Introduction to Uniform Circular Motion and Gravitation

This Australian Grand Prix Formula 1 race car moves in a circular path as it makes the turn. Its wheels also spin rapidly—the latter completing many revolutions, the former only part of one (a circular arc). The same physical principles are involved in each. (credit: Richard Munckton)

Many motions, such as the arc of a bird's flight or Earth's path around the Sun, are curved. Recall that Newton's first law tells us that motion is along a straight line at constant speed unless there is a net external force. We will therefore study not only motion along curves, but also the forces that cause it, including gravitational forces. In some ways, this chapter is a continuation of Dynamics: Newton's Laws of Motion as we study more applications of Newton's laws of motion.

This chapter deals with the simplest form of curved motion, **uniform circular motion**, motion in a circular path at constant speed. Studying this topic illustrates most concepts associated with rotational motion and leads to the study of many new topics we group under the name *rotation*. Pure *rotational motion* occurs when points in an object move in circular paths centered on one point. Pure *translational motion* is motion with no rotation. Some motion combines both types, such as a rotating hockey puck moving along ice.

Glossary

uniform circular motion

the motion of an object in a circular path at constant speed

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Rotation Angle and Angular Velocity

- Define arc length, rotation angle, radius of curvature and angular velocity.
- Calculate the angular velocity of a car wheel spin.

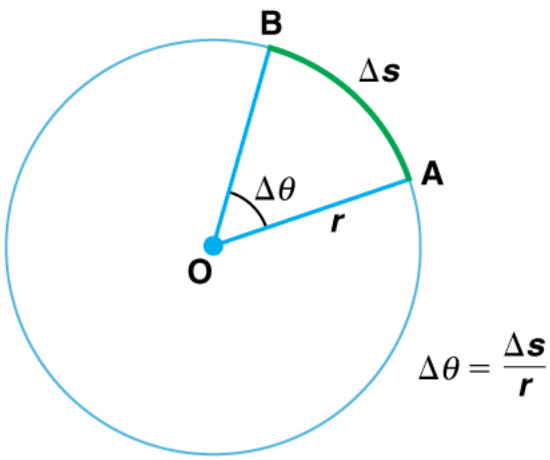
In Kinematics, we studied motion along a straight line and introduced such concepts as displacement, velocity, and acceleration. Two-Dimensional Kinematics dealt with motion in two dimensions. Projectile motion is a special case of two-dimensional kinematics in which the object is projected into the air, while being subject to the gravitational force, and lands a distance away. In this chapter, we consider situations where the object does not land but moves in a curve. We begin the study of uniform circular motion by defining two angular quantities needed to describe rotational motion.

Rotation Angle

When objects rotate about some axis—for example, when the CD (compact disc) in Figure 1 rotates about its center—each point in the object follows a circular arc. Consider a line from the center of the CD to its edge. Each **pit** used to record sound along this line moves through the same angle in the same amount of time. The rotation angle is the amount of rotation and is analogous to linear distance. We define the **rotation angle** $\Delta\theta$ to be the ratio of the arc length to the radius of curvature:

$$\Delta\theta=\Delta sr.$$

All points on a CD travel in circular arcs. The pits along a line from the center to the edge all move through the same angle $\Delta\theta$



The radius of a circle is rotated through an angle $\Delta\theta$. The arc length Δs is described on the circumference.

The **arc length** Δs is the distance traveled along a circular path as shown in Figure 2 Note that r is the **radius of curvature** of the circular path.

We know that for one complete revolution, the arc length is the circumference of a circle of radius r. The circumference of a circle is $2\pi r$. Thus for one complete revolution the rotation angle is

$$\Delta\theta = \frac{2\pi r}{r} = 2\pi.$$

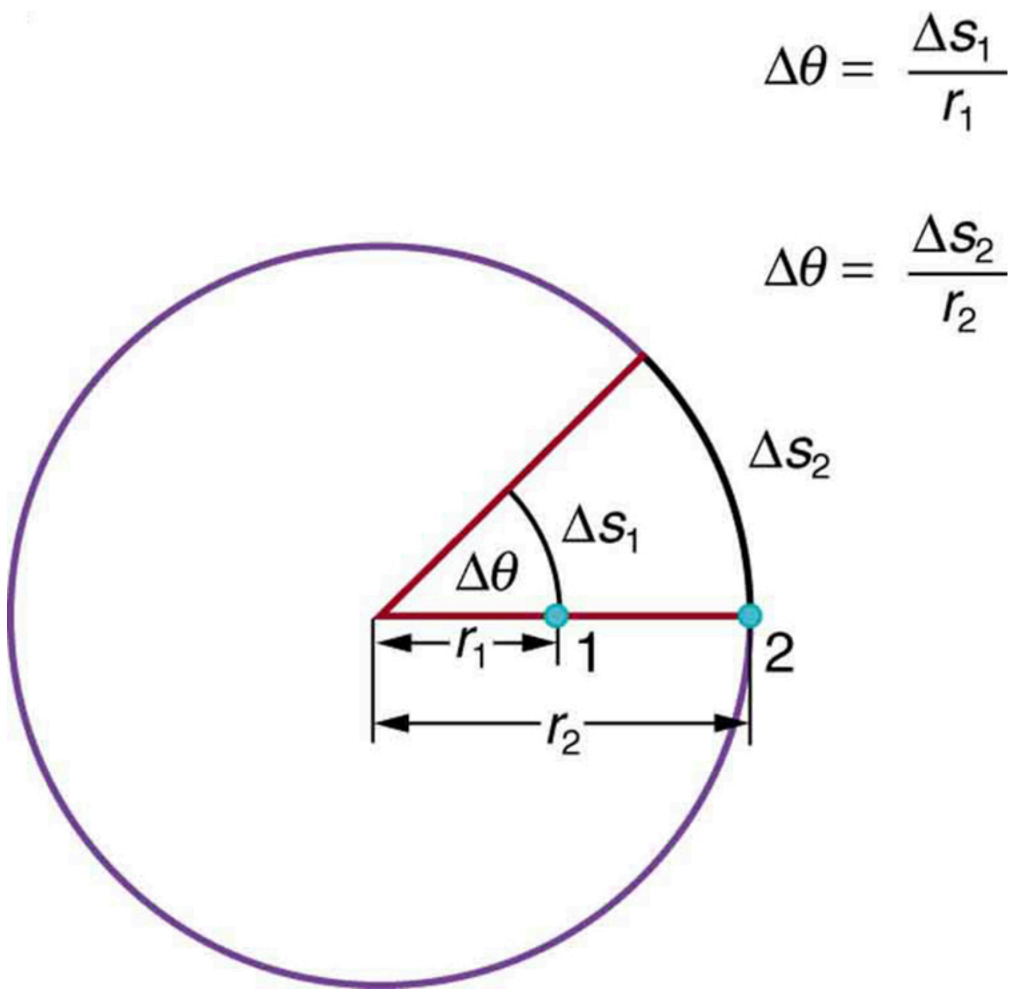
This result is the basis for defining the units used to measure rotation angles, $\Delta\theta$ to be **radians** (rad), defined so that

$$2\pi \text{ rad} = 1 \text{ revolution}.$$

A comparison of some useful angles expressed in both degrees and radians is shown in Table 1.

Table 1: Comparison of Angular units

Degree Measures	Radian Measure
30º	$\pi/6$
60º	$\pi/3$
90º	$\pi/2$
120º	$2\pi/3$
135º	$3\pi/4$
180º	π



Points 1 and 2 rotate through the same angle $\Delta\theta$, but point 2 moves through a greater arc length Δs because it is at a greater distance from the center of rotation r

If $\Delta\theta = 2\pi$ rad, then the CD has made one complete revolution, and every point on the CD is back at its original position. Because there are 360° in a circle or one revolution, the relationship between radians and degrees is thus

$$2\pi \text{ rad} = 360^\circ$$

so that

$$1 \text{ rad} = \frac{360^\circ}{2\pi} \approx 57.3^\circ.$$

Angular Velocity

How fast is an object rotating? We define **angular velocity** ω as the rate of change of an angle. In symbols, this is

$$\omega = \frac{\Delta\theta}{\Delta t},$$

where an angular rotation $\Delta\theta$ takes place in a time Δt . The greater the rotation angle in a given amount of time, the greater the angular velocity. The units for angular velocity are radians per second (rad/s).

Angular velocity ω is analogous to linear velocity v. To get the precise relationship between angular and linear velocity, we again consider a pit on the rotating CD. This pit moves an arc length Δs in a time Δt, and so it has a linear velocity

$$v = \frac{\Delta s}{\Delta t}.$$

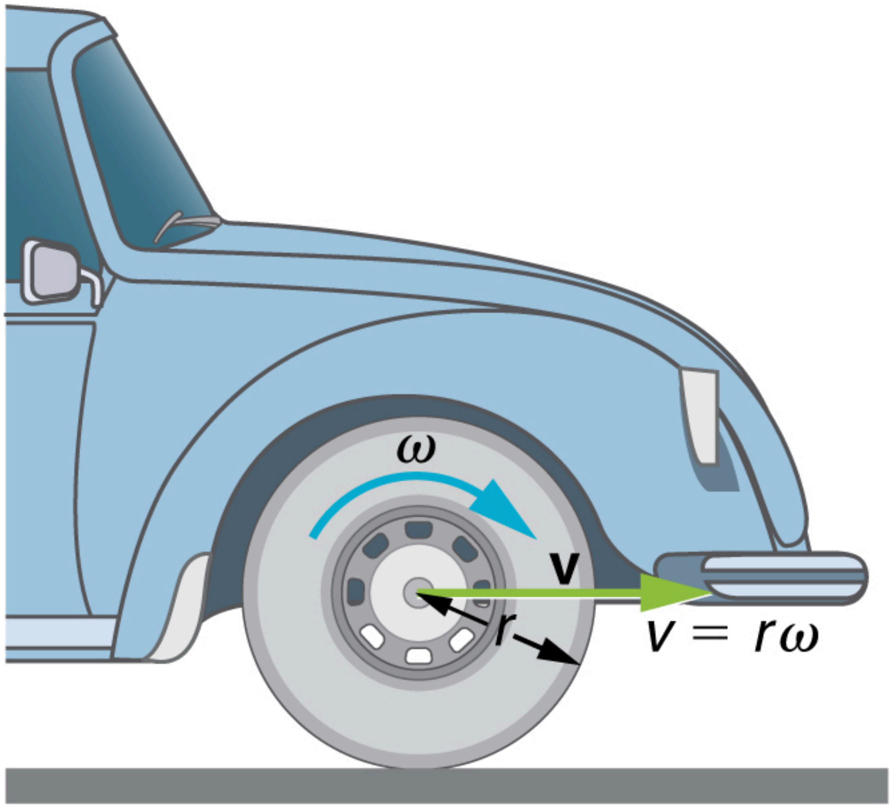
From $\Delta\theta = \frac{\Delta s}{r}$ we see that $\Delta s = r\Delta\theta$. Substituting this into the expression for v gives

$$v = \frac{r\Delta\theta}{\Delta t} = r\omega.$$

We write this relationship in two different ways and gain two different insights:

$v = r\omega$ or $\omega = vr$.

The first relationship in $v = r\omega$ or $\omega = vr$ states that the linear velocity v is proportional to the distance from the center of rotation, thus, it is largest for a point on the rim (largest r), as you might expect. We can also call this linear speed v of a point on the rim the *tangential speed*. The second relationship in $v = r\omega$ or $\omega = vr$ can be illustrated by considering the tire of a moving car. Note that the speed of a point on the rim of the tire is the same as the speed v of the car. See Figure 4. So the faster the car moves, the faster the tire spins—large v means a large ω, because $v = r\omega$. Similarly, a larger-radius tire rotating at the same angular velocity (ω) will produce a greater linear speed (v) for the car.



A car moving at a velocity v to the right has a tire rotating with an angular velocity ω. The speed of the tread of the tire relative to the axle is v , the same as if the car were jacked up. Thus the car moves forward at linear velocity $r\omega$, where r is the tire radius. A larger angular velocity for the tire means a greater velocity for the car.

How Fast Does a Car Tire Spin?

Calculate the angular velocity of a 0.300 m radius car tire when the car travels at 15.0m/s (about 54km/h). See Figure 4.

Strategy

Because the linear speed of the tire rim is the same as the speed of the car, we have $v = 15.0 \text{m/s}$. The radius of the tire is given to be $r = 0.300 \text{m}$. Knowing v and r, we can use the second relationship in $v = r\omega$, $\omega = vr$ to calculate the angular velocity.

Solution

To calculate the angular velocity, we will use the following relationship:

$\omega = vr$.

Substituting the knowns,

$\omega = 15.0 \text{m/s} 0.300 \text{m} = 50.0 \text{rad/s}$.

Discussion

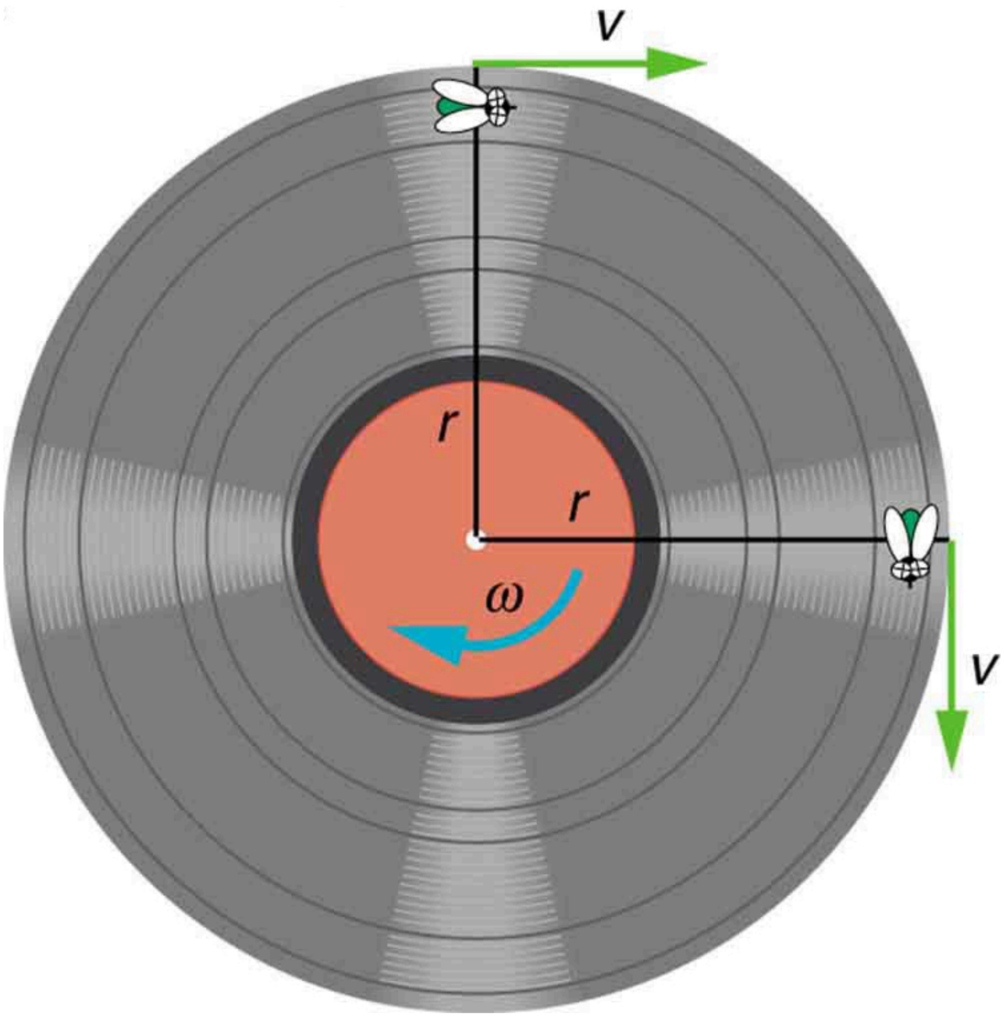
When we cancel units in the above calculation, we get 50.0/s. But the angular velocity must have units of rad/s. Because radians are actually unitless (radians are defined as a ratio of distance), we can simply insert them into the answer for the angular velocity. Also note that if an earth mover with much larger tires, say 1.20 m in radius, were moving at the same speed of 15.0 m/s, its tires would rotate more slowly. They would have an angular velocity

$\omega = (15.0\text{m/s})/(1.20\text{m}) = 12.5\text{rad/s}$.

Both ω and v have directions (hence they are angular and linear *velocities*, respectively). Angular velocity has only two directions with respect to the axis of rotation—it is either clockwise or counterclockwise. Linear velocity is tangent to the path, as illustrated in Figure 5.

Take-Home Experiment

Tie an object to the end of a string and swing it around in a horizontal circle above your head (swing at your wrist). Maintain uniform speed as the object swings and measure the angular velocity of the motion. What is the approximate speed of the object? Identify a point close to your hand and take appropriate measurements to calculate the linear speed at this point. Identify other circular motions and measure their angular velocities.



As an object moves in a circle, here a fly on the edge of an old-fashioned vinyl record, its instantaneous velocity is always tangent to the circle. The direction of the angular velocity is clockwise in this case.

Ladybug Revolution

Ladybug Revolution

Join the ladybug in an exploration of rotational motion. Rotate the merry-go-round to change its angle, or choose a constant angular velocity or angular acceleration. Explore how circular motion relates to the bug's x,y position, velocity, and acceleration using vectors or graphs.

Section Summary

- Uniform circular motion is motion in a circle at constant speed. The rotation angle $\Delta\theta$ is defined as the ratio of the arc length to the radius of curvature:
  $\Delta\theta = \frac{\Delta s}{r}$,

where arc length Δs is distance traveled along a circular path and r is the radius of curvature of the circular path. The quantity $\Delta\theta$ is measured in units of radians (rad), for which

$$2\pi \text{ rad} = 360^\circ = 1 \text{ revolution.}$$

- The conversion between radians and degrees is $1 \text{ rad} = 57.3^\circ$.
- Angular velocity ω is the rate of change of an angle,
  $\omega = \frac{\Delta\theta}{\Delta t}$,

where a rotation $\Delta\theta$ takes place in a time Δt. The units of angular velocity are radians per second (rad/s). Linear velocity v and angular velocity ω are related by

$$v = r\omega \text{ or } \omega = \frac{v}{r}.$$

Conceptual Questions

There is an analogy between rotational and linear physical quantities. What rotational quantities are analogous to distance and velocity?

Problem Exercises

Semi-trailer trucks have an odometer on one hub of a trailer wheel. The hub is weighted so that it does not rotate, but it contains gears to count the number of wheel revolutions—it then calculates the distance traveled. If the wheel has a 1.15 m diameter and goes through 200 000 rotations, how many kilometers should the odometer read?

Show Solution

Strategy

Each complete rotation of the wheel covers a distance equal to the wheel's circumference. We'll calculate the circumference from the diameter, then multiply by the number of rotations to find the total distance traveled.

Solution

The circumference of the wheel is:

$$C = \pi d = \pi (1.15 \text{ m}) = 3.61 \text{ m}$$

The total distance traveled is:

$$s = (\text{number of rotations}) \times C = (200{,}000)(3.61\ \text{m}) = 7.23 \times 10^5\ \text{m}$$

Convert to kilometers:

$$s = 7.23 \times 10^5\ \text{m} \times \frac{1\ \text{km}}{1000\ \text{m}} = 723\ \text{km}$$

Discussion

The odometer should read 723 km, which is a reasonable distance for a semi-trailer truck to travel before needing maintenance or inspection. This calculation demonstrates the relationship between rotational motion (number of wheel revolutions) and linear distance traveled, which is fundamental to how mechanical odometers work.

Microwave ovens rotate at a rate of about 6 rev/min. What is this in revolutions per second? What is the angular velocity in radians per second?

Show Solution

Strategy

We need to convert from revolutions per minute to revolutions per second, and then to radians per second. We'll use the conversion factors: 1 minute = 60 seconds and 1 revolution = 2π radians.

Solution

First, convert revolutions per minute to revolutions per second:

$$6 \frac{\text{rev}}{\text{min}} \times \frac{1\ \text{min}}{60\ \text{s}} = 0.10\ \text{rev/s}$$

Next, convert revolutions per second to radians per second:

$$0.10 \frac{\text{rev}}{\text{s}} \times \frac{2\pi\ \text{rad}}{1\ \text{rev}} = 0.628\ \text{rad/s}$$

Discussion

The microwave oven turntable rotates at 0.10 revolutions per second, which corresponds to an angular velocity of 0.628 rad/s or approximately 0.63 rad/s.

An automobile with 0.260 m radius tires travels 80 000 km before wearing them out. How many revolutions do the tires make, neglecting any backing up and any change in radius due to wear?

Show Solution

Strategy

Each revolution of the tire covers a distance equal to the circumference. We'll find the circumference, convert the total distance to meters, then divide the total distance by the circumference to find the number of revolutions.

Solution

The circumference of the tire is:

$$C = 2\pi r = 2\pi(0.260\ \text{m}) = 1.63\ \text{m}$$

Convert the total distance to meters:

$$s = 80{,}000\ \text{km} \times \frac{1000\ \text{m}}{1\ \text{km}} = 8.00 \times 10^7\ \text{m}$$

The number of revolutions is:

$$N = \frac{s}{C} = \frac{8.00 \times 10^7\ \text{m}}{1.63\ \text{m}} = 4.91 \times 10^7\ \text{rotations} \approx 5 \times 10^7\ \text{rotations}$$

Discussion

The tires make approximately 50 million revolutions before wearing out after traveling 80,000 km. This enormous number of rotations explains why tire tread wears down over time and why tires need to be replaced periodically. The calculation also assumes the tire radius remains constant, though in reality, tire wear would slightly reduce the radius, affecting the actual number of revolutions.

(a) What is the period of rotation of Earth in seconds? (b) What is the angular velocity of Earth? (c) Given that Earth has a radius of $6.4 \times 10^6\text{m}$ at its equator, what is the linear velocity at Earth's surface?

Show Solution

Strategy

(a) Earth completes one full rotation (one day) in 24 hours. We'll convert this to seconds. (b) Angular velocity is found using $\omega = \frac{\Delta\theta}{\Delta t}$, where one complete rotation is 2π radians. (c) We'll use the relationship $v = r\omega$ to find the linear velocity at the equator.

Solution

(a) Convert the period from hours to seconds:

$$T = 24\ \text{h} \times \frac{60\ \text{min}}{1\ \text{h}} \times \frac{60\ \text{s}}{1\ \text{min}} = 86{,}400\ \text{s}$$

(b) Calculate the angular velocity:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi\ \text{rad}}{86{,}400\ \text{s}} = 7.27 \times 10^{-5}\ \text{rad/s}$$

(c) Calculate the linear velocity at Earth's surface:

$$v = r\omega = (6.4 \times 10^{6}\ \text{m})(7.27 \times 10^{-5}\ \text{rad/s}) = 465\ \text{m/s}$$

Discussion

Earth completes one rotation in 86,400 seconds (24 hours) with an angular velocity of 7.27×10^{-5} rad/s. At the equator, the linear velocity of Earth's surface is approximately 465 m/s or about 1674 km/h, which is quite substantial but not noticeable in everyday life because we rotate along with Earth.

A baseball pitcher brings his arm forward during a pitch, rotating the forearm about the elbow. If the velocity of the ball in the pitcher's hand is 35.0 m/s and the ball is 0.300 m from the elbow joint, what is the angular velocity of the forearm?

Show Solution

Strategy

We need to find the angular velocity given the linear velocity and radius. We'll use the relationship $v = r\omega$, which we can rearrange to solve for $\omega = \frac{v}{r}$.

Solution

Using the relationship between linear and angular velocity:

$$\omega = \frac{v}{r} = \frac{35.0\ \text{m/s}}{0.300\ \text{m}} = 117\ \text{rad/s}$$

Discussion

The angular velocity of the forearm is 117 rad/s, which is quite large. To put this in perspective, we can convert to revolutions per second: $\omega = 117\ \text{rad/s} \times \frac{1\ \text{rev}}{2\pi\ \text{rad}} = 18.6\ \text{rev/s}$, or about 19 revolutions per second. This high angular velocity, combined with the radius from the elbow to the ball, produces the high linear velocity needed for a fast pitch. Professional pitchers achieve these impressive velocities through coordinated whole-body motion, not just arm rotation.

In lacrosse, a ball is thrown from a net on the end of a stick by rotating the stick and forearm about the elbow. If the angular velocity of the ball about the elbow joint is 30.0 rad/s and the ball is 1.30 m from the elbow joint, what is the velocity of the ball?

Show Solution

Strategy

We need to find the linear velocity of the ball given its angular velocity and distance from the axis of rotation. We'll use the relationship $v = r\omega$.

Solution

Using the relationship between linear and angular velocity:

$$v = r\omega = (1.30\ \text{m})(30.0\ \text{rad/s}) = 39.0\ \text{m/s}$$

Discussion

The velocity of the lacrosse ball is 39.0 m/s, which is greater than the baseball pitcher's velocity of 35.0 m/s in the earlier problem, despite having a lower angular velocity (30.0 rad/s compared to 117 rad/s). This is because the ball is much farther from the axis of rotation (1.30 m compared to 0.300 m), demonstrating that linear velocity is directly proportional to both angular velocity and radius.

A truck with 0.420-m-radius tires travels at 32.0 m/s. What is the angular velocity of the rotating tires in radians per second? What is this in rev/min?

Show Solution

Strategy

We need to find the angular velocity from the linear velocity and radius. We'll use $\omega = vr$ to find the angular velocity in rad/s, then convert to rev/min using appropriate conversion factors.

Solution

Calculate the angular velocity in rad/s:

$$\omega = vr = 32.0\text{ m/s}0.420\text{ m} = 76.2\text{ rad/s}$$

Convert to rev/min:

$$\omega = 76.2\text{rads} \times 1\text{ rev}2\pi\text{ rad} \times 60\text{ s}1\text{ min} = 76.2 \times 602\pi\text{ rev/min} = 45726.28\text{ rev/min} = 728\text{ rpm}$$

Discussion

The angular velocity of the truck's tires is 76.2 rad/s or 728 rpm. This rotation rate is typical for vehicles traveling at highway speeds. Notice that the angular velocity depends on both the linear speed and the tire radius - larger tires would rotate more slowly at the same linear speed, while smaller tires would rotate faster.

Integrated Concepts When kicking a football, the kicker rotates his leg about the hip joint.

(a) If the velocity of the tip of the kicker's shoe is 35.0 m/s and the hip joint is 1.05 m from the tip of the shoe, what is the shoe tip's angular velocity?

(b) The shoe is in contact with the initially stationary 0.500 kg football for 20.0 ms. What average force is exerted on the football to give it a velocity of 20.0 m/s?

(c) Find the maximum range of the football, neglecting air resistance.

Show Solution

Strategy

(a) We'll use $\omega = vr$ to find the angular velocity of the leg. (b) We'll use the impulse-momentum theorem: $F\Delta t = m\Delta v$ to find the average force. (c) Maximum range occurs at a 45° launch angle. We'll use the range formula $R = v^2\sin(2\theta)g$.

Solution

(a) Calculate the angular velocity:

$$\omega = vr = 35.0\text{ m/s}1.05\text{ m} = 33.3\text{ rad/s}$$

(b) Calculate the average force using the impulse-momentum theorem:

$$F\Delta t = m\Delta v$$
$$F = m\Delta v\Delta t = (0.500\text{ kg})(20.0\text{ m/s} - 0)20.0 \times 10^{-3}\text{ s} = 10.0\text{ kg}\cdot\text{m/s}0.0200\text{ s} = 500\text{ N}$$

(c) Calculate the maximum range (at 45° launch angle):

$$R = v^2\sin(2\theta)g = (20.0\text{ m/s})^2\sin(90°)9.80\text{ m/s}^2 = 400 \times 19.80 = 40.8\text{ m}$$

Discussion

The angular velocity of the kicker's leg is 33.3 rad/s, which combined with the 1.05 m radius produces a shoe tip velocity of 35.0 m/s. The average force on the football during the brief 20 ms contact is 500 N (about 112 lb), which accelerates the football to 20.0 m/s. At the optimal 45° launch angle, the football can travel a maximum distance of 40.8 m (about 45 yards), which is a reasonable distance for a punt or field goal attempt in American football.

Construct Your Own Problem

Consider an amusement park ride in which participants are rotated about a vertical axis in a cylinder with vertical walls. Once the angular velocity reaches its full value, the floor drops away and friction between the walls and the riders prevents them from sliding down. Construct a problem in which you calculate the necessary angular velocity that assures the riders will not slide down the wall. Include a free body diagram of a single rider. Among the variables to consider are the radius of the cylinder and the coefficients of friction between the riders' clothing and the wall.

Glossary

arc length
: Δs, the distance traveled by an object along a circular path

pit
: a tiny indentation on the spiral track moulded into the top of the polycarbonate layer of CD

rotation angle
: the ratio of the arc length to the radius of curvature on a circular path: $\Delta\theta = \Delta sr$

radius of curvature
: radius of a circular path

radians
: a unit of angle measurement

angular velocity
: ω, the rate of change of the angle with which an object moves on a circular path

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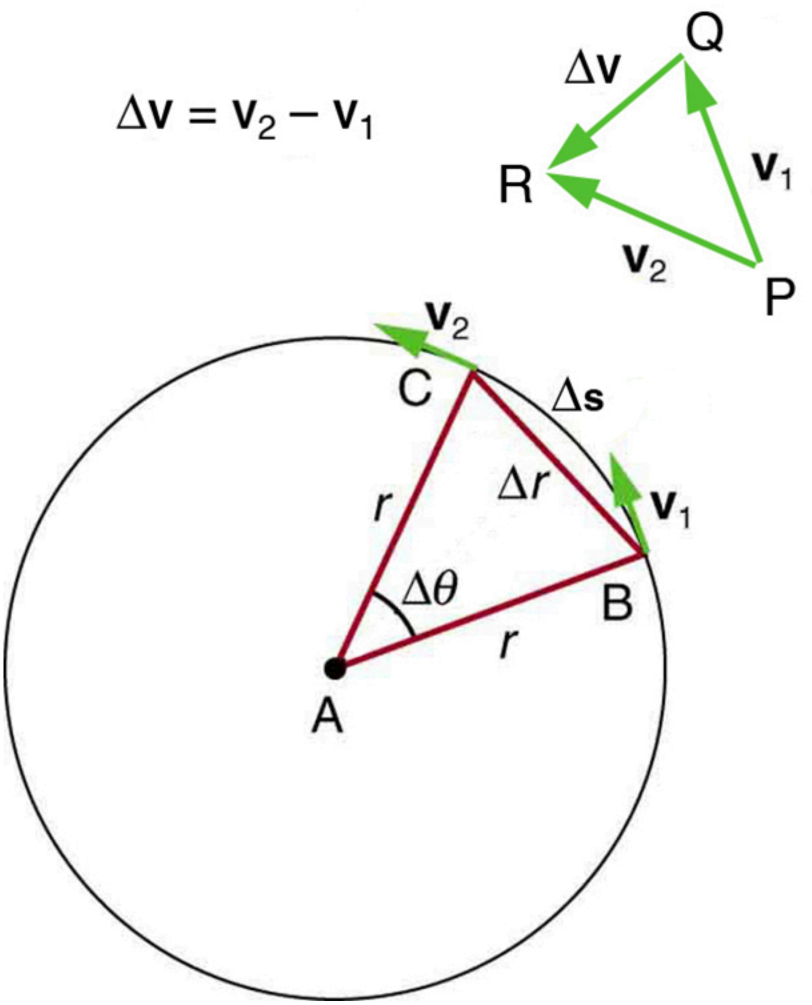
Centripetal Acceleration

- Establish the expression for centripetal acceleration.
- Explain the centrifuge.

We know from kinematics that acceleration is a change in velocity, either in its magnitude or in its direction, or both. In uniform circular motion, the direction of the velocity changes constantly, so there is always an associated acceleration, even though the magnitude of the velocity might be constant. You experience this acceleration yourself when you turn a corner in your car. (If you hold the wheel steady during a turn and move at constant speed, you are in uniform circular motion.) What you notice is a sideways acceleration because you and the car are changing direction. The sharper the curve and the greater your speed, the more noticeable this acceleration will become. In this section we examine the direction and magnitude of that acceleration.

Figure 1 shows an object moving in a circular path at constant speed. The direction of the instantaneous velocity is shown at two points along the path. Acceleration is in the direction of the change in velocity, which points directly toward the center of rotation (the center of the circular path). This pointing is shown with the vector diagram in the figure. We call the acceleration of an object moving in uniform circular motion (resulting from a net external force) the **centripetal acceleration** (a_c); centripetal means "toward the center" or "center seeking."



The directions of the velocity of an object at two different points are shown, and the change in velocity $\vec{\Delta \mathbf{v}}$ is seen to point directly toward the center of curvature. (See small inset.) Because $\vec{\mathbf{a}}_c = \frac{\vec{\Delta \mathbf{v}}}{\Delta t}$, the acceleration is also toward the center; $\vec{\mathbf{a}}_c$ is called centripetal acceleration. (Because $\Delta\theta$ is very small, the arc length Δs is equal to the chord length Δr for small time differences.)

The direction of centripetal acceleration is toward the center of curvature, but what is its magnitude? Note that the triangle formed by the velocity vectors and the one formed by the radii r and Δs are similar. Both the triangles ABC and PQR are isosceles triangles (two equal sides). The two equal sides of the velocity vector triangle are the speeds $v_1 = v_2 = v$. Using the properties of two similar triangles, we obtain

$$\frac{\Delta v}{v}=\frac{\Delta s}{r}.$$

Acceleration is $\Delta v/\Delta t$, and so we first solve this expression for Δv:

$$\Delta v=\frac{v}{r}\Delta s.$$

Then we divide this by Δt, yielding

$$\Delta v \Delta t = v r \times \Delta s \Delta t.$$

Finally, noting that $\Delta v / \Delta t = a_\mathrm{c}$ and that $\Delta s / \Delta t = v$, the linear or tangential speed, we see that the magnitude of the centripetal acceleration is

$$a_\mathrm{c} = v^2 r,$$

which is the acceleration of an object in a circle of radius r at a speed v. So, centripetal acceleration is greater at high speeds and in sharp curves (smaller radius), as you have noticed when driving a car. But it is a bit surprising that a_c is proportional to speed squared, implying, for example, that it is four times as hard to take a curve at 100 km/h than at 50 km/h. A sharp corner has a small radius, so that a_c is greater for tighter turns, as you have probably noticed.

It is also useful to express a_c in terms of angular velocity. Substituting $v = r\omega$ into the above expression, we find $a_\mathrm{c} = (r\omega)^2 / r = r\omega^2$. We can express the magnitude of centripetal acceleration using either of two equations:

$$a_\mathrm{c} = v^2 r \text{ and } a_\mathrm{c} = r\omega^2.$$

Recall that the direction of a_c is toward the center. You may use whichever expression is more convenient, as illustrated in examples below.

A **centrifuge** (see Figure 2b) is a rotating device used to separate specimens of different densities. High centripetal acceleration significantly decreases the time it takes for separation to occur, and makes separation possible with small samples. Centrifuges are used in a variety of applications in science and medicine, including the separation of single cell suspensions such as bacteria, viruses, and blood cells from a liquid medium and the separation of macromolecules, such as DNA and protein, from a solution. Centrifuges are often rated in terms of their centripetal acceleration relative to acceleration due to gravity (g); maximum centripetal acceleration of several hundred thousand g is possible in a vacuum. Human centrifuges, extremely large centrifuges, have been used to test the tolerance of astronauts to the effects of accelerations larger than that of Earth's gravity.

How Does the Centripetal Acceleration of a Car Around a Curve Compare with That Due to Gravity?

What is the magnitude of the centripetal acceleration of a car following a curve of radius 500 m at a speed of 25.0 m/s (about 90 km/h)? Compare the acceleration with that due to gravity for this fairly gentle curve taken at highway speed. See Figure 2(a).

Strategy

Because v and r are given, the first expression in $a_\mathrm{c} = v^2 r; a_\mathrm{c} = r\omega^2$ is the most convenient to use.

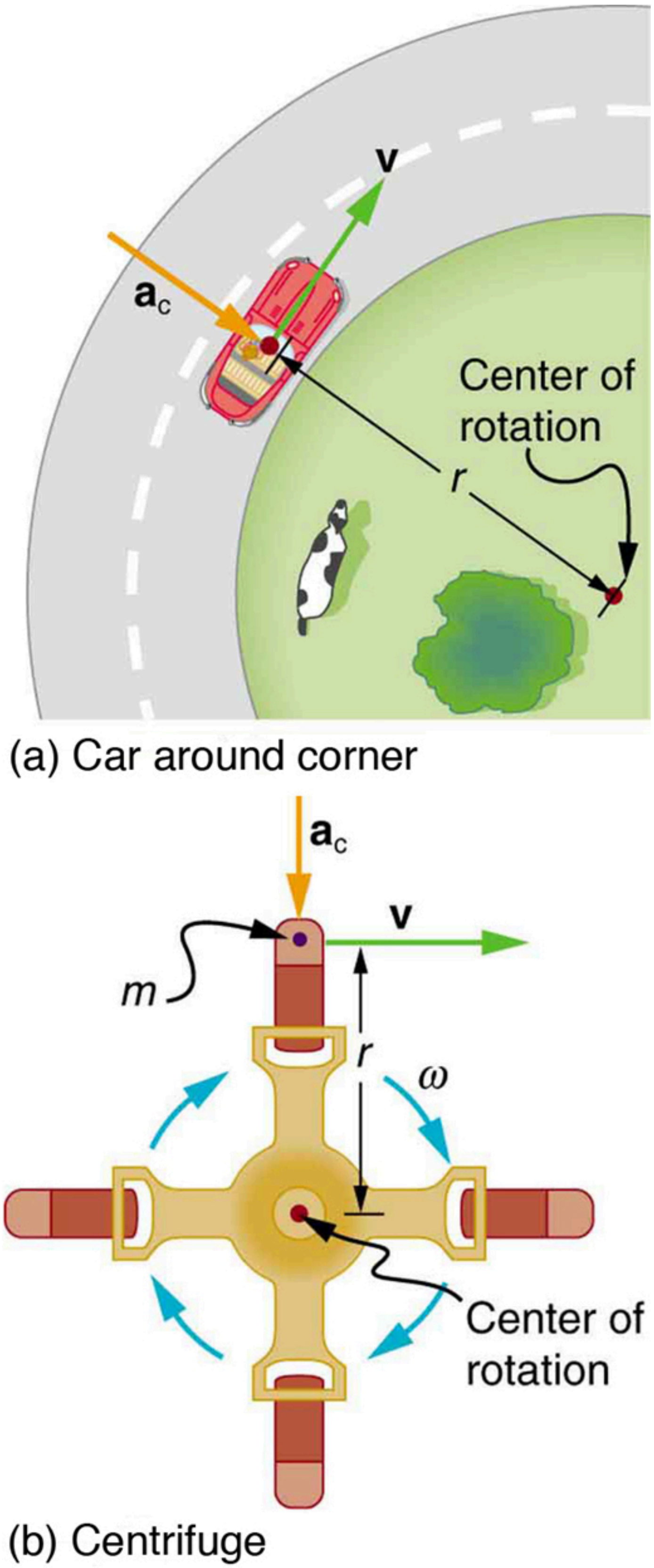
Solution

Entering the given values of $v = 25.0\,\mathrm{m/s}$ and $r = 500\,\mathrm{m}$ into the first expression for a_c gives

$$a_\mathrm{c} = v^2 r = (25.0\,\mathrm{m/s})^2 500\,\mathrm{m} = 1.25\,\mathrm{m/s}^2.$$

Discussion

To compare this with the acceleration due to gravity $(g = 9.80\,\mathrm{m/s}^2)$, we take the ratio of $a_\mathrm{c} / g = (1.25\,\mathrm{m/s}^2)/(9.80\,\mathrm{m/s}^2) = 0.128$. Thus, $a_\mathrm{c} = 0.128g$ and is noticeable especially if you were not wearing a seat belt.



(a) The car following a circular path at constant speed is accelerated perpendicular to its velocity, as shown. The magnitude of this centripetal acceleration is found in [Example 1](). (b) A particle of mass in a centrifuge is rotating at constant angular velocity . It must be accelerated perpendicular to its velocity or it would continue in a straight line. The magnitude of the necessary acceleration is found in [Example 2]().

How Big Is the Centripetal Acceleration in an Ultracentrifuge?

Calculate the centripetal acceleration of a point 7.50 cm from the axis of an **ultracentrifuge** spinning at $7.5\times10^4\ \text{rev/min}$. Determine the ratio of this acceleration to that due to gravity. See Figure 2(b).

Strategy

The term rev/min stands for revolutions per minute. By converting this to radians per second, we obtain the angular velocity ω. Because r is given, we can use the second expression in the equation $a_c=\frac{v^2}{r};\ a_c=r\omega^2$ to calculate the centripetal acceleration.

Solution

To convert $7.50\times10^4\ \text{rev/min}$ to radians per second, we use the facts that one revolution is $2\pi\ \text{rad}$ and one minute is 60.0 s. Thus,

$$\omega=7.50\times10^4\ \frac{\text{rev}}{\text{min}}\times\frac{2\pi\ \text{rad}}{1\ \text{rev}}\times\frac{1\ \text{min}}{60.0\ \text{s}}=7854\ \text{rad/s}.$$

Now the centripetal acceleration is given by the second expression in $a_c=\frac{v^2}{r};\ a_c=r\omega^2$ as

$$a_c=r\omega^2.$$

Converting 7.50 cm to meters and substituting known values gives

$$a_c=(0.0750\ \text{m})(7854\ \text{rad/s})^2=4.63\times10^6\ \text{m/s}^2.$$

Note that the unitless radians are discarded in order to get the correct units for centripetal acceleration. Taking the ratio of a_c to g yields

$$\frac{a_c}{g}=\frac{4.63\times10^6}{9.80}=4.72\times10^5.$$

Discussion

This last result means that the centripetal acceleration is 472 000 times as strong as g. It is no wonder that such high ω centrifuges are called ultracentrifuges. The extremely large accelerations involved greatly decrease the time needed to cause the sedimentation of blood cells or other materials.

Of course, a net external force is needed to cause any acceleration, just as Newton proposed in his second law of motion. So a net external force is needed to cause a centripetal acceleration. In Centripetal Force, we will consider the forces involved in circular motion.

Ladybug Motion 2D

Learn about position, velocity and acceleration vectors. Move the ladybug by setting the position, velocity or acceleration, and see how the vectors change. Choose linear, circular or elliptical motion, and record and playback the motion to analyze the behavior.

Ladybug Motion 2D

Section Summary

- Centripetal acceleration a_c is the acceleration experienced while in uniform circular motion. It always points toward the center of rotation. It is perpendicular to the linear velocity v and has the magnitude
  $a_c = v^2 r; a_c = r\omega^2.$
- The unit of centripetal acceleration is $\mathrm{m/s^2}$.

Conceptual Questions

Can centripetal acceleration change the speed of circular motion? Explain.

Problem Exercises

A fairground ride spins its occupants inside a flying saucer-shaped container. If the horizontal circular path the riders follow has an 8.00 m radius, at how many revolutions per minute will the riders be subjected to a centripetal acceleration whose magnitude is 1.50 times that due to gravity?

Show Solution

Strategy

We need to find the angular velocity that produces a centripetal acceleration of $a_c = 1.50g$. We'll use $a_c = r\omega^2$ and solve for ω, then convert from rad/s to rev/min.

Solution

The centripetal acceleration is:

$$a_c = 1.50g = 1.50(9.80\ \mathrm{m/s^2}) = 14.7\ \mathrm{m/s^2}$$

Using $a_c = r\omega^2$ and solving for ω:

$$\omega^2 = a_c r = 14.7\ \mathrm{m/s^2} 8.00\ \mathrm{m} = 1.84\ \mathrm{rad^2/s^2}$$
$$\omega = \sqrt{1.84} = 1.36\ \mathrm{rad/s}$$

Convert to rev/min:

$$\omega = 1.36 \mathrm{rads} \times 1\ \mathrm{rev} 2\pi\ \mathrm{rad} \times 60\ \mathrm{s} 1\ \mathrm{min} = 1.36 \times 60 2\pi\ \mathrm{rev/min} = 12.9\ \mathrm{rev/min}$$

Discussion

The fairground ride must rotate at 12.9 revolutions per minute to subject riders to a centripetal acceleration of 1.50g. This is about one revolution every 4.6 seconds, which is slow enough to be comfortable but fast enough to create the sensation of increased "weight" pushing riders outward against the walls of the container.

A runner taking part in the 200 m dash must run around the end of a track that has a circular arc with a radius of curvature of 30 m. If she completes the 200 m dash in 23.2 s and runs at constant speed throughout the race, what is the magnitude of her centripetal acceleration as she runs the curved portion of the track?

Show Solution

Strategy

First, we need to find the runner's constant speed by dividing the total distance by the total time. Then we'll use the centripetal acceleration formula $a_c = v^2 r$ with the radius of curvature given for the curved portion of the track.

Solution

Calculate the runner's speed:

$$v = dt = 200\ \mathrm{m} 23.2\ \mathrm{s} = 8.62\ \mathrm{m/s}$$

Now calculate the centripetal acceleration on the curved portion:

$$a_c = v^2 r = (8.62\ \mathrm{m/s})^2 30\ \mathrm{m} = 74.3\ \mathrm{m^2/s^2} 30\ \mathrm{m} = 2.48\ \mathrm{m/s^2}$$

Discussion

The magnitude of the runner's centripetal acceleration as she runs the curved portion of the track is 2.48 m/s² or approximately 2.5 m/s². This is about one-quarter of the acceleration due to gravity, which the runner would definitely feel as she rounds the curve.

Taking the age of Earth to be about 4×10^9 years and assuming its orbital radius of 1.5×10^{11} m has not changed and is circular, calculate the approximate total distance Earth has traveled since its birth (in a frame of reference stationary with respect to the Sun).

Show Solution

Strategy

Earth completes one orbit around the Sun each year, traveling a distance equal to the circumference of its orbit. We'll calculate the circumference, then multiply by the number of years Earth has existed.

Solution

The circumference of Earth's orbit is:

$$C = 2\pi r = 2\pi(1.5\times10^{11}\text{ m}) = 9.42\times10^{11}\text{ m}$$

The total distance traveled in 4×10^9 years is:

$$s = (\text{number of orbits})\times C = (4\times10^9)(9.42\times10^{11}\text{ m}) = 3.77\times10^{21}\text{ m} \approx 4\times10^{21}\text{ m}$$

Discussion

Earth has traveled approximately 4×10^{21} meters (4 billion trillion meters) since its birth 4 billion years ago. This staggering distance is about 400 trillion kilometers, or roughly 25 million light-years. Despite this enormous journey through space, Earth remains in a stable orbit due to the gravitational force of the Sun providing the necessary centripetal force.

The propeller of a World War II fighter plane is 2.30 m in diameter.

(a) What is its angular velocity in radians per second if it spins at 1200 rev/min?

(b) What is the linear speed of its tip at this angular velocity if the plane is stationary on the tarmac?

(c) What is the centripetal acceleration of the propeller tip under these conditions? Calculate it in meters per second squared and convert to multiples of g.

Show Solution

Strategy

(a) We'll convert from rev/min to rad/s using the conversion factors: 1 revolution = 2π radians and 1 minute = 60 seconds. (b) We'll use $v = r\omega$ where the radius is half the diameter. (c) We'll use $a_c = r\omega^2$ to find the centripetal acceleration, then divide by $g = 9.80$ m/s² to express it as a multiple of g.

Solution

(a) Convert angular velocity to rad/s:

$$\omega = 1200\,\frac{\text{rev}}{\text{min}}\times\frac{2\pi\text{ rad}}{1\text{ rev}}\times\frac{1\text{ min}}{60\text{ s}} = 126\text{ rad/s}$$

(b) Calculate the linear speed at the tip. The radius is half the diameter:

$$r = \frac{d}{2} = \frac{2.30\text{ m}}{2} = 1.15\text{ m}$$
$$v = r\omega = (1.15\text{ m})(126\text{ rad/s}) = 145\text{ m/s}$$

(c) Calculate the centripetal acceleration:

$$a_c = r\omega^2 = (1.15\text{ m})(126\text{ rad/s})^2 = (1.15\text{ m})(15{,}876\text{ rad}^2/\text{s}^2) = 1.83\times10^4\text{ m/s}^2$$

Express as a multiple of g:

$$\frac{a_c}{g} = \frac{1.83\times10^4\text{ m/s}^2}{9.80\text{ m/s}^2} = 1.87\times10^3 = 1870$$

Discussion

The propeller tip has an angular velocity of 126 rad/s and moves at a linear speed of 145 m/s. The centripetal acceleration at the tip is 1.83×10^4 m/s², which is approximately 1870 times the acceleration due to gravity. This enormous acceleration places tremendous stress on the propeller blades, which is why they must be constructed from very strong materials.

An ordinary workshop grindstone has a radius of 7.50 cm and rotates at 6500 rev/min.

(a) Calculate the magnitude of the centripetal acceleration at its edge in meters per second squared and convert it to multiples of g.

(b) What is the linear speed of a point on its edge?

Show Solution

Strategy

(a) We'll convert the angular velocity from rev/min to rad/s, then use $a_c = r\omega^2$ to find the centripetal acceleration. (b) We'll use $v = r\omega$ to find the linear speed.

Solution

Convert angular velocity to rad/s:

$$\omega = 6500 \frac{\text{rev}}{\text{min}} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} \times \frac{1 \text{ min}}{60 \text{ s}} = \frac{6500 \times 2\pi}{60} \text{ rad/s} = 681 \text{ rad/s}$$

(a) Calculate the centripetal acceleration (with radius in meters: $r = 0.0750$ m):

$$a_c = r\omega^2 = (0.0750 \text{ m})(681 \text{ rad/s})^2 = (0.0750)(464{,}000) = 3.47 \times 10^4 \text{ m/s}^2$$

Express as a multiple of g:

$$\frac{a_c}{g} = \frac{3.47 \times 10^4 \text{ m/s}^2}{9.80 \text{ m/s}^2} = 3.55 \times 10^3 = 3550$$

(b) Calculate the linear speed:

$$v = r\omega = (0.0750 \text{ m})(681 \text{ rad/s}) = 51.1 \text{ m/s}$$

Discussion

The centripetal acceleration at the edge of the grindstone is 3.47×10^4 m/s², which is 3550 times the acceleration due to gravity. The linear speed at the edge is 51.1 m/s (about 184 km/h or 114 mi/h). These extreme values demonstrate why grindstones must be made from very strong materials and why safety guards are essential - if a piece broke off, it would be ejected at high speed with devastating force.

Helicopter blades withstand tremendous stresses. In addition to supporting the weight of a helicopter, they are spun at rapid rates and experience large centripetal accelerations, especially at the tip.

(a) Calculate the magnitude of the centripetal acceleration at the tip of a 4.00 m long helicopter blade that rotates at 300 rev/min.

(b) Compare the linear speed of the tip with the speed of sound (taken to be 340 m/s).

Show Solution

Strategy

(a) First, we'll convert the angular velocity from rev/min to rad/s. Then we'll use $a_c = r\omega^2$ to find the centripetal acceleration at the tip. (b) We'll calculate the linear speed using $v = r\omega$ and compare it to the speed of sound.

Solution

(a) Convert angular velocity to rad/s:

$$\omega = 300 \frac{\text{rev}}{\text{min}} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} \times \frac{1 \text{ min}}{60 \text{ s}} = 31.4 \text{ rad/s}$$

Calculate the centripetal acceleration:

$$a_c = r\omega^2 = (4.00 \text{ m})(31.4 \text{ rad/s})^2 = (4.00 \text{ m})(986 \text{ rad}^2/\text{s}^2) = 3.95 \times 10^3 \text{ m/s}^2$$

(b) Calculate the linear speed at the tip:

$$v = r\omega = (4.00 \text{ m})(31.4 \text{ rad/s}) = 126 \text{ m/s}$$

Compare to the speed of sound:

$$\frac{v}{v_{\text{sound}}} = \frac{126 \text{ m/s}}{340 \text{ m/s}} = 0.371$$

Discussion

The magnitude of the centripetal acceleration at the tip of the helicopter blade is 3.95×10^3 m/s² or approximately 3950 m/s². The linear speed of the tip is 126 m/s, which is about 37% of the speed of sound. This is well below the speed of sound, which is important because if the blade tips approached or exceeded the speed of sound, they would create shock waves that would dramatically reduce efficiency and increase noise and vibration.

Olympic ice skaters are able to spin at about 5 rev/s.

(a) What is their angular velocity in radians per second?

(b) What is the centripetal acceleration of the skater's nose if it is 0.120 m from the axis of rotation?

(c) An exceptional skater named Dick Button was able to spin much faster in the 1950s than anyone since—at about 9 rev/s. What was the centripetal acceleration of the tip of his nose, assuming it is at 0.120 m radius?

(d) Comment on the magnitudes of the accelerations found. It is reputed that Button ruptured small blood vessels during his spins.

Show Solution

Strategy

(a) We'll convert from rev/s to rad/s using the conversion factor: 1 revolution = 2π radians. (b) and (c) We'll use $a_c = r\omega^2$ to find the centripetal acceleration for each skater. (d) We'll compare these accelerations to $g = 9.80$ m/s² to understand the physical stress.

Solution

(a) Convert angular velocity to rad/s:

$$\omega = 5 \frac{\text{rev}}{\text{s}} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} = 10\pi \text{ rad/s} = 31.4 \text{ rad/s}$$

(b) Calculate the centripetal acceleration for the Olympic skater:

$$a_c = r\omega^2 = (0.120 \text{ m})(31.4 \text{ rad/s})^2 = (0.120)(986) = 118 \text{ m/s}^2$$

(c) For Dick Button spinning at 9 rev/s:

$$\omega = 9 \frac{\text{rev}}{\text{s}} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} = 18\pi \text{ rad/s} = 56.5 \text{ rad/s}$$
$$a_c = r\omega^2 = (0.120 \text{ m})(56.5 \text{ rad/s})^2 = (0.120)(3192) = 383 \text{ m/s}^2$$

(d) Compare to gravity:

For the Olympic skater:

$$\frac{a_c}{g} = \frac{118 \text{ m/s}^2}{9.80 \text{ m/s}^2} = 12.0$$

For Dick Button:

$$\frac{a_c}{g} = \frac{383 \text{ m/s}^2}{9.80 \text{ m/s}^2} = 39.1$$

Discussion

The centripetal acceleration felt by Olympic skaters spinning at 5 rev/s is 12 times the acceleration due to gravity—that's quite substantial. Dick Button's faster spin rate of 9 rev/s produced a centripetal acceleration 39 times larger than gravity. At such extreme accelerations, blood vessels experience tremendous stress, especially the small capillaries near the surface. The pressure difference between the center and periphery of the body becomes so large that it's not surprising Button ruptured small blood vessels during his record-breaking spins. This is why modern figure skaters rarely attempt such extreme spin rates—the physiological stress is simply too great.

What percentage of the acceleration at Earth's surface is the acceleration due to gravity at the position of a satellite located 300 km above Earth?

Show Solution

Strategy

We'll use the relationship $g = \frac{GM}{r^2}$ to find the acceleration due to gravity at the satellite's altitude. The acceleration is inversely proportional to the square of the distance from Earth's center. We'll need to use Earth's radius (approximately 6380 km) and add the satellite's altitude to get the total distance from Earth's center.

Solution

The acceleration due to gravity at any distance r from Earth's center is:

$$g = \frac{GM}{r^2}$$

Taking the ratio of gravity at the satellite's position to gravity at Earth's surface:

$$\frac{g_{\text{satellite}}}{g_{\text{surface}}} = \frac{GM/r^2_{\text{satellite}}}{GM/r^2_{\text{Earth}}} = \frac{r^2_{\text{Earth}}}{r^2_{\text{satellite}}}$$

where:

- r_Earth $= 6.38 \times 10^6$ m
- r_satellite $= 6.38 \times 10^6 + 3.00 \times 10^5 = 6.68 \times 10^6$ m

$$\frac{g_{\text{satellite}}}{g_{\text{surface}}} = \frac{(6.38 \times 10^6\ \text{m})^2}{(6.68 \times 10^6\ \text{m})^2} = \frac{4.07 \times 10^{13}}{4.46 \times 10^{13}} = 0.912$$

Convert to percentage:

$$0.912 \times 100\% = 91.2\%$$

Discussion

The acceleration due to gravity at 300 km above Earth's surface is 91.2% of the acceleration at Earth's surface. This shows that gravity doesn't decrease very rapidly with altitude - even at 300 km above the surface, gravity is still over 90% as strong as it is on the ground. The "weightlessness" experienced by astronauts at this altitude is not due to the absence of gravity, but rather to the fact that they are in continuous free fall around Earth.

Verify that the linear speed of an ultracentrifuge is about 0.50 km/s, and Earth in its orbit is about 30 km/s by calculating:

(a) The linear speed of a point on an ultracentrifuge 0.100 m from its center, rotating at 50 000 rev/min.

(b) The linear speed of Earth in its orbit about the Sun (use data from the text on the radius of Earth's orbit and approximate it as being circular).

Show Solution

Strategy

(a) We'll convert the angular velocity to rad/s and use $v = r\omega$ to find the linear speed. (b) We'll use $v = \frac{2\pi r}{T}$ where $r = 1.5 \times 10^{11}$ m (Earth's orbital radius) and $T = 1$ year.

Solution

(a) Convert angular velocity to rad/s:

$$\omega = 50{,}000 \frac{\text{rev}}{\text{min}} \times \frac{2\pi\ \text{rad}}{1\ \text{rev}} \times \frac{1\ \text{min}}{60\ \text{s}} = \frac{50{,}000 \times 2\pi}{60} = 5236\ \text{rad/s}$$

Calculate the linear speed:

$$v = r\omega = (0.100\ \text{m})(5236\ \text{rad/s}) = 524\ \text{m/s} = 0.524\ \text{km/s}$$

(b) Calculate Earth's orbital speed. First, convert the period to seconds:

$$T = 1\ \text{year} \times \frac{365.25\ \text{days}}{1\ \text{year}} \times \frac{24\ \text{h}}{1\ \text{day}} \times \frac{3600\ \text{s}}{1\ \text{h}} = 3.156 \times 10^7\ \text{s}$$

Calculate the linear speed:

$$v = \frac{2\pi r}{T} = \frac{2\pi(1.5 \times 10^{11}\ \text{m})}{3.156 \times 10^7\ \text{s}} = \frac{9.42 \times 10^{11}}{3.156 \times 10^7} = 2.99 \times 10^4\ \text{m/s} = 29.9\ \text{km/s}$$

Discussion

The ultracentrifuge has a linear speed of 0.524 km/s at 0.100 m from its center, which verifies the approximate value of 0.50 km/s. Earth's orbital speed is 29.9 km/s (about 30 km/s), which is approximately 60 times faster than the ultracentrifuge. Despite Earth's enormous mass and distance from the Sun, it travels at this remarkable speed to maintain its stable circular orbit.

A rotating space station is said to create "artificial gravity"—a loosely-defined term used for an acceleration that would be crudely similar to gravity. The outer wall of the rotating space station would become a floor for the astronauts, and centripetal acceleration supplied by the floor would allow astronauts to exercise and maintain muscle and bone strength more naturally than in non-rotating space environments. If the space station is 200 m in diameter, what angular velocity would produce an "artificial gravity" of 9.80m/s^2 at the rim?

Show Solution

Strategy

We need to find the angular velocity that produces a centripetal acceleration equal to $g = 9.80$ m/s² at the rim of the space station. We'll use the formula $a_c = r\omega^2$ and solve for ω. The radius is half the diameter.

Solution

The radius of the space station is:

$$r = \frac{d}{2} = \frac{200\ \text{m}}{2} = 100\ \text{m}$$

Using the centripetal acceleration formula and solving for ω:

$$a_c = r\omega^2$$
$$\omega^2 = \frac{a_c}{r} = \frac{9.80\ \text{m/s}^2}{100\ \text{m}} = 0.0980\ \text{rad}^2/\text{s}^2$$
$$\omega = \sqrt{0.0980\ \text{rad}^2/\text{s}^2} = 0.313\ \text{rad/s}$$

Discussion

The space station must rotate with an angular velocity of 0.313 rad/s to produce artificial gravity equal to Earth's gravity at the rim. This corresponds to about 3.0 revolutions per minute. At this rotation rate, astronauts standing on the outer wall would experience a centripetal acceleration pushing them toward the floor that would feel very similar to normal gravity, allowing them to maintain muscle and bone mass during extended space missions.

At takeoff, a commercial jet has a 60.0 m/s speed. Its tires have a diameter of 0.850 m.

(a) At how many rev/min are the tires rotating?

(b) What is the centripetal acceleration at the edge of the tire?

(c) With what force must a determined $1.00 \times 10^{-15}\text{kg}$ bacterium cling to the rim?

(d) Take the ratio of this force to the bacterium's weight.

Show Solution

Strategy

(a) We'll use $\omega = \frac{v}{r}$ to find angular velocity, then convert to rev/min. (b) We'll use $a_c = \frac{v^2}{r}$ to find the centripetal acceleration. (c) We'll use $F_c = ma_c$ to find the centripetal force. (d) We'll calculate weight using $w = mg$ and find the ratio.

Solution

The radius is half the diameter: $r = \frac{0.850}{2} = 0.425$ m.

(a) Calculate the angular velocity:

$$\omega = \frac{v}{r} = \frac{60.0\ \text{m/s}}{0.425\ \text{m}} = 141\ \text{rad/s}$$

Convert to rev/min:

$$\omega = 141 \frac{\text{rad}}{\text{s}} \times \frac{1\ \text{rev}}{2\pi\ \text{rad}} \times \frac{60\ \text{s}}{1\ \text{min}} = \frac{141 \times 60}{2\pi} = 1.35 \times 10^3\ \text{rpm}$$

(b) Calculate the centripetal acceleration:

$$a_c = \frac{v^2}{r} = \frac{(60.0\ \text{m/s})^2}{0.425\ \text{m}} = \frac{3600}{0.425} = 8.47 \times 10^3\ \text{m/s}^2$$

(c) Calculate the centripetal force on the bacterium:

$$F_c = ma_c = (1.00 \times 10^{-15}\ \text{kg})(8.47 \times 10^3\ \text{m/s}^2) = 8.47 \times 10^{-12}\ \text{N}$$

(d) Calculate the bacterium's weight and the ratio:

$$w = mg = (1.00 \times 10^{-15}\ \text{kg})(9.80\ \text{m/s}^2) = 9.80 \times 10^{-15}\ \text{N}$$
$$\frac{F_c}{w} = \frac{8.47 \times 10^{-12}\ \text{N}}{9.80 \times 10^{-15}\ \text{N}} = 865$$

Discussion

At takeoff, the jet's tires rotate at 1,350 rpm with a centripetal acceleration of 8,470 m/s² (about 865g) at the edge. A bacterium clinging to the rim must withstand a force 865 times its weight to avoid being flung off. This enormous centripetal acceleration demonstrates the extreme forces experienced by rotating objects at high speeds, which is why tire integrity is critical for aircraft safety.

Integrated Concepts

Riders in an amusement park ride shaped like a Viking ship hung from a large pivot are rotated back and forth like a rigid pendulum. Sometime near the middle of the ride, the ship is momentarily motionless at the top of its circular arc. The ship then swings down under the influence of gravity. The radius of the circular arc is 39.2 m, and the speed at the bottom of the arc is 39.2 m/s.

(a) What is the centripetal acceleration at the bottom of the arc?

(b) Draw a free body diagram of the forces acting on a rider at the bottom of the arc.

(c) Find the force exerted by the ride on a 60.0 kg rider and compare it to their weight.

(d) Discuss whether the answer seems reasonable.

Show Solution

Strategy

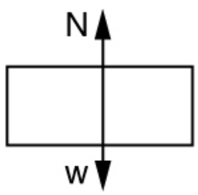
(a) We'll use $a_C = v^2 r$ to find the centripetal acceleration with the given values v =39.2 m/s and r =39.2 m. (b) We'll draw forces acting on the rider: weight downward and normal force upward. (c) We'll use Newton's second law in the radial direction: $N - mg = ma_C$. (d) We'll evaluate whether the result makes physical sense.

Solution

(a) Using the relationship $a_C = v^2 r$ with v =39.2 m/s and r =39.2 m:

$$a_C = v^2 r = (39.2 \text{ m/s})^2 39.2 \text{ m} = 1536.6439.2 = 39.2 \text{ m/s}^2$$

(b) Free body diagram:



The rider experiences two forces: the normal force N upward from the seat, and weight $w = mg$ downward.

(c) Applying Newton's second law in the vertical direction (upward is positive):

$$N - mg = ma_C$$
$$N = mg + ma_C = m(g + a_C)$$
$$N = (60.0 \text{ kg})(9.80 \text{ m/s}^2 + 39.2 \text{ m/s}^2) = (60.0)(49.0) = 2.94 \times 10^3 \text{ N}$$

Compare to weight:

$$w = mg = (60.0 \text{ kg})(9.80 \text{ m/s}^2) = 588 \text{ N}$$
$$N w = 2940588 = 5.00$$

The normal force is **5.00 times the rider's weight**.

Discussion

This answer is very reasonable. At the bottom of the arc, the rider experiences an apparent weight five times their normal weight—they feel pressed into the seat with a force of 2,940 N. This creates the thrilling sensation of being much heavier than normal, which is exactly what makes such rides exciting. The centripetal acceleration of 39.2 m/s² (about 4g) combined with Earth's gravity (1g) produces a total apparent acceleration of 5g toward the center of the circular path. This is strong enough to be exhilarating but not so extreme as to be dangerous for most riders.

Unreasonable Results

A mother pushes her child on a swing so that his speed is 9.00 m/s at the lowest point of his path. The swing is suspended 2.00 m above the child's center of mass.

(a) What is the magnitude of the centripetal acceleration of the child at the low point?

(b) What is the magnitude of the force the child exerts on the seat if his mass is 18.0 kg?

(c) What is unreasonable about these results?

(d) Which premises are unreasonable or inconsistent?

Show Solution

Strategy

(a) We'll use $a_C = v^2 r$ to find the centripetal acceleration. (b) We'll use Newton's second law: $N - mg = ma_C$ to find the normal force, which equals the force the child exerts on the seat by Newton's third law. (c) We'll examine whether the results are physically reasonable. (d) We'll identify which assumptions or values are problematic.

Solution

(a) Calculate the centripetal acceleration with $v = 9.00$ m/s and $r = 2.00$ m:

$$a_c = \frac{v^2}{r} = \frac{(9.00\text{ m/s})^2}{2.00\text{ m}} = \frac{81.0}{2.00} = 40.5\text{ m/s}^2$$

(b) Apply Newton's second law at the bottom of the swing (upward positive):

$$N - mg = ma_c$$

$$N = m(g + a_c) = (18.0\text{ kg})(9.80\text{ m/s}^2 + 40.5\text{ m/s}^2) = (18.0)(50.3) = 905\text{ N}$$

(c) **Analysis of unreasonable results:**

The centripetal acceleration of 40.5 m/s² is about **4.1 times gravity**—this is an extreme acceleration for a playground swing. The force of 905 N is equivalent to the weight of a 92 kg person, meaning an 18 kg child would need to exert a force more than **5 times their own weight** on the seat. This would be extremely uncomfortable and potentially dangerous.

(d) **Unreasonable premises:**

The speed of 9.00 m/s at the bottom is far too large for a typical swing. To check this, we can use energy conservation. At the bottom, the child has kinetic energy:

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}(18.0)(9.00)^2 = 729\text{ J}$$

For the child to swing up to the top of a circular arc (height $h = 2r = 4.00$ m above the bottom), the potential energy needed would be:

$$PE = mgh = (18.0)(9.80)(4.00) = 706\text{ J}$$

Since the kinetic energy (729 J) exceeds the potential energy needed to reach the top (706 J), the child would actually have enough energy to go completely over the top of the swing, which would cause the swing to go slack and the child to be in free fall—a very dangerous situation!

Discussion

The fundamental problem is that the specified speed of 9.00 m/s is unrealistically high for a 2.00 m long swing. A typical playground swing might reach speeds of 3-4 m/s at most. The given speed would create forces and accelerations that are not only uncomfortable but also structurally unsafe, and would result in the physically impossible situation of the child looping completely over the top. This problem illustrates the importance of checking whether calculated results make physical sense in real-world contexts.

Glossary

centripetal acceleration
: the acceleration of an object moving in a circle, directed toward the center

ultracentrifuge
: a centrifuge optimized for spinning a rotor at very high speeds

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Centripetal Force

- Calculate coefficient of friction on a car tire.
- Calculate ideal speed and angle of a car on a turn.

Any force or combination of forces can cause a centripetal or radial acceleration. Just a few examples are the tension in the rope on a tether ball, the force of Earth's gravity on the Moon, friction between roller skates and a rink floor, a banked roadway's force on a car, and forces on the tube of a spinning centrifuge.

Any net force causing uniform circular motion is called a **centripetal force**. The direction of a centripetal force is toward the center of curvature, the same as the direction of centripetal acceleration. According to Newton's second law of motion, net force is mass times acceleration: $\vec{\mathbf{F}}_{\text{net}} = m\vec{\mathbf{a}}$. For uniform circular motion, the acceleration is the centripetal acceleration— $\vec{\mathbf{a}} = \vec{\mathbf{a}}_{\text{C}}$. Thus, the magnitude of centripetal force $|\vec{\mathbf{F}}_{\mathbf{c}}|$ is

$$|\vec{\mathbf{F}}_{\mathbf{c}}| = m|\vec{\mathbf{a}}_{\text{C}}|.$$

By using the expressions for the magnitude of the centripetal acceleration $|\vec{\mathbf{a}}_{\text{C}}|$ from $a_{\text{C}} = v^2 r; a_{\text{C}} = r\omega^2$, we get two expressions for the magnitude of the centripetal force $|\vec{\mathbf{F}}_{\mathbf{c}}|$ in terms of mass, velocity, angular velocity, and radius of curvature:

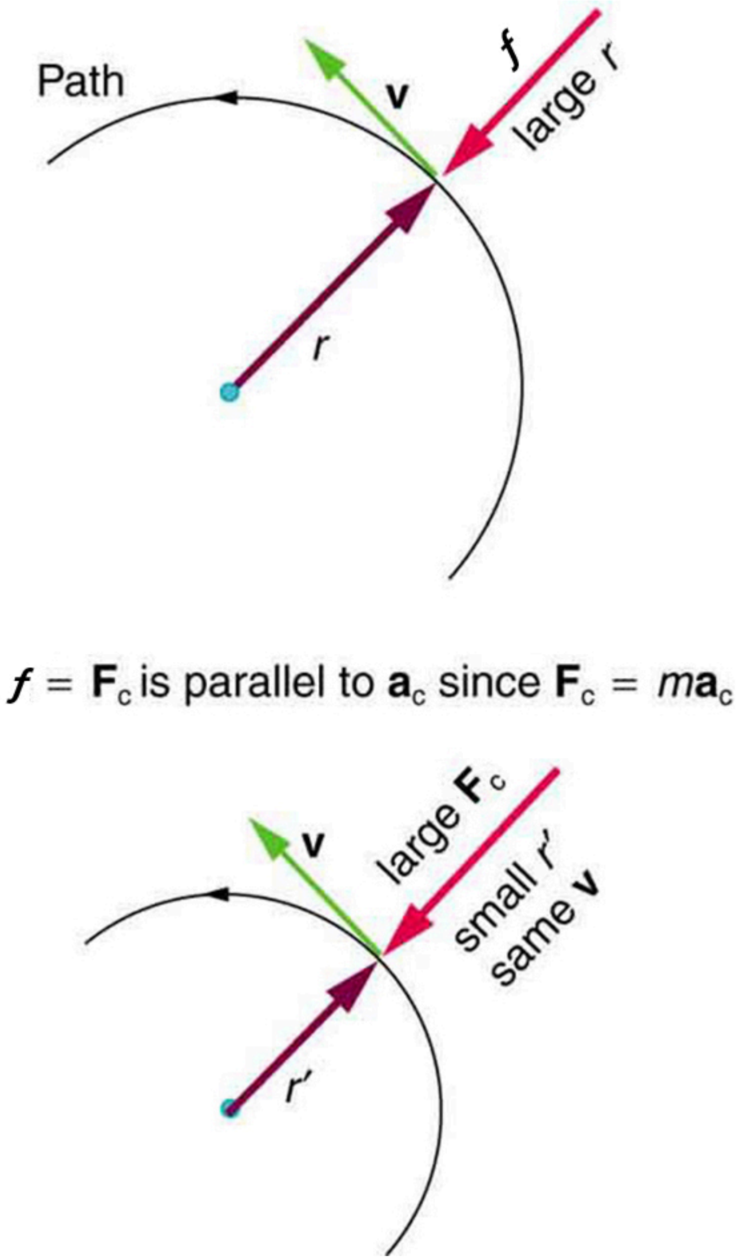
$$F_{\text{C}} = mv^2 r; F_{\text{C}} = mr\omega^2.$$

You may use whichever expression for centripetal force is more convenient. Centripetal force $\vec{\mathbf{F}}_{\text{C}}$ is always perpendicular to the path and pointing to the center of curvature, because $\vec{\mathbf{a}}_{\text{C}}$ is perpendicular to the velocity and pointing to the center of curvature.

Note that if you solve the first expression for r, you get

$$r = mv^2 F_{\text{C}}.$$

This implies that for a given mass and velocity, a large centripetal force causes a small radius of curvature—that is, a tight curve.



The frictional force supplies the centripetal force and is numerically equal to it. Centripetal force is perpendicular to velocity and causes uniform circular motion. The larger the F_c , the smaller the radius of curvature r and the sharper the curve. The second curve has the same v , but a larger F_c produces a smaller $r^\prime$.

What Coefficient of Friction Do Car Tires Need on a Flat Curve?

(a) Calculate the centripetal force exerted on a 900 kg car that negotiates a 500 m radius curve at 25.0 m/s.

(b) Assuming an unbanked curve, find the minimum static coefficient of friction, between the tires and the road, static friction being the reason that keeps the car from slipping (see Figure 2).

Strategy and Solution for (a)

We know that F_c $=mv^2r$. Thus,

$F_c=mv^2r=(900\text{kg})(25.0\text{m/s})^2(500\text{m})=1125\text{N}.$

Strategy for (b)

Figure 2 shows the forces acting on the car on an unbanked (level ground) curve. Friction is to the left, keeping the car from slipping, and because it is the only horizontal force acting on the car, the friction is the centripetal force in this case. We know that the maximum static friction (at which the tires roll but do not slip) is $\mu_S N$, where μ_S is the static coefficient of friction and N is the normal force. The normal force equals the car's weight on level ground, so that $N = mg$. Thus the centripetal force in this situation is

$$F_C = f = \mu_S N = \mu_S mg.$$

Now we have a relationship between centripetal force and the coefficient of friction. Using the first expression for F_C from the equation

$$F_C = mv^2 r \quad F_C = mr\omega^2 ,$$

$$mv^2 r = \mu_S mg.$$

We solve this for μ_s, noting that mass cancels, and obtain

$$\mu_S = v^2 r g.$$

Solution for (b)

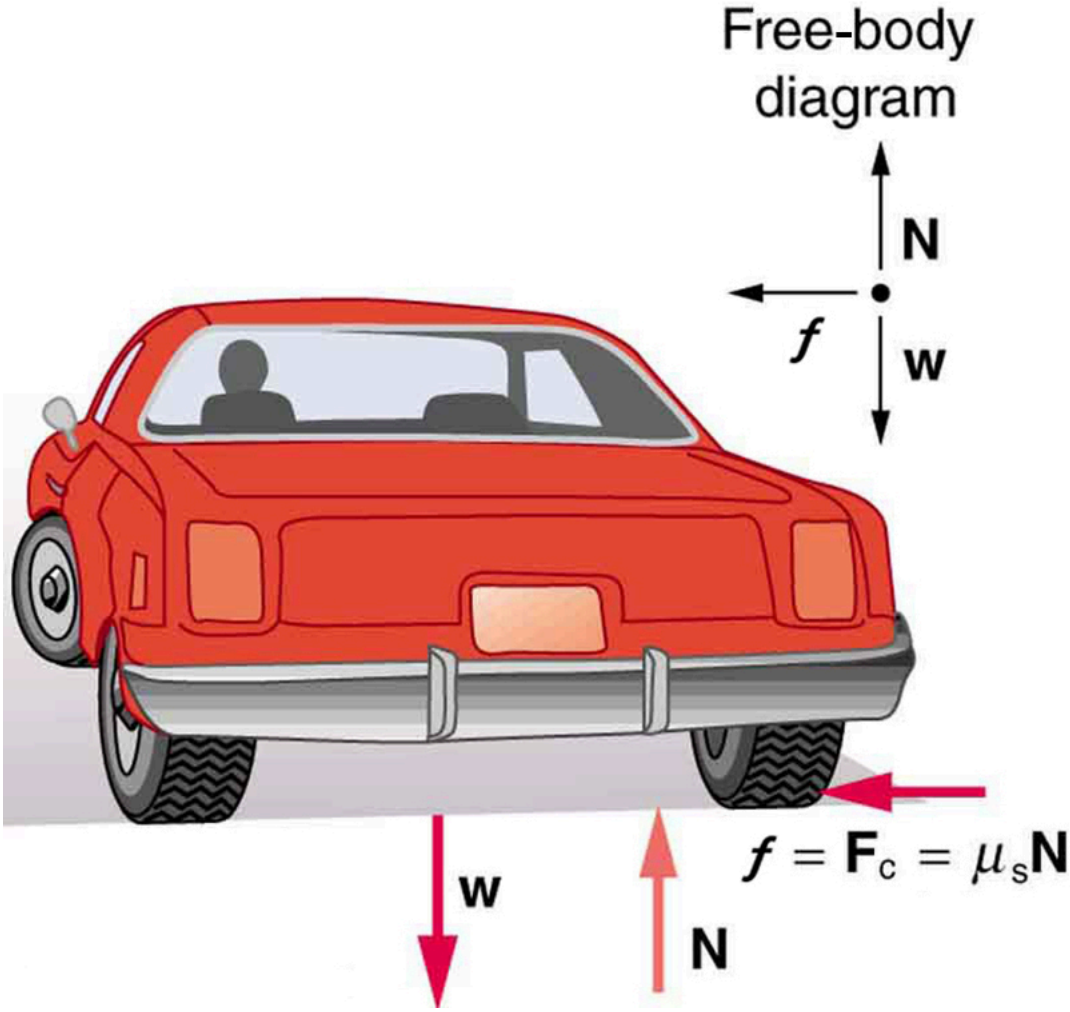
Substituting the knowns,

$$\mu_S = (25.0\text{m/s})^2 (500\text{m})(9.80\text{m/s}^2) = 0.13.$$

(Because coefficients of friction are approximate, the answer is given to only two digits.)

Discussion

We could also solve part (a) using the first expression in $F_C = mv^2 r \quad F_C = mr\omega^2$ }, because m, v, and r are given. The coefficient of friction found in part (b) is much smaller than is typically found between tires and roads. The car will still negotiate the curve if the coefficient is greater than 0.13, because static friction is a responsive force, being able to assume a value less than but no more than $\mu_S N$. A higher coefficient would also allow the car to negotiate the curve at a higher speed, but if the coefficient of friction is less, the safe speed would be less than 25 m/s. Note that mass cancels, implying that in this example, it does not matter how heavily loaded the car is to negotiate the turn. Mass cancels because friction is assumed proportional to the normal force, which in turn is proportional to mass. If the surface of the road were banked, the normal force would be less as will be discussed below.



This car on level ground is moving away and turning to the left. The centripetal force causing the car to turn in a circular path is due to friction between the tires and the road. A minimum coefficient of friction is needed, or the car will move in a larger-radius curve and leave the roadway.

Let us now consider **banked curves**, where the slope of the road helps you negotiate the curve. See Figure 3. The greater the angle θ, the faster you can take the curve. Race tracks for bikes as well as cars, for example, often have steeply banked curves. In an "ideally banked curve," the angle θ is such that you can negotiate the curve at a certain speed without the aid of friction between the tires and the road. We will derive an expression for θ for an ideally banked curve and consider an example related to it.

For **ideal banking**, the net external force equals the horizontal centripetal force in the absence of friction. The components of the normal force N in the horizontal and vertical directions must equal the centripetal force and the weight of the car, respectively. In cases in which forces are not parallel, it is most convenient to consider components along perpendicular axes—in this case, the vertical and horizontal directions.

Figure 3 shows a free body diagram for a car on a frictionless banked curve. If the angle θ is ideal for the speed and radius, then the net external force will equal the necessary centripetal force. The only two external forces acting on the car are its weight $\vec{\mathbf{w}}$ and the normal force of the road $\vec{\mathbf{N}}$. (A frictionless surface can only exert a force perpendicular to the surface—that is, a normal force.) These two forces must add to give a net external force that is horizontal toward the center of curvature and has magnitude mv^2r. Because this is the crucial force and it is horizontal, we use a coordinate system with vertical and horizontal axes. Only the normal force has a horizontal component, and so this must equal the centripetal force—that is,

$$N\sin\theta=mv^2r.$$

Because the car does not leave the surface of the road, the net vertical force must be zero, meaning that the vertical components of the two external forces must be equal in magnitude and opposite in direction. From the figure, we see that the vertical component of the normal force is $N\cos\theta$, and the only other vertical force is the car's weight. These must be equal in magnitude; thus,

$$N\cos\theta=mg.$$

Now we can combine the last two equations to eliminate N and get an expression for θ, as desired. Solving the second equation for $N=mg/(\cos\theta)$, and substituting this into the first yields

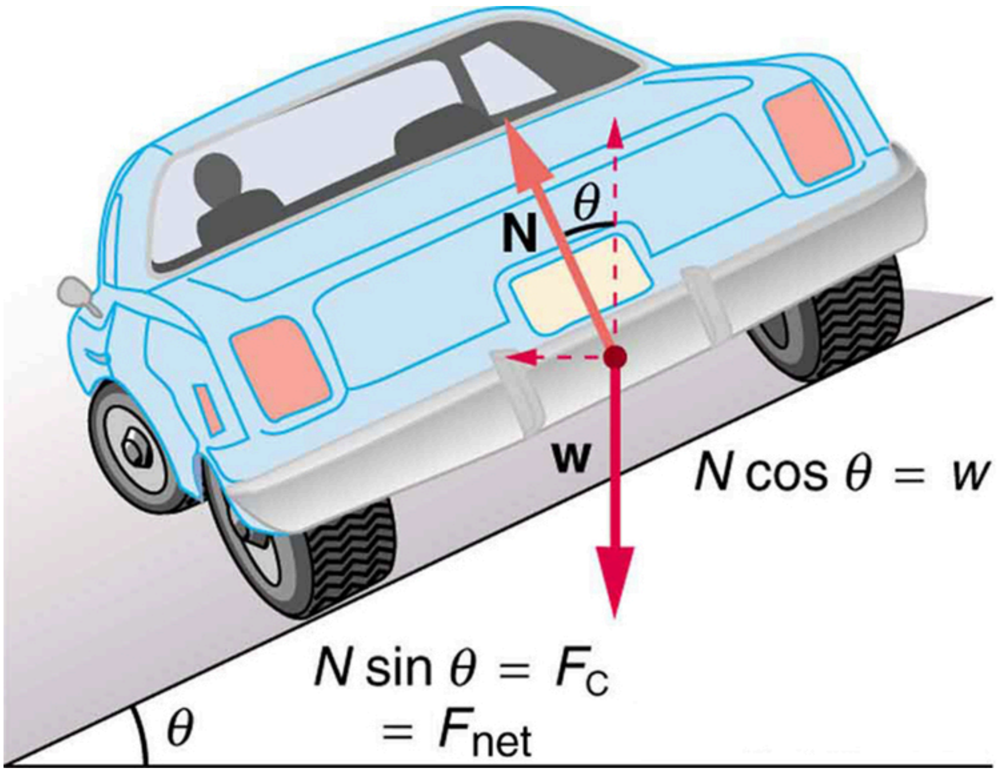
$mg\sin\theta\cos\theta = mv^2r$

$mg\tan(\theta) = mv^2r \quad \tan\theta = v^2rg.$

Taking the inverse tangent gives

$\theta = \tan^{-1}(v^2rg)$(ideally banked curve, no friction).

This expression can be understood by considering how θ depends on v and r. A large θ will be obtained for a large v and a small r. That is, roads must be steeply banked for high speeds and sharp curves. Friction helps, because it allows you to take the curve at greater or lower speed than if the curve is frictionless. Note that θ does not depend on the mass of the vehicle.



The car on this banked curve is moving away and turning to the left.

What Is the Ideal Speed to Take a Steeply Banked Tight Curve?

Curves on some test tracks and race courses, such as the Daytona International Speedway in Florida, are very steeply banked. This banking, with the aid of tire friction and very stable car configurations, allows the curves to be taken at very high speed. To illustrate, calculate the speed at which a 100 m radius curve banked at 65.0° should be driven if the road is frictionless.

Strategy

We first note that all terms in the expression for the ideal angle of a banked curve except for speed are known; thus, we need only rearrange it so that speed appears on the left-hand side and then substitute known quantities.

Solution

Starting with

$\tan\theta = v^2rg$

we get

$v = \sqrt{rg\tan\theta}.$

Noting that $\tan 65.0^\circ = 2.14$, we obtain

$v = \sqrt{(100\text{m})(9.80\text{m/s}^2)(2.14)} \quad v = 45.8\text{m/s}.$

Discussion

This is just about 165 km/h, consistent with a very steeply banked and rather sharp curve. Tire friction enables a vehicle to take the curve at significantly higher speeds.

Calculations similar to those in the preceding examples can be performed for a host of interesting situations in which centripetal force is involved—a number of these are presented in this chapter’s Problems and Exercises.
Take-Home Experiment

Ask a friend or relative to swing a golf club or a tennis racquet. Take appropriate measurements to estimate the centripetal acceleration of the end of the club or racquet. You may choose to do this in slow motion.

Section Summary

- Centripetal force $\vec{\mathbf{F}}$_c is any force causing uniform circular motion. It is a “center-seeking” force that always points toward the center of rotation. It is perpendicular to linear velocity v and has magnitude

  $$F_c = ma_c,$$

which can also be expressed as

$$F_c = mv^2r \quad \text{or} \quad F_c = mr\omega^2$$

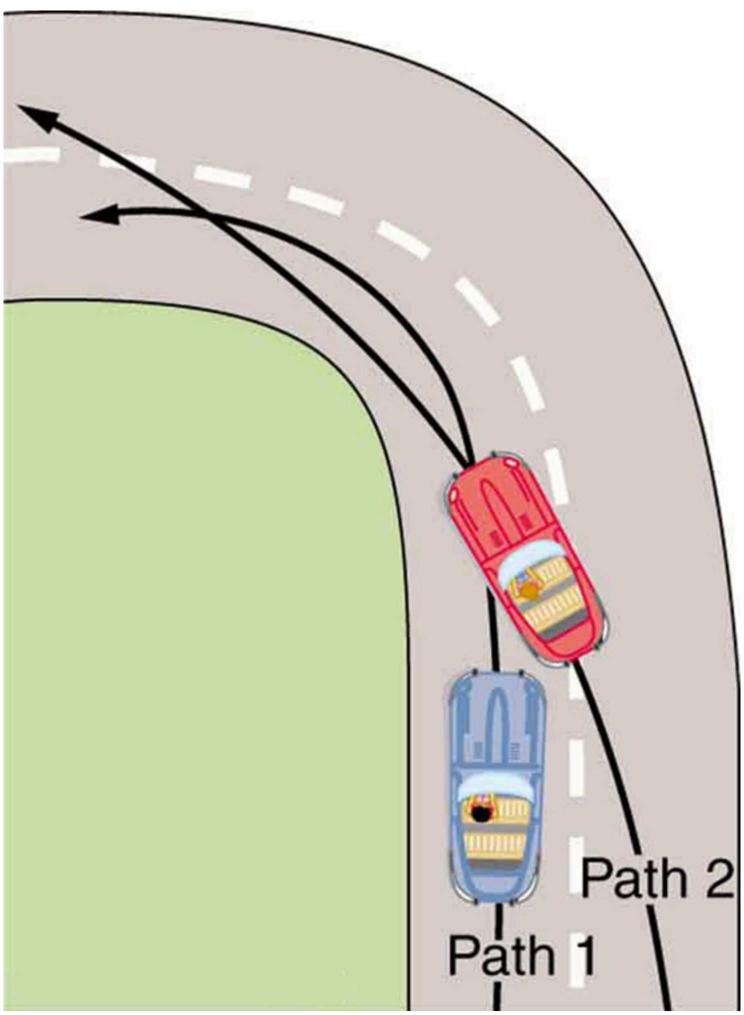
Conceptual Questions

If you wish to reduce the stress (which is related to centripetal force) on high-speed tires, would you use large- or small-diameter tires? Explain.

Define centripetal force. Can any type of force (for example, tension, gravitational force, friction, and so on) be a centripetal force? Can any combination of forces be a centripetal force?

If centripetal force is directed toward the center, why do you feel that you are ‘thrown’ away from the center as a car goes around a curve? Explain.

Race car drivers routinely cut corners as shown in Figure 4. Explain how this allows the curve to be taken at the greatest speed.

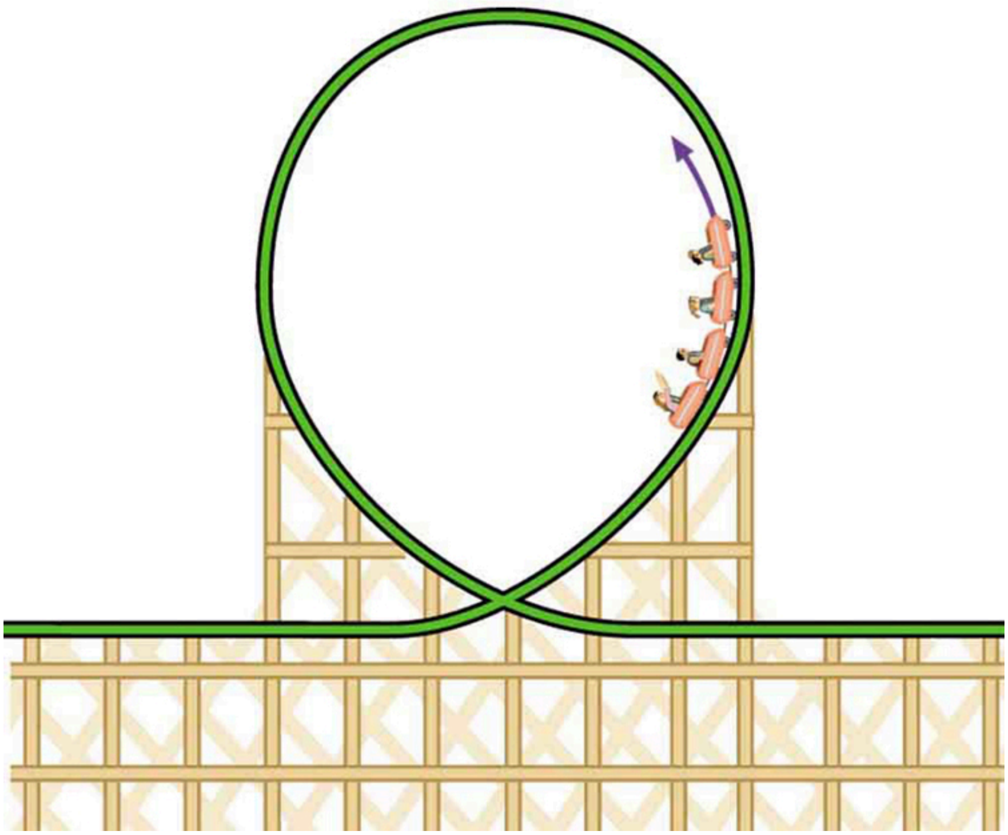


Two paths around a race track curve are shown. Race car drivers will take the inside path (called cutting the corner) whenever possible because it allows them to take the curve at the highest speed.

A number of amusement parks have rides that make vertical loops like the one shown in Figure 5. For safety, the cars are attached to the rails in such a way that they cannot fall off. If the car goes over the top at just the right speed, gravity alone will supply the centripetal force. What other force acts and what is its direction if:

(a) The car goes over the top at faster than this speed?

(b)The car goes over the top at slower than this speed?



Amusement rides with a vertical loop are an example of a form of curved motion.

What is the direction of the force exerted by the car on the passenger as the car goes over the top of the amusement ride pictured in Figure 5 under the following circumstances:

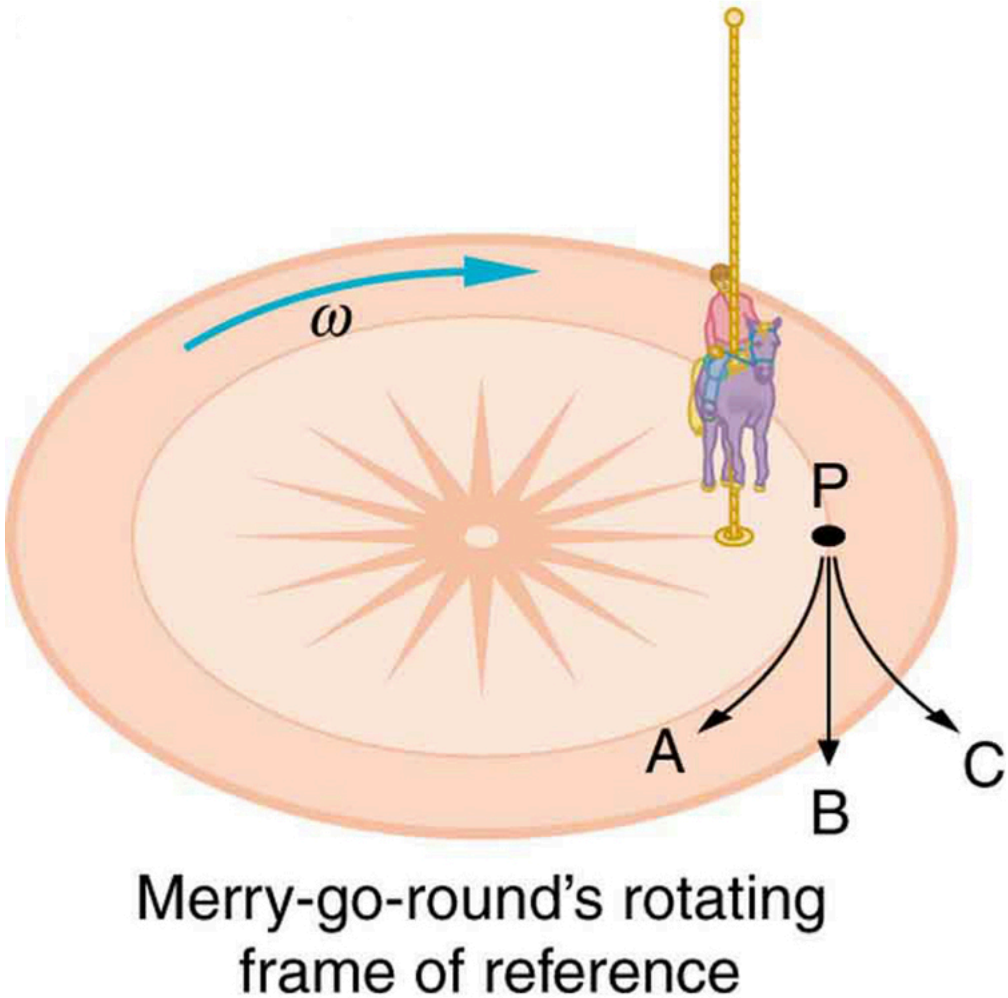
(a) The car goes over the top at such a speed that the gravitational force is the only force acting?

(b) The car goes over the top faster than this speed?

(c) The car goes over the top slower than this speed?

As a skater forms a circle, what force is responsible for making her turn? Use a free body diagram in your answer.

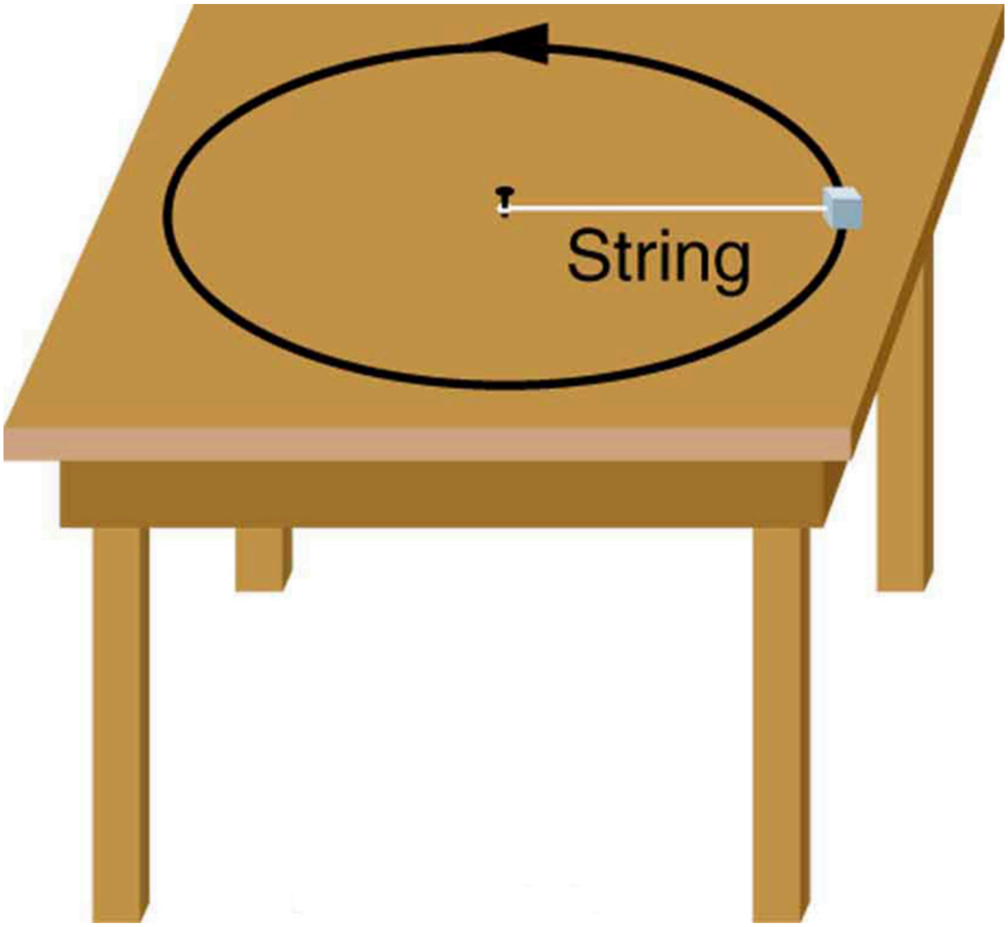
Suppose a child is riding on a merry-go-round at a distance about halfway between its center and edge. She has a lunch box resting on wax paper, so that there is very little friction between it and the merry-go-round. Which path shown in Figure 6 will the lunch box take when she lets go? The lunch box leaves a trail in the dust on the merry-go-round. Is that trail straight, curved to the left, or curved to the right? Explain your answer.



A child riding on a merry-go-round releases her lunch box at point P. This is a view from above the clockwise rotation. Assuming it slides with negligible friction, will it follow path A, B, or C, as viewed from Earth's frame of reference? What will be the shape of the path it leaves in the dust on the merry-go-round?

Do you feel yourself thrown to either side when you negotiate a curve that is ideally banked for your car's speed? What is the direction of the force exerted on you by the car seat?

Suppose a mass is moving in a circular path on a frictionless table as shown in figure. In the Earth's frame of reference, there is no centrifugal force pulling the mass away from the centre of rotation, yet there is a very real force stretching the string attaching the mass to the nail. Using concepts related to centripetal force and Newton's third law, explain what force stretches the string, identifying its physical origin.



A mass attached to a nail on a frictionless table moves in a circular path. The force stretching the string is real and not fictional. What is the physical origin of the force on the string?

Problems Exercise

(a) A 22.0 kg child is riding a playground merry-go-round that is rotating at 40.0 rev/min. What centripetal force must she exert to stay on if she is 1.25 m from its center?

(b) What centripetal force does she need to stay on an amusement park merry-go-round that rotates at 3.00 rev/min if she is 8.00 m from its center?

(c) Compare each force with her weight.

Show Solution

Strategy

We'll convert angular velocities to rad/s, then use $F_c = mr\omega^2$ to calculate the centripetal force. We'll compare each to the child's weight $w = mg$.

Solution

(a) Convert angular velocity and calculate centripetal force:

ω=40.0revmin×2π rad1 rev×1 min60 s=4.19 rad/s
F_c=$mr\omega^2$=(22.0 kg)(1.25 m)(4.19 rad/s)^2=(22.0)(1.25)(17.6)=483 N

(b) For the amusement park ride:

ω=3.00revmin×2π rad1 rev×1 min60 s=0.314 rad/s
F_c=$mr\omega^2$=(22.0 kg)(8.00 m)(0.314 rad/s)^2=(22.0)(8.00)(0.0987)=17.4 N

(c) Calculate the child's weight and compare:

w=mg=(22.0 kg)(9.80 m/s^2)=216 N
$F_{c,a}$$w$=483 N216 N=2.24

$$\frac{F_{c,b}}{w}=\frac{17.4\text{ N}}{216\text{ N}}=0.0807$$

Discussion

On the playground merry-go-round, the child must exert a centripetal force of 483 N, which is 2.24 times her weight - she would feel significantly “heavier” and need to hold on tightly. On the slower amusement park ride, she needs only 17.4 N, which is just 0.0807 times her weight - barely noticeable. The much larger radius partially compensates for the slower rotation rate in the second case.

Calculate the centripetal force on the end of a 100 m (radius) wind turbine blade that is rotating at 0.5 rev/s. Assume the mass is 4 kg.

Show Solution

Strategy

We need to find the centripetal force using $F_c = mr\omega^2$. First, we’ll convert the angular velocity from rev/s to rad/s, then calculate the centripetal force.

Solution

Convert angular velocity to rad/s:

$$\omega=0.5\frac{\text{rev}}{\text{s}}\times\frac{2\pi\text{ rad}}{1\text{ rev}}=\pi\text{ rad/s}=3.14\text{ rad/s}$$

Calculate the centripetal force:

$$F_c=mr\omega^2=(4\text{ kg})(100\text{ m})(3.14\text{ rad/s})^2=(4)(100)(9.87)=3.95\times10^3\text{ N}$$

Discussion

The centripetal force on the end of the wind turbine blade is approximately 3950 N or about 3.95 kN. This substantial force must be withstood by the blade structure and mounting points, which is why wind turbine blades are constructed from strong, lightweight composite materials. Even though the mass at the tip is only 4 kg, the large radius and rotational speed create a significant centripetal force.

What is the ideal banking angle for a gentle turn of 1.20 km radius on a highway with a 105 km/h speed limit (about 65 mi/h), assuming everyone travels at the limit?

Show Solution

Strategy

For an ideally banked curve, we use $\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$. We’ll need to convert the speed from km/h to m/s.

Solution

Convert the speed to m/s:

$$v=105\frac{\text{km}}{\text{h}}\times\frac{1000\text{ m}}{1\text{ km}}\times\frac{1\text{ h}}{3600\text{ s}}=29.2\text{ m/s}$$

Convert the radius to meters:

$$r=1.20\text{ km}\times1000=1200\text{ m}$$

Calculate the ideal banking angle:

$$\theta=\tan^{-1}\left(\frac{v^2}{rg}\right)=\tan^{-1}\left(\frac{(29.2\text{ m/s})^2}{(1200\text{ m})(9.80\text{ m/s}^2)}\right)$$
$$\theta=\tan^{-1}\left(\frac{853}{11{,}760}\right)=\tan^{-1}(0.0725)=4.14°$$

Discussion

The ideal banking angle for this highway curve is 4.14°, which is a very gentle slope. At this angle, cars traveling at exactly 105 km/h would not need any friction between the tires and road to negotiate the turn - the normal force from the banked road would provide exactly the right amount of centripetal force. In practice, highway curves are typically banked at conservative angles to accommodate a range of speeds safely.

What is the ideal speed to take a 100 m radius curve banked at a 20.0° angle?

Show Solution

Strategy

For an ideally banked curve (no friction needed), we use the relationship $\tan\theta = \frac{v^2}{rg}$. We’ll solve for v and substitute the known values.

Solution

Starting with the ideal banking equation:

$\tan\theta = v^2 r\, g$

Solve for v:

$v^2 = r\, g \tan\theta$
$v = \sqrt{r\, g \tan\theta}$

Substitute known values (r =100 m, g =9.80 m/s², θ =20.0 °):

$v = \sqrt{(100\text{ m})(9.80\text{ m/s}^2)\tan(20.0°)} = \sqrt{(980)(0.364)} = \sqrt{357} = 18.9\text{ m/s}$

Discussion

The ideal speed to take the 100 m radius curve banked at 20.0° is 18.9 m/s (about 68 km/h or 42 mi/h). At this speed, no friction between the tires and road is needed to maintain the circular path - the normal force from the banked road provides exactly the right amount of centripetal force. At speeds higher or lower than this ideal speed, friction would be required to prevent the car from sliding up or down the banked surface.

(a) What is the radius of a bobsled turn banked at 75.0° and taken at 30.0 m/s, assuming it is ideally banked?

(b) Calculate the centripetal acceleration.

(c) Does this acceleration seem large to you?

Show Solution

Strategy

For an ideally banked curve (no friction required), we use the relationship $\tan\theta = v^2 r\, g$. We'll solve for the radius r, then calculate the centripetal acceleration using $a_c = v^2 r$ and compare it to g.

Solution

Given:

- Banking angle: θ =75.0 °
- Speed: v =30.0m/s

(a) Radius of the turn:

From the ideal banking formula:

$\tan\theta = v^2 r\, g$

Solving for r:

$r = v^2 g \tan\theta$

Substitute values:

$r = (30.0\text{ m/s})^2 (9.80\text{ m/s}^2)\tan(75.0°) = 900 (9.80)(3.732) = 900 36.6 = 24.6\text{ m}$

(b) Centripetal acceleration:

$a_c = v^2 r = (30.0\text{ m/s})^2 24.6\text{ m} = 900 24.6 = 36.6\text{ m/s}^2$

(c) Comparison to g:

$a_c\, g = 36.6\text{ m/s}^2 9.80\text{ m/s}^2 = 3.73$

So a_c =3.73g.

Discussion

The centripetal acceleration of 3.73g means that the bobsledders experience a force nearly 4 times their normal weight pushing them into the side of the bobsled. While this might not seem extreme compared to fighter pilots who experience up to 9g, bobsledders must maintain this level of force throughout the turn while also controlling their sled. The 75° banking angle is quite steep - nearly vertical - which is typical for bobsled tracks where tight, fast turns are needed. The small radius of just 24.6 m allows for exciting, high-speed turns while the steep banking keeps the forces manageable. Bobsledders train to handle these forces and must have strong core and neck muscles to maintain proper positioning throughout the run.

Answer

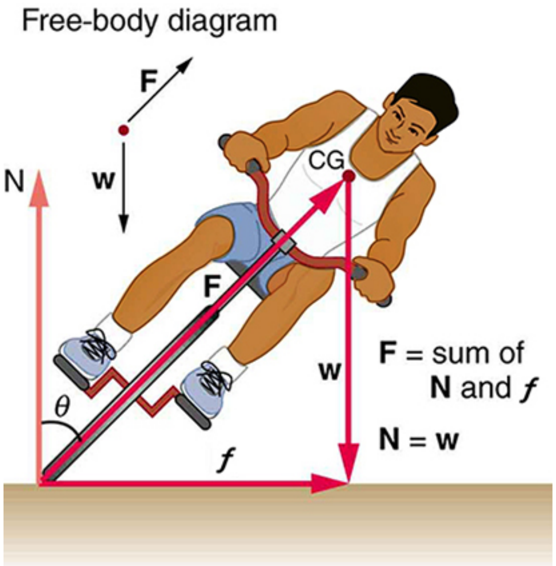
(a) The radius of the bobsled turn is **24.6 m**.

(b) The centripetal acceleration is **36.6 m/s²**.

(c) This acceleration is **3.73g**, which is significant but typical for bobsled runs. It requires athletic conditioning but is within human tolerance for short durations.

Part of riding a bicycle involves leaning at the correct angle when making a turn, as seen in Figure 8. To be stable, the force exerted by the ground must be on a line going through the center of gravity. The force on the bicycle wheel can be resolved into two perpendicular components—friction parallel to the road (this must supply the centripetal force), and the vertical normal force (which must equal the system's weight).

(a) Show that θ (as defined in the figure) is related to the speed v and radius of curvature r of the turn in the same way as for an ideally banked roadway —that is, $\theta = \tan^{-1} v^2/r g$ (b) Calculate θ for a 12.0 m/s turn of radius 30.0 m (as in a race).



A bicyclist negotiating a turn on level ground must lean at the correct angle—the ability to do this becomes instinctive. The force of the ground on the wheel needs to be on a line through the center of gravity. The net external force on the system is the centripetal force. The vertical component of the force on the wheel cancels the weight of the system while its horizontal component must supply the centripetal force. This process produces a relationship among the angle θ, the speed v, and the radius of curvature r of the turn similar to that for the ideal banking of roadways.

Show Solution

Strategy

(a) We'll analyze the forces on the bicycle and rider system. The net force from the ground has components: horizontal (friction providing centripetal force) and vertical (normal force balancing weight). The resultant must pass through the center of gravity, making angle θ with the vertical. (b) We'll use the derived formula to calculate the angle for the given speed and radius.

Solution

(a) The horizontal component of the ground force is the friction f, which provides the centripetal force:

$$f = mv^2 r$$

The vertical component is the normal force N, which equals the weight:

$$N = mg$$

The angle θ that the resultant force makes with the vertical is:

$$\tan\theta = f N = mv^2/r mg = v^2 r g$$

Therefore:

$$\theta = \tan^{-1}(v^2 r g)$$

This is identical to the formula for an ideally banked curve.

(b) Calculate θ for v =12.0 m/s and r =30.0 m:

$$\theta=\tan^{-1}\left(\frac{v^2}{rg}\right)=\tan^{-1}\left(\frac{(12.0\text{ m/s})^2}{(30.0\text{ m})(9.80\text{ m/s}^2)}\right)$$
$$\theta=\tan^{-1}\left(\frac{144}{294}\right)=\tan^{-1}(0.490)=26.1°$$

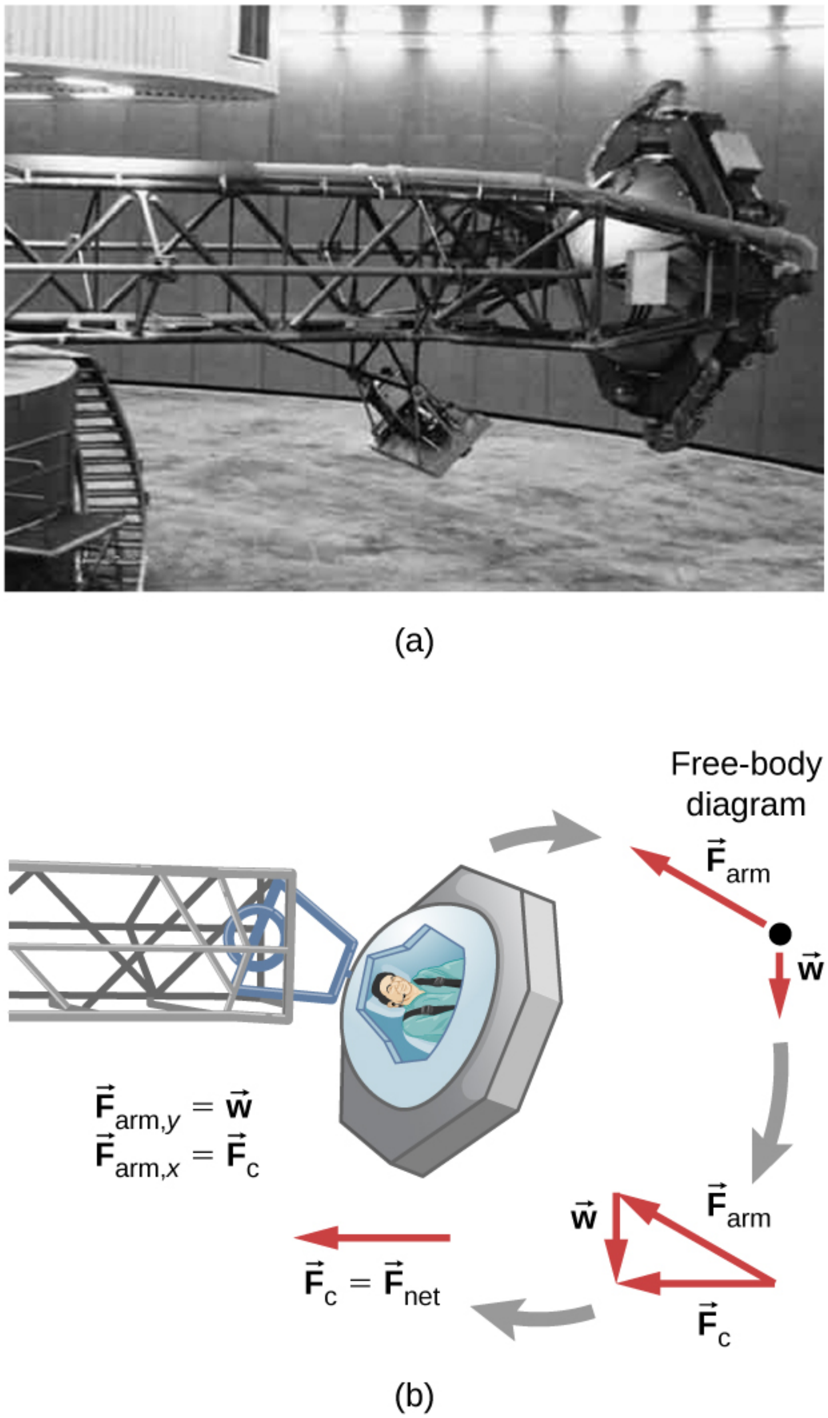
Discussion

We have shown that the bicycle lean angle follows the same relationship as an ideally banked curve. For the racing turn at 12.0 m/s with a 30.0 m radius, the cyclist must lean at 26.1° from the vertical. This significant lean angle is necessary to keep the line of action of the ground force passing through the center of gravity, ensuring stability. Experienced cyclists make these adjustments instinctively.

A large centrifuge, like the one shown in Figure 9(a), is used to expose aspiring astronauts to accelerations similar to those experienced in rocket launches and atmospheric reentries.

(a) At what angular velocity is the centripetal acceleration $10g$ if the rider is 15.0 m from the center of rotation?

(b) The rider's cage hangs on a pivot at the end of the arm, allowing it to swing outward during rotation as shown in Figure 9(b). At what angle θ below the horizontal will the cage hang when the centripetal acceleration is $10g$? (Hint: The arm supplies centripetal force and supports the weight of the cage. Draw a free body diagram of the forces to see what the angle θ should be.)



(a) NASA centrifuge used to subject trainees to accelerations similar to those experienced in rocket launches and reentries. (credit: NASA) (b) Rider in cage showing how the cage pivots outward during rotation. This allows the total force exerted on the rider by the cage to be along its axis at all times.

Show Solution

Strategy

(a) We'll use $a_c = r\omega^2$ with $a_c = 10g = 98.0$ m/s² and solve for ω. (b) We'll draw a free body diagram and use the fact that the cage hangs at an angle where the arm force can simultaneously provide the centripetal force and support the weight. Using geometry, $\tan\theta = g a_c$.

Solution

(a) Using $a_c = r\omega^2$, solve for ω:

$$a_c = 10g = 10(9.80\ \text{m/s}^2) = 98.0\ \text{m/s}^2$$
$$\omega^2 = \frac{a_c}{r} = \frac{98.0\ \text{m/s}^2}{15.0\ \text{m}} = 6.53\ \text{rad}^2/\text{s}^2$$
$$\omega = \sqrt{6.53} = 2.56\ \text{rad/s}$$

(b) Draw a free body diagram: The cage experiences its weight mg downward and the arm force F at angle θ below horizontal. The horizontal component of F provides centripetal force, while the vertical component balances weight:

$$F\cos\theta = ma_c \quad \text{and} \quad F\sin\theta = mg$$

Dividing the second equation by the first:

$$\frac{F\sin\theta}{F\cos\theta} = \frac{mg}{ma_c}$$
$$\tan\theta = \frac{g}{a_c} = \frac{9.80}{98.0} = 0.100$$

$$\theta = \arctan(0.100) = 5.71^\circ$$

Discussion

The centrifuge must rotate at 2.56 rad/s (about 24.4 rpm) to produce 10g acceleration at a radius of 15.0 m. The cage hangs only 5.71° below horizontal because the centripetal acceleration (10g) is much larger than gravitational acceleration (1g). This small angle means the arm force is nearly horizontal, pointing mostly toward the center to provide the large centripetal force needed. As rotation increases, the angle becomes even smaller, approaching horizontal. This pivoting cage design ensures the rider always experiences force along the cage axis, preventing uncomfortable sideways forces.

Integrated Concepts

If a car takes a banked curve at less than the ideal speed, friction is needed to keep it from sliding toward the inside of the curve (a real problem on icy mountain roads). (a) Calculate the ideal speed to take a 100 m radius curve banked at 15.0°. (b) What is the minimum coefficient of friction needed for a frightened driver to take the same curve at 20.0 km/h?

Show Solution

Strategy

(a) For the ideal speed (no friction needed), the horizontal component of the normal force provides exactly the needed centripetal force. We'll use $v = \sqrt{rg\tan\theta}$. (b) At a speed lower than ideal, friction must prevent sliding inward. We'll analyze forces to find the minimum coefficient of friction.

Solution

(a) Calculate the ideal speed using the formula for a banked curve:

$$v = \sqrt{rg\tan\theta} = \sqrt{(100\ \text{m})(9.80\ \text{m/s}^2)\tan(15.0^\circ)}$$
$$v = \sqrt{(980)(0.268)} = \sqrt{262.6} = 16.2\ \text{m/s}$$

(b) Convert the actual speed to m/s:

$$v = 20.0\,\frac{\text{km}}{\text{h}} \times \frac{1000\ \text{m}}{1\ \text{km}} \times \frac{1\ \text{h}}{3600\ \text{s}} = 5.56\ \text{m/s}$$

At this slower speed, the car tends to slide down the bank. The required centripetal force is less than what the banked curve would naturally provide, so friction must act up the slope. Using force balance equations for a banked curve with friction:

$$\mu_s = \frac{\tan\theta - \frac{v^2}{rg}}{1 + \frac{v^2\tan\theta}{rg}}$$

Calculate $\frac{v^2}{rg}$:

$$\frac{v^2}{rg} = \frac{(5.56)^2}{(100)(9.80)} = \frac{30.9}{980} = 0.0316$$

Now calculate μ_s:

$$\mu_s = \frac{0.268 - 0.0316}{1 + (0.0316)(0.268)} = \frac{0.236}{1.0085} = 0.234$$

Discussion

The ideal speed for this banked curve is 16.2 m/s (about 58 km/h), at which no friction is needed. However, at the much slower speed of 20.0 km/h (5.56 m/s), the car requires a minimum coefficient of friction of 0.234 to prevent sliding down toward the inside of the curve. This is why icy mountain roads with banked curves are particularly dangerous—if the coefficient of friction drops below this value due to ice, slow-moving vehicles will slide inward regardless of driver skill. The problem illustrates why banked curves are designed for a specific “ideal” speed, and driving significantly slower than this speed can be just as problematic as driving too fast.

Modern roller coasters have vertical loops like the one shown in Figure 10. The radius of curvature is smaller at the top than on the sides so that the downward centripetal acceleration at the top will be greater than the acceleration due to gravity, keeping the passengers pressed firmly into their seats. What is the speed of the roller coaster at the top of the loop if the radius of curvature there is 15.0 m and the downward acceleration of the car is 1.50 g?

![A teardrop shaped loop of a roller coaster is shown. The car of the roller coaster starts from the point A near the right of the base and covers the teardrop portion of the roller coaster and move to a point D at the left of base. Near the top of tear drop portion an upward arrow is shown labeled as r-minimum. Also at a point near the base toward A there is a label called r-maximum. The wire frame of the base is also shown.](../resources/Figure_06_03_10.jpg 'Teardrop-shaped loops are used in the latest roller coasters so that the radius of curvature gradually decreases to a minimum at the top. This means that the centripetal acceleration builds from zero to a maximum at the top and gradually decreases again. A circular loop would cause a jolting change in acceleration at entry, a disadvantage discovered long ago in railroad curve design. With a small radius of curvature at the top, the centripetal acceleration can more easily be kept greater than

g

so that the passengers do not lose contact with their seats nor do they need seat belts to keep them in place.')

Show Solution

Strategy

We need to find the speed at the top of the loop given the centripetal acceleration and radius of curvature. We'll use $a_c = v^2 r$ and solve for v. The centripetal acceleration is given as 1.50g, where g =9.80 m/s².

Solution

The centripetal acceleration is:

$$a_c=1.50g=1.50(9.80\ \text{m/s}^2)=14.7\ \text{m/s}^2$$

Using the centripetal acceleration formula and solving for v:

$$a_c=v^2 r$$
$$v^2=a_c r=(14.7\ \text{m/s}^2)(15.0\ \text{m})=221\ \text{m}^2/\text{s}^2$$
$$v=\sqrt{221}\ \text{m}^2/\text{s}^2=14.9\ \text{m/s}$$

Discussion

The speed of the roller coaster at the top of the loop is 14.9 m/s (about 54 km/h or 33 mi/h). At this speed with a 15.0 m radius of curvature, the centripetal acceleration is 1.50g, which means passengers experience a downward force 1.50 times their normal weight. This keeps them firmly pressed into their seats even though they are upside down at the top of the loop, creating the thrilling sensation that makes roller coasters exciting while maintaining safety.

Unreasonable Results

(a) Calculate the minimum coefficient of friction needed for a car to negotiate an unbanked 50.0 m radius curve at 30.0 m/s.

(b) What is unreasonable about the result?

(c) Which premises are unreasonable or inconsistent?

Show Solution

Strategy

For an unbanked curve, friction provides the centripetal force. We'll set up the equation where friction equals the required centripetal force, then solve for the coefficient of friction. We'll then analyze whether the result is physically reasonable.

Solution

(a) For circular motion on an unbanked curve:

The centripetal force is provided by friction:

$$f=\mu_s N=\mu_s mg$$

This must equal the required centripetal force:

$$\mu_s mg=mv^2 r$$

Solving for the coefficient of friction (mass cancels):

$$\mu_s=v^2 r g=(30.0\ \text{m/s})^2(50.0\ \text{m})(9.80\ \text{m/s}^2)=900490=1.84$$

(b) A coefficient of friction of 1.84 is unreasonable. Typical coefficients of static friction between rubber tires and dry pavement range from 0.7 to 1.0. Values significantly greater than 1.0 are not physically achievable with normal road surfaces and tires.

(c) The assumed speed of 30.0 m/s (108 km/h or 67 mph) is too great for a 50.0 m radius curve. At this tight radius, safer speeds would be around 15-20 m/s. Alternatively, banking the curve would reduce the required friction coefficient.

Discussion

This problem illustrates why highway curves are designed with specific speed limits based on their radius. For a 50.0 m radius curve with $\mu = 0.7$ (wet conditions), the maximum safe speed would be: $v = \sqrt{\mu r g} = \sqrt{(0.7)(50.0)(9.80)} = 18.5\text{ m/s} \approx 67\text{ km/h}$

This is why tight curves on roads have reduced speed limits and are often banked.

Glossary

centripetal force
: any net force causing uniform circular motion

ideal banking
: the sloping of a curve in a road, where the angle of the slope allows the vehicle to negotiate the curve at a certain speed without the aid of friction between the tires and the road; the net external force on the vehicle equals the horizontal centripetal force in the absence of friction

ideal speed
: the maximum safe speed at which a vehicle can turn on a curve without the aid of friction between the tire and the road

ideal angle
: the angle at which a car can turn safely on a steep curve, which is in proportion to the ideal speed

banked curve
: the curve in a road that is sloping in a manner that helps a vehicle negotiate the curve

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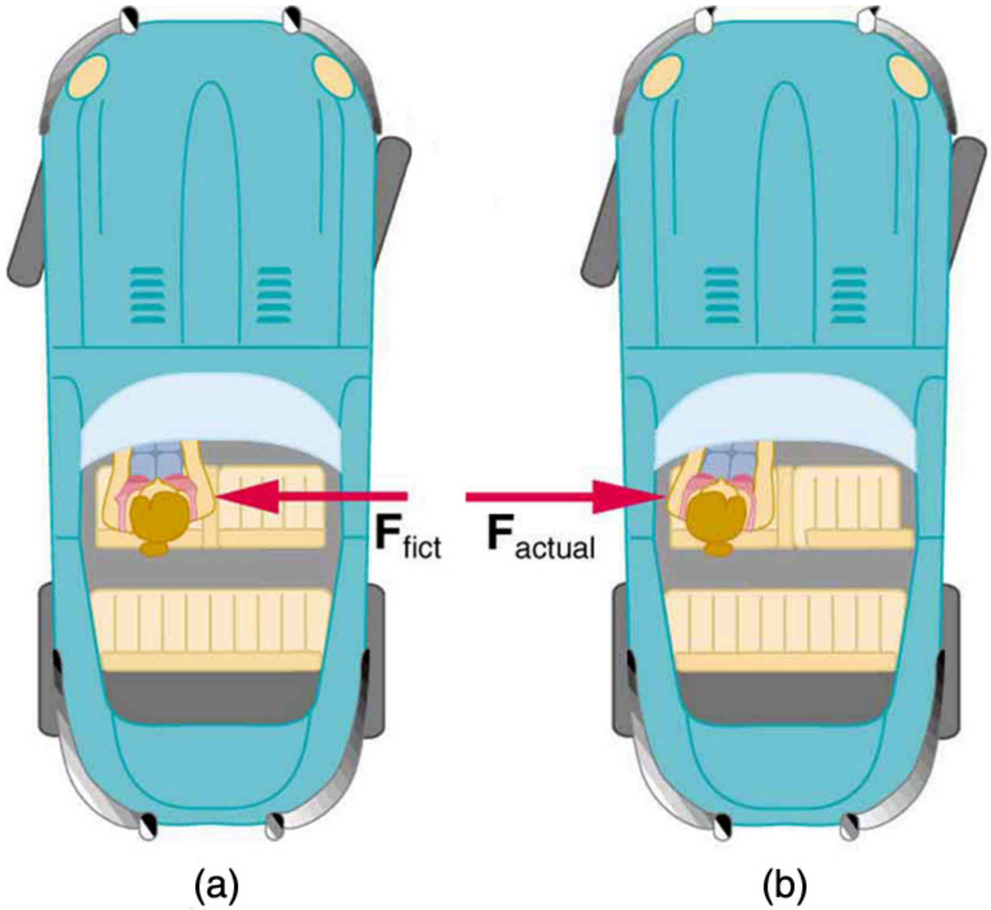
Fictitious Forces and Non-inertial Frames: The Coriolis Force

- Discuss the inertial frame of reference.
- Discuss the non-inertial frame of reference.
- Describe the effects of the Coriolis force.

What do taking off in a jet airplane, turning a corner in a car, riding a merry-go-round, and the circular motion of a tropical cyclone have in common? Each exhibits fictitious forces—unreal forces that arise from motion and may *seem* real, because the observer's frame of reference is accelerating or rotating.

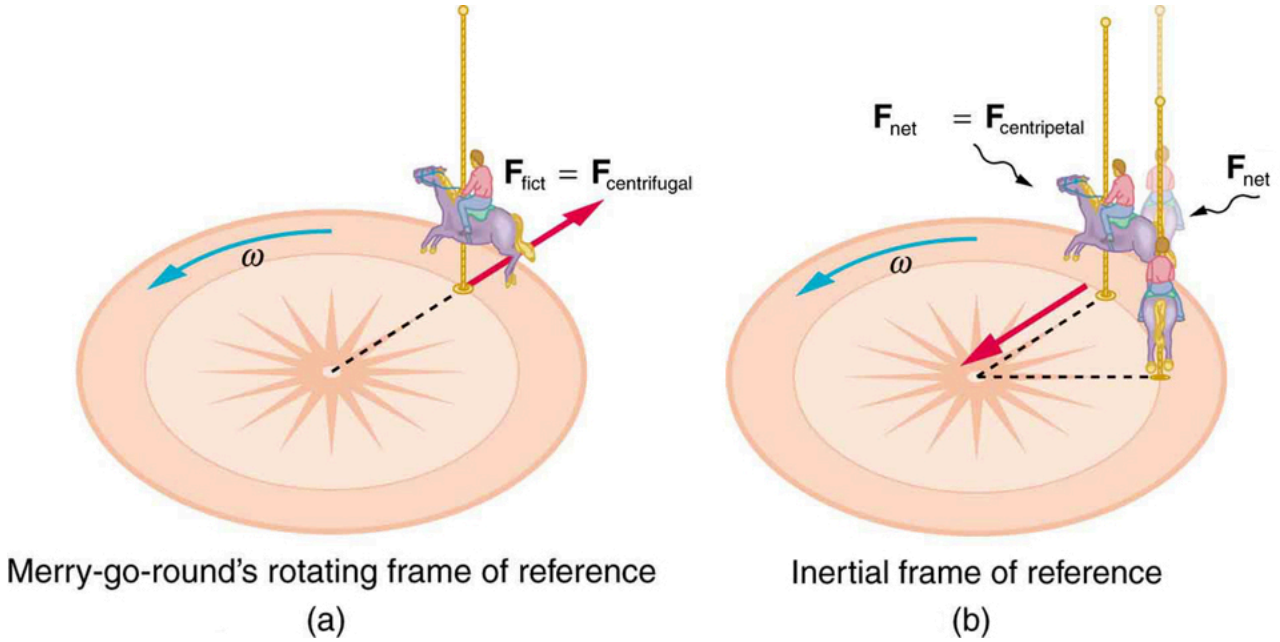
When taking off in a jet, most people would agree it feels as if you are being pushed back into the seat as the airplane accelerates down the runway. Yet a physicist would say that *you* tend to remain stationary while the *seat* pushes forward on you, and there is no real force backward on you. An even more common experience occurs when you make a tight curve in your car—say, to the right. You feel as if you are thrown (that is, *forced*) toward the left relative to the car. Again, a physicist would say that *you* are going in a straight line but the *car* moves to the right, and there is no real force on you to the left. Recall Newton's first law.



(a) The car driver feels herself forced to the left relative to the car when she makes a right turn. This is a fictitious force arising from the use of the car as a frame of reference. (b) In the Earth's frame of reference, the driver moves in a straight line, obeying Newton's first law, and the car moves to the right. There is no real force to the left on the driver relative to Earth. There is a real force to the right on the car to make it turn.

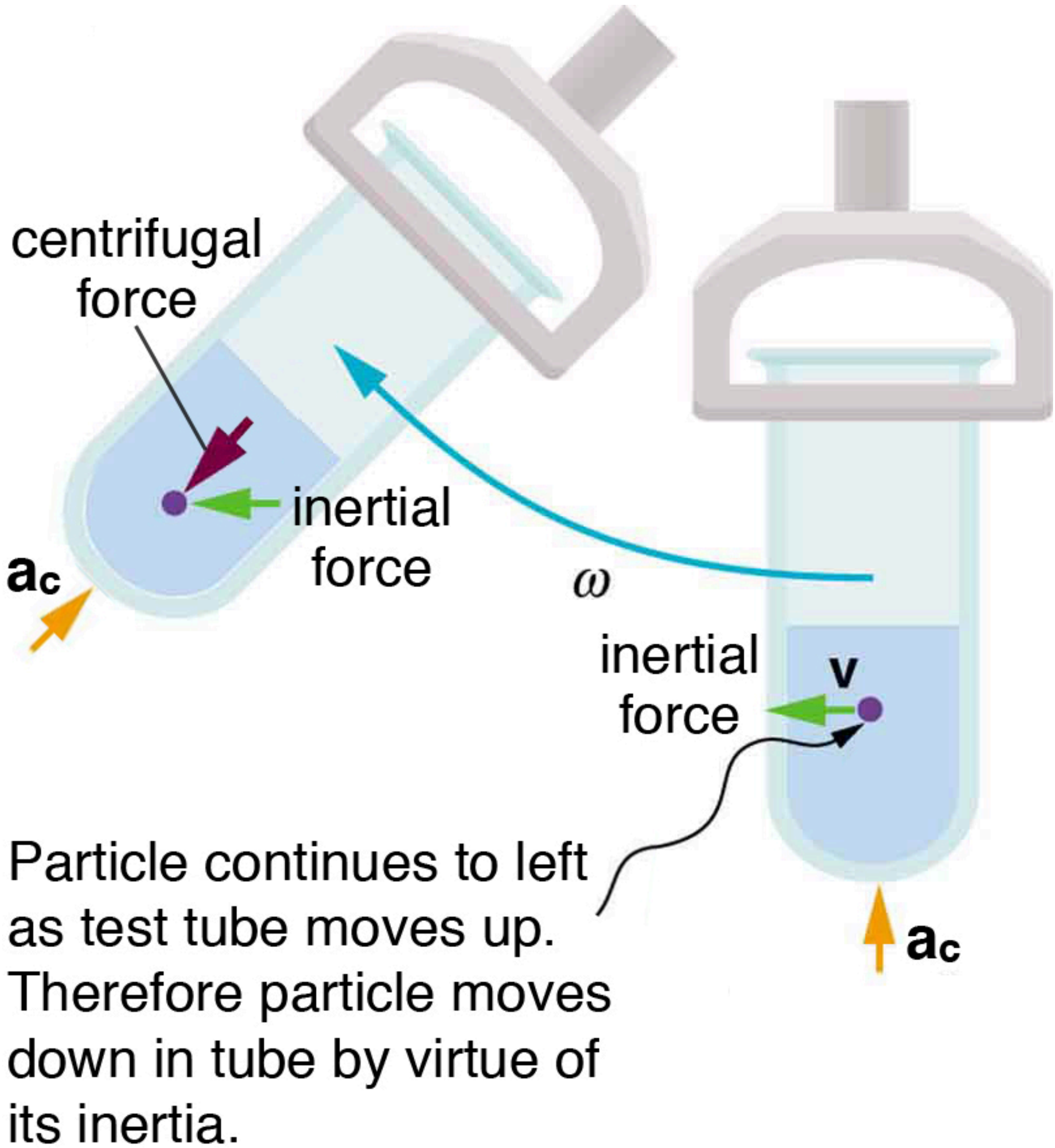
We can reconcile these points of view by examining the frames of reference used. Let us concentrate on people in a car. Passengers instinctively use the car as a frame of reference, while a physicist uses Earth. The physicist chooses Earth because it is very nearly an inertial frame of reference—one in which all forces are real (that is, in which all forces have an identifiable physical origin). In such a frame of reference, Newton's laws of motion take the form given in Dynamics: Newton's Laws of Motion The car is a **non-inertial frame of reference** because it is accelerated to the side. The force to the left sensed by car passengers is a **fictitious force** having no physical origin. There is nothing real pushing them left—the car, as well as the driver, is actually accelerating to the right.

Let us now take a mental ride on a merry-go-round—specifically, a rapidly rotating playground merry-go-round. You take the merry-go-round to be your frame of reference because you rotate together. In that non-inertial frame, you feel a fictitious force, named **centrifugal force** not to be confused with centripetal force, trying to throw you off. You must hang on tightly to counteract the centrifugal force. In Earth's frame of reference, there is no force trying to throw you off. Rather you must hang on to make yourself go in a circle because otherwise you would go in a straight line, right off the merry-go-round.



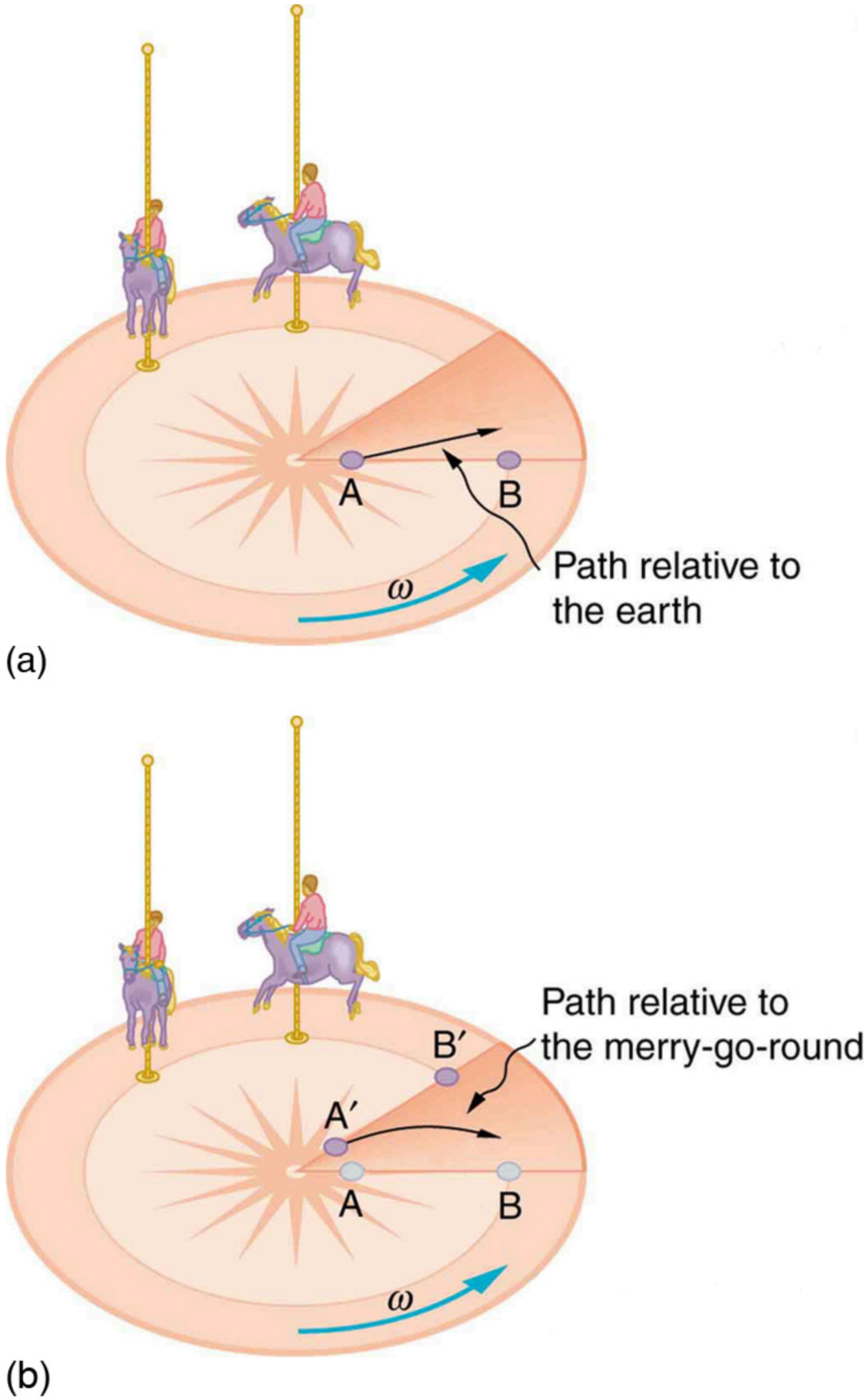
(a) A rider on a merry-go-round feels as if he is being thrown off. This fictitious force is called the centrifugal force—it explains the rider's motion in the rotating frame of reference. (b) In an inertial frame of reference and according to Newton's laws, it is his inertia that carries him off and not a real force (the unshaded rider has $F_{\text{net}} = 0$ and heads in a straight line). A real force, $F_{\text{centripetal}}$, is needed to cause a circular path.

This inertial effect, carrying you away from the center of rotation if there is no centripetal force to cause circular motion, is put to good use in centrifuges (see Figure 3). A centrifuge spins a sample very rapidly, as mentioned earlier in this chapter. Viewed from the rotating frame of reference, the fictitious centrifugal force throws particles outward, hastening their sedimentation. The greater the angular velocity, the greater the centrifugal force. But what really happens is that the inertia of the particles carries them along a line tangent to the circle while the test tube is forced in a circular path by a centripetal force.



Centrifuges use inertia to perform their task. Particles in the fluid sediment come out because their inertia carries them away from the center of rotation. The large angular velocity of the centrifuge quickens the sedimentation. Ultimately, the particles will come into contact with the test tube walls, which will then supply the centripetal force needed to make them move in a circle of constant radius.

Let us now consider what happens if something moves in a frame of reference that rotates. For example, what if you slide a ball directly away from the center of the merry-go-round, as shown in Figure 4? The ball follows a straight path relative to Earth (assuming negligible friction) and a path curved to the right on the merry-go-round's surface. A person standing next to the merry-go-round sees the ball moving straight and the merry-go-round rotating underneath it. In the merry-go-round's frame of reference, we explain the apparent curve to the right by using a fictitious force, called the **Coriolis force**, that causes the ball to curve to the right. The fictitious Coriolis force can be used by anyone in that frame of reference to explain why objects follow curved paths and allows us to apply Newton's Laws in non-inertial frames of reference.

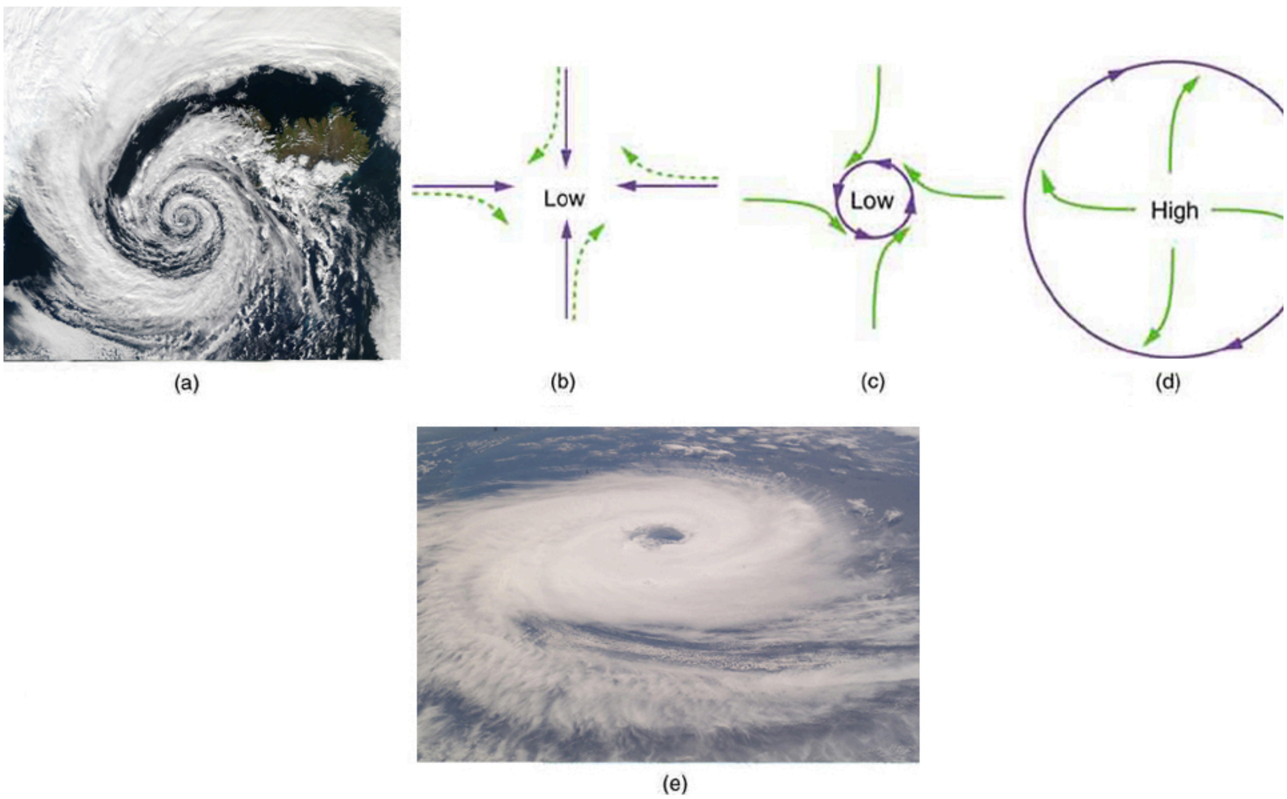


Looking down on the counterclockwise rotation of a merry-go-round, we see that a ball slid straight toward the edge follows a path curved to the right. The person slides the ball toward point B, starting at point A. Both points rotate to the shaded positions (A' and B') shown in the time that the ball follows the curved path in the rotating frame and a straight path in Earth's frame.

Up until now, we have considered Earth to be an inertial frame of reference with little or no worry about effects due to its rotation. Yet such effects *do* exist—in the rotation of weather systems, for example. Most consequences of Earth's rotation can be qualitatively understood by analogy with the merry-go-round. Viewed from above the North Pole, Earth rotates counterclockwise, as does the merry-go-round in Figure 4. As on the merry-go-round, any motion in Earth's northern hemisphere experiences a Coriolis force to the right. Just the opposite occurs in the southern hemisphere; there, the force is to the left. Because Earth's angular velocity is small, the Coriolis force is usually negligible, but for large-scale motions, such as wind patterns, it has substantial effects.

The Coriolis force causes hurricanes in the northern hemisphere to rotate in the counterclockwise direction, while the tropical cyclones (what hurricanes are called below the equator) in the southern hemisphere rotate in the clockwise direction. The terms hurricane, typhoon, and tropical storm are regionally-specific names for tropical cyclones, storm systems characterized by low pressure centers, strong winds, and heavy rains. Figure 5 helps show how these rotations take place. Air flows toward any region of low pressure, and tropical cyclones contain particularly low pressures. Thus winds flow toward the center of a tropical cyclone or a low-pressure weather system at the surface. In the northern hemisphere, these inward winds are deflected to the right, as shown in the figure, producing a counterclockwise circulation at the surface for low-pressure zones of any type. Low pressure at the surface is associated with rising air, which also produces cooling and cloud formation, making low-pressure patterns quite visible from space. Conversely, wind circulation around high-pressure zones is clockwise in the northern hemisphere but is less visible because high pressure is associated with sinking air, producing clear skies.

The rotation of tropical cyclones and the path of a ball on a merry-go-round can just as well be explained by inertia and the rotation of the system underneath. When non-inertial frames are used, fictitious forces, such as the Coriolis force, must be invented to explain the curved path. There is no identifiable physical source for these fictitious forces. In an inertial frame, inertia explains the path, and no force is found to be without an identifiable source. Either view allows us to describe nature, but a view in an inertial frame is the simplest and truest, in the sense that all forces have real origins and explanations.



(a) The counterclockwise rotation of this northern hemisphere hurricane is a major consequence of the Coriolis force. (credit: NASA) (b) Without the Coriolis force, air would flow straight into a low-pressure zone, such as that found in tropical cyclones. (c) The Coriolis force deflects the winds to the right, producing a counterclockwise rotation. (d) Wind flowing away from a high-pressure zone is also deflected to the right, producing a clockwise rotation. (e) The opposite direction of rotation is produced by the Coriolis force in the southern hemisphere, leading to tropical cyclones. (credit: NASA)

Section Summary

- Rotating and accelerated frames of reference are non-inertial.
- Fictitious forces, such as the Coriolis force, are needed to explain motion in such frames.

Conceptual Questions

When a toilet is flushed or a sink is drained, the water (and other material) begins to rotate about the drain on the way down. Assuming no initial rotation and a flow initially directly straight toward the drain, explain what causes the rotation and which direction it has in the northern hemisphere. (Note that this is a small effect and in most toilets the rotation is caused by directional water jets.) Would the direction of rotation reverse if water were forced up the drain?

Is there a real force that throws water from clothes during the spin cycle of a washing machine? Explain how the water is removed.

In one amusement park ride, riders enter a large vertical barrel and stand against the wall on its horizontal floor. The barrel is spun up and the floor drops away. Riders feel as if they are pinned to the wall by a force something like the gravitational force. This is a fictitious force sensed and used by the riders to explain events in the rotating frame of reference of the barrel. Explain in an inertial frame of reference (Earth is nearly one) what pins the riders to the wall, and identify all of the real forces acting on them.

Action at a distance, such as is the case for gravity, was once thought to be illogical and therefore untrue. What is the ultimate determinant of the truth in physics, and why was this action ultimately accepted?

Two friends are having a conversation. Anna says a satellite in orbit is in freefall because the satellite keeps falling toward Earth. Tom says a satellite in orbit is not in freefall because the acceleration due to gravity is not 9.80 m/s^2. Who do you agree with and why?

A non-rotating frame of reference placed at the center of the Sun is very nearly an inertial one. Why is it not exactly an inertial frame?

Glossary

fictitious force
: a force having no physical origin

centrifugal force
: a fictitious force that tends to throw an object off when the object is rotating in a non-inertial frame of reference

Coriolis force
: the fictitious force causing the apparent deflection of moving objects when viewed in a rotating frame of reference

non-inertial frame of reference
: an accelerated frame of reference

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Newton’s Universal Law of Gravitation

- Explain Earth’s gravitational force.
- Describe the gravitational effect of the Moon on Earth.
- Discuss weightlessness in space.
- Examine the Cavendish experiment

What do aching feet, a falling apple, and the orbit of the Moon have in common? Each is caused by the gravitational force. Our feet are strained by supporting our weight—the force of Earth’s gravity on us. An apple falls from a tree because of the same force acting a few meters above Earth’s surface. And the Moon orbits Earth because gravity is able to supply the necessary centripetal force at a distance of hundreds of millions of meters. In fact, the same force causes planets to orbit the Sun, stars to orbit the center of the galaxy, and galaxies to cluster together. Gravity is another example of underlying simplicity in nature. It is the weakest of the four basic forces found in nature, and in some ways the least understood. It is a force that acts at a distance, without physical contact, and is expressed by a formula that is valid everywhere in the universe, for masses and distances that vary from the tiny to the immense.

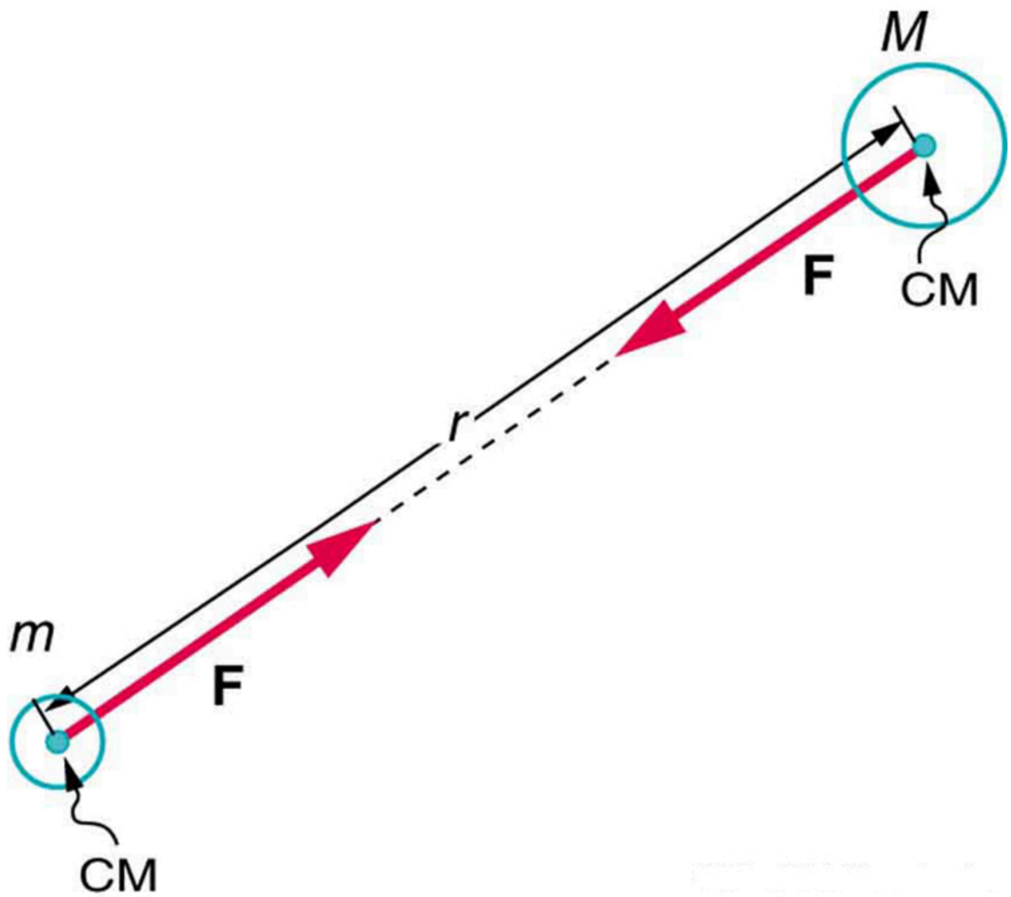
Sir Isaac Newton was the first scientist to precisely define the gravitational force, and to show that it could explain both falling bodies and astronomical motions. See Figure 1. But Newton was not the first to suspect that the same force caused both our weight and the motion of planets. His forerunner Galileo Galilei had contended that falling bodies and planetary motions had the same cause. Some of Newton’s contemporaries, such as Robert Hooke, Christopher Wren, and Edmund Halley, had also made some progress toward understanding gravitation. But Newton was the first to propose an exact mathematical form and to use that form to show that the motion of heavenly bodies should be conic sections—circles, ellipses, parabolas, and hyperbolas. This theoretical prediction was a major triumph—it had been known for some time that moons, planets, and comets follow such paths, but no one had been able to propose a mechanism that caused them to follow these paths and not others. Other prominent scientists and mathematicians of the time, particularly those outside of England, were reluctant to accept Newton’s principles. It took the work of another prominent philosopher, writer, and scientist, Émilie du Châtelet, to establish the Newtonian gravitation as the accurate and overarching law. Du Châtelet, who had earlier laid the foundation for the understanding of conservation of energy as well as the principle that light had no mass, translated and augmented Newton’s key work. She also utilized calculus to explain gravity, which helped lead to its acceptance.

(a)

(b)

According to early accounts, Newton was inspired to make the connection between falling bodies and astronomical motions when he saw an apple fall from a tree and realized that if the gravitational force could extend above the ground to a tree, it might also reach the Sun. The inspiration of Newton's apple is a part of worldwide folklore and may even be based in fact. Great importance is attached to it because Newton's universal law of gravitation and his laws of motion answered very old questions about nature and gave tremendous support to the notion of underlying simplicity and unity in nature. Like many revolutionary discoveries, it was not immediately accepted. Prominent French scientist and philosopher Émilie du Châtelet helped establish Newton's theory in France and mainland Europe.

The gravitational force is relatively simple. It is always attractive, and it depends only on the masses involved and the distance between them. Stated in modern language, **Newton's universal law of gravitation** states that every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.



Gravitational attraction is along a line joining the centers of mass of these two bodies. The magnitude of the force is the same on each, consistent with Newton's third law.

Misconception Alert

The magnitude of the force on each object (one has larger mass than the other) is the same, consistent with Newton's third law.

The bodies we are dealing with tend to be large. To simplify the situation we assume that the body acts as if its entire mass is concentrated at one specific point called the **center of mass** (CM) , which will be further explored in Linear Momentum and Collisions . For two bodies having masses m and M with a distance r between their centers of mass, the equation for Newton's universal law of gravitation is

$$F=GmMr^2,$$

where F is the magnitude of the gravitational force and G is a proportionality factor called the **gravitational constant**. G is a universal gravitational constant—that is, it is thought to be the same everywhere in the universe. It has been measured experimentally to be

$$G=6.674\times10^{-11}\mathrm{N{\cdot}m^2kg^2}$$

in SI units. Note that the units of G are such that a force in newtons is obtained from $F=GmMr^2$, when considering masses in kilograms and distance in meters. For example, two 1.000 kg masses separated by 1.000 m will experience a gravitational attraction of $6.674\times10^{-11}\mathrm{N}$. This is an extraordinarily small force. The small magnitude of the gravitational force is consistent with everyday experience. We are unaware that even large objects like mountains exert gravitational forces on us. In fact, our body weight is the force of attraction of the *entire Earth* on us with a mass of $6\times10^{24}\mathrm{kg}$.

Recall that the acceleration due to gravity g is about $9.80\mathrm{m/s}^2$ on Earth. We can now determine why this is so. The weight of an object *mg* is the gravitational force between it and Earth. Substituting *mg* for F in Newton's universal law of gravitation gives

$$mg=GmMr^2,$$

where m is the mass of the object, M is the mass of Earth, and r is the distance to the center of Earth (the distance between the centers of mass of the object and Earth). See Figure 3. The mass m of the object cancels, leaving an equation for g:

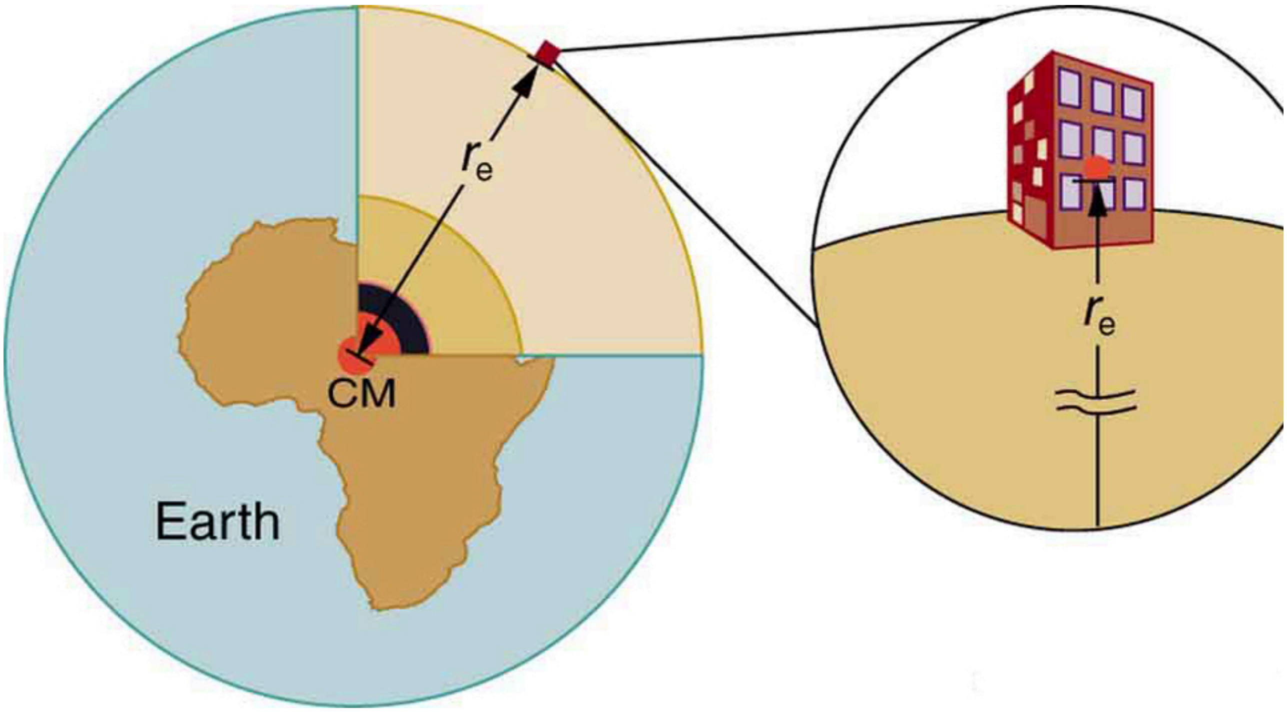
$$g=GMr^2.$$

Substituting known values for Earth's mass and radius (to three significant figures),

$$g=(6.67\times10^{-11}\,\mathrm{N\cdot m^2 kg^2})\times5.98\times10^{24}\,\mathrm{kg}(6.38\times10^{6}\,\mathrm{m})^2,$$

and we obtain a value for the acceleration of a falling body:

$$g=9.80\,\mathrm{m/s}^2.$$



The distance between the centers of mass of Earth and an object on its surface is very nearly the same as the radius of Earth, because Earth is so much larger than the object.

This is the expected value *and is independent of the body's mass*. Newton's law of gravitation takes Galileo's observation that all masses fall with the same acceleration a step further, explaining the observation in terms of a force that causes objects to fall—in fact, in terms of a universally existing force of attraction between masses.

Take-Home Experiment

Take a marble, a ball, and a spoon and drop them from the same height. Do they hit the floor at the same time? If you drop a piece of paper as well, does it behave like the other objects? Explain your observations.

Making Connections

Attempts are still being made to understand the gravitational force. Modern physics is exploring the connections of gravity to other forces, space, and time. General relativity alters our view of gravitation, leading us to think of gravitation as bending space and time.

In the following example, we make a comparison similar to one made by Newton himself. He noted that if the gravitational force caused the Moon to orbit Earth, then the acceleration due to gravity should equal the centripetal acceleration of the Moon in its orbit. Newton found that the two accelerations agreed "pretty nearly."

Earth's Gravitational Force Is the Centripetal Force Making the Moon Move in a Curved Path

(a) Find the acceleration due to Earth's gravity at the distance of the Moon.

(b) Calculate the centripetal acceleration needed to keep the Moon in its orbit (assuming a circular orbit about a fixed Earth), and compare it with the value of the acceleration due to Earth's gravity that you have just found.

Strategy for (a)

This calculation is the same as the one finding the acceleration due to gravity at Earth's surface, except that r is the distance from the center of Earth to the center of the Moon. The radius of the Moon's nearly circular orbit is

3.84 \times 10^{8}\m $$.

Solution for (a)

Substituting known values into the expression for g found above, remembering that M is the mass of Earth not the Moon, yields

$$g = \frac{GM}{r^2} = \frac{\left(6.67\times10^{-11}\,\mathrm{N\cdot m^2/kg^2}\right)\times 5.98\times10^{24}\,\mathrm{kg}}{\left(3.84\times10^{8}\,\mathrm{m}\right)^2} \qquad g = 2.70\times10^{-3}\,\mathrm{m/s^2}.$$

Strategy for (b)

Centripetal acceleration can be calculated using either form of

$$a_c = \frac{v^2}{r} \qquad a_c = r\omega^2 \ \}.$$

We choose to use the second form:

$$a_c = r\omega^2,$$

where ω is the angular velocity of the Moon about Earth.

Solution for (b)

Given that the period (the time it takes to make one complete rotation) of the Moon's orbit is 27.3 days, (d) and using

$$1\,d\times\frac{24\,\mathrm{hr}}{\mathrm{d}}\times\frac{60\,min}{\mathrm{hr}}\times\frac{60\,s}{\mathrm{min}} = 86400\,\mathrm{s}$$

we see that

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi\,\mathrm{rad}}{(27.3\,\mathrm{d})(86400\,\mathrm{s/d})} = 2.66\times10^{-6}\,\frac{\mathrm{rad}}{\mathrm{s}}.$$

The centripetal acceleration is

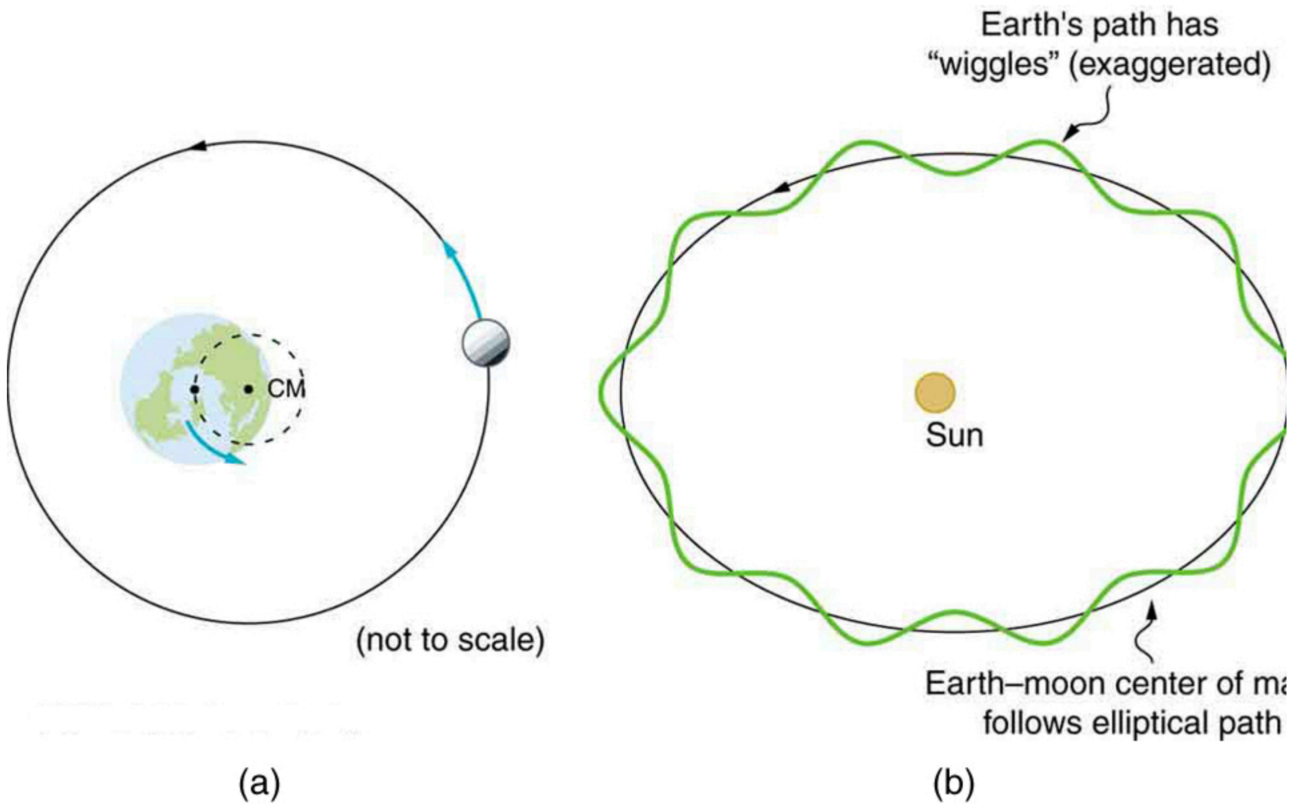
$$a_c = r\omega^2 = \left(3.84\times10^{8}\,\mathrm{m}\right)\left(2.66\times10^{-6}\,\mathrm{rad/s}\right)^2 \qquad a_c = 2.72\times10^{-3}\,\mathrm{m/s^2}.$$

The direction of the acceleration is toward the center of the Earth.

Discussion

The centripetal acceleration of the Moon found in (b) differs by less than 1% from the acceleration due to Earth's gravity found in (a). This agreement is approximate because the Moon's orbit is slightly elliptical, and Earth is not stationary (rather the Earth-Moon system rotates about its center of mass, which is located some 1700 km below Earth's surface). The clear implication is that Earth's gravitational force causes the Moon to orbit Earth.

Why does Earth not remain stationary as the Moon orbits it? This is because, as expected from Newton's third law, if Earth exerts a force on the Moon, then the Moon should exert an equal and opposite force on Earth (see Figure 4). We do not sense the Moon's effect on Earth's motion, because the Moon's gravity moves our bodies right along with Earth but there are other signs on Earth that clearly show the effect of the Moon's gravitational force as discussed in Satellites and Kepler's Laws: An Argument for Simplicity .



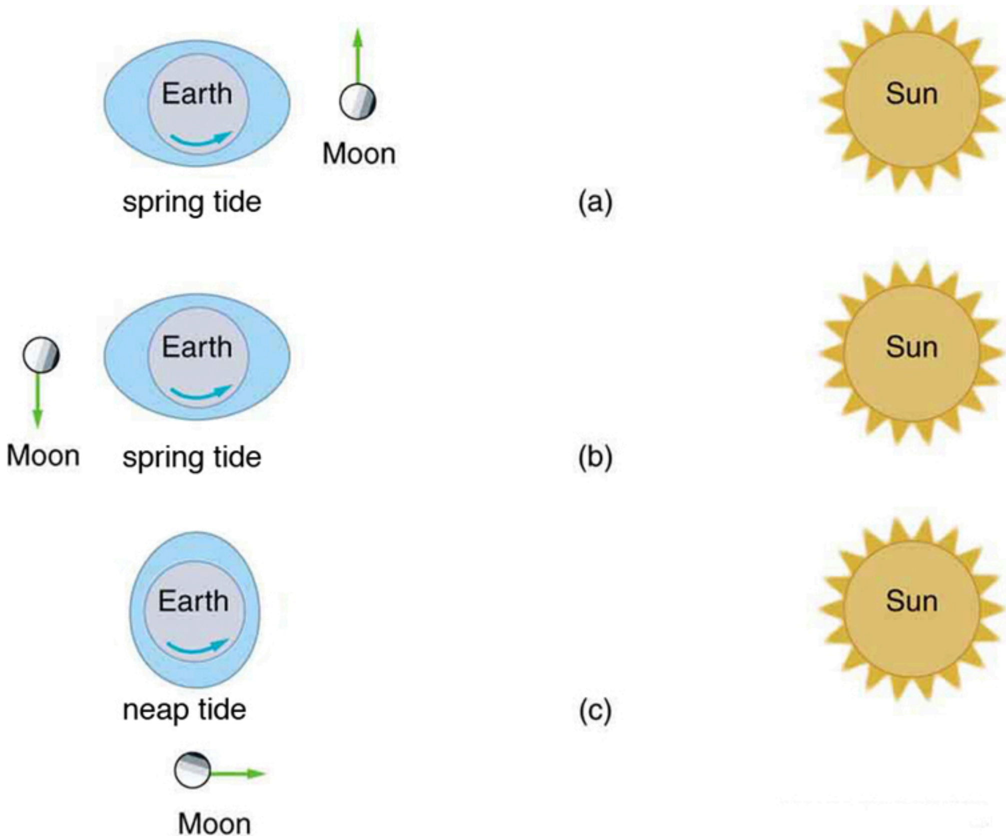
(a) Earth and the Moon rotate approximately once a month around their common center of mass. (b) Their center of mass orbits the Sun in an elliptical orbit, but Earth's path around the Sun has "wiggles" in it. Similar wiggles in the paths of stars have been observed and are considered direct evidence of planets orbiting those stars. This is important because the planets' reflected light is often too dim to be observed.

Tides

Ocean tides are one very observable result of the Moon's gravity acting on Earth. Figure 5 is a simplified drawing of the Moon's position relative to the tides. Because water easily flows on Earth's surface, a high tide is created on the side of Earth nearest to the Moon, where the Moon's gravitational pull is strongest. Why is there also a high tide on the opposite side of Earth? The answer is that Earth is pulled toward the Moon more than the water on the far side, because Earth is closer to the Moon. So the water on the side of Earth closest to the Moon is pulled away from Earth, and Earth is pulled away from water on the far side. As Earth rotates, the tidal bulge (an effect of the tidal forces between an orbiting natural satellite and the primary planet that it orbits) keeps its orientation with the Moon. Thus there are two tides per day (the actual tidal period is about 12 hours and 25.2 minutes), because the Moon moves in its orbit each day as well).

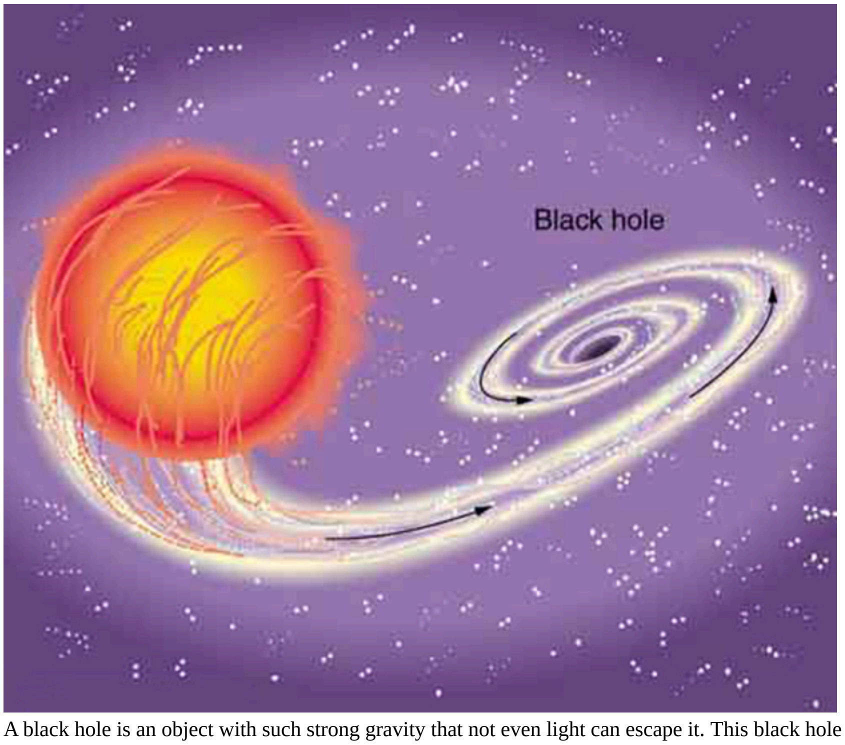
The Moon causes ocean tides by attracting the water on the near side more than Earth, and by attracting Earth more than the water on the far side. The distances and sizes are not to scale. For this simplified representation of the Earth-Moon system, there are two high and two low tides per day at any location, because Earth rotates under the tidal bulge.

The Sun also affects tides, although it has about half the effect of the Moon. However, the largest tides, called spring tides, occur when Earth, the Moon, and the Sun are aligned. The smallest tides, called neap tides, occur when the Sun is at a 90° angle to the Earth-Moon alignment.



(a, b) Spring tides: The highest tides occur when Earth, the Moon, and the Sun are aligned. (c) Neap tide: The lowest tides occur when the Sun lies at 90 degrees to the Earth-Moon alignment. Note that this figure is not drawn to scale.

Tides are not unique to Earth but occur in many astronomical systems. The most extreme tides occur where the gravitational force is the strongest and varies most rapidly, such as near black holes (see Figure 7). A few likely candidates for black holes have been observed in our galaxy. These have masses greater than the Sun but have diameters only a few kilometers across. The tidal forces near them are so great that they can actually tear matter from a companion star.



A black hole is an object with such strong gravity that not even light can escape it. This black hole was created by the supernova of one star in a two-star system. The tidal forces created by the black hole are so great that it tears matter from the companion star. This matter is compressed and heated as it is sucked into the black hole, creating light and X-rays observable from Earth.

"Weightlessness" and Microgravity

In contrast to the tremendous gravitational force near black holes is the apparent gravitational field experienced by astronauts orbiting Earth. What is the effect of "weightlessness" upon an astronaut who is in orbit for months? Or what about the effect of weightlessness upon plant growth? Weightlessness doesn't mean that an astronaut is not being acted upon by the gravitational force. There is no "zero gravity" in an astronaut's orbit. The term just means that the astronaut is in free-fall, accelerating with the acceleration due to gravity. If an elevator cable breaks, the passengers inside will be in free fall and will experience weightlessness. You can experience short periods of weightlessness in some rides in amusement parks.

Astronauts experiencing weightlessness on board the International Space Station. (credit: NASA)

Microgravity refers to an environment in which the apparent net acceleration of a body is small compared with that produced by Earth at its surface. Many interesting biology and physics topics have been studied over the past three decades in the presence of microgravity. Of immediate concern is the effect on astronauts of extended times in outer space, such as at the International Space Station. Researchers have observed that muscles will atrophy (waste away) in this environment. There is also a corresponding loss of bone mass. Study continues on cardiovascular adaptation to space flight. On Earth, blood pressure is usually higher in the feet than in the head, because the higher column of blood exerts a downward force on it, due to gravity. When standing, 70% of your blood is below the level of the heart, while in a horizontal position, just the opposite occurs. What difference does the absence of this pressure differential have upon the heart?

Some findings in human physiology in space can be clinically important to the management of diseases back on Earth. On a somewhat negative note, spaceflight is known to affect the human immune system, possibly making the crew members more vulnerable to infectious diseases. Experiments flown in space also have shown that some bacteria grow faster in microgravity than they do on Earth. However, on a positive note, studies indicate that microbial antibiotic production can increase by a factor of two in space-grown cultures. One hopes to be able to understand these mechanisms so that similar successes can be achieved on the ground. In another area of physics space research, inorganic crystals and protein crystals have been grown in outer space that have much higher quality than any grown on Earth, so crystallography studies on their structure can yield much better results.

Plants have evolved with the stimulus of gravity and with gravity sensors. Roots grow downward and shoots grow upward. Plants might be able to provide a life support system for long duration space missions by regenerating the atmosphere, purifying water, and producing food. Some studies have indicated that plant growth and development are not affected by gravity, but there is still uncertainty about structural changes in plants grown in a microgravity environment.

The Cavendish Experiment: Then and Now

As previously noted, the universal gravitational constant G is determined experimentally. This definition was first done accurately by Henry Cavendish (1731–1810), an English scientist, in 1798, more than 100 years after Newton published his universal law of gravitation. The measurement of G is very basic and important because it determines the strength of one of the four forces in nature. Cavendish's experiment was very difficult because he measured the tiny gravitational attraction between two ordinary-sized masses (tens of kilograms at most), using apparatus like that in Figure 9. Remarkably, his value for G differs by less than 1% from the best modern value.

One important consequence of knowing G was that an accurate value for Earth's mass could finally be obtained. This was done by measuring the acceleration due to gravity as accurately as possible and then calculating the mass of Earth M from the relationship Newton's universal law of gravitation gives

$$mg=GmMr^2,$$

where m is the mass of the object, M is the mass of Earth, and r is the distance to the center of Earth (the distance between the centers of mass of the object and Earth). See Figure 2. The mass m of the object cancels, leaving an equation for g:

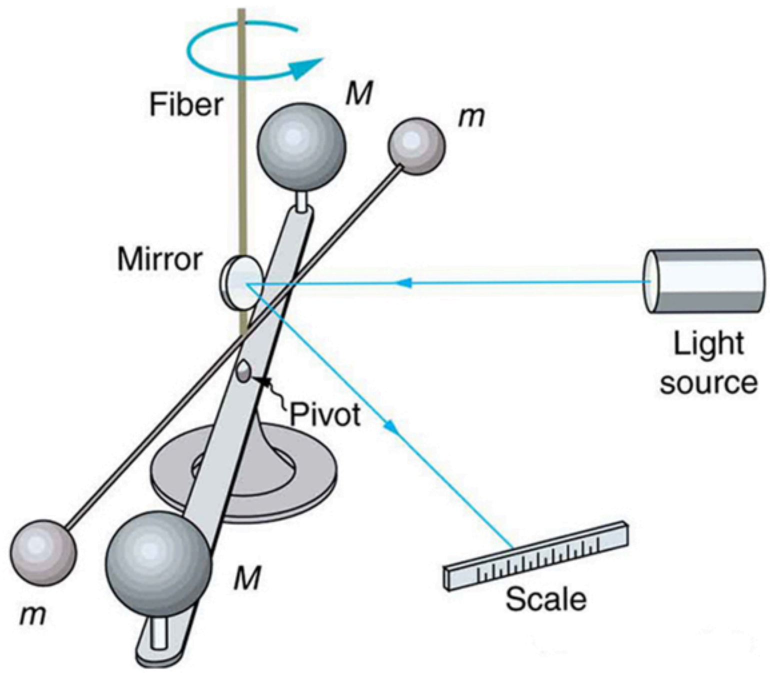
$$g=GMr^2.$$

Rearranging to solve for M yields

$$M=gr^2G.$$

So M can be calculated because all quantities on the right, including the radius of Earth r, are known from direct measurements. We shall see in Satellites and Kepler's Laws: An Argument for Simplicity that knowing G also allows for the determination of astronomical masses. Interestingly, of all the fundamental constants in physics, G is by far the least well determined.

The Cavendish experiment is also used to explore other aspects of gravity. One of the most interesting questions is whether the gravitational force depends on substance as well as mass—for example, whether one kilogram of lead exerts the same gravitational pull as one kilogram of water. A Hungarian scientist named Roland von Eötvös pioneered this inquiry early in the 20th century. He found, with an accuracy of five parts per billion, that the gravitational force does not depend on the substance. Such experiments continue today, and have improved upon Eötvös' measurements. Cavendish-type experiments such as those of Eric Adelberger and others at the University of Washington, have also put severe limits on the possibility of a fifth force and have verified a major prediction of general relativity—that gravitational energy contributes to rest mass. Ongoing measurements there use a torsion balance and a parallel plate (not spheres, as Cavendish used) to examine how Newton's law of gravitation works over sub-millimeter distances. On this small-scale, do gravitational effects depart from the inverse square law? So far, no deviation has been observed.



Cavendish used an apparatus like this to measure the gravitational attraction between the two suspended spheres (\(m \)) and the two on the stand (\(M \)) by observing the amount of torsion (twisting) created in the fiber. Distance between the masses can be varied to check the dependence of the force on distance. Modern experiments of this type continue to explore gravity.

Gravity Force Lab

Visualize the gravitational force that two objects exert on each other. Change properties of the objects in order to see how it changes the gravity force.

Gravity Force Lab

Section Summary

- Newton's universal law of gravitation: Every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. In equation form, this is $F = GmMr^2$,

where F is the magnitude of the gravitational force. G is the gravitational constant, given by $G = 6.674 \times 10^{-11} \text{N} \cdot \text{m}^2/kg^2$.

- Newton's law of gravitation applies universally.

Conceptual Questions

Action at a distance, such as is the case for gravity, was once thought to be illogical and therefore untrue. What is the ultimate determinant of the truth in physics, and why was this action ultimately accepted?

Two friends are having a conversation. Anna says a satellite in orbit is in freefall because the satellite keeps falling toward Earth. Tom says a satellite in orbit is not in freefall because the acceleration due to gravity is not 9.80m/s^2. Who do you agree with and why?

Draw a free body diagram for a satellite in an elliptical orbit showing why its speed increases as it approaches its parent body and decreases as it moves away.

Newton's laws of motion and gravity were among the first to convincingly demonstrate the underlying simplicity and unity in nature. Many other examples have since been discovered, and we now expect to find such underlying order in complex situations. Is there proof that such order will always be found in new explorations?

Problem Exercises

(a) Calculate Earth's mass given the acceleration due to gravity at the North Pole is 9.807m/s^2

and the radius of the Earth is 6372 km from center to pole.

(b) Compare this with the accepted value of $5.972 \times 10^{24} \text{kg}$.

Show Solution

Strategy

(a) We'll use Newton's law of universal gravitation in the form $g = GMr^2$ and solve for Earth's mass M. We know $g = 9.807$ m/s² at the North Pole and the radius $r = 6372$ km. (b) We'll compare our calculated value with the accepted value.

Solution

(a) Start with the gravitational acceleration formula:

$$g=\frac{GM}{r^2}$$

Solve for Earth's mass M:

$$M=\frac{gr^2}{G}$$

Convert radius to meters:

$$r=6372\text{ km}=6.372\times10^{6}\text{ m}$$

Substitute values with $G=6.674\times10^{-11}$ N·m²/kg²:

$$M=\frac{(9.807\text{ m/s}^2)(6.372\times10^{6}\text{ m})^2}{6.674\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2}$$
$$M=\frac{(9.807)(4.060\times10^{13})}{6.674\times10^{-11}}=\frac{3.982\times10^{14}}{6.674\times10^{-11}}=5.972\times10^{24}\text{ kg}$$

(b) Compare with the accepted value:

The accepted value is 5.972×10^{24} kg. Our calculated value of 5.972×10^{24} kg is **identical to the accepted value to three significant figures**. This perfect agreement confirms the accuracy of both Newton's law of universal gravitation and our measurement of Earth's radius and surface gravity.

Discussion

This problem demonstrates how Newton's law of universal gravitation can be used to determine the mass of Earth using only surface measurements. The fact that we can calculate Earth's mass so accurately from surface gravity and radius measurements is a powerful validation of Newton's gravitational theory. This same technique has been used to determine the masses of other planets, moons, and even the Sun, making universal gravitation one of the most important tools in astronomy and planetary science.

(a) Calculate the magnitude of the acceleration due to gravity on the surface of Earth due to the Moon.

(b) Calculate the magnitude of the acceleration due to gravity at Earth due to the Sun.

(c) Take the ratio of the Moon's acceleration to the Sun's and comment on why the tides are predominantly due to the Moon in spite of this number.

Show Solution

Strategy

We'll use Newton's law of gravitation $F=\frac{GmM}{r^2}$ to find the gravitational force, then divide by mass to get acceleration $a=\frac{F}{m}=\frac{GM}{r^2}$. We need the masses and distances for both the Moon and Sun.

Solution

(a) Calculate acceleration due to the Moon. Using:

- Mass of Moon: $M_{\text{Moon}}=7.35\times10^{22}$ kg
- Earth-Moon distance: $r_{\text{Moon}}=3.84\times10^{8}$ m
- $G=6.67\times10^{-11}$ N·m²/kg²

$$a_{\text{Moon}}=\frac{GM_{\text{Moon}}}{r^2_{\text{Moon}}}=\frac{(6.67\times10^{-11})7.35\times10^{22}}{(3.84\times10^{8})^2}$$
$$a_{\text{Moon}}=\frac{4.90\times10^{12}}{1.47\times10^{17}}=3.33\times10^{-5}\text{ m/s}^2$$

(b) Calculate acceleration due to the Sun. Using:

- Mass of Sun: $M_{\text{Sun}}=1.99\times10^{30}$ kg
- Earth-Sun distance: $r_{\text{Sun}}=1.50\times10^{11}$ m

$$a_{\text{Sun}}=\frac{GM_{\text{Sun}}}{r^2_{\text{Sun}}}=\frac{(6.67\times10^{-11})1.99\times10^{30}}{(1.50\times10^{11})^2}$$
$$a_{\text{Sun}}=\frac{1.33\times10^{20}}{2.25\times10^{22}}=5.90\times10^{-3}\text{ m/s}^2$$

(c) Calculate the ratio:

$$\frac{a_{\text{Moon}}}{a_{\text{Sun}}}=\frac{3.33\times10^{-5}}{5.90\times10^{-3}}=5.64\times10^{-3}\approx0.006$$

Discussion

The acceleration due to the Moon's gravity at Earth is 3.33×10^{-5} m/s², while that due to the Sun is 5.90×10^{-3} m/s². The Sun's gravitational acceleration on Earth is about 177 times stronger than the Moon's. However, tides are caused by the *gradient* (variation) in gravitational force across Earth's diameter, not the absolute magnitude. Because the Moon is much closer to Earth than the Sun, the difference in gravitational pull between the near and far sides of Earth is proportionally much greater for the Moon, making it the dominant cause of tides.

(a) What is the acceleration due to gravity on the surface of the Moon?

(b) On the surface of Mars? The mass of Mars is 6.418×10^{23} kg and its radius is 3.38×10^{6} m.

Show Solution

Strategy

For both parts, we'll use $g = \frac{GM}{r^2}$ to calculate surface gravity. We'll need the mass and radius of each body.

Solution

(a) Calculate the acceleration due to gravity on the Moon's surface. Using:

- Mass of Moon: $M_{\text{Moon}} = 7.35 \times 10^{22}$ kg
- Radius of Moon: $r_{\text{Moon}} = 1.74 \times 10^{6}$ m
- $G = 6.674 \times 10^{-11}$ N·m²/kg²

$$g_{\text{Moon}} = \frac{GM_{\text{Moon}}}{r^2_{\text{Moon}}} = \frac{(6.674 \times 10^{-11})7.35 \times 10^{22}}{(1.74 \times 10^{6})^2}$$
$$g_{\text{Moon}} = \frac{4.91 \times 10^{12}}{3.03 \times 10^{12}} = 1.62 \text{ m/s}^2$$

(b) Calculate the acceleration due to gravity on Mars' surface. Using:

- Mass of Mars: $M_{\text{Mars}} = 6.418 \times 10^{23}$ kg
- Radius of Mars: $r_{\text{Mars}} = 3.38 \times 10^{6}$ m

$$g_{\text{Mars}} = \frac{GM_{\text{Mars}}}{r^2_{\text{Mars}}} = \frac{(6.674 \times 10^{-11})6.418 \times 10^{23}}{(3.38 \times 10^{6})^2}$$
$$g_{\text{Mars}} = \frac{4.28 \times 10^{13}}{1.14 \times 10^{13}} = 3.75 \text{ m/s}^2$$

Discussion

The Moon's surface gravity (1.62 m/s²) is about 16.5% of Earth's gravity (9.80 m/s²). This is why astronauts could make such dramatic leaps on the lunar surface—their weight was only about 1/6 of what it is on Earth. Mars' surface gravity (3.75 m/s²) is about 38% of Earth's gravity, making it significantly stronger than the Moon's but still much weaker than Earth's. This has important implications for future Mars missions: astronauts would weigh roughly 2/5 of their Earth weight, making movement easier than on Earth but not as dramatically different as on the Moon. The relatively low gravity on both bodies also explains why neither has been able to retain a substantial atmosphere—the escape velocity is lower, allowing atmospheric gases to more easily escape into space.

(a) Calculate the acceleration due to gravity on the surface of the Sun.

(b) By what factor would your weight increase if you could stand on the Sun? (Never mind that you cannot.)

Show Solution

Strategy

(a) We'll use $g = \frac{GM}{r^2}$ with the Sun's mass and radius. (b) Weight is proportional to gravitational acceleration, so the ratio of weights equals the ratio of g_{Sun} to g_{Earth}.

Solution

(a) Calculate the acceleration due to gravity on the Sun's surface. Using:

- Mass of Sun: $M_{\text{Sun}} = 1.99 \times 10^{30}$ kg
- Radius of Sun: $r_{\text{Sun}} = 6.96 \times 10^{8}$ m
- $G = 6.67 \times 10^{-11}$ N·m²/kg²

$$g_{\text{Sun}} = \frac{GM_{\text{Sun}}}{r^2_{\text{Sun}}} = \frac{(6.67 \times 10^{-11})1.99 \times 10^{30}}{(6.96 \times 10^{8})^2}$$

$$g_{Sun} = \frac{1.33\times10^{20}}{4.84\times10^{17}} = 275\ \text{m/s}^2$$

(b) Calculate the factor by which weight would increase:

$$\frac{g_{Sun}}{g_{Earth}} = \frac{275\ \text{m/s}^2}{9.80\ \text{m/s}^2} = 28.1$$

Discussion

The acceleration due to gravity on the Sun's surface is 275 m/s², which is about 28 times stronger than Earth's gravity. If you could somehow stand on the Sun's surface, your weight would increase by a factor of 28.1. This means that if you weigh 70 kg (about 154 lb) on Earth, you would weigh nearly 2000 kg (about 4300 lb) on the Sun - far too much for your body to support, even before considering the intense heat and radiation!

The Moon and Earth rotate about their common center of mass, which is located about 4700 km from the center of Earth. (This is 1690 km below the surface.) (a) Calculate the magnitude of the acceleration due to the Moon's gravity at that point.

(b) Calculate the magnitude of the centripetal acceleration of the center of Earth as it rotates about that point once each lunar month (about 27.3 d) and compare it with the acceleration found in part (a). Comment on whether or not they are equal and why they should or should not be.

Show Solution

Strategy

(a) We'll use $a = \frac{GM}{r^2}$ to find the Moon's gravitational acceleration at the center of mass, using the distance from the center of mass to the Moon. (b) We'll calculate the centripetal acceleration using $a_c = r\omega^2$ where $r = 4700$ km is the radius of Earth's orbit around the center of mass.

Solution

(a) Calculate the acceleration due to Moon's gravity at the center of mass.

The distance from the center of mass to the Moon is:

- Earth-Moon distance: 3.84×10^8 m
- Distance from Earth's center to center of mass: $r_1 = 4.70\times10^6$ m
- Distance from center of mass to Moon: $r_2 = 3.84\times10^8 - 4.70\times10^6 = 3.793\times10^8$ m

Using:

- Mass of Moon: $M_{Moon} = 7.35\times10^{22}$ kg
- $G = 6.674\times10^{-11}$ N·m²/kg²

$$a_{gravity} = \frac{GM_{Moon}}{r_2^2} = \frac{(6.674\times10^{-11})7.35\times10^{22}}{(3.793\times10^8)^2}$$
$$a_{gravity} = \frac{4.91\times10^{12}}{1.439\times10^{17}} = 3.41\times10^{-5}\ \text{m/s}^2$$

(b) Calculate the centripetal acceleration of Earth's center as it orbits the center of mass.

The angular velocity is:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{27.3\ \text{days}} = \frac{2\pi}{27.3\times24\times3600\ \text{s}} = \frac{2\pi}{2.36\times10^6} = 2.66\times10^{-6}\ \text{rad/s}$$

The centripetal acceleration is:

$$a_c = r_1\omega^2 = (4.70\times10^6\ \text{m})(2.66\times10^{-6}\ \text{rad/s})^2$$
$$a_c = (4.70\times10^6)(7.08\times10^{-12}) = 3.33\times10^{-5}\ \text{m/s}^2$$

Comparison:

$$\frac{a_{gravity}}{a_c} = \frac{3.41\times10^{-5}}{3.33\times10^{-5}} = 1.024$$

The values are nearly identical (within 2.4%), which is expected.

Discussion

The gravitational acceleration due to the Moon (3.41×10^{-5} m/s²) and the centripetal acceleration of Earth's center (3.33×10^{-5} m/s²) are nearly equal, as they must be. At the center of mass, the gravitational attraction from the Moon provides exactly the centripetal force needed to keep Earth orbiting around that point. The small difference (about 2%) is due to rounding in our calculations and the approximation that the orbit is perfectly circular.

This problem beautifully demonstrates that the Earth-Moon system is truly a binary system, with both bodies orbiting their common center of mass, rather than the Moon simply orbiting Earth. The center of mass lies inside Earth (about 1690 km below the surface) because Earth is much more massive than the Moon, but both objects orbit this point.

Solve part (b) of Example 1 using $a_c = v^2/r$.

Show Solution

Strategy

Example 1 part (b) calculated the centripetal acceleration of the Moon using $a_c = r\omega^2$. Now we'll solve it using $a_c = v^2 r$ instead. We need to find the Moon's orbital velocity first using $v = 2\pi r T$.

Solution

First, calculate the Moon's orbital velocity. Using:

- Orbital radius: $r = 3.84 \times 10^8$ m
- Orbital period: $T = 27.3$ days $= 27.3 \times 86{,}400$ s $= 2.36 \times 10^6$ s

$$v = 2\pi r T = 2\pi (3.84 \times 10^8 \text{ m}) 2.36 \times 10^6 \text{ s} = 2.41 \times 10^9 2.36 \times 10^6 = 1.02 \times 10^3 \text{ m/s}$$

Now calculate the centripetal acceleration:

$$a_c = v^2 r = (1.02 \times 10^3 \text{ m/s})^2 3.84 \times 10^8 \text{ m} = 1.04 \times 10^6 3.84 \times 10^8 = 2.71 \times 10^{-3} \text{ m/s}^2$$

Discussion

Using the alternative formula $a_c = v^2 r$, we obtain 2.71×10^{-3} m/s² for the Moon's centripetal acceleration, which matches (within rounding error) the value of 2.72×10^{-3} m/s² found in Example 1 using $a_c = r\omega^2$. This confirms that both formulas for centripetal acceleration are equivalent and yield the same result, as expected.

Astrology, that unlikely and vague pseudoscience, makes much of the position of the planets at the moment of one's birth. The only known force a planet exerts on Earth is gravitational.

(a) Calculate the magnitude of the gravitational force exerted on a 4.20 kg baby by a 100 kg father 0.200 m away at birth (he is assisting, so he is close to the child).

(b) Calculate the magnitude of the force on the baby due to Jupiter if it is at its closest distance to Earth, some 6.29×10^{11}m away. How does the force of Jupiter on the baby compare to the force of the father on the baby? Other objects in the room and the hospital building also exert similar gravitational forces. (Of course, there could be an unknown force acting, but scientists first need to be convinced that there is even an effect, much less that an unknown force causes it.)

Show Solution

Strategy

For both parts, we'll use Newton's law of universal gravitation: $F = G m_1 m_2 r^2$. We'll compare the gravitational force from the father (nearby) to the force from Jupiter (far away).

Solution

(a) Calculate the gravitational force between father and baby:

Given:

- Mass of baby: $m_1 = 4.20$ kg
- Mass of father: $m_2 = 100$ kg
- Distance: $r = 0.200$ m
- $G = 6.674 \times 10^{-11}$ N·m²/kg²

$$F_{\text{father}} = G m_1 m_2 r^2 = (6.674 \times 10^{-11})(4.20)(100)(0.200)^2$$
$$F_{\text{father}} = (6.674 \times 10^{-11})(420) 0.0400 = 2.80 \times 10^{-8} 0.0400 = 7.01 \times 10^{-7} \text{ N}$$

(b) Calculate the gravitational force from Jupiter:

Given:

- Mass of baby: $m_1 = 4.20$ kg
- Mass of Jupiter: $M_{\text{Jupiter}} = 1.90 \times 10^{27}$ kg
- Distance: $r = 6.29 \times 10^{11}$ m

$$F_{\text{Jupiter}} = \frac{G m_1 M_{\text{Jupiter}}}{r^2} = \frac{(6.674 \times 10^{-11})(4.20)(1.90 \times 10^{27})}{(6.29 \times 10^{11})^2}$$
$$F_{\text{Jupiter}} = \frac{(6.674 \times 10^{-11})(7.98 \times 10^{27})}{3.96 \times 10^{23}} = \frac{5.33 \times 10^{17}}{3.96 \times 10^{23}} = 1.35 \times 10^{-6} \text{ N}$$

Compare the two forces:

$$\frac{F_{\text{father}}}{F_{\text{Jupiter}}} = \frac{7.01 \times 10^{-7} \text{ N}}{1.35 \times 10^{-6} \text{ N}} = 0.521$$

Discussion

The father’s gravitational pull on the baby (7.01×10^{-7} N) is about half the gravitational pull from Jupiter (1.35×10^{-6} N), even though Jupiter is enormously more massive. This is because the father is so much closer - distance matters enormously in gravitational interactions due to the inverse-square law.

This calculation thoroughly debunks astrological claims about planetary influences at birth. If gravitational force were the mechanism by which planets influence people, then the father standing nearby would have nearly the same effect as Jupiter! Furthermore, the doctor, the hospital building, and even nearby furniture would exert comparable gravitational forces. Both forces are incredibly tiny anyway - about a millionth of a newton, which is completely negligible compared to other forces acting on the baby. This demonstrates that if planets do influence human affairs (which has never been scientifically demonstrated), it cannot be through gravitational force.

The existence of the dwarf planet Pluto was proposed based on irregularities in Neptune’s orbit. Pluto was subsequently discovered near its predicted position. But it now appears that the discovery was fortuitous, because Pluto is small and the irregularities in Neptune’s orbit were not well known. To illustrate that Pluto has a minor effect on the orbit of Neptune compared with the closest planet to Neptune:

(a) Calculate the acceleration due to gravity at Neptune due to Pluto when they are 4.50×10^{12}m apart, as they are at present. The mass of Pluto is 1.4×10^{22}kg.

(b) Calculate the acceleration due to gravity at Neptune due to Uranus, presently about 2.50×10^{12}m apart, and compare it with that due to Pluto. The mass of Uranus is 8.62×10^{25}kg.

Show Solution

Strategy

We’ll use $a = \frac{GM}{r^2}$ to calculate the gravitational acceleration at Neptune due to both Pluto and Uranus, then compare the two values.

Solution

(a) Calculate acceleration at Neptune due to Pluto:

$$a_{\text{Pluto}} = \frac{G M_{\text{Pluto}}}{r^2} = \frac{(6.67 \times 10^{-11}) 1.4 \times 10^{22}}{(4.50 \times 10^{12})^2}$$
$$a_{\text{Pluto}} = \frac{9.34 \times 10^{11}}{2.03 \times 10^{25}} = 4.6 \times 10^{-14} \text{ m/s}^2$$

(b) Calculate acceleration at Neptune due to Uranus:

$$a_{\text{Uranus}} = \frac{G M_{\text{Uranus}}}{r^2} = \frac{(6.67 \times 10^{-11}) 8.62 \times 10^{25}}{(2.50 \times 10^{12})^2}$$
$$a_{\text{Uranus}} = \frac{5.75 \times 10^{15}}{6.25 \times 10^{24}} = 9.2 \times 10^{-10} \text{ m/s}^2$$

Compare the two accelerations:

$$\frac{a_{\text{Uranus}}}{a_{\text{Pluto}}} = \frac{9.2 \times 10^{-10}}{4.6 \times 10^{-14}} = 2.0 \times 10^4 = 20{,}000$$

Discussion

The acceleration due to gravity at Neptune caused by Pluto is 4.6×10^{-14} m/s², while that caused by Uranus is 9.2×10^{-10} m/s². Uranus produces an acceleration at Neptune that is about 20,000 times greater than that of Pluto. This demonstrates that Pluto’s gravitational effect on Neptune is indeed negligible compared to Uranus. The discovery of Pluto near its predicted position was fortuitous, as the observed irregularities in Neptune’s orbit could not have been caused primarily by such a small body.

(a) The Sun orbits the Milky Way galaxy once each 2.60×10^8y, with a roughly circular orbit averaging 3.00×10^4 light years in radius. (A light year is the distance traveled by light in 1 y.) Calculate the centripetal acceleration of the Sun in its galactic orbit. Does your result support the contention that a nearly inertial frame of reference can be located at the Sun?

(b) Calculate the average speed of the Sun in its galactic orbit. Does the answer surprise you?
Show Solution

Strategy

(a) Use the relationship for centripetal acceleration $a_c = \frac{v^2}{r} = \frac{4\pi^2 r}{T^2}$, converting all units appropriately. (b) Use $v = \frac{2\pi r}{T}$ to find the orbital speed.

Solution

Given:

- Orbital period: $T = 2.60 \times 10^8$ years
- Orbital radius: $r = 3.00 \times 10^4$ light years

First, convert to SI units.

Convert period to seconds:

$$T = 2.60 \times 10^8 \text{ y} \times 3.156 \times 10^7 \text{ s/y} = 8.21 \times 10^{15} \text{ s}$$

Convert radius to meters:

One light year is the distance light travels in one year:

$$1 \text{ ly} = c \times 1 \text{ y} = (3.00 \times 10^8 \text{ m/s})(3.156 \times 10^7 \text{ s}) = 9.47 \times 10^{15} \text{ m}$$
$$r = 3.00 \times 10^4 \text{ ly} \times 9.47 \times 10^{15} \text{ m/ly} = 2.84 \times 10^{20} \text{ m}$$

(a) Centripetal acceleration:

$$a_c = \frac{4\pi^2 r}{T^2} = \frac{4\pi^2 (2.84 \times 10^{20} \text{ m})}{(8.21 \times 10^{15} \text{ s})^2}$$
$$a_c = \frac{4(9.87)(2.84 \times 10^{20})}{6.74 \times 10^{31}} = \frac{1.12 \times 10^{22}}{6.74 \times 10^{31}} = 1.66 \times 10^{-10} \text{ m/s}^2$$

Compare to g:

$$\frac{a_c}{g} = \frac{1.66 \times 10^{-10}}{9.80} = 1.69 \times 10^{-11}$$

The centripetal acceleration is about $1.7 \times 10^{-11} g$, which is incredibly small - about 17 billionths of g. This strongly supports the use of the Sun as a nearly inertial reference frame.

(b) Average orbital speed:

$$v = \frac{2\pi r}{T} = \frac{2\pi (2.84 \times 10^{20} \text{ m})}{8.21 \times 10^{15} \text{ s}}$$
$$v = \frac{1.78 \times 10^{21}}{8.21 \times 10^{15}} = 2.17 \times 10^5 \text{ m/s} = 217 \text{ km/s}$$

Discussion

The Sun orbits the center of the Milky Way galaxy at an impressive speed of 217 km/s (about 780,000 km/h or 485,000 mph)! Despite this tremendous speed, the centripetal acceleration is incredibly small (1.66×10^{-10} m/s^2) because the orbital radius is so enormous. This tiny acceleration - about 17 billionths of the acceleration due to gravity on Earth - confirms that the Sun can be treated as an excellent inertial reference frame for almost all practical purposes.

The Sun's orbital speed might seem surprisingly high, but consider that at this speed, it still takes 260 million years to complete one orbit around the galaxy. The Sun has completed only about 18 orbits since the solar system formed 4.6 billion years ago. This motion carries the entire solar system with it - all the planets, moons, asteroids, and comets move together at 217 km/s through the galaxy.

Answer

(a) The centripetal acceleration of the Sun is 1.66×10^{-10} m/s^2. This is about $1.7 \times 10^{-11} g$, which strongly supports using the Sun as a nearly inertial reference frame.

(b) The average speed of the Sun in its galactic orbit is $2.17 \times 10^5 \text{ m/s} = 217 \text{ km/s}$. This is surprisingly fast - about 0.07% of the speed of light!

Unreasonable Result

A mountain 10.0 km from a person exerts a gravitational force on him equal to 2.00% of his weight.

(a) Calculate the mass of the mountain.

(b) Compare the mountain’s mass with that of Earth.

(c) What is unreasonable about these results?

(d) Which premises are unreasonable or inconsistent? (Note that accurate gravitational measurements can easily detect the effect of nearby mountains and variations in local geology.)

Show Solution

Strategy

(a) We’ll use Newton’s law of universal gravitation $F = G\frac{m_1 m_2}{r^2}$ where the force is 2.00% of the person’s weight. We’ll solve for the mountain’s mass. (b) We’ll compare the calculated mass to Earth’s mass. (c) and (d) We’ll evaluate whether the results and premises are reasonable.

Solution

(a) The gravitational force from the mountain is given as 2.00% of the person’s weight:

$$F_{\text{mountain}} = 0.0200mg$$

Using Newton’s law of universal gravitation:

$$F_{\text{mountain}} = \frac{GmM_{\text{mountain}}}{r^2}$$

Setting these equal and solving for M_{mountain}:

$$0.0200mg = \frac{GmM_{\text{mountain}}}{r^2}$$
$$M_{\text{mountain}} = \frac{0.0200gr^2}{G}$$

Substitute values with $r = 10.0\text{ km} = 1.00\times10^4$ m:

$$M_{\text{mountain}} = \frac{0.0200(9.80\text{ m/s}^2)(1.00\times10^4\text{ m})^2}{6.674\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2}$$
$$M_{\text{mountain}} = \frac{0.0200(9.80)(1.00\times10^8)}{6.674\times10^{-11}} = \frac{1.96\times10^7}{6.674\times10^{-11}} = 2.94\times10^{17}\text{ kg}$$

(b) Compare to Earth’s mass ($M_{\text{Earth}} = 5.972\times10^{24}$ kg):

$$\frac{M_{\text{mountain}}}{M_{\text{Earth}}} = \frac{2.94\times10^{17}\text{ kg}}{5.972\times10^{24}\text{ kg}} = 4.92\times10^{-8}$$

The mountain would be 4.92×10^{-8} (about 0.00000492%) of Earth’s mass.

(c) **What is unreasonable:**

The calculated mass of 2.94×10^{17} kg is absurdly large for a mountain. For comparison, Mount Everest has a mass of approximately 8.1×10^{14} kg - about 360 times less than our calculated value. The fraction of Earth’s mass (4.92×10^{-8}) may seem small, but it’s still unreasonably large for a single mountain.

(d) **Which premises are unreasonable:**

The unreasonable premise is that the mountain exerts a gravitational force equal to 2.00% of a person’s weight. This is far too large. In reality, even massive mountains exert gravitational forces that are typically less than 0.001% of a person’s weight at distances of 10 km. The force would need to be about 1000 times smaller to be realistic. While sensitive gravitational measurements can detect the effect of nearby mountains (which is how geologists map underground structures), these effects are much more subtle than the problem suggests.

Discussion

This problem illustrates the importance of checking whether calculated results match physical reality. The enormous mass required for a mountain to exert 2.00% of a person’s weight demonstrates that gravitational forces between everyday objects are extremely weak. Only objects with planetary masses create gravitational forces strong enough to dominate our daily experience. This is why we don’t notice the gravitational attraction between people, buildings, or even mountains—these forces are millions of times weaker than Earth’s gravitational pull.

Glossary

gravitational constant, *G*
: a proportionality factor used in the equation for Newton’s universal law of gravitation; it is a universal constant—that is, it is thought to be the same everywhere in the universe

center of mass
: the point where the entire mass of an object can be thought to be concentrated

microgravity
: an environment in which the apparent net acceleration of a body is small compared with that produced by Earth at its surface

Newton’s universal law of gravitation
: every particle in the universe attracts every other particle with a force along a line joining them; the force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them

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Satellites and Kepler’s Laws: An Argument for Simplicity

- State Kepler’s laws of planetary motion.
- Derive the third Kepler’s law for circular orbits.
- Discuss the Ptolemaic model of the universe.

Examples of gravitational orbits abound. Hundreds of artificial satellites orbit Earth together with thousands of pieces of debris. The Moon’s orbit about Earth has intrigued humans from time immemorial. The orbits of planets, asteroids, meteors, and comets about the Sun are no less interesting. If we look further, we see almost unimaginable numbers of stars, galaxies, and other celestial objects orbiting one another and interacting through gravity.

All these motions are governed by gravitational force, and it is possible to describe them to various degrees of precision. Precise descriptions of complex systems must be made with large computers. However, we can describe an important class of orbits without the use of computers, and we shall find it instructive to study them. These orbits have the following characteristics:

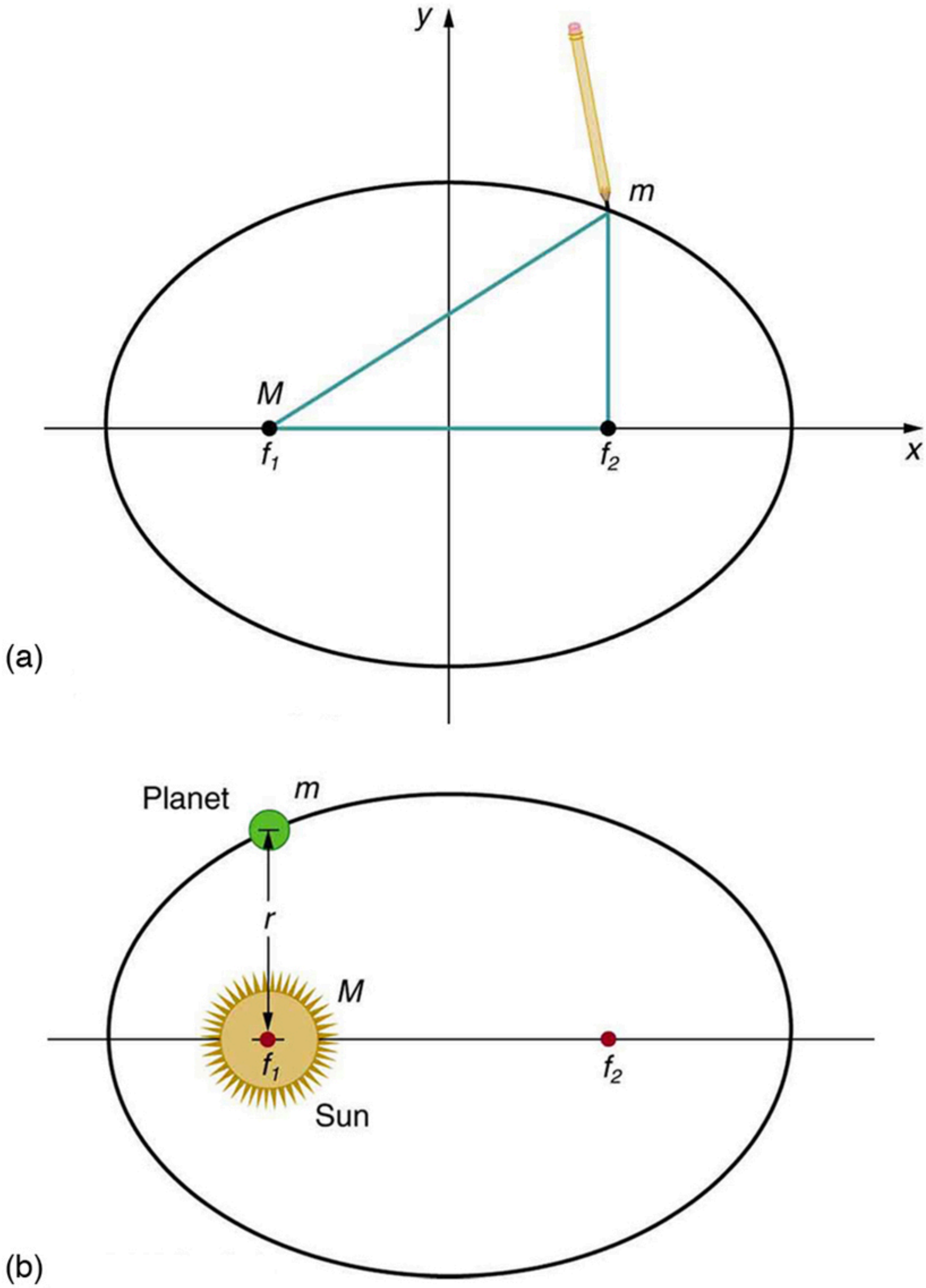
1. A small mass m orbits a much larger mass M. This allows us to view the motion as if M were stationary—in fact, as if from an inertial frame of reference placed on M —without significant error. Mass m is the satellite of M, if the orbit is gravitationally bound.
2. *The system is isolated from other masses*. This allows us to neglect any small effects due to outside masses.

The conditions are satisfied, to good approximation, by Earth’s satellites (including the Moon), by objects orbiting the Sun, and by the satellites of other planets. Historically, planets were studied first, and there is a classical set of three laws, called Kepler’s laws of planetary motion, that describe the orbits of all bodies satisfying the two previous conditions (not just planets in our solar system). These descriptive laws are named for the German astronomer Johannes Kepler (1571–1630), who devised them after careful study (over some 20 years) of a large amount of meticulously recorded observations of planetary motion done by Tycho Brahe (1546–1601). Such careful collection and detailed recording of methods and data are hallmarks of good science. Data constitute the evidence from which new interpretations and meanings can be constructed.

Kepler’s Laws of Planetary Motion

Kepler’s First Law

The orbit of each planet about the Sun is an ellipse with the Sun at one focus.



(a) An ellipse is a closed curve such that the sum of the distances from a point on the curve to the two foci $(f_1$ and f_2) is a constant. You can draw an ellipse as shown by putting a pin at each focus, and then placing a string around a pencil and the pins and tracing a line on paper. A circle is a special case of an ellipse in which the two foci coincide (thus any point on the circle is the same distance from the center). (b) For any closed gravitational orbit, m follows an elliptical path with M at one focus. Kepler’s first law states this fact for planets orbiting the Sun.

Kepler’s Second Law

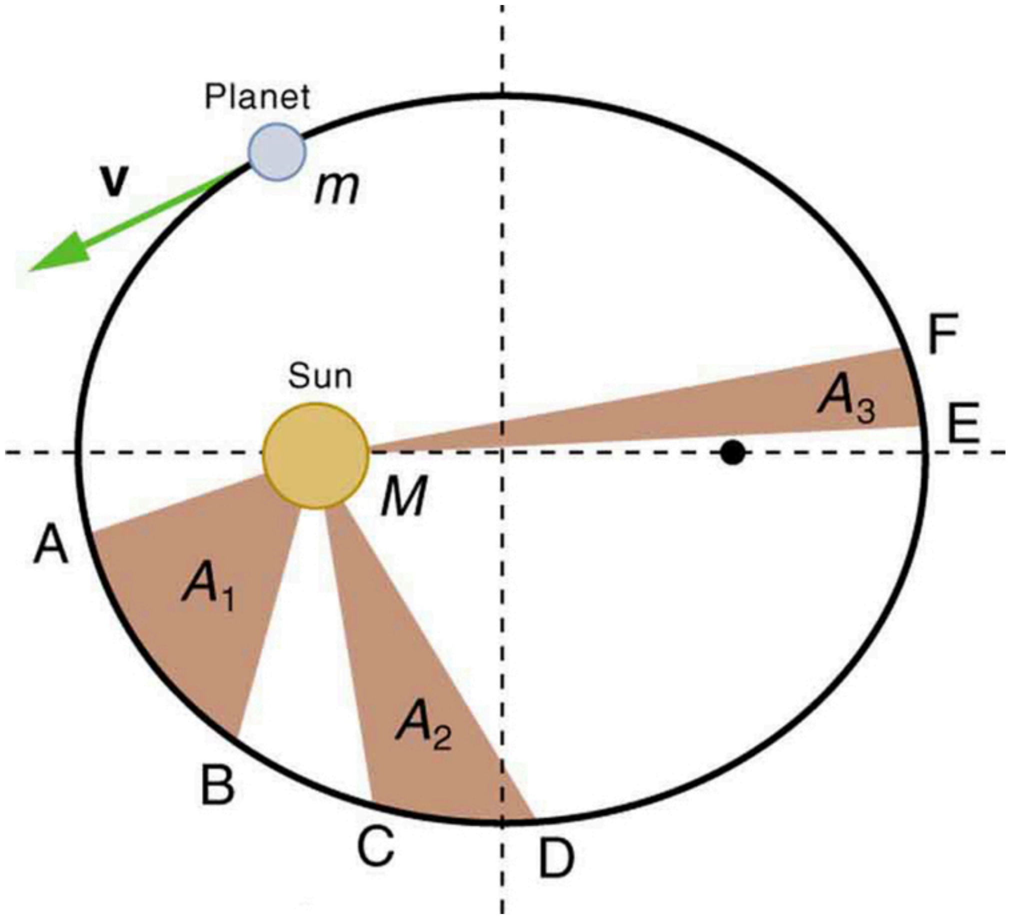
Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal times (see Figure 2).

Kepler’s Third Law

The ratio of the squares of the periods of any two planets about the Sun is equal to the ratio of the cubes of their average distances from the Sun. In equation form, this is

$$T_{21}T_{22}=r_{31}r_{32},$$

where T is the period (time for one orbit) and r is the average radius. This equation is valid only for comparing two small masses orbiting the same large one. Most importantly, this is a descriptive equation only, giving no information as to the cause of the equality.



The shaded regions have equal areas. It takes equal times for m to go from A to B, from C to D, and from E to F. The mass m moves fastest when it is closest to M . Kepler's second law was originally devised for planets orbiting the Sun, but it has broader validity.

Note again that while, for historical reasons, Kepler's laws are stated for planets orbiting the Sun, they are actually valid for all bodies satisfying the two previously stated conditions.

Find the Time for One Orbit of an Earth Satellite

Given that the Moon orbits Earth each 27.3 d and that it is an average distance of $3.84\times10^{8}\text{m}$ from the center of Earth, calculate the period of an artificial satellite orbiting at an average altitude of 1500 km above Earth's surface.

Strategy

The period, or time for one orbit, is related to the radius of the orbit by Kepler's third law, given in mathematical form in $T_{21}T_{22}=r_{31}r_{32}$. Let us use the subscript 1 for the Moon and the subscript 2 for the satellite. We are asked to find T_2. The given information tells us that the orbital radius of the Moon is $r_1=3.84\times10^{8}\text{m}$, and that the period of the Moon is $T_1=27.3\text{d}$. The height of the artificial satellite above Earth's surface is given, and so we must add the radius of Earth (6380 km) to get $r_2=(1500+6380)\text{km}=7880\text{km}$. Now all quantities are known, and so T_2 can be found.

Solution

Kepler's third law is

$$T_{21}T_{22}=r_{31}r_{32}.$$

To solve for T_2, we cross-multiply and take the square root, yielding

$$T_2^2 = T_1^2 \left(\frac{r_2}{r_1}\right)^3$$

$$T_2 = T_1 \left(\frac{r_2}{r_1}\right)^{3/2}.$$

Substituting known values yields

$$T_2 = 27.3\text{ d} \times \frac{24.0\text{ h}}{\text{d}} \times \left(\frac{7880\text{ km}}{3.84 \times 10^5\text{ km}}\right)^{3/2} \qquad T_2 = 1.93\text{ h}.$$

Discussion This is a reasonable period for a satellite in a fairly low orbit. It is interesting that any satellite at this altitude will orbit in the same amount of time. This fact is related to the condition that the satellite's mass is small compared with that of Earth.

People immediately search for deeper meaning when broadly applicable laws, like Kepler's, are discovered. It was Newton who took the next giant step when he proposed the law of universal gravitation. While Kepler was able to discover *what* was happening, Newton discovered that gravitational force was the cause.

Derivation of Kepler's Third Law for Circular Orbits

We shall derive Kepler's third law, starting with Newton's laws of motion and his universal law of gravitation. The point is to demonstrate that the force of gravity is the cause for Kepler's laws (although we will only derive the third one).

Let us consider a circular orbit of a small mass m around a large mass M, satisfying the two conditions stated at the beginning of this section. Gravity supplies the centripetal force to mass m. Starting with Newton's second law applied to circular motion,

$$F_{\text{net}} = ma_c = m\frac{v^2}{r}.$$

The net external force on mass m is gravity, and so we substitute the force of gravity for F_net:

$$G\frac{mM}{r^2} = m\frac{v^2}{r}.$$

The mass m cancels, yielding

$$G\frac{M}{r} = v^2.$$

The fact that m cancels out is another aspect of the oft-noted fact that at a given location all masses fall with the same acceleration. Here we see that at a given orbital radius r, all masses orbit at the same speed. (This was implied by the result of the preceding worked example.) Now, to get at Kepler's third law, we must get the period T into the equation. By definition, period T is the time for one complete orbit. Now the average speed v is the circumference divided by the period—that is,

$$v = \frac{2\pi r}{T}.$$

Substituting this into the previous equation gives

$$G\frac{M}{r} = \frac{4\pi^2 r^2}{T^2}.$$

Solving for T^2 yields

$$T^2 = \frac{4\pi^2}{GM} r^3.$$

Using subscripts 1 and 2 to denote two different satellites, and taking the ratio of the last equation for satellite 1 to satellite 2 yields

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}.$$

This is Kepler's third law. Note that Kepler's third law is valid only for comparing satellites of the same parent body, because only then does the mass of the parent body M cancel.

Now consider what we get if we solve $T^2 = \frac{4\pi^2}{GM} r^3$ for the ratio r^3/T^2. We obtain a relationship that can be used to determine the mass M of a parent body from the orbits of its satellites:

$$\frac{r^3}{T^2} = \frac{G}{4\pi^2} M.$$

If r and T are known for a satellite, then the mass M of the parent can be calculated. This principle has been used extensively to find the masses of heavenly bodies that have satellites. Furthermore, the ratio r^3/T^2 should be a constant for all satellites of the same parent body (because $\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$). (See Table 1).

It is clear from Table 1 that the ratio of r^3/T^2 is constant, at least to the third digit, for all listed satellites of the Sun, and for those of Jupiter. Small variations in that ratio have two causes—uncertainties in the r and T data, and perturbations of the orbits due to other bodies. Interestingly, those perturbations can be—and have been—used to predict the location of new planets and moons. This is another verification of Newton's universal law of gravitation.

Making Connections

Newton's universal law of gravitation is modified by Einstein's general theory of relativity. Newton's gravity is not seriously in error—it was and still is an extremely good approximation for most situations. Einstein's modification is most noticeable in extremely large gravitational fields, such as near black holes. However, general relativity also explains such phenomena as small but long-known deviations of the orbit of the planet Mercury from classical predictions.

The Case for Simplicity

The development of the universal law of gravitation by Newton played a pivotal role in the history of ideas. While it is beyond the scope of this text to cover that history in any detail, we note some important points. The definition of planet set in 2006 by the International Astronomical Union (IAU) states that in the solar system, a planet is a celestial body that:

1. is in orbit around the Sun,
2. has sufficient mass to assume hydrostatic equilibrium and
3. has cleared the neighborhood around its orbit.

A non-satellite body fulfilling only the first two of the above criteria is classified as "dwarf planet."

In 2006, Pluto was demoted to a 'dwarf planet' after scientists revised their definition of what constitutes a "true" planet.

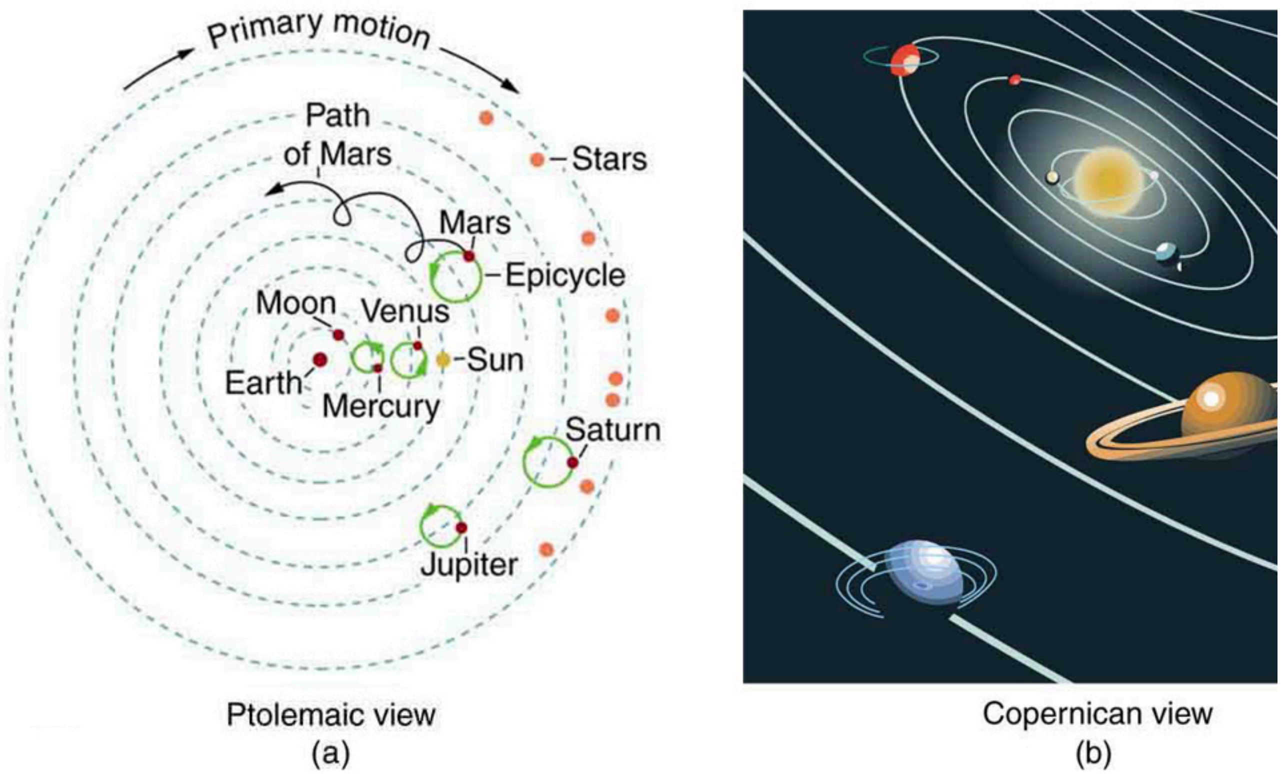
Table 1: Orbital Data and Kepler's Third Law

Parent	Satellite	Average orbital radius r(km)	Period T(y)	r^3/T^2, (km^3/y^2)
Earth	Moon	3.84×10^5	0.07481	1.01×10^{19}
Sun	Mercury	5.79×10^7	0.2409	3.34×10^{24}
Sun	Venus	1.082×10^8	0.6150	3.35×10^{24}
Sun	Earth	1.496×10^8	1.000	3.35×10^{24}
Sun	Mars	2.279×10^8	1.881	3.35×10^{24}
Sun	Jupiter	7.783×10^8	11.86	3.35×10^{24}
Sun	Saturn	1.427×10^9	29.46	3.35×10^{24}
Sun	Neptune	4.497×10^9	164.8	3.35×10^{24}
Sun	Pluto	5.90×10^9	248.3	3.33×10^{24}
Jupiter	Io	4.22×10^5	0.00485 (1.77 d)	3.19×10^{21}
Jupiter	Europa	6.71×10^5	0.00972 (3.55 d)	3.20×10^{21}
Jupiter	Ganymede	1.07×10^6	0.0196 (7.16 d)	3.19×10^{21}
Jupiter	Callisto	1.88×10^6	0.0457 (16.19 d)	3.20×10^{21}

The universal law of gravitation is a good example of a physical principle that is very broadly applicable. That single equation for the gravitational force describes all situations in which gravity acts. It gives a cause for a vast number of effects, such as the orbits of the planets and moons in the solar system. It epitomizes the underlying unity and simplicity of physics.

Before the discoveries of Kepler, Copernicus, Galileo, Newton, and others, the solar system was thought to revolve around Earth as shown in Figure 3(a). This is called the Ptolemaic view, for the Greek philosopher who lived in the second century AD. This model is characterized by a list of facts for the motions of planets with no cause and effect explanation. There tended to be a different rule for each heavenly body and a general lack of simplicity.

Figure 3(b) represents the modern or Copernican model. In this model, a small set of rules and a single underlying force explain not only all motions in the solar system, but all other situations involving gravity. The breadth and simplicity of the laws of physics are compelling. As our knowledge of nature has grown, the basic simplicity of its laws has become ever more evident.



(a) The Ptolemaic model of the universe has Earth at the center with the Moon, the planets, the Sun, and the stars revolving about it in complex superpositions of circular paths. This geocentric model, which can be made progressively more accurate by adding more circles, is purely descriptive, containing no hints as to what are the causes of these motions. (b) The Copernican model has the Sun at the center of the solar system. It is fully explained by a small number of laws of physics, including Newton's universal law of gravitation.

Gravity and Orbits

Move the sun, earth, moon and space station to see how it affects their gravitational forces and orbital paths. Visualize the sizes and distances between different heavenly bodies, and turn off gravity to see what would happen without it!

Gravity and Orbits

Section Summary

- Kepler's laws are stated for a small mass m orbiting a larger mass M in near-isolation. Kepler's laws of planetary motion are then as follows:

  Kepler's first law

The orbit of each planet about the Sun is an ellipse with the Sun at one focus.

Kepler's second law

Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal times.

Kepler's third law

The ratio of the squares of the periods of any two planets about the Sun is equal to the ratio of the cubes of their average distances from the Sun:

$$\frac{T_1^2}{T_2^2}=\frac{r_1^3}{r_2^3},$$

where T is the period (time for one orbit) and r is the average radius of the orbit.

- The period and radius of a satellite's orbit about a larger body M are related by
  $$T^2=\frac{4\pi^2}{GM}r^3$$

or

$$\frac{r^3}{T^2}=\frac{G}{4\pi^2}M.$$

Conceptual Questions

In what frame(s) of reference are Kepler's laws valid? Are Kepler's laws purely descriptive, or do they contain causal information?

Problem Exercises

A geosynchronous Earth satellite is one that has an orbital period of precisely 1 day. Such orbits are useful for communication and weather observation because the satellite remains above the same point on Earth (provided it orbits in the equatorial plane in the same direction as Earth's rotation). Calculate the radius of such an orbit based on the data for the moon in Table 1.

Show Solution

Strategy

We'll use Kepler's third law: $\frac{T_1^2}{T_2^2}=\frac{r_1^3}{r_2^3}$. We'll use the Moon's data as reference (subscript 1) and solve for the geosynchronous satellite's orbital radius (subscript 2).

Solution

From Table 1, the Moon has:

- Orbital radius: $r_1=3.84\times10^5$ km
- Orbital period: $T_1=27.3$ days

For the geosynchronous satellite:

- Orbital period: $T_2=1.00$ day
- Orbital radius: $r_2=?$

Using Kepler's third law and solving for r_2:

$$\frac{T_1^2}{T_2^2}=\frac{r_1^3}{r_2^3}$$

$$r_2^3=r_1^3\left(\frac{T_2}{T_1}\right)^2$$

$$r_2=r_1\left(\frac{T_2}{T_1}\right)^{2/3}=(3.84\times10^5\text{ km})\left(\frac{1.00}{27.3}\right)^{2/3}$$

$$r_2=(3.84\times10^5)(0.0366)^{2/3}=(3.84\times10^5)(0.110)=4.23\times10^4\text{ km}$$

Discussion

The orbital radius of a geosynchronous satellite is approximately 4.23×10^4 km or 42,300 km from Earth's center. This corresponds to an altitude of about 35,900 km above Earth's surface (subtracting Earth's radius of 6,400 km). At this altitude and with a 24-hour orbital period, the satellite remains fixed above the same point on Earth's equator, making it ideal for communications and weather monitoring.

Calculate the mass of the Sun based on data for Earth's orbit and compare the value obtained with the Sun's actual mass.

Show Solution

Strategy

We'll use the relationship $M = \frac{4\pi^2 r^3}{GT^2}$ derived from Kepler's third law. We know Earth's orbital radius is 1.50×10^{11} m and its period is 1 year.

Solution

Convert Earth's period to seconds:

$$T = 1\text{ year} \times \frac{365.25\text{ days}}{1\text{ year}} \times \frac{24\text{ h}}{1\text{ day}} \times \frac{3600\text{ s}}{1\text{ h}} = 3.156 \times 10^{7}\text{ s}$$

Calculate the Sun's mass:

$$M = \frac{4\pi^2 r^3}{GT^2} = \frac{4\pi^2 (1.50 \times 10^{11}\text{ m})^3}{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(3.156 \times 10^{7}\text{ s})^2}$$
$$M = \frac{4\pi^2 (3.375 \times 10^{33})}{(6.67 \times 10^{-11})(9.96 \times 10^{14})} = \frac{1.33 \times 10^{35}}{6.64 \times 10^{4}} = 2.00 \times 10^{30}\text{ kg}$$

The actual mass of the Sun is 1.989×10^{30} kg.

Discussion

The calculated mass of the Sun based on Earth's orbital data is 2.00×10^{30} kg, which agrees remarkably well with the accepted value of 1.989×10^{30} kg. This excellent agreement validates both Newton's law of universal gravitation and Kepler's laws, demonstrating that we can determine the mass of distant celestial bodies simply by observing the orbital motion of objects around them.

Find the mass of Jupiter based on data for the orbit of one of its moons, and compare your result with its actual mass.

Show Solution

Strategy

We'll use the relationship $\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$ derived from Kepler's third law to find Jupiter's mass M. We can use data for any of Jupiter's moons from Table 1. Let's use Io's data.

Solution

From Table 1, Io has:

- Orbital radius: $r = 4.22 \times 10^{5}$ km $= 4.22 \times 10^{8}$ m
- Orbital period: $T = 0.00485$ years

Convert the period to seconds:

$$T = 0.00485\text{ y} \times \frac{365.25\text{ d}}{1\text{ y}} \times \frac{24\text{ h}}{1\text{ d}} \times \frac{3600\text{ s}}{1\text{ h}} = 1.53 \times 10^{5}\text{ s}$$

Solve for Jupiter's mass using $M = \frac{4\pi^2 r^3}{GT^2}$:

$$M = \frac{4\pi^2 r^3}{GT^2} = \frac{4\pi^2 (4.22 \times 10^{8})^3}{(6.67 \times 10^{-11})(1.53 \times 10^{5})^2}$$
$$M = \frac{4\pi^2 (7.51 \times 10^{25})}{(6.67 \times 10^{-11})(2.34 \times 10^{10})} = \frac{2.97 \times 10^{27}}{1.56 \times 10^{0}} = 1.90 \times 10^{27}\text{ kg}$$

The actual mass of Jupiter is 1.90×10^{27} kg.

Discussion

The calculated mass of Jupiter based on Io's orbital data is 1.90×10^{27} kg, which matches Jupiter's actual mass perfectly (to three significant figures). This demonstrates the power of Kepler's laws and Newton's law of gravitation - we can determine the mass of a distant planet simply by observing the motion of its satellites. This same technique has been used to find the masses of all planets with moons, as well as stars with orbiting planets.

Find the ratio of the mass of Jupiter to that of Earth based on data in Table 1.

Show Solution

Strategy

We'll use Kepler's third law in the form $\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$ for both Jupiter (with its moon) and Earth (with its moon). Taking the ratio will allow G and $4\pi^2$ to cancel, leaving us with the mass ratio.

Solution

From Kepler's third law:

For Earth and its Moon:

$$\frac{r^3_{\text{Moon}}}{T^2_{\text{Moon}}} = \frac{GM_{\text{Earth}}}{4\pi^2}$$

For Jupiter and one of its moons (using Io):

$$\frac{r^3_{\text{Io}}}{T^2_{\text{Io}}} = \frac{GM_{\text{Jupiter}}}{4\pi^2}$$

Taking the ratio:

$$\frac{M_{\text{Jupiter}}}{M_{\text{Earth}}} = \frac{r^3_{\text{Io}}/T^2_{\text{Io}}}{r^3_{\text{Moon}}/T^2_{\text{Moon}}}$$

From Table 1, we can read the r^3/T^2 values directly:

- Moon: $r^3/T^2 = 1.01 \times 10^{19}$ km³/y²
- Io: $r^3/T^2 = 3.19 \times 10^{21}$ km³/y²

$$\frac{M_{\text{Jupiter}}}{M_{\text{Earth}}} = \frac{3.19 \times 10^{21}}{1.01 \times 10^{19}} = 316$$

Discussion

Jupiter's mass is approximately 316 times Earth's mass. This enormous mass difference reflects Jupiter's status as a gas giant - it contains more than twice the mass of all other planets in our solar system combined. The fact that we can calculate this mass ratio simply from observational data in Table 1 demonstrates the elegance and power of Kepler's laws. Remarkably, Jupiter is so massive that the Sun-Jupiter center of mass actually lies slightly outside the Sun's surface, making it almost a binary star system rather than a simple planet-star system.

Astronomical observations of our Milky Way galaxy indicate that it has a mass of about 8.0×10^{11} solar masses. A star orbiting on the galaxy's periphery is about 6.0×10^4 light years from its center. (a) What should the orbital period of that star be? (b) If its period is 6.0×10^7 years instead, what is the mass of the galaxy? Such calculations are used to imply the existence of "dark matter" in the universe and have indicated, for example, the existence of very massive black holes at the centers of some galaxies.

Show Solution

Strategy

(a) We'll use $T^2 = \frac{4\pi^2}{GM} r^3$ to find the orbital period. We need to convert units: 1 solar mass $= 1.99 \times 10^{30}$ kg, and 1 light year $= 9.46 \times 10^{15}$ m.
(b) We'll use the same equation solved for mass: $M = \frac{4\pi^2 r^3}{GT^2}$.

Solution

(a) Convert the given values to SI units:

- Galaxy mass: $M = 8.0 \times 10^{11} \times 1.99 \times 10^{30} = 1.59 \times 10^{42}$ kg
- Orbital radius: $r = 6.0 \times 10^4 \times 9.46 \times 10^{15} = 5.68 \times 10^{20}$ m

Calculate the orbital period:

$$T^2 = \frac{4\pi^2 r^3}{GM} = \frac{4\pi^2 (5.68 \times 10^{20})^3}{(6.67 \times 10^{-11})(1.59 \times 10^{42})}$$
$$T^2 = \frac{7.25 \times 10^{63}}{1.06 \times 10^{32}} = 6.84 \times 10^{31} \text{ s}^2$$
$$T = 2.62 \times 10^{15} \text{ s} = 2.62 \times 10^{15} \text{ s} \times \frac{1 \text{ y}}{3.156 \times 10^7 \text{ s}} = 8.3 \times 10^7 \text{ years}$$

(b) If the actual period is 6.0×10^7 years, calculate the implied mass:

Convert period to seconds: $T = 6.0 \times 10^7 \times 3.156 \times 10^7 = 1.89 \times 10^{15}$ s

$$M = \frac{4\pi^2 r^3}{GT^2} = \frac{4\pi^2 (5.68 \times 10^{20})^3}{(6.67 \times 10^{-11})(1.89 \times 10^{15})^2}$$
$$M = \frac{7.25 \times 10^{63}}{2.38 \times 10^{31}} = 3.05 \times 10^{42} \text{ kg}$$

Convert to solar masses: $M = \frac{3.05 \times 10^{42}}{1.99 \times 10^{30}} = 1.5 \times 10^{12}$ solar masses

Discussion

(a) Based on the observed mass of 8.0×10^{11} solar masses, the star should have an orbital period of about 8.3×10^{7} years. (b) However, if the actual observed period is 6.0×10^{7} years (faster than expected), this implies the galaxy has a mass of about 1.5×10^{12} solar masses - nearly twice the visible mass. This discrepancy is evidence for “dark matter” - matter that doesn’t emit light but exerts gravitational force. Such calculations have revolutionized our understanding of galactic structure and the composition of the universe.

Integrated Concepts

Space debris left from old satellites and their launchers is becoming a hazard to other satellites. (a) Calculate the speed of a satellite in an orbit 900 km above Earth’s surface. (b) Suppose a loose rivet is in an orbit of the same radius that intersects the satellite’s orbit at an angle of 90° relative to Earth. What is the velocity of the rivet relative to the satellite just before striking it? (c) Given the rivet is 3.00 mm in size, how long will its collision with the satellite last? (d) If its mass is 0.500 g, what is the average force it exerts on the satellite? (e) How much energy in joules is generated by the collision? (The satellite’s velocity does not change appreciably, because its mass is much greater than the rivet’s.)

Show Solution

Strategy

(a) We’ll use $v=\sqrt{GMr}$ where r is the distance from Earth’s center. (b) Since the orbits intersect at 90°, the velocities are perpendicular, so we’ll use the Pythagorean theorem: $v_{\text{rel}}=\sqrt{v_{21}+v_{22}}$. (c) Collision time is $\Delta t=dv_{\text{rel}}$. (d) We’ll use impulse-momentum: $F_{\text{avg}}=\Delta p\Delta t$. (e) We’ll calculate the kinetic energy lost by the rivet.

Solution

(a) Calculate satellite orbital speed. The orbital radius is:

$$r=R_{\text{Earth}}+h=6.38\times10^{6}\text{ m}+9.00\times10^{5}\text{ m}=7.28\times10^{6}\text{ m}$$
$$v=\sqrt{GM_{\text{Earth}}r}=\sqrt{(6.674\times10^{-11})(5.97\times10^{24})7.28\times10^{6}}=\sqrt{5.48\times10^{7}}=7.40\times10^{3}\text{ m/s}$$

(b) The rivet has the same orbital speed but at 90°. The relative velocity is:

$$v_{\text{rel}}=\sqrt{v^2+v^2}=v\sqrt{2}=(7.40\times10^{3})\sqrt{2}=1.05\times10^{4}\text{ m/s}$$

(c) Collision duration (time for rivet to pass through):

$$\Delta t=dv_{\text{rel}}=3.00\times10^{-3}\text{ m}1.05\times10^{4}\text{ m/s}=2.86\times10^{-7}\text{ s}$$

(d) Average force using impulse-momentum theorem:

$$F_{\text{avg}}=\Delta p\Delta t=mv_{\text{rel}}\Delta t=(0.500\times10^{-3}\text{ kg})(1.05\times10^{4}\text{ m/s})2.86\times10^{-7}\text{ s}$$
$$F_{\text{avg}}=5.252.86\times10^{-7}=1.84\times10^{7}\text{ N}$$

(e) Energy generated (kinetic energy of rivet in satellite’s reference frame):

$$E=12mv_{2\text{rel}}=12(0.500\times10^{-3})(1.05\times10^{4})^{2}=(2.50\times10^{-4})(1.10\times10^{8})=2.76\times10^{4}\text{ J}$$

Discussion

This problem illustrates the serious hazard posed by space debris. Even though the rivet is tiny (3 mm, 0.5 g), the relative velocity of 10.5 km/s creates enormous forces and energy. The average force of 18.4 million newtons during the microsecond-long collision is equivalent to the weight of about 1,880 tons! The 27.6 kJ of energy released is comparable to a small explosive. This is why even paint flecks in orbit can damage spacecraft windows, and why tracking and avoiding space debris is critical for satellite and space station safety. The problem also demonstrates why the International Space Station has shielding and why astronauts performing spacewalks face real danger from hypervelocity impacts.

Unreasonable Results

(a) Based on Kepler’s laws and information on the orbital characteristics of the Moon, calculate the orbital radius for an Earth satellite having a period of 1.00 h. (b) What is unreasonable about this result? (c) What is unreasonable or inconsistent about the premise of a 1.00 h orbit?

Show Solution

Strategy

Use Kepler’s third law to relate the orbital period and radius. We know the Moon’s orbital characteristics, so we can set up a ratio or use the derived form of Kepler’s third law with Earth’s mass.

Solution

(a) Calculate the orbital radius:

Using Kepler's third law for Earth satellites:

$$T^2 = 4\pi^2 G M_{\text{Earth}} r^3$$

Solving for r:

$$r = (G M_{\text{Earth}} T^2 4\pi^2)^{1/3}$$

Given:

- Period: T =1.00 h =3600 s
- $G = 6.674 \times 10^{-11}$ N·m^2/kg^2
- $M_{\text{Earth}} = 5.972 \times 10^{24}$ kg

$$r = ((6.674 \times 10^{-11})(5.972 \times 10^{24})(3600)^2 4\pi^2)^{1/3}$$
$$r = ((3.985 \times 10^{14})(1.296 \times 10^{7}) 39.48)^{1/3}$$
$$r = (5.17 \times 10^{21} 39.48)^{1/3} = (1.31 \times 10^{20})^{1/3}$$
$$r = 5.08 \times 10^{6} \text{ m} = 5.08 \times 10^{3} \text{ km} = 5080 \text{ km}$$

(b) What is unreasonable about this result?

The calculated orbital radius of 5080 km is **less than Earth's radius** of 6371 km. This means the satellite would need to orbit inside the Earth, which is physically impossible. An orbit can only exist if the satellite is above Earth's surface.

(c) What is unreasonable about the premise?

The premise of a 1-hour orbital period is physically impossible for an Earth satellite. The **minimum possible orbital period** occurs when a satellite orbits just above Earth's surface (ignoring atmospheric drag). This minimum period can be calculated:

$$T_{\text{min}} = 2\pi\sqrt{R^3_{\text{Earth}} G M_{\text{Earth}}} = 2\pi\sqrt{(6.371 \times 10^{6})^3 3.985 \times 10^{14}}$$
$$T_{\text{min}} = 2\pi\sqrt{2.586 \times 10^{20} 3.985 \times 10^{14}} = 2\pi\sqrt{6.49 \times 10^{5}} = 2\pi(806) = 5063 \text{ s} \approx 84.4 \text{ min}$$

Therefore, the minimum orbital period for an Earth satellite is about 84.4 minutes. A 60-minute (1-hour) orbit would require the satellite to be inside Earth, which is impossible.

Discussion

This problem illustrates an important physical constraint: there is a minimum orbital period for any spherical body determined by its density. For Earth, this minimum is about 84-85 minutes. The International Space Station, orbiting at about 400 km altitude, has a period of approximately 92 minutes. Satellites in lower orbits have shorter periods, but they can never go below the ~84-minute minimum. Interestingly, this minimum period depends only on the planet's average density - a larger planet with the same density would have the same minimum period. This is why the minimum period can be expressed as $T_{\text{min}} = \sqrt{3\pi G \rho}$, where ρ is the average density.

Answer

(a) The orbital radius would be 5.08×10^3 km (5080 km).

(b) This radius is unreasonable because it is **less than Earth's radius** (6371 km), meaning the satellite would have to orbit inside the Earth.

(c) The premise of a 1-hour orbit is physically impossible. The **minimum orbital period** for an Earth satellite is about **84.4 minutes**, which occurs at Earth's surface. No satellite can orbit faster than this around Earth.

Construct Your Own Problem

On February 14, 2000, the NEAR spacecraft was successfully inserted into orbit around Eros, becoming the first artificial satellite of an asteroid. Construct a problem in which you determine the orbital speed for a satellite near Eros. You will need to find the mass of the asteroid and consider such things as a safe distance for the orbit. Although Eros is not spherical, calculate the acceleration due to gravity on its surface at a point an average distance from its center of mass. Your instructor may also wish to have you calculate the escape velocity from this point on Eros.

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