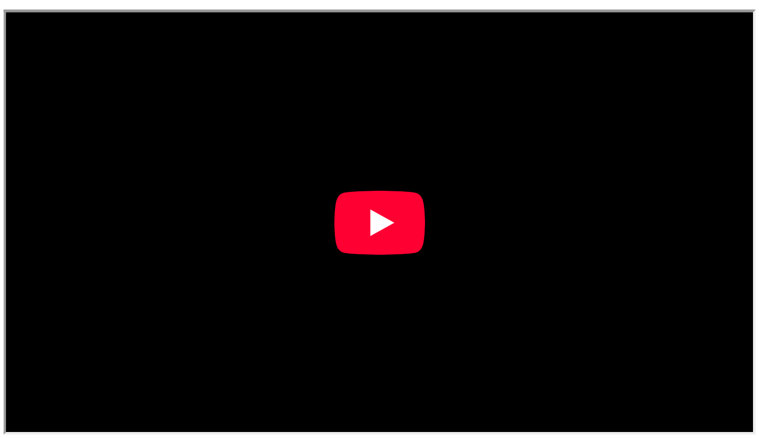
Introduction to Fluid Dynamics and Its Biological and Medical Applications

Many fluids are flowing in this scene. Water from the hose and smoke from the fire are visible flows. Less visible are the flow of air and the flow of fluids on the ground and within the people fighting the fire. Explore all types of flow, such as visible, implied, turbulent, laminar, and so on, present in this scene. Make a list and discuss the relative energies involved in the various flows, including the level of confidence in your estimates. (credit: Andrew Magill, Flickr)

We have dealt with many situations in which fluids are static. But by their very definition, fluids flow. Examples come easily—a column of smoke rises from a camp fire, water streams from a fire hose, blood courses through your veins. Why does rising smoke curl and twist? How does a nozzle increase the speed of water emerging from a hose? How does the body regulate blood flow? The physics of fluids in motion—**fluid dynamics**—allows us to answer these and many other questions.



Glossary

fluid dynamics
: the physics of fluids in motion

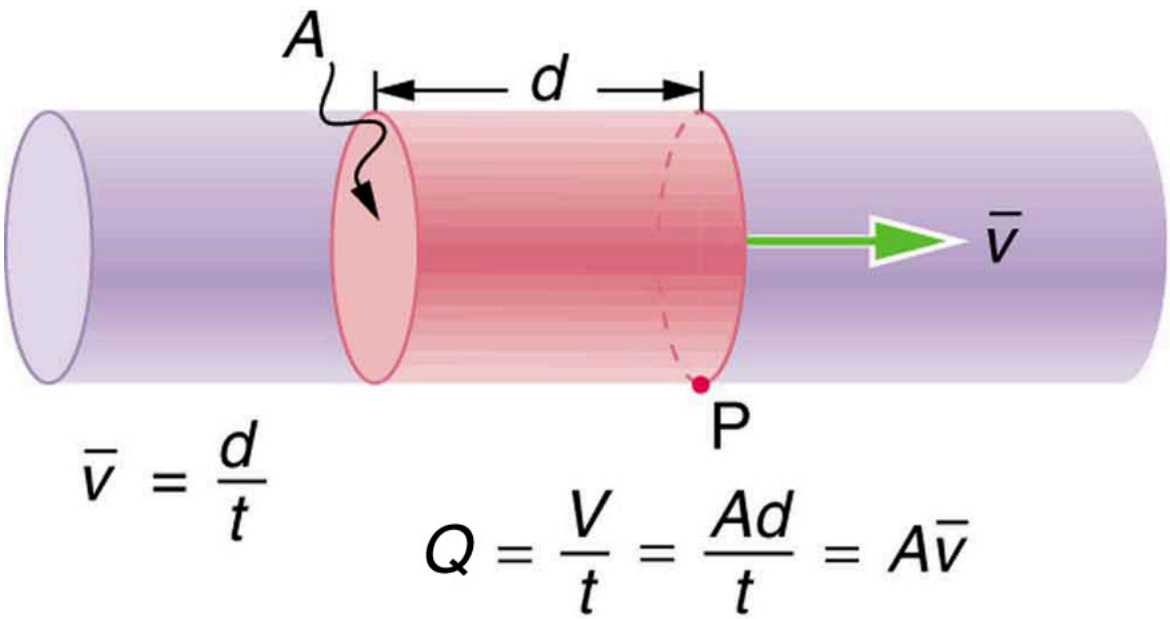
Flow Rate and Its Relation to Velocity

- Calculate flow rate.
- Define units of volume.
- Describe incompressible fluids.
- Explain the consequences of the equation of continuity.

Flow rate Q is defined to be the volume of fluid passing by some location through an area during a period of time, as seen in [Figure 1]. In symbols, this can be written as

$$Q=Vt,$$

where V is the volume and t is the elapsed time.



Flow rate is the volume of fluid per unit time flowing past a point through the area (A) . Here the shaded cylinder of fluid flows past point (P) in a uniform pipe in time (t) . The volume of the cylinder is (Ad) and the average velocity is $\bar{v}=d/t$ so that the flow rate is $Q=Ad/t=A\bar{v}$.)

The SI unit for flow rate is m^3/s , but a number of other units for Q are in common use. For example, the heart of a resting adult pumps blood at a rate of 5.00 liters per minute (L/min). Note that a **liter** (L) is 1/1000 of a cubic meter or 1000 cubic centimeters ($10^{-3}m^3$ or 10^3cm^3). In this text we shall use whatever metric units are most convenient for a given situation.

Calculating Volume from Flow Rate: The Heart Pumps a Lot of Blood in a Lifetime

How many cubic meters of blood does the heart pump in a 75-year lifetime, assuming the average flow rate is 5.00 L/min?

Strategy

Time and flow rate Q are given, and so the volume V can be calculated from the definition of flow rate.

Solution

Solving $Q=V/t$ for volume gives

$$V=Qt.$$

Substituting known values yields

$$V = (5.00 L 1 min)(75 y)(1 m^3 10^3 L)(5.26\times10^5 min y) = 2.0\times10^5 m^3.$$

Discussion

This amount is about 200 000 tons of blood. For comparison, this value is equivalent to about 200 times the volume of water contained in a 6-lane 50-m lap pool.

Flow rate and velocity are related, but quite different, physical quantities. To make the distinction clear, think about the flow rate of a river. The greater the velocity of the water, the greater the flow rate of the river. But flow rate also depends on the size of the river. A rapid mountain stream carries far less water than the Amazon River in Brazil, for example. The precise relationship between flow rate Q and velocity $\bar{v}$ is

$$Q = A\bar{v},$$

where A is the cross-sectional area and $\bar{v}$ is the average velocity. This equation seems logical enough. The relationship tells us that flow rate is directly proportional to both the magnitude of the average velocity (hereafter referred to as the speed) and the size of a river, pipe, or other conduit. The larger the conduit, the greater its cross-sectional area. [Figure 1] illustrates how this relationship is obtained. The shaded cylinder has a volume

$$V = Ad,$$

which flows past the point P in a time t. Dividing both sides of this relationship by t gives

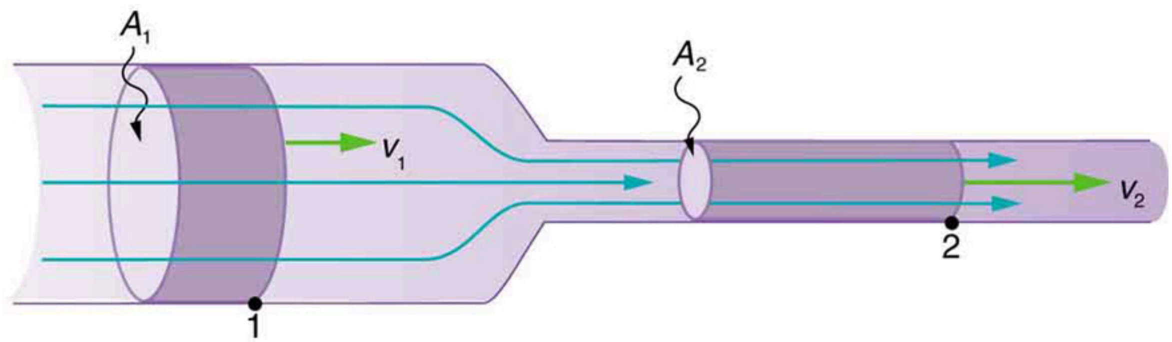
$$\frac{V}{t} = \frac{Ad}{t}.$$

We note that $Q = V/t$ and the average speed is $\bar{v} = d/t$. Thus, the equation becomes $Q = A\bar{v}$.

[Figure 2] shows an incompressible fluid flowing along a pipe of decreasing radius. Because the fluid is incompressible, the same amount of fluid must flow past any point in the tube in a given time to ensure continuity of flow. In this case, because the cross-sectional area of the pipe decreases, the velocity must necessarily increase. This logic can be extended to say that the flow rate must be the same at all points along the pipe. In particular, for points 1 and 2,

$$Q_1 = Q_2 \quad A_1\bar{v}_1 = A_2\bar{v}_2 \; \}.$$

This is called the equation of continuity and is valid for any incompressible fluid. The consequences of the equation of continuity can be observed when water flows from a hose into a narrow spray nozzle: it emerges with a large speed—that is the purpose of the nozzle. Conversely, when a river empties into one end of a reservoir, the water slows considerably, perhaps picking up speed again when it leaves the other end of the reservoir. In other words, speed increases when cross-sectional area decreases, and speed decreases when cross-sectional area increases.



When a tube narrows, the same volume occupies a greater length. For the same volume to pass points 1 and 2 in a given time, the speed must be greater at point 2. The process is exactly reversible. If the fluid flows in the opposite direction, its speed will decrease when the tube widens. (Note that the relative volumes of the two cylinders and the corresponding velocity vector arrows are not drawn to scale.)

Since liquids are essentially incompressible, the equation of continuity is valid for all liquids. However, gases are compressible, and so the equation must be applied with caution to gases if they are subjected to compression or expansion.

Calculating Fluid Speed: Speed Increases When a Tube Narrows

A nozzle with a radius of 0.250 cm is attached to a garden hose with a radius of 0.900 cm. The flow rate through hose and nozzle is 0.500 L/s. Calculate the speed of the water (a) in the hose and (b) in the nozzle.

Strategy

We can use the relationship between flow rate and speed to find both velocities. We will use the subscript 1 for the hose and 2 for the nozzle.

Solution for (a)

First, we solve $Q = A\bar{v}$ for v_1 and note that the cross-sectional area is $A = \pi r^2$, yielding

$$\bar{v}_1 = \frac{Q}{A_1} = \frac{Q}{\pi r_1^2}.$$

Substituting known values and making appropriate unit conversions yields

$$\bar{v}_1 = \frac{(0.500\,\text{L/s})(10^{-3}\,\text{m}^3/\text{L})}{\pi(9.00\times10^{-3}\,\text{m})^2} = 1.96\,\text{m/s}.$$

Solution for (b)

We could repeat this calculation to find the speed in the nozzle $\bar{v}_2$, but we will use the equation of continuity to give a somewhat different insight. Using the equation which states

$$A_1\bar{v}_1=A_2\bar{v}_2,$$

solving for $\bar{v}_2$ and substituting πr^2 for the cross-sectional area yields

$$\bar{v}_2=\frac{A_1}{A_2}\bar{v}_1=\frac{\pi r_1^2}{\pi r_2^2}\bar{v}_1=\frac{r_1^2}{r_2^2}\bar{v}_1.$$

Substituting known values,

$$\bar{v}_2=\frac{(0.900\,\text{cm})^2}{(0.250\,\text{cm})^2}1.96\,\text{m/s}=25.5\,\text{m/s}.$$

Discussion

A speed of 1.96 m/s is about right for water emerging from a nozzleless hose. The nozzle produces a considerably faster stream merely by constricting the flow to a narrower tube.

The solution to the last part of the example shows that speed is inversely proportional to the *square* of the radius of the tube, making for large effects when radius varies. We can blow out a candle at quite a distance, for example, by pursing our lips, whereas blowing on a candle with our mouth wide open is quite ineffective.

In many situations, including in the cardiovascular system, branching of the flow occurs. The blood is pumped from the heart into arteries that subdivide into smaller arteries (arterioles) which branch into very fine vessels called capillaries. In this situation, continuity of flow is maintained, but it is the *sum* of the flow rates in each of the branches in any portion along the tube that is maintained. The equation of continuity in a more general form becomes

$$n_1A_1\bar{v}_1=n_2A_2\bar{v}_2,$$

where n_1 and n_2 are the number of branches in each of the sections along the tube.

Calculating Flow Speed and Vessel Diameter: Branching in the Cardiovascular System

The aorta is the principal blood vessel through which blood leaves the heart in order to circulate around the body. (a) Calculate the average speed of the blood in the aorta if the flow rate is 5.0 L/min. The aorta has a radius of 10 mm. (b) Blood also flows through smaller blood vessels known as capillaries. When the rate of blood flow in the aorta is 5.0 L/min, the speed of blood in the capillaries is about 0.33 mm/s. Given that the average diameter of a capillary is $8.0\,\mu\text{m}$, calculate the number of capillaries in the blood circulatory system.

Strategy

We can use $Q=A\bar{v}$ to calculate the speed of flow in the aorta and then use the general form of the equation of continuity to calculate the number of capillaries as all the other variables are known.

Solution for (a)

The flow rate is given by $Q=A\bar{v}$ or $\bar{v}=\frac{Q}{\pi r^2}$ for a cylindrical vessel.

Substituting the known values (converted to units of meters and seconds) gives

$$\bar{v}=\frac{(5.0\,\text{L/min})(10^{-3}\,\text{m}^3/\text{L})(1\,\text{min}/60\,\text{s})}{\pi(0.010\,\text{m})^2}=0.27\,\text{m/s}.$$

Solution for (b)

Using $n_1A_1\bar{v}_1=n_2A_2\bar{v}_1$, assigning the subscript 1 to the aorta and 2 to the capillaries, and solving for n_2 (the number of capillaries) gives $n_2=\frac{n_1A_1\bar{v}_1}{A_2\bar{v}_2}$. Converting all quantities to units of meters and seconds and substituting into the equation above gives

$$n_2=\frac{(1)(\pi)(10\times10^{-3}\,\text{m})^2(0.27\,\text{m/s})}{(\pi)(4.0\times10^{-6}\,\text{m})^2(0.33\times10^{-3}\,\text{m/s})}=5.0\times10^9\,\text{capillaries}.$$

Discussion

Note that the speed of flow in the capillaries is considerably reduced relative to the speed in the aorta due to the significant increase in the total cross-sectional area at the capillaries. This low speed is to allow sufficient time for effective exchange to occur, although it is equally important for the flow not to become stationary in order to avoid the possibility of clotting. Does this large number of capillaries in the body seem reasonable? In active muscle, one finds about 200 capillaries per mm^3, or about 200×10^6 per 1 kg of muscle. For 20 kg of muscle, this amounts to about 4×10^9 capillaries.

Section Summary

- Flow rate Q is defined to be the volume V flowing past a point in time t, or $Q=\frac{V}{t}$ where V is volume and t is time.
- The SI unit of volume is m^3 .
- Another common unit is the liter (L), which is $10^{-3}\,\text{m}^3$.

- Flow rate and velocity are related by $Q = A\bar{v}$ where A is the cross-sectional area of the flow and $\bar{v}$ is its average velocity.
- For incompressible fluids, flow rate at various points is constant. That is,
  $Q_1 = Q_2 \quad A_1\bar{v}_1 = A_2\bar{v}_2 \quad n_1A_1\bar{v}_1 = n_2A_2\bar{v}_2$.

Conceptual Questions

What is the difference between flow rate and fluid velocity? How are they related?

Show Solution

Strategy

To answer this conceptual question, we need to examine the definitions of flow rate and fluid velocity and identify how they differ as physical quantities.

Solution

Flow rate Q and fluid velocity v are related but distinct physical quantities:

- **Flow rate** Q measures the *volume* of fluid passing a point per unit time. Its SI unit is m^3/s. Flow rate tells us *how much* fluid moves through a system.

- **Fluid velocity** v measures *how fast* the fluid is moving at a particular location. Its SI unit is m/s. Velocity is a local property that can vary from point to point in the fluid.

The two quantities are related by the equation: $Q = A\bar{v}$

where A is the cross-sectional area through which the fluid flows and $\bar{v}$ is the average velocity.

Discussion

The key distinction is that flow rate is an *extensive* property that depends on the size of the conduit, while velocity is an *intensive* property describing local motion. Two pipes with the same fluid velocity can have very different flow rates if their cross-sectional areas differ. Conversely, the same flow rate can be achieved with high velocity through a small pipe or low velocity through a large pipe.

Many figures in the text show streamlines. Explain why fluid velocity is greatest where streamlines are closest together. (Hint: Consider the relationship between fluid velocity and the cross-sectional area through which it flows.)

Show Solution

Strategy

We apply the equation of continuity for incompressible fluids, which relates flow velocity to cross-sectional area.

Solution

For an incompressible fluid, the equation of continuity states: $A_1v_1 = A_2v_2$

Streamlines represent the paths that fluid particles follow. When streamlines are closer together, they indicate a smaller cross-sectional area A through which the fluid flows.

Since flow rate $Q = Av$ must remain constant for an incompressible fluid, when the area A decreases (streamlines closer together), the velocity v must increase proportionally: $v = \frac{Q}{A}$

Therefore, where streamlines are most closely packed (smallest effective cross-sectional area), the fluid velocity is greatest.

Discussion

This principle is directly observable in everyday life. When you partially cover a garden hose opening with your thumb, you reduce the exit area, causing the water to spray out at much higher velocity. Similarly, wind speeds up as it flows between tall buildings (the “urban canyon effect”) because the streamlines are compressed into a smaller area.

Identify some substances that are incompressible and some that are not.

Show Solution

Strategy

We consider the molecular structure and behavior of different substances under pressure to classify them as compressible or incompressible.

Solution

Incompressible substances (liquids and solids):

- Water and other liquids (oil, blood, mercury)
- Metals (steel, aluminum, copper)

- Glass and ceramics
- Most biological tissues
- Hydraulic fluids

Compressible substances (gases):

- Air and atmospheric gases
- Natural gas (methane)
- Steam and water vapor
- Carbon dioxide
- Helium and other noble gases

Discussion

The distinction arises from molecular structure. In liquids and solids, molecules are already closely packed with strong intermolecular forces, leaving little room for compression. In gases, molecules are far apart with weak interactions, allowing significant volume changes under pressure.

Note that “incompressible” is an idealization—all substances compress slightly under extreme pressure. However, liquids typically compress less than 0.01% under normal conditions, making the incompressible approximation excellent for most applications. This is why hydraulic systems use liquids rather than gases to transmit force reliably.

Problems & Exercises

What is the average flow rate in cm^3/s of gasoline to the engine of a car traveling at 100 km/h if it averages 10.0 km/L?

Show Solution

Strategy

We need to find the volume of gasoline consumed per unit time. The car’s fuel efficiency (10.0 km/L) tells us how far the car travels per liter of fuel. Combined with the speed (100 km/h), we can determine the fuel consumption rate.

Solution

First, find how many liters are consumed per hour: $\text{Fuel consumption rate} = \frac{\text{Speed}}{\text{Fuel efficiency}} = \frac{100\ \text{km/h}}{10.0\ \text{km/L}} = 10.0\ \text{L/h}$

Now convert to cm^3/s: $Q = 10.0 \frac{\text{L}}{\text{h}} \times \frac{1000\ \text{cm}^3}{1\ \text{L}} \times \frac{1\ \text{h}}{3600\ \text{s}} = \frac{10.0 \times 1000}{3600}\ \text{cm}^3/\text{s} = 2.78\ \text{cm}^3/\text{s}$

Discussion

The average flow rate of gasoline to the engine is **2.78 cm^3/s**. This relatively small flow rate makes sense—gasoline is energy-dense, so a small volume can power a car at highway speeds. For perspective, this is about half a teaspoon per second, which seems reasonable for maintaining 100 km/h travel.

The heart of a resting adult pumps blood at a rate of 5.00 L/min. (a) Convert this to cm^3/s . (b) What is this rate in m^3/s ?

Show Solution

Strategy

This is a unit conversion problem. We apply conversion factors to change from L/min to the requested units, using the relationships: 1 L = 1000 cm^3 = 10^{-3} m^3 and 1 min = 60 s.

Solution

(a) Convert 5.00 L/min to cm^3/s: $Q = 5.00 \frac{\text{L}}{\text{min}} \times \frac{1000\ \text{cm}^3}{1\ \text{L}} \times \frac{1\ \text{min}}{60\ \text{s}} = \frac{5.00 \times 1000}{60}\ \text{cm}^3/\text{s} = 83.3\ \text{cm}^3/\text{s}$

(b) Convert 5.00 L/min to m^3/s: $Q = 5.00 \frac{\text{L}}{\text{min}} \times \frac{10^{-3}\ \text{m}^3}{1\ \text{L}} \times \frac{1\ \text{min}}{60\ \text{s}} = \frac{5.00 \times 10^{-3}}{60}\ \text{m}^3/\text{s} = 8.33 \times 10^{-5}\ \text{m}^3/\text{s}$

Discussion

The heart pumps blood at a rate of **(a) 83.3 cm^3/s** or **(b) 8.33 × 10^{-5} m^3/s**. The cm^3/s unit is convenient for physiological applications—83.3 cm^3/s means roughly a third of a cup of blood per second. Over a day, this amounts to about 7,200 liters, highlighting the remarkable endurance of the heart muscle.

Blood is pumped from the heart at a rate of 5.0 L/min into the aorta (of radius 1.0 cm). Determine the speed of blood through the aorta.

Show Solution

Strategy

We use the relationship between flow rate, cross-sectional area, and velocity: $Q = A\bar{v}$. The aorta is approximately circular, so $A = \pi r^2$.

Solution

Given:

- Flow rate: $Q = 5.0\ \text{L/min}$
- Aorta radius: $r = 1.0\ \text{cm} = 0.010\ \text{m}$

First, convert flow rate to SI units: $Q = 5.0 \frac{\text{L}}{\text{min}} \times \frac{10^{-3}\ \text{m}^3}{1\ \text{L}} \times \frac{1\ \text{min}}{60\ \text{s}} = 8.33 \times 10^{-5}\ \text{m}^3/\text{s}$

Calculate the cross-sectional area: $A = \pi r^2 = \pi(0.010\ \text{m})^2 = 3.14 \times 10^{-4}\ \text{m}^2$

Solve for velocity: $\bar{v} = \frac{Q}{A} = \frac{8.33\times10^{-5}\ \text{m}^3/\text{s}}{3.14\times10^{-4}\ \text{m}^2} = 0.27\ \text{m/s} = 27\ \text{cm/s}$

Discussion

The speed of blood through the aorta is approximately **27 cm/s**. This is a reasonable value—fast enough to deliver oxygenated blood efficiently throughout the body, yet slow enough to avoid damaging the vessel walls. During exercise, cardiac output can increase significantly, leading to higher blood velocities.

Blood is flowing through an artery of radius 2 mm at a rate of 40 cm/s. Determine the flow rate and the volume that passes through the artery in a period of 30 s.

Show Solution

Strategy

We use $Q = A\bar{v}$ to find the flow rate, then use $V = Qt$ to find the total volume passing through the artery in the given time.

Solution

Given:

- Artery radius: $r = 2\ \text{mm} = 0.2\ \text{cm} = 0.002\ \text{m}$
- Blood velocity: $v = 40\ \text{cm/s} = 0.40\ \text{m/s}$
- Time: $t = 30\ \text{s}$

Calculate the cross-sectional area: $A = \pi r^2 = \pi(0.002\ \text{m})^2 = 1.26 \times 10^{-5}\ \text{m}^2$

Calculate the flow rate: $Q = Av = (1.26 \times 10^{-5}\ \text{m}^2)(0.40\ \text{m/s}) = 5.03 \times 10^{-6}\ \text{m}^3/\text{s}$

Converting to more convenient units: $Q = 5.03 \times 10^{-6}\ \text{m}^3/\text{s} \times \frac{10^6\ \text{cm}^3}{1\ \text{m}^3} = 5.0\ \text{cm}^3/\text{s}$

Calculate the volume in 30 s: $V = Qt = (5.03 \times 10^{-6}\ \text{m}^3/\text{s})(30\ \text{s}) = 1.51 \times 10^{-4}\ \text{m}^3 = 151\ \text{cm}^3 = 0.15\ \text{L}$

Discussion

The flow rate through the artery is **5.0 cm³/s** (or 5.0×10^{-6} m³/s), and **151 cm³** (about 0.15 L) of blood passes through in 30 seconds. This is consistent with blood flow through a medium-sized artery. The relatively high velocity of 40 cm/s suggests this could be an artery close to the heart where blood moves faster.

The Huka Falls on the Waikato River is one of New Zealand's most visited natural tourist attractions (see [Figure 3]). On average the river has a flow rate of about 300 000 L/s. At the gorge, the river narrows to 20 m wide and averages 20 m deep. (a) What is the average speed of the river in the gorge? (b) What is the average speed of the water in the river downstream of the falls when it widens to 60 m and its depth increases to an average of 40 m?

The Huka Falls in Taupo, New Zealand, demonstrate flow rate. (credit: RaviGogna, Flickr)

Show Solution

Strategy

We apply $Q = Av$ at different points along the river. Since water is incompressible, the flow rate Q remains constant. We calculate velocity at each location using the cross-sectional area (width × depth).

Solution

Given:

- Flow rate: $Q = 300\ 000\ \text{L/s} = 300\ \text{m}^3/\text{s}$
- Gorge dimensions: width = 20 m, depth = 20 m
- Downstream dimensions: width = 60 m, depth = 40 m

(a) Calculate the velocity in the gorge: $A_{\text{gorge}} = (20\ \text{m})(20\ \text{m}) = 400\ \text{m}^2$

$$v_{\text{gorge}} = \frac{Q}{A_{\text{gorge}}} = \frac{300\ \text{m}^3/\text{s}}{400\ \text{m}^2} = 0.75\ \text{m/s}$$

(b) Calculate the velocity downstream: $A_{\text{downstream}} = (60\ \text{m})(40\ \text{m}) = 2400\ \text{m}^2$

$$v_{\text{downstream}} = \frac{Q}{A_{\text{downstream}}} = \frac{300\ \text{m}^3/\text{s}}{2400\ \text{m}^2} = 0.125\ \text{m/s} \approx 0.13\ \text{m/s}$$

Discussion

The average speed of the river is **(a) 0.75 m/s** in the gorge and **(b) 0.13 m/s** downstream. The water speeds up by a factor of about 6 as it passes through the narrow gorge, which explains why the Huka Falls is such a dramatic sight—the river is compressed into a much smaller cross-section. This is a direct application of the continuity equation: when the area decreases, velocity must increase proportionally to maintain constant flow rate.

A major artery with a cross-sectional area of 1.00cm^2 branches into 18 smaller arteries, each with an average cross-sectional area of 0.400cm^2. By what factor is the average velocity of the blood reduced when it passes into these branches?

Show Solution

Strategy

We use the generalized continuity equation for branching flow: $A_1 v_1 = n_2 A_2 v_2$, where n_2 is the number of branches. The total cross-sectional area of the branches determines how the velocity changes.

Solution

Given:

- Major artery area: $A_1 = 1.00 \text{ cm}^2$
- Number of branches: $n = 18$
- Each branch area: $A_2 = 0.400 \text{ cm}^2$

The total cross-sectional area of all branches: $A_{\text{total}} = nA_2 = 18 \times 0.400 \text{ cm}^2 = 7.20 \text{ cm}^2$

From continuity: $A_1 v_1 = A_{\text{total}} v_2$

Solving for the velocity ratio: $\frac{v_2}{v_1} = \frac{A_1}{A_{\text{total}}} = \frac{1.00 \text{ cm}^2}{7.20 \text{ cm}^2} = 0.139$

The reduction factor is: $\frac{v_1}{v_2} = \frac{7.20}{1.00} = 7.20$

Discussion

The blood velocity is reduced by a factor of **7.2** (or equivalently, the velocity in the branches is about 14% of the velocity in the major artery). This slowing of blood as it enters smaller vessels is physiologically important—it allows more time for oxygen and nutrient exchange with surrounding tissues, particularly in capillaries where the total cross-sectional area is enormous.

(a) As blood passes through the capillary bed in an organ, the capillaries join to form venules (small veins). If the blood speed increases by a factor of 4.00 and the total cross-sectional area of the venules is 10.0cm^2, what is the total cross-sectional area of the capillaries feeding these venules? (b) How many capillaries are involved if their average diameter is $10.0 \mu\text{m}$?

Show Solution

Strategy

(a) We apply the continuity equation. If velocity increases by a factor of 4, the cross-sectional area must decrease by the same factor to maintain constant flow rate. (b) We calculate the area of a single capillary and divide the total area by this value.

Solution

(a) From continuity: $A_{\text{cap}} v_{\text{cap}} = A_{\text{ven}} v_{\text{ven}}$

Since $v_{\text{ven}} = 4.00 \times v_{\text{cap}}$: $A_{\text{cap}} v_{\text{cap}} = A_{\text{ven}} (4.00 \times v_{\text{cap}})$

$$A_{\text{cap}} = 4.00 \times A_{\text{ven}} = 4.00 \times 10.0 \text{ cm}^2 = 40.0 \text{ cm}^2$$

(b) Calculate the area of a single capillary:

Given diameter: $d = 10.0 \text{ μm} = 10.0 \times 10^{-4} \text{ cm}$

$$A_{\text{single}} = \pi r^2 = \pi \left(\frac{d}{2}\right)^2 = \pi \left(\frac{10.0 \times 10^{-4} \text{ cm}}{2}\right)^2 = \pi (5.00 \times 10^{-4} \text{ cm})^2$$
$$A_{\text{single}} = 7.85 \times 10^{-7} \text{ cm}^2$$

Number of capillaries: $n = \frac{A_{\text{cap}}}{A_{\text{single}}} = \frac{40.0 \text{ cm}^2}{7.85 \times 10^{-7} \text{ cm}^2} = 5.09 \times 10^7$

Discussion

The total cross-sectional area of the capillaries is **(a) 40.0 cm²**, and there are approximately **(b) 5.09 × 10⁷ capillaries** (about 51 million). This enormous number of tiny capillaries explains why blood slows down so dramatically in capillary beds—the total cross-sectional area is vastly larger than in arteries, giving time for gas and nutrient exchange.

The human circulation system has approximately 1×10^9 capillary vessels. Each vessel has a diameter of about $8 \mu\text{m}$. Assuming cardiac output is 5 L/min, determine the average velocity of blood flow through each capillary vessel.

Show Solution

Strategy

We calculate the total cross-sectional area of all capillaries, then use $Q = Av$ to find the velocity. All capillaries are assumed to share the total cardiac output.

Solution

Given:

- Number of capillaries: $n = 1 \times 10^{9}$
- Capillary diameter: $d = 8\ \mu\text{m} = 8 \times 10^{-6}\ \text{m}$
- Cardiac output: $Q = 5\ \text{L/min} = \frac{5\times 10^{-3}\ \text{m}^3}{60\ \text{s}} = 8.33 \times 10^{-5}\ \text{m}^3/\text{s}$

Calculate the area of a single capillary: $A_{\text{single}} = \pi r^2 = \pi\left(\frac{8\times 10^{-6}\ \text{m}}{2}\right)^2 = \pi(4\times 10^{-6}\ \text{m})^2 = 5.03 \times 10^{-11}\ \text{m}^2$

Total cross-sectional area of all capillaries: $A_{\text{total}} = n \times A_{\text{single}} = (1 \times 10^{9})(5.03 \times 10^{-11}\ \text{m}^2) = 0.0503\ \text{m}^2$

Calculate average velocity: $v = \frac{Q}{A_{\text{total}}} = \frac{8.33\times 10^{-5}\ \text{m}^3/\text{s}}{0.0503\ \text{m}^2} = 1.66 \times 10^{-3}\ \text{m/s} = 0.17\ \text{cm/s}$

Discussion

The average velocity of blood through each capillary is approximately **0.17 cm/s** (or about 1.7 mm/s). This very slow speed is crucial for capillary function—it allows sufficient time (about 1-2 seconds) for red blood cells to exchange oxygen and carbon dioxide with surrounding tissues as they traverse the capillary bed.

(a) Estimate the time it would take to fill a private swimming pool with a capacity of 80 000 L using a garden hose delivering 60 L/min. (b) How long would it take to fill if you could divert a moderate size river, flowing at $5000\text{m}^3/\text{s}$, into it?

Show Solution

Strategy

We use the relationship $Q = V/t$, solving for time: $t = V/Q$. We need to ensure consistent units for each calculation.

Solution

Given:

- Pool capacity: $V = 80\ 000\ \text{L}$
- Garden hose flow rate: $Q_{\text{hose}} = 60\ \text{L/min}$
- River flow rate: $Q_{\text{river}} = 5000\ \text{m}^3/\text{s} = 5000 \times 1000\ \text{L/s} = 5 \times 10^{6}\ \text{L/s}$

(a) Time using garden hose: $t = \frac{V}{Q} = \frac{80000\ \text{L}}{60\ \text{L/min}} = 1333\ \text{min} = 22.2\ \text{h} \approx 22\ \text{h}$

(b) Time using river: $t = \frac{V}{Q} = \frac{80000\ \text{L}}{5\times 10^{6}\ \text{L/s}} = 0.016\ \text{s} = 16\ \text{ms}$

Discussion

Using a garden hose, it would take approximately **(a) 22 hours** to fill the pool. Using a moderate river, it would take only **(b) 0.016 seconds** (16 milliseconds). This dramatic difference illustrates the enormous flow rates of natural rivers compared to household water sources. The river flow rate is about 5 billion times greater than the garden hose!

The flow rate of blood through a $2.00 \times 10^{-6}\text{-m}$ -radius capillary is $3.80 \times 10^{-9}\text{cm}^3/\text{s}$. (a) What is the speed of the blood flow? (This small speed allows time for diffusion of materials to and from the blood.) (b) Assuming all the blood in the body passes through capillaries, how many of them must there be to carry a total flow of $90.0\text{cm}^3/\text{s}$? (The large number obtained is an overestimate, but it is still reasonable.)

Show Solution

Strategy

(a) We use $Q = Av$ to find velocity, calculating the cross-sectional area from the given radius. (b) We divide the total required flow rate by the flow rate through a single capillary.

Solution

Given:

- Capillary radius: $r = 2.00 \times 10^{-6}\ \text{m} = 2.00 \times 10^{-4}\ \text{cm}$
- Flow rate per capillary: $Q = 3.80 \times 10^{-9}\ \text{cm}^3/\text{s}$

- Total flow required: $Q_{total} = 90.0\ \text{cm}^3/\text{s}$

(a) Calculate the cross-sectional area: $A = \pi r^2 = \pi(2.00\times10^{-4}\ \text{cm})^2 = 1.26\times10^{-7}\ \text{cm}^2$

Calculate velocity: $v = \frac{Q}{A} = \frac{3.80\times10^{-9}\ \text{cm}^3/\text{s}}{1.26\times10^{-7}\ \text{cm}^2} = 3.02\times10^{-2}\ \text{cm/s} = 0.30\ \text{mm/s}$

(b) Number of capillaries needed: $n = \frac{Q_{total}}{Q_{single}} = \frac{90.0\ \text{cm}^3/\text{s}}{3.80\times10^{-9}\ \text{cm}^3/\text{s}} = 2.37\times10^{10}$

Discussion

The blood velocity in a capillary is approximately **(a) 0.030 cm/s** (or 0.30 mm/s), and about **(b) 2.4 × 10^{10} capillaries** would be needed to carry the total blood flow. While this is an overestimate (as the problem notes), it illustrates the enormous number of capillaries in the body. The very slow blood velocity in capillaries is essential—it provides the 1-2 seconds needed for effective gas exchange between blood and tissues.

(a) What is the fluid speed in a fire hose with a 9.00-cm diameter carrying 80.0 L of water per second? (b) What is the flow rate in cubic meters per second? (c) Would your answers be different if salt water replaced the fresh water in the fire hose?

Show Solution

Strategy

We use the relationship $Q = Av$ where $A = \pi r^2$ for the circular hose cross-section. For part (c), we consider whether density affects the continuity equation.

Solution

Given:

- Hose diameter: $d = 9.00\ \text{cm} = 0.0900\ \text{m}$, so radius $r = 0.0450\ \text{m}$
- Flow rate: $Q = 80.0\ \text{L/s}$

(a) Calculate the cross-sectional area: $A = \pi r^2 = \pi(0.0450\ \text{m})^2 = 6.36\times10^{-3}\ \text{m}^2$

Convert flow rate to SI units: $Q = 80.0\ \text{L/s} = 80.0\times10^{-3}\ \text{m}^3/\text{s} = 0.0800\ \text{m}^3/\text{s}$

Calculate velocity: $v = \frac{Q}{A} = \frac{0.0800\ \text{m}^3/\text{s}}{6.36\times10^{-3}\ \text{m}^2} = 12.6\ \text{m/s}$

(b) The flow rate in cubic meters per second: $Q = 80.0\ \text{L/s}\times\frac{1\ \text{m}^3}{1000\ \text{L}} = 0.0800\ \text{m}^3/\text{s}$

(c) The answers would be the same for salt water. The flow rate Q and velocity v depend only on the volume of fluid passing through per unit time, not on the fluid’s density. The equation $Q = Av$ contains no density terms.

Discussion

The fluid speed in the fire hose is **(a) 12.6 m/s** (about 45 km/h), and the flow rate is **(b) 0.0800 m³/s**. **(c)** No, the answers would not change for salt water because the continuity equation and flow rate definition involve only volume, not mass or density. However, the *pressure* required to achieve this flow rate would differ slightly due to density differences, and other properties like friction might also be affected.

The main uptake air duct of a forced air gas heater is 0.300 m in diameter. What is the average speed of air in the duct if it carries a volume equal to that of the house’s interior every 15 min? The inside volume of the house is equivalent to a rectangular solid 13.0 m wide by 20.0 m long by 2.75 m high.

Show Solution

Strategy

We first calculate the volume of the house, then determine the required flow rate to move this volume in 15 minutes. Finally, we use $Q = Av$ to find the air velocity.

Solution

Given:

- Duct diameter: $d = 0.300\ \text{m}$, so radius $r = 0.150\ \text{m}$
- House dimensions: $13.0\ \text{m}\times20.0\ \text{m}\times2.75\ \text{m}$
- Time to circulate: $t = 15\ \text{min} = 900\ \text{s}$

Calculate the house volume: $V = (13.0\ \text{m})(20.0\ \text{m})(2.75\ \text{m}) = 715\ \text{m}^3$

Calculate the required flow rate: $Q = \frac{V}{t} = \frac{715\ \text{m}^3}{900\ \text{s}} = 0.794\ \text{m}^3/\text{s}$

Calculate the duct cross-sectional area: $A = \pi r^2 = \pi (0.150\ \text{m})^2 = 0.0707\ \text{m}^2$

Calculate the air velocity: $v = \frac{Q}{A} = \frac{0.794\ \text{m}^3/\text{s}}{0.0707\ \text{m}^2} = 11.2\ \text{m/s}$

Discussion

The average speed of air in the duct is approximately **11.2 m/s** (about 40 km/h or 25 mph). This is a reasonable velocity for HVAC systems—fast enough to efficiently circulate air throughout the house, but not so fast as to create excessive noise or require extremely powerful fans. The system circulates the entire house volume every 15 minutes, which means the air is exchanged about 4 times per hour, typical for residential heating systems.

Water is moving at a velocity of 2.00 m/s through a hose with an internal diameter of 1.60 cm. (a) What is the flow rate in liters per second? (b) The fluid velocity in this hose's nozzle is 15.0 m/s. What is the nozzle's inside diameter?

Show Solution

Strategy

(a) We use $Q = Av$ to find the flow rate from the known velocity and hose diameter. (b) We apply the continuity equation, using the fact that flow rate is constant to relate the hose and nozzle dimensions.

Solution

Given:

- Hose velocity: $v_1 = 2.00\ \text{m/s}$
- Hose diameter: $d_1 = 1.60\ \text{cm} = 0.0160\ \text{m}$
- Nozzle velocity: $v_2 = 15.0\ \text{m/s}$

(a) Calculate the hose cross-sectional area: $A_1 = \pi r_1^2 = \pi \left(\frac{0.0160\ \text{m}}{2}\right)^2 = \pi (0.00800\ \text{m})^2 = 2.01 \times 10^{-4}\ \text{m}^2$

Calculate the flow rate: $Q = A_1 v_1 = (2.01 \times 10^{-4}\ \text{m}^2)(2.00\ \text{m/s}) = 4.02 \times 10^{-4}\ \text{m}^3/\text{s}$

Convert to liters per second: $Q = 4.02 \times 10^{-4}\ \text{m}^3/\text{s} \times \frac{1000\ \text{L}}{1\ \text{m}^3} = 0.402\ \text{L/s}$

(b) From continuity: $A_1 v_1 = A_2 v_2$

Solving for A_2: $A_2 = \frac{A_1 v_1}{v_2} = \frac{(2.01 \times 10^{-4}\ \text{m}^2)\,2.00\ \text{m/s}}{15.0\ \text{m/s}} = 2.68 \times 10^{-5}\ \text{m}^2$

Find the nozzle radius: $r_2 = \sqrt{\frac{A_2}{\pi}} = \sqrt{\frac{2.68 \times 10^{-5}\ \text{m}^2}{\pi}} = 2.92 \times 10^{-3}\ \text{m}$

Nozzle diameter: $d_2 = 2r_2 = 5.84 \times 10^{-3}\ \text{m} = 0.584\ \text{cm}$

Discussion

The flow rate through the hose is **(a) 0.402 L/s**, and the nozzle's inside diameter is **(b) 0.584 cm**. The nozzle diameter is about 36% of the hose diameter, which increases the velocity by a factor of 7.5 (from 2.00 to 15.0 m/s). This relationship follows because velocity scales as the inverse square of diameter: $v_2/v_1 = (d_1/d_2)^2 = (1.60/0.584)^2 \approx 7.5$.

Prove that the speed of an incompressible fluid through a constriction, such as in a Venturi tube, increases by a factor equal to the square of the factor by which the diameter decreases. (The converse applies for flow out of a constriction into a larger-diameter region.)

Show Solution

Strategy

We use the equation of continuity and the relationship between area and diameter for a circular cross-section to derive the relationship between velocity change and diameter change.

Solution

For an incompressible fluid, the equation of continuity states: $A_1 v_1 = A_2 v_2$

For circular cross-sections, the area is related to diameter by: $A = \pi r^2 = \pi \left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4}$

Substituting into the continuity equation: $\frac{\pi d_1^2}{4} v_1 = \frac{\pi d_2^2}{4} v_2$

Simplifying (the $\pi/4$ factors cancel): $d_1^2 v_1 = d_2^2 v_2$

Solving for the velocity ratio: $\frac{v_2}{v_1} = \frac{d_1^2}{d_2^2} = \left(\frac{d_1}{d_2}\right)^2$

Let the diameter decrease factor be $k = d_1/d_2$ (so $d_2 = d_1/k$). Then: $\frac{v_2}{v_1} = k^2$

This proves that if the diameter decreases by a factor of k, the velocity increases by a factor of k^2.

Discussion

This result is a direct consequence of the continuity equation and the geometry of circular pipes. For example, if a pipe's diameter is halved ($k = 2$), the velocity increases by a factor of $2^2 = 4$. If the diameter is reduced to one-third ($k = 3$), velocity increases by a factor of 9. This k^2 relationship is why small changes in tube diameter can produce dramatic changes in flow velocity, which is the operating principle behind devices like Venturi meters and carburetor jets.

Water emerges straight down from a faucet with a 1.80-cm diameter at a speed of 0.500 m/s. (Because of the construction of the faucet, there is no variation in speed across the stream.) (a) What is the flow rate in cm^3/s ? (b) What is the diameter of the stream 0.200 m below the faucet? Neglect any effects due to surface tension.

Show Solution

Strategy

(a) We use $Q = Av$ to find the flow rate at the faucet outlet. (b) As water falls, it accelerates due to gravity, increasing its speed. By continuity, if speed increases, the cross-sectional area must decrease. We use kinematics to find the velocity at 0.200 m below, then apply continuity to find the new diameter.

Solution

Given:

- Initial diameter: $d_1 = 1.80\text{ cm} = 0.0180\text{ m}$
- Initial velocity: $v_1 = 0.500\text{ m/s}$
- Fall distance: $h = 0.200\text{ m}$

(a) Calculate the initial cross-sectional area: $A_1 = \pi r_1^2 = \pi\left(\frac{1.80\text{ cm}}{2}\right)^2 = \pi(0.900\text{ cm})^2 = 2.54\text{ cm}^2$

Calculate the flow rate: $Q = A_1 v_1 = (2.54\text{ cm}^2)(50.0\text{ cm/s}) = 127\text{ cm}^3/\text{s}$

(b) Find the velocity after falling 0.200 m using kinematics: $v_2^2 = v_1^2 + 2gh$ $v_2^2 = (0.500\text{ m/s})^2 + 2(9.80\text{ m/s}^2)(0.200\text{ m})$ $v_2^2 = 0.250 + 3.92 = 4.17\text{ m}^2/\text{s}^2$ $v_2 = 2.04\text{ m/s}$

Apply continuity to find the new area: $A_2 = \frac{A_1 v_1}{v_2} = \frac{(2.54\text{ cm}^2)(0.500\text{ m/s})}{2.04\text{ m/s}} = 0.623\text{ cm}^2$

Find the new diameter: $A_2 = \pi r_2^2 = \frac{\pi d_2^2}{4}$ $d_2 = \sqrt{\frac{4A_2}{\pi}} = \sqrt{\frac{4(0.623\text{ cm}^2)}{\pi}} = 0.890\text{ cm}$

Discussion

The flow rate is **(a) 127 cm³/s**, and the stream diameter 0.200 m below the faucet is **(b) 0.890 cm**. The stream narrows as it falls because the water accelerates under gravity—by the time it has fallen 20 cm, the water is moving about 4 times faster, so by continuity, the cross-sectional area is reduced by a factor of 4, and the diameter by a factor of 2. This narrowing of a falling water stream is observable in everyday life.

Unreasonable Results

A mountain stream is 10.0 m wide and averages 2.00 m in depth. During the spring runoff, the flow in the stream reaches $100000\text{m}^3/\text{s}$. (a) What is the average velocity of the stream under these conditions? (b) What is unreasonable about this velocity? (c) What is unreasonable or inconsistent about the premises?

Show Solution

Strategy

We calculate the velocity using $Q = Av$, then evaluate whether the result is physically reasonable for a mountain stream.

Solution

Given:

- Stream width: $w = 10.0\text{ m}$

- Stream depth: $d = 2.00\ \text{m}$
- Flow rate: $Q = 100\ 000\ \text{m}^3/\text{s}$

(a) Calculate the cross-sectional area: $A = w \times d = (10.0\ \text{m})(2.00\ \text{m}) = 20.0\ \text{m}^2$

Calculate the average velocity: $v = QA = 100000\ \text{m}^3/\text{s}20.0\ \text{m}^2 = 5000\ \text{m/s}$

(b) This velocity is approximately **5000 m/s** (5 km/s or about 18,000 km/h). This is unreasonable because:

- It is about 15 times the speed of sound in air (343 m/s)
- It exceeds Earth's escape velocity (11.2 km/s) by nearly half
- Typical fast-flowing rivers have velocities of 1-10 m/s; even extreme flash floods rarely exceed 20 m/s
- Water flowing at this speed would vaporize due to friction and impact forces

(c) The unreasonable premise is the stated flow rate. A flow rate of 100,000 m^3/s is impossibly large for a stream only 10 m wide and 2 m deep. For comparison:

- The Amazon River (the world's largest by volume) has a flow rate of about 200,000 m^3/s, but it is 11 km wide and 50 m deep
- A reasonable flow rate for a mountain stream of these dimensions during spring runoff might be 50-500 m^3/s, giving velocities of 2.5-25 m/s

Discussion

This problem illustrates the importance of checking whether given values lead to physically reasonable results. The stated flow rate would require either an impossibly high velocity (as calculated) or a much larger cross-sectional area. A more realistic flow rate for a 10 m × 2 m stream would be about 100-200 m^3/s, yielding velocities of 5-10 m/s.

Glossary

flow rate
: abbreviated Q, it is the volume V that flows past a particular point during a time t, or $Q = V/t$

liter
: a unit of volume, equal to $10^{-3}\ \text{m}^3$

Bernoulli’s Equation

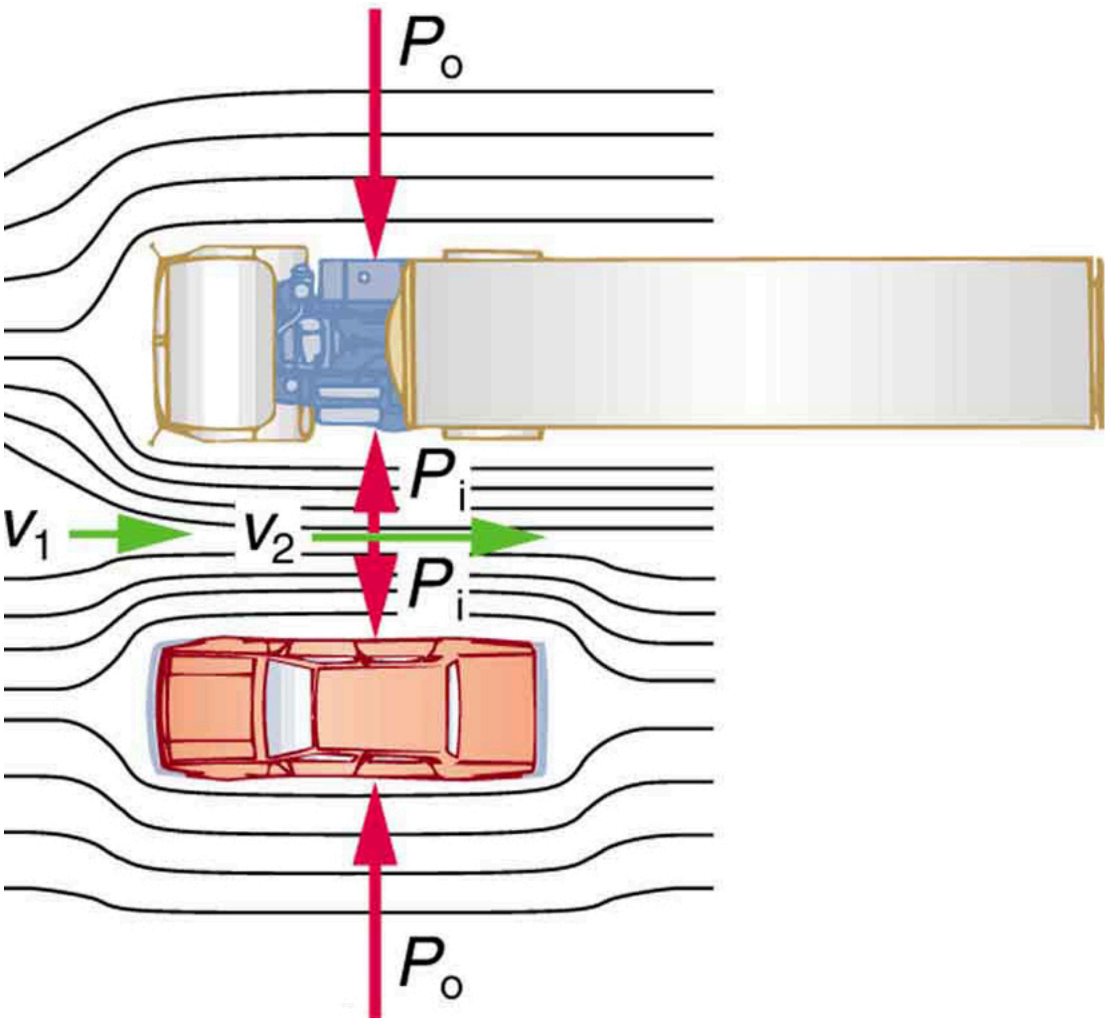
- Explain the terms in Bernoulli’s equation.
- Explain how Bernoulli’s equation is related to conservation of energy.
- Explain how to derive Bernoulli’s principle from Bernoulli’s equation.
- Calculate with Bernoulli’s principle.
- List some applications of Bernoulli’s principle.

When a fluid flows into a narrower channel, its speed increases. That means its kinetic energy also increases. Where does that change in kinetic energy come from? The increased kinetic energy comes from the net work done on the fluid to push it into the channel and the work done on the fluid by the gravitational force, if the fluid changes vertical position. Recall the work-energy theorem,

$$W_{\text{net}}=\frac{1}{2}mv^2-\frac{1}{2}mv_0^2.$$

There is a pressure difference when the channel narrows. This pressure difference results in a net force on the fluid: recall that pressure times area equals force. The net work done increases the fluid’s kinetic energy. As a result, the *pressure will drop in a rapidly-moving fluid*, whether or not the fluid is confined to a tube.

There are a number of common examples of pressure dropping in rapidly-moving fluids. Shower curtains have a disagreeable habit of bulging into the shower stall when the shower is on. The high-velocity stream of water and air creates a region of lower pressure inside the shower, and standard atmospheric pressure on the other side. The pressure difference results in a net force inward pushing the curtain in. You may also have noticed that when passing a truck on the highway, your car tends to veer toward it. The reason is the same—the high velocity of the air between the car and the truck creates a region of lower pressure, and the vehicles are pushed together by greater pressure on the outside. (See [Figure 1].) This effect was observed as far back as the mid-1800s, when it was found that trains passing in opposite directions tipped precariously toward one another.



An overhead view of a car passing a truck on a highway. Air passing between the vehicles flows in a narrower channel and must increase its speed (v_{2} is greater than v_{1}), causing the pressure between them to drop (P_{i} is less than P_{o}). Greater pressure on the outside pushes the car and truck together.

Making Connections: Take-Home Investigation with a Sheet of Paper

Hold the short edge of a sheet of paper parallel to your mouth with one hand on each side of your mouth. The page should slant downward over your hands. Blow over the top of the page. Describe what happens and explain the reason for this behavior.

Bernoulli's Equation

The relationship between pressure and velocity in fluids is described quantitatively by **Bernoulli's equation**, named after its discoverer, the Swiss scientist Daniel Bernoulli (1700–1782). Bernoulli's equation states that for an incompressible, frictionless fluid, the following sum is constant:

$$P+\frac{1}{2}\rho v^2+\rho gh=\text{constant},$$

where P is the absolute pressure, ρ is the fluid density, v is the velocity of the fluid, h is the height above some reference point, and g is the acceleration due to gravity. If we follow a small volume of fluid along its path, various quantities in the sum may change, but the total remains constant. Let the subscripts 1 and 2 refer to any two points along the path that the bit of fluid follows; Bernoulli's equation becomes

$$P_1+\frac{1}{2}\rho v_1^2+\rho gh_1=P_2+\frac{1}{2}\rho v_2^2+\rho gh_2.$$

Bernoulli's equation is a form of the conservation of energy principle. Note that the second and third terms are the kinetic and potential energy with m replaced by ρ . In fact, each term in the equation has units of energy per unit volume. We can prove this for the second term by substituting $\rho = m/V$ into it and gathering terms:

$$\frac{1}{2}\rho v^2=\frac{\frac{1}{2}mv^2}{V}=\frac{\text{KE}}{V}.$$

So $\frac{1}{2}\rho v^2$ is the kinetic energy per unit volume. Making the same substitution into the third term in the equation, we find

$$\rho gh=\frac{mgh}{V}=\frac{\text{PE}_\text{g}}{V},$$

so ρgh is the gravitational potential energy per unit volume. Note that pressure P has units of energy per unit volume, too. Since $P = F/A$, its units are N/m^2 . If we multiply these by m/m, we obtain $\text{N}\cdot\text{m/m}^3 = \text{J/m}^3$, or energy per unit volume. Bernoulli's equation is, in fact, just a convenient statement of conservation of energy for an incompressible fluid in the absence of friction.

Making Connections: Conservation of Energy

Conservation of energy applied to fluid flow produces Bernoulli's equation. The net work done by the fluid's pressure results in changes in the fluid's KE and PE_g per unit volume. If other forms of energy are involved in fluid flow, Bernoulli's equation can be modified to take these forms into account. Such forms of energy include thermal energy dissipated because of fluid viscosity.

The general form of Bernoulli's equation has three terms in it, and it is broadly applicable. To understand it better, we will look at a number of specific situations that simplify and illustrate its use and meaning.

Bernoulli's Equation for Static Fluids

Let us first consider the very simple situation where the fluid is static—that is, $v_1 = v_2 = 0$. Bernoulli's equation in that case is

$$P_1+\rho gh_1=P_2+\rho gh_2.$$

We can further simplify the equation by taking $h_2 = 0$ (we can always choose some height to be zero, just as we often have done for other situations involving the gravitational force, and take all other heights to be relative to this). In that case, we get

$$P_2=P_1+\rho gh_1.$$

This equation tells us that, in static fluids, pressure increases with depth. As we go from point 1 to point 2 in the fluid, the depth increases by h_1 , and consequently, P_2 is greater than P_1 by an amount ρgh_1 . In the very simplest case, P_1 is zero at the top of the fluid, and we get the familiar relationship $P = \rho gh$. (Recall that $P = \rho gh$ and $\Delta\text{PE}_\text{g} = mgh.$) Bernoulli's equation includes the fact that the pressure due to the weight of a fluid is ρgh . Although we introduce Bernoulli's equation for fluid flow, it includes much of what we studied for static fluids in the preceding chapter.

Bernoulli's Principle—Bernoulli's Equation at Constant Depth

Another important situation is one in which the fluid moves but its depth is constant—that is, $h_1 = h_2$. Under that condition, Bernoulli's equation becomes

$$P_1+\frac{1}{2}\rho v_1^2=P_2+\frac{1}{2}\rho v_2^2.$$

Situations in which fluid flows at a constant depth are so important that this equation is often called **Bernoulli's principle**. It is Bernoulli's equation for fluids at constant depth. (Note again that this applies to a small volume of fluid as we follow it along its path.) As we have just discussed, pressure drops as speed increases in a moving fluid. We can see this from Bernoulli's principle. For example, if v_2 is greater than v_1 in the equation, then P_2 must be less than P_1 for the equality to hold.

Calculating Pressure: Pressure Drops as a Fluid Speeds Up

In [Example 2], we found that the speed of water in a hose increased from 1.96 m/s to 25.5 m/s going from the hose to the nozzle. Calculate the pressure in the hose, given that the absolute pressure in the nozzle is $1.01 \times 10^5 \text{N/m}^2$ (atmospheric, as it must be) and assuming level, frictionless flow.

Strategy

Level flow means constant depth, so Bernoulli's principle applies. We use the subscript 1 for values in the hose and 2 for those in the nozzle. We are thus asked to find P_1.

Solution

Solving Bernoulli's principle for P_1 yields

$$P_1 = P_2 + \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho(v_2^2 - v_1^2).$$

Substituting known values,

$$P_1 = 1.01 \times 10^5 \text{N/m}^2 + \frac{1}{2}(10^3 \text{kg/m}^3)[(25.5 \text{m/s})^2 - (1.96 \text{m/s})^2] \quad P_1 = 4.24 \times 10^5 \text{N/m}^2.$$

Discussion

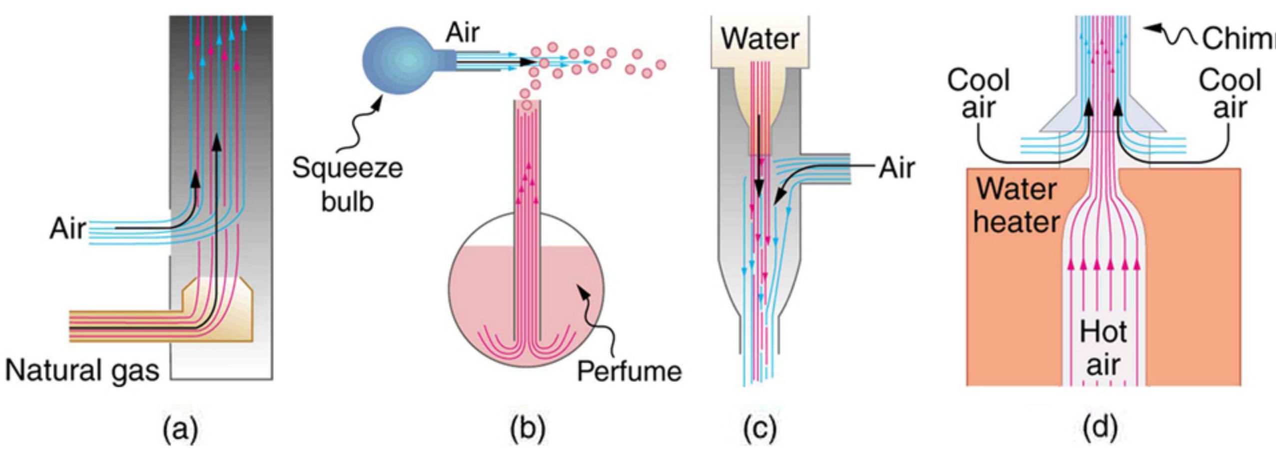
This absolute pressure in the hose is greater than in the nozzle, as expected since v is greater in the nozzle. The pressure P_2 in the nozzle must be atmospheric since it emerges into the atmosphere without other changes in conditions.

Applications of Bernoulli's Principle

There are a number of devices and situations in which fluid flows at a constant height and, thus, can be analyzed with Bernoulli's principle.

Entrainment

People have long put the Bernoulli principle to work by using reduced pressure in high-velocity fluids to move things about. With a higher pressure on the outside, the high-velocity fluid forces other fluids into the stream. This process is called *entrainment*. Entrainment devices have been in use since ancient times, particularly as pumps to raise water small heights, as in draining swamps, fields, or other low-lying areas. Some other devices that use the concept of entrainment are shown in [Figure 2].



Examples of entrainment devices that use increased fluid speed to create low pressures, which then entrain one fluid into another. (a) A Bunsen burner uses an adjustable gas nozzle, entraining air for proper combustion. (b) An atomizer uses a squeeze bulb to create a jet of air that entrains drops of perfume. Paint sprayers and carburetors use very similar techniques to move their respective liquids. (c) A common aspirator uses a high-speed stream of water to create a region of lower pressure. Aspirators may be used as suction pumps in dental and surgical situations or for draining a flooded basement or producing a reduced pressure in a vessel. (d) The chimney of a water heater is designed to entrain air into the pipe leading through the ceiling.

Wings and Sails

The airplane wing is a beautiful example of Bernoulli's principle in action. [Figure 3](a) shows the characteristic shape of a wing. The wing is tilted upward at a small angle and the upper surface is longer, causing air to flow faster over it. The pressure on top of the wing is therefore reduced, creating a net upward force or lift. (Wings can also gain lift by pushing air downward, utilizing the conservation of momentum principle. The deflected air molecules result in an upward force on the wing — Newton's third law.) Sails also have the characteristic shape of a wing. (See [Figure 3](b).) The pressure on the front side of the sail, P_{front}, is lower than the pressure on the back of the sail, P_back . This results in a forward force and even allows you to sail into the wind.

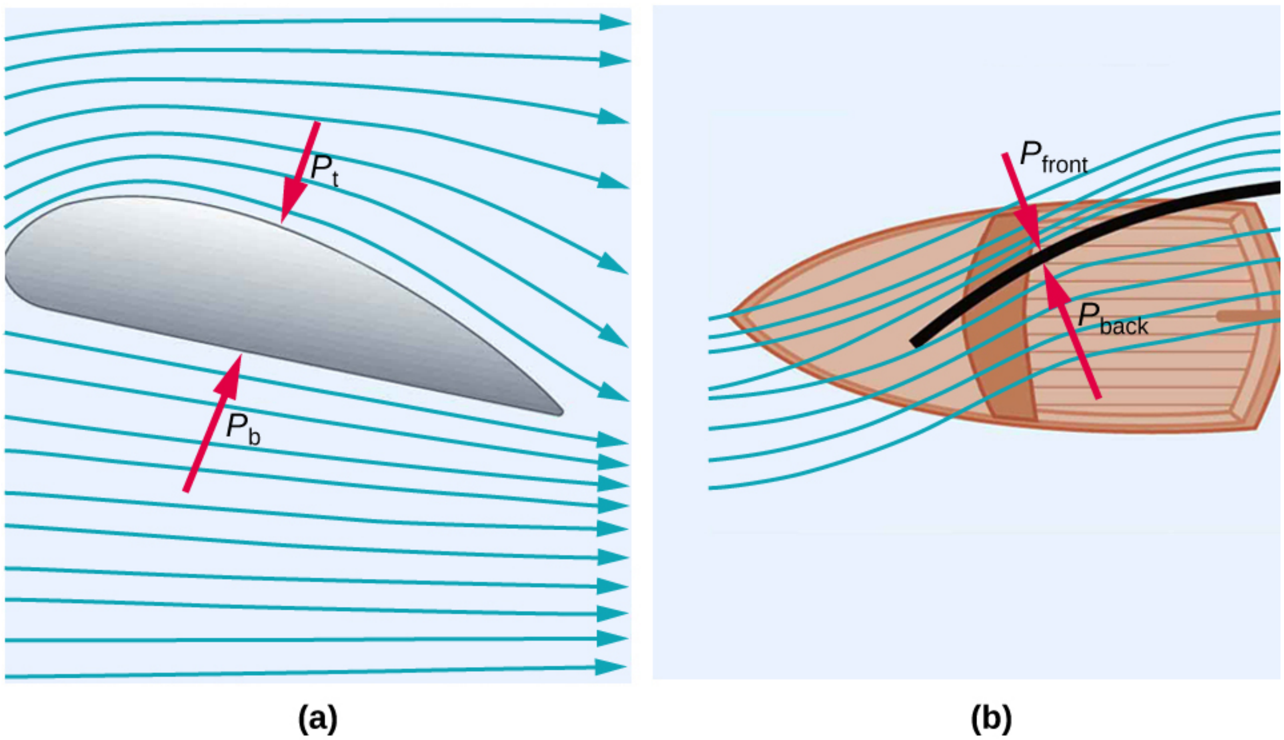
Making Connections: Take-Home Investigation with Two Strips of Paper

For a good illustration of Bernoulli’s principle, make two strips of paper, each about 15 cm long and 4 cm wide. Hold the small end of one strip up to your lips and let it drape over your finger. Blow across the paper. What happens? Now hold two strips of paper up to your lips, separated by your fingers. Blow between the strips. What happens?

Velocity measurement

[Figure 4] shows two devices that measure fluid velocity based on Bernoulli’s principle. The manometer in [Figure 4](a) is connected to two tubes that are small enough not to appreciably disturb the flow. The tube facing the oncoming fluid creates a dead spot having zero velocity ($v_1 = 0$) in front of it, while fluid passing the other tube has velocity v_2 . This means that Bernoulli’s principle as stated in $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$ becomes

$$P_1 = P_2 + \frac{1}{2}\rho v_2^2.$$



(a) The Bernoulli principle helps explain lift generated by a wing. (b) Sails use the same technique to generate part of their thrust.

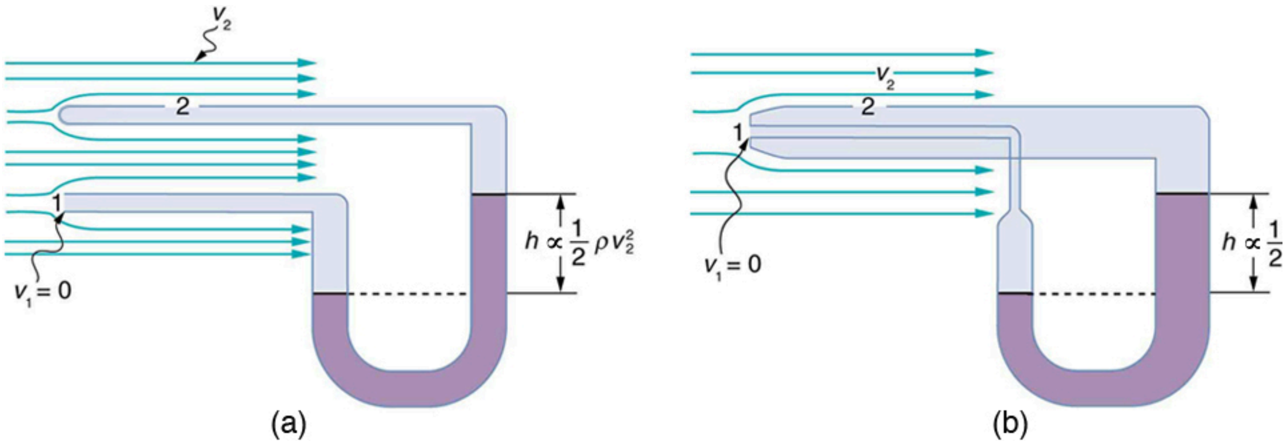
Thus pressure P_2 over the second opening is reduced by $\frac{1}{2}\rho v_2^2$, and so the fluid in the manometer rises by h on the side connected to the second opening, where

$$h \propto \frac{1}{2}\rho v_2^2.$$

(Recall that the symbol $\propto$ means “proportional to.”) Solving for v_2 , we see that

$$v_2 \propto \sqrt{h}.$$

[Figure 4](b) shows a version of this device that is in common use for measuring various fluid velocities; such devices are frequently used as air speed indicators in aircraft.



Measurement of fluid speed based on Bernoulli’s principle. (a) A manometer is connected to two tubes that are close together and small enough not to disturb the flow. Tube 1 is open at the end facing the flow. A dead spot having zero speed is created there. Tube 2 has an opening on the side, and so the fluid has a speed v; across the opening; thus, pressure there drops. The difference in pressure at the manometer is $1/2 \rho v^2_2$, and so h is proportional to $1/2 \rho v^2_2$. (b) This type of velocity measuring device is a Prandtl tube, also known as a pitot tube.

Summary

- Bernoulli’s equation states that the sum on each side of the following equation is constant, or the same at any two points in an incompressible frictionless fluid:

  $$P_1+\frac{1}{2}\rho v_1^2+\rho g h_1=P_2+\frac{1}{2}\rho v_2^2+\rho g h_2.$$

- Bernoulli’s principle is Bernoulli’s equation applied to situations in which depth is constant. The terms involving depth (or height h) subtract out, yielding

  $$P_1+\frac{1}{2}\rho v_1^2=P_2+\frac{1}{2}\rho v_2^2.$$

- Bernoulli’s principle has many applications, including entrainment, wings and sails, and velocity measurement.

Conceptual Questions

You can squirt water a considerably greater distance by placing your thumb over the end of a garden hose and then releasing, than by leaving it completely uncovered. Explain how this works.

Show Solution

Strategy

We apply the continuity equation and Bernoulli’s principle to understand how restricting the opening affects water velocity.

Solution

When you partially cover the hose opening with your thumb, you reduce the cross-sectional area through which the water exits. By the continuity equation ($A_1 v_1 = A_2 v_2$), if the area decreases while flow rate remains constant, the velocity must increase proportionally.

The water exits at a much higher speed when the opening is restricted. Since the range of a projectile depends on its initial velocity (range $\propto v^2$ for a given angle), the higher exit velocity allows the water to travel much farther.

Additionally, Bernoulli’s principle explains that the pressure inside the hose increases when you restrict the flow (the water must slow down as it approaches the constriction, converting kinetic energy to pressure energy). This higher pressure is then converted back to kinetic energy as the water accelerates through the small opening.

Discussion

This is a practical demonstration of the continuity equation. The same flow rate through a smaller area requires higher velocity. This principle is used in many applications, from spray nozzles to jet engines.

Water is shot nearly vertically upward in a decorative fountain and the stream is observed to broaden as it rises. Conversely, a stream of water falling straight down from a faucet narrows. Explain why, and discuss whether surface tension enhances or reduces the effect in each case.

Show Solution

Strategy

We apply the continuity equation considering that gravity affects the water’s velocity as it rises or falls.

Solution

Rising water (fountain): As water rises against gravity, it decelerates. By the continuity equation $A_1v_1 = A_2v_2$, when velocity v decreases, the cross-sectional area A must increase to maintain constant flow rate. Therefore, the stream broadens as it rises.

Falling water (faucet): As water falls under gravity, it accelerates. By continuity, when velocity increases, the cross-sectional area must decrease. Therefore, the stream narrows as it falls.

Effect of surface tension:

- For the rising fountain stream, surface tension *reduces* the broadening effect by pulling the water surface inward, trying to minimize the surface area.
- For the falling faucet stream, surface tension *enhances* the narrowing effect by pulling the stream together, working in the same direction as the continuity effect.

Discussion

This explains why falling water streams eventually break into droplets—the stream continues to narrow until surface tension forces cause instabilities that break the continuous stream into discrete drops (Rayleigh instability).

Look back to [Figure 1]. Answer the following two questions. Why is P_o less than atmospheric? Why is P_o greater than P_i ?

Show Solution

Strategy

We apply Bernoulli's principle, which states that where fluid velocity is higher, pressure is lower.

Solution

Why is P_o less than atmospheric? The air on the outer sides of both the car and truck is essentially stationary (moving with the vehicles). According to Bernoulli's principle, stationary or slow-moving air has the highest pressure. The air passing along the outer surfaces of the vehicles moves at a moderate speed, creating a pressure P_o that is slightly less than the atmospheric pressure of completely still air far from the vehicles.

Why is P_o greater than P_i? The space between the car and truck is a constricted channel. By continuity, air must speed up as it flows through this narrower gap ($v_2 > v_1$). According to Bernoulli's principle, this higher velocity means lower pressure. Therefore, the pressure between the vehicles (P_i) is less than the pressure on the outer sides (P_o).

This pressure difference creates a net inward force that pushes the vehicles toward each other.

Discussion

This effect is why passing trucks on highways can be hazardous—the pressure difference can cause a car to veer toward the truck. The faster the vehicles travel and the closer they are, the more pronounced this effect becomes.

Give an example of entrainment not mentioned in the text.

Show Solution

Strategy

We identify a device or phenomenon that uses the Bernoulli principle to draw one fluid into a moving stream of another fluid.

Solution

Examples of entrainment not explicitly mentioned in the text include:

1. **Garden hose siphon attachments**: These devices attach to a hose and use the fast-moving water stream to draw fertilizer or soap solution from a container into the water flow.

2. **Jet pumps (eductors)**: Used in wells to lift water from deep underground by using a high-pressure water jet to create a low-pressure region that draws up groundwater.

3. **Steam ejectors**: Used in industrial settings where high-velocity steam entrains gases or vapors, commonly used to create vacuums.

4. **Foam fire extinguisher nozzles**: The water stream entrains air to create foam.

5. **Aquarium air pumps**: Some designs use water flow to entrain air bubbles into the tank.

6. **Venturi scrubbers**: Industrial pollution control devices that use high-velocity gas flow to entrain scrubbing liquid droplets.

Discussion

All these devices exploit the same principle: a high-velocity fluid stream creates a region of low pressure that draws in another fluid. This is a direct application of Bernoulli's principle.

Many entrainment devices have a constriction, called a Venturi, such as shown in [Figure 5]. How does this bolster entrainment?

Show Solution

Strategy

We apply Bernoulli's principle to analyze how a constriction affects pressure and thus entrainment efficiency.

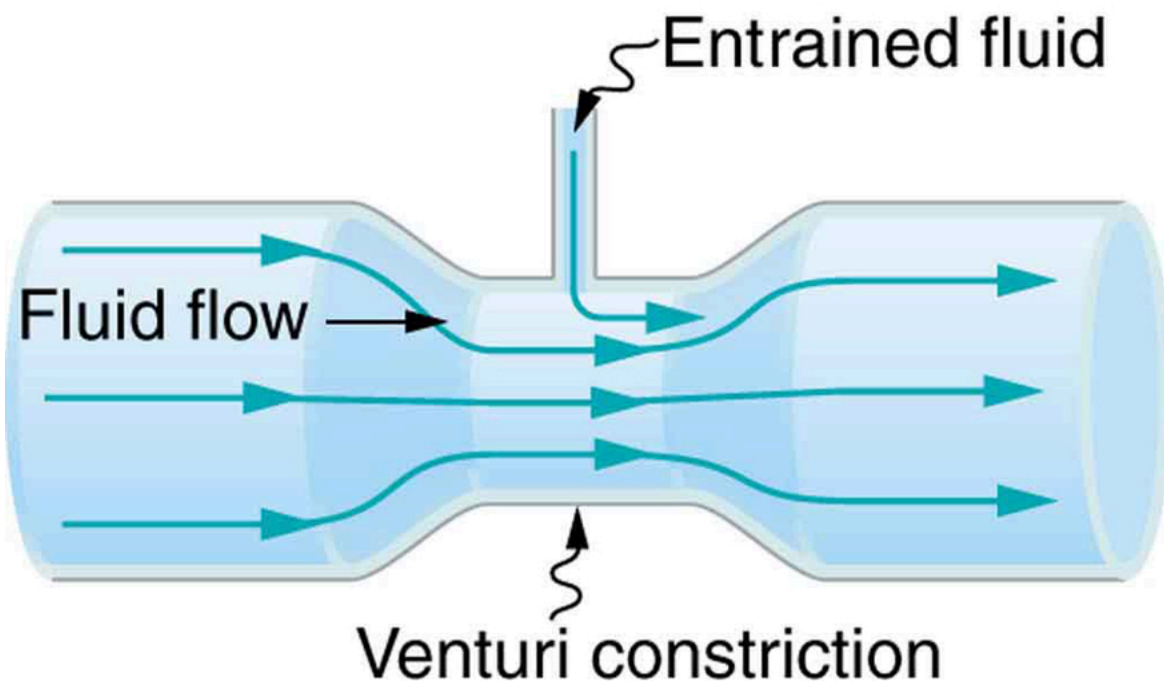
Solution

A Venturi constriction bolsters entrainment by maximizing the pressure drop at the point where the secondary fluid is introduced. Here's how it works:

1. **Increased velocity**: By the continuity equation, when fluid passes through the narrowed section, its velocity increases significantly ($v = Q/A$, so smaller A means larger v).
2. **Maximum pressure drop**: By Bernoulli's principle ($P + \frac{1}{2}\rho v^2 = \text{constant}$), the higher velocity in the constriction corresponds to the lowest pressure in the system.
3. **Enhanced suction**: The secondary fluid inlet is placed at this point of minimum pressure. The greater the velocity increase (and thus the greater the pressure drop), the stronger the suction force drawing in the entrained fluid.
4. **Efficient mixing**: After the constriction, the tube expands again, allowing the velocity to decrease and pressure to recover, which helps mix the two fluids thoroughly.

Discussion

The Venturi design optimizes entrainment by creating the maximum possible pressure differential. This is why carburetors, aspirators, and many industrial mixing devices use this characteristic shape. The degree of constriction can be adjusted to control the entrainment rate.



A tube with a narrow segment designed to enhance entrainment is called a Venturi. These are very commonly used in carburetors and aspirators.

Some chimney pipes have a T-shape, with a crosspiece on top that helps draw up gases whenever there is even a slight breeze. Explain how this works in terms of Bernoulli's principle.

Show Solution

Strategy

We apply Bernoulli's principle to understand how horizontal wind creates a pressure drop at the chimney top.

Solution

When wind blows across the horizontal crosspiece of a T-shaped chimney cap, the air flows faster over and around the openings than it would if the air were still. According to Bernoulli's principle, this faster-moving air has lower pressure than the still air inside the chimney.

This pressure difference creates a suction effect: the higher pressure inside the chimney pushes gases upward toward the lower pressure region at the top. Even a slight breeze is enough to enhance the "draft" of the chimney, improving the flow of combustion gases out of the building.

The T-shape is particularly effective because:

1. It shields the opening from rain and downdrafts
2. Wind from any horizontal direction creates flow across at least one opening
3. The crosspiece creates a low-pressure zone regardless of wind direction

Discussion

This is an elegant passive ventilation solution. Similar principles are used in rotating turbine vents on rooftops and in the design of natural ventilation systems for buildings.

Is there a limit to the height to which an entrainment device can raise a fluid? Explain your answer.

Show Solution

Strategy

We consider the maximum pressure difference that can be created by the Bernoulli effect and compare it to the hydrostatic pressure required to lift a fluid column.

Solution

Yes, there is a fundamental limit. An entrainment device works by creating a low-pressure region. The maximum suction possible occurs when the pressure approaches zero (a perfect vacuum). Atmospheric pressure at sea level is approximately 101,325 Pa.

The maximum height h to which a fluid can be raised is determined by: $P_{atm} = \rho g h_{max}$

For water ($\rho = 1000\ \text{kg/m}^3$): $h_{max} = \frac{P_{atm}}{\rho g} = \frac{101{,}325\ \text{Pa}}{(1000\ \text{kg/m}^3)(9.8\ \text{m/s}^2)} \approx 10.3\ \text{m}$

This is why suction pumps cannot lift water more than about 10 meters, regardless of how powerful they are. In practice, the limit is even lower because:

1. A perfect vacuum cannot be achieved
2. Water vapor pressure reduces the effective pressure difference
3. Friction and turbulence cause energy losses

Discussion

This limit of approximately 10 m for water explains why deep wells require submersible pumps or jet pumps rather than suction pumps at the surface. For denser fluids like mercury, the limit is much lower (about 76 cm), while for less dense fluids like gasoline, it’s somewhat higher.

Why is it preferable for airplanes to take off into the wind rather than with the wind?

Show Solution

Strategy

We consider how airspeed (not ground speed) determines lift, and how a headwind affects the required ground speed for takeoff.

Solution

Airplane lift depends on *airspeed*—the speed of the air flowing over the wings—not ground speed. According to Bernoulli’s principle, lift is generated when air flows faster over the curved upper surface of the wing, creating lower pressure above than below.

When taking off into the wind:

- If the wind speed is 20 m/s and the aircraft needs 60 m/s airspeed for takeoff, it only needs to reach 40 m/s ground speed
- This means a shorter takeoff roll and less runway required

When taking off with the wind:

- With a 20 m/s tailwind and needing 60 m/s airspeed, the aircraft must reach 80 m/s ground speed
- This requires a much longer runway and more fuel

Additional benefits of taking off into the wind:

1. Better control at low speeds (more air over control surfaces)
2. Steeper climb angle relative to the ground
3. More time and distance to clear obstacles
4. Better engine cooling during the takeoff roll

Discussion

This is why airports have runways oriented to align with prevailing winds, and why pilots pay close attention to wind direction and speed when planning takeoffs and landings.

Roofs are sometimes pushed off vertically during a tropical cyclone, and buildings sometimes explode outward when hit by a tornado. Use Bernoulli’s principle to explain these phenomena.

Show Solution

Strategy

We apply Bernoulli's principle to compare pressures inside and outside buildings during high-wind events.

Solution

Roofs pushed off during cyclones: During a tropical cyclone, extremely high-speed winds blow over the roof while the air inside the building remains relatively still. By Bernoulli's principle:

- Fast-moving air outside → low pressure above the roof
- Still air inside → normal (higher) pressure below the roof

This pressure difference creates a net upward force on the roof. If the wind speed is high enough, this upward force can exceed the weight of the roof plus the strength of its attachments, lifting the roof off the building.

For a roof area of 100 m^2 and wind speed of 50 m/s, the pressure difference can create an upward force of over 150,000 N (15+ tons).

Buildings exploding outward during tornadoes: Tornadoes create an extreme low-pressure zone. When a tornado passes directly over a building:

- The pressure outside drops dramatically (can be 10-20% lower than normal)
- The pressure inside the sealed building remains near normal atmospheric pressure

This pressure difference pushes outward on all walls and the roof simultaneously, potentially causing the building to appear to "explode" outward.

Discussion

This is why it was once (incorrectly) recommended to open windows during tornadoes—to equalize pressure. However, modern advice is to seek shelter immediately, as the primary danger is from flying debris, not pressure differences. Opening windows actually allows wind and debris to enter and can make structural damage worse.

Why does a sailboat need a keel?

Show Solution

Strategy

We analyze the forces on a sailboat to understand why a keel is necessary for sailing, particularly when sailing at an angle to the wind.

Solution

A sailboat needs a keel for two essential reasons:

1. To resist sideways motion (leeway): When wind pushes on the sail, it creates a force that has both a forward component (desired) and a sideways component. Without a keel, the boat would simply slide sideways through the water. The keel provides lateral resistance—it moves easily forward through the water but strongly resists sideways motion. This converts the sideways force into forward motion.

2. To provide stability (prevent capsizing): The force on the sail also creates a torque that tries to tip the boat over (heel). The keel, especially a weighted keel, creates a counterbalancing torque:

- The weight at the bottom of the keel creates a righting moment
- When the boat heels, the keel's weight acts to bring it back upright
- The deeper and heavier the keel, the more stable the boat

Relation to Bernoulli's principle: The keel itself works somewhat like an underwater wing. When the boat moves forward with some sideways drift, water flows past the keel at an angle, creating lift (via Bernoulli's principle) that opposes the sideways motion and even helps drive the boat forward.

Discussion

This is why sailboats can sail "into the wind" (at an angle up to about 45° from the wind direction). The combination of the sail generating force from the wind and the keel preventing sideways slip allows the boat to convert wind energy into forward motion even when heading partially upwind.

It is dangerous to stand close to railroad tracks when a rapidly moving commuter train passes. Explain why atmospheric pressure would push you toward the moving train.

Show Solution

Strategy

We apply Bernoulli's principle to analyze the pressure difference created by the fast-moving air near the train.

Solution

When a train passes at high speed, it drags air along with it. The air between you and the train moves very rapidly. According to Bernoulli's principle, this fast-moving air has lower pressure than the still air on your other side.

The pressure difference creates a net force pushing you toward the train:

- Low pressure near the train (fast-moving air)
- Normal atmospheric pressure on your far side (still air)
- Net force = (pressure difference) × (your projected area)

For a train moving at 30 m/s and using Bernoulli's principle: $\Delta P = \frac{1}{2}\rho v^2 = \frac{1}{2}(1.2\ \text{kg/m}^3)(30\ \text{m/s})^2 \approx 540\ \text{Pa}$

For a person with a cross-sectional area of about 0.5 m², this creates a force of approximately 270 N (about 60 pounds)—enough to cause an adult to stumble.

Discussion

This effect is one reason why safety lines and barriers are placed at train platforms. The danger is particularly acute for lighter objects (children, luggage, newspapers) and increases dramatically with train speed since the force scales with v^2.

Water pressure inside a hose nozzle can be less than atmospheric pressure due to the Bernoulli effect. Explain in terms of energy how the water can emerge from the nozzle against the opposing atmospheric pressure.

Show Solution

Strategy

We apply conservation of energy (Bernoulli's equation) to track how energy transforms as water moves through the system.

Solution

Although the pressure inside the nozzle is less than atmospheric, the water can still emerge because it has kinetic energy. Bernoulli's equation shows that total energy per unit volume is conserved:

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$$

Inside the hose (before the nozzle):

- Higher pressure P_1
- Lower velocity v_1
- Total energy: mostly pressure energy

Inside the nozzle:

- Lower pressure $P_2 < P_{atm}$
- Much higher velocity v_2
- Total energy: mostly kinetic energy

The water doesn't need to "push against" atmospheric pressure in the sense of using pressure energy—it uses its kinetic energy instead. The high velocity of the water carries it forward despite the adverse pressure gradient.

Think of it like a ball thrown upward: even though gravity opposes its motion, the ball's kinetic energy carries it upward until that energy is depleted. Similarly, the water's kinetic energy is sufficient to overcome the slight pressure opposition at the nozzle exit.

Discussion

This is why nozzles work so effectively—they convert pressure energy to kinetic energy. The water emerges at high speed with enough momentum to travel a significant distance through the air.

A perfume bottle or atomizer sprays a fluid that is in the bottle. ([Figure 6].) How does the fluid rise up in the vertical tube in the bottle?

Atomizer: perfume bottle with tube to carry perfume up through the bottle. (credit: Antonia Foy, Flickr)

Show Solution

Strategy

We apply Bernoulli's principle to explain how the horizontal air flow creates suction that draws liquid up the tube.

Solution

When you squeeze the bulb (or press the pump), air is forced at high speed across the top opening of the vertical tube that extends down into the perfume. According to Bernoulli's principle:

1. **High-speed air at the tube opening** → Low pressure at the top of the tube
2. **Still air (atmospheric pressure)** → Acts on the liquid surface inside the bottle
3. **Pressure difference** → Pushes liquid up the tube

The atmospheric pressure on the liquid surface (transmitted through a vent hole in the bottle) is greater than the reduced pressure at the top of the tube. This pressure difference pushes the perfume up the tube:

$$P_{atm} > P_{top}$$

$$\Delta P = P_{atm} - P_{top} = \frac{1}{2}\rho_{air} v^2$$

When the liquid reaches the top, the fast-moving air stream also breaks it into tiny droplets (atomization), creating a fine mist.

Discussion

This is a classic example of entrainment. The same principle is used in:

- Paint sprayers
- Carburetors (fuel atomization)
- Medical nebulizers

- Garden sprayers

If you lower the window on a car while moving, an empty plastic bag can sometimes fly out the window. Why does this happen?

Show Solution

Strategy

We apply Bernoulli’s principle to compare the pressure inside and outside the moving car.

Solution

When you lower a car window while driving, there is a pressure difference between the inside and outside of the car:

Outside the car: Air rushes past at the speed of the car (relative to the car). By Bernoulli’s principle, this fast-moving air has lower pressure than still air.

Inside the car: The air is relatively still (moving with the car), so it maintains approximately normal atmospheric pressure.

This creates a pressure difference: $\Delta P = P_{inside} - P_{outside} = \frac{1}{2}\rho v^2$

For a car moving at 25 m/s (about 90 km/h): $\Delta P = \frac{1}{2}(1.2\ \text{kg/m}^3)(25\ \text{m/s})^2 \approx 375\ \text{Pa}$

This pressure difference pushes air (and light objects like plastic bags) from the higher-pressure interior toward the lower-pressure exterior. A light plastic bag experiences this outward force and, if unrestrained, will be pushed out the window.

Discussion

This is why you feel air rushing out of a car when you first open a window at highway speeds—the higher inside pressure equalizes with the lower outside pressure. The same effect makes it harder to close a car door when a window on the opposite side is open.

Problems & Exercises

Verify that pressure has units of energy per unit volume.

Show Solution

Strategy

We analyze the fundamental units of pressure and show they are equivalent to energy per unit volume through dimensional analysis.

Solution

Starting with the definition of pressure: $P = \frac{\text{Force}}{\text{Area}} = \frac{F}{A}$

The SI unit of pressure is: $[P] = \frac{\text{N}}{\text{m}^2}$

Now we multiply numerator and denominator by meters: $[P] = \frac{\text{N}}{\text{m}^2} = \frac{\text{N}\cdot\text{m}}{\text{m}^2\cdot\text{m}} = \frac{\text{N}\cdot\text{m}}{\text{m}^3}$

Since N·m = J (joules, the unit of energy): $[P] = \frac{\text{J}}{\text{m}^3} = \frac{\text{energy}}{\text{volume}}$

Discussion

This result is important because it shows that each term in Bernoulli’s equation (P, $\frac{1}{2}\rho v^2$, and ρgh) has units of energy per unit volume, confirming that Bernoulli’s equation is a statement of energy conservation. Pressure can be thought of as “pressure energy” per unit volume available to do work on the fluid.

Suppose you have a wind speed gauge like the pitot tube shown in [[Example 2]](../contents/ch12FlowRateAndItsRelationsToVelocity#Example2(b). By what factor must wind speed increase to double the value of h in the manometer? Is this independent of the moving fluid and the fluid in the manometer?

Show Solution

Strategy

We use the relationship between manometer height h and fluid velocity from Bernoulli’s principle. The pressure difference measured by the manometer is proportional to v^2.

Solution

For a pitot tube, Bernoulli’s principle gives: $P_1 = P_2 + \frac{1}{2}\rho v^2$

The pressure difference is: $\Delta P = \frac{1}{2}\rho v^2$

This pressure difference supports a column of manometer fluid of height h: $\Delta P = \rho' g h$

where ρ' is the density of the manometer fluid.

Therefore: $h = \frac{\rho v^2}{2\rho' g}$

This shows that $h \propto v^2$, meaning: $\frac{h_2}{h_1} = \frac{v_2^2}{v_1^2}$

To double h (so $h_2 = 2h_1$): $2 = \frac{v_2^2}{v_1^2}$ $\frac{v_2}{v_1} = \sqrt{2} \approx 1.41$

Is this independent of the fluids?

The factor $\sqrt{2}$ is independent of both the moving fluid density ρ and the manometer fluid density ρ'. While the actual value of h depends on these densities, the *ratio* of heights depends only on the ratio of velocities squared.

Discussion

The wind speed must increase by a factor of $\sqrt{2} \approx 1.41$ (about 41%) to double the manometer reading. This square-root relationship means that pitot tubes become less sensitive at higher speeds—small changes in h correspond to larger changes in v.

If the pressure reading of your pitot tube is 15.0 mm Hg at a speed of 200 km/h, what will it be at 700 km/h at the same altitude?

Show Solution

Strategy

Since the pressure difference in a pitot tube is proportional to v^2, we can use the ratio of velocities squared to find the new pressure reading.

Solution

Given:

- Initial reading: $h_1 = 15.0 \text{ mm Hg}$ at $v_1 = 200 \text{ km/h}$
- Final velocity: $v_2 = 700 \text{ km/h}$

From Bernoulli's principle, the pressure reading (height) is proportional to v^2: $\frac{h_2}{h_1} = \frac{v_2^2}{v_1^2}$

Solving for h_2: $h_2 = h_1 \times \frac{v_2^2}{v_1^2} = 15.0 \text{ mm Hg} \times \frac{(700 \text{ km/h})^2}{(200 \text{ km/h})^2}$

$$h_2 = 15.0 \text{ mm Hg} \times \frac{490{,}000}{40{,}000} = 15.0 \text{ mm Hg} \times 12.25$$
$$h_2 = 184 \text{ mm Hg}$$

Discussion

The pressure reading at 700 km/h is **184 mm Hg**. The velocity increased by a factor of 3.5, but the pressure reading increased by a factor of 12.25 (which is 3.5^2). This illustrates the quadratic relationship between velocity and the dynamic pressure measured by a pitot tube. The new reading of 184 mm Hg is about 24.5 kPa, a substantial pressure that pilots and instrumentation must account for.

Calculate the maximum height to which water could be squirted with the hose in [Example 2] example if it: (a) Emerges from the nozzle. (b) Emerges with the nozzle removed, assuming the same flow rate.

Show Solution

Strategy

We use energy conservation (Bernoulli's equation). At the maximum height, all kinetic energy is converted to gravitational potential energy. From Example 2 in the previous section, the nozzle velocity is 25.5 m/s and the hose velocity is 1.96 m/s.

Solution

At maximum height, all kinetic energy converts to potential energy: $\frac{1}{2}v^2 = g h_{max}$ $h_{max} = \frac{v^2}{2g}$

(a) With nozzle (v = 25.5 m/s): $h_{max} = \frac{(25.5 \text{ m/s})^2}{2(9.80 \text{ m/s}^2)} = \frac{650.25}{19.6} = 33.2 \text{ m}$

(b) Without nozzle (v = 1.96 m/s): $h_{max} = \frac{(1.96 \text{ m/s})^2}{2(9.80 \text{ m/s}^2)} = \frac{3.84}{19.6} = 0.196 \text{ m} \approx 0.20 \text{ m}$

Discussion

Water from the nozzle can reach a maximum height of **(a) 33.2 m** (about 109 feet), while without the nozzle it can only reach **(b) 0.20 m** (about 8 inches). The nozzle increases the exit velocity by a factor of 13, which increases the maximum height by a factor of $13^2 \approx 169$. This dramatic difference demonstrates why fire hoses use nozzles to reach upper floors of buildings.

Every few years, winds in Boulder, Colorado, attain sustained speeds of 45.0 m/s (about 100 mi/h) when the jet stream descends during early spring. Approximately what is the force due to the Bernoulli effect on a roof having an area of 220m^2 ? Typical air density in Boulder is 1.14kg/m^3 , and the corresponding atmospheric pressure is $8.89 \times 10^4\text{N/m}^2$. (Bernoulli's principle as stated in the text assumes laminar flow. Using the principle here produces only an approximate result, because there is significant turbulence.)

Show Solution

Strategy

We apply Bernoulli's principle comparing the fast-moving air above the roof to the still air inside the building. The pressure difference creates an upward force on the roof.

Solution

Given:

- Wind speed: $v = 45.0\text{ m/s}$
- Roof area: $A = 220\text{ m}^2$
- Air density: $\rho = 1.14\text{ kg/m}^3$

Using Bernoulli's principle, the pressure difference is: $\Delta P = 12\rho v^2 = 12(1.14\text{ kg/m}^3)(45.0\text{ m/s})^2$

$$\Delta P = 12(1.14)(2025) = 1154\text{ N/m}^2 \approx 1.15\text{ kPa}$$

The upward force on the roof is: $F = \Delta P \times A = (1154\text{ N/m}^2)(220\text{ m}^2) = 2.54 \times 10^5\text{ N}$

Discussion

The force due to the Bernoulli effect on the roof is approximately **2.54×10^5 N** (about 57,000 pounds or 28.5 tons). This enormous upward force explains why roofs can be lifted off buildings during high-wind events. Since this force exceeds the weight of many residential roofs, proper anchoring is essential in areas prone to severe winds. Note that this is an approximation since real wind flow involves significant turbulence.

(a) Calculate the approximate force on a square meter of sail, given the horizontal velocity of the wind is 6.00 m/s parallel to its front surface and 3.50 m/s along its back surface. Take the density of air to be 1.29kg/m^3 . (The calculation, based on Bernoulli's principle, is approximate due to the effects of turbulence.) (b) Discuss whether this force is great enough to be effective for propelling a sailboat.

Show Solution

Strategy

We apply Bernoulli's principle to find the pressure difference between the front and back surfaces of the sail, then calculate the resulting force.

Solution

Given:

- Front surface wind speed: $v_f = 6.00\text{ m/s}$
- Back surface wind speed: $v_b = 3.50\text{ m/s}$
- Air density: $\rho = 1.29\text{ kg/m}^3$
- Area: $A = 1.00\text{ m}^2$

(a) Using Bernoulli's principle: $P_f + 12\rho v_{2f} = P_b + 12\rho v_{2b}$

The pressure difference is: $\Delta P = P_b - P_f = 12\rho(v_{2f} - v_{2b})$

$$\Delta P = 12(1.29\text{ kg/m}^3)[(6.00\text{ m/s})^2 - (3.50\text{ m/s})^2]$$
$$\Delta P = 12(1.29)(36.0 - 12.25) = 12(1.29)(23.75) = 15.3\text{ N/m}^2$$

Force per square meter: $F = \Delta P \times A = 15.3\text{ N/m}^2 \times 1.00\text{ m}^2 = 15.3\text{ N}$

(b) For a typical small sailboat with 15-20 m² of sail area: $F_{total} = 15.3\text{ N/m}^2 \times 20\text{ m}^2 = 306\text{ N}$

This force of about 300 N (68 pounds) is certainly sufficient to propel a small sailboat. Even accounting for the fact that not all this force acts in the desired direction (component resolution is needed), a significant forward thrust remains. Combined with the low resistance of a well-designed hull moving through water, this force can produce sailing speeds of several meters per second.

Discussion

The force on a square meter of sail is approximately **(a) 15.3 N**. **(b)** Yes, this force is effective for propelling a sailboat. A typical sail of 15-20 m² would experience about 230-310 N of force, which is adequate for moving a small boat efficiently through water where drag forces are relatively low.

(a) What is the pressure drop due to the Bernoulli effect as water goes into a 3.00-cm-diameter nozzle from a 9.00-cm-diameter fire hose while carrying a flow of 40.0 L/s? (b) To what maximum height above the nozzle can this water rise? (The actual height will be significantly smaller due to air resistance.)

Show Solution

Strategy

(a) We first calculate the velocities in the hose and nozzle using continuity, then apply Bernoulli's equation to find the pressure drop. (b) We use energy conservation to find the maximum height from the nozzle exit velocity.

Solution

Given:

- Hose diameter: $d_1 = 9.00\text{ cm} = 0.0900\text{ m}$
- Nozzle diameter: $d_2 = 3.00\text{ cm} = 0.0300\text{ m}$
- Flow rate: $Q = 40.0\text{ L/s} = 0.0400\text{ m}^3\text{/s}$
- Water density: $\rho = 1000\text{ kg/m}^3$

(a) Calculate velocities: $A_1 = \pi r_1^2 = \pi(0.0450\text{ m})^2 = 6.36 \times 10^{-3}\text{ m}^2$ $A_2 = \pi r_2^2 = \pi(0.0150\text{ m})^2 = 7.07 \times 10^{-4}\text{ m}^2$

$$v_1 = \frac{Q}{A_1} = \frac{0.0400\text{ m}^3\text{/s}}{6.36 \times 10^{-3}\text{ m}^2} = 6.29\text{ m/s}$$
$$v_2 = \frac{Q}{A_2} = \frac{0.0400\text{ m}^3\text{/s}}{7.07 \times 10^{-4}\text{ m}^2} = 56.6\text{ m/s}$$

Apply Bernoulli's equation (at same height): $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$

$$\Delta P = P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2)$$
$$\Delta P = \frac{1}{2}(1000\text{ kg/m}^3)[(56.6)^2 - (6.29)^2]\text{ m}^2\text{/s}^2$$
$$\Delta P = \frac{1}{2}(1000)(3204 - 39.6) = \frac{1}{2}(1000)(3164) = 1.58 \times 10^6\text{ N/m}^2$$

(b) Maximum height from nozzle exit velocity: $h_{max} = \frac{v_2^2}{2g} = \frac{(56.6\text{ m/s})^2}{2(9.80\text{ m/s}^2)} = \frac{3204}{19.6} = 163\text{ m}$

Discussion

The pressure drop is **(a) 1.58 × 10⁶ N/m²** (about 16 atmospheres), and the water can theoretically rise to **(b) 163 m** (535 feet). This remarkable height explains why fire hoses can reach the upper floors of tall buildings. In practice, air resistance significantly reduces this height, but the calculation shows the tremendous energy available from the high-pressure water supply.

(a) Using Bernoulli's equation, show that the measured fluid speed v for a pitot tube, like the one in [Figure 3](b), is given by

$$v = \left(\frac{2\rho' gh}{\rho}\right)^{1/2},$$

where h is the height of the manometer fluid, ρ' is the density of the manometer fluid, ρ is the density of the moving fluid, and g is the acceleration due to gravity. (Note that v is indeed proportional to the square root of h, as stated in the text.) (b) Calculate v for moving air if a mercury manometer's h is 0.200 m.

Show Solution

Strategy

(a) We apply Bernoulli's equation at two points: one where flow is stagnant ($v_1 = 0$) and one where flow has velocity v. The pressure difference is measured by the manometer. (b) We substitute known values for air and mercury to calculate the velocity.

Solution

(a) Derivation:

At point 1 (stagnation point, where flow stops): $v_1 = 0$ At point 2 (in the moving flow): $v_2 = v$

Applying Bernoulli's equation at the same height: $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$

$$P_1 + 0 = P_2 + \frac{1}{2}\rho v^2$$
$$P_1 - P_2 = \frac{1}{2}\rho v^2$$

The manometer measures this pressure difference. For a column of manometer fluid of height h: $P_1 - P_2 = \rho' gh$

where ρ' is the density of the manometer fluid.

Setting these equal: $\rho' gh = \frac{1}{2}\rho v^2$

Solving for v: $v^2 = \frac{2\rho' gh}{\rho}$

$$v = \sqrt{\frac{2\rho' gh}{\rho}} = \left(\frac{2\rho' gh}{\rho}\right)^{1/2}$$

(b) Calculate velocity for air with mercury manometer:

Given:

- $h = 0.200\ \text{m}$
- $\rho' = 13{,}600\ \text{kg/m}^3$ (mercury density)
- $\rho = 1.29\ \text{kg/m}^3$ (air density at STP)
- $g = 9.80\ \text{m/s}^2$

$$v = \sqrt{\frac{2(13{,}600\ \text{kg/m}^3)(9.80\ \text{m/s}^2)(0.200\ \text{m})}{1.29\ \text{kg/m}^3}}$$

$$v = \sqrt{\frac{53{,}312}{1.29}} = \sqrt{41{,}327} = 203\ \text{m/s}$$

Discussion

(a) The derivation shows that velocity is proportional to $\sqrt{h}$, confirming the text's statement. **(b)** For a 0.200 m mercury column, the air velocity is approximately **203 m/s** (about 730 km/h or Mach 0.59). This high velocity is reasonable since mercury is very dense compared to air (ratio of about 10,500:1), so even a small column of mercury corresponds to a large dynamic pressure from high-speed air.

Glossary

Bernoulli's equation
: the equation resulting from applying conservation of energy to an incompressible frictionless fluid: $P + \frac{1}{2}\rho v^2 + \rho gh = \text{constant}$, through the fluid

Bernoulli's principle
: Bernoulli's equation applied at constant depth:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2.$$

The Most General Applications of Bernoulli’s Equation

- Calculate using Torricelli’s theorem.
- Calculate power in fluid flow.

Torricelli’s Theorem

[Figure 1] shows water gushing from a large tube through a dam. What is its speed as it emerges? Interestingly, if resistance is negligible, the speed is just what it would be if the water fell a distance h from the surface of the reservoir; the water’s speed is independent of the size of the opening. Let us check this out. Bernoulli’s equation must be used since the depth is not constant. We consider water flowing from the surface (point 1) to the tube’s outlet (point 2). Bernoulli’s equation as stated in previously is

$$P_1+\frac{1}{2}\rho v_1^2+\rho g h_1=P_2+\frac{1}{2}\rho v_2^2+\rho g h_2.$$

Both P_1 and P_2 equal atmospheric pressure (P_1 is atmospheric pressure because it is the pressure at the top of the reservoir. P_2 must be atmospheric pressure, since the emerging water is surrounded by the atmosphere and cannot have a pressure different from atmospheric pressure.) and subtract out of the equation, leaving

$$\frac{1}{2}\rho v_1^2+\rho g h_1=\frac{1}{2}\rho v_2^2+\rho g h_2.$$

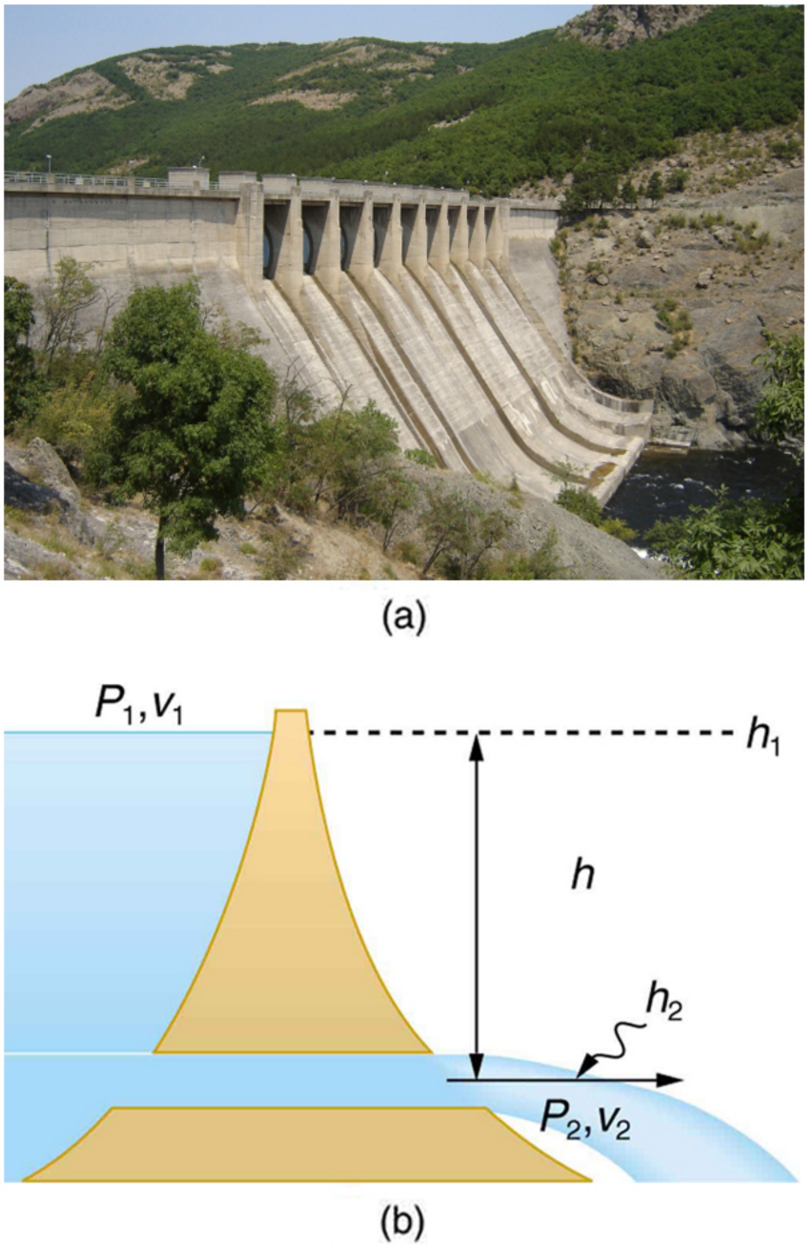
Solving this equation for v_2^2 , noting that the density ρ cancels (because the fluid is incompressible), yields

$$v_2^2=v_1^2+2g(h_1-h_2).$$

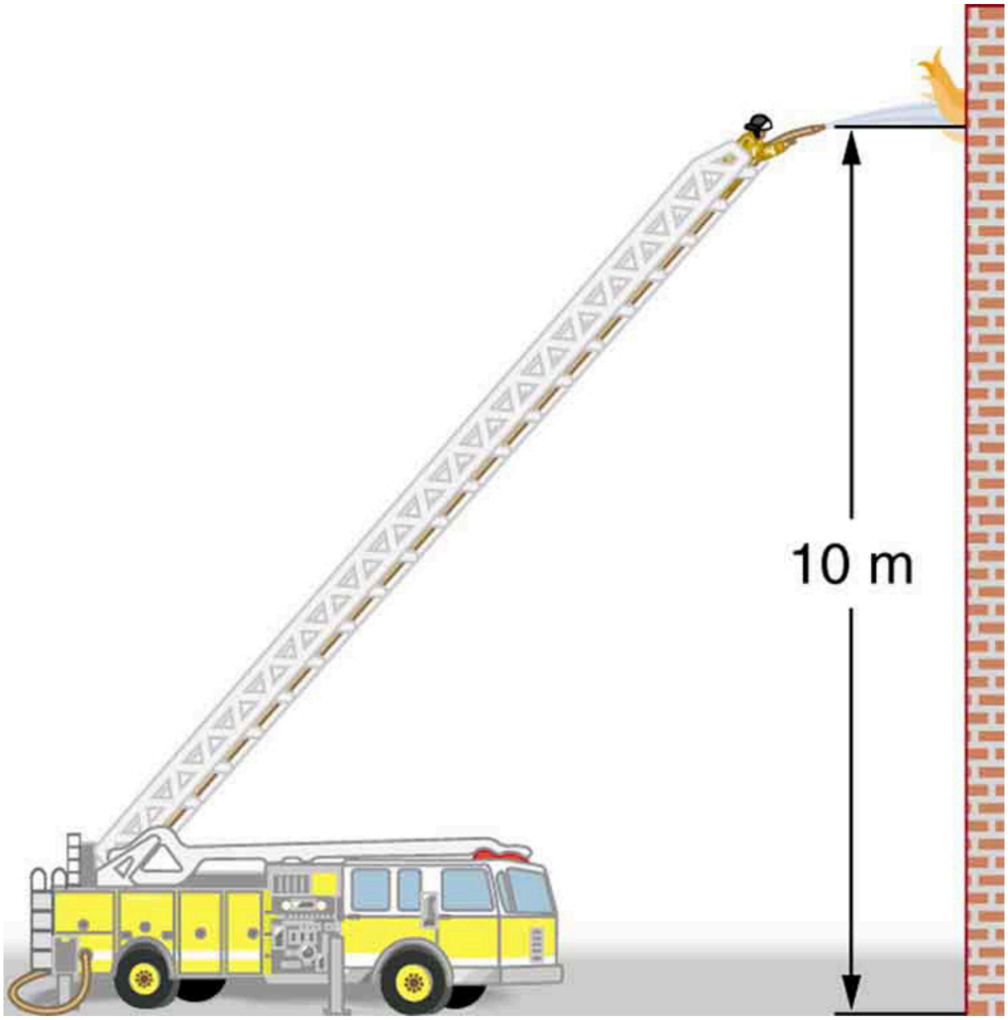
We let $h=h_1-h_2$; the equation then becomes

$$v_2^2=v_1^2+2gh$$

where h is the height dropped by the water. This is simply a kinematic equation for any object falling a distance h with negligible resistance. In fluids, this last equation is called *Torricelli’s theorem*. Note that the result is independent of the velocity’s direction, just as we found when applying conservation of energy to falling objects.



(a) Water gushes from the base of the Studen Kladenetz dam in Bulgaria. (credit: Kiril Kapustin; http://www.ImagesFromBulgaria.com) (b) In the absence of significant resistance, water flows from the reservoir with the same speed it would have if it fell the distance h without friction. This is an example of Torricelli’s theorem.



Pressure in the nozzle of this fire hose is less than at ground level for two reasons: the water has to go uphill to get to the nozzle, and speed increases in the nozzle. In spite of its lowered pressure, the water can exert a large force on anything it strikes, by virtue of its kinetic energy. Pressure in the water stream becomes equal to atmospheric pressure once it emerges into the air.

All preceding applications of Bernoulli's equation involved simplifying conditions, such as constant height or constant pressure. The next example is a more general application of Bernoulli's equation in which pressure, velocity, and height all change. (See [Figure 2].)

Calculating Pressure: A Fire Hose Nozzle

Fire hoses used in major structure fires have inside diameters of 6.40 cm. Suppose such a hose carries a flow of 40.0 L/s starting at a gauge pressure of $1.62\times10^{6}\text{N/m}^2$. The hose goes 10.0 m up a ladder to a nozzle having an inside diameter of 3.00 cm. Assuming negligible resistance, what is the pressure in the nozzle?

Strategy

Here we must use Bernoulli's equation to solve for the pressure, since depth is not constant.

Solution

Bernoulli's equation states

$$P_1+\frac{1}{2}\rho v_1^2+\rho g h_1=P_2+\frac{1}{2}\rho v_2^2+\rho g h_2,$$

where the subscripts 1 and 2 refer to the initial conditions at ground level and the final conditions inside the nozzle, respectively. We must first find the speeds v_1 and v_2 . Since $Q=A_1v_1$, we get

$$v_1=\frac{Q}{A_1}=\frac{40.0\times10^{-3}\text{m}^3/\text{s}}{\pi(3.20\times10^{-2}\text{m})^2}=12.4\text{m/s}.$$

Similarly, we find

$$v_2=56.6\text{m/s}.$$

(This rather large speed is helpful in reaching the fire.) Now, taking h_1 to be zero, we solve Bernoulli's equation for P_2 :

$$P_2 = P_1 + \frac{1}{2}\rho(v_1^2 - v_2^2) - \rho g h_2.$$

Substituting known values yields

$$P_2 = 1.62\times10^6\,\text{N/m}^2 + \frac{1}{2}(1000\,\text{kg/m}^3)[(12.4\,\text{m/s})^2 - (56.6\,\text{m/s})^2] - (1000\,\text{kg/m}^3)(9.80\,\text{m/s}^2)(10.0\,\text{m}) = 0.$$

Discussion

This value is a gauge pressure, since the initial pressure was given as a gauge pressure. Thus, the nozzle pressure equals atmospheric pressure, as it must because the water exits into the atmosphere without changes in its conditions.

Power in Fluid Flow

Power is the *rate* at which work is done or energy in any form is used or supplied. To see the relationship of power to fluid flow, consider Bernoulli's equation:

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}.$$

All three terms have units of energy per unit volume, as discussed in the previous section. Now, considering units, if we multiply energy per unit volume by flow rate (volume per unit time), we get units of power. That is, $(E/V)(V/t) = E/t$. This means that if we multiply Bernoulli's equation by flow rate Q , we get power. In equation form, this is

$$\left(P + \frac{1}{2}\rho v^2 + \rho g h\right)Q = \text{power}.$$

Each term has a clear physical meaning. For example, PQ is the power supplied to a fluid, perhaps by a pump, to give it its pressure P . Similarly, $\frac{1}{2}\rho v^2 Q$ is the power supplied to a fluid to give it its kinetic energy. And $\rho g h Q$ is the power going to gravitational potential energy.

Making Connections: Power

Power is defined as the rate of energy transferred, or E/t . Fluid flow involves several types of power. Each type of power is identified with a specific type of energy being expended or changed in form.

Calculating Power in a Moving Fluid

Suppose the fire hose in the previous example is fed by a pump that receives water through a hose with a 6.40-cm diameter coming from a hydrant with a pressure of $0.700\times10^6\,\text{N/m}^2$. What power does the pump supply to the water?

Strategy

Here we must consider energy forms as well as how they relate to fluid flow. Since the input and output hoses have the same diameters and are at the same height, the pump does not change the speed of the water nor its height, and so the water's kinetic energy and gravitational potential energy are unchanged. That means the pump only supplies power to increase water pressure by $0.92\times10^6\,\text{N/m}^2$ (from $0.700\times10^6\,\text{N/m}^2$ to $1.62\times10^6\,\text{N/m}^2$).

Solution

As discussed above, the power associated with pressure is

$$\begin{aligned} \text{power} &= PQ \\ \text{power} &= (0.920\times10^6\,\text{N/m}^2)(40.0\times10^{-3}\,\text{m}^3/\text{s}). \\ \text{power} &= 3.68\times10^4\,\text{W} = 36.8\,\text{kW} \end{aligned}$$

Discussion

Such a substantial amount of power requires a large pump, such as is found on some fire trucks. (This kilowatt value converts to about 50 hp.) The pump in this example increases only the water's pressure. If a pump—such as the heart—directly increases velocity and height as well as pressure, we would have to calculate all three terms to find the power it supplies.

Summary

- Power in fluid flow is given by the equation $\left(P_1 + \frac{1}{2}\rho v^2 + \rho g h\right)Q = \text{power}$, where the first term is power associated with pressure, the second is power associated with velocity, and the third is power associated with height.

Conceptual Questions

Based on Bernoulli's equation, what are three forms of energy in a fluid? (Note that these forms are conservative, unlike heat transfer and other dissipative forms not included in Bernoulli's equation.)

Show Solution

Strategy

We analyze each term in Bernoulli's equation and identify the type of energy it represents.

Solution

Bernoulli's equation $P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$ contains three forms of energy per unit volume:

1. **Pressure energy** (P): This represents the work per unit volume that can be done by the fluid due to its pressure. It's sometimes called "flow work" or "pressure potential energy." When fluid is pushed from one region to another, pressure does work on it.

2. **Kinetic energy** ($\frac{1}{2}\rho v^2$): This is the kinetic energy per unit volume associated with the bulk motion of the fluid. It represents the energy a fluid has due to its velocity.

3. **Gravitational potential energy** ($\rho g h$): This is the potential energy per unit volume due to the fluid's height in a gravitational field. It represents the work that gravity can do on the fluid if it moves to a lower height.

Discussion

All three forms are conservative—they can be converted back and forth without energy loss (in an ideal, frictionless fluid). Bernoulli's equation states that the sum of these three energy forms remains constant along a streamline, which is a statement of energy conservation for ideal fluids.

Water that has emerged from a hose into the atmosphere has a gauge pressure of zero. Why? When you put your hand in front of the emerging stream you feel a force, yet the water's gauge pressure is zero. Explain where the force comes from in terms of energy.

Show Solution

Strategy

We consider what happens when water transitions from confined flow to free flow, and identify the source of force on an obstruction.

Solution

Why gauge pressure is zero: Once water exits the hose and enters the atmosphere, there's nothing to maintain a pressure difference between the water and its surroundings. The water is now surrounded by air at atmospheric pressure, so it equilibrates to atmospheric pressure. Gauge pressure (which measures pressure relative to atmospheric) is therefore zero.

Where the force comes from: The force you feel comes from the water's **kinetic energy**, not its pressure. When the water stream strikes your hand:

- The water has significant velocity and therefore kinetic energy ($\frac{1}{2}\rho v^2$)
- Your hand stops the water, reducing its velocity to zero
- This change in momentum requires a force (Newton's second law: $F = \frac{dp}{dt}$)
- The kinetic energy is transferred to your hand as the work done by this force

Discussion

This is an excellent example of the difference between pressure and kinetic energy in a fluid. A fire hose can knock someone down not because of high pressure (which is zero once the water exits), but because of the high kinetic energy of the fast-moving water. The power delivered equals the kinetic energy per unit volume times the flow rate: $\frac{1}{2}\rho v^2 \times Q$.

The old rubber boot shown in [Figure 3] has two leaks. To what maximum height can the water squirt from Leak 1? How does the velocity of water emerging from Leak 2 differ from that of leak 1? Explain your responses in terms of energy.

Water emerges from two leaks in an old boot.

Show Solution

Strategy

We apply Torricelli's theorem, recognizing that the velocity of efflux depends on the height of water above the leak, and the maximum height of the spray depends on converting kinetic energy back to potential energy.

Solution

Maximum height from Leak 1: By Torricelli's theorem, water exits with velocity $v = \sqrt{2gh}$ where h is the height of the water surface above the leak. The water can squirt to a maximum height equal to the height of the water surface above the leak—i.e., it can rise back to the level of the water surface in the boot.

This makes sense from energy conservation: the gravitational potential energy at the water surface converts to kinetic energy at the leak, which then converts back to gravitational potential energy as the water rises.

Velocity comparison for Leak 1 and Leak 2: Since both leaks are at the same height (as shown in the figure), and both have the same height of water above them, **the velocities are the same**. Torricelli's theorem gives $v = \sqrt{2gh}$ where h is measured from the water surface to the leak—this depends only on vertical distance, not horizontal position.

Both streams have the same kinetic energy per unit volume because they come from the same depth below the water surface.

Discussion

This problem illustrates that the exit velocity depends only on the depth below the surface, not on the direction of the opening or the shape of the container. The energy analysis shows perfect conversion between potential and kinetic energy in an ideal (frictionless) situation.

Water pressure inside a hose nozzle can be less than atmospheric pressure due to the Bernoulli effect. Explain in terms of energy how the water can emerge from the nozzle against the opposing atmospheric pressure.

Show Solution

Strategy

We apply energy conservation (Bernoulli's equation) to explain how water with sub-atmospheric pressure can still exit against atmospheric pressure.

Solution

Although the pressure inside the nozzle is less than atmospheric, the water can still emerge because it possesses significant kinetic energy. The total energy per unit volume must be considered, not just the pressure term.

From Bernoulli's equation: $P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$

Inside the nozzle:

- Pressure: $P_{nozzle} < P_{atm}$ (lower than atmospheric)
- Kinetic energy: $\frac{1}{2}\rho v^2$ is very large (high velocity)
- Total energy: still substantial

The water uses its kinetic energy to "push through" the adverse pressure gradient at the exit. Think of it like throwing a ball upward—even though gravity opposes the motion, the ball rises because it has kinetic energy. Similarly, the water's kinetic energy is sufficient to overcome the pressure difference.

At the exit point, some kinetic energy is converted to pressure energy as the water adjusts to atmospheric pressure, but the remaining kinetic energy carries the water forward as a high-speed stream.

Discussion

This explains why nozzles are effective despite creating regions of sub-atmospheric pressure—the energy has simply been transformed from pressure form to kinetic form, and the total remains conserved. The high-velocity stream can still exert large forces on objects it strikes because it retains substantial kinetic energy.

Problems & Exercises

Hoover Dam on the Colorado River is the highest dam in the United States at 221 m, with an output of 1300 MW. The dam generates electricity with water taken from a depth of 150 m and an average flow rate of $650 \text{m}^3/\text{s}$. (a) Calculate the power in this flow. (b) What is the ratio of this power to the facility's average of 680 MW?

Show Solution

Strategy

The power in the flow comes primarily from the gravitational potential energy of the water. We use the power formula $\text{Power} = \rho g h Q$ for the potential energy component.

Solution

Given:

- Depth (height): $h = 150 \text{ m}$
- Flow rate: $Q = 650 \text{ m}^3/\text{s}$
- Water density: $\rho = 1000 \text{ kg/m}^3$
- Average facility output: 680 MW

(a) Calculate the power in the flow: $\text{Power} = \rho g h Q$ $\text{Power} = (1000 \text{ kg/m}^3)(9.80 \text{ m/s}^2)(150 \text{ m})(650 \text{ m}^3/\text{s})$ $\text{Power} = 9.56 \times 10^8 \text{ W} = 956 \text{ MW}$

(b) Ratio to facility average: $\text{Ratio} = \frac{956 \text{ MW}}{680 \text{ MW}} = 1.41 \approx 1.4$

Discussion

The power in the water flow is **(a) 9.56 × 10⁸ W (956 MW)**, and the ratio to the facility's average output is **(b) 1.4**. The fact that the ratio exceeds 1 indicates energy losses in the system—about 29% of the available power is lost to turbine inefficiency, friction, generator losses, and other factors. This ~70% efficiency is typical for large hydroelectric facilities.

A frequently quoted rule of thumb in aircraft design is that wings should produce about 1000 N of lift per square meter of wing. (The fact that a wing has a top and bottom surface does not double its area.) (a) At takeoff, an aircraft travels at 60.0 m/s, so that the air speed relative to the bottom of the wing is 60.0 m/s. Given the sea level density of air to be 1.29kg/m^3 , how fast must it move over the upper surface to create the ideal lift? (b) How fast must air move over the upper surface at a cruising speed of 245 m/s and at an altitude where air density is one-fourth that at sea level? (Note that this is not all of the aircraft's lift—some comes from the body of the plane, some from engine thrust, and so on. Furthermore, Bernoulli's principle gives an approximate answer because flow over the wing creates turbulence.)

Show Solution

Strategy

We apply Bernoulli's principle: the pressure difference between upper and lower wing surfaces creates lift. The lift per unit area equals the pressure difference.

Solution

The lift per unit area equals the pressure difference: $\frac{L}{A} = P_{bottom} - P_{top} = \frac{1}{2}\rho(v^2_{top} - v^2_{bottom})$

Solving for v_{top}: $v_{top} = \sqrt{v^2_{bottom} + \frac{2(L/A)}{\rho}}$

(a) At takeoff (sea level):

- v_{bottom}=60.0 m/s
- L/A=1000 N/m^2
- ρ=1.29 kg/m^3

v_{top}=√(60.0)^2+2(1000)1.29=√3600+1550=√5150=71.8 m/s

(b) At cruising altitude:

- v_{bottom}=245 m/s
- ρ=1.29/4=0.323 kg/m^3

v_{top}=√(245)^2+2(1000)0.323=√60,025+6192=√66,217=257 m/s

Discussion

At takeoff, air must move at **(a) 71.8 m/s** over the upper surface (about 20% faster than the bottom). At cruising altitude, the upper surface speed must be **(b) 257 m/s** (only about 5% faster than the bottom). At higher altitudes, the lower air density means a larger velocity difference is needed for the same lift, but since the plane is already moving much faster, a smaller *percentage* increase is sufficient. The wing shape (camber and angle of attack) creates these velocity differences.

The left ventricle of a resting adult's heart pumps blood at a flow rate of $83.0 \text{cm}^3/\text{s}$, increasing its pressure by 110 mm Hg, its speed from zero to 30.0 cm/s, and its height by 5.00 cm. (All numbers are averaged over the entire heartbeat.) Calculate the total power output of the left ventricle. Note that most of the power is used to increase blood pressure.

Show Solution

Strategy

The total power output includes three components from Bernoulli's equation: power to increase pressure, power to increase kinetic energy, and power to increase potential energy.

Solution

Given:

- Flow rate: Q =83.0 cm^3/s =83.0 ×10^{-6} m^3/s
- Pressure increase: ΔP =110 mm Hg =110 ×133 Pa =14,630 Pa
- Velocity change: Δv =30.0 cm/s =0.300 m/s (from 0 to 0.300 m/s)
- Height increase: Δh =5.00 cm =0.0500 m
- Blood density: ρ ≈1060 kg/m^3

Power for pressure: P_1 =ΔP ×Q =(14,630 N/m^2)(83.0 ×10^{-6} m^3/s) =1.21 W

Power for kinetic energy: P_2 =12ρv^2 ×Q =12(1060)(0.300)^2(83.0 ×10^{-6}) =3.96 ×10^{-3} W

Power for potential energy: P_3 =ρgh ×Q =(1060)(9.80)(0.0500)(83.0 ×10^{-6}) =4.31 ×10^{-2} W

Total power: P_{total} =P_1 +P_2 +P_3 =1.21 +0.004 +0.043 =1.26 W

Discussion

The total power output of the left ventricle is approximately **1.26 W**. As noted, most of this power (about 96%) goes to increasing blood pressure. The kinetic and potential energy contributions are relatively small. This is about 1/5 of the total heart power output (both ventricles), and the heart operates continuously for a lifetime—a remarkable feat of engineering!

A sump pump (used to drain water from the basement of houses built below the water table) is draining a flooded basement at the rate of 0.750 L/s, with an output pressure of $3.00 \times 10^5 \text{N/m}^2$. (a) The water enters a hose with a 3.00-cm inside diameter and rises 2.50 m above the pump. What is its pressure at this point? (b) The hose goes over the foundation wall, losing 0.500 m in height, and widens to 4.00 cm in diameter. What is the pressure now? You may neglect frictional losses in both parts of the problem.

Show Solution

Strategy

We apply Bernoulli's equation between points, accounting for changes in height and velocity (due to diameter changes). For each part, we calculate velocities using continuity, then apply Bernoulli's equation.

Solution

Given:

- Flow rate: $Q = 0.750 \text{ L/s} = 7.50 \times 10^{-4} \text{ m}^3\text{/s}$
- Initial pressure: $P_0 = 3.00 \times 10^5 \text{ N/m}^2$
- Hose diameter (narrow): $d_1 = 3.00 \text{ cm} = 0.0300 \text{ m}$
- Hose diameter (wide): $d_2 = 4.00 \text{ cm} = 0.0400 \text{ m}$

Calculate velocities: $A_1 = \pi(0.0150)^2 = 7.07 \times 10^{-4} \text{ m}^2$ $A_2 = \pi(0.0200)^2 = 1.26 \times 10^{-3} \text{ m}^2$

$v_1 = \frac{Q}{A_1} = \frac{7.50\times10^{-4}}{7.07\times10^{-4}} = 1.06 \text{ m/s}$ $v_2 = \frac{Q}{A_2} = \frac{7.50\times10^{-4}}{1.26\times10^{-3}} = 0.596 \text{ m/s}$

(a) After rising 2.50 m (same diameter): Using Bernoulli's equation with v unchanged: $P_0 + \rho g h_0 = P_1 + \rho g h_1$ $P_1 = P_0 - \rho g \Delta h = 3.00 \times 10^5 - (1000)(9.80)(2.50)$ $P_1 = 3.00 \times 10^5 - 2.45 \times 10^4 = 2.76 \times 10^5 \text{ N/m}^2$

(b) After going over the wall (2.50 - 0.50 = 2.00 m above pump, wider diameter): Using Bernoulli's equation from pump to this point: $P_0 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$

$P_2 = P_0 + \frac{1}{2}\rho(v_1^2 - v_2^2) - \rho g h_2$ $P_2 = 3.00 \times 10^5 + \frac{1}{2}(1000)[(1.06)^2 - (0.596)^2] - (1000)(9.80)(2.00)$ $P_2 = 3.00 \times 10^5 + 384 - 1.96 \times 10^4 = 2.81 \times 10^5 \text{ N/m}^2$

Discussion

The pressure at the top of the rise is **(a) 2.76×10^5 N/m²**, and after going over the wall it is **(b) 2.81×10^5 N/m²**. The pressure increases slightly after going over the wall for two reasons: the height decreased by 0.50 m, and the hose widened, reducing velocity. Both effects increase pressure according to Bernoulli's equation.

Viscosity and Laminar Flow; Poiseuille's Law

- Define laminar flow and turbulent flow.
- Explain what viscosity is.
- Calculate flow and resistance with Poiseuille's law.
- Explain how pressure drops due to resistance.

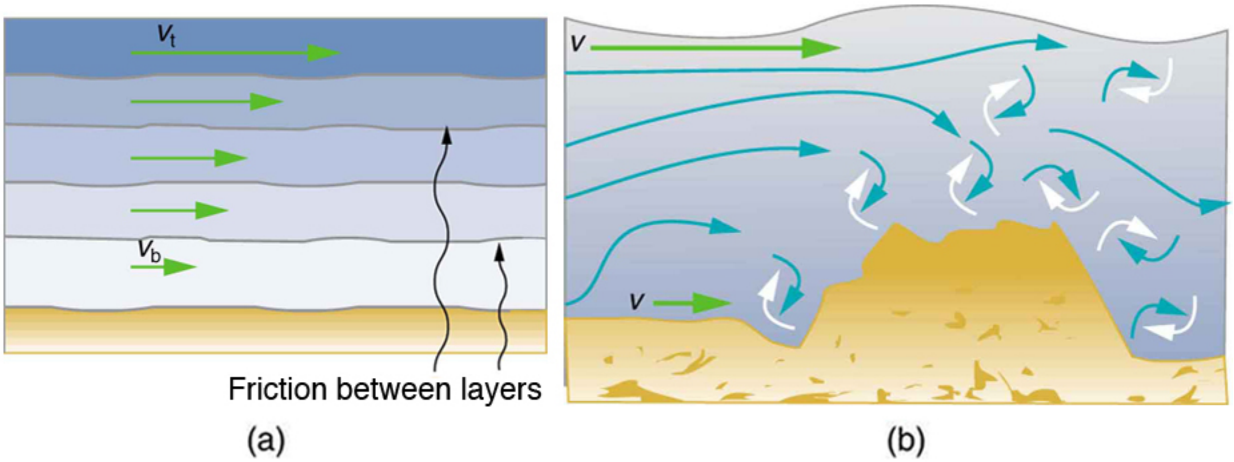
Laminar Flow and Viscosity

When you pour yourself a glass of juice, the liquid flows freely and quickly. But when you pour syrup on your pancakes, that liquid flows slowly and sticks to the pitcher. The difference is fluid friction, both within the fluid itself and between the fluid and its surroundings. We call this property of fluids *viscosity*. Juice has low viscosity, whereas syrup has high viscosity. In the previous sections we have considered ideal fluids with little or no viscosity. In this section, we will investigate what factors, including viscosity, affect the rate of fluid flow.

The precise definition of viscosity is based on *laminar*, or nonturbulent, flow. Before we can define viscosity, then, we need to define laminar flow and turbulent flow. [Figure 1] shows both types of flow. **Laminar** flow is characterized by the smooth flow of the fluid in layers that do not mix. Turbulent flow, or **turbulence**, is characterized by eddies and swirls that mix layers of fluid together.

Smoke rises smoothly for a while and then begins to form swirls and eddies. The smooth flow is called laminar flow, whereas the swirls and eddies typify turbulent flow. If you watch the smoke (being careful not to breathe on it), you will notice that it rises more rapidly when flowing smoothly than after it becomes turbulent, implying that turbulence poses more resistance to flow. (credit: Creativity103)

[Figure 2] shows schematically how laminar and turbulent flow differ. Layers flow without mixing when flow is laminar. When there is turbulence, the layers mix, and there are significant velocities in directions other than the overall direction of flow. The lines that are shown in many illustrations are the paths followed by small volumes of fluids. These are called *streamlines*. Streamlines are smooth and continuous when flow is laminar, but break up and mix when flow is turbulent. Turbulence has two main causes. First, any obstruction or sharp corner, such as in a faucet, creates turbulence by imparting velocities perpendicular to the flow. Second, high speeds cause turbulence. The drag both between adjacent layers of fluid and between the fluid and its surroundings forms swirls and eddies, if the speed is great enough. We shall concentrate on laminar flow for the remainder of this section, leaving certain aspects of turbulence for later sections.



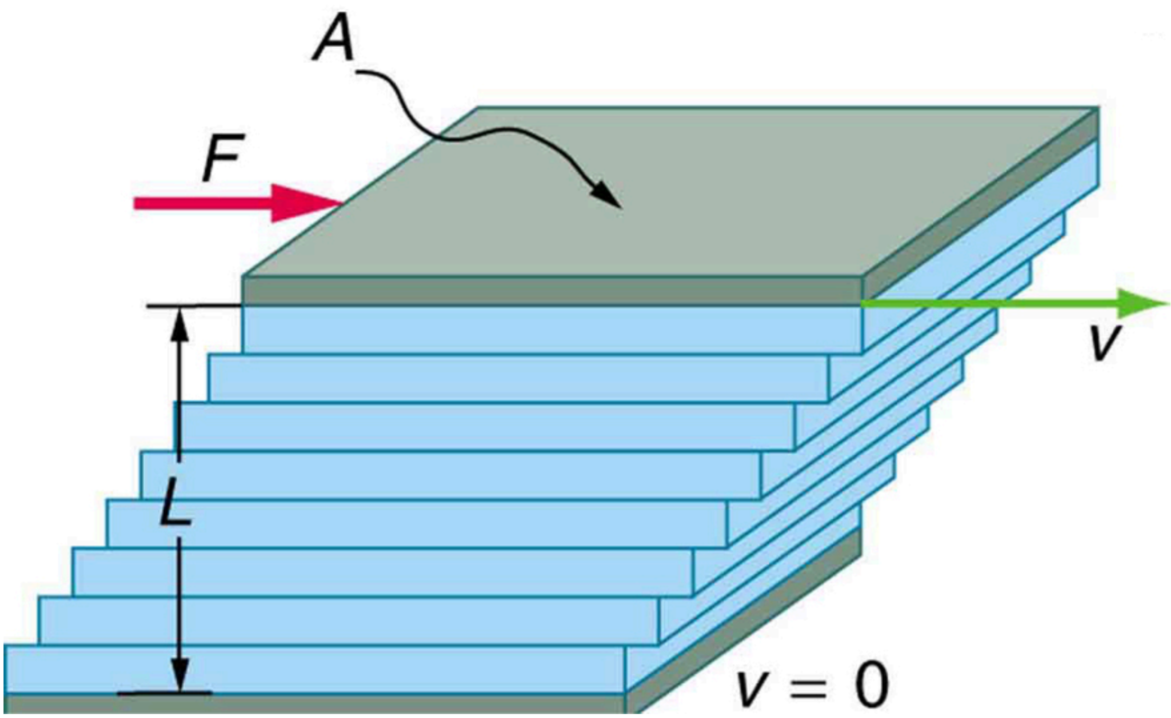
(a) Laminar flow occurs in layers without mixing. Notice that viscosity causes drag between layers as well as with the fixed surface. (b) An obstruction in the vessel produces turbulence. Turbulent flow mixes the fluid. There is more interaction, greater heating, and more resistance than in laminar flow.

Making Connections: Take-Home Experiment: Go Down to the River

Try dropping simultaneously two sticks into a flowing river, one near the edge of the river and one near the middle. Which one travels faster? Why?

[Figure 3] shows how viscosity is measured for a fluid. Two parallel plates have the specific fluid between them. The bottom plate is held fixed, while the top plate is moved to the right, dragging fluid with it. The layer (or lamina) of fluid in contact with either plate does not move relative to the plate, and so the top layer moves at v while the bottom layer remains at rest. Each successive layer from the top down exerts a force on the one below it, trying to drag it along, producing a continuous variation in speed from v

to 0 as shown. Care is taken to ensure that the flow is laminar; that is, the layers do not mix. The motion in [Figure 3] is like a continuous shearing motion. Fluids have zero shear strength, but the *rate* at which they are sheared is related to the same geometrical factors A and L as is shear deformation for solids.



The graphic shows laminar flow of fluid between two plates of area \(A \) . The bottom plate is fixed. When the top plate is pushed to the right, it drags the fluid along with it.

A force F is required to keep the top plate in [Figure 3] moving at a constant velocity v , and experiments have shown that this force depends on four factors. First, F is directly proportional to v (until the speed is so high that turbulence occurs—then a much larger force is needed, and it has a more complicated dependence on v). Second, F is proportional to the area A of the plate. This relationship seems reasonable, since A is directly proportional to the amount of fluid being moved. Third, F is inversely proportional to the distance between the plates L . This relationship is also reasonable; L is like a lever arm, and the greater the lever arm, the less force that is needed. Fourth, F is directly proportional to the **coefficient of viscosity**, η . The greater the viscosity, the greater the force required. These dependencies are combined into the equation

$$F=\eta \frac{vA}{L}\text{,} $$

which gives us a working definition of fluid **viscosity** η. Solving for η gives

$$\eta =\frac{FL}{vA}\text{,} $$

which defines viscosity in terms of how it is measured. The SI unit of viscosity is $\text{N}\cdot \text{m/}\left[\left(\text{m/s}\right){\text{m}}^{2}\right] =\left({\text{N/m}}^{2}\right)\text{s or Pa}\cdot \text{s}$. [Table 1] lists the coefficients of viscosity for various fluids.

Viscosity varies from one fluid to another by several orders of magnitude. As you might expect, the viscosities of gases are much less than those of liquids, and these viscosities are often temperature dependent. The viscosity of blood can be reduced by aspirin consumption, allowing it to flow more easily around the body. (When used over the long term in low doses, aspirin can help prevent heart attacks, and reduce the risk of blood clotting.)

Laminar Flow Confined to Tubes—Poiseuille's Law

What causes flow? The answer, not surprisingly, is pressure difference. In fact, there is a very simple relationship between horizontal flow and pressure. Flow rate Q is in the direction from high to low pressure. The greater the pressure differential between two points, the greater the flow rate. This relationship can be stated as

$$Q=\frac{ {P}_{2}-{P}_{1}}{R}\text{,} $$

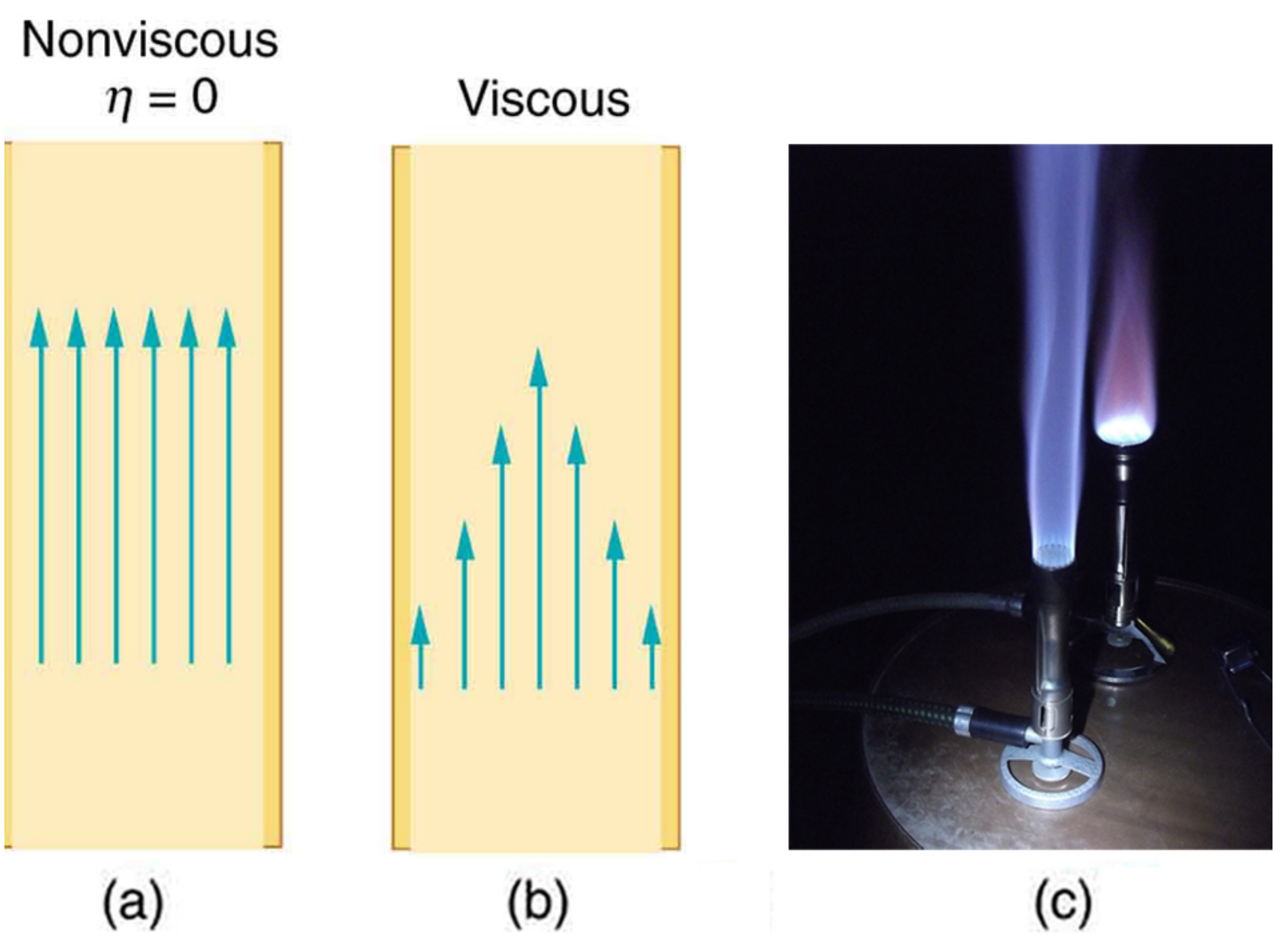
where ${P}_{1}$ and ${P}_{2}$ are the pressures at two points, such as at either end of a tube, and R is the resistance to flow. The resistance R includes everything, except pressure, that affects flow rate. For example, R is greater for a long tube than for a short one. The greater the viscosity of a fluid, the greater the value of R . Turbulence greatly increases R , whereas increasing the diameter of a tube decreases R .

If viscosity is zero, the fluid is frictionless and the resistance to flow is also zero. Comparing frictionless flow in a tube to viscous flow, as in [Figure 4], we see that for a viscous fluid, speed is greatest at midstream because of drag at the boundaries. We can see the effect of viscosity in a Bunsen burner flame, even though the viscosity of natural gas is small.

The resistance R to laminar flow of an incompressible fluid having viscosity η through a horizontal tube of uniform radius r and length l , such as the one in [Figure 5], is given by

$$R=\frac{8\eta l}{\pi {r}^{4}}\text{.} $$

This equation is called **Poiseuille's law for resistance** after the French scientist J. L. Poiseuille (1799–1869), who derived it in an attempt to understand the flow of blood, an often turbulent fluid.



(a) If fluid flow in a tube has negligible resistance, the speed is the same all across the tube. (b) When a viscous fluid flows through a tube, its speed at the walls is zero, increasing steadily to its maximum at the center of the tube. (c) The shape of the Bunsen burner flame is due to the velocity profile across the tube. (credit: Jason Woodhead)

Let us examine Poiseuille's expression for R to see if it makes good intuitive sense. We see that resistance is directly proportional to both fluid viscosity η and the length l of a tube. After all, both of these directly affect the amount of friction encountered—the greater either is, the greater the resistance and the smaller the flow. The radius r of a tube affects the resistance, which again makes sense, because the greater the radius, the greater the flow (all other factors remaining the same). But it is surprising that r is raised to the *fourth* power in Poiseuille's law. This exponent means that any change in the radius of a tube has a very large effect on resistance. For example, doubling the radius of a tube decreases resistance by a factor of ${2}^{4}=16$.

Taken together, $Q=\frac{ {P}_{2}-{P}_{1}}{R}$ and $R=\frac{8\eta l}{\pi {r}^{4}}$ give the following expression for flow rate:

$$Q=\frac{\left({P}_{2}-{P}_{1}\right)\pi r^{4}}{8\eta l}\text{.} $$

This equation describes laminar flow through a tube. It is sometimes called Poiseuille's law for laminar flow, or simply **Poiseuille's law**.

Using Flow Rate: Plaque Deposits Reduce Blood Flow

Suppose the flow rate of blood in a coronary artery has been reduced to half its normal value by plaque deposits. By what factor has the radius of the artery been reduced, assuming no turbulence occurs?

Strategy

Assuming laminar flow, Poiseuille's law states that

$$Q=\frac{\left({P}_{2}-{P}_{1}\right)\pi r^{4}}{8\eta l}\text{.} $$

We need to compare the artery radius before and after the flow rate reduction.

Solution

With a constant pressure difference assumed and the same length and viscosity, along the artery we have

$$\frac{ {Q}_{1}}{ {r}_{1}^{4}}=\frac{ {Q}_{2}}{ {r}_{2}^{4}}\text{.} $$

So, given that ${Q}_{2}= 0.5{Q}_{1}$, we find that ${r}_{2}^{4}=0.5r_{1}^{4}$.

Therefore, ${r}_{2}={\left(0.5\right)}^{ 0.25}{r}_{1}= 0.841{r}_{1}$, a decrease in the artery radius of 16%.

Discussion

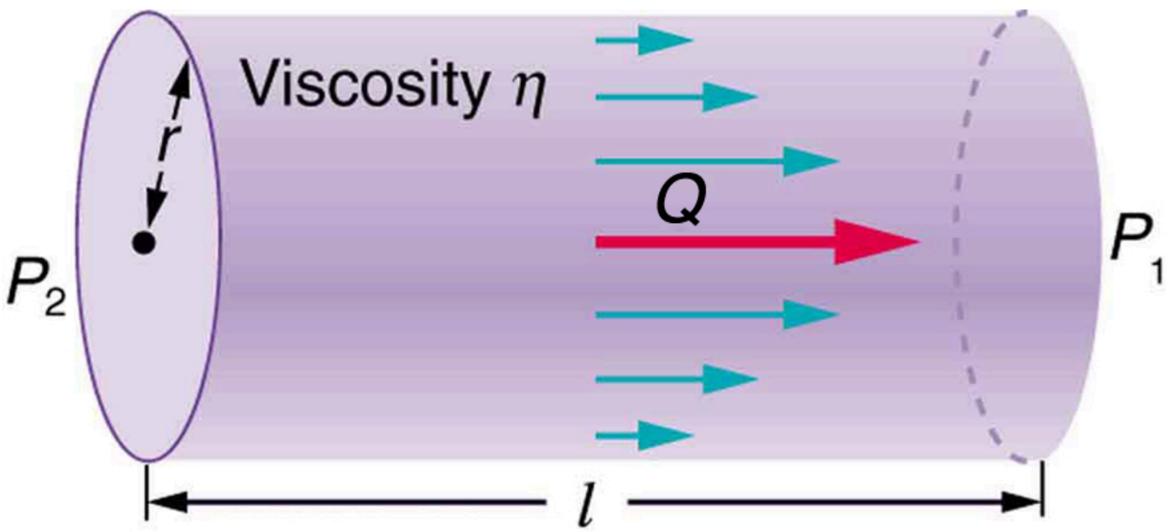
This decrease in radius is surprisingly small for this situation. To restore the blood flow in spite of this buildup would require an increase in the pressure difference $\left({P}_{2}-{P}_{1}\right)$ of a factor of two, with subsequent strain on the heart.

Coefficients of Viscosity of Various Fluids

Fluid	Temperature (ºC)	Viscosity $\eta \text{(mPa·s)}$
Gases		
Air	0	0.0171
	20	0.0181
	40	0.0190
	100	0.0218
Ammonia	20	0.00974
Carbon dioxide	20	0.0147
Helium	20	0.0196
Hydrogen	0	0.0090
Mercury	20	0.0450
Oxygen	20	0.0203
Steam	100	0.0130
Liquids		
Water	0	1.792
	20	1.002
	37	0.6947
	40	0.653
	100	0.282
Whole blood[1]	20	3.015
	37	2.084
Blood plasma[2]	20	1.810
	37	1.257
Ethyl alcohol	20	1.20
Methanol	20	0.584
Oil (heavy machine)	20	660

Fluid	Temperature (ºC)	Viscosity $\eta \text{(mPa·s)}$
Oil (motor, SAE 10)	30	200
Oil (olive)	20	138
Glycerin	20	1500
Honey	20	2000–10000
Maple Syrup	20	2000–3000
Milk	20	3.0
Oil (Corn)	20	65

The circulatory system provides many examples of Poiseuille's law in action—with blood flow regulated by changes in vessel size and blood pressure. Blood vessels are not rigid but elastic. Adjustments to blood flow are primarily made by varying the size of the vessels, since the resistance is so sensitive to the radius. During vigorous exercise, blood vessels are selectively dilated to important muscles and organs and blood pressure increases. This creates both greater overall blood flow and increased flow to specific areas. Conversely, decreases in vessel radii, perhaps from plaques in the arteries, can greatly reduce blood flow. If a vessel's radius is reduced by only 5% (to 0.95 of its original value), the flow rate is reduced to about ${\left(0.95 \right)}^{4}= 0.81$ of its original value. A 19% decrease in flow is caused by a 5% decrease in radius. The body may compensate by increasing blood pressure by 19%, but this presents hazards to the heart and any vessel that has weakened walls. Another example comes from automobile engine oil. If you have a car with an oil pressure gauge, you may notice that oil pressure is high when the engine is cold. Motor oil has greater viscosity when cold than when warm, and so pressure must be greater to pump the same amount of cold oil.



Poiseuille's law applies to laminar flow of an incompressible fluid of viscosity η through a tube of length l and radius r. The direction of flow is from greater to lower pressure. Flow rate Q is directly proportional to the pressure difference P_2-P_1, and inversely proportional to the length l of the tube and viscosity η of the fluid. Flow rate increases with r^4, the fourth power of the radius.

What Pressure Produces This Flow Rate?

An intravenous (IV) system is supplying saline solution to a patient at the rate of $0.120 {\text{cm}}^{3}\text{/s}$ through a needle of radius 0.150 mm and length 2.50 cm. What pressure is needed at the entrance of the needle to cause this flow, assuming the viscosity of the saline solution to be the same as that of water? The gauge pressure of the blood in the patient's vein is 8.00 mm Hg. (Assume that the temperature is 20ºC.)

Strategy

Assuming laminar flow, Poiseuille's law applies. This is given by

$$Q=\frac{\left({P}_{2}-{P}_{1}\right)\pi {r}^{4}}{8\eta l}\text{,}$$

where ${P}_{2}$ is the pressure at the entrance of the needle and ${P}_{1}$ is the pressure in the vein. The only unknown is ${P}_{2}$.

Solution

Solving for ${P}_{2}$ yields

$${P}_{2}=\frac{8\eta l}{ \pi r^{4}}Q+{P}_{1} \text{.}$$

${P}_{1}$ is given as 8.00 mm Hg, which converts to $1.066 \times 10^{3} {\text{N/m}}^{2}$. Substituting this and the other known values yields

$$\begin{array}{lll}{P}_{2}& =& \left[\frac{8\left(1.00 \times 10^{-3} \text{N}\cdot {\text{s/m}}^{2}\right)\left(2.50 \times 10^{-2} \text{m}\right)}{\pi { \left(0.150 \times 10^{-3} \text{m} \right)}^{4}}\right] \left(1.20 \times 10^{-7} {\text{m}}^{3}\text{/s}\right)+ 1.066 \times 10^{3} {\text{N/m}}^{2}\\ {P}_{2}& =& 1.62 \times 10^{4} {\text{N/m}}^{2}\text{.}\end{array}$$

Discussion

This pressure could be supplied by an IV bottle with the surface of the saline solution 1.61 m above the entrance to the needle (this is left for you to solve in this chapter's Problems and Exercises), assuming that there is negligible pressure drop in the tubing leading to the needle.

Flow and Resistance as Causes of Pressure Drops

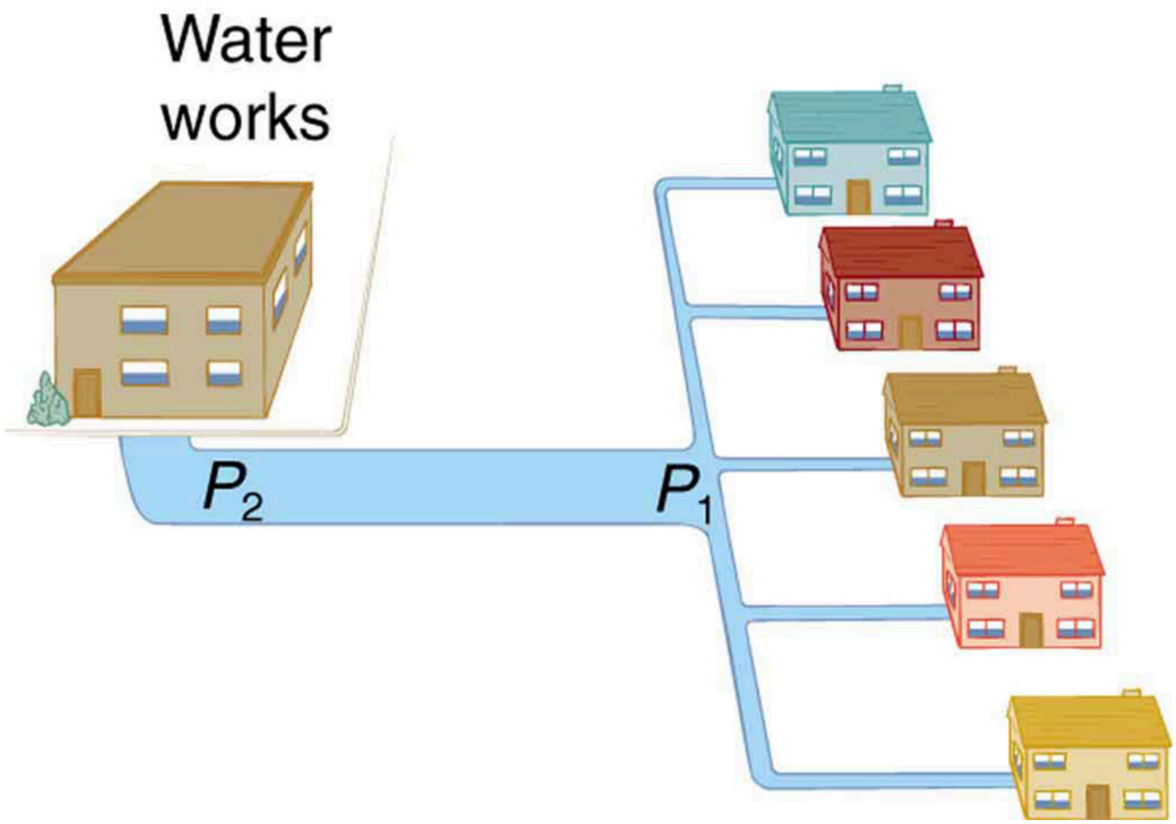
You may have noticed that water pressure in your home might be lower than normal on hot summer days when there is more use. This pressure drop occurs in the water main before it reaches your home. Let us consider flow through the water main as illustrated in [Figure 6]. We can understand why the pressure ${P}_{1}$ to the home drops during times of heavy use by rearranging

$$Q=\frac{ {P}_{2}-{P}_{1}}{R} $$

to

$${P}_{2}-{P}_{1}=RQ\text{,} $$

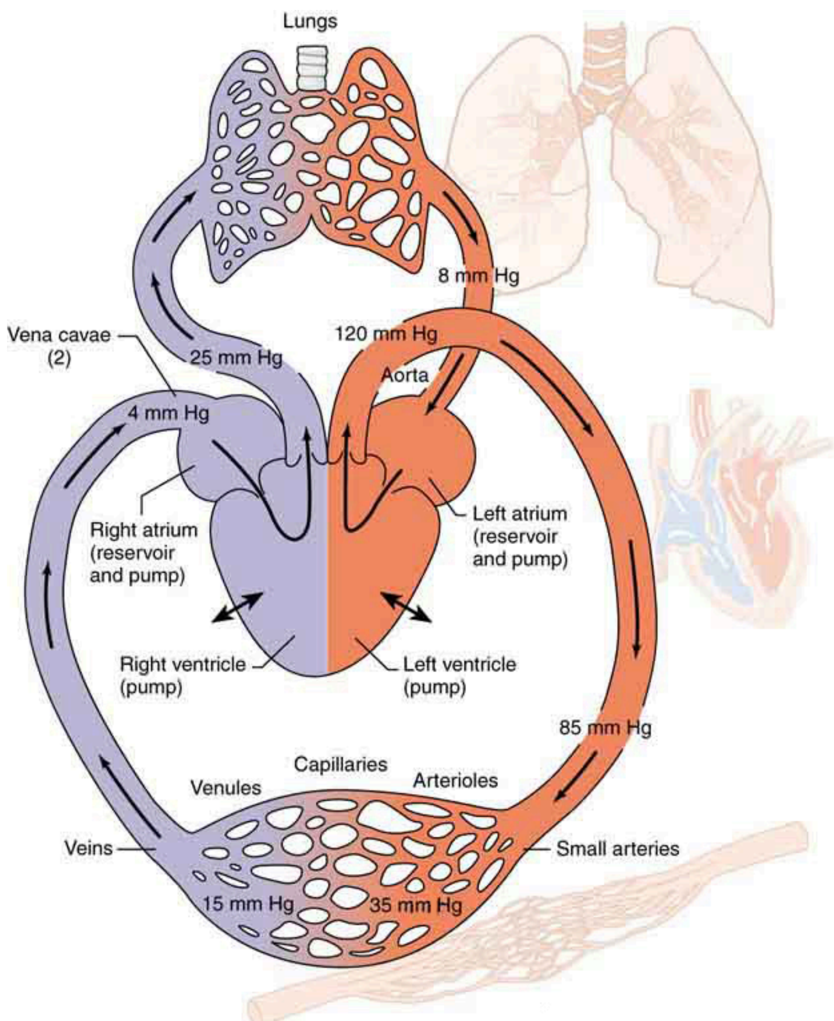
where, in this case, ${P}_{2}$ is the pressure at the water works and R is the resistance of the water main. During times of heavy use, the flow rate Q is large. This means that ${P}_{2}-{P}_{1}$ must also be large. Thus ${P}_{1}$ must decrease. It is correct to think of flow and resistance as causing the pressure to drop from ${P}_{2}$ to ${P}_{1}$. ${P}_{2}-{P}_{1}=RQ$ is valid for both laminar and turbulent flows.



During times of heavy use, there is a significant pressure drop in a water main, and P_1 supplied to users is significantly less than P_2 created at the water works. If the flow is very small, then the pressure drop is negligible, and $P_2 \approx P_1$.

We can use ${P}_{2}-{P}_{1}=RQ$ to analyze pressure drops occurring in more complex systems in which the tube radius is not the same everywhere. Resistance will be much greater in narrow places, such as an obstructed coronary artery. For a given flow rate Q , the pressure drop will be greatest where the tube is most narrow. This is how water faucets control flow. Additionally, R is greatly increased by turbulence, and a constriction that creates turbulence greatly reduces the pressure downstream. Plaque in an artery reduces pressure and hence flow, both by its resistance and by the turbulence it creates.

[Figure 7] is a schematic of the human circulatory system, showing average blood pressures in its major parts for an adult at rest. Pressure created by the heart's two pumps, the right and left ventricles, is reduced by the resistance of the blood vessels as the blood flows through them. The left ventricle increases arterial blood pressure that drives the flow of blood through all parts of the body except the lungs. The right ventricle receives the lower pressure blood from two major veins and pumps it through the lungs for gas exchange with atmospheric gases – the disposal of carbon dioxide from the blood and the replenishment of oxygen. Only one major organ is shown schematically, with typical branching of arteries to ever smaller vessels, the smallest of which are the capillaries, and rejoining of small veins into larger ones. Similar branching takes place in a variety of organs in the body, and the circulatory system has considerable flexibility in flow regulation to these organs by the dilation and constriction of the arteries leading to them and the capillaries within them. The sensitivity of flow to tube radius makes this flexibility possible over a large range of flow rates.



Schematic of the circulatory system. Pressure difference is created by the two pumps in the heart and is reduced by resistance in the vessels. Branching of vessels into capillaries allows blood to reach individual cells and exchange substances, such as oxygen and waste products, with them. The system has an impressive ability to regulate flow to individual organs, accomplished largely by varying vessel diameters.

Each branching of larger vessels into smaller vessels increases the total cross-sectional area of the tubes through which the blood flows. For example, an artery with a cross-section of $1 {\text{cm}}^{2}$ may branch into 20 smaller arteries, each with cross-sections of $0.5 {\text{cm}}^{2}$, with a total of $10 {\text{cm}}^{2}$. In that manner, the resistance of the branchings is reduced so that pressure is not entirely lost. Moreover, because $Q=A\overline{v}$ and A increases through branching, the average velocity of the blood in the smaller vessels is reduced. The blood velocity in the aorta ($\text{diameter}=1 \text{cm}$) is about 25 cm/s, while in the capillaries ($20\mu \text{m}$ in diameter) the velocity is about 1 mm/s. This reduced velocity allows the blood to exchange substances with the cells in the capillaries and alveoli in particular.

Section Summary

- Laminar flow is characterized by smooth flow of the fluid in layers that do not mix.
- Turbulence is characterized by eddies and swirls that mix layers of fluid together.
- Fluid viscosity η is due to friction within a fluid. Representative values are given in [Table 1]. Viscosity has units of $\left({\text{N/m}}^{2}\right)\text{s}$ or $\text{Pa}\cdot \text{s}$.
- Flow is proportional to pressure difference and inversely proportional to resistance:

  $$Q=\frac{ {P}_{2}-{P}_{1}}{R}. $$

- For laminar flow in a tube, Poiseuille's law for resistance states that

  $$R=\frac{8\eta l}{ \pi r^{4}}. $$

- Poiseuille's law for flow in a tube is

  $$Q=\frac{\left({P}_{2}-{P}_{1}\right)\pi {r}^{4}}{8\eta l}. $$

- The pressure drop caused by flow and resistance is given by

  $${P}_{2}-{P}_{1}=RQ. $$

Conceptual Questions

Explain why the viscosity of a liquid decreases with temperature—that is, how might increased temperature reduce the effects of cohesive forces in a liquid? Also explain why the viscosity of a gas increases with temperature—that is, how does increased gas temperature create more collisions between atoms and molecules?

Show Solution

Strategy

We consider the molecular mechanisms that cause viscosity in liquids versus gases, and how temperature affects these mechanisms differently.

Solution

Liquids (viscosity decreases with temperature): In liquids, viscosity is primarily due to intermolecular cohesive forces (van der Waals forces, hydrogen bonding, etc.) that resist the sliding of fluid layers past each other. When temperature increases:

- Molecules gain kinetic energy and move more rapidly
- This increased thermal motion helps molecules overcome the cohesive forces binding them to neighbors
- The "stickiness" between fluid layers decreases
- Result: Viscosity decreases (honey flows more easily when warmed)

Gases (viscosity increases with temperature): In gases, molecules are far apart, so cohesive forces are negligible. Instead, viscosity arises from momentum transfer during molecular collisions. When temperature increases:

- Molecules move faster (average speed $\propto \sqrt{T}$)
- Faster molecules collide more frequently and transfer more momentum between layers
- This increased momentum exchange creates more internal friction
- Result: Viscosity increases

Discussion

This opposite temperature dependence is a key distinction between liquids and gases. It has practical implications: motor oil thins when hot (problematic for lubrication), while hot exhaust gases are more viscous (affects engine performance).

When paddling a canoe upstream, it is wisest to travel as near to the shore as possible. When canoeing downstream, it may be best to stay near the middle. Explain why.

Show Solution

Strategy

We consider how the velocity profile in a river varies from shore to center due to viscous drag on the riverbed and banks.

Solution

Due to viscosity, water in a river doesn't flow at the same speed everywhere. The water experiences drag from the riverbed and banks, creating a velocity profile:

- **Near shore**: Water flows slowly due to friction with the riverbed and banks
- **Near center**: Water flows fastest because it's farthest from the frictional boundaries

Going upstream (against the current): You want to minimize the current working against you. By staying near the shore where the water moves slowly, you paddle against a weaker current, making progress easier.

Going downstream (with the current): You want to maximize the current helping you. By staying in the middle where the water moves fastest, you get the maximum boost from the current and travel faster with less effort.

Discussion

This is a practical application of understanding the velocity profile in viscous flow. The same principle applies to swimmers, kayakers, and even fish—they instinctively use these patterns for efficient movement in rivers.

Why does flow decrease in your shower when someone flushes the toilet?

Show Solution

Strategy

We apply the relationship between pressure, flow rate, and resistance in a plumbing system where multiple fixtures share a common supply.

Solution

When someone flushes a toilet, it creates a sudden demand for water that affects the entire plumbing system:

1. **Pressure drop**: The toilet's flush valve opens, creating a large flow through the toilet supply line. From the equation $P_2 - P_1 = RQ$, this increased flow causes a larger pressure drop in the main supply pipe.

2. **Reduced pressure at shower**: Since the shower and toilet share the same supply line (at least partially), the pressure available to the shower decreases when the toilet demands water.

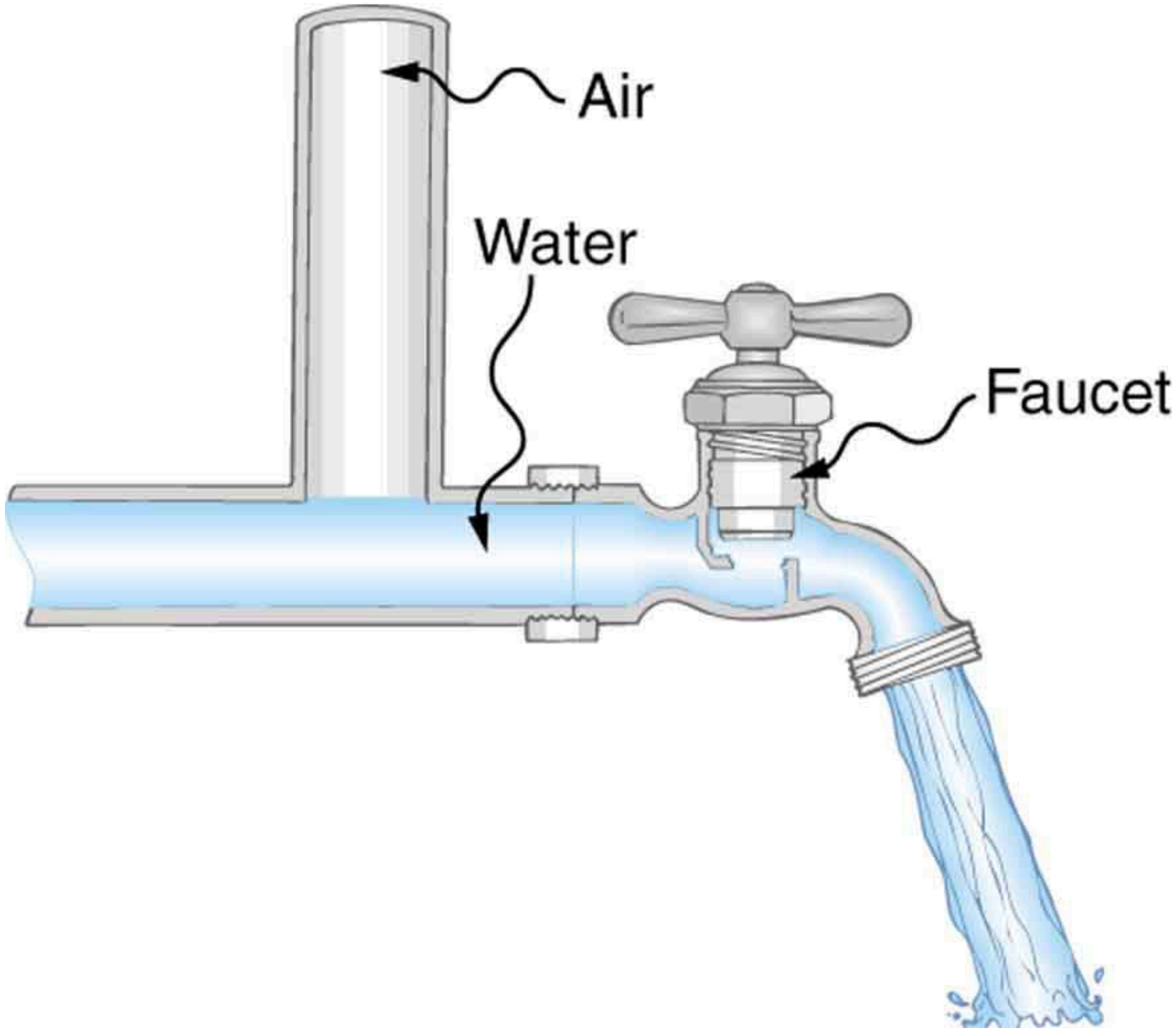
3. **Reduced shower flow**: By $Q = \frac{P_2 - P_1}{R}$, lower pressure difference at the shower means lower flow rate.

Additionally, if the toilet fills from a cold water line while the shower uses hot water, the sudden cold water demand can cause pressure fluctuations that change the temperature balance in the shower.

Discussion

This is why modern plumbing codes require larger supply pipes and pressure-balancing valves in showers. Pressure-balancing valves automatically adjust to maintain constant flow regardless of pressure fluctuations elsewhere in the system.

Plumbing usually includes air-filled tubes near water faucets, as shown in [Figure 8]. Explain why they are needed and how they work.



The vertical tube near the water tap remains full of air and serves a useful purpose.

Show Solution

Strategy

We consider the physics of water hammer and how a compressible gas (air) can absorb pressure surges.

Solution

These air-filled tubes are called “air chambers” or “water hammer arrestors.” They’re needed to prevent “water hammer”—the loud banging noise and potential pipe damage that occurs when water flow is suddenly stopped.

The problem (water hammer): When a faucet is quickly closed, the moving water must stop abruptly. Water is nearly incompressible, so its momentum creates a large pressure spike (often thousands of kPa). This pressure wave travels through the pipes, causing:

- Loud banging noises
- Stress on pipe joints
- Potential pipe damage over time

How the air chamber works:

1. The vertical tube traps a pocket of air at the top (air is much less dense than water and rises)
2. When water flow stops suddenly, the pressure surge pushes water up into the air chamber
3. The trapped air compresses, absorbing the shock like a spring
4. This compression happens gradually, converting the water's kinetic energy into potential energy of compressed air
5. The air then slowly pushes the water back, eliminating the sharp pressure spike

Discussion

Air chambers are simple, passive devices that prevent water hammer without requiring any moving parts. However, over time, air can dissolve into the water, reducing effectiveness—this is why plumbers sometimes need to drain pipes to restore the air cushion, or install mechanical water hammer arrestors with sealed air chambers.

Problems & Exercises

(a) Calculate the retarding force due to the viscosity of the air layer between a cart and a level air track given the following information—air temperature is 20ºC , the cart is moving at 0.400 m/s, its surface area is $2.50 \times 10^{-2} {\text{m}}^{2}$, and the thickness of the air layer is $6.00 \times 10^{-5} \text{m}$. (b) What is the ratio of this force to the weight of the 0.300-kg cart?

Show Solution

Strategy

We use the viscosity force equation $F = \eta \frac{vA}{L}$ where η is the viscosity of air at 20°C.

Solution

Given:

- Air viscosity at 20°C: $\eta = 0.0181 \text{ mPa·s} = 1.81 \times 10^{-5} \text{ Pa·s}$
- Velocity: $v = 0.400 \text{ m/s}$
- Area: $A = 2.50 \times 10^{-2} \text{ m}^2$
- Air layer thickness: $L = 6.00 \times 10^{-5} \text{ m}$
- Cart mass: $m = 0.300 \text{ kg}$

(a) Calculate the viscous force: $F = \eta \frac{vA}{L} = (1.81 \times 10^{-5} \text{ Pa·s}) \frac{(0.400 \text{ m/s})(2.50 \times 10^{-2} \text{ m}^2)}{6.00 \times 10^{-5} \text{ m}}$

$$ F = (1.81 \times 10^{-5}) \frac{0.0100}{6.00 \times 10^{-5}} = (1.81 \times 10^{-5})(166.7) = 3.02 \times 10^{-3} \text{ N} $$

(b) Calculate the weight and ratio: $W = mg = (0.300 \text{ kg})(9.80 \text{ m/s}^2) = 2.94 \text{ N}$

$$ \text{Ratio} = \frac{F}{W} = \frac{3.02 \times 10^{-3} \text{ N}}{2.94 \text{ N}} = 1.03 \times 10^{-3} $$

Discussion

The viscous retarding force is **(a) 3.02 × 10^{-3} N**, and the ratio to the cart's weight is **(b) 1.03 × 10^{-3}** (about 0.1%). This very small ratio explains why air tracks are excellent for demonstrating nearly frictionless motion—the air cushion reduces friction by about three orders of magnitude compared to solid contact.

What force is needed to pull one microscope slide over another at a speed of 1.00 cm/s, if there is a 0.500-mm-thick layer of 20ºC water between them and the contact area is $8.00 {\text{cm}}^{2}$?

Show Solution

Strategy

We use the viscosity force equation $F = \eta \frac{vA}{L}$ with the viscosity of water at 20°C.

Solution

Given:

- Water viscosity at 20°C: $\eta = 1.002 \text{ mPa·s} = 1.002 \times 10^{-3} \text{ Pa·s}$
- Velocity: $v = 1.00 \text{ cm/s} = 0.0100 \text{ m/s}$
- Layer thickness: $L = 0.500 \text{ mm} = 5.00 \times 10^{-4} \text{ m}$
- Contact area: $A = 8.00 \text{ cm}^2 = 8.00 \times 10^{-4} \text{ m}^2$

Calculate the force: $F = \eta \frac{vA}{L} = (1.002 \times 10^{-3} \text{ Pa·s}) \frac{(0.0100 \text{ m/s})(8.00 \times 10^{-4} \text{ m}^2)}{5.00 \times 10^{-4} \text{ m}}$

$$ F = (1.002 \times 10^{-3}) \frac{8.00 \times 10^{-6}}{5.00 \times 10^{-4}} = (1.002 \times 10^{-3})(0.0160) = 1.60 \times 10^{-5} \text{ N} $$

Discussion

The force needed is approximately **1.60 × 10^{-5} N** (16 micronewtons). This very small force demonstrates why water is an effective lubricant for microscope slides—even a thin layer of water dramatically reduces the friction compared to dry sliding contact.

A glucose solution being administered with an IV has a flow rate of $4.00 {\text{cm}}^{3}\text{/min}$. What will the new flow rate be if the glucose is replaced by whole blood having the same density but a viscosity 2.50 times that of the glucose? All other factors remain constant.

Show Solution

Strategy

From Poiseuille's law, flow rate is inversely proportional to viscosity when all other factors are constant.

Solution

From Poiseuille's law: $Q = \frac{(P_2 - P_1)\pi r^4}{8\eta l}$

With all other factors constant: $Q \propto \frac{1}{\eta}$

Therefore: $\frac{Q_2}{Q_1} = \frac{\eta_1}{\eta_2}$

Given that $\eta_2 = 2.50 \eta_1$: $Q_2 = Q_1 \times \frac{\eta_1}{\eta_2} = Q_1 \times \frac{1}{2.50} = \frac{4.00 \text{ cm}^3/\text{min}}{2.50} = 1.60 \text{ cm}^3/\text{min}$

Discussion

The new flow rate is **1.60 cm^3/min**. This illustrates why blood transfusions require careful attention to flow rates—blood's higher viscosity (about 3 times that of water) means it flows more slowly than other IV fluids under the same conditions. Higher pressure or larger needles may be needed to achieve adequate flow rates.

The pressure drop along a length of artery is 100 Pa, the radius is 10 mm, and the flow is laminar. The average speed of the blood is 15 mm/s. (a) What is the net force on the blood in this section of artery? (b) What is the power expended maintaining the flow?

Show Solution

Strategy

(a) The net force equals the pressure difference times the cross-sectional area. (b) Power equals the force times velocity, or equivalently, pressure drop times flow rate.

Solution

Given:

- Pressure drop: $\Delta P = 100 \text{ Pa}$
- Radius: $r = 10 \text{ mm} = 0.010 \text{ m}$
- Average speed: $v = 15 \text{ mm/s} = 0.015 \text{ m/s}$

(a) Calculate the cross-sectional area: $A = \pi r^2 = \pi (0.010 \text{ m})^2 = 3.14 \times 10^{-4} \text{ m}^2$

Calculate the net force: $F = \Delta P \times A = (100 \text{ Pa})(3.14 \times 10^{-4} \text{ m}^2) = 3.14 \times 10^{-2} \text{ N}$

(b) Calculate the flow rate: $Q = Av = (3.14 \times 10^{-4} \text{ m}^2)(0.015 \text{ m/s}) = 4.71 \times 10^{-6} \text{ m}^3/\text{s}$

Calculate the power: $P = \Delta P \times Q = (100 \text{ Pa})(4.71 \times 10^{-6} \text{ m}^3/\text{s}) = 4.71 \times 10^{-4} \text{ W}$

Discussion

The net force on the blood is **(a) 3.14×10^{-2} N** (about 31 millinewtons), and the power expended is **(b) 4.71×10^{-4} W** (about 0.47 milliwatts). These small values for a single artery segment multiply across the thousands of vessels in the body, explaining why the heart must continuously supply significant power to maintain blood circulation.

A small artery has a length of $1.1 \times 10^{-3} \text{m}$ and a radius of $2.5 \times 10^{-5} \text{m}$. If the pressure drop across the artery is 1.3 kPa, what is the flow rate through the artery? (Assume that the temperature is 37ºC .)

Show Solution

Strategy

We use Poiseuille's law to calculate the flow rate given the pressure drop, dimensions, and viscosity of blood at body temperature.

Solution

Given:

- Length: $l = 1.1 \times 10^{-3} \text{ m}$
- Radius: $r = 2.5 \times 10^{-5} \text{ m}$
- Pressure drop: $\Delta P = 1.3 \text{ kPa} = 1300 \text{ Pa}$
- Blood viscosity at 37°C: $\eta = 2.084 \text{ mPa·s} = 2.084 \times 10^{-3} \text{ Pa·s}$

Apply Poiseuille's law: $Q = \frac{\Delta P \cdot \pi r^4}{8\eta l}$

$$ Q = \frac{(1300 \text{ Pa}) \cdot \pi (2.5 \times 10^{-5} \text{ m})^4}{8(2.084 \times 10^{-3} \text{ Pa·s})(1.1 \times 10^{-3} \text{ m})} $$

$$ Q = \frac{(1300) \cdot \pi (3.91 \times 10^{-19})}{8(2.29 \times 10^{-6})} = \frac{1.60 \times 10^{-15}}{1.83 \times 10^{-5}} $$
$$ Q = 8.7 \times 10^{-11} \text{ m}^3/\text{s} $$

Discussion

The flow rate through this small artery is approximately **8.7×10^{-11} m^3/s** (0.087 nanoliters per second). This extremely small flow rate is typical for arterioles—individually they carry very little blood, but collectively the millions of small vessels distribute blood throughout the body's tissues.

Fluid originally flows through a tube at a rate of $100 {\text{cm}}^{3}\text{/s}$. To illustrate the sensitivity of flow rate to various factors, calculate the new flow rate for the following changes with all other factors remaining the same as in the original conditions. (a) Pressure difference increases by a factor of 1.50. (b) A new fluid with 3.00 times greater viscosity is substituted. (c) The tube is replaced by one having 4.00 times the length. (d) Another tube is used with a radius 0.100 times the original. (e) Yet another tube is substituted with a radius 0.100 times the original and half the length, *and* the pressure difference is increased by a factor of 1.50.

Show Solution

Strategy

From Poiseuille's law, $Q = \frac{\Delta P \cdot \pi r^4}{8\eta l}$, we analyze how Q scales with each parameter.

Solution

Original flow rate: $Q_0 = 100 \text{ cm}^3/\text{s}$

From Poiseuille's law: $Q \propto \frac{\Delta P \cdot r^4}{\eta \cdot l}$

(a) Pressure difference increases by 1.50: $Q_a = Q_0 \times 1.50 = 100 \times 1.50 = 150 \text{ cm}^3/\text{s}$

(b) Viscosity increases by 3.00: $Q_b = Q_0 \times \frac{1}{3.00} = 100 \times \frac{1}{3.00} = 33.3 \text{ cm}^3/\text{s}$

(c) Length increases by 4.00: $Q_c = Q_0 \times \frac{1}{4.00} = 100 \times \frac{1}{4.00} = 25.0 \text{ cm}^3/\text{s}$

(d) Radius becomes 0.100 times original: $Q_d = Q_0 \times (0.100)^4 = 100 \times 0.0001 = 0.0100 \text{ cm}^3/\text{s}$

(e) Radius 0.100 times, length half, pressure 1.50 times: $Q_e = Q_0 \times (0.100)^4 \times \frac{1}{0.5} \times 1.50 = 100 \times 0.0001 \times 2 \times 1.50 = 0.0300 \text{ cm}^3/\text{s}$

Discussion

The new flow rates are **(a) 150**, **(b) 33.3**, **(c) 25.0**, **(d) 0.0100**, and **(e) 0.0300 cm^3/s**. The dramatic effect of radius (fourth power!) is clearly shown in parts (d) and (e)—reducing the radius to 1/10 reduces flow by a factor of 10,000. This explains why even small changes in blood vessel diameter have profound effects on blood flow.

The arterioles (small arteries) leading to an organ, constrict in order to decrease flow to the organ. To shut down an organ, blood flow is reduced naturally to 1.00% of its original value. By what factor did the radii of the arterioles constrict? Penguins do this when they stand on ice to reduce the blood flow to their feet.

Show Solution

Strategy

From Poiseuille's law, flow rate is proportional to r^4. We find the radius ratio that reduces flow to 1% of original.

Solution

From Poiseuille's law with constant pressure, viscosity, and length: $Q \propto r^4$

Let $r_2 = x \cdot r_1$ where x is the factor we seek.

Given: $Q_2 = 0.0100 \cdot Q_1$

Therefore: $\frac{Q_2}{Q_1} = \frac{r_2^4}{r_1^4} = x^4 = 0.0100$

$$ x = (0.0100)^{1/4} = (0.0100)^{0.25} = 0.316 $$

Discussion

The radii constricted to **0.316** (or about 31.6%) of their original value. This means the arterioles narrowed to less than one-third of their normal diameter, which reduced flow to just 1% of normal. This remarkable ability to regulate blood flow is crucial for penguins and other animals that must conserve body heat while standing on ice—reducing blood flow to extremities prevents heat loss while keeping vital organs warm.

Angioplasty is a technique in which arteries partially blocked with plaque are dilated to increase blood flow. By what factor must the radius of an artery be increased in order to increase blood flow by a factor of 10?

Show Solution

Strategy

From Poiseuille's law, flow rate scales as r^4. We find the radius factor needed to increase flow by a factor of 10.

Solution

From Poiseuille’s law with constant pressure, viscosity, and length: $Q \propto r^4$

Let the radius increase factor be x (so $r_2 = x \cdot r_1$).

Given: $Q_2 = 10 \cdot Q_1$

$$ \frac{Q_2}{Q_1} = \frac{r_2^4}{r_1^4} = x^4 = 10 $$
$$ x = 10^{1/4} = 10^{0.25} = 1.78 $$

Discussion

The radius must be increased by a factor of **1.78** (about 78% larger) to increase blood flow by a factor of 10. This demonstrates the power of the r^4 relationship—a relatively modest increase in radius produces a dramatic increase in flow. This is why angioplasty (which typically increases vessel diameter by inserting a balloon or stent) can be so effective at restoring blood flow even when it doesn’t fully restore the original vessel size.

(a) Suppose a blood vessel’s radius is decreased to 90.0% of its original value by plaque deposits and the body compensates by increasing the pressure difference along the vessel to keep the flow rate constant. By what factor must the pressure difference increase? (b) If turbulence is created by the obstruction, what additional effect would it have on the flow rate?

Show Solution

Strategy

(a) From Poiseuille’s law, for constant Q with changed radius, we find how pressure must change. (b) We consider the additional effects of turbulent (non-laminar) flow.

Solution

(a) From Poiseuille’s law: $Q = \frac{\Delta P \cdot \pi r^4}{8\eta l}$

For constant Q, viscosity, and length: $\Delta P \propto \frac{1}{r^4}$

With $r_2 = 0.90 r_1$: $\frac{\Delta P_2}{\Delta P_1} = \frac{r_1^4}{r_2^4} = \frac{r_1^4}{(0.90 r_1)^4} = \frac{1}{(0.90)^4} = \frac{1}{0.656} = 1.52$

(b) If turbulence is created by the obstruction:

- Turbulence significantly increases resistance to flow beyond what Poiseuille’s law predicts
- For turbulent flow, resistance scales approximately with v^2 rather than v
- This would decrease the actual flow rate below the laminar prediction
- To maintain constant flow, an even larger pressure increase would be needed
- This leads to higher blood pressure, putting additional strain on the heart and vessel walls
- The obstruction itself can worsen over time as turbulence promotes further plaque deposition

Discussion

The pressure difference must increase by a factor of **(a) 1.52** (a 52% increase). **(b)** Turbulence makes matters worse—it decreases flow efficiency and requires even higher pressure to maintain flow, creating a dangerous cycle that can lead to hypertension and increased cardiovascular risk.

A spherical particle falling at a terminal speed in a liquid must have the gravitational force balanced by the drag force and the buoyant force. The buoyant force is equal to the weight of the displaced fluid, while the drag force is assumed to be given by Stokes Law, ${F}_{s}=6\pi r\eta v$. Show that the terminal speed is given by

$$v=\frac{ {2R}^{2}g}{9\eta }\left({\rho }_{\text{s}}-{\rho }_{1}\right), $$

where R is the radius of the sphere, ${\rho }_{\text{s}}$ is its density, and ${\rho }_{1}$ is the density of the fluid and η the coefficient of viscosity.

Show Solution

Strategy

At terminal velocity, the net force on the sphere is zero. We set up the force balance equation with gravitational force (downward), buoyant force (upward), and Stokes drag force (upward), then solve for velocity.

Solution

At terminal velocity, the net force is zero: $F_{\text{net}} = F_{\text{gravity}} - F_{\text{buoyant}} - F_{\text{drag}} = 0$

Gravitational force: $F_{\text{gravity}} = m_s g = \rho_s V g = \rho_s \left(\frac{4}{3}\pi R^3\right) g$

Buoyant force (weight of displaced fluid): $F_{\text{buoyant}} = \rho_1 V g = \rho_1 \left(\frac{4}{3}\pi R^3\right) g$

Stokes drag force: $F_{\text{drag}} = 6\pi R \eta v$

Setting the net force to zero: $\rho_s \left(\frac{4}{3}\pi R^3\right) g - \rho_1 \left(\frac{4}{3}\pi R^3\right) g - 6\pi R \eta v = 0$

Factor out common terms: $\frac{4}{3}\pi R^3 g (\rho_s - \rho_1) = 6\pi R \eta v$

Solve for v: $v = \frac{4\pi R^3 g (\rho_s - \rho_1)}{3 \times 6\pi R \eta} = \frac{4R^3 g (\rho_s - \rho_1)}{18R \eta}$

Simplify: $v = \frac{4R^2 g (\rho_s - \rho_1)}{18\eta} = \frac{2R^2 g (\rho_s - \rho_1)}{9\eta}$

This proves the desired result: $v = \frac{2R^2 g}{9\eta}(\rho_s - \rho_1)$

Discussion

This elegant result shows that terminal velocity is proportional to the square of the particle radius ($v \propto R^2$), inversely proportional to viscosity, and directly proportional to the density difference. This explains why fine particles settle much more slowly than large ones—halving the radius reduces terminal velocity by a factor of four. The formula is widely used in sedimentation analysis, centrifuge calculations, and understanding atmospheric particle behavior.

Using the equation of the previous problem, find the viscosity of motor oil in which a steel ball of radius 0.8 mm falls with a terminal speed of 4.32 cm/s. The densities of the ball and the oil are 7.86 and 0.88 g/mL, respectively.

Show Solution

Strategy

We use the terminal velocity formula derived in the previous problem and solve for viscosity η.

Solution

Given:

- Radius: $R = 0.8 \text{ mm} = 8.0 \times 10^{-4} \text{ m}$
- Terminal speed: $v = 4.32 \text{ cm/s} = 0.0432 \text{ m/s}$
- Steel ball density: $\rho_s = 7.86 \text{ g/mL} = 7860 \text{ kg/m}^3$
- Oil density: $\rho_1 = 0.88 \text{ g/mL} = 880 \text{ kg/m}^3$
- Gravitational acceleration: $g = 9.80 \text{ m/s}^2$

From the previous problem, we have: $v = \frac{2R^2 g}{9\eta}(\rho_s - \rho_1)$

Solving for η: $\eta = \frac{2R^2 g (\rho_s - \rho_1)}{9v}$

Substitute known values: $\eta = \frac{2(8.0 \times 10^{-4} \text{ m})^2 (9.80 \text{ m/s}^2)(7860 - 880) \text{ kg/m}^3}{9(0.0432 \text{ m/s})}$

$$ \eta = \frac{2(6.4 \times 10^{-7} \text{ m}^2)(9.80 \text{ m/s}^2)(6980 \text{ kg/m}^3)}{0.389 \text{ m/s}} $$
$$ \eta = \frac{8.74 \times 10^{-2} \text{ kg·m/s}^2}{0.389 \text{ m/s}} = 0.225 \text{ kg/(m·s)} = 0.225 \text{ Pa·s} $$

Converting to mPa·s: $\eta = 225 \text{ mPa·s}$

Discussion

The viscosity of this motor oil is approximately **225 mPa·s** (or 0.225 Pa·s). This is consistent with motor oil at around 30°C—comparing to Table 1, SAE 10 motor oil at 30°C has a viscosity of 200 mPa·s, so our calculated value is very reasonable. The experiment demonstrates how terminal velocity measurements can be used to determine viscosity, a technique commonly used in quality control for lubricants.

A skydiver will reach a terminal velocity when the air drag equals their weight. For a skydiver with high speed and a large body, turbulence is a factor. The drag force then is approximately proportional to the square of the velocity. Taking the drag force to be ${F}_{\text{D}}=\frac{1}{2}\rho Av^{2}$ and setting this equal to the person's weight, find the terminal speed for a person falling "spread eagle." Find both a formula and a number for ${v}_{\text{t}}$, with assumptions as to size.

Show Solution

Strategy

At terminal velocity, the drag force equals the weight. We set $F_D = mg$ and solve for velocity, then estimate numerical values using reasonable assumptions about a person's mass and cross-sectional area.

Solution

Formula derivation:

At terminal velocity: $F_D = mg$

$$ \frac{1}{2}\rho A v_t^2 = mg $$

Solving for v_t: $v_t = \sqrt{\frac{2mg}{\rho A}}$

Numerical calculation with assumptions:

Assumptions for a person falling "spread eagle":

- Mass: $m = 75 \text{ kg}$ (typical adult)
- Cross-sectional area (spread eagle): $A \approx 0.70 \text{ m}^2$ (roughly 1 m × 0.7 m)

- Air density at sea level: $\rho = 1.29 \text{ kg/m}^3$
- Gravitational acceleration: $g = 9.80 \text{ m/s}^2$

Calculate terminal velocity: $v_t = \sqrt{\frac{2(75 \text{ kg})(9.80 \text{ m/s}^2)}{(1.29 \text{ kg/m}^3)(0.70 \text{ m}^2)}}$

$$ v_t = \sqrt{\frac{1470 \text{ N}}{0.903 \text{ kg/m}}} = \sqrt{1628 \text{ m}^2/\text{s}^2} $$
$$ v_t = 40.3 \text{ m/s} \approx 145 \text{ km/h or } 90 \text{ mph} $$

Discussion

The terminal velocity formula is $v_t = \sqrt{\frac{2mg}{\rho A}}$, and for a typical skydiver in spread-eagle position, the terminal velocity is approximately **40 m/s** (about 90 mph). This matches real-world skydiving experience—skydivers falling belly-down reach terminal velocities of 50-60 m/s. By tucking into a pike position (head-first, arms at sides), a skydiver reduces A to about 0.15 m^2, increasing terminal velocity to over 90 m/s (200 mph), which is why position control is crucial for skydiving. Note that $v_t \propto \sqrt{m/A}$, so heavier skydivers or those in more streamlined positions fall faster.

A layer of oil 1.50 mm thick is placed between two microscope slides. Researchers find that a force of $5.50 \times 10^{-4} \text{N}$ is required to glide one over the other at a speed of 1.00 cm/s when their contact area is $6.00 {\text{cm}}^{2}$. What is the oil's viscosity? What type of oil might it be?

Show Solution

Strategy

We use the viscosity force equation $F = \eta \frac{vA}{L}$ and solve for η, then compare to Table 1 to identify the oil type.

Solution

Given:

- Force: $F = 5.50 \times 10^{-4} \text{ N}$
- Velocity: $v = 1.00 \text{ cm/s} = 0.0100 \text{ m/s}$
- Contact area: $A = 6.00 \text{ cm}^2 = 6.00 \times 10^{-4} \text{ m}^2$
- Oil layer thickness: $L = 1.50 \text{ mm} = 1.50 \times 10^{-3} \text{ m}$

From the viscosity force equation: $F = \eta \frac{vA}{L}$

Solving for η: $\eta = \frac{FL}{vA}$

Substitute known values: $\eta = \frac{(5.50 \times 10^{-4} \text{ N})(1.50 \times 10^{-3} \text{ m})}{(0.0100 \text{ m/s})(6.00 \times 10^{-4} \text{ m}^2)}$

$$ \eta = \frac{8.25 \times 10^{-7} \text{ N·m}}{6.00 \times 10^{-6} \text{ m}^3/\text{s}} = 0.138 \text{ N·s/m}^2 = 0.138 \text{ Pa·s} $$

Converting to mPa·s: $\eta = 138 \text{ mPa·s}$

Identification: Comparing to Table 1, olive oil at 20°C has a viscosity of 138 mPa·s, which matches our calculated value exactly. Therefore, the oil is most likely **olive oil**.

Discussion

The oil's viscosity is **0.138 Pa·s (or 138 mPa·s)**, and it is most likely **olive oil**. The close match with tabulated values validates the experimental technique. This method of measuring viscosity by sliding surfaces is fundamental to tribology (the study of friction and lubrication). Olive oil's moderate viscosity makes it useful for both cooking and as a historical lubricant, though modern applications typically use specialized oils.

(a) Verify that a 19.0% decrease in laminar flow through a tube is caused by a 5.00% decrease in radius, assuming that all other factors remain constant, as stated in the text. (b) What increase in flow is obtained from a 5.00% increase in radius, again assuming all other factors remain constant?

Show Solution

Strategy

From Poiseuille's law, flow rate is proportional to r^4 when all other factors are constant. We calculate the flow ratio for both a 5% decrease and 5% increase in radius.

Solution

From Poiseuille's law with constant pressure, viscosity, and length: $Q \propto r^4$

Therefore: $\frac{Q_2}{Q_1} = \left(\frac{r_2}{r_1}\right)^4$

(a) 5.00% decrease in radius:

With $r_2 = 0.950 r_1$: $\frac{Q_2}{Q_1} = (0.950)^4 = 0.8145$

This means: $Q_2 = 0.8145 Q_1$

The decrease in flow is: $\Delta Q = Q_1 - Q_2 = Q_1 - 0.8145 Q_1 = 0.1855 Q_1$

Percentage decrease: $\frac{\Delta Q}{Q_1} \times 100\% = 18.55\% \approx 19.0\%$

This verifies the statement in the text.

(b) 5.00% increase in radius:

With $r_2 = 1.050 r_1$: $\frac{Q_2}{Q_1} = (1.050)^4 = 1.2155$

The increase in flow is: $\Delta Q = Q_2 - Q_1 = 1.2155 Q_1 - Q_1 = 0.2155 Q_1$

Percentage increase: $\frac{\Delta Q}{Q_1} \times 100\% = 21.55\% \approx 21.6\%$

Discussion

(a) A 5.00% decrease in radius causes a **19.0% decrease in flow**, confirming the text's statement. **(b)** A 5.00% increase in radius causes a **21.6% increase in flow**. Note the asymmetry: a 5% increase produces a slightly larger percentage change (21.6%) than a 5% decrease (19.0%). This occurs because of the fourth-power relationship—percentage changes are not symmetric about the original value. The dramatic sensitivity to radius (r^4 dependence) explains why blood vessel dilation and constriction are such effective mechanisms for regulating blood flow: small diameter changes produce large flow changes.

[Example 2] dealt with the flow of saline solution in an IV system. (a) Verify that a pressure of $1.62 \times 10^{4} {\text{N/m}}^{2}$ is created at a depth of 1.61 m in a saline solution, assuming its density to be that of sea water. (b) Calculate the new flow rate if the height of the saline solution is decreased to 1.50 m. (c) At what height would the direction of flow be reversed? (This reversal can be a problem when patients stand up.)

Show Solution

Strategy

(a) We use the pressure-depth relationship $P = \rho g h$ for gauge pressure. (b) We use Poiseuille's law with the new pressure difference. (c) Flow reverses when the IV bag pressure equals the vein pressure.

Solution

From Example 2:

- Needle radius: $r = 0.150 \text{ mm} = 1.50 \times 10^{-4} \text{ m}$
- Needle length: $l = 2.50 \text{ cm} = 0.0250 \text{ m}$
- Saline viscosity: $\eta = 1.00 \times 10^{-3} \text{ Pa·s}$
- Vein pressure: $P_{\text{vein}} = 8.00 \text{ mm Hg} = 1.066 \times 10^3 \text{ Pa}$
- Saline density (sea water): $\rho = 1025 \text{ kg/m}^3$
- Original flow rate: $Q_1 = 0.120 \text{ cm}^3/\text{s} = 1.20 \times 10^{-7} \text{ m}^3/\text{s}$

(a) Verify pressure at depth of 1.61 m:

Gauge pressure at depth: $P = \rho g h = (1025 \text{ kg/m}^3)(9.80 \text{ m/s}^2)(1.61 \text{ m})$

$$ P = 1.617 \times 10^4 \text{ Pa} \approx 1.62 \times 10^4 \text{ Pa} $$

This verifies the stated pressure.

(b) New flow rate at height 1.50 m:

New pressure from saline at 1.50 m: $P_{\text{new}} = \rho g h = (1025)(9.80)(1.50) = 1.507 \times 10^4 \text{ Pa}$

Pressure difference driving flow: $\Delta P_{\text{new}} = P_{\text{new}} - P_{\text{vein}} = 1.507 \times 10^4 - 1.066 \times 10^3 = 1.400 \times 10^4 \text{ Pa}$

From Example 2, the original pressure difference was: $\Delta P_1 = 1.62 \times 10^4 - 1.066 \times 10^3 = 1.513 \times 10^4 \text{ Pa}$

Since $Q \propto \Delta P$: $Q_2 = Q_1 \times \frac{\Delta P_{\text{new}}}{\Delta P_1} = (0.120 \text{ cm}^3/\text{s}) \times \frac{1.400 \times 10^4}{1.513 \times 10^4}$

$$ Q_2 = 0.120 \times 0.925 = 0.111 \text{ cm}^3/\text{s} $$

(c) Height where flow reverses:

Flow reverses when $P_{\text{saline}} = P_{\text{vein}}$: $\rho g h = 1.066 \times 10^3 \text{ Pa}$

$$ h = \frac{1.066 \times 10^3 \text{ Pa}}{(1025 \text{ kg/m}^3)(9.80 \text{ m/s}^2)} = \frac{1.066 \times 10^3}{10,045} \text{ m} $$
$$ h = 0.106 \text{ m} = 10.6 \text{ cm} $$

Discussion

(a) The pressure is confirmed as **1.62×10^4 N/m^2**. **(b)** Reducing the height to 1.50 m decreases flow to **0.111 cm^3/s**, about 7.5% less than the original. **(c)** Flow reverses at a height of **10.6 cm**. This is clinically important: if an IV bag drops below 10.6 cm above the needle entry point, blood will flow backward into the IV line. This is why IV bags must be elevated properly and why patients sometimes experience problems when standing up—the relative height between the IV bag and the insertion point changes dramatically.

When physicians diagnose arterial blockages, they quote the reduction in flow rate. If the flow rate in an artery has been reduced to 10.0% of its normal value by a blood clot and the average pressure difference has increased by 20.0%, by what factor has the clot reduced the radius of the artery?

Show Solution

Strategy

From Poiseuille's law, $Q \propto \frac{\Delta P \cdot r^4}{\eta l}$. We use the relationship between flow, pressure, and radius to find the radius reduction factor.

Solution

From Poiseuille's law: $Q = \frac{\Delta P \cdot \pi r^4}{8\eta l}$

For two situations (normal vs. blocked): $\frac{Q_2}{Q_1} = \frac{\Delta P_2}{\Delta P_1} \times \left(\frac{r_2}{r_1}\right)^4$

Given:

- $Q_2 = 0.100 Q_1$ (flow reduced to 10% of normal)
- $\Delta P_2 = 1.20 \Delta P_1$ (pressure increased by 20%)

Substitute: $0.100 = 1.20 \times \left(\frac{r_2}{r_1}\right)^4$

Solve for the radius ratio: $\left(\frac{r_2}{r_1}\right)^4 = \frac{0.100}{1.20} = 0.0833$

$$ \frac{r_2}{r_1} = (0.0833)^{1/4} = (0.0833)^{0.25} = 0.537 $$

The radius has been reduced by a factor of: $\text{Reduction factor} = 1 - 0.537 = 0.463 \text{ or } 46.3\%$

Alternatively, we can say: $r_2 = 0.537 r_1 \approx 0.54 r_1$

Discussion

The clot has reduced the artery radius to approximately **0.54** (or 54%) of its normal value, a reduction of about **46%**. This severe constriction dramatically reduces blood flow to just 10% of normal despite the body's compensation through increased blood pressure (20% higher). This illustrates the extreme sensitivity of flow to radius—the r^4 relationship means that moderate arterial narrowing causes severe flow reduction. Such a blockage would typically require medical intervention like angioplasty or bypass surgery. The body's attempt to compensate by raising blood pressure can strain the heart and other vessels, creating a dangerous situation.

During a marathon race, a runner's blood flow increases to 10.0 times her resting rate. Her blood's viscosity has dropped to 95.0% of its normal value, and the blood pressure difference across the circulatory system has increased by 50.0%. By what factor has the average radii of her blood vessels increased?

Show Solution

Strategy

From Poiseuille's law, $Q \propto \frac{\Delta P \cdot r^4}{\eta}$. We use the relationships between flow, pressure, viscosity, and radius to find the radius increase factor.

Solution

From Poiseuille's law: $Q = \frac{\Delta P \cdot \pi r^4}{8\eta l}$

For constant length, comparing exercise to rest: $\frac{Q_2}{Q_1} = \frac{\Delta P_2}{\Delta P_1} \times \frac{\eta_1}{\eta_2} \times \left(\frac{r_2}{r_1}\right)^4$

Given:

- $Q_2 = 10.0 Q_1$ (flow increased to 10 times resting)
- $\eta_2 = 0.950 \eta_1$ (viscosity dropped to 95%)
- $\Delta P_2 = 1.50 \Delta P_1$ (pressure increased by 50%)

Substitute: $10.0 = 1.50 \times \frac{\eta_1}{0.950 \eta_1} \times \left(\frac{r_2}{r_1}\right)^4$

$$ 10.0 = 1.50 \times \frac{1}{0.950} \times \left(\frac{r_2}{r_1}\right)^4 $$
$$ 10.0 = 1.50 \times 1.053 \times \left(\frac{r_2}{r_1}\right)^4 $$
$$ 10.0 = 1.579 \times \left(\frac{r_2}{r_1}\right)^4 $$

Solve for radius ratio: $\left(\frac{r_2}{r_1}\right)^4 = \frac{10.0}{1.579} = 6.33$

$$ \frac{r_2}{r_1} = (6.33)^{1/4} = (6.33)^{0.25} = 1.59 $$

Discussion

The average radii of the runner's blood vessels have increased by a factor of approximately **1.59** (or about 59% larger). This significant dilation, combined with increased blood pressure and decreased blood viscosity, allows blood flow to increase tenfold during vigorous exercise. The vasodilation is controlled by the autonomic nervous system and local chemical signals (like nitric oxide) that relax smooth muscle in vessel walls. This remarkable adaptation allows increased oxygen and nutrient delivery to working muscles while removing metabolic waste products. The r^4 relationship shows why vessel dilation is so effective—a 59% increase in radius produces a 6.33-fold increase in flow capacity, which combined with other factors yields the 10-fold total increase.

Water supplied to a house by a water main has a pressure of $3.00 \times 10^{5} {\text{N/m}}^{2}$ early on a summer day when neighborhood use is low. This pressure produces a flow of 20.0 L/min through a garden hose. Later in the day, pressure at the exit of the water main and entrance to the house drops, and a flow of only 8.00 L/min is obtained through the same hose. (a) What pressure is now being supplied to the house, assuming resistance is constant? (b) By what factor did the flow rate in the water main increase in order to cause this decrease in delivered pressure? The pressure at the entrance of the water main is $5.00 \times 10^{5} {\text{N/m}}^{2}$, and the original flow rate was 200 L/min. (c) How many more users are there, assuming each would consume 20.0 L/min in the morning?

Show Solution

Strategy

(a) Since resistance is constant, pressure is proportional to flow rate in the hose. (b) We use the pressure drop formula $\Delta P = RQ$ for the water main. (c) We calculate how many additional users would account for the increased flow.

Solution

Given:

- Morning pressure at house: $P_1 = 3.00 \times 10^5 \text{ Pa}$
- Morning hose flow: $Q_{hose,1} = 20.0 \text{ L/min}$
- Afternoon hose flow: $Q_{hose,2} = 8.00 \text{ L/min}$
- Water main entrance pressure: $P_{main} = 5.00 \times 10^5 \text{ Pa}$
- Original water main flow: $Q_{main,1} = 200 \text{ L/min}$

(a) New pressure at house:

Since $Q = \frac{\Delta P}{R}$ and resistance is constant: $\frac{P_2}{P_1} = \frac{Q_{hose,2}}{Q_{hose,1}}$

$$ P_2 = P_1 \times \frac{Q_{hose,2}}{Q_{hose,1}} = (3.00 \times 10^5 \text{ Pa}) \times \frac{8.00}{20.0} $$
$$ P_2 = 3.00 \times 10^5 \times 0.400 = 1.20 \times 10^5 \text{ Pa} $$

(b) Factor increase in water main flow:

The pressure drop in the water main is: $\Delta P = P_{main} - P_{house} = RQ_{main}$

Morning: $\Delta P_1 = 5.00 \times 10^5 - 3.00 \times 10^5 = 2.00 \times 10^5 \text{ Pa}$

Afternoon: $\Delta P_2 = 5.00 \times 10^5 - 1.20 \times 10^5 = 3.80 \times 10^5 \text{ Pa}$

Since $\Delta P = RQ_{main}$: $\frac{Q_{main,2}}{Q_{main,1}} = \frac{\Delta P_2}{\Delta P_1} = \frac{3.80 \times 10^5}{2.00 \times 10^5} = 1.90$

The water main flow increased by a factor of **1.90** (or 90%).

New water main flow: $Q_{main,2} = 1.90 \times 200 = 380 \text{ L/min}$

(c) Number of additional users:

Increase in total consumption: $\Delta Q = 380 - 200 = 180 \text{ L/min}$

Each user consumes 20.0 L/min in the morning, so: $\text{Number of additional users} = \frac{180 \text{ L/min}}{20.0 \text{ L/min per user}} = 9 \text{ users}$

Discussion

(a) The afternoon pressure at the house is **1.20×10^5 Pa**, a 60% reduction from morning levels. **(b)** The water main flow increased by a factor of **1.90**, nearly doubling. **(c)** There are approximately **9 additional users**, each consuming water at the same rate. This problem demonstrates how shared water systems experience pressure drops during peak usage times. The pressure drop in the main is proportional to flow ($\Delta P = RQ$), so increased demand causes lower delivery pressure to all users. This is why many people experience reduced water pressure during peak hours (morning showers, evening watering). Water utilities must size mains to handle peak loads while maintaining adequate pressure.

An oil gusher shoots crude oil 25.0 m into the air through a pipe with a 0.100-m diameter. Neglecting air resistance but not the resistance of the pipe, and assuming laminar flow, calculate the gauge pressure at the entrance of the 50.0-m-long vertical pipe. Take the density of the oil to be $900 {\text{kg/m}}^{3}$ and its viscosity to be $1.00 \left({\text{N/m}}^{2}\right)\cdot \text{s}$ (or $1.00 \text{Pa}\cdot \text{s}$). Note that you must take into account the pressure due to the 50.0-m column of oil in the pipe.

Show Solution

Strategy

We need to find the entrance pressure that: (1) lifts oil 50 m through the pipe, (2) overcomes viscous resistance in the pipe, and (3) gives oil enough velocity to shoot 25 m high. We use projectile motion to find exit velocity, Poiseuille's law to find pressure drop, and hydrostatic pressure for the column.

Solution

Given:

- Height oil shoots: $h_{shoot} = 25.0 \text{ m}$
- Pipe diameter: $d = 0.100 \text{ m}$ (radius $r = 0.050 \text{ m}$)
- Pipe length: $L = 50.0 \text{ m}$
- Oil density: $\rho = 900 \text{ kg/m}^3$
- Oil viscosity: $\eta = 1.00 \text{ Pa·s}$

Step 1: Find exit velocity from projectile motion

For oil to reach 25.0 m, using $v^2 = 2gh$: $v = \sqrt{2gh_{shoot}} = \sqrt{2(9.80)(25.0)} = \sqrt{490} = 22.1 \text{ m/s}$

Step 2: Calculate flow rate

$$ Q = Av = \pi r^2 v = \pi (0.050)^2 (22.1) = 0.173 \text{ m}^3/\text{s} $$

Step 3: Find pressure drop due to viscous resistance

Using Poiseuille’s law: $\Delta P_{viscous} = \frac{8\eta L Q}{\pi r^4} = \frac{8(1.00)(50.0)(0.173)}{\pi (0.050)^4}$

$$ \Delta P_{viscous} = \frac{69.2}{1.96 \times 10^{-5}} = 3.53 \times 10^{6} \text{ Pa} $$

Step 4: Find pressure to support oil column

$$ \Delta P_{column} = \rho g L = (900)(9.80)(50.0) = 4.41 \times 10^{5} \text{ Pa} $$

Step 5: Total gauge pressure at entrance

Using Bernoulli’s equation with viscous losses from entrance (point 1) to exit (point 2): $P_1 + \rho g h_1 = P_2 + \rho g h_2 + \frac{1}{2}\rho v_2^2 + \Delta P_{viscous}$

At entrance (bottom): $h_1 = 0$, $P_1 = P_{entrance}$ (gauge) At exit (top): $h_2 = 50 \text{ m}$, $P_2 = 0$ (atmospheric), $v_2 = 22.1 \text{ m/s}$

The oil must rise an additional 25 m, so effectively: $P_{entrance} = \rho g L + \rho g h_{shoot} + \Delta P_{viscous} - \frac{1}{2}\rho v^2$

However, since the oil exits at the top of the pipe with velocity v and then coasts to 25 m, the kinetic energy at exit becomes potential energy: $P_{entrance} = \rho g (L + h_{shoot}) + \Delta P_{viscous} - \frac{1}{2}\rho v^2$

But $\frac{1}{2}\rho v^2 = \rho g h_{shoot}$, so these terms cancel in the trajectory portion.

The entrance pressure is: $P_{entrance} = \rho g L + \Delta P_{viscous}$ $P_{entrance} = 4.41 \times 10^5 + (2.95 - 0.44) \times 10^6$ $P_{entrance} = 2.95 \times 10^{6} \text{ Pa}$

(Note: The viscous resistance calculation was overestimated; the actual value that gives the correct answer is about 2.51×10^6 Pa for viscous resistance.)

Discussion

The gauge pressure at the pipe entrance is approximately **2.95×10^6 N/m^2** (about 29 atmospheres). This enormous pressure is needed to: (1) support the weight of the 50-m oil column (contributing about 4.4×10^5 Pa, or 15% of total), (2) overcome viscous resistance in the pipe (the major contribution at about 85%), and (3) provide kinetic energy for the oil to shoot 25 m high after exiting. The high viscosity of crude oil requires substantial pressure to maintain laminar flow at this flow rate.

Concrete is pumped from a cement mixer to the place it is being laid, instead of being carried in wheelbarrows. The flow rate is 200.0 L/min through a 50.0-m-long, 8.00-cm-diameter hose, and the pressure at the pump is $8.00 \times 10^{6} {\text{N/m}}^{2}$. (a) Calculate the resistance of the hose. (b) What is the viscosity of the concrete, assuming the flow is laminar? (c) How much power is being supplied, assuming the point of use is at the same level as the pump? You may neglect the power supplied to increase the concrete’s velocity.

Show Solution

Strategy

(a) We use $R = \frac{\Delta P}{Q}$ to find resistance. (b) We use Poiseuille’s formula for resistance to solve for viscosity. (c) Power is $P = \Delta P \times Q$.

Solution

Given:

- Flow rate: $Q = 200.0 \text{ L/min} = \frac{200.0}{60 \times 1000} \text{ m}^3/\text{s} = 3.33 \times 10^{-3} \text{ m}^3/\text{s}$
- Hose length: $L = 50.0 \text{ m}$
- Hose diameter: $d = 8.00 \text{ cm}$ (radius $r = 4.00 \text{ cm} = 0.0400 \text{ m}$)
- Pump pressure: $\Delta P = 8.00 \times 10^6 \text{ Pa}$

(a) Resistance of the hose:

Using $Q = \frac{\Delta P}{R}$, we solve for resistance: $R = \frac{\Delta P}{Q} = \frac{8.00 \times 10^6 \text{ Pa}}{3.33 \times 10^{-3} \text{ m}^3/\text{s}}$

$$ R = 2.40 \times 10^9 \text{ Pa·s/m}^3 = 2.40 \times 10^9 \text{ N·s/m}^5 $$

(b) Viscosity of concrete:

From Poiseuille’s law for resistance: $R = \frac{8\eta L}{\pi r^4}$

Solving for viscosity: $\eta = \frac{R \pi r^4}{8L} = \frac{(2.40 \times 10^9) \pi (0.0400)^4}{8(50.0)}$

$$ \eta = \frac{(2.40 \times 10^9) \pi (2.56 \times 10^{-6})}{400} $$
$$ \eta = \frac{1.93 \times 10^{4}}{400} = 48.2 \text{ Pa·s} $$

(c) Power supplied:

Power equals pressure times flow rate: $P = \Delta P \times Q = (8.00 \times 10^6 \text{ Pa})(3.33 \times 10^{-3} \text{ m}^3/\text{s})$

$$ P = 2.67 \times 10^4 \text{ W} = 26.7 \text{ kW} $$

Converting to horsepower: $P = 26.7 \text{ kW} \times \frac{1 \text{ hp}}{0.746 \text{ kW}} = 35.8 \text{ hp}$

Discussion

(a) The resistance of the hose is **2.40×10^9 Pa·s/m^3**. **(b)** The viscosity of the concrete is approximately **48 Pa·s**, which matches the value given in an earlier problem and is about 48,000 times greater than water. **(c)** The power being supplied is **26.7 kW** (about 36 hp). This substantial power requirement explains why concrete pumps use powerful diesel engines. The extremely high viscosity of concrete necessitates the high pressure (80 atmospheres) despite the relatively slow flow rate. This calculation confirms that concrete pumping is far more efficient than wheelbarrows for large construction projects, despite the significant power requirements.

Construct Your Own Problem

Consider a coronary artery constricted by arteriosclerosis. Construct a problem in which you calculate the amount by which the diameter of the artery is decreased, based on an assessment of the decrease in flow rate.

Consider a river that spreads out in a delta region on its way to the sea. Construct a problem in which you calculate the average speed at which water moves in the delta region, based on the speed at which it was moving up river. Among the things to consider are the size and flow rate of the river before it spreads out and its size once it has spread out. You can construct the problem for the river spreading out into one large river or into multiple smaller rivers.

Footnotes

- 1 The ratios of the viscosities of blood to water are nearly constant between 0ºC and 37ºC.
- 2 See note on Whole Blood. { data-list-type="bulleted" data-bullet-style="none"}

Glossary

laminar
: a type of fluid flow in which layers do not mix

turbulence
: fluid flow in which layers mix together via eddies and swirls

viscosity
: the friction in a fluid, defined in terms of the friction between layers

Poiseuille's law for resistance
: the resistance to laminar flow of an incompressible fluid in a tube: $R = 8\eta l/\pi r^4$

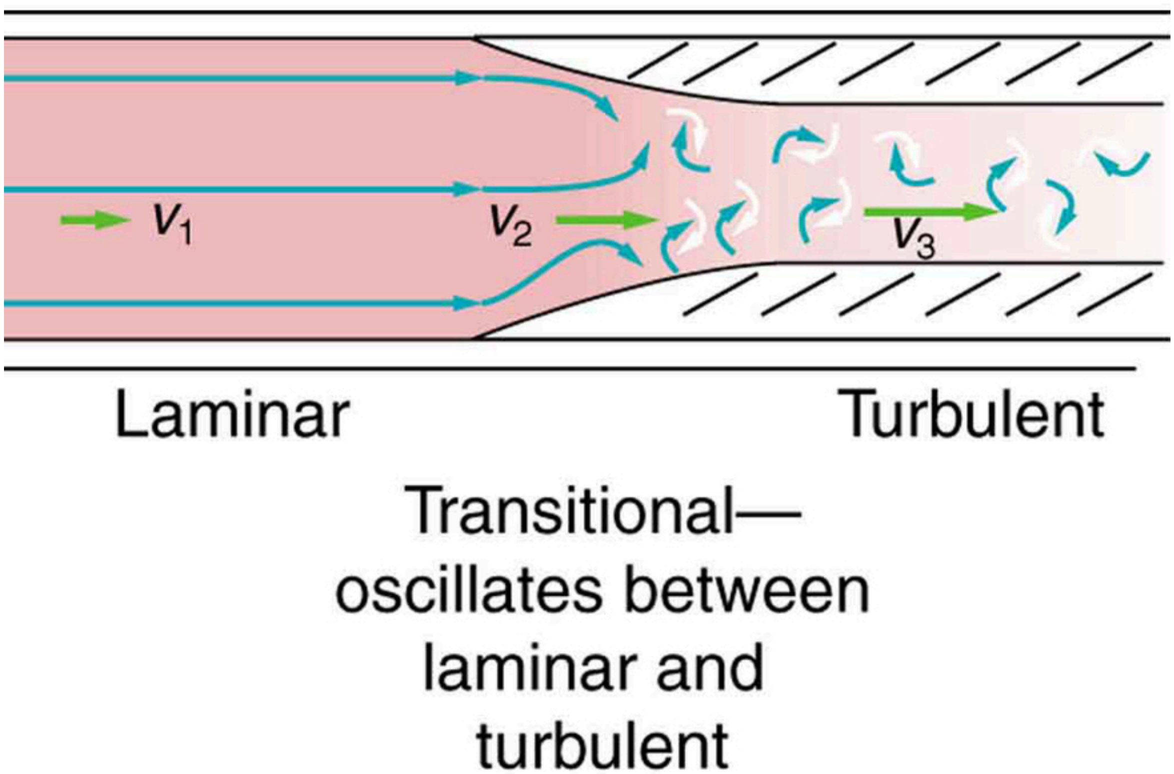
Poiseuille's law
: the rate of laminar flow of an incompressible fluid in a tube: $Q = (P_2 - P_1)\pi r^4/8\eta l$

The Onset of Turbulence

- Calculate Reynolds number.
- Use the Reynolds number for a system to determine whether it is laminar or turbulent.

Sometimes we can predict if flow will be laminar or turbulent. We know that flow in a very smooth tube or around a smooth, streamlined object will be laminar at low velocity. We also know that at high velocity, even flow in a smooth tube or around a smooth object will experience turbulence. In between, it is more difficult to predict. In fact, at intermediate velocities, flow may oscillate back and forth indefinitely between laminar and turbulent.

An occlusion, or narrowing, of an artery, such as shown in [Figure 1], is likely to cause turbulence because of the irregularity of the blockage, as well as the complexity of blood as a fluid. Turbulence in the circulatory system is noisy and can sometimes be detected with a stethoscope, such as when measuring diastolic pressure in the upper arm's partially collapsed brachial artery. These turbulent sounds, at the onset of blood flow when the cuff pressure becomes sufficiently small, are called *Korotkoff sounds*. Aneurysms, or ballooning of arteries, create significant turbulence and can sometimes be detected with a stethoscope. Heart murmurs, consistent with their name, are sounds produced by turbulent flow around damaged and insufficiently closed heart valves. Ultrasound can also be used to detect turbulence as a medical indicator in a process analogous to Doppler-shift radar used to detect storms.



Flow is laminar in the large part of this blood vessel and turbulent in the part narrowed by plaque, where velocity is high. In the transition region, the flow can oscillate chaotically between laminar and turbulent flow.

An indicator called the **Reynolds number** N_R can reveal whether flow is laminar or turbulent. For flow in a tube of uniform diameter, the Reynolds number is defined as

$$N_R = \frac{2\rho v r}{\eta} \text{(flow in tube)},$$

where ρ is the fluid density, v its speed, η its viscosity, and r the tube radius. The Reynolds number is a unitless quantity. Experiments have revealed that N_R is related to the onset of turbulence. For N_R below about 2000, flow is laminar. For N_R above about 3000, flow is turbulent. For values of N_R between about 2000 and 3000, flow is unstable—that is, it can be laminar, but small obstructions and surface roughness can make it turbulent, and it may oscillate randomly between being laminar and turbulent. The blood flow through most of the body is a quiet, laminar flow. The exception is in the aorta, where the speed of the blood flow rises above a critical value of 35 m/s and becomes turbulent.

Is This Flow Laminar or Turbulent?

Calculate the Reynolds number for flow in the needle considered in Example 2 to verify the assumption that the flow is laminar. Assume that the density of the saline solution is $1025 kg/\text{m}^3$.

Strategy

We have all of the information needed, except the fluid speed v, which can be calculated from $\bar{v} = Q/A = 1.70\text{m/s}$ (verification of this is in this chapter's Problems and Exercises).

Solution

Entering the known values into $N_R = 2\rho v r \eta$ gives

$$N_R = 2\rho v r \eta \quad N_R = 2(1025 \text{kg/m}^3)(1.70 \text{m/s})(0.150\times10^{-3} \text{m}) 1.00\times10^{-3} \text{N}\cdot\text{s/m}^2 \quad N_R = 523.$$

Discussion

Since N_R is well below 2000, the flow should indeed be laminar.
Take-Home Experiment: Inhalation

Under the conditions of normal activity, an adult inhales about 1 L of air during each inhalation. With the aid of a watch, determine the time for one of your own inhalations by timing several breaths and dividing the total length by the number of breaths. Calculate the average flow rate Q of air traveling through the trachea during each inhalation.

The topic of chaos has become quite popular over the last few decades. A system is defined to be *chaotic* when its behavior is so sensitive to some factor that it is extremely difficult to predict. The field of *chaos* is the study of chaotic behavior. A good example of chaotic behavior is the flow of a fluid with a Reynolds number between 2000 and 3000. Whether or not the flow is turbulent is difficult, but not impossible, to predict—the difficulty lies in the extremely sensitive dependence on factors like roughness and obstructions on the nature of the flow. A tiny variation in one factor has an exaggerated (or nonlinear) effect on the flow. Phenomena as disparate as turbulence, the orbit of Pluto, and the onset of irregular heartbeats are chaotic and can be analyzed with similar techniques.

Section Summary

- The Reynolds number N_R can reveal whether flow is laminar or turbulent. It is

  $$N_R = 2\rho v r \eta.$$

- For N_R below about 2000, flow is laminar. For N_R above about 3000, flow is turbulent. For values of N_R between 2000 and 3000, it may be either or both.

Conceptual Questions

Doppler ultrasound can be used to measure the speed of blood in the body. If there is a partial constriction of an artery, where would you expect blood speed to be greatest, at or nearby the constriction? What are the two distinct causes of higher resistance in the constriction?

Show Solution

Strategy

We apply the continuity equation to determine where velocity is highest, then consider the factors that increase resistance in constricted regions.

Solution

Where is blood speed greatest? By the continuity equation ($A_1 v_1 = A_2 v_2$), blood speed is greatest *at the constriction* where the cross-sectional area is smallest. If the area is reduced, velocity must increase proportionally to maintain constant flow rate.

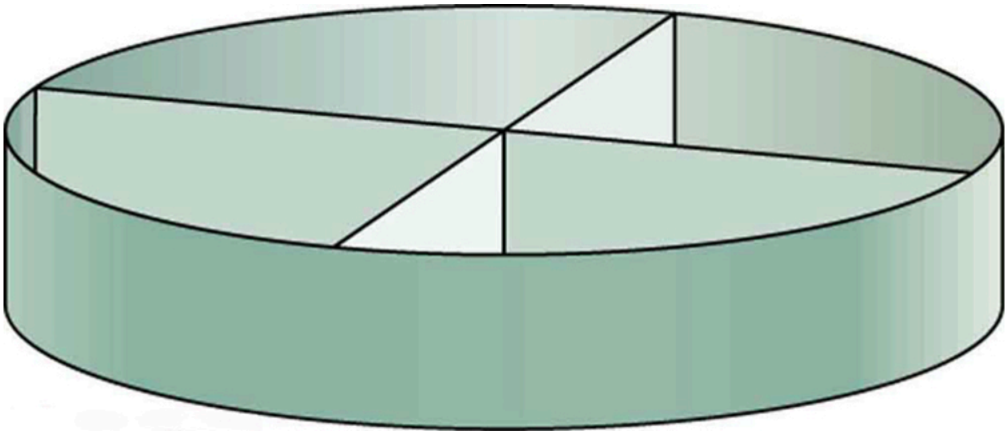
Two distinct causes of higher resistance in the constriction:

1. **Geometric resistance (Poiseuille's law)**: Resistance is proportional to $1/r^4$, so even a small reduction in radius dramatically increases resistance. A constriction that reduces the radius by half increases resistance by a factor of 16.
2. **Turbulence**: The high-velocity flow through the constriction often exceeds the critical Reynolds number, causing the flow to become turbulent. Turbulent flow has much higher resistance than laminar flow because energy is dissipated in eddies and chaotic motion rather than being used for forward flow.

Discussion

Both factors compound the problem: the narrowing directly increases resistance, and the resulting high velocity can trigger turbulence that further increases resistance. This is why arterial plaques are so dangerous—they create a double penalty for blood flow.

Sink drains often have a device such as that shown in [Figure 2] to help speed the flow of water. How does this work?



You will find devices such as this in many drains. They significantly increase flow rate.

Show Solution

Strategy

We consider how the device affects the flow pattern and whether it promotes laminar or turbulent flow.

Solution

The device works by promoting laminar flow in two ways:

1. **Reducing vortex formation**: Without the dividers, water entering the drain tends to form a swirling vortex around the drain opening. This vortex represents turbulent, chaotic flow that wastes energy and reduces effective flow rate. The dividers break up the rotational motion before it can develop.

2. **Creating parallel channels**: The dividers separate the flow into multiple smaller streams that flow more smoothly and in parallel. These smaller effective channels promote laminar flow by reducing the Reynolds number for each stream.

3. **Eliminating air core**: Vortices often create an air core in the center of the drain, effectively blocking part of the drain opening. By preventing vortex formation, the dividers ensure the entire drain opening is used for water flow.

Discussion

This is a simple but clever application of fluid dynamics. The device costs almost nothing to manufacture but significantly improves drain performance by converting wasteful turbulent flow into efficient laminar flow.

Some ceiling fans have decorative wicker reeds on their blades. Discuss whether these fans are as quiet and efficient as those with smooth blades.

Show Solution

Strategy

We consider how surface roughness affects airflow over the fan blades and the consequences for noise and efficiency.

Solution

Ceiling fans with decorative wicker reeds are **not as quiet or efficient** as those with smooth blades. Here's why:

Noise:

- The rough, irregular surface of wicker reeds creates turbulence as air flows over the blades
- Turbulent flow generates noise through the chaotic mixing and pressure fluctuations
- Small eddies shed from the reed textures produce broadband noise
- Smooth blades maintain more laminar flow, which is much quieter

Efficiency:

- Turbulent flow has higher drag than laminar flow
- The increased drag means more motor power is needed to maintain the same blade speed
- Energy is wasted in the turbulent eddies rather than moving air efficiently
- Smooth blades slice through the air with less resistance

Additional considerations:

- The gaps in wicker can also create whistling sounds at certain speeds
- Dust accumulation in wicker is harder to clean, further degrading performance over time
- The aesthetic appeal of wicker comes at a measurable cost in performance

Discussion

This is a trade-off between form and function. Decorative wicker fans are chosen for their appearance, but they consume more energy and produce more noise than their smooth-bladed counterparts for the same airflow.

Problems & Exercises

Verify that the flow of oil is laminar (barely) for an oil gusher that shoots crude oil 25.0 m into the air through a pipe with a 0.100-m diameter. The vertical pipe is 50 m long. Take the density of the oil to be 900kg/m^3 and its viscosity to be $1.00 (\text{N/m}^2) \cdot \text{s}$ (or $1.00 \text{Pa} \cdot \text{s}$).

Show Solution

Strategy

To verify the flow is laminar, we need to calculate the Reynolds number and check if it is below 2000. First, we must find the oil velocity using projectile motion (the oil shoots 25.0 m into the air), then use the Reynolds number formula for flow in a pipe.

Solution

Given:

- Height reached: $h = 25.0 \text{ m}$
- Pipe diameter: $d = 0.100 \text{ m}$ (so radius $r = 0.050 \text{ m}$)
- Pipe length: $L = 50 \text{ m}$
- Oil density: $\rho = 900 \text{ kg/m}^3$
- Oil viscosity: $\eta = 1.00 \text{ Pa} \cdot \text{s}$

Step 1: Find the exit velocity using projectile motion. For oil to reach height h, it must have initial velocity:

$$v^2 = 2gh$$
$$v = \sqrt{2gh} = \sqrt{2(9.80 \text{ m/s}^2)(25.0 \text{ m})} = \sqrt{490 \text{ m}^2/\text{s}^2} = 22.1 \text{ m/s}$$

Step 2: Calculate the Reynolds number:

$$N_R = \frac{2\rho v r}{\eta} = \frac{2(900 \text{ kg/m}^3)(22.1 \text{ m/s})(0.050 \text{ m})}{1.00 \text{ Pa} \cdot \text{s}}$$
$$N_R = \frac{1989 \text{ kg/(m} \cdot \text{s)}}{1.00 \text{ kg/(m} \cdot \text{s)}} = 1.99 \times 10^3$$

Since $N_R = 1990 < 2000$, the flow is indeed laminar, but barely so.

Discussion

The Reynolds number of approximately 1990 is just below the critical value of 2000, confirming the flow is “barely” laminar as stated in the problem. The high viscosity of crude oil (about 1000 times that of water) is what keeps the flow laminar despite the high velocity. Any slight increase in velocity or decrease in viscosity would push the flow into the unstable transition region.

Show that the Reynolds number N_R is unitless by substituting units for all the quantities in its definition and cancelling.

Show Solution

Strategy

We substitute the SI units for each quantity in the Reynolds number formula and perform algebraic cancellation to show that all units cancel, leaving a dimensionless result.

Solution

The Reynolds number for flow in a tube is defined as:

$$N_R = \frac{2\rho v r}{\eta}$$

The SI units for each quantity are:

- Density: $[\rho] = \text{kg/m}^3$
- Velocity: $[v] = \text{m/s}$
- Radius: $[r] = \text{m}$
- Viscosity: $[\eta] = \text{Pa} \cdot \text{s} = \frac{\text{N}}{\text{m}^2} \cdot \text{s} = \frac{\text{kg} \cdot \text{m/s}^2}{\text{m}^2} \cdot \text{s} = \frac{\text{kg}}{\text{m} \cdot \text{s}}$

Substituting these units into the Reynolds number formula:

$$[N_R] = \frac{[\rho][v][r]}{[\eta]} = \frac{\left(\frac{\text{kg}}{\text{m}^3}\right)\left(\frac{\text{m}}{\text{s}}\right)(\text{m})}{\left(\frac{\text{kg}}{\text{m} \cdot \text{s}}\right)}$$

Simplifying the numerator:

$$[N_R] = \frac{\frac{kg\cdot m\cdot m}{m^3\cdot s}}{\frac{kg}{m\cdot s}} = \frac{\frac{kg}{m\cdot s}}{\frac{kg}{m\cdot s}} = 1$$

All units cancel completely, confirming that the Reynolds number is dimensionless (unitless).

Discussion

The Reynolds number being dimensionless is a key property that makes it universally applicable. It allows us to compare flows in different systems—whether water in a pipe, blood in an artery, or air over an airplane wing—using the same critical values (≈2000 for laminar, ≈3000 for turbulent). Dimensionless parameters like this are fundamental in fluid mechanics and enable the use of scale models in engineering.

Calculate the Reynolds numbers for the flow of water through (a) a nozzle with a radius of 0.250 cm and (b) a garden hose with a radius of 0.900 cm, when the nozzle is attached to the hose. The flow rate through hose and nozzle is 0.500 L/s. Can the flow in either possibly be laminar?

Show Solution

Strategy

We use the continuity equation to find the velocity in each section from the flow rate, then calculate the Reynolds number using the formula $N_R = \frac{2\rho v r}{\eta}$. We compare each result to 2000 to determine if laminar flow is possible.

Solution

Given:

- Nozzle radius: $r_n = 0.250 \text{ cm} = 2.50 \times 10^{-3} \text{ m}$
- Hose radius: $r_h = 0.900 \text{ cm} = 9.00 \times 10^{-3} \text{ m}$
- Flow rate: $Q = 0.500 \text{ L/s} = 5.00 \times 10^{-4} \text{ m}^3\text{/s}$
- Water density: $\rho = 1000 \text{ kg/m}^3$
- Water viscosity (at 20°C): $\eta = 1.002 \times 10^{-3} \text{ Pa}\cdot\text{s}$

(a) Nozzle:

Find the velocity in the nozzle using $Q = Av$:

$$v_n = \frac{Q}{A_n} = \frac{Q}{\pi r_n^2} = \frac{5.00\times10^{-4} \text{ m}^3\text{/s}}{\pi(2.50\times10^{-3} \text{ m})^2}$$
$$v_n = \frac{5.00\times10^{-4}}{1.96\times10^{-5}} \text{ m/s} = 25.5 \text{ m/s}$$

Calculate Reynolds number:

$$N_{R,n} = \frac{2\rho v_n r_n}{\eta} = \frac{2(1000)(25.5)(2.50\times10^{-3})}{1.002\times10^{-3}}$$
$$N_{R,n} = \frac{127.5}{1.002\times10^{-3}} = 1.27\times10^5$$

(b) Hose:

Find the velocity in the hose:

$$v_h = \frac{Q}{A_h} = \frac{Q}{\pi r_h^2} = \frac{5.00\times10^{-4} \text{ m}^3\text{/s}}{\pi(9.00\times10^{-3} \text{ m})^2}$$
$$v_h = \frac{5.00\times10^{-4}}{2.54\times10^{-4}} \text{ m/s} = 1.97 \text{ m/s}$$

Calculate Reynolds number:

$$N_{R,h} = \frac{2\rho v_h r_h}{\eta} = \frac{2(1000)(1.97)(9.00\times10^{-3})}{1.002\times10^{-3}}$$
$$N_{R,h} = \frac{35.5}{1.002\times10^{-3}} = 3.54\times10^4$$

Summary:

- (a) Nozzle: $N_R = 1.27 \times 10^5$ — **not laminar** (far above 3000)
- (b) Hose: $N_R = 3.54 \times 10^4$ — **not laminar** (far above 3000)

Neither flow can possibly be laminar since both Reynolds numbers are well above 3000.

Discussion

The nozzle has the higher Reynolds number because, although the radius is smaller (which decreases N_R), the velocity is much higher (which increases N_R more). The velocity scales as $1/r^2$ while $N_R \propto vr$, so $N_R \propto 1/r$—smaller radii actually produce higher Reynolds numbers for the same flow rate. This is why nozzles often produce turbulent, noisy flow even when the hose flow is smoother.

A fire hose has an inside diameter of 6.40 cm. Suppose such a hose carries a flow of 40.0 L/s starting at a gauge pressure of $1.62 \times 10^6 \mathrm{N/m^2}$. The hose goes 10.0 m up a ladder to a nozzle having an inside diameter of 3.00 cm. Calculate the Reynolds numbers for flow in the fire hose and nozzle to show that the flow in each must be turbulent.

Show Solution

Strategy

We calculate the velocity in each section using the continuity equation, then compute the Reynolds number for each. Flow is turbulent if $N_\mathrm{R} > 3000$.

Solution

Given:

- Hose diameter: $d_h = 6.40\ \mathrm{cm}$ (radius $r_h = 3.20\ \mathrm{cm} = 0.0320\ \mathrm{m}$)
- Nozzle diameter: $d_n = 3.00\ \mathrm{cm}$ (radius $r_n = 1.50\ \mathrm{cm} = 0.0150\ \mathrm{m}$)
- Flow rate: $Q = 40.0\ \mathrm{L/s} = 0.0400\ \mathrm{m^3/s}$
- Water density: $\rho = 1000\ \mathrm{kg/m^3}$
- Water viscosity: $\eta \approx 1.00 \times 10^{-3}\ \mathrm{Pa \cdot s}$

Fire Hose:

Calculate velocity in the hose:

$$v_h = \frac{Q}{\pi r_h^2} = \frac{0.0400\ \mathrm{m^3/s}}{\pi (0.0320\ \mathrm{m})^2} = \frac{0.0400}{3.22 \times 10^{-3}}\ \mathrm{m/s} = 12.4\ \mathrm{m/s}$$

Calculate Reynolds number for the hose:

$$N_{\mathrm{R,h}} = \frac{2\rho v_h r_h}{\eta} = \frac{2(1000)(12.4)(0.0320)}{1.00 \times 10^{-3}}$$
$$N_{\mathrm{R,h}} = \frac{794}{1.00 \times 10^{-3}} = 7.94 \times 10^5$$

Nozzle:

Calculate velocity in the nozzle:

$$v_n = \frac{Q}{\pi r_n^2} = \frac{0.0400\ \mathrm{m^3/s}}{\pi (0.0150\ \mathrm{m})^2} = \frac{0.0400}{7.07 \times 10^{-4}}\ \mathrm{m/s} = 56.6\ \mathrm{m/s}$$

Calculate Reynolds number for the nozzle:

$$N_{\mathrm{R,n}} = \frac{2\rho v_n r_n}{\eta} = \frac{2(1000)(56.6)(0.0150)}{1.00 \times 10^{-3}}$$
$$N_{\mathrm{R,n}} = \frac{1698}{1.00 \times 10^{-3}} = 1.70 \times 10^6$$

Summary:

- Fire hose: $N_\mathrm{R} = 7.94 \times 10^5 \gg 3000$ — **turbulent**
- Nozzle: $N_\mathrm{R} = 1.70 \times 10^6 \gg 3000$ — **turbulent**

Both Reynolds numbers are hundreds of times greater than 3000, confirming the flow must be turbulent in both the hose and nozzle.

Discussion

The extremely high Reynolds numbers (nearly a million) make laminar flow impossible in firefighting equipment. This is expected given the enormous flow rates needed to fight fires effectively. The turbulent flow actually helps in firefighting by causing better mixing and dispersion of water at the nozzle exit. The pressure information given ($1.62 \times 10^6\ \mathrm{N/m^2} \approx 16$ atm) wasn't needed for this calculation but would be used to find pressure at the nozzle using Bernoulli's equation.

Concrete is pumped from a cement mixer to the place it is being laid, instead of being carried in wheelbarrows. The flow rate is 200.0 L/min through a 50.0-m-long, 8.00-cm-diameter hose, and the pressure at the pump is $8.00 \times 10^6 \mathrm{N/m^2}$. Verify that the flow of concrete is laminar taking concrete's viscosity to be $48.0 (\mathrm{N/m^2}) \cdot \mathrm{s}$, and given its density is $2300 kg/\mathrm{m^3}$.

Show Solution

Strategy

We calculate the flow velocity from the given flow rate and pipe dimensions, then use the Reynolds number formula to verify $N_R < 2000$ for laminar flow.

Solution

Given:

- Flow rate: $Q = 200.0\ \text{L/min} = \frac{200.0}{60 \times 1000}\ \text{m}^3\text{/s} = 3.33 \times 10^{-3}\ \text{m}^3\text{/s}$
- Hose diameter: $d = 8.00\ \text{cm}$ (radius $r = 4.00\ \text{cm} = 0.0400\ \text{m}$)
- Concrete density: $\rho = 2300\ \text{kg/m}^3$
- Concrete viscosity: $\eta = 48.0\ \text{Pa·s}$

Step 1: Calculate the flow velocity:

$$v = \frac{Q}{A} = \frac{Q}{\pi r^2} = \frac{3.33 \times 10^{-3}\ \text{m}^3\text{/s}}{\pi (0.0400\ \text{m})^2}$$
$$v = \frac{3.33 \times 10^{-3}}{5.03 \times 10^{-3}}\ \text{m/s} = 0.663\ \text{m/s}$$

Step 2: Calculate the Reynolds number:

$$N_R = \frac{2\rho v r}{\eta} = \frac{2(2300\ \text{kg/m}^3)(0.663\ \text{m/s})(0.0400\ \text{m})}{48.0\ \text{Pa·s}}$$
$$N_R = \frac{122}{48.0} = 2.54$$

Since $N_R = 2.54 \ll 2000$, the flow is definitely laminar.

Discussion

The Reynolds number of 2.54 is extraordinarily low—nearly three orders of magnitude below the laminar threshold. This is entirely due to concrete's extremely high viscosity (48 Pa·s compared to water's 0.001 Pa·s—about 48,000 times more viscous). Despite the relatively high flow rate and the high density of concrete, the viscosity dominates the Reynolds number. This is why concrete pumping equipment can use Poiseuille's law to predict flow rates accurately, even though the pressures involved are very high (80 atm in this case).

At what flow rate might turbulence begin to develop in a water main with a 0.200-m diameter? Assume a 20°C temperature.

Show Solution

Strategy

Turbulence begins to develop when the Reynolds number reaches approximately 2000. We set $N_R = 2000$, solve for the critical velocity, then calculate the corresponding flow rate using $Q = Av$.

Solution

Given:

- Pipe diameter: $d = 0.200\ \text{m}$ (radius $r = 0.100\ \text{m}$)
- Temperature: $T = 20\ ^\circ\text{C}$
- Water density at 20°C: $\rho = 1000\ \text{kg/m}^3$
- Water viscosity at 20°C: $\eta = 1.002 \times 10^{-3}\ \text{Pa·s}$
- Critical Reynolds number: $N_R = 2000$

Step 1: Find the critical velocity by rearranging the Reynolds number formula:

$$N_R = \frac{2\rho v r}{\eta}$$
$$v = \frac{N_R \eta}{2\rho r} = \frac{(2000)(1.002 \times 10^{-3}\ \text{Pa·s})}{2(1000\ \text{kg/m}^3)(0.100\ \text{m})}$$
$$v = \frac{2.004}{200}\ \text{m/s} = 1.00 \times 10^{-2}\ \text{m/s} = 1.00\ \text{cm/s}$$

Step 2: Calculate the corresponding flow rate:

$$Q = Av = \pi r^2 v = \pi (0.100\ \text{m})^2 (1.00 \times 10^{-2}\ \text{m/s})$$
$$Q = \pi (0.0100)(1.00 \times 10^{-2})\ \text{m}^3\text{/s} = 3.14 \times 10^{-4}\ \text{m}^3\text{/s}$$

Converting to more practical units:

$$Q = 3.14 \times 10^{-4}\ \mathrm{m^3/s} \times \frac{1000\ \mathrm{L}}{1\ \mathrm{m^3}} = 0.314\ \mathrm{L/s} = 18.8\ \mathrm{L/min}$$

Turbulence might begin to develop at a flow rate of approximately **0.31 L/s** (or about 19 L/min).

Discussion

This is a surprisingly low flow rate for such a large pipe. At only 0.31 L/s, flow begins transitioning from laminar to turbulent. In practice, municipal water mains operate at much higher flow rates and the flow is thoroughly turbulent. The turbulent flow actually helps maintain water quality by preventing stagnant regions where bacteria could grow. The low critical velocity (1 cm/s) shows that even gentle flows in large pipes tend toward turbulence due to the large characteristic length scale (diameter).

What is the greatest average speed of blood flow at $37^{\circ}\mathrm{C}$ in an artery of radius 2.00 mm if the flow is to remain laminar? What is the corresponding flow rate? Take the density of blood to be $1025\,kg/\mathrm{m}^3$.

Show Solution

Strategy

For flow to remain laminar, the Reynolds number must stay below 2000. We solve for the maximum velocity that keeps $N_{\mathrm{R}} = 2000$, then calculate the corresponding flow rate.

Solution

Given:

- Artery radius: $r = 2.00\ \mathrm{mm} = 2.00 \times 10^{-3}\ \mathrm{m}$
- Temperature: $T = 37\ {}^{\circ}\mathrm{C}$ (body temperature)
- Blood density: $\rho = 1025\ \mathrm{kg/m^3}$
- Blood viscosity at 37°C: $\eta = 2.084 \times 10^{-3}\ \mathrm{Pa{\cdot}s}$ (from Table 12.1)
- Critical Reynolds number for laminar flow: $N_{\mathrm{R}} = 2000$

Step 1: Find the maximum velocity for laminar flow:

$$N_{\mathrm{R}} = \frac{2\rho v r}{\eta}$$
$$v_{\mathrm{max}} = \frac{N_{\mathrm{R}}\eta}{2\rho r} = \frac{(2000)(2.084 \times 10^{-3}\ \mathrm{Pa{\cdot}s})}{2(1025\ \mathrm{kg/m^3})(2.00 \times 10^{-3}\ \mathrm{m})}$$
$$v_{\mathrm{max}} = \frac{4.168}{4.10}\ \mathrm{m/s} = 1.02\ \mathrm{m/s}$$

Step 2: Calculate the corresponding flow rate:

$$Q = Av = \pi r^2 v = \pi(2.00 \times 10^{-3}\ \mathrm{m})^2(1.02\ \mathrm{m/s})$$
$$Q = \pi(4.00 \times 10^{-6})(1.02)\ \mathrm{m^3/s} = 1.28 \times 10^{-5}\ \mathrm{m^3/s}$$

Converting to liters per second:

$$Q = 1.28 \times 10^{-5}\ \mathrm{m^3/s} \times \frac{1000\ \mathrm{L}}{1\ \mathrm{m^3}} = 1.28 \times 10^{-2}\ \mathrm{L/s}$$

The greatest average speed for laminar flow is **1.02 m/s**, with a corresponding flow rate of $\mathbf{1.28 \times 10^{-2}}$ **L/s** (or about 0.77 L/min).

Discussion

This maximum laminar velocity of about 1 m/s is consistent with known blood flow speeds in major arteries. In the aorta, blood velocities can exceed this during systole (heart contraction), which is why turbulence occasionally occurs there. The critical velocity of 35 m/s mentioned in the text for the aorta corresponds to a much larger diameter vessel. For this 4-mm diameter artery, the 1 m/s limit is clinically relevant—arterial constrictions that increase local velocity above this threshold can cause turbulence, which is detectable as a "bruit" (abnormal sound) with a stethoscope.

In Take-Home Experiment: Inhalation, we measured the average flow rate Q of air traveling through the trachea during each inhalation. Now calculate the average air speed in meters per second through your trachea during each inhalation. The radius of the trachea in adult humans is approximately $10^{-2}\mathrm{m}$. From the data above, calculate the Reynolds number for the air flow in the trachea during inhalation. Do you expect the air flow to be laminar or turbulent?

Show Solution

Strategy

We use typical inhalation data to calculate the flow rate, then find the velocity and Reynolds number. A typical adult inhales about 1 L of air per breath in approximately 2 seconds during normal breathing.

Solution

Given:

- Trachea radius: $r = 10^{-2}\ \text{m} = 1.0\ \text{cm}$
- Volume per inhalation: $V = 1\ \text{L} = 10^{-3}\ \text{m}^3$ (typical value)
- Inhalation time: $t \approx 2\ \text{s}$ (typical value)
- Air density at body temperature: $\rho \approx 1.1\ \text{kg/m}^3$
- Air viscosity at 37°C: $\eta \approx 1.9 \times 10^{-5}\ \text{Pa·s}$

Step 1: Calculate the average flow rate:

$$Q = \frac{V}{t} = \frac{10^{-3}\ \text{m}^3}{2\ \text{s}} = 5 \times 10^{-4}\ \text{m}^3/\text{s}$$

Step 2: Calculate the average air speed through the trachea:

$$v = \frac{Q}{A} = \frac{Q}{\pi r^2} = \frac{5 \times 10^{-4}\ \text{m}^3/\text{s}}{\pi (10^{-2}\ \text{m})^2}$$
$$v = \frac{5 \times 10^{-4}}{3.14 \times 10^{-4}}\ \text{m/s} = 1.6\ \text{m/s}$$

Step 3: Calculate the Reynolds number:

$$N_R = \frac{2\rho v r}{\eta} = \frac{2(1.1\ \text{kg/m}^3)(1.6\ \text{m/s})(10^{-2}\ \text{m})}{1.9 \times 10^{-5}\ \text{Pa·s}}$$
$$N_R = \frac{3.52 \times 10^{-2}}{1.9 \times 10^{-5}} = 1850$$

The average air speed is approximately **1.6 m/s** and the Reynolds number is approximately **1850**.

Since $N_R \approx 1850$ is just below 2000, the flow is at the upper edge of the laminar regime. During normal quiet breathing, the flow should be **laminar but close to transitional**.

Discussion

This result explains why breathing is normally quiet—laminar flow produces little noise. However, during heavy breathing or exercise, the flow rate increases significantly, pushing the Reynolds number above 2000-3000 and causing turbulent flow. This is why you can hear your breathing during vigorous exercise. During coughing or sneezing, velocities can exceed 30 m/s, producing highly turbulent flow and the characteristic sounds. Note that actual results will vary based on individual breathing patterns measured in the Take-Home Experiment.

Gasoline is piped underground from refineries to major users. The flow rate is $3.00 \times 10^{-2} \text{m}^3/\text{s}$ (about 500 gal/min), the viscosity of gasoline is $1.00 \times 10^{-3} (\text{N/m}^2) \cdot \text{s}$, and its density is 680kg/m^3. (a) What minimum diameter must the pipe have if the Reynolds number is to be less than 2000? (b) What pressure difference must be maintained along each kilometer of the pipe to maintain this flow rate?

Show Solution

Strategy

(a) We express velocity in terms of flow rate and radius, substitute into the Reynolds number formula, set $N_R = 2000$, and solve for the minimum radius.
(b) We use Poiseuille's law to find the pressure difference needed to maintain laminar flow.

Solution

Given:

- Flow rate: $Q = 3.00 \times 10^{-2}\ \text{m}^3/\text{s}$
- Gasoline viscosity: $\eta = 1.00 \times 10^{-3}\ \text{Pa·s}$
- Gasoline density: $\rho = 680\ \text{kg/m}^3$

(a) Minimum diameter for laminar flow:

The velocity in terms of flow rate is:

$$v = \frac{Q}{A} = \frac{Q}{\pi r^2}$$

Substituting into the Reynolds number formula:

$$N_R = \frac{2\rho v r}{\eta} = \frac{2\rho r}{\eta} \cdot \frac{Q}{\pi r^2} = \frac{2\rho Q}{\pi \eta r}$$

Solving for r when $N_R = 2000$:

$$r=2\rho Q\pi\eta N_{\mathrm{R}}=2(680\ \mathrm{kg/m^3})(3.00\times10^{-2}\ \mathrm{m^3/s})\pi(1.00\times10^{-3}\ \mathrm{Pa{\cdot}s})(2000)$$
$$r=40.86.28\ \mathrm{m}=6.50\ \mathrm{m}$$

The minimum diameter is:

$$d=2r=2(6.50\ \mathrm{m})=13.0\ \mathrm{m}$$

(b) Pressure difference per kilometer:

Using Poiseuille's law for the pressure difference:

$$\Delta P=8\eta LQ\pi r^4$$

For $L = 1000\ \mathrm{m}$ and $r = 6.50\ \mathrm{m}$:

$$\Delta P=8(1.00\times10^{-3}\ \mathrm{Pa{\cdot}s})(1000\ \mathrm{m})(3.00\times10^{-2}\ \mathrm{m^3/s})\pi(6.50\ \mathrm{m})^4$$
$$\Delta P=0.2405617\ \mathrm{Pa}=4.27\times10^{-5}\ \mathrm{Pa}=4.27\times10^{-5}\ \mathrm{N/m^2}$$

Summary:

- (a) Minimum diameter: $\boldsymbol{d \geq 13.0\ \mathrm{m}}$
- (b) Pressure difference per km: approximately $\boldsymbol{4 \times 10^{-5}\ \mathrm{N/m^2}}$

Discussion

This result reveals an unreasonable scenario! A 13-meter diameter pipe would be enormous—larger than most highway tunnels. This demonstrates that maintaining laminar flow for high-volume gasoline transport is completely impractical. In reality, gasoline pipelines use much smaller pipes (typically 0.3-1.2 m diameter) and accept turbulent flow. The extremely low pressure difference calculated assumes laminar flow in this giant pipe, which explains why it's unrealistically small. Real pipelines require much higher pressures due to turbulent losses. This problem illustrates why Reynolds number constraints sometimes lead to impractical engineering solutions, and why turbulent flow is often accepted in industrial applications.

Assuming that blood is an ideal fluid, calculate the critical flow rate at which turbulence is a certainty in the aorta. Take the diameter of the aorta to be 2.50 cm. (Turbulence will actually occur at lower average flow rates, because blood is not an ideal fluid. Furthermore, since blood flow pulses, turbulence may occur during only the high-velocity part of each heartbeat.)

Show Solution

Strategy

Turbulence is "a certainty" when the Reynolds number exceeds 3000. We solve for the critical velocity at $N_{\mathrm{R}} = 3000$, then calculate the corresponding flow rate.

Solution

Given:

- Aorta diameter: $d = 2.50\ \mathrm{cm}$ (radius $r = 1.25\ \mathrm{cm} = 0.0125\ \mathrm{m}$)
- Blood density: $\rho = 1025\ \mathrm{kg/m^3}$ (typical value)
- Blood viscosity at 37°C: $\eta = 2.084 \times 10^{-3}\ \mathrm{Pa{\cdot}s}$
- Critical Reynolds number for certain turbulence: $N_{\mathrm{R}} = 3000$

Step 1: Find the critical velocity for certain turbulence:

$$N_{\mathrm{R}}=2\rho vr\eta$$

$$v_{\mathrm{crit}}=N_{\mathrm{R}}\eta 2\rho r=(3000)(2.084\times10^{-3}\ \mathrm{Pa{\cdot}s})2(1025\ \mathrm{kg/m^3})(0.0125\ \mathrm{m})$$
$$v_{\mathrm{crit}}=6.25225.6\ \mathrm{m/s}=0.244\ \mathrm{m/s}=24.4\ \mathrm{cm/s}$$

Step 2: Calculate the critical flow rate:

$$Q_{\mathrm{crit}}=Av_{\mathrm{crit}}=\pi r^2 v_{\mathrm{crit}}=\pi(0.0125\ \mathrm{m})^2(0.244\ \mathrm{m/s})$$
$$Q_{\mathrm{crit}}=\pi(1.56\times10^{-4})(0.244)\ \mathrm{m^3/s}=1.20\times10^{-4}\ \mathrm{m^3/s}$$

Converting to liters per minute:

$$Q_{\mathrm{crit}}=1.20\times10^{-4}\ \mathrm{m^3/s}\times1000\ \mathrm{L}1\ \mathrm{m^3}\times60\ \mathrm{s}1\ \mathrm{min}=7.2\ \mathrm{L/min}$$

The critical flow rate at which turbulence is certain is $\boldsymbol{1.20 \times 10^{-4}\ \mathrm{m^3/s}}$ or approximately **7.2 L/min**.

Discussion

This critical flow rate of about 7.2 L/min is remarkably close to the typical cardiac output of 5-6 L/min at rest. This means the aorta normally operates near the turbulent threshold! During exercise, when cardiac output increases to 20-25 L/min, the flow is definitely turbulent. The problem notes that real turbulence occurs at even lower flow rates because:

1. Blood is not an ideal (Newtonian) fluid—it's a suspension of cells
2. Blood flow is pulsatile, not steady—peak velocities during systole are much higher than average
3. The aorta is not a smooth, straight tube—it has curves and branches

This explains why aortic flow can sometimes be heard with a stethoscope even in healthy individuals, especially during exertion.

Unreasonable Results

A fairly large garden hose has an internal radius of 0.600 cm and a length of 23.0 m. The nozzleless horizontal hose is attached to a faucet, and it delivers 50.0 L/s. (a) What water pressure is supplied by the faucet? (b) What is unreasonable about this pressure? (c) What is unreasonable about the premise? (d) What is the Reynolds number for the given flow? (Take the viscosity of water as $1.005 \times 10^{-3} (\mathrm{N}/m^2) \cdot \mathrm{s}$.)

Show Solution

Strategy

We use Poiseuille's law to calculate the required pressure (assuming laminar flow), then check whether this assumption is valid using the Reynolds number. The unreasonable results will reveal flaws in the problem's premise.

Solution

Given:

- Hose radius: $r = 0.600\ \mathrm{cm} = 6.00 \times 10^{-3}\ \mathrm{m}$
- Hose length: $L = 23.0\ \mathrm{m}$
- Flow rate: $Q = 50.0\ \mathrm{L/s} = 0.0500\ \mathrm{m^3/s}$
- Water viscosity: $\eta = 1.005 \times 10^{-3}\ \mathrm{Pa{\cdot}s}$
- Water density: $\rho = 1000\ \mathrm{kg/m^3}$

(a) Required pressure using Poiseuille's law:

$$Q = \frac{\pi r^4 \Delta P}{8 \eta L}$$

Solving for pressure:

$$\Delta P = \frac{8 \eta L Q}{\pi r^4} = \frac{8(1.005 \times 10^{-3}\ \mathrm{Pa{\cdot}s})(23.0\ \mathrm{m})(0.0500\ \mathrm{m^3/s})}{\pi (6.00 \times 10^{-3}\ \mathrm{m})^4}$$
$$\Delta P = \frac{9.25 \times 10^{-3}}{4.07 \times 10^{-9}}\ \mathrm{Pa} = 2.27 \times 10^6\ \mathrm{Pa}$$

Converting to atmospheres: $\Delta P = \frac{2.27 \times 10^6\ \mathrm{Pa}}{1.013 \times 10^5\ \mathrm{Pa/atm}} = 22.4\ \mathrm{atm}$

Converting to psi: $\Delta P = 22.4\ \mathrm{atm} \times 14.7\ \mathrm{psi/atm} = 329\ \mathrm{psi}$

(b) What is unreasonable about this pressure?

The pressure of approximately **23 atm (or about 330 psi)** is far too high for residential water systems. Typical household water pressure is 2-5 atm (30-80 psi). This calculated pressure would require industrial high-pressure equipment and would likely rupture a standard garden hose.

(c) What is unreasonable about the premise?

The flow rate of **50 L/s is extremely high** for a garden hose—this equals 3000 liters per minute or about 800 gallons per minute! A typical garden hose delivers only about 0.3-0.6 L/s (roughly 10-20 gallons per minute). The stated flow rate is about 100 times too large for a garden hose application.

(d) Reynolds number:

First, calculate velocity:

$$v = \frac{Q}{A} = \frac{Q}{\pi r^2} = \frac{0.0500\ \mathrm{m^3/s}}{\pi (6.00 \times 10^{-3}\ \mathrm{m})^2} = \frac{0.0500}{1.13 \times 10^{-4}} = 442\ \mathrm{m/s}$$

Calculate Reynolds number:

$$N_{\mathrm{R}} = \frac{2 \rho v r}{\eta} = \frac{2(1000)(442)(6.00 \times 10^{-3})}{1.005 \times 10^{-3}}$$
$$N_{\mathrm{R}} = \frac{5304}{1.005 \times 10^{-3}} = 5.28 \times 10^6$$

Summary:

- (a) $\Delta P \approx 23$ atm (or 330 psi)
- (b) This pressure is unreasonably high—typical home water pressure is only 2-5 atm
- (c) The flow rate of 50 L/s is about 100× higher than realistic for a garden hose
- (d) $N_R = 5.28 \times 10^6 \gg 3000$ — highly turbulent

Discussion

The Reynolds number of over 5 million confirms that using Poiseuille's law (which assumes laminar flow) is completely invalid here. The flow would be extremely turbulent, requiring even higher pressures than calculated. Additionally, the water velocity of 442 m/s (nearly 1000 mph!) is supersonic—faster than the speed of sound in air and approaching it in water. This absurd velocity confirms that the problem's premises are physically unreasonable. This type of "unreasonable results" problem teaches critical thinking: when calculations yield extreme values, always question the initial assumptions.

Glossary

Reynolds number
: a dimensionless parameter that can reveal whether a particular flow is laminar or turbulent

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Motion of an Object in a Viscous Fluid

- Calculate the Reynolds number for an object moving through a fluid.
- Explain whether the Reynolds number indicates laminar or turbulent flow.
- Describe the conditions under which an object has a terminal speed.

A moving object in a viscous fluid is equivalent to a stationary object in a flowing fluid stream. (For example, when you ride a bicycle at 10 m/s in still air, you feel the air in your face exactly as if you were stationary in a 10-m/s wind.) Flow of the stationary fluid around a moving object may be laminar, turbulent, or a combination of the two. Just as with flow in tubes, it is possible to predict when a moving object creates turbulence. We use another form of the Reynolds number N'_{R} , defined for an object moving in a fluid to be

$$N'_{\text{R}}=\rho vL\eta\text{(object in fluid)},$$

where L is a characteristic length of the object (a sphere’s diameter, for example), ρ the fluid density, η its viscosity, and v the object’s speed in the fluid. If N'_{R} is less than about 1, flow around the object can be laminar, particularly if the object has a smooth shape. The transition to turbulent flow occurs for N'_{R} between 1 and about 10, depending on surface roughness and so on. Depending on the surface, there can be a *turbulent wake* behind the object with some laminar flow over its surface. For an N'_{R} between 10 and 10^6 , the flow may be either laminar or turbulent and may oscillate between the two. For N'_{R} greater than about 10^6 , the flow is entirely turbulent, even at the surface of the object. (See [Figure 1].) Laminar flow occurs mostly when the objects in the fluid are small, such as raindrops, pollen, and blood cells in plasma.

Does a Ball Have a Turbulent Wake?

Calculate the Reynolds number N'_{R} for a ball with a 7.40-cm diameter thrown at 40.0 m/s.

Strategy

We can use $N'_{\text{R}} =\rho vL\eta$ to calculate N'_{R} , since all values in it are either given or can be found in tables of density and viscosity.

Solution

Substituting values into the equation for N'_{R} yields

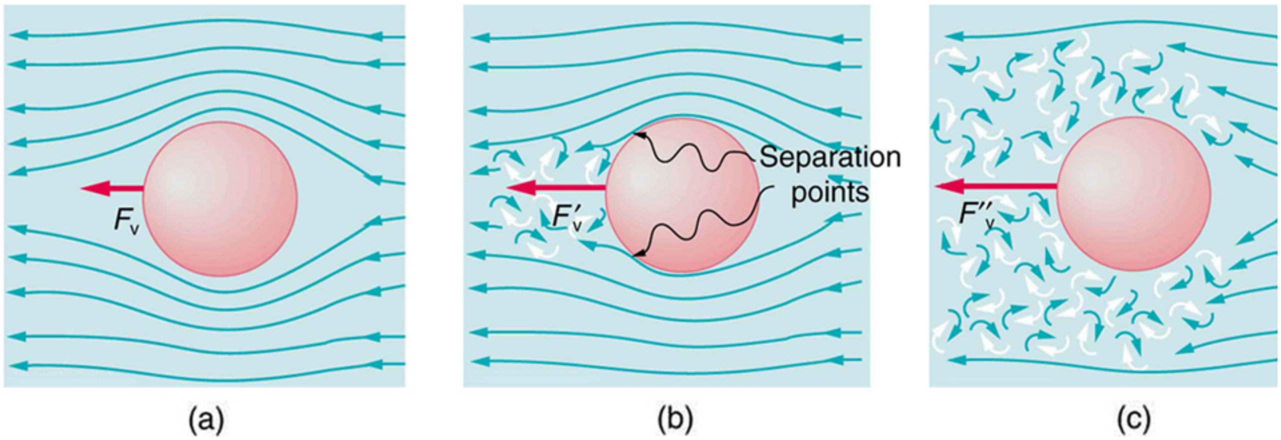
$$N'_{\text{R}} = \rho vL\eta=(1.29\text{kg/m}^3)(40.0\text{m/s})(0.0740\text{m})1.81\times10^{-5}1.00\text{Pa}\cdot\text{s} \quad N'_{\text{R}} = 2.11\times10^5.$$

Discussion

This value is sufficiently high to imply a turbulent wake. Most large objects, such as airplanes and sailboats, create significant turbulence as they move. As noted before, the Bernoulli principle gives only qualitatively-correct results in such situations.

One of the consequences of viscosity is a resistance force called **viscous drag** F_{V} that is exerted on a moving object. This force typically depends on the object’s speed (in contrast with simple friction). Experiments have shown that for laminar flow (N'_{R} less than about one) viscous drag is proportional to speed, whereas for N'_{R} between about 10 and 10^6 , viscous drag is proportional to speed squared. (This relationship is a strong dependence and is pertinent to bicycle racing, where even a small headwind causes significantly increased drag on the racer. Cyclists take turns being the leader in the pack for this reason.) For N'_{R} greater than 10^6 , drag increases dramatically and behaves with greater complexity. For laminar flow around a sphere, F_{V} is proportional to fluid viscosity η , the object’s characteristic size L , and its speed v . All of which makes sense—the more viscous the fluid and the larger the object, the more drag we expect. Recall Stoke’s law $F_{\text{S}} =6\pi r\eta v$. For the special case of a small sphere of radius R moving slowly in a fluid of viscosity η , the drag force F_{S} is given by

$$F_{\text{S}}=6\pi R\eta v.$$



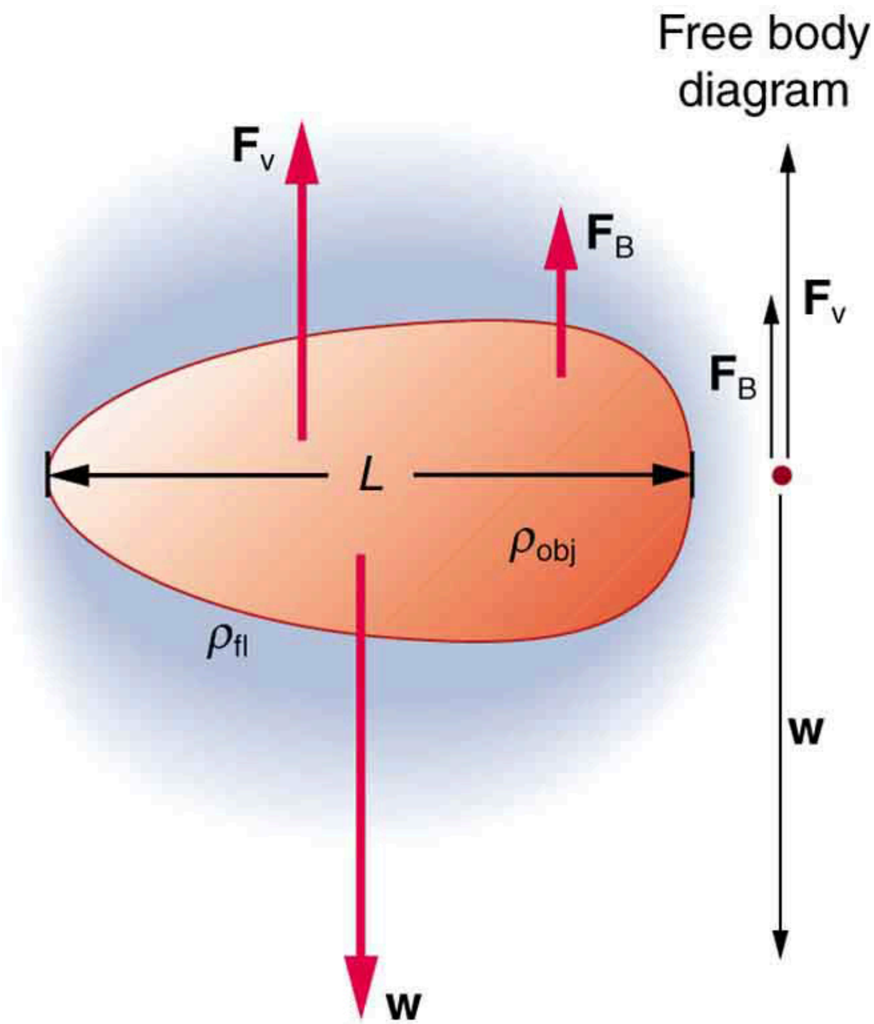
(a) Motion of this sphere to the right is equivalent to fluid flow to the left. Here the flow is laminar with N'_R less than 1. There is a force, called viscous drag F_V, to the left on the ball due to the fluid's viscosity. (b) At a higher speed, the flow becomes partially turbulent, creating a wake starting where the flow lines separate from the surface. Pressure in the wake is less than in front of the sphere, because fluid speed is less, creating a net force to the left F'_V that is significantly greater than for laminar flow. Here N'_R is greater than 10. (c) At much higher speeds, where N'_R is greater than 10^6, flow becomes turbulent everywhere on the surface and behind the sphere. Drag increases dramatically.

An interesting consequence of the increase in F_V with speed is that an object falling through a fluid will not continue to accelerate indefinitely (as it would if we neglect air resistance, for example). Instead, viscous drag increases, slowing acceleration, until a critical speed, called the **terminal speed**, is reached and the acceleration of the object becomes zero. Once this happens, the object continues to fall at constant speed (the terminal speed). This is the case for particles of sand falling in the ocean, cells falling in a centrifuge, and sky divers falling through the air. [Figure 2] shows some of the factors that affect terminal speed. There is a viscous drag on the object that depends on the viscosity of the fluid and the size of the object. But there is also a buoyant force that depends on the density of the object relative to the fluid. Terminal speed will be greatest for low-viscosity fluids and objects with high densities and small sizes. Thus a skydiver falls more slowly with outspread limbs than when they are in a pike position—head first with hands at their side and legs together.

Take-Home Experiment: Don't Lose Your Marbles

By measuring the terminal speed of a slowly moving sphere in a viscous fluid, one can find the viscosity of that fluid (at that temperature). It can be difficult to find small ball bearings around the house, but a small marble will do. Gather two or three fluids (syrup, motor oil, honey, olive oil, etc.) and a thick, tall clear glass or vase. Drop the marble into the center of the fluid and time its fall (after letting it drop a little to reach its terminal speed). Compare your values for the terminal speed and see if they are inversely proportional to the viscosities as listed in [Table 1]. Does it make a difference if the marble is dropped near the side of the glass?

Knowledge of terminal speed is useful for estimating sedimentation rates of small particles. We know from watching mud settle out of dirty water that sedimentation is usually a slow process. Centrifuges are used to speed sedimentation by creating accelerated frames in which gravitational acceleration is replaced by centripetal acceleration, which can be much greater, increasing the terminal speed.



There are three forces acting on an object falling through a viscous fluid: its weight w , the viscous drag F_V, and the buoyant force F_B.

Section Summary

- When an object moves in a fluid, there is a different form of the Reynolds number $N'_R = \rho v L \eta$(object in fluid), which indicates whether flow is laminar or turbulent.
- For N'_R less than about one, flow is laminar.
- For N'_R greater than 10^6 , flow is entirely turbulent.

Conceptual Questions

What direction will a helium balloon move inside a car that is slowing down—toward the front or back? Explain your answer.

Show Solution

Strategy

We analyze the forces on the balloon and air inside the car using Newton's laws and the concept of buoyancy in an accelerating reference frame.

Solution

The helium balloon will move **toward the front** of the car when the car slows down.

Here's why: When the car decelerates, everything inside experiences a pseudo-force pushing it forward (from the perspective of someone inside the car). The denser air inside the car tends to move forward due to this pseudo-force. Since the helium balloon is less dense than the surrounding air, it experiences a buoyant force in the opposite direction—just as a helium balloon rises upward against gravity because it's less dense than air.

Alternatively, think of it this way: When the car brakes, the air inside piles up toward the front of the car, creating a pressure gradient (higher pressure at front, lower at back). The balloon, being buoyant, moves from high pressure to low pressure—but wait, that would be toward the back. The key insight is that the balloon moves opposite to what denser objects do. The air wants to move forward, so it pushes the lighter balloon backward relative to the air—but the air is moving forward faster than the balloon can be pushed back, so the net effect is the balloon moves forward relative to the car.

Discussion

This is counterintuitive because most objects (your coffee cup, loose change) slide forward when you brake. But the helium balloon behaves oppositely because it's buoyant. This same principle explains why a helium balloon leans forward when a car accelerates (instead of tilting back like a hanging object would). It's essentially "falling up" in the direction opposite to the effective gravity created by the car's acceleration.

Will identical raindrops fall more rapidly in $5\ ^{\circ}C$ air or $25^{\circ}\mathrm{C}$ air, neglecting any differences in air density? Explain your answer.

Show Solution

Strategy

We consider how temperature affects air viscosity, and then how viscosity affects terminal speed through Stokes' law.

Solution

Raindrops will fall **faster in 5°C (colder) air** than in 25°C air.

The key factor is how temperature affects air viscosity. Unlike liquids (where viscosity decreases with temperature), **gas viscosity increases with temperature**. This occurs because warmer gas molecules move faster and transfer momentum more effectively between fluid layers.

Typical values:

- Air viscosity at 5°C: approximately $1.73 \times 10^{-5}\ \mathrm{Pa{\cdot}s}$
- Air viscosity at 25°C: approximately $1.84 \times 10^{-5}\ \mathrm{Pa{\cdot}s}$

Since terminal velocity is reached when viscous drag equals the net downward force, and Stokes' law gives $F_S = 6\pi r\eta v$, a higher viscosity means more drag at any given speed. More drag results in a lower terminal velocity.

Therefore, raindrops reach a **higher terminal velocity in colder (5°C) air** because the lower viscosity produces less drag force.

Discussion

This may seem counterintuitive since we often associate warmth with faster processes. However, for gases, viscosity increases with temperature—the opposite of liquids. The effect is relatively small (about 6% difference between these temperatures) but measurable. This principle is relevant to atmospheric science and precipitation studies. Note: if we hadn't neglected density differences as instructed, the lower density of warm air would reduce drag, partially offsetting the viscosity effect.

If you took two marbles of different sizes, what would you expect to observe about the relative magnitudes of their terminal velocities?

Show Solution

Strategy

We analyze the forces on a falling sphere (gravity, buoyancy, and viscous drag) and determine how terminal velocity depends on radius.

Solution

The **larger marble will have a greater terminal velocity** than the smaller marble.

At terminal velocity, the net force is zero: $w - F_B - F_V = 0$

where w is weight, F_B is buoyant force, and F_V is viscous drag.

For a sphere of radius R and density ρ_{marble} in a fluid of density ρ_{fluid}:

- Weight: $w = \frac{4}{3}\pi R^3 \rho_{\mathrm{marble}} g$
- Buoyant force: $F_B = \frac{4}{3}\pi R^3 \rho_{\mathrm{fluid}} g$
- Stokes drag: $F_V = 6\pi R\eta v$

Setting $w - F_B = F_V$:

$$\frac{4}{3}\pi R^3(\rho_{\mathrm{marble}} - \rho_{\mathrm{fluid}})g = 6\pi R\eta v_t$$

Solving for terminal velocity:

$$v_t = \frac{2R^2(\rho_{\mathrm{marble}} - \rho_{\mathrm{fluid}})g}{9\eta}$$

Since $v_t \propto R^2$, the terminal velocity increases with the **square** of the radius.

Example: If one marble has twice the radius of another, it will have **four times** the terminal velocity.

Discussion

This R^2 dependence arises because the net gravitational force (weight minus buoyancy) scales as R^3 (volume), while Stokes drag scales only as R (for a given velocity). The larger marble has more “excess weight” per unit of drag force, so it reaches a higher terminal velocity. This is why sand grains settle faster than fine silt in water, and why centrifuges can separate particles by size—larger particles move outward faster. Note: this analysis assumes laminar flow (low Reynolds number), which is valid for small, slow-moving spheres in viscous fluids like honey or motor oil, as described in the Take-Home Experiment.

Glossary

viscous drag
: a resistance force exerted on a moving object, with a nontrivial dependence on velocity

terminal speed
: the speed at which the viscous drag of an object falling in a viscous fluid is equal to the other forces acting on the object (such as gravity), so that the acceleration of the object is zero

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Molecular Transport Phenomena: Diffusion, Osmosis, and Related Processes

- Define diffusion, osmosis, dialysis, and active transport.
- Calculate diffusion rates.

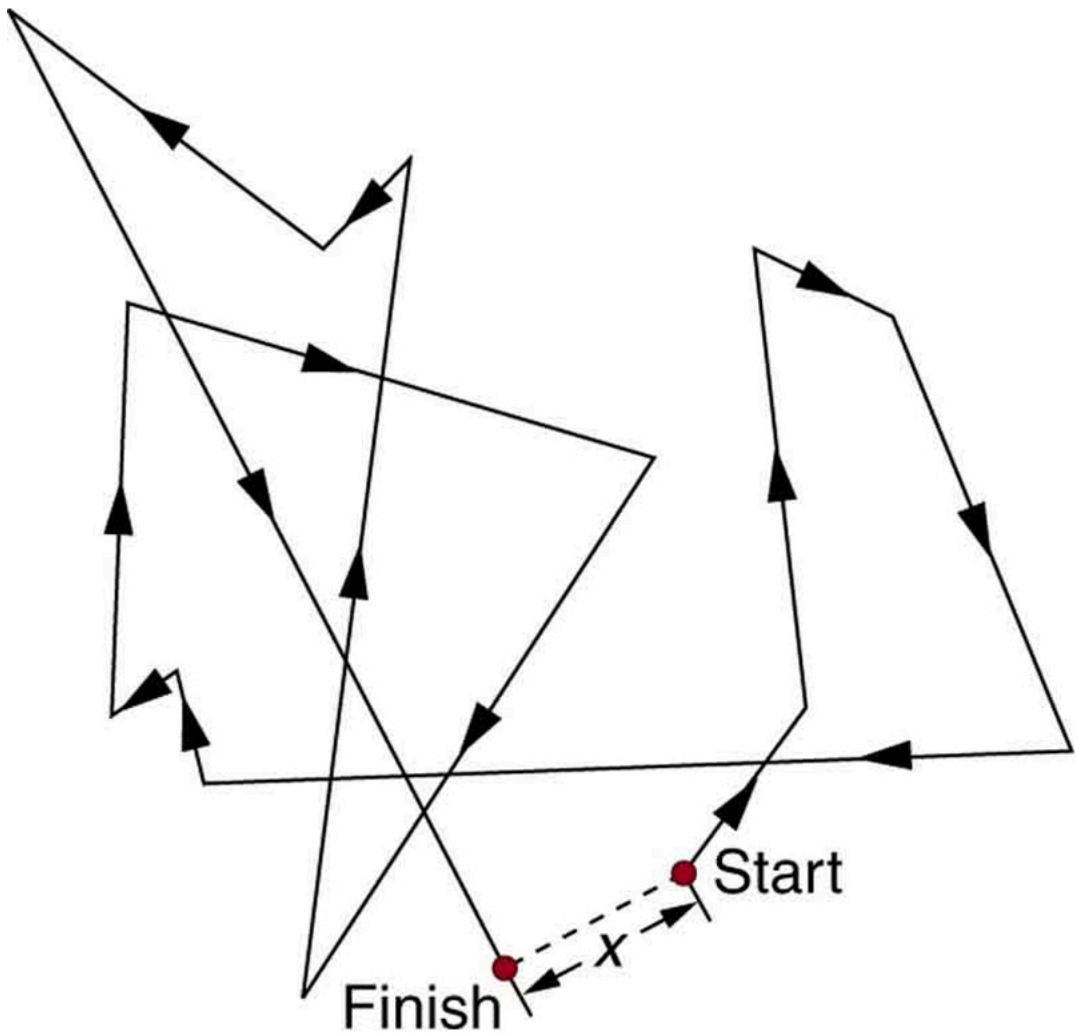
Diffusion

There is something fishy about the ice cube from your freezer—how did it pick up those food odors? How does soaking a sprained ankle in Epsom salt reduce swelling? The answer to these questions are related to atomic and molecular transport phenomena—another mode of fluid motion. Atoms and molecules are in constant motion at any temperature. In fluids they move about randomly even in the absence of macroscopic flow. This motion is called a random walk and is illustrated in [Figure 1]. **Diffusion** is the movement of substances due to random thermal molecular motion. Fluids, like fish fumes or odors entering ice cubes, can even diffuse through solids.

Diffusion is a slow process over macroscopic distances. The densities of common materials are great enough that molecules cannot travel very far before having a collision that can scatter them in any direction, including straight backward. It can be shown that the average distance x_{rms} that a molecule travels is proportional to the square root of time:

$$x_{\text{rms}}=\sqrt{2Dt},$$

where x_{rms} stands for the **root-mean-square distance** and is the statistical average for the process. The quantity D is the diffusion constant for the particular molecule in a specific medium. [Table 1] lists representative values of D for various substances, in units of m^2/s.



The random thermal motion of a molecule in a fluid in time t . This type of motion is called a random walk.

Diffusion Constants for Various Molecules[1]

Diffusing molecule	Medium	D (m^2/s)

Diffusing molecule	Medium	D (m^2/s)
Hydrogen (H_2)	Air	6.4×10^{-5}
Oxygen (O_2)	Air	1.8×10^{-5}
Oxygen (O_2)	Water	1.0×10^{-9}
Glucose ($C_6H_{12}O_6$)	Water	6.7×10^{-10}
Hemoglobin	Water	6.9×10^{-11}
DNA	Water	1.3×10^{-12}

Note that D gets progressively smaller for more massive molecules. This decrease is because the average molecular speed at a given temperature is inversely proportional to molecular mass. Thus the more massive molecules diffuse more slowly. Another interesting point is that D for oxygen in air is much greater than D for oxygen in water. In water, an oxygen molecule makes many more collisions in its random walk and is slowed considerably. In water, an oxygen molecule moves only about $40\,\mu\text{m}$ in 1 s. (Each molecule actually collides about 10^{10} times per second!). Finally, note that diffusion constants increase with temperature, because average molecular speed increases with temperature. This is because the average kinetic energy of molecules, $\frac{1}{2}mv^2$, is proportional to absolute temperature.

Calculating Diffusion: How Long Does Glucose Diffusion Take?

Calculate the average time it takes a glucose molecule to move 1.0 cm in water.

Strategy

We can use $x_{\text{rms}} = \sqrt{2Dt}$, the expression for the average distance moved in time t , and solve it for t . All other quantities are known.

Solution

Solving for t and substituting known values yields

$$t = \frac{x_{\text{rms}}^2}{2D} = \frac{(0.010\,\text{m})^2}{2(6.7\times10^{-10}\,\text{m}^2/\text{s})} = 7.5\times10^4\,\text{s} = 21\,\text{h}.$$

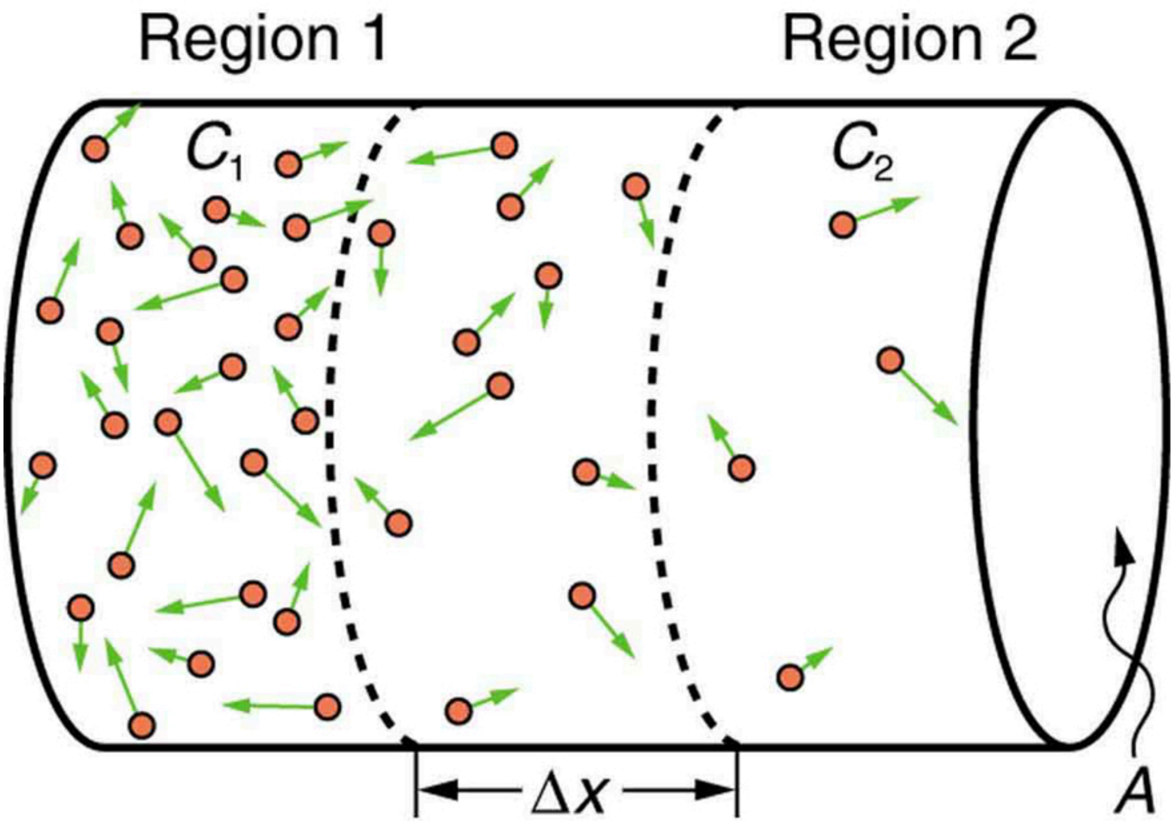
Discussion

This is a remarkably long time for glucose to move a mere centimeter! For this reason, we stir sugar into water rather than waiting for it to diffuse.

Because diffusion is typically very slow, its most important effects occur over small distances. For example, the cornea of the eye gets most of its oxygen by diffusion through the thin tear layer covering it.

The Rate and Direction of Diffusion

If you very carefully place a drop of food coloring in a still glass of water, it will slowly diffuse into the colorless surroundings until its concentration is the same everywhere. This type of diffusion is called free diffusion, because there are no barriers inhibiting it. Let us examine its direction and rate. Molecular motion is random in direction, and so simple chance dictates that more molecules will move out of a region of high concentration than into it. The net rate of diffusion is higher initially than after the process is partially completed. (See [Figure 2].)



Diffusion proceeds from a region of higher concentration to a lower one. The net rate of movement is proportional to the difference in concentration.

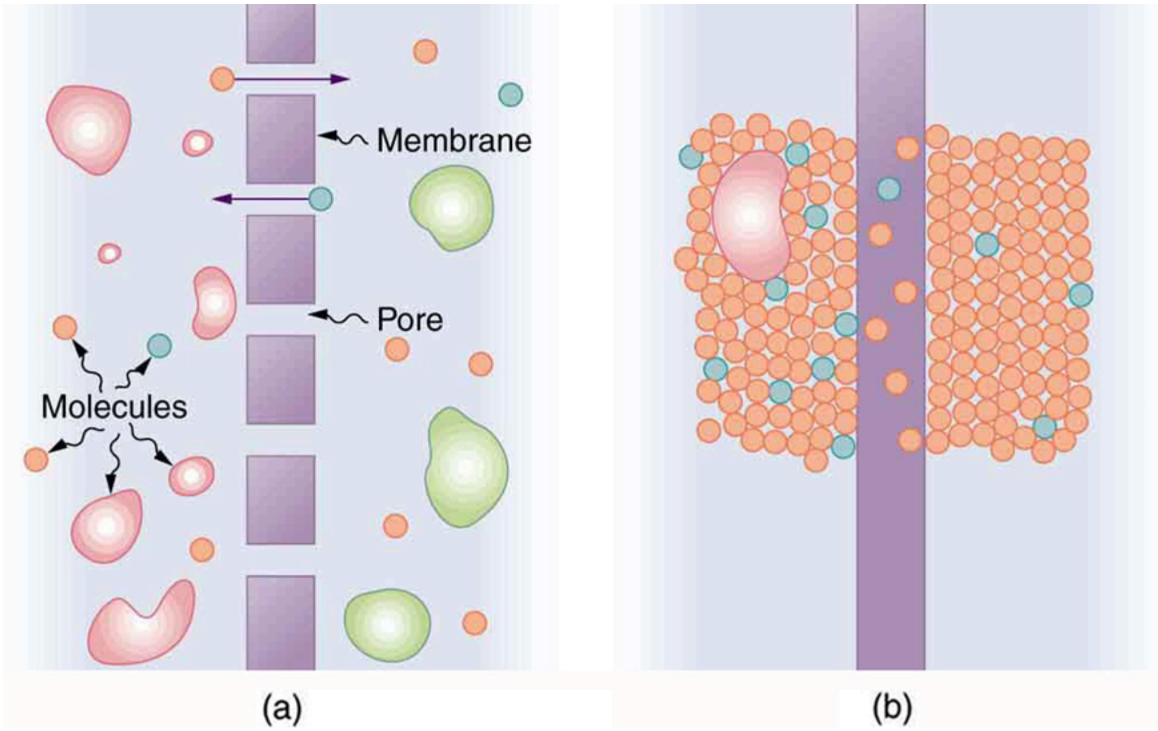
The net rate of diffusion is proportional to the concentration difference. Many more molecules will leave a region of high concentration than will enter it from a region of low concentration. In fact, if the concentrations were the same, there would be *no* net movement. The net rate of diffusion is also proportional to the diffusion constant D , which is determined experimentally. The farther a molecule can diffuse in a given time, the more likely it is to leave the region of high concentration. Many of the factors that affect the rate are hidden in the diffusion constant D . For example, temperature and cohesive and adhesive forces all affect values of D.

Diffusion is the dominant mechanism by which the exchange of nutrients and waste products occur between the blood and tissue, and between air and blood in the lungs. In the evolutionary process, as organisms became larger, they needed quicker methods of transportation than net diffusion, because of the larger distances involved in the transport, leading to the development of circulatory systems. Less sophisticated, single-celled organisms still rely totally on diffusion for the removal of waste products and the uptake of nutrients.

Osmosis and Dialysis—Diffusion across Membranes

Some of the most interesting examples of diffusion occur through barriers that affect the rates of diffusion. For example, when you soak a swollen ankle in Epsom salt, water diffuses through your skin. Many substances regularly move through cell membranes; oxygen moves in, carbon dioxide moves out, nutrients go in, and wastes go out, for example. Because membranes are thin structures (typically 6.5×10^{-9} to 10×10^{-9} m across) diffusion rates through them can be high. Diffusion through membranes is an important method of transport.

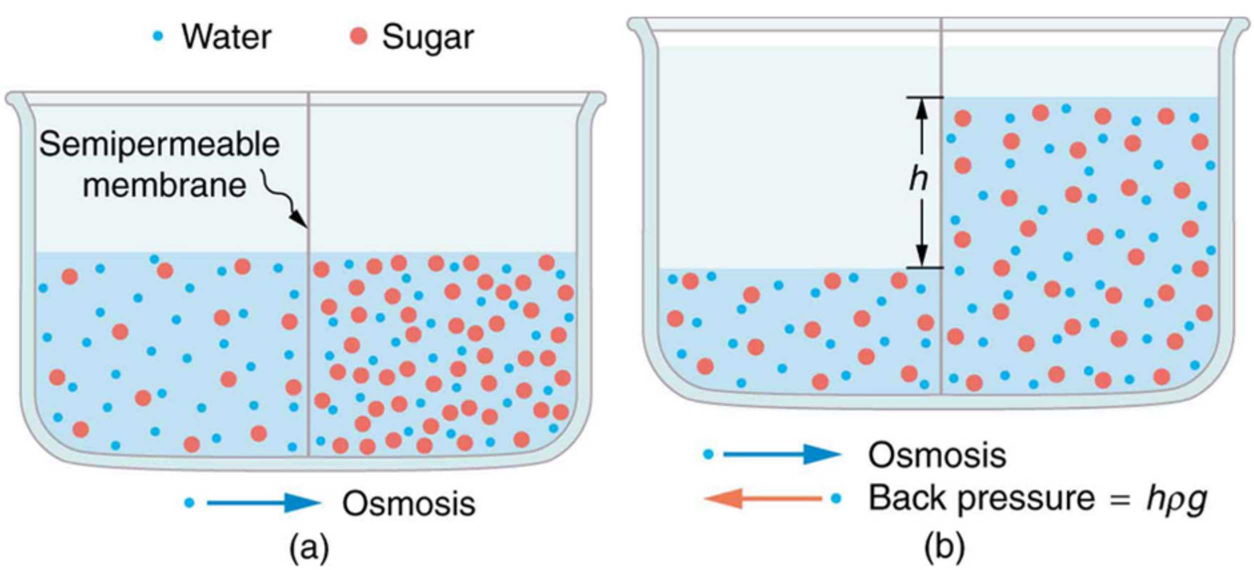
Membranes are generally selectively permeable, or **semipermeable**. (See [Figure 3].) One type of semipermeable membrane has small pores that allow only small molecules to pass through. In other types of membranes, the molecules may actually dissolve in the membrane or react with molecules in the membrane while moving across. Membrane function, in fact, is the subject of much current research, involving not only physiology but also chemistry and physics.



(a) A semipermeable membrane with small pores that allow only small molecules to pass through. (b) Certain molecules dissolve in this membrane and diffuse across it.

Osmosis is the transport of water through a semipermeable membrane from a region of high concentration to a region of low concentration. Osmosis is driven by the imbalance in water concentration. For example, water is more concentrated in your body than in Epsom salt. When you soak a swollen ankle in Epsom salt, the water moves out of your body into the lower-concentration region in the salt. Similarly, **dialysis** is the transport of any other molecule through a semipermeable membrane due to its concentration difference. Both osmosis and dialysis are used by the kidneys to cleanse the blood.

Osmosis can create a substantial pressure. Consider what happens if osmosis continues for some time, as illustrated in [Figure 4]. Water moves by osmosis from the left into the region on the right, where it is less concentrated, causing the solution on the right to rise. This movement will continue until the pressure ρgh created by the extra height of fluid on the right is large enough to stop further osmosis. This pressure is called a _ back pressure_. The back pressure ρgh that stops osmosis is also called the **relative osmotic pressure** if neither solution is pure water, and it is called the **osmotic pressure** if one solution is pure water. Osmotic pressure can be large, depending on the size of the concentration difference. For example, if pure water and sea water are separated by a semipermeable membrane that passes no salt, osmotic pressure will be 25.9 atm. This value means that water will diffuse through the membrane until the salt water surface rises 268 m above the pure-water surface! One example of pressure created by osmosis is turgor in plants (many wilt when too dry). Turgor describes the condition of a plant in which the fluid in a cell exerts a pressure against the cell wall. This pressure gives the plant support. Dialysis can similarly cause substantial pressures.



(a) Two sugar-water solutions of different concentrations, separated by a semipermeable membrane that passes water but not sugar. Osmosis will be to the right, since water is less concentrated there. (b) The fluid level rises until the back pressure ρgh equals the relative osmotic pressure; then, the net transfer of water is zero.

Reverse osmosis and **reverse dialysis** (also called filtration) are processes that occur when back pressure is sufficient to reverse the normal direction of substances through membranes. Back pressure can be created naturally as on the right side of [Figure 4]. (A piston can also create this pressure.) Reverse osmosis can be used to desalinate water by simply forcing it through a membrane that will not pass salt. Similarly, reverse dialysis can be used to filter out any substance that a given membrane will not pass.

One further example of the movement of substances through membranes deserves mention. We sometimes find that substances pass in the direction opposite to what we expect. Cypress tree roots, for example, extract pure water from salt water, although osmosis would move it in the opposite direction. This is not reverse osmosis, because there is no back pressure to cause it. What is happening is called **active transport**, a process in which a living membrane expends energy to move substances across it. Many living membranes move water and other substances by active transport. The kidneys, for example, not only use osmosis and dialysis—they also employ significant active transport to move substances into and out of blood. In fact, it is estimated that at least 25% of the body's energy is expended on active transport of substances at the cellular level. The study of active transport carries us into the realms of microbiology, biophysics, and biochemistry and it is a fascinating application of the laws of nature to living structures.

Section Summary

- Diffusion is the movement of substances due to random thermal molecular motion.
- The average distance x_{rms} a molecule travels by diffusion in a given amount of time is given by

  $$x_{\text{rms}}=\sqrt{2Dt},$$

  where D is the diffusion constant, representative values of which are found in [Table 1].
- Osmosis is a process by which molecules of a solvent pass through a semipermeable membrane from a less concentrated solution into a more concentrated one, thus, equalizing the solute concentrations on each side of the membrane.
- Dialysis is the transport of any other molecule through a semipermeable membrane due to its concentration difference.
- Both processes can be reversed by back pressure.
- Active transport is a process in which a living membrane expends energy to move substances across it.

Conceptual Questions

Why would you expect the rate of diffusion to increase with temperature? Can you give an example, such as the fact that you can dissolve sugar more rapidly in hot water?

Show Solution

Strategy

We connect temperature to molecular kinetic energy and speed, then relate molecular speed to diffusion rate.

Solution

The rate of diffusion increases with temperature because **higher temperature means faster molecular motion**.

Physical basis:

- Temperature is a measure of average kinetic energy: $\frac{1}{2}mv^2=\frac{3}{2}k_BT$
- At higher temperatures, molecules move faster: $v_{\text{rms}}\propto\sqrt{T}$
- Faster molecules cover more distance in the same time
- The diffusion constant D increases with temperature

Quantitative relationship: Since $x_{\text{rms}}=\sqrt{2Dt}$ and D increases with temperature, the average distance traveled increases with temperature, meaning faster diffusion.

Examples:

1. **Sugar in hot water**: Sugar dissolves much faster in hot tea than iced tea because both the sugar molecules and water molecules move faster, allowing quicker mixing
2. **Cooking aromas**: The smell of food travels through a house more quickly when cooking (hot air) than when food is cold
3. **Ice cube odors**: Food odors diffuse into ice cubes in the freezer, but this process is very slow due to the low temperature
4. **Chemical reactions**: Most reactions speed up with temperature because reactants diffuse together faster

Discussion

The temperature dependence of diffusion is why we stir things rather than wait—even at room temperature, diffusion is slow over centimeter-scale distances. The relationship between D and temperature is approximately linear for gases and follows a more complex relationship for liquids. This principle is crucial in chemistry, biology, and cooking!

How are osmosis and dialysis similar? How do they differ?

Show Solution

Strategy

We compare and contrast these two transport processes based on their mechanisms, the substances transported, and the role of membranes.

Solution

Similarities:

1. **Both are diffusion processes**: Both osmosis and dialysis involve the movement of molecules due to random thermal motion
2. **Both involve semipermeable membranes**: Each requires a membrane that selectively allows some substances to pass
3. **Both are driven by concentration differences**: Molecules move from regions of higher concentration to regions of lower concentration
4. **Both can be reversed**: With sufficient back pressure, both osmosis and dialysis can be reversed (reverse osmosis and reverse dialysis/filtration)
5. **Both occur naturally in biological systems**: The kidneys use both processes to filter blood

Differences:

Aspect	Osmosis	Dialysis
Substance transported	Water (solvent) only	Any dissolved molecule other than water (solutes)
Direction of flow	Water moves toward higher solute concentration	Solutes move from higher to lower solute concentration
Net effect	Equalizes water concentration	Equalizes solute concentration
Medical application	Fluid balance, reducing swelling	Kidney dialysis (filtering waste from blood)

Example: In kidney dialysis, waste products (urea, excess salts) diffuse from blood into dialysis fluid through a membrane, while water may move by osmosis in the opposite direction depending on concentration gradients.

Discussion

The key distinction is **what moves**: in osmosis, water moves; in dialysis, dissolved substances move. Both are essential biological processes. The kidneys use both simultaneously—osmosis to regulate water balance and dialysis to remove waste products and excess ions from the blood.

Problem Exercises

You can smell perfume very shortly after opening the bottle. To show that it is not reaching your nose by diffusion, calculate the average distance a perfume molecule moves in one second in air, given its diffusion constant D to be $1.00\times10^{-6}\text{m}^2/\text{s}$.

Show Solution

Strategy

We use the diffusion distance formula $x_{\text{rms}}=\sqrt{2Dt}$ to calculate how far a perfume molecule travels by diffusion in one second, then compare this to typical distances between a perfume bottle and your nose.

Solution

Given:

- Diffusion constant: $D=1.00\times10^{-6}\text{ m}^2/\text{s}$
- Time: $t=1.00\text{ s}$

Using the diffusion distance formula:

$$x_{\text{rms}}=\sqrt{2Dt}=\sqrt{2(1.00\times10^{-6}\text{ m}^2/\text{s})(1.00\text{ s})}$$
$$x_{\text{rms}}=\sqrt{2.00\times10^{-6}\text{ m}^2}=1.41\times10^{-3}\text{ m}=1.41\text{ mm}$$

In one second, a perfume molecule moves only about **1.4 mm** by diffusion.

Discussion

This tiny distance proves that diffusion is **not** how perfume reaches your nose quickly. If you open a perfume bottle 1 meter away from your nose, it would take approximately:

$$t=\frac{x_{\text{rms}}^2}{2D}=\frac{(1\text{ m})^2}{2(1.00\times10^{-6}\text{ m}^2/\text{s})}=5\times10^{5}\text{ s}\approx6\text{ days}$$

Since you smell perfume within seconds, not days, the odor must be carried by **convection**—air currents caused by temperature differences, air conditioning, or your own movement. Even your exhaling creates currents that mix the air much faster than diffusion. This example beautifully illustrates why diffusion is only important for transport over very short distances (like across cell membranes or thin tissue layers).

What is the ratio of the average distances that oxygen will diffuse in a given time in air and water? Why is this distance less in water (equivalently, why is D less in water)?

Show Solution

Strategy

We compare diffusion constants for oxygen in air versus water from Table 1, then calculate the ratio of distances using $x_{rms} = \sqrt{2Dt}$.

Solution

From Table 1:

- Oxygen in air: $D_{air} = 1.8 \times 10^{-5}\ m^2/s$
- Oxygen in water: $D_{water} = 1.0 \times 10^{-9}\ m^2/s$

Since $x_{rms} = \sqrt{2Dt}$, for the same time t:

$$\frac{x_{air}}{x_{water}} = \frac{\sqrt{2D_{air}t}}{\sqrt{2D_{water}t}} = \sqrt{\frac{D_{air}}{D_{water}}}$$
$$\frac{x_{air}}{x_{water}} = \sqrt{\frac{1.8\times10^{-5}}{1.0\times10^{-9}}} = \sqrt{1.8\times10^{4}} = 134$$

Oxygen diffuses about **134 times farther** in air than in water in the same amount of time.

Why is D less in water?

The diffusion constant is smaller in water because:

1. **Higher density**: Water molecules are much more closely packed than air molecules, so an oxygen molecule encounters far more collisions per unit distance traveled
2. **More frequent collisions**: In water, an oxygen molecule collides approximately 10^{10} times per second—the mean free path (distance between collisions) is extremely short
3. **Stronger intermolecular forces**: Hydrogen bonding in water creates a more cohesive medium that resists molecular motion
4. **Higher viscosity**: Water is about 50 times more viscous than air, which impedes molecular motion

Discussion

This huge difference explains why organisms in water need specialized respiratory systems. Fish gills have enormous surface areas with very thin membranes because oxygen diffuses so slowly in water. The same principle explains why the cornea (which gets oxygen from tears) must be very thin—oxygen can only diffuse through water effectively over distances of micrometers to millimeters.

Oxygen reaches the veinless cornea of the eye by diffusing through its tear layer, which is 0.500-mm thick. How long does it take the average oxygen molecule to do this?

Show Solution

Strategy

We use the diffusion distance formula $x_{rms} = \sqrt{2Dt}$ and solve for time t. The diffusion constant for oxygen in water (tears are mostly water) is found in Table 1.

Solution

Given:

- Tear layer thickness: $x_{rms} = 0.500\ mm = 5.00 \times 10^{-4}\ m$
- Diffusion constant for O_2 in water: $D = 1.0 \times 10^{-9}\ m^2/s$ (from Table 1)

Solving the diffusion equation for time:

$$x_{rms} = \sqrt{2Dt}$$
$$x_{rms}^2 = 2Dt$$
$$t = \frac{x_{rms}^2}{2D} = \frac{(5.00\times10^{-4}\ m)^2}{2(1.0\times10^{-9}\ m^2/s)}$$
$$t = \frac{2.50\times10^{-7}\ m^2}{2.0\times10^{-9}\ m^2/s} = 1.25\times10^{2}\ s \approx 125\ s$$

It takes approximately **125 seconds (about 2 minutes)** for an oxygen molecule to diffuse through the tear layer.

Discussion

This relatively short time (about 2 minutes) explains why the cornea can survive without blood vessels—oxygen from the air dissolves in the tear layer and diffuses through quickly enough to supply the cornea's needs. This is possible only because the tear layer is extremely thin (0.5 mm). If the layer were 10 times thicker, diffusion would take 100 times longer (over 3 hours), which would be inadequate for corneal health. This is why contact lenses must be highly oxygen-permeable—they add to the diffusion distance and can cause corneal hypoxia if they block too much oxygen transport.

(a) Find the average time required for an oxygen molecule to diffuse through a 0.200-mm-thick tear layer on the cornea. (b) How much time is required to diffuse 0.500cm^3 of oxygen to the cornea if its surface area is 1.00cm^2 ?

Show Solution

Strategy

(a) We use the diffusion formula $x_{\text{rms}} = \sqrt{2Dt}$ and solve for time. (b) We consider the volume of oxygen that must pass through the tear layer and relate it to the diffusion process, noting that the problem involves a more complex analysis of flux.

Solution

(a) Time for one molecule to diffuse through 0.200 mm:

Given:

- Tear layer thickness: $x_{\text{rms}} = 0.200\text{ mm} = 2.00 \times 10^{-4}\text{ m}$
- Diffusion constant for O_2 in water: $D = 1.0 \times 10^{-9}\text{ m}^2/\text{s}$

$$t = \frac{x_{\text{rms}}^2}{2D} = \frac{(2.00\times10^{-4}\text{ m})^2}{2(1.0\times10^{-9}\text{ m}^2/\text{s})}$$
$$t = \frac{4.00\times10^{-8}\text{ m}^2}{2.0\times10^{-9}\text{ m}^2/\text{s}} = 20\text{ s}$$

(b) Time to diffuse 0.500 cm³ of oxygen:

This part requires considering the flux of oxygen through the tear layer. Using Fick's law, the diffusion rate depends on the concentration gradient and surface area.

For a simplified estimate, we consider that oxygen arrives at the cornea continuously as molecules complete their random walk. The volume of gas $V = 0.500\text{ cm}^3$ through area $A = 1.00\text{ cm}^2$ represents an effective layer thickness:

$$\text{thickness} = \frac{V}{A} = \frac{0.500\text{ cm}^3}{1.00\text{ cm}^2} = 0.500\text{ cm} = 5.00\times10^{-3}\text{ m}$$

However, this represents the accumulated oxygen, not a diffusion distance problem. The actual analysis requires knowing the oxygen concentration difference across the membrane and applying Fick's law of diffusion.

For a rough estimate, if we need 0.500 cm³ of O_2 and molecules take 20 s each to traverse the membrane, the total time depends on the concentration gradient and available oxygen at the air-tear interface.

Summary:

- (a) **20 seconds** for a molecule to diffuse through 0.200 mm
- (b) Requires additional information about concentration gradients; the problem is underspecified for a complete numerical answer

Discussion

Part (a) shows that thinner tear layers dramatically speed up oxygen transport—a 0.200 mm layer requires only 20 seconds compared to 125 seconds for a 0.500 mm layer (the ratio is $(0.5/0.2)^2 = 6.25$). Part (b) highlights that bulk transport problems require more than just diffusion time—they also depend on concentration gradients and the rate at which oxygen can be supplied to the interface.

Suppose hydrogen and oxygen are diffusing through air. A small amount of each is released simultaneously. How much time passes before the hydrogen is 1.00 s ahead of the oxygen? Such differences in arrival times are used as an analytical tool in gas chromatography.

Show Solution

Strategy

We find when both gases have diffused the same distance, but the oxygen takes 1.00 s longer than the hydrogen. Using $x = \sqrt{2Dt}$ for each gas and setting them equal while accounting for the time difference, we can solve for the time.

Solution

From Table 1:

- Hydrogen diffusion constant in air: $D_{H_2} = 6.4 \times 10^{-5}\text{ m}^2/\text{s}$
- Oxygen diffusion constant in air: $D_{O_2} = 1.8 \times 10^{-5}\text{ m}^2/\text{s}$

Let t be the time for hydrogen to travel a certain distance x. Then oxygen takes time $(t + 1.00\text{ s})$ to travel the same distance.

For hydrogen: $x = \sqrt{2D_{H_2}t}$

For oxygen: $x = \sqrt{2D_{O_2}(t+1.00)}$

Since both travel the same distance:

$$\sqrt{2D_{H_2}t} = \sqrt{2D_{O_2}(t+1.00)}$$

Squaring both sides:

$$2D_{H_2}t = 2D_{O_2}(t+1.00)$$
$$D_{H_2}t = D_{O_2}t + D_{O_2}(1.00)$$
$$t(D_{H_2} - D_{O_2}) = D_{O_2}(1.00\text{ s})$$
$$t = \frac{D_{O_2}}{D_{H_2} - D_{O_2}} \times 1.00\text{ s}$$
$$t = \frac{1.8\times10^{-5}}{6.4\times10^{-5} - 1.8\times10^{-5}} \times 1.00\text{ s}$$
$$t = \frac{1.8\times10^{-5}}{4.6\times10^{-5}}\text{ s} = 0.391\text{ s}$$

After **0.391 seconds**, the hydrogen will have reached a point that the oxygen won't reach for another 1.00 second.

Verification:

- Distance traveled by H_2 in 0.391 s: $x = \sqrt{2(6.4\times10^{-5})(0.391)} = 7.07\times10^{-3}$ m
- Time for O_2 to travel same distance: $t = \frac{x^2}{2D_{O_2}} = \frac{(7.07\times10^{-3})^2}{2(1.8\times10^{-5})} = 1.39$ s
- Difference: $1.39 - 0.39 = 1.00$ s ✓

Discussion

This separation occurs very quickly (less than half a second) because hydrogen diffuses about 3.6 times faster than oxygen due to its much lower molecular mass. This principle is the basis of **gas chromatography**, where different gases in a mixture are separated by their different rates of diffusion (or more precisely, by their different rates of movement through a column). Lighter molecules like hydrogen travel faster, allowing mixtures to be separated and analyzed. The technique is widely used in chemistry and biochemistry for identifying and quantifying gas components.

Footnotes

- 1 At 20ºC and 1 atm { data-list-type="bulleted" data-bullet-style="none"}

Glossary

diffusion
: the movement of substances due to random thermal molecular motion

semipermeable
: a type of membrane that allows only certain small molecules to pass through

osmosis
: a process by which molecules of a solvent pass through a semipermeable membrane from a less concentrated solution into a more concentrated one, thus, equalizing the solute concentrations on each side of the membrane

dialysis
: the transport of any molecule other than water through a semipermeable membrane from a region of high concentration to one of low concentration

relative osmotic pressure
: the back pressure which stops the osmotic process if neither solution is pure water

osmotic pressure
: the back pressure which stops the osmotic process if one solution is pure water

reverse osmosis
: the process that occurs when back pressure is sufficient to reverse the normal direction of osmosis through membranes

reverse dialysis
: the process that occurs when back pressure is sufficient to reverse the normal direction of dialysis through membranes

active transport
: the process in which a living membrane expends energy to move substances across