$$P(x) = \int_{-\infty}^{x} p(x')\,dx'$$

The random number generator

$$\text{ran} \rightarrow r \in [0,1]$$

generates a sequence of random numbers $r_1, r_2, r_3, \ldots, r_n, \ldots$ that are uncorrelated with uniform probability on the unit interval:

$$p(r) = 1 \quad , \quad r \in [0,1]$$

Cumulative probability:

$$\underline{P(r)} = \int_0^r p(r')\,dr' = \int_0^r dr' = \underline{r.}$$

$$\Longrightarrow$$

$$dP = \underset{\substack{\uparrow \\ p(r)=1.}}{1} \cdot dr$$

Since $P = r$, generating a sequence $r_1, r_2, \ldots, r_n, \ldots$ of random numbers corresponds to generating a sequence of numbers $P_1, P_2, \ldots, P_n, \ldots$.

This observation makes it possible to generate <u>any distribution $p(x)$</u> of numbers x.

$$P(x) = \int_{-\infty}^{x} p(x')\,dx'$$

$$dP = p(x)\,dx$$

Generate sequence $r_1, r_2, \ldots, r_n, \ldots$

$$r_1 = P_1 = P(x_1) \Rightarrow x_1 = x(P_1) = P^{-1}(r_1)$$

$$r_2 = P_2 = P(x_2) \Rightarrow x_2 = x(P_2) = P^{-1}(r_2)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Hence,

1) Generate r

2) Calculate $P^{-1}(r) = x$.

This hinges on having an explicit expression for P^{-1}.

Example:

$$P(x) = x^{\beta} \qquad 0 < x < 1$$

$$\beta > 0.$$

$$p(x) = \frac{d}{dx} P(x) = \beta x^{\beta - 1}$$

$$\underline{x} = X(P) = P^{-1}(r) = \underline{r^{1/\beta}}$$

Example:

$$P(x) = 1 - x^{-\alpha} \qquad 1 < x < \infty$$
$$\alpha > 0$$

$$p(x) = \alpha x^{-\alpha-1}$$

$$x(P) = (1-P)^{-1/\alpha} = (1-r)^{-1/\alpha}$$

$$\longrightarrow r^{-1/\alpha}$$

$1-r \to r.$

Example:

$$P(x) = 1 - e^{-x} \qquad 0 < x < \infty.$$

$$p(x) = e^{-x}$$

$$x(P) = -\ln(1-P) = -\ln(1-r)$$

$$\longrightarrow -\ln r$$

$1-r \to r.$

But, it is not always possible to invert $P(x)$.

Example: Gaussian.

How to generate random numbers that follow a Gaussian distribution.

Probability density:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-m)^2/2\sigma}$$

Mean m

Standard deviation $\sqrt{\sigma}$.

Cumulative distribution:

$$P(x) = \int_{-\infty}^{x} p(x')\,dx' = \int_{-\infty}^{x} \frac{dx'}{\sqrt{2\pi\sigma}} e^{-(x'-m)^2/2\sigma}.$$

Cannot be inverted.