

$f$  at regular intervals, and in such a way that we know the errors we are making and how to improve on them in a systematic way.

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We start with a set of definitions:

$f(x)$  is a continuous function.

$$f_k = f(x_k) \text{ where } x_k = x_0 + k\Delta x$$

$\uparrow$

This is the only information we have about the function.

## Forward difference

$$\Delta f_k = f_{k+1} - f_k$$

We illustrate this definition

$$\Delta^2 f_k = \Delta f_{k+1} - \Delta f_k = f_{k+2} - 2f_{k+1} + f_k$$

$$\Delta^2 f_k = \sum_{i=0}^2 (-1)^i \binom{2}{i} f_{k+2-i}$$

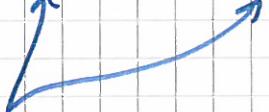
## Backwards difference

$$\nabla f_k = f_k - f_{k-1}$$

We illustrate the definition

$$\nabla^2 f_k = \sum_{i=0}^1 (-1)^i \binom{2}{i} f_{k-1+i}$$

## Central difference

$$\delta f_k = f_{k+\frac{1}{2}} - f_{k-\frac{1}{2}}$$


But, these points do not exist!

Again we make the definition

$$\delta^{-1} \delta^n$$

$$\delta^n f_k = \sum_{i=0}^n (-1)^i \binom{n}{i} f_{k-\frac{n}{2}+i}$$

No problem when n is even.

To get a definition that also works for odd n, we introduce the central average:

Central average:

$$\mu f_k = \frac{1}{2} (f_{k+\frac{1}{2}} + f_{k-\frac{1}{2}})$$

$$\mu^2 f_k = \frac{1}{2} (\mu f_{k+\frac{1}{2}} + \mu f_{k-\frac{1}{2}})$$

$$= \frac{1}{4} (f_{k+1} + 2f_k + f_{k-1})$$

Substitute all odd central differences  
by central averages of central  
differences.

$$\delta f_k = f_{k+\frac{1}{2}} - f_{k-\frac{1}{2}} \rightarrow$$

$$\mu \delta f_k = \frac{1}{2} (f_{k+1} - f_{k-1})$$


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From derivatives to finite differences :

This is based on  
interpolation formulas.

We only know  $f(x)$  for  $x_k = f(x_k)$ .

How can we get as closely as possible to  $f(x)$  for any  $x$ ?

Define  $u = \frac{x - x_0}{\Delta x}$

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$$x_k = x_0 + k \Delta x$$

NGF - Newton - Gregory - forward  
 interpolation formula.

$$\binom{u}{l} = \frac{u(u-1) \dots (u-l+1)}{l!}$$

$$\begin{aligned}
 F_m(x) &= f_k + \binom{m}{1} \Delta f_k + \binom{m}{2} \Delta^2 f_k + \dots \\
 &= f_k + \sum_{l=1}^m \binom{m}{l} \Delta^l f_k \\
 &\quad + O(\Delta x^{m+1})
 \end{aligned}$$

$F_m(x)$  is a polynomial of order  $m$  that passes through  $m$  tabulated points  $(x_{k+i}, f_{k+i})$ ,  $i=0, \dots, m-1$ .

Example:

$$\begin{aligned}
 F_2(x) &= f_k + \frac{\Delta f_k}{\Delta x} (x - x_k) \\
 &\quad + \frac{1}{2} \frac{\Delta^2 f_k}{\Delta x^2} (x - x_k)(x - x_{k+1}) \\
 &\quad + O(\Delta x^3) \\
 &= f_k + \frac{1}{\Delta x} (f_{k+1} - f_k) (x - x_k) \\
 &\quad + \frac{1}{2\Delta x^2} (f_{k+2} - 2f_{k+1} + f_k) \\
 &\quad (x - x_k)(x - x_{k+1}) + O(\Delta x^3)
 \end{aligned}$$

Hence,

$$F_2(x_k) = f_k$$

$$F_2(x_{k+1}) = f_{k+1}$$

$$F_2(x_{k+2}) = f_{k+2}$$

NGB - Newton-Gregory - Backwards interpolation scheme.

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$$F_m(x) = f_k + \sum_{l=1}^m \binom{m+l-1}{l} \nabla^l f_k + O(\Delta x^{m+1}).$$

Example:

$$\begin{aligned} F_2(x) &= f_k + \frac{\nabla f_k}{\Delta x} (x - x_k) \\ &\quad + \frac{1}{2!} \frac{\nabla^2 f_k}{\Delta x^2} (x - x_k)(x - x_{k-1}) \\ &\quad + O(\Delta x^3). \end{aligned}$$