



## Vehicle Tutorial Chapter 4: Property-Driven Training

Today's presentors: Ekaterina Komendantskaya and Luca Arnaboldi (live),  
Matthew Daggitt (online), on behalf of the Vehicle team

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- ▶ ... why and how training is a part of verification of neural networks
- ▶ ... what choices exist for implementing this, generally
- ▶ ... what choices **Vehicle** makes in this respect
- ▶ ... theoretical and practical issues with the chosen methods, and **Vehicle's** take on them

## Recap: four PL problems



- $I^O$  Interoperability – properties are not portable between training/counter-example search/ verification.
- $I^P$  Interpretability – code is not easy to understand.
- $I^f$  Integration – properties of networks cannot be linked to larger control system properties.
- $E^G$  Embedding gap – little support for translation between problem space and input space.

# Why Training is a part of Verification?



```
vehicle verify \  
  --specification acasXu.vcl \  
  --verifier Marabou \  
  --network acasXu:acasXu_1_7.onnx \  
  --property property3
```

Verifying properties:

```
property3 [=====] 1/1 queries  
  result: counterexample found  
  x: [1799.9886669999978, 1.9509286320000003e-2,  
      3.09999732192, 980.0, 1017.6036]
```

# Why Training is a part of Verification?



Verifying a Fashion MNIST network on 500 samples we get:

	$\epsilon = 0.01$	$\epsilon = 0.05$	$\epsilon = 0.1$	$\epsilon = 0.5$
Success rate:	82.6 % (413/500)	29.8 % (149/500)	3.8 % (19/500)	0 % (0/500)




# A few words on the context



1943 Perceptron by McCulloch and Pitts

90-2000 Rise of machine learning applications

2013  C. Szegedy, W. Zaremba, I. Sutskever, J. Bruna, D. Erhan, I. Goodfellow, and R. Fergus. Intriguing properties of neural networks. 2013. (10000+ citations on GS)

2013-.. Tens of thousands of papers on adversarial training  
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2017 First Neural network verification attempts

 G. Katz, C.W. Barrett, D.L. Dill, K. Julian, M.J. Kochenderfer: Reluplex: An Efficient SMT Solver for Verifying Deep Neural Networks. CAV (1) 2017: 97-117.

 X. Huang, M. Kwiatkowska, S. Wang, M. Wu. Safety Verification of Deep Neural Networks. CAV (1) 2017: 3-29.

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## Training for Robustness



## Training for Robustness

Training generally:

1. depends on data
2. depends on loss functions
3. some other parameters like shape of the functions

# 1. Data Augmentation

Suppose we are given a data set  $\mathcal{D} = \{(\mathbf{x}_1, \mathbf{y}_1), \dots, (\mathbf{x}_n, \mathbf{y}_n)\}$ .

Prior to training, generate new training data samples close to existing data and label them with the same output as the original data.



C. Shorten, T.M. Khoshgoftaar: A survey on image data augmentation for deep learning. J. Big Data 6, 60 (2019)

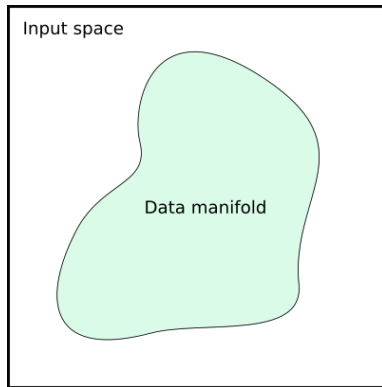


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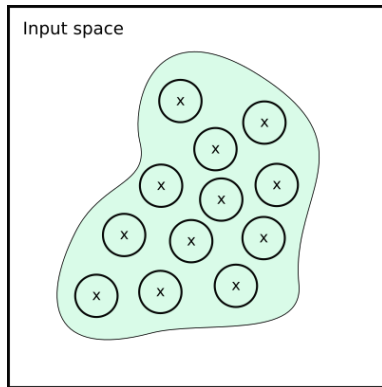
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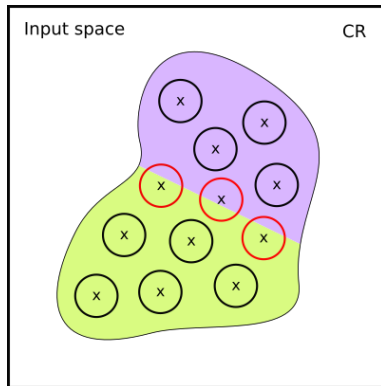
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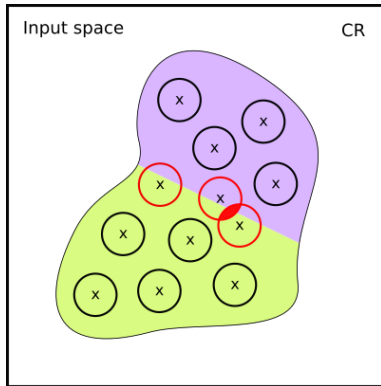


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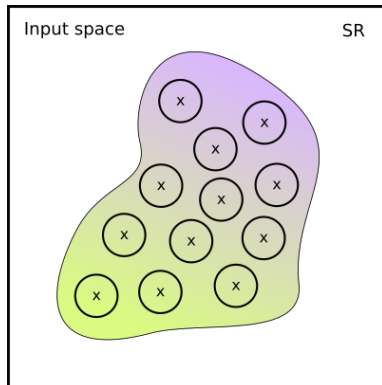




However,



# Adversarial Training



I.J. Goodfellow, J. Shlens, C. Szegedy: Explaining and harnessing adversarial examples. 3rd International Conference on Learning Representations, ICLR 2015, San Diego, CA, USA, May 7-9, 2015, Conference Track Proceedings (2015)

## 2. Solutions Involving Loss Functions



Given a data set  $\mathcal{D}$ ,

a function  $f_{\theta} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  (with optimisation parameters  $\theta$ ),

a loss function  $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$  computes a penalty proportional to the difference between the output of  $f_{\theta}$  on a training input  $\hat{\mathbf{x}}$  and a desired output  $\mathbf{y}$ .

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### Example (Cross Entropy Loss Function)

Given a function  $f_{\theta} : \mathbb{R}^n \rightarrow [0, 1]^m$ , the cross-entropy loss is defined as

$$\mathcal{L}_{ce}(\hat{\mathbf{x}}, \mathbf{y}) = - \sum_{i=1}^m \mathbf{y}_i \log(f_{\theta}(\hat{\mathbf{x}})_i) \quad (1)$$

where  $\mathbf{y}_i$  is the true probability for class  $i$  and  $f_{\theta}(\hat{\mathbf{x}})_i$  is the probability for class  $i$  as predicted by  $f_{\theta}$  when applied to  $\hat{\mathbf{x}}$ .

## 2. Adversarial Training for Robustness



- ▶ gradient descent minimises loss  $\mathcal{L}(\hat{\mathbf{x}}, \mathbf{y})$  between the predicted value  $f_{\theta}(\hat{\mathbf{x}})$  and the true value  $\mathbf{y}$ , for each entry  $(\hat{\mathbf{x}}, \mathbf{y})$  in  $\mathcal{D}$ . It thus solves the optimisation problem:

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$$\min_{\theta} \mathcal{L}(\hat{\mathbf{x}}, \mathbf{y})$$

- ▶ For adversarial training, we instead minimise the loss with respect to the worst-case perturbation of each sample in  $\mathcal{D}$ .
  - ▶ Replace the standard training objective with:

$$\min_{\theta} \left[ \max_{\mathbf{x}: |\mathbf{x} - \hat{\mathbf{x}}| \leq \epsilon} \mathcal{L}(\mathbf{x}, \mathbf{y}) \right]$$

- ▶ the inner maximisation is done by “projected gradient descent” (PGD), that “projects” the gradient of  $\mathcal{L}$  on  $\hat{\mathbf{x}}$  in order to perturb it and get the worst  $\mathbf{x}$ .




# Lipshitz Continuity



Optimise for:

$$\forall \mathbf{x} : |\mathbf{x} - \hat{\mathbf{x}}| \leq \epsilon \Rightarrow |f(\mathbf{x}) - f(\hat{\mathbf{x}})| \leq L|\mathbf{x} - \hat{\mathbf{x}}|$$

 P. Pauli, A. Koch, J. Berberich, P. Kohler, F. Allgower: Training robust neural networks using Lipschitz bounds. IEEE Control Systems Letters (2021)

 H. Gouk, E. Frank, B. Pfahringer, M.J. Cree: Regularisation of neural networks by enforcing Lipschitz continuity. Machine Learning 110(2), 393–416 (2021)

and much more...

### 3. Other ways to do property-driven training



More sophisticated approaches to property driven training include:

- ▶ Tailoring the neural network architectures
- ▶ Tailoring the activation functions
- ▶ Including probabilistic or deterministic solvers into neural network layers
- ▶ ...



Eleonora Giunchiglia, Mihaela Catalina Stoian, and Thomas Lukasiewicz. Deep learning with logical constraints. IJCAI 2022, pages 5478–5485. [ijcai.org](https://ijcai.org), 2022





Ok, great!

Machine Learning Community knows how to make our networks more robust, and maybe even verifiable!



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But remember:

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$I^P$  Interpretability ...

$I^f$  Integration ...

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# Interpretation of adversarial training:



Recall the epsilon ball robustness:

$$\forall \mathbf{x} \in \mathbb{B}(\hat{\mathbf{x}}, \epsilon). \text{ *robust*}(f(\mathbf{x}))$$

We can map different kinds of adversarial training to formal properties:

Training style	Definition of <i>robust</i>
Data Augmentation	$\operatorname{argmax} [f(\mathbf{x})] = i$
DL2 training	$f(\mathbf{x})_i \geq \eta$
Adversarial Training	$ f(\mathbf{x}) - f(\hat{\mathbf{x}})  \leq \delta$
Lipschitz Continuity	$ f(\mathbf{x}) - f(\hat{\mathbf{x}})  \leq L \mathbf{x} - \hat{\mathbf{x}} $
...	...

 M. Casadio, E. Komendantskaya, M. L. Daggitt, W. Kokke, G. Katz, G. Amir, and I. Rafaeli. 2022. Neural Network Robustness as a Verification Property: A Principled Case Study. CAV'22.

## Consequences of this finding:



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### Example

In majority of ML + verification papers, adversarial robustness is used for training (it encourages standard robustness of networks), while verification is done for classification robustness. We show that these two types of robustness are not in any relation: i.e. increasing one does not generally increase the other.

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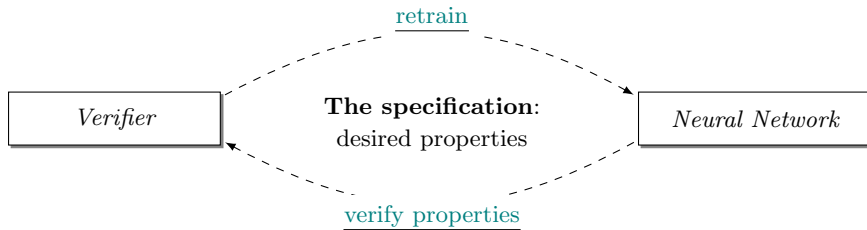
- ▶ And what to do with properties that are not  $\epsilon$ -ball robustness?  
Out-of-the-box PGD training only works with  $\epsilon$ -balls around data points.

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NB

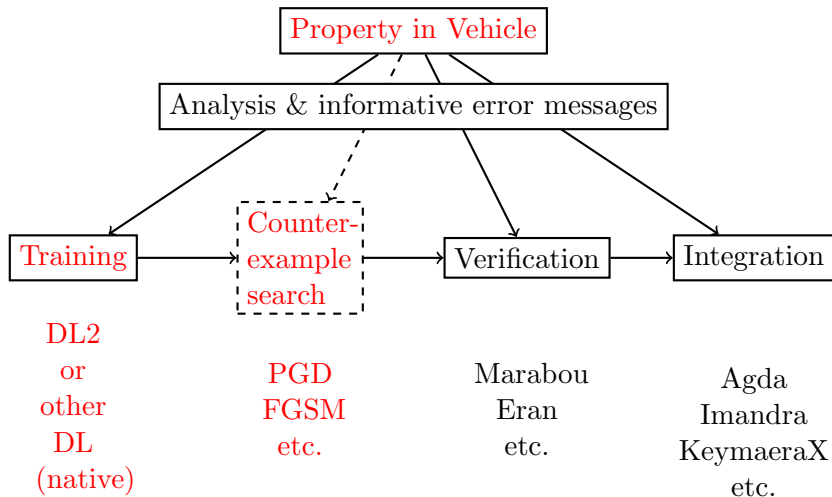
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In Vehicle terms,



# Vehicle's formula



Property-driven training =

Differentiable Logic + (PGD) optimisation

## Example - differentiable logic



We define a very simple differentiable logic on a toy language

$$p := p \mid a \leq a \mid p \wedge p \mid p \Rightarrow p$$

based on Gödel fuzzy logic [van Krieken 2022].

$$\mathbf{T}(a_1 \leq a_2) := 1 - \max\left(\frac{a_1 - a_2}{a_1 + a_2}, 0\right)$$

$$\mathbf{T}(p_1 \wedge p_2) := \min(\mathbf{T}(p_1), \mathbf{T}(p_2))$$

$$\mathbf{T}(p_1 \Rightarrow p_2) := \max(1 - \mathbf{T}(p_1), \mathbf{T}(p_2))$$

## Example - translation



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# Different existing DLs



## ► DL2



Marc Fischer, Mislav Balunovic, Dana Drachsler-Cohen, Timon Gehr, Ce Zhang, and Martin Vechev. DL2: Training and querying neural networks with logic. In ICML'19, pp. 1931–1941.

## ► fuzzy DLs such as: Godel, Łukasiewicz, Yager, product and others



Emile van Krieken, Erman Acar, and Frank van Harmelen. Analyzing differentiable fuzzy logic operators. Artificial Intelligence, 302:103602, 2022.

## ► Signal Temporal Logic DL



Peter Varnai and Dimos V. Dimarogonas. On robustness metrics for learning stl tasks. In 2020 American Control Conference (ACC), pp. 5394–5399, 2020.

## ► Formalising and comparing them all



Natalia Slusarz, Ekaterina Komendantskaya, Matthew L. Daggitt, Robert J. Stewart, Kathrin Stark: Logic of Differentiable Logics: Towards a Uniform Semantics of DL. LPAR 2023: 473-493

# Design Space for Differentiable Logics



<b>Properties:</b>	DL2	Gödel	Łukasiewicz	Yager	Product	STL
<b>Geometric:</b>						
Weak Smoothness	yes*	no	no	no	yes*	<b>yes</b>
Shadow-lifting	yes	no	no	no	yes	<b>yes</b>
Scale invariance	yes	yes	no	no	no	<b>yes</b>
<b>Logical:</b>						
Idempotence	no	<b>yes</b>	<b>no</b>	<b>no</b>	<b>no</b>	<b>yes</b>
Commutativity	yes	<b>yes</b>	<b>yes</b>	<b>yes</b>	<b>yes</b>	<b>yes</b>
Associativity	yes	<b>yes</b>	<b>yes</b>	<b>yes</b>	<b>yes</b>	<b>no</b>
Quantifier commutativity	no	yes	no	no	no	no
Soundness	yes	yes	yes	yes	yes	no



# Vehicle's formula



Property-driven training =

Differentiable Logic + (PGD) optimisation



Recall: With “projected gradient descent” (PGD),

we minimise the loss wrt the worst-case perturbation of each sample in  $\mathcal{D}$ .

- Replace the standard training objective with:

$$\min_{\theta} \left[ \max_{\mathbf{x}: |\mathbf{x} - \hat{\mathbf{x}}| \leq \epsilon} \mathcal{L}(\mathbf{x}, \mathbf{y}) \right]$$



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- in Vehicle, given a property  $\forall \mathbf{x}. \mathcal{P}(\mathbf{x}) \Rightarrow \mathcal{S}(\mathbf{x})$ , we replace the above training objective with

$$\min_{\theta} \left[ \max_{\mathbf{x} \in \mathbb{H}_{\mathcal{P}(\mathbf{x})}} \mathcal{L}_{\mathcal{S}(\mathbf{x})}(\mathbf{x}, \mathbf{y}) \right]$$

where  $\mathbb{H}_{\mathcal{P}(\mathbf{x})}$  is a hyper-shape (usually a hyper-rectangle) that corresponds to the pre-condition  $\mathcal{P}(\mathbf{x})$  of the property and  $\mathcal{L}_{\mathcal{S}(\mathbf{x})}$  is obtained by DL-translation of  $\mathcal{S}(\mathbf{x})$ .

## Examples: AcasXU



```
@property
property3 : Bool
property3 = forall x . validInput x and
                    directlyAhead x and
                    movingTowards x =>
                    not (minimalScore clearOfConflict x)
```

- ▶  $\mathbb{H}_{\mathcal{P}(\mathbf{x})}$  is given by (the normalised) vector bounds given by:

$$1500 \leq \rho \leq 1800$$

$$-0.06 \leq \theta \leq 0.06$$

$$\psi \geq 3.10$$

$$v_{own} \geq 980$$

$$v_{int} \geq 960$$

- ▶ Recall that for Acas Xu networks,  $\mathbf{x} = [\rho, \theta, \psi, v_{own}, v_{int}]^{norm}$

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- ▶ and  $\mathcal{L}_{\mathcal{S}(\mathbf{x})} = \mathbf{T}(\text{not } (\text{minimalScore clearOfConflict } \mathbf{x}))$ .

## Examples: Degenerate case



- When we have

$$\forall \mathbf{x}. \mathcal{S}(\mathbf{x})$$

instead of  $\forall \mathbf{x}. \mathcal{P}(\mathbf{x}) \Rightarrow \mathcal{S}(\mathbf{x})$

Recall we solve the optimisation problem

$$\min_{\theta} \left[ \max_{\mathbf{x} \in \mathbb{H}_{\mathcal{P}(\mathbf{x})}} \mathcal{L}_{\mathcal{S}(\mathbf{x})}(\mathbf{x}, \mathbf{y}) \right]$$

- $\mathbb{H}_{\mathcal{P}(\mathbf{x})} = \mathbb{H}_{\mathbf{x}}$  is the domain of  $\mathbf{x}$  (usually constrained by normalisation boundaries, e.g. by  $[0, 0, \dots, 0]$  and  $[1, 1, \dots, 1]$ .)

## Examples: standard robustness



- ▶ When we have

$$\forall \mathbf{x}. |\mathbf{x} - \hat{\mathbf{x}}| \leq \epsilon \Rightarrow |f(\mathbf{x}) - f(\hat{\mathbf{x}})| \leq \delta$$

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- ▶  $\mathbb{H}_{\mathcal{P}(x)}$  is the  $\epsilon$ -cube around  $\hat{\mathbf{x}}$
- ▶ and, in case we use the Gödel DL, we get

$$\begin{aligned} \mathcal{L}_{\mathcal{S}(\mathbf{x})} &= \mathbf{T}(\|f(\mathbf{x}) - f(\hat{\mathbf{x}})\| \leq \delta) = \\ &1 - \max\left(\frac{\mathbf{T}(\|f(\mathbf{x}) - f(\hat{\mathbf{x}})\| - \delta)}{\mathbf{T}(\|f(\mathbf{x}) - f(\hat{\mathbf{x}})\| + \delta)}, 0\right) \end{aligned}$$

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## In Conclusion,

