_{n+1} &= y_n + \Delta u_n \\ u_{n+1} &= u_n - \Delta\omega_0^2 y_n \end{aligned} \tag{2}$$

wheere $\Delta = x_{n+1} - x_n$. The program looks like this:

```
      program explicit
c Integrate the wave equation using explicit Euler
      parameter(y0=1.,u0=1.,itemx=100000,del=0.0001,om=1.)
      yn=y0
      un=u0
      deo=del*om**2
      do ite=1,itemx
      ynp1=yn+del*un
      unp1=un-deo*yn
      write(*,*) ynp1,unp1
      yn=ynp1
      un=unp1
      enddo
      end
```

Implisit Euler integration gives

$$\begin{aligned} y_{n+1} &= y_n + \Delta u_{n+1} \\ u_{n+1} &= u_n - \Delta\omega_0^2 y_{n+1} \ . \end{aligned} \tag{3}$$

This can be rewritten on matrix form as

$$\begin{pmatrix} 1 & -\Delta \\ \Delta\omega_0^2 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_{n+1} \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} y_n \\ u_n \end{pmatrix} \ . \tag{4}$$

We invert and get

$$\begin{pmatrix} y_{n+1} \\ u_{n+1} \end{pmatrix} = \frac{1}{1+\Delta^2\omega_0^2} \begin{pmatrix} 1 & \Delta \\ -\Delta^2\omega_0^2 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_n \\ u_n \end{pmatrix} \ . \tag{5}$$

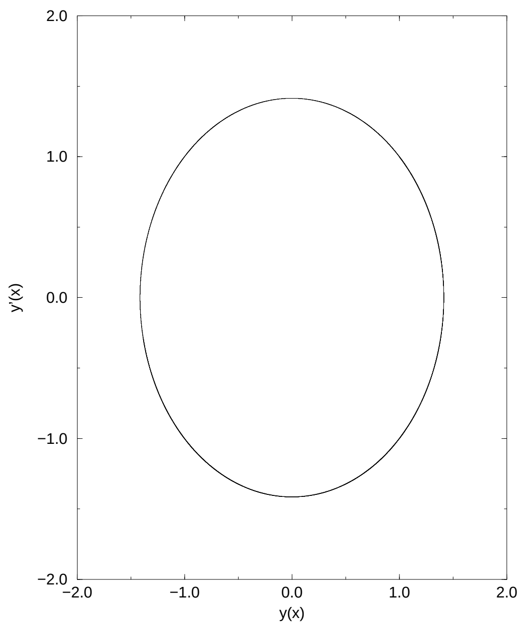
Written on component form, this is

$$\begin{aligned} y_{n+1} &= (y_n + \Delta u_n)/(1 + \Delta^2 \omega_0^2) \\ u_{n+1} &= (-\Delta \omega_0^2 y_n + u_n)/(1 + \Delta^2 \omega_0^2)\ . \end{aligned} \tag{6}$$

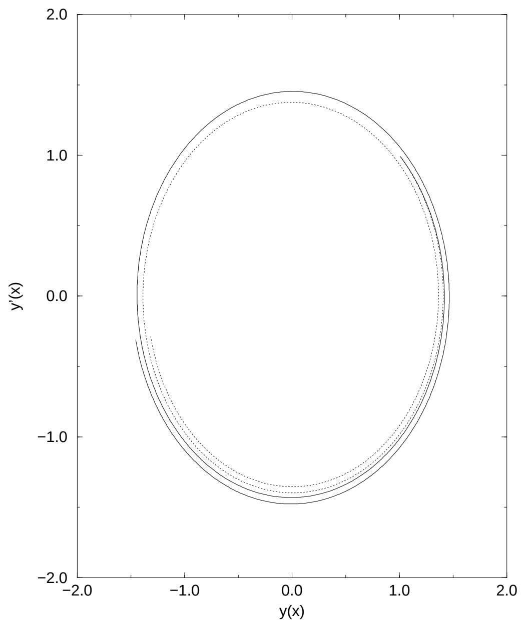
Here is the program:

```
      program implicit
c Integrate the wave equation using implicit Euler
      parameter(y0=1.,u0=1.,itemx=100000,del=0.0001,om=1.)
      yn=y0
      un=u0
      rat=1./(1.+del**2*om**2)
      deo=del*om**2
      do ite=1,itemx
      ynp1=rat*(yn+del*un)
      unp1=rat*(-deo*yn+un)
      write(*,*) ynp1,unp1
      yn=ynp1
      un=unp1
      enddo
      end
```

In the linear case, there is only one parameter, ω_0, and it sets the scale for the integration variable x. Hence, it is not necessary to change this as there will be no qualitaive changes in the solution. However, it does influence the solution to change the initial conditions. I have chosen only to study the $y_0 = u_0 = 1$ case. Here is the result of both the explicit and implicit integration:



We set $\Delta = 0.001$. No visible difference exists between the two algorithms. However, setting $\Delta = 0.01$ results in a difference. In the figure below the unbroken curve is based on the explicit integration, while the broken curve shows the implicit integration. This figure illustrates the important point that implicit integration makes the process stable, but not more accurate. We see that both solutions spiral away from the exact solution, the explicit outwards towards larger values and the implicit inwards towards zero.



The non-linear wave equation gives rise to the following explicit Euler scheme:

$$\begin{aligned} y_{n+1} &= y_n + \Delta u_n \\ u_{n+1} &= u_n - \Delta\omega_0^2 y_n - \Delta\beta y_n^3 \ . \end{aligned} \tag{7}$$

The implicit scheme usually cannot be implemented as we did in the linear case. The reason for that is that it is not possible to explicitly solve the equations[1]

$$\begin{aligned} y_{n+1} &= y_n + \Delta u_{n+1} \\ u_{n+1} &= u_n - \Delta\omega_0^2 y_{n+1} - \Delta\beta y_{n+1}^3 \end{aligned} \tag{8}$$

with respect to y_{n+1} og u_{n+1}. This has to be done numerically.

Here is an explicit Euler program for the non-linear case:

```
      program nonlin
c Integrate non-linear wave equation with explicit Euler
      parameter(y0=1.,u0=1.,itemx=100000,del=0.0001,om=1.,bet=-1.1)
      yn=y0
```

[1] However, in this particular example, they may actually be solved.

```
un=u0
deo=del*om**2
beo=del*bet
do ite=1,itemx
ynp1=yn+del*un
unp1=un-deo*yn-beo*yn**3
write(*,*) ynp1,unp1
yn=ynp1
un=unp1
enddo
end
```

Now there is a parameter in the problem, β/ω_0. I have set $\omega_0 = 1$, so that we only need to adjust β. The values for β that I have used in the figure below is -0.1, 0, 1.1 og 4.1. Along the negative abscissa, the values follw from left to right. In all cases, I have used $y(0) = 1$ and $u(0) = 1$ as initial values.