f at regular intervals, and in such a way that we know the errors we are making and how to improve on them in a systematic way.

We start with a set of definitions:

$f(x)$ is a continuous function.

$f_k = f(x_k)$ where $x_k = x_0 + k\Delta x$

This is the only information we have about the function.

Forward difference

$$\boxed{\Delta f_k \equiv f_{k+1} - f_k}$$

We iterate this definition

$$\Delta^2 f_k = \Delta f_{k+1} - \Delta f_k = f_{k+2} - 2f_{k+1} + f_k$$

$$\vdots$$

$$\Delta^n f_k = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f_{k+n-i}$$

Backwards difference

$$\boxed{\nabla f_k = f_k - f_{k-1}}$$

We iterate the definition

$$\nabla^n f_k = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f_{k-n+i}$$

Central difference

$$\delta f_k = f_{k+\frac{1}{2}} - f_{k-1/2}$$

But, these points do not exist!

Again we iterate the definition

$\delta \to \delta^n$

$$\delta^n f_k = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f_{k-\frac{n}{2}+i}$$

No problem when n is even.

To get a definition that also works for odd n, we introduce the central average:

Central average:

$$\mu f_k = \frac{1}{2}\left(f_{k+\frac{1}{2}} + f_{k-\frac{1}{2}}\right)$$

$$\mu^2 f_k = \frac{1}{2}\left(\mu f_{k+\frac{1}{2}} + \mu f_{k-\frac{1}{2}}\right)$$

$$= \frac{1}{4}\left(f_{k+1} + 2 f_k + f_{k-1}\right)$$

Substitute all odd central differences by central averages of central differences.

$$\delta f_k = f_{k+\frac{1}{2}} - f_{k-\frac{1}{2}} \longrightarrow$$

$$\mu \delta f_k = \frac{1}{2}\left(f_{k+1} - f_{k-1}\right)$$

From derivatives to finite differences:

This is based on interpolation formulas.

We only know $f(x)$ for $f_k = f(x_k)$.
How can we get as closely as possible to $f(x)$ for any x?

Define $u = \frac{x - x_k}{\Delta x}$

$x_k = x_0 + k\Delta x$.

NGF - Newton - Gregory - forward interpolation formula.

$$\binom{u}{l} = \frac{u(u-1)\cdots(u-l+1)}{l!}$$

$$F_m(x) = f_k + \binom{u}{1}\Delta f_k + \binom{u}{2}\Delta^2 f_k + \dots$$

$$= f_k + \sum_{l=1}^{m} \binom{u}{l} \Delta^l f_k$$

$$+ O(\Delta x^{m+1})$$

$F_m(x)$ is a polynomial of order m that passes through m tabulated points (x_{k+i}, f_{k+i}), $i = 0, \dots, m-1$.

Example:

$$F_2(x) = f_k + \frac{\Delta f_k}{\Delta x}(x - x_k)$$

$$+ \frac{1}{2}\frac{\Delta^2 f_k}{\Delta x^2}(x - x_k)(x - x_{k+1})$$

$$+ O(\Delta x^3)$$

$$= f_k + \frac{1}{\Delta x}(f_{k+1} - f_k)(x - x_k)$$

$$+ \frac{1}{2\Delta x^2}(f_{k+2} - 2f_{k+1} + f_k)$$
$$(x - x_k)(x - x_{k+1}) + O(\Delta x^3)$$

Hence,

$$F_2(x_k) = f_k$$

$$F_2(x_{k+1}) = f_{k+1}$$

$$F_2(x_{k+2}) = f_{k+2}$$

NGB - Newton - Gregory - backwards interpolation scheme.

$$F_m(x) = f_k + \sum_{l=1}^{m} \binom{u+l-1}{l} \nabla^l f_k + O(\Delta x^{m+1}).$$

<u>Example:</u>

$$F_2(x) = f_k + \frac{\nabla f_k}{\Delta x}(x - x_k) + \frac{1}{2}\frac{\nabla^2 f_k}{\Delta x^2}(x - x_k)(x - x_{k-1}) + O(\Delta x^3).$$