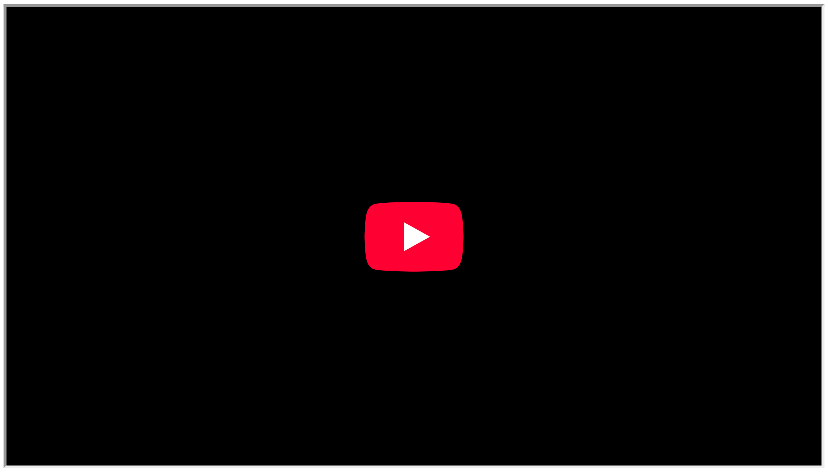
Introduction to Rotational Motion and Angular Momentum

The mention of a tornado conjures up images of raw destructive power. Tornadoes blow houses away as if they were made of paper and have been known to pierce tree trunks with pieces of straw. They descend from clouds in funnel-like shapes that spin violently, particularly at the bottom where they are most narrow, producing winds as high as 500 km/h. (credit: Daphne Zaras, U.S. National Oceanic and Atmospheric Administration)

Why do tornadoes spin at all? And why do tornadoes spin so rapidly? The answer is that air masses that produce tornadoes are themselves rotating, and when the radii of the air masses decrease, their rate of rotation increases. An ice skater increases her spin in an exactly analogous manner as seen in Figure 2. The skater starts her rotation with outstretched limbs and increases her spin by pulling them in toward her body. The same physics describes the exhilarating spin of a skater and the wrenching force of a tornado.

Clearly, force, energy, and power are associated with rotational motion. These and other aspects of rotational motion are covered in this chapter. We shall see that all important aspects of rotational motion either have already been defined for linear motion or have exact analogs in linear motion. First, we look at angular acceleration—the rotational analog of linear acceleration.

This figure skater increases her rate of spin by pulling her arms and her extended leg closer to her axis of rotation. (credit: Luu, Wikimedia Commons)



Angular Acceleration

- Describe uniform circular motion.
- Explain non-uniform circular motion.
- Calculate angular acceleration of an object.
- Observe the link between linear and angular acceleration.

Uniform Circular Motion and Gravitation discussed only uniform circular motion, which is motion in a circle at constant speed and, hence, constant angular velocity. Recall that angular velocity ω was defined as the time rate of change of angle θ:

$$\omega = \Delta\theta\Delta t,$$

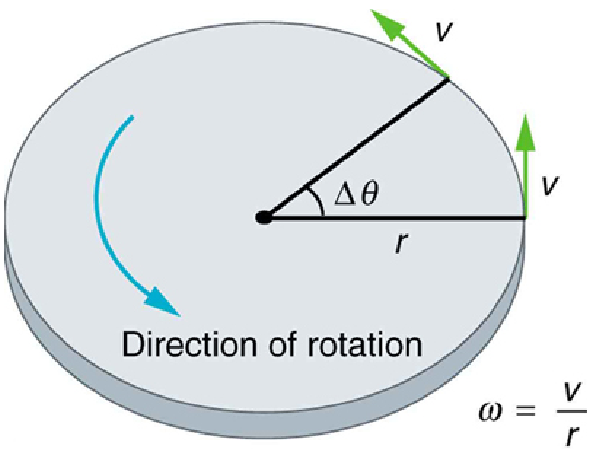
where θ is the angle of rotation as seen in Figure 1. The relationship between angular velocity ω and linear velocity v was also defined in Rotation Angle and Angular Velocity as

$$v = r\omega$$

or

$$\omega = vr,$$

where r is the radius of curvature, also seen in Figure 1. According to the sign convention, the counter clockwise direction is considered as positive direction and clockwise direction as negative



This figure shows uniform circular motion and some of its defined quantities.

Angular velocity is not constant when a skater pulls in her arms, when a child starts up a merry-go-round from rest, or when a computer's hard disk slows to a halt when switched off. In all these cases, there is an **angular acceleration**, in which ω changes. The faster the change occurs, the greater the angular acceleration. Angular acceleration α is defined as the rate of change of angular velocity. In equation form, angular acceleration is expressed as follows:

$$\alpha = \Delta\omega\Delta t,$$

where $\Delta\omega$ is the **change in angular velocity** and Δt is the change in time. The units of angular acceleration are $(\text{rad/s})/\text{s}$, or rad/s^2. If ω increases, then α is positive. If ω decreases, then α is negative.

Calculating the Angular Acceleration and Deceleration of a Bike Wheel

Suppose a teenager puts her bicycle on its back and starts the rear wheel spinning from rest to a final angular velocity of 250 rpm in 5.00 s. (a) Calculate the angular acceleration in rad/s^2. (b) If she now slams on the brakes, causing an angular acceleration of -87.3rad/s^2, how long does it take the wheel to stop?

Strategy for (a)

The angular acceleration can be found directly from its definition in $\alpha = \Delta\omega\Delta t$ because the final angular velocity and time are given. We see that $\Delta\omega$ is 250 rpm and Δt is 5.00 s.

Solution for (a)

Entering known information into the definition of angular acceleration, we get

$$\alpha = \frac{\Delta\omega}{\Delta t} \quad \alpha = \frac{250\,\text{rpm}}{5.00\,\text{s}}.$$

Because $\Delta\omega$ is in revolutions per minute (rpm) and we want the standard units of rad/s^2 for angular acceleration, we need to convert $\Delta\omega$ from rpm to rad/s:

$$\Delta\omega = 250\frac{\text{rev}}{\text{min}}\cdot\frac{2\pi\,\text{rad}}{\text{rev}}\cdot\frac{1\,\text{min}}{60\,\text{sec}} \quad \Delta\omega = 26.2\frac{\text{rad}}{\text{s}}.$$

Entering this quantity into the expression for α, we get

$$\alpha = \frac{\Delta\omega}{\Delta t} \quad \alpha = \frac{26.2\,\text{rad/s}}{5.00\,\text{s}} \quad \alpha = 5.24\,\text{rad/s}^2.$$

Strategy for (b)

In this part, we know the angular acceleration and the initial angular velocity. We can find the stoppage time by using the definition of angular acceleration and solving for Δt, yielding

$$\Delta t = \frac{\Delta\omega}{\alpha}.$$

Solution for (b)

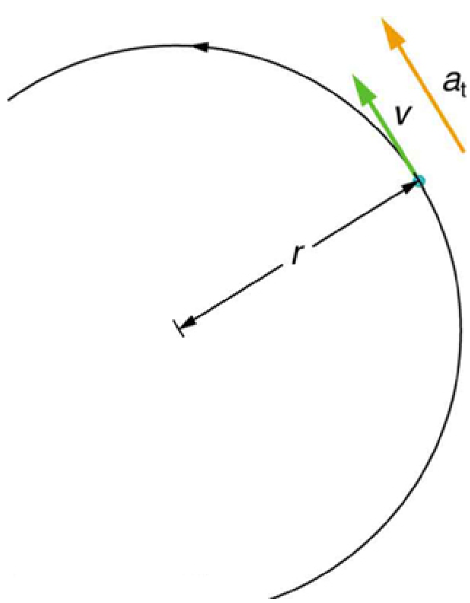
Here the angular velocity decreases from $26.2\,\text{rad/s}$ (250 rpm) to zero, so that $\Delta\omega$ is $-26.2\,\text{rad/s}$, and α is given to be $-87.3\,\text{rad/s}^2$. Thus,

$$\Delta t = \frac{-26.2\,\text{rad/s}}{-87.3\,\text{rad/s}^2} \quad \Delta t = 0.300\,\text{s}.$$

Discussion

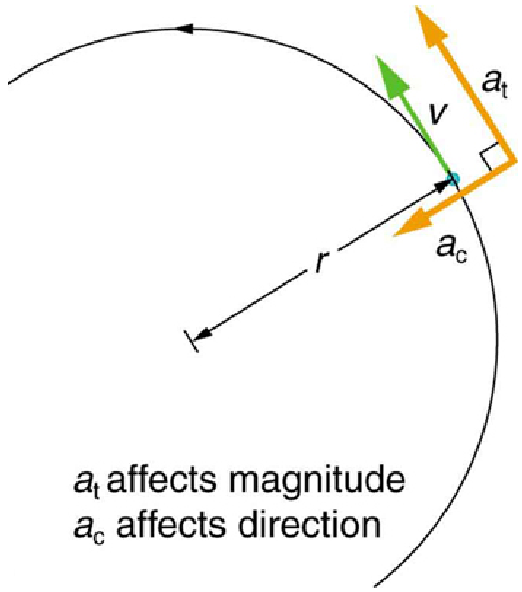
Note that the angular acceleration as the girl spins the wheel is small and positive; it takes 5 s to produce an appreciable angular velocity. When she hits the brake, the angular acceleration is large and negative. The angular velocity quickly goes to zero. In both cases, the relationships are analogous to what happens with linear motion. For example, there is a large deceleration when you crash into a brick wall—the velocity change is large in a short time interval.

If the bicycle in the preceding example had been on its wheels instead of upside-down, it would first have accelerated along the ground and then come to a stop. This connection between circular motion and linear motion needs to be explored. For example, it would be useful to know how linear and angular acceleration are related. In circular motion, linear acceleration is *tangent* to the circle at the point of interest, as seen in Figure 2. Thus, linear acceleration is called **tangential acceleration** a_t.



In circular motion, linear acceleration a , occurs as the magnitude of the velocity changes: a is tangent to the motion. In the context of circular motion, linear acceleration is also called tangential acceleration a_t .

Linear or tangential acceleration refers to changes in the magnitude of velocity but not its direction. We know from Uniform Circular Motion and Gravitation that in circular motion centripetal acceleration, a_c, refers to changes in the direction of the velocity but not its magnitude. An object undergoing circular motion experiences centripetal acceleration, as seen in Figure 3. Thus, a_t and a_c are perpendicular and independent of one another. Tangential acceleration a_t is directly related to the angular acceleration α and is linked to an increase or decrease in the velocity, but not its direction.



Centripetal acceleration a_c occurs as the direction of velocity changes; it is perpendicular to the circular motion. Centripetal and tangential acceleration are thus perpendicular to each other.

Now we can find the exact relationship between linear acceleration a_t and angular acceleration α. Because linear acceleration is proportional to a change in the magnitude of the velocity, it is defined (as it was in One-Dimensional Kinematics) to be

$$a_t=\frac{\Delta v}{\Delta t}.$$

For circular motion, note that $v=r\omega$, so that

$$a_t=\frac{\Delta(r\omega)}{\Delta t}.$$

The radius r is constant for circular motion, and so $\Delta(r\omega)=r(\Delta\omega)$. Thus,

$$a_t=r\frac{\Delta\omega}{\Delta t}.$$

By definition, $\alpha=\frac{\Delta\omega}{\Delta t}$. Thus,

$$a_t=r\alpha,$$

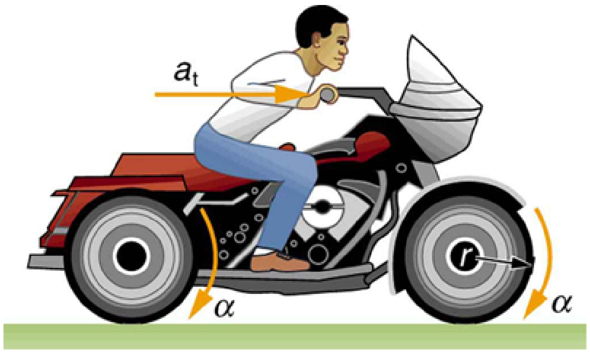
or

$$\alpha=\frac{a_t}{r}.$$

These equations mean that linear acceleration and angular acceleration are directly proportional. The greater the angular acceleration is, the larger the linear (tangential) acceleration is, and vice versa. For example, the greater the angular acceleration of a car's drive wheels, the greater the acceleration of the car. The radius also matters. For example, the smaller a wheel, the smaller its linear acceleration for a given angular acceleration α.

Calculating the Angular Acceleration of a Motorcycle Wheel

A powerful motorcycle can accelerate from 0 to 30.0 m/s (about 108 km/h) in 4.20 s. What is the angular acceleration of its 0.320-m-radius wheels? (See Figure 4.)



The linear acceleration of a motorcycle is accompanied by an angular acceleration of its wheels.

Strategy

We are given information about the linear velocities of the motorcycle. Thus, we can find its linear acceleration a_t. Then, the expression $\alpha = a_t r$ can be used to find the angular acceleration.

Solution

The linear acceleration is

$$a_t \quad = \quad \Delta v \Delta t \quad a_t \quad = \quad 30.0\text{m/s}4.20\text{s} \quad a_t \quad = \quad 7.14\text{m/s}^2.$$

We also know the radius of the wheels. Entering the values for a_t and r into $\alpha = a_t r$, we get

$$\alpha \quad = \quad a_t r \quad \alpha \quad = \quad 7.14\text{m/s}^2 0.320\text{m} \quad \alpha \quad = \quad 22.3\text{rad/s}^2.$$

Discussion

Units of radians are dimensionless and appear in any relationship between angular and linear quantities.

So far, we have defined three rotational quantities— θ, ω, and α. These quantities are analogous to the translational quantities x, v and a. Table 1 displays rotational quantities, the analogous translational quantities, and the relationships between them.

Table: Rotational and Translational Quantities

Rotational	**Translational**	**Relationship**
θ	x	$\theta = xr$
ω	v	$\omega = vr$
α	a	$\alpha = ar$

Making Connections: Take-Home Experiment

Sit down with your feet on the ground on a chair that rotates. Lift one of your legs such that it is unbent (straightened out). Using the other leg, begin to rotate yourself by pushing on the ground. Stop using your leg to push the ground but allow the chair to rotate. From the origin where you began, sketch the angle, angular velocity, and angular acceleration of your leg as a function of time in the form of three separate graphs. Estimate the magnitudes of these quantities.

Check Your Understanding

Angular acceleration is a vector, having both magnitude and direction. How do we denote its magnitude and direction? Illustrate with an example.

Show Solution

The magnitude of angular acceleration is α and its most common units are rad/s^2. The direction of angular acceleration along a fixed axis is denoted by a + or a – sign, just as the direction of linear acceleration in one dimension is denoted by a + or a – sign. For example, consider a gymnast doing a forward flip. Her angular momentum would be parallel to the mat and to her left. The magnitude of her angular acceleration would be proportional to her angular velocity (spin rate) and her moment of inertia about her spin axis.

Ladybug Revolution

Join the ladybug in an exploration of rotational motion. Rotate the merry-go-round to change its angle, or choose a constant angular velocity or angular acceleration. Explore how circular motion relates to the bug's x,y position, velocity, and acceleration using vectors or graphs.

Ladybug Revolution

Section Summary

- Uniform circular motion is the motion with a constant angular velocity $\omega = \frac{\Delta\theta}{\Delta t}$.
- In non-uniform circular motion, the velocity changes with time and the rate of change of angular velocity (i.e. angular acceleration) is $\alpha = \frac{\Delta\omega}{\Delta t}$.
- Linear or tangential acceleration refers to changes in the magnitude of velocity but not its direction, given as $a_t = \frac{\Delta v}{\Delta t}$.
- For circular motion, note that $v = r\omega$, so that
  $$a_t = \frac{\Delta(r\omega)}{\Delta t}.$$
- The radius r is constant for circular motion, and so $\Delta(r\omega) = r\Delta\omega$. Thus,
  $$a_t = r\frac{\Delta\omega}{\Delta t}.$$
- By definition, $\Delta\omega/\Delta t = \alpha$. Thus,
  $$a_t = r\alpha$$

or

$$\alpha = \frac{a_t}{r}.$$

Conceptual Questions

Analogies exist between rotational and translational physical quantities. Identify the rotational term analogous to each of the following: acceleration, force, mass, work, translational kinetic energy, linear momentum, impulse.

Explain why centripetal acceleration changes the direction of velocity in circular motion but not its magnitude.

In circular motion, a tangential acceleration can change the magnitude of the velocity but not its direction. Explain your answer.

Suppose a piece of food is on the edge of a rotating microwave oven plate. Does it experience nonzero tangential acceleration, centripetal acceleration, or both when: (a) The plate starts to spin? (b) The plate rotates at constant angular velocity? (c) The plate slows to a halt?

Problems & Exercises

At its peak, a tornado is 60.0 m in diameter and carries 500 km/h winds. What is its angular velocity in revolutions per second?

Show Solution

Strategy

The wind speed represents the tangential velocity at the outer edge of the tornado. We can find angular velocity using $\omega = v/r$, where r is the radius (half the diameter).

Solution

First, convert the wind speed to m/s:

$$v = 500\ \text{km/h} \times \frac{1000\,\text{m}}{1\ \text{km}} \times \frac{1\ \text{h}}{3600\,\text{s}} = 139\ \text{m/s}$$

The radius is half the diameter:

$$r = \frac{60.0\,\text{m}}{2} = 30.0\,\text{m}$$

Calculate the angular velocity in rad/s:

$$\omega = \frac{v}{r} = \frac{139\ \text{m/s}}{30.0\,\text{m}} = 4.63\ \text{rad/s}$$

Convert to revolutions per second:

$$\omega = 4.63\ \text{rad/s} \times \frac{1\ \text{rev}}{2\pi\ \text{rad}} = 0.737\ \text{rev/s}$$

Discussion

The tornado completes nearly three-quarters of a revolution every second. This rapid rotation, combined with the large diameter, produces the devastating 500 km/h winds at the edge. The angular velocity of 0.737 rev/s means the tornado rotates about 44 times per minute, creating the characteristic funnel shape through conservation of angular momentum as air spirals inward.

Answer

The angular velocity of the tornado is **0.737 rev/s** (or **4.63 rad/s**).

Integrated Concepts

An ultracentrifuge accelerates from rest to 100 000 rpm in 2.00 min. (a) What is its angular acceleration in rad/s^2 ? (b) What is the tangential acceleration of a point 9.50 cm from the axis of rotation? (c) What is the radial acceleration in m/s^2 and multiples of g of this point at full rpm?

Show Solution

Strategy

We use rotational kinematics and the relationships between angular and linear quantities. Angular acceleration is found from $\alpha = \Delta\omega/\Delta t$. Tangential acceleration is $a_t = r\alpha$, and radial (centripetal) acceleration is $a_c = \omega^2 r$.

Solution

(a) First, convert 100,000 rpm to rad/s:

$$\omega = 100000\ \text{rpm} \times \frac{2\pi\ \text{rad}}{1\ \text{rev}} \times \frac{1\ \text{min}}{60\,\text{s}} = 10472\ \text{rad/s}$$

The angular acceleration (starting from rest, over 2.00 min = 120 s):

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{10472\ \text{rad/s} - 0}{120\,\text{s}} = 87.3\,\text{rad/s}^2$$

(b) The tangential acceleration at r = 9.50 cm = 0.0950 m:

$$a_t = r\alpha = (0.0950\,\text{m})(87.3\,\text{rad/s}^2) = 8.29\ \text{m/s}^2$$

(c) The radial (centripetal) acceleration at full speed:

$$a_c = \omega^2 r = (10472\ \text{rad/s})^2 (0.0950\,\text{m}) = 1.04 \times 10^7\ \text{m/s}^2$$

In multiples of g:

$$a_c = \frac{1.04\times10^7\ \text{m/s}^2}{9.80\ \text{m/s}^2} = 1.06\times10^6\ \text{g}$$

Discussion

The centripetal acceleration is over a million times g! This enormous acceleration is why ultracentrifuges can separate molecules by mass—even tiny differences in mass result in significant differences in the centripetal force needed to keep particles in circular motion. The tangential acceleration of 8.29 m/s² is comparable to Earth's gravity, which is modest compared to the final centripetal acceleration. This shows how the sustained angular acceleration over 2 minutes builds up to an extreme final rotational speed.

Answer

(a) The angular acceleration is **87.3 rad/s²**.

(b) The tangential acceleration is **8.29 m/s²**.

(c) The radial acceleration is **1.04×10^7 m/s²**, or **1.06×10^6 g**.

Integrated Concepts

You have a grindstone (a disk) that is 90.0 kg, has a 0.340-m radius, and is turning at 90.0 rpm, and you press a steel axe against it with a radial force of 20.0 N. (a) Assuming the kinetic coefficient of friction between steel and stone is 0.20, calculate the angular acceleration of the grindstone. (b) How many turns will the stone make before coming to rest?

Show Solution

Strategy

The friction force creates a torque that decelerates the grindstone. We find the friction force using f = μN, then the torque, and finally the angular acceleration using τ = Iα. For part (b), we use rotational kinematics to find the number of revolutions.

Solution

(a) The friction force is:

$$f = \mu_k N = (0.20)(20.0\text{N}) = 4.0\text{N}$$

The torque from friction (acting at the rim, R = 0.340 m):

$$\tau = fR = (4.0\text{N})(0.340\text{m}) = 1.36\text{N}\cdot\text{m}$$

The moment of inertia of the disk:

$$I = \frac{1}{2}MR^2 = \frac{1}{2}(90.0\text{kg})(0.340\text{m})^2 = 5.20\text{kg}\cdot\text{m}^2$$

The angular acceleration (negative because it opposes the rotation):

$$\alpha = -\frac{\tau}{I} = -\frac{1.36\text{N}\cdot\text{m}}{5.20\text{kg}\cdot\text{m}^2} = -0.26\text{rad/s}^2$$

(b) First, convert the initial angular velocity:

$$\omega_0 = 90.0\ \text{rpm} \times \frac{2\pi\ \text{rad}}{60\text{s}} = 9.42\ \text{rad/s}$$

Using $\omega^2 = \omega_0^2 + 2\alpha\theta$ with $\omega = 0$:

$$\theta = -\frac{\omega_0^2}{2\alpha} = -\frac{(9.42)^2}{2(-0.26)} = 171\ \text{rad}$$

Converting to revolutions:

$$\theta = \frac{171\ \text{rad}}{2\pi\ \text{rad/rev}} = 27\ \text{rev}$$

Discussion

The grindstone makes 27 complete turns before stopping. The small friction force (only 4 N) produces a modest deceleration, allowing for gradual stopping that's useful for controlled grinding operations. The angular deceleration of 0.26 rad/s² is relatively gentle, taking about 36 seconds to bring the stone to rest. This controlled deceleration prevents sudden jolts that could damage the axe or stone.

Answer

(a) The angular acceleration is **−0.26 rad/s²** (negative indicating deceleration).

(b) The grindstone makes **27 revolutions** before coming to rest.

Unreasonable Results

You are told that a basketball player spins the ball with an angular acceleration of 100rad/s^2. (a) What is the ball's final angular velocity if the ball starts from rest and the acceleration lasts 2.00 s? (b) What is unreasonable about the result? (c) Which premises are unreasonable or inconsistent?

Show Solution

Strategy

We use the rotational kinematic equation $\omega = \omega_0 + \alpha t$ to find the final angular velocity, then analyze whether the result is physically reasonable.

Solution

(a) Using $\omega = \omega_0 + \alpha t$ with $\omega_0 = 0$:

$$\omega = 0 + (100 \text{rad/s}^2)(2.00 \text{s}) = 200 \text{ rad/s}$$

Converting to more familiar units:

$$\omega = 200 \text{ rad/s} \times \frac{1 \text{ rev}}{2\pi \text{ rad}} \times \frac{60 \text{s}}{1 \text{ min}} = 1910 \text{ rpm}$$

(b) This result is unreasonable. A basketball spinning at nearly 2000 rpm (about 32 revolutions per second) would be impossible for a human to control or even perceive as spinning on a fingertip. Professional players typically spin balls at only 3-5 rev/s.

(c) The angular acceleration of 100 rad/s² is unreasonable. A more realistic value would be around 5-10 rad/s² for a ball spun by hand, limited by the friction between the player's hand and the ball surface.

Discussion

The calculation reveals the absurdity of the given angular acceleration. A basketball spinning at 1910 rpm would be a blur, impossible to balance on a fingertip, and the air resistance would quickly slow it down. Real basketball spins are limited by human physiology—the torque a player can apply with their hand and the time they have contact with the ball during the spin. Professional players achieve impressive spins of 3-5 rev/s, which is less than 3% of the calculated result.

Answer

(a) The final angular velocity would be **200 rad/s** (or **1910 rpm**).

(b) This result is unreasonable because no player could spin a basketball this fast, and it would be impossible to control at such high angular velocity.

(c) The angular acceleration of 100 rad/s² is unreasonably high; realistic values are 5-10 rad/s².

Glossary

angular acceleration
: the rate of change of angular velocity with time

change in angular velocity
: the difference between final and initial values of angular velocity

tangential acceleration
: the acceleration in a direction tangent to the circle at the point of interest in circular motion

Kinematics of Rotational Motion

- Observe the kinematics of rotational motion.
- Derive rotational kinematic equations.
- Evaluate problem solving strategies for rotational kinematics.

Just by using our intuition, we can begin to see how rotational quantities like θ, ω, and α are related to one another. For example, if a motorcycle wheel has a large angular acceleration for a fairly long time, it ends up spinning rapidly and rotates through many revolutions. In more technical terms, if the wheel's angular acceleration α is large for a long period of time t, then the final angular velocity ω and angle of rotation θ are large. The wheel's rotational motion is exactly analogous to the fact that the motorcycle's large translational acceleration produces a large final velocity, and the distance traveled will also be large.

Kinematics is the description of motion. The **kinematics of rotational motion** describes the relationships among rotation angle, angular velocity, angular acceleration, and time. Let us start by finding an equation relating ω, α, and t. To determine this equation, we recall a familiar kinematic equation for translational, or straight-line, motion:

$$v=v_0+at\,(\text{constant } a)$$

Note that in rotational motion $a=a_t$, and we shall use the symbol a for tangential or linear acceleration from now on. As in linear kinematics, we assume a is constant, which means that angular acceleration α is also a constant, because $a=r\alpha$. Now, let us substitute $v=r\omega$ and $a=r\alpha$ into the linear equation above:

$$r\omega=r\omega_0+r\alpha t.$$

The radius r cancels in the equation, yielding

$$\omega=\omega_0+at\,(\text{constant } a),$$

where ω_0 is the initial angular velocity. This last equation is a *kinematic relationship* among ω, α, and t —that is, it describes their relationship without reference to forces or masses that may affect rotation. It is also precisely analogous in form to its translational counterpart.

Making Connections

Kinematics for rotational motion is completely analogous to translational kinematics, first presented in One-Dimensional Kinematics. Kinematics is concerned with the description of motion without regard to force or mass. We will find that translational kinematic quantities, such as displacement, velocity, and acceleration have direct analogs in rotational motion.

Starting with the four kinematic equations we developed in One-Dimensional Kinematics, we can derive the following four rotational kinematic equations (presented together with their translational counterparts):

Table: Rotational Kinematic Equations

Rotational	Translational	Conditions
$\theta=\bar{\omega}t$	$x=\bar{v}t$	
$\omega=\omega_0+\alpha t$	$v=v_0+at$	constant α, constant a
$\theta=\omega_0 t+\frac{1}{2}\alpha t^2$	$x=v_0 t+\frac{1}{2}at^2$	constant α, constant a
$\omega^2=\omega_0^2+2\alpha\theta$	$v^2=v_0^2+2ax$	constant α, constant a

In these equations, the subscript 0 denotes initial values (θ_0, x_0, and t_0 are initial values), and the average angular velocity $\bar{\omega}$ and average velocity $\bar{v}$ are defined as follows:

$$\bar{\omega}=\frac{\omega_0+\omega}{2};\quad\bar{v}=\frac{v_0+v}{2}.$$

The equations given above in Table 1 can be used to solve any rotational or translational kinematics problem in which a and α are constant.

Problem-Solving Strategy for Rotational Kinematics

1. *Examine the situation to determine that rotational kinematics (rotational motion) is involved*. Rotation must be involved, but without the need to consider forces or masses that affect the motion.

2. *Identify exactly what needs to be determined in the problem (identify the unknowns)*. A sketch of the situation is useful.
3. *Make a list of what is given or can be inferred from the problem as stated (identify the knowns)*.
4. *Solve the appropriate equation or equations for the quantity to be determined (the unknown)*. It can be useful to think in terms of a translational analog because by now you are familiar with such motion.
5. *Substitute the known values along with their units into the appropriate equation, and obtain numerical solutions complete with units*. Be sure to use units of radians for angles.
6. *Check your answer to see if it is reasonable: Does your answer make sense*?

Calculating the Acceleration of a Fishing Reel

A deep-sea fisherman hooks a big fish that swims away from the boat pulling the fishing line from his fishing reel. The whole system is initially at rest and the fishing line unwinds from the reel at a radius of 4.50 cm from its axis of rotation. The reel is given an angular acceleration of 110rad/s^2 for 2.00 s as seen in Figure 1.

(a) What is the final angular velocity of the reel?

(b) At what speed is fishing line leaving the reel after 2.00 s elapses?

(c) How many revolutions does the reel make?

(d) How many meters of fishing line come off the reel in this time?

Strategy

In each part of this example, the strategy is the same as it was for solving problems in linear kinematics. In particular, known values are identified and a relationship is then sought that can be used to solve for the unknown.

Solution for (a)

Here α and t are given and ω needs to be determined. The most straightforward equation to use is $\omega = \omega_0 + \alpha t$ because the unknown is already on one side and all other terms are known. That equation states that

$$\omega = \omega_0 + \alpha t.$$

We are also given that $\omega_0 = 0$ (it starts from rest), so that

$$\omega = 0 + (110 \text{rad/s}^2)(2.00 \text{s}) = 220 \text{rad/s}.$$

Solution for (b)

Now that ω is known, the speed v can most easily be found using the relationship

$$v = r\omega,$$

where the radius r of the reel is given to be 4.50 cm; thus,

$$v = (0.0450 \text{m})(220 \text{rad/s}) = 9.90 \text{m/s}.$$

Note again that radians must always be used in any calculation relating linear and angular quantities. Also, because radians are dimensionless, we have $\text{m} \times \text{rad} = \text{m}$.

Solution for (c)

Here, we are asked to find the number of revolutions. Because $1 \text{rev} = 2\pi \text{rad}$, we can find the number of revolutions by finding θ in radians. We are given α and t, and we know ω_0 is zero, so that θ can be obtained using $\theta = \omega_0 t + 12\alpha t^2$.

$$\theta = \omega_0 t + 12\alpha t^2 \quad \theta = 0 + (0.500)(110 \text{rad/s}^2)(2.00 \text{s})^2 = 220 \text{rad}.$$

Converting radians to revolutions gives

$$\theta = (220 \text{rad}) 1 \text{rev} 2\pi \text{rad} = 35.0 \text{rev}.$$

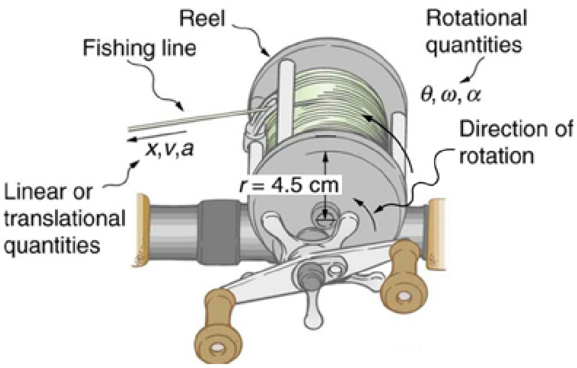
Solution for (d)

The number of meters of fishing line is x, which can be obtained through its relationship with θ:

$$x = r\theta = (0.0450\,\text{m})(220\,\text{rad}) = 9.90\,\text{m}.$$

Discussion

This example illustrates that relationships among rotational quantities are highly analogous to those among linear quantities. We also see in this example how linear and rotational quantities are connected. The answers to the questions are realistic. After unwinding for two seconds, the reel is found to spin at 220 rad/s, which is 2100 rpm. (No wonder reels sometimes make high-pitched sounds.) The amount of fishing line played out is 9.90 m, about right for when the big fish bites.



Fishing line coming off a rotating reel moves linearly. Example 1 and Example 2 consider relationships between rotational and linear quantities associated with a fishing reel.

Calculating the Duration When the Fishing Reel Slows Down and Stops

Now let us consider what happens if the fisherman applies a brake to the spinning reel, achieving an angular acceleration of $-300\,\text{rad/s}^2$. How long does it take the reel to come to a stop?

Strategy

We are asked to find the time t for the reel to come to a stop. The initial and final conditions are different from those in the previous problem, which involved the same fishing reel. Now we see that the initial angular velocity is $\omega_0 = 220\,\text{rad/s}$ and the final angular velocity ω is zero. The angular acceleration is given to be $\alpha = -300\,\text{rad/s}^2$. Examining the available equations, we see all quantities but *t* are known in $\omega = \omega_0 + \alpha t$, making it easiest to use this equation.

Solution

The equation states

$$\omega = \omega_0 + \alpha t.$$

We solve the equation algebraically for *t*, and then substitute the known values as usual, yielding

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - 220\,\text{rad/s}}{-300\,\text{rad/s}^2} = 0.733\,\text{s}.$$

Discussion

Note that care must be taken with the signs that indicate the directions of various quantities. Also, note that the time to stop the reel is fairly small because the acceleration is rather large. Fishing lines sometimes snap because of the accelerations involved, and fishermen often let the fish swim for a while before applying brakes on the reel. A tired fish will be slower, requiring a smaller acceleration.

Calculating the Slow Acceleration of Trains and Their Wheels

Large freight trains accelerate very slowly. Suppose one such train accelerates from rest, giving its 0.350-m-radius wheels an angular acceleration of $0.250\,\text{rad/s}^2$. After the wheels have made 200 revolutions (assume no slippage): (a) How far has the train moved down the track? (b) What are the final angular velocity of the wheels and the linear velocity of the train?

Strategy

In part (a), we are asked to find x, and in (b) we are asked to find ω and v. We are given the number of revolutions θ, the radius of the wheels r, and the angular acceleration α.

Solution for (a)

The distance x is very easily found from the relationship between distance and rotation angle:

$$\theta = x r.$$

Solving this equation for x yields

$$x = r\theta.$$

Before using this equation, we must convert the number of revolutions into radians, because we are dealing with a relationship between linear and rotational quantities:

$$\theta = (200 \text{rev}) 2\pi \text{rad} 1 \text{rev} = 1257 \text{rad}.$$

Now we can substitute the known values into $x = r\theta$ to find the distance the train moved down the track:

$$x = r\theta = (0.350 \text{m})(1257 \text{rad}) = 440 \text{m}.$$

Solution for (b)

We cannot use any equation that incorporates t to find ω, because the equation would have at least two unknown values. The equation $\omega^2 = \omega\_0^2 + 2\alpha\theta$ will work, because we know the values for all variables except ω:

$$\omega^2 = \omega_{20} + 2\alpha\theta$$

Taking the square root of this equation and entering the known values gives

$$\omega = [0 + 2(0.250 \text{rad/s}^2)(1257 \text{rad})]^{1/2} \quad \omega = 25.1 \text{rad/s}.$$

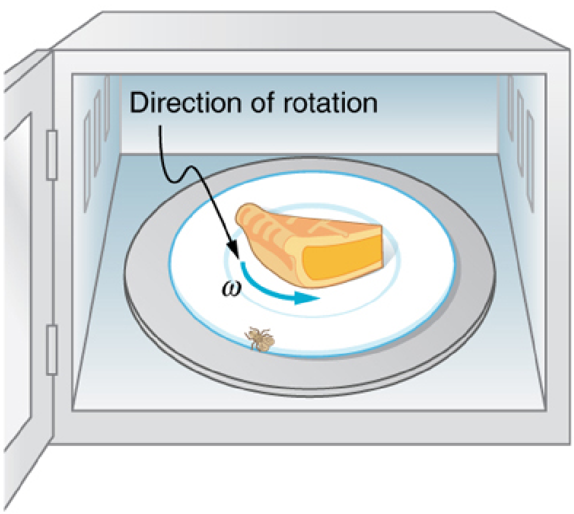
We can find the linear velocity of the train, v, through its relationship to ω:

$$v = r\omega = (0.350 \text{m})(25.1 \text{rad/s}) = 8.77 \text{m/s}.$$

Discussion

The distance traveled is fairly large and the final velocity is fairly slow (just under 32 km/h).

There is translational motion even for something spinning in place, as the following example illustrates. Figure 2 shows a fly on the edge of a rotating microwave oven plate. The example below calculates the total distance it travels.



The image shows a microwave plate. The fly makes revolutions while the food is heated (along with the fly).

Calculating the Distance Traveled by a Fly on the Edge of a Microwave Oven Plate

A person decides to use a microwave oven to reheat some lunch. In the process, a fly accidentally flies into the microwave and lands on the outer edge of the rotating plate and remains there. If the plate has a radius of 0.15 m and rotates at 6.0 rpm, calculate the total distance traveled by the fly during a 2.0-min cooking period. (Ignore the start-up and slow-down times.)

Strategy

First, find the total number of revolutions θ, and then the linear distance x traveled. $\theta = \bar{\omega} t$ can be used to find θ because $\bar{\omega}$ is given to be 6.0 rpm.

Solution

Entering known values into $\theta = \bar{\omega} t$ gives

$$\theta = \bar{\omega} t = (6.0\text{ rpm})(2.0\text{ min}) = 12\text{ rev}.$$

As always, it is necessary to convert revolutions to radians before calculating a linear quantity like x from an angular quantity like θ:

$$\theta = (12\text{ rev})\left(\frac{2\pi\text{ rad}}{1\text{ rev}}\right) = 75.4\text{ rad}.$$

Now, using the relationship between x and θ, we can determine the distance traveled:

$$x = r\theta = (0.15\text{ m})(75.4\text{ rad}) = 11\text{ m}.$$

Discussion

Quite a trip (if it survives)! Note that this distance is the total distance traveled by the fly. Displacement is actually zero for complete revolutions because they bring the fly back to its original position. The distinction between total distance traveled and displacement was first noted in One-Dimensional Kinematics.
Check Your Understanding

Rotational kinematics has many useful relationships, often expressed in equation form. Are these relationships laws of physics or are they simply descriptive? (Hint: the same question applies to linear kinematics.)

Show Solution

Rotational kinematics (just like linear kinematics) is descriptive and does not represent laws of nature. With kinematics, we can describe many things to great precision but kinematics does not consider causes. For example, a large angular acceleration describes a very rapid change in angular velocity without any consideration of its cause.

Section Summary

- Kinematics is the description of motion.
- The kinematics of rotational motion describes the relationships among rotation angle, angular velocity, angular acceleration, and time.
- Starting with the four kinematic equations we developed in the One-Dimensional Kinematics, we can derive the four rotational kinematic equations (presented together with their translational counterparts) seen in Table 1.
- In these equations, the subscript 0 denotes initial values (x_0 and t_0 are initial values), and the average angular velocity $\bar{\omega}$ and average velocity $\bar{v}$ are defined as follows:
  $\bar{\omega} = \frac{\omega_0 + \omega}{2}$ and $\bar{v} = \frac{v_0 + v}{2}$.

Problems & Exercises

With the aid of a string, a gyroscope is accelerated from rest to 32 rad/s in 0.40 s.

(a) What is its angular acceleration in rad/s^2?

(b) How many revolutions does it go through in the process?

Show Solution

Strategy

For part (a), we use the definition of angular acceleration: $\alpha = \Delta\omega/\Delta t$. For part (b), we use a rotational kinematic equation to find the angular displacement, then convert to revolutions.

Solution

(a) The angular acceleration is:

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{32\text{ rad/s} - 0}{0.40\text{ s}} = 80\text{ rad/s}^2$$

(b) Using the kinematic equation $\theta = \omega_0 t + 12 \alpha t^2$ with $\omega_0 = 0$:

$$\theta = 0 + 12(80 \text{rad/s}^2)(0.40 \text{s})^2 = 12(80)(0.16) = 6.4 \text{ rad}$$

Converting to revolutions:

$$\theta = 6.4 \text{ rad} 2\pi \text{ rad/rev} = 1.0 \text{ rev}$$

Discussion

The gyroscope makes exactly one complete revolution while spinning up to 32 rad/s. The angular acceleration of 80 rad/s² is quite large, reflecting the rapid spin-up from the string pull. This is a typical way to start a gyroscope—a quick pull on a string wrapped around its axis.

Answer

(a) The angular acceleration is **80 rad/s²**.

(b) The gyroscope makes **1.0 revolution** during spin-up.

Suppose a piece of dust finds itself on a CD. If the spin rate of the CD is 500 rpm, and the piece of dust is 4.3 cm from the center, what is the total distance traveled by the dust in 3 minutes? (Ignore accelerations due to getting the CD rotating.)

Show Solution

Strategy

The CD rotates at a constant angular velocity of 500 rpm for 3 minutes. We find the total number of revolutions, convert to radians, and then use the relationship between arc length and angle to find the total distance traveled.

Solution

First, calculate the total number of revolutions:

$$\theta = \omega t = (500 \text{ rpm})(3 \text{ min}) = 1500 \text{ rev}$$

Convert to radians:

$$\theta = 1500 \text{ rev} \times 2\pi \text{ rad} 1 \text{ rev} = 9425 \text{ rad}$$

The distance traveled is:

$$x = r\theta = (0.043 \text{m})(9425 \text{ rad}) = 405 \text{m}$$

Discussion

The dust particle travels about 405 meters (roughly a quarter mile) while riding on the spinning CD for just 3 minutes. This illustrates how even a modest angular velocity can produce significant linear distances when sustained over time.

Answer

The dust particle travels a total distance of **405 m** in 3 minutes.

A gyroscope slows from an initial rate of 32.0 rad/s at a rate of 0.700rad/s^2.

(a) How long does it take to come to rest?

(b) How many revolutions does it make before stopping?

Show Solution

Strategy

The gyroscope is decelerating, so $\alpha = -0.700$ rad/s². For part (a), we use $\omega = \omega_0 + \alpha t$ with $\omega = 0$. For part (b), we use $\omega^2 = \omega_0^2 + 2\alpha\theta$ to find the angular displacement.

Solution

(a) Using $\omega = \omega_0 + \alpha t$ with $\omega = 0$:

$$t=\frac{\omega-\omega_0}{\alpha}=\frac{0-32.0\text{ rad/s}}{-0.700\text{rad/s}^2}=45.7\text{s}$$

(b) Using $\omega^2=\omega_0^2+2\alpha\theta$ with $\omega=0$:

$$\theta=\frac{-\omega_0^2}{2\alpha}=\frac{-(32.0\text{ rad/s})^2}{2(-0.700\text{rad/s}^2)}=\frac{-1024}{-1.40}=731\text{ rad}$$

Converting to revolutions:

$$\theta=\frac{731\text{ rad}}{2\pi\text{ rad/rev}}=116\text{ rev}$$

Discussion

The gyroscope takes 45.7 seconds to come to rest, making 116 complete revolutions in that time. The relatively small angular deceleration (0.700 rad/s²) compared to the initial angular velocity (32.0 rad/s) means the gyroscope coasts for a long time and distance. This is typical of gyroscopes, which have low friction bearings and can spin for extended periods.

Answer

(a) The gyroscope takes **45.7 s** to come to rest.

(b) It makes **116 revolutions** before stopping.

During a very quick stop, a car decelerates at 7.00m/s^2.

(a) What is the angular acceleration of its 0.280-m-radius tires, assuming they do not slip on the pavement?

(b) How many revolutions do the tires make before coming to rest, given their initial angular velocity is 95.0rad/s ?

(c) How long does the car take to stop completely?

(d) What distance does the car travel in this time?

(e) What was the car’s initial velocity?

(f) Do the values obtained seem reasonable, considering that this stop happens very quickly?

Yo-yos are amusing toys that display significant physics and are engineered to enhance performance based on physical laws. (credit: Beyond Neon, Flickr)

Show Solution

Strategy

We use the relationship between linear and angular quantities, along with rotational kinematics. Since the tires don’t slip, we have $a=r\alpha$ and $v=r\omega$.

Solution

(a) The angular acceleration is related to linear acceleration by:

$$\alpha=\frac{a}{r}=\frac{-7.00\text{ m/s}^2}{0.280\text{m}}=-25.0\text{rad/s}^2$$

The negative sign indicates the tires are decelerating.

(b) Using $\omega^2 = \omega_{20} + 2\alpha\theta$ with $\omega = 0$:

$$\theta = -\omega_{20} 2\alpha = -(95.0 \text{ rad/s})^2 2(-25.0 \text{rad/s}^2) = 180.5 \text{ rad}$$

Converting to revolutions:

$$\theta = 180.5 \text{ rad} 2\pi \text{ rad/rev} = 28.7 \text{ rev}$$

(c) Using $\omega = \omega_0 + \alpha t$ with $\omega = 0$:

$$t = -\omega_0 \alpha = -95.0 \text{ rad/s} -25.0 \text{rad/s}^2 = 3.80 \text{s}$$

(d) The distance traveled is:

$$x = r\theta = (0.280 \text{m})(180.5 \text{ rad}) = 50.5 \text{m}$$

(e) The car’s initial velocity was:

$$v_0 = r\omega_0 = (0.280 \text{m})(95.0 \text{ rad/s}) = 26.6 \text{ m/s}$$

(f) Yes, the values are reasonable. The initial velocity of 26.6 m/s is about 96 km/h (60 mph), which is a typical highway speed. Stopping in 3.80 s over a distance of 50.5 m (about 166 feet) with a deceleration of 7.00 m/s² (about 0.7g) represents a hard but achievable emergency stop for a car with good brakes and tires.

Discussion

This problem demonstrates the power of connecting rotational and translational motion. By analyzing the tire rotation, we can determine all aspects of the car’s motion. The no-slip condition (where the tire’s contact point with the pavement has zero velocity) is crucial for this analysis and explains why anti-lock braking systems work—they prevent wheel lockup to maintain this no-slip condition, allowing maximum braking force.

Answer

(a) Angular acceleration: **−25.0 rad/s²**

(b) Number of revolutions: **28.7 rev**

(c) Time to stop: **3.80 s**

(d) Distance traveled: **50.5 m**

(e) Initial velocity: **26.6 m/s** (about 96 km/h or 60 mph)

(f) Yes, these values are reasonable for a hard emergency stop from highway speed.

Everyday application: Suppose a yo-yo has a center shaft that has a 0.250 cm radius and that its string is being pulled.

(a) If the string is stationary and the yo-yo accelerates away from it at a rate of 1.50m/s^2, what is the angular acceleration of the yo-yo?

(b) What is the angular velocity after 0.750 s if it starts from rest?

(c) The outside radius of the yo-yo is 3.50 cm. What is the tangential acceleration of a point on its edge?

Show Solution

Strategy

The string wraps around the center shaft, so the linear acceleration of the yo-yo relates to angular acceleration by a = rα, where r is the shaft radius. For part (b), we use $\omega = \omega_0 + \alpha t$. For part (c), we use a_t = Rα with the outer radius.

Solution

(a) The shaft radius is r = 0.250 cm = 0.00250 m. Using $a = r\alpha$:

$$\alpha = \frac{a}{r} = \frac{1.50 \text{m/s}^2}{0.00250 \text{m}} = 600 \text{rad/s}^2$$

(b) Starting from rest ($\omega_0 = 0$), after t = 0.750 s:

$$\omega = \omega_0 + \alpha t = 0 + (600 \text{rad/s}^2)(0.750 \text{s}) = 450 \text{ rad/s}$$

(c) At the outer edge, R = 3.50 cm = 0.0350 m:

$$a_t = R\alpha = (0.0350 \text{m})(600 \text{rad/s}^2) = 21.0 \text{m/s}^2$$

Discussion

The angular acceleration is quite large (600 rad/s²) because the shaft radius is so small. After only 0.750 s, the yo-yo is spinning at 450 rad/s (about 4,300 rpm). The tangential acceleration at the outer edge (21.0 m/s²) is much larger than the yo-yo's linear acceleration (1.50 m/s²) because the outer radius is 14 times larger than the shaft radius. This large difference in accelerations at different radii is characteristic of rotating objects.

Answer

(a) The angular acceleration is **600 rad/s²**.

(b) The angular velocity after 0.750 s is **450 rad/s** (about 4,300 rpm).

(c) The tangential acceleration at the outer edge is **21.0 m/s²**.

Glossary

kinematics of rotational motion
: describes the relationships among rotation angle, angular velocity, angular acceleration, and time

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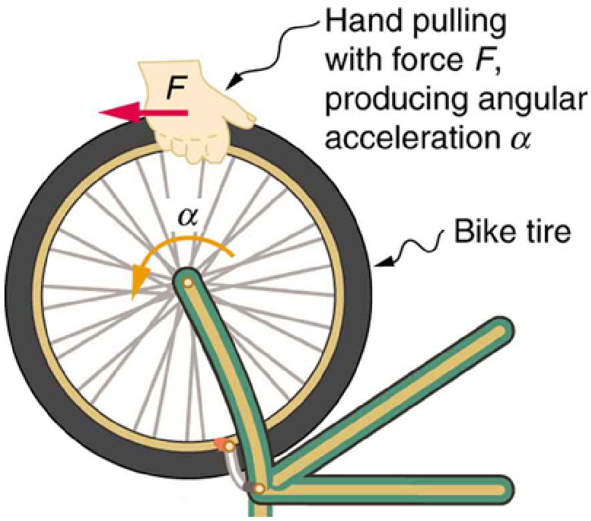
Dynamics of Rotational Motion: Rotational Inertia

- Understand the relationship between force, mass and acceleration.
- Study the turning effect of force.
- Study the analogy between force and torque, mass and moment of inertia, and linear acceleration and angular acceleration.

If you have ever spun a bike wheel or pushed a merry-go-round, you know that force is needed to change angular velocity as seen in Figure 1. In fact, your intuition is reliable in predicting many of the factors that are involved. For example, we know that a door opens slowly if we push too close to its hinges. Furthermore, we know that the more massive the door, the more slowly it opens. The first example implies that the farther the force is applied from the pivot, the greater the angular acceleration; another implication is that angular acceleration is inversely proportional to mass. These relationships should seem very similar to the familiar relationships among force, mass, and acceleration embodied in Newton's second law of motion. There are, in fact, precise rotational analogs to both force and mass.



Force is required to spin the bike wheel. The greater the force, the greater the angular acceleration produced. The more massive the wheel, the smaller the angular acceleration. If you push on a spoke closer to the axle, the angular acceleration will be smaller.

To develop the precise relationship among force, mass, radius, and angular acceleration, consider what happens if we exert a force F on a point mass m that is at a distance r from a pivot point, as shown in Figure 2. Because the force is perpendicular to r, an acceleration $a = Fm$ is obtained in the direction of F. We can rearrange this equation such that $F = ma$ and then look for ways to relate this expression to expressions for rotational quantities. We note that $a = r\alpha$, and we substitute this expression into $F = ma$, yielding

$$F = mr\alpha.$$

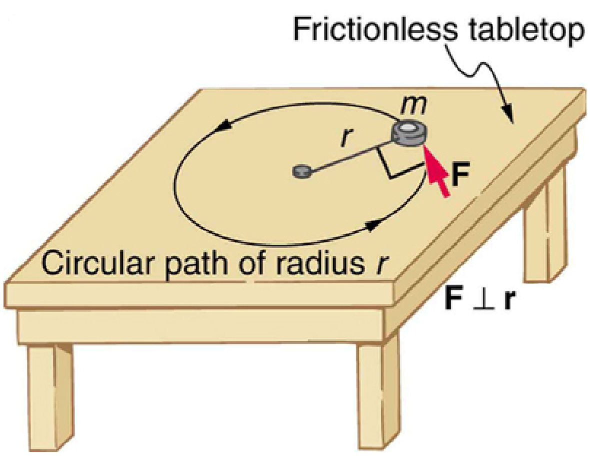
Recall that **torque** is the turning effectiveness of a force. In this case, because $\vec{\mathbf{F}}$ is perpendicular to r, torque is simply $\tau = Fr$. So, if we multiply both sides of the equation above by r, we get torque on the left-hand side. That is,

$$rF = mr^2\alpha$$

or

$$\tau = mr^2\alpha.$$

This last equation is the rotational analog of Newton's second law ($\vec{\mathbf{F}} = m\vec{\mathbf{a}}$), where torque is analogous to force, angular acceleration is analogous to translational acceleration, and mr^2 is analogous to mass (or inertia). The quantity mr^2 is called the **rotational inertia** or **moment of inertia** of a point mass m a distance r from the center of rotation.



An object is supported by a horizontal frictionless table and is attached to a pivot point by a cord that supplies centripetal force. A force F is applied to the object perpendicular to the radius r , causing it to accelerate about the pivot point. The force is kept perpendicular to r .

Making Connections: Rotational Motion Dynamics

Dynamics for rotational motion is completely analogous to linear or translational dynamics. Dynamics is concerned with force and mass and their effects on motion. For rotational motion, we will find direct analogs to force and mass that behave just as we would expect from our earlier experiences.

Rotational Inertia and Moment of Inertia

Before we can consider the rotation of anything other than a point mass like the one in Figure 2, we must extend the idea of rotational inertia to all types of objects. To expand our concept of rotational inertia, we define the **moment of inertia** I of an object to be the sum of mr^2 for all the point masses of which it is composed. That is, $I = \sum mr^2$. Here I is analogous to m in translational motion. Because of the distance r, the moment of inertia for any object depends on the chosen axis. Actually, calculating I is beyond the scope of this text except for one simple case—that of a hoop, which has all its mass at the same distance from its axis. A hoop's moment of inertia around its axis is therefore MR^2, where M is its total mass and R its radius. (We use M and R for an entire object to distinguish them from m and r for point masses.) In all other cases, we must consult Figure 3 (note that the table is piece of artwork that has shapes as well as formulae) for formulas for I that have been derived from integration over the continuous body. Note that I has units of mass multiplied by distance squared ($\text{kg} \cdot \text{m}^2$), as we might expect from its definition.

The general relationship among torque, moment of inertia, and angular acceleration is

$$\text{net } \tau = I\alpha$$

or

$$\alpha = \text{net } \tau I,$$

where net τ is the total torque from all forces relative to a chosen axis. For simplicity, we will only consider torques exerted by forces in the plane of the rotation. Such torques are either positive or negative and add like ordinary numbers. The relationship in $\tau = I\alpha, \alpha = \text{net } \tau I$ is the rotational analog to Newton's second law and is very generally applicable. This equation is actually valid for *any* torque, applied to *any* object, relative to *any* axis.

As we might expect, the larger the torque is, the larger the angular acceleration is. For example, the harder a child pushes on a merry-go-round, the faster it accelerates. Furthermore, the more massive a merry-go-round, the slower it accelerates for the same torque. The basic relationship between moment of inertia and angular acceleration is that the larger the moment of inertia, the smaller is the angular acceleration. But there is an additional twist. The moment of inertia depends not only on the mass of an object, but also on its *distribution* of mass relative to the axis around which it rotates. For example, it will be much easier to accelerate a merry-go-round full of children if they stand close to its axis than if they all stand at the outer edge. The mass is the same in both cases; but the moment of inertia is much larger when the children are at the edge.

Take-Home Experiment

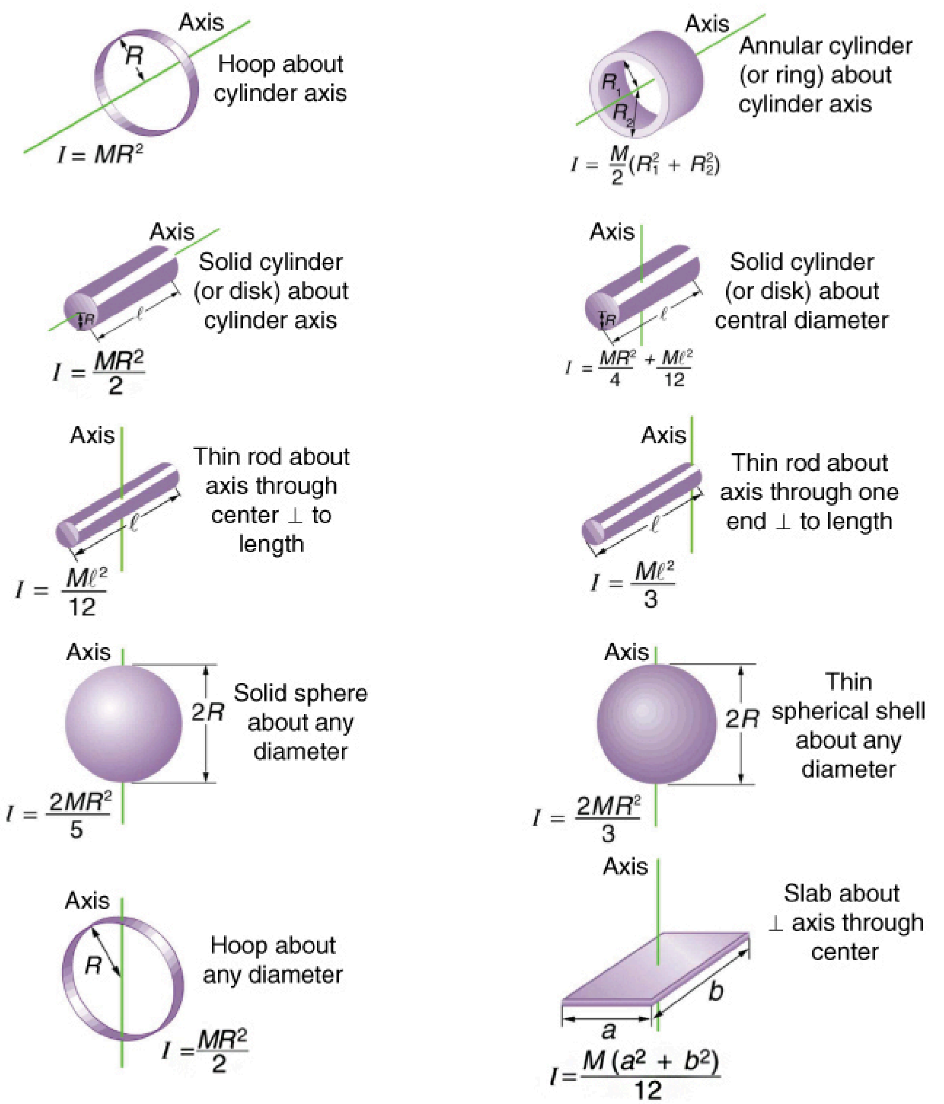
Cut out a circle that has about a 10 cm radius from stiff cardboard. Near the edge of the circle, write numbers 1 to 12 like hours on a clock face. Position the circle so that it can rotate freely about a horizontal axis through its center, like a wheel. (You could loosely nail the circle to a wall.) Hold the circle stationary and with the number 12 positioned at the top, attach a lump of blue putty (sticky material used for fixing posters to walls) at the number 3. How large does the lump need to be to just rotate the circle? Describe how you can change the moment of

inertia of the circle. How does this change affect the amount of blue putty needed at the number 3 to just rotate the circle? Change the circle's moment of inertia and then try rotating the circle by using different amounts of blue putty. Repeat this process several times.
Problem-Solving Strategy for Rotational Dynamics

1. *Examine the situation to determine that torque and mass are involved in the rotation*. Draw a careful sketch of the situation.
2. *Determine the system of interest*.
3. *Draw a free body diagram*. That is, draw and label all external forces acting on the system of interest.
4. *Apply* $\text{net}\,\tau = I\alpha$, $\alpha = \text{net}\,\tau I$, *the rotational equivalent of Newton's second law, to solve the problem*. Care must be taken to use the correct moment of inertia and to consider the torque about the point of rotation.
5. *As always, check the solution to see if it is reasonable*.

Making Connections

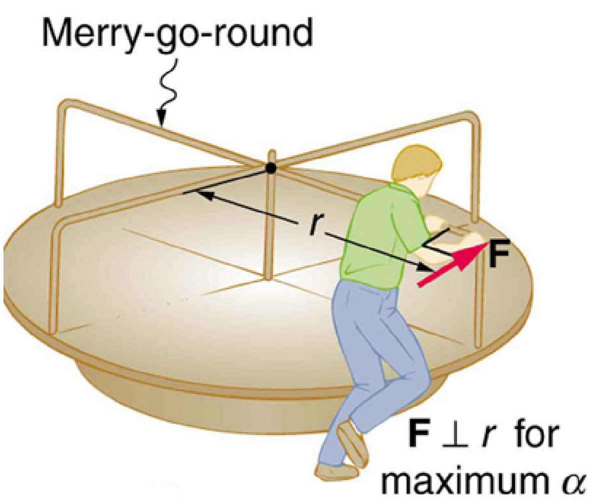
In statics, the net torque is zero, and there is no angular acceleration. In rotational motion, net torque is the cause of angular acceleration, exactly as in Newton's second law of motion for rotation.



Some rotational inertias.

Calculating the Effect of Mass Distribution on a Merry-Go-Round

Consider the father pushing a playground merry-go-round in Figure 4. He exerts a force of 250 N at the edge of the 50.0-kg merry-go-round, which has a 1.50 m radius. Calculate the angular acceleration produced (a) when no one is on the merry-go-round and (b) when an 18.0-kg child sits 1.25 m away from the center. Consider the merry-go-round itself to be a uniform disk with negligible retarding friction.



A father pushes a playground merry-go-round at its edge and perpendicular to its radius to achieve maximum torque.

Strategy

Angular acceleration is given directly by the expression α =net τI:

$$\alpha=\tau I.$$

To solve for α, we must first calculate the torque τ (which is the same in both cases) and moment of inertia I (which is greater in the second case). To find the torque, we note that the applied force is perpendicular to the radius and friction is negligible, so that

$$\tau=rF\sin\theta=(1.50\text{m})(250\text{N})=375\text{N}\cdot\text{m}.$$

Solution for (a)

The moment of inertia of a solid disk about this axis is given in Figure 3 to be

$$12MR^2,$$

where M =50.0kg and R =1.50m, so that

$$I=(0.500)(50.0\text{kg})(1.50\text{m})^2=56.25\text{kg}\cdot\text{m}^2.$$

Now, after we substitute the known values, we find the angular acceleration to be

$$\alpha=\tau I=375\text{N}\cdot\text{m}56.25\text{kg}\cdot\text{m}^2=6.67\text{rad/s}^2.$$

Solution for (b)

We expect the angular acceleration for the system to be less in this part, because the moment of inertia is greater when the child is on the merry-go-round. To find the total moment of inertia I, we first find the child's moment of inertia I_c by considering the child to be equivalent to a point mass at a distance of 1.25 m from the axis. Then,

$$I_c=MR^2=(18.0\text{kg})(1.25\text{m})^2=28.13\text{kg}\cdot\text{m}^2.$$

The total moment of inertia is the sum of moments of inertia of the merry-go-round and the child (about the same axis). To justify this sum to yourself, examine the definition of I:

$$I=28.13\text{kg}\cdot\text{m}^2+56.25\text{kg}\cdot\text{m}^2=84.38\text{kg}\cdot\text{m}^2.$$

Substituting known values into the equation for α gives

$$\alpha=\tau I=375\text{N}\cdot\text{m}84.38\text{kg}\cdot\text{m}^2=4.44\text{rad/s}^2.$$

Discussion

The angular acceleration is less when the child is on the merry-go-round than when the merry-go-round is empty, as expected. The angular accelerations found are quite large, partly due to the fact that friction was considered to be negligible. If, for example, the father kept pushing perpendicularly for 2.00 s, he would give the merry-go-round an angular velocity of 13.3 rad/s when it is empty but only 8.89 rad/s when the child is on it. In terms of revolutions per second, these angular velocities are 2.12 rev/s and 1.41 rev/s, respectively. The father would end up running at about 50 km/h in the first case. Summer Olympics, here he comes! Confirmation of these numbers is left as an exercise for the reader.
Check Your Understanding

Torque is the analog of force and moment of inertia is the analog of mass. Force and mass are physical quantities that depend on only one factor. For example, mass is related solely to the numbers of atoms of various types in an object. Are torque and moment of inertia similarly simple?

Show Solution

No. Torque depends on three factors: force magnitude, force direction, and point of application. Moment of inertia depends on both mass and its distribution relative to the axis of rotation. So, while the analogies are precise, these rotational quantities depend on more factors.

Section Summary

- The farther the force is applied from the pivot, the greater is the angular acceleration; angular acceleration is inversely proportional to mass.
- If we exert a force F on a point mass m that is at a distance r from a pivot point and because the force is perpendicular to r, an acceleration $a = F/m$ is obtained in the direction of F. We can rearrange this equation such that
  $F=ma,$

  and then look for ways to relate this expression to expressions for rotational quantities. We note that $a = r\alpha$, and we substitute this expression into $F = ma$, yielding

  $F=mr\alpha$
- Torque is the turning effectiveness of a force. In this case, because F is perpendicular to r, torque is simply $\tau = rF$. If we multiply both sides of the equation above by r, we get torque on the left-hand side. That is,
  $rF=mr^2\alpha$

or

$\tau=mr^2\alpha.$

- The moment of inertia I of an object is the sum of MR^2 for all the point masses of which it is composed. That is,
  $I=\sum mr^2.$
- The general relationship among torque, moment of inertia, and angular acceleration is
  $\tau=I\alpha$

or

α=net τ$I.$

Conceptual Questions

The moment of inertia of a long rod spun around an axis through one end perpendicular to its length is $ML^2/3$. Why is this moment of inertia greater than it would be if you spun a point mass M at the location of the center of mass of the rod (at $L/2$)? (That would be $ML^2/4$.)

Why is the moment of inertia of a hoop that has a mass M and a radius R greater than the moment of inertia of a disk that has the same mass and radius? Why is the moment of inertia of a spherical shell that has a mass M and a radius R greater than that of a solid sphere that has the same mass and radius?

Give an example in which a small force exerts a large torque. Give another example in which a large force exerts a small torque.

While reducing the mass of a racing bike, the greatest benefit is realized from reducing the mass of the tires and wheel rims. Why does this allow a racer to achieve greater accelerations than would an identical reduction in the mass of the bicycle's frame?

The image shows a side view of a racing bicycle. Can you see evidence in the design of the wheels on this racing bicycle that their moment of inertia has been purposely reduced? (credit: Jesús Rodriguez)

A ball slides up a frictionless ramp. It is then rolled without slipping and with the same initial velocity up another frictionless ramp (with the same slope angle). In which case does it reach a greater height, and why?

Problems & Exercises

This problem considers additional aspects of example Calculating the Effect of Mass Distribution on a Merry-Go-Round. (a) How long does it take the father to give the merry-go-round an angular velocity of 1.50 rad/s? (b) How many revolutions must he go through to generate this velocity? (c) If he exerts a slowing force of 300 N at a radius of 1.35 m, how long would it take him to stop them?

Show Solution

Strategy

This problem builds on the merry-go-round example. We'll use the angular acceleration found there ($\alpha = 4.44\ \text{rad/s}^2$ with the child on board) and the total moment of inertia ($I = 84.38\ \text{kg}\cdot\text{m}^2$). We apply rotational kinematics for parts (a) and (b), and for part (c), we calculate a new torque and angular acceleration for the stopping motion.

Solution

(a) To find the time to reach $\omega = 1.50$ rad/s starting from rest:

Using the rotational kinematic equation:

$$\omega = \omega_0 + \alpha t$$

With $\omega_0 = 0$:

$$t = \frac{\omega}{\alpha} = \frac{1.50\ \text{rad/s}}{4.44\ \text{rad/s}^2} = 0.338\ \text{s}$$

(b) To find the angular displacement (in revolutions):

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2}(4.44\ \text{rad/s}^2)(0.338\ \text{s})^2 = 0.253\ \text{rad}$$

Converting to revolutions:

$$\theta = \frac{0.253\ \text{rad}}{2\pi\ \text{rad/rev}} = 0.0403\ \text{rev}$$

(c) For stopping with $F = 300$ N at $r = 1.35$ m:

The opposing torque is:

$$\tau = Fr = (300\ \text{N})(1.35\ \text{m}) = 405\ \text{N}\cdot\text{m}$$

The angular deceleration is:

$$\alpha = \frac{\tau}{I} = \frac{405\ \text{N}\cdot\text{m}}{84.38\ \text{kg}\cdot\text{m}^2} = 4.80\ \text{rad/s}^2$$

Time to stop from $\omega = 1.50$ rad/s:

$$t=\frac{\omega}{\alpha}=\frac{1.50\text{ rad/s}}{4.80\text{rad/s}^2}=0.313\text{s}$$

Discussion

Notice that applying a larger force (300 N vs 250 N) at a slightly smaller radius (1.35 m vs 1.50 m) produces a larger torque (405 N·m vs 375 N·m), resulting in a faster deceleration than the original acceleration. This is why the stopping time (0.313 s) is slightly less than the time needed to reach the same angular velocity (0.338 s).

Answer

(a) It takes **0.338 s** to reach 1.50 rad/s.

(b) He goes through **0.0403 revolutions** (about 14.5 degrees).

(c) It takes **0.313 s** to stop them.

Calculate the moment of inertia of a skater given the following information. (a) The 60.0-kg skater is approximated as a cylinder that has a 0.110-m radius. (b) The skater with arms extended is approximately a cylinder that is 52.5 kg, has a 0.110-m radius, and has two 0.900-m-long arms which are 3.75 kg each and extend straight out from the cylinder like rods rotated about their ends.

Show Solution

Strategy

We calculate the moment of inertia using the formulas for standard shapes. For part (a), we use the formula for a solid cylinder rotating about its central axis: I = ½MR². For part (b), we add the moment of inertia of the cylinder (body) to the moments of inertia of the two arms, treating each arm as a rod rotating about one end: I = ⅓ML².

Solution

(a) For the skater approximated as a cylinder (M = 60.0 kg, R = 0.110 m):

$$I=\frac{1}{2}MR^2=\frac{1}{2}(60.0\text{kg})(0.110\text{m})^2=0.363\text{kg}\cdot\text{m}^2$$

(b) For the skater with arms extended:

First, find the moment of inertia of the body (cylinder with M = 52.5 kg, R = 0.110 m):

$$I_{\text{body}}=\frac{1}{2}MR^2=\frac{1}{2}(52.5\text{kg})(0.110\text{m})^2=0.318\text{kg}\cdot\text{m}^2$$

Next, find the moment of inertia of each arm (rod with m = 3.75 kg, L = 0.900 m, rotating about one end):

$$I_{\text{arm}}=\frac{1}{3}mL^2=\frac{1}{3}(3.75\text{kg})(0.900\text{m})^2=1.01\text{kg}\cdot\text{m}^2$$

For two arms:

$$I_{\text{arms}}=2\times1.01\text{kg}\cdot\text{m}^2=2.03\text{kg}\cdot\text{m}^2$$

Total moment of inertia:

$$I_{\text{total}}=I_{\text{body}}+I_{\text{arms}}=0.318\text{kg}\cdot\text{m}^2+2.03\text{kg}\cdot\text{m}^2=2.34\text{kg}\cdot\text{m}^2$$

Discussion

Extending the arms increases the moment of inertia by a factor of about 6.4 (from 0.363 to 2.34 kg·m²). This dramatic increase is why figure skaters spin faster when they pull their arms in—angular momentum is conserved, so reducing I causes ω to increase proportionally.

Answer

(a) The moment of inertia as a cylinder is **0.363 kg·m²**.

(b) With arms extended, the moment of inertia is **2.34 kg·m²**.

The triceps muscle in the back of the upper arm extends the forearm. This muscle in a professional boxer exerts a force of $2.00\times10^3\text{N}$ with an effective perpendicular lever arm of 3.00 cm, producing an angular acceleration of the forearm of 120rad/s^2. What is the moment of inertia of the boxer's forearm?

Show Solution

Strategy

We use the rotational form of Newton's second law, τ = Iα. The torque is produced by the muscle force acting at a perpendicular distance (lever arm) from the pivot point (elbow). We can solve for the moment of inertia I = τ/α.

Solution

First, calculate the torque produced by the triceps muscle:

$$\tau = Fr = (2.00\times10^3\text{N})(0.0300\text{m}) = 60.0\text{N}\cdot\text{m}$$

Now solve for the moment of inertia using τ = Iα:

$$I = \tau\alpha = 60.0\text{N}\cdot\text{m}120\text{rad/s}^2 = 0.50\text{kg}\cdot\text{m}^2$$

Discussion

This moment of inertia is reasonable for a human forearm. The small lever arm (3.00 cm) requires a large muscle force (2000 N) to produce significant torque, which is characteristic of the human body's biomechanics—muscles insert close to joints for compact design, trading mechanical advantage for range of motion.

Answer

The moment of inertia of the boxer's forearm is **0.50 kg·m²**.

A soccer player extends her lower leg in a kicking motion by exerting a force with the muscle above the knee in the front of her leg. She produces an angular acceleration of $30.00\backslash\text{rad/s}^2$ and her lower leg has a moment of inertia of $0.750\text{kg}\ \cdot\text{m}^2$. What is the force exerted by the muscle if its effective perpendicular lever arm is 1.90 cm?

Show Solution

Strategy

This problem is the reverse of the boxer problem—here we know the moment of inertia and angular acceleration, and we need to find the force. We first calculate the required torque using τ = Iα, then find the force using τ = Fr.

Solution

First, calculate the torque needed to produce the angular acceleration:

$$\tau = I\alpha = (0.750\text{kg}\cdot\text{m}^2)(30.00\text{rad/s}^2) = 22.5\text{N}\cdot\text{m}$$

Now solve for the muscle force:

$$F = \tau r = 22.5\text{N}\cdot\text{m}0.0190\text{m} = 1.18\times10^3\text{N}$$

Discussion

The quadriceps muscle must exert about 1200 N (roughly 270 lbs of force) to produce this angular acceleration of the lower leg. As in the boxer example, the small lever arm (1.90 cm) means the muscle must exert a large force. This is typical of biological systems, where muscles sacrifice mechanical advantage for compact anatomy and a greater range of motion.

Answer

The muscle force is **1.18 × 10³ N** (or **1180 N**).

Suppose you exert a force of 180 N tangential to a 0.280-m-radius 75.0-kg grindstone (a solid disk). (a) What torque is exerted? (b) What is the angular acceleration assuming negligible opposing friction? (c) What is the angular acceleration if there is an opposing frictional force of 20.0 N exerted 1.50 cm from the axis?

Show Solution

Strategy

For part (a), we calculate torque directly using τ = Fr. For parts (b) and (c), we need the moment of inertia of the grindstone (a solid disk: I = ½MR²) to find the angular acceleration using α = τ/I. In part (c), friction creates an opposing torque that reduces the net torque.

Solution

(a) The torque exerted by the tangential force:

$$\tau = Fr = (180\text{N})(0.280\text{m}) = 50.4\text{N}\cdot\text{m}$$

(b) First, calculate the moment of inertia of the solid disk:

$$I = 12MR^2 = 12(75.0\text{kg})(0.280\text{m})^2 = 2.94\text{kg}\cdot\text{m}^2$$

The angular acceleration with no friction:

$$\alpha = \tau I = 50.4\text{N}\cdot\text{m}2.94\text{kg}\cdot\text{m}^2 = 17.1\text{rad/s}^2$$

(c) The friction force creates an opposing torque:

$$\tau_{\text{friction}} = F_f r_f = (20.0\text{N})(0.0150\text{m}) = 0.300\text{N}\cdot\text{m}$$

The net torque is:

$$\tau_{\text{net}} = \tau - \tau_{\text{friction}} = 50.4\text{N}\cdot\text{m} - 0.300\text{N}\cdot\text{m} = 50.1\text{N}\cdot\text{m}$$

The angular acceleration with friction:

$$\alpha = \tau_{\text{net}} I = 50.1\text{N}\cdot\text{m}2.94\text{kg}\cdot\text{m}^2 = 17.0\text{rad/s}^2$$

Discussion

The friction barely affects the angular acceleration because it acts very close to the axis (1.50 cm). Even though 20 N is a significant force, the small lever arm means the friction torque is only 0.300 N·m—less than 1% of the applied torque. Friction would be much more effective at slowing the grindstone if it acted at a larger radius.

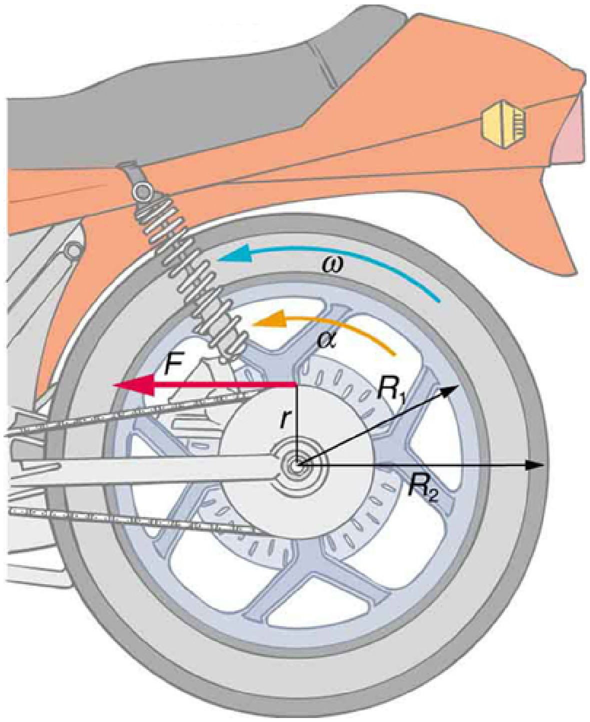
Answer

(a) The torque is **50.4 N·m**.

(b) With no friction, the angular acceleration is **17.1 rad/s²**.

(c) With friction, the angular acceleration is **17.0 rad/s²**.

Consider the 12.0 kg motorcycle wheel shown in Figure 6. Assume it to be approximately an annular ring with an inner radius of 0.280 m and an outer radius of 0.330 m. The motorcycle is on its center stand, so that the wheel can spin freely. (a) If the drive chain exerts a force of 2200 N at a radius of 5.00 cm, what is the angular acceleration of the wheel? (b) What is the tangential acceleration of a point on the outer edge of the tire? (c) How long, starting from rest, does it take to reach an angular velocity of 80.0 rad/s?



A motorcycle wheel has a moment of inertia approximately that of an annular ring.

Show Solution

Strategy

The motorcycle wheel is approximated as an annular ring (a ring with inner and outer radii). Its moment of inertia is $I = \frac{1}{2}M(R_1^2 + R_2^2)$. We calculate the torque from the chain, find the angular acceleration, then use rotational kinematics.

Solution

(a) First, calculate the moment of inertia of the annular ring:

$$I=\frac{1}{2}M(R_1^2+R_2^2)=\frac{1}{2}(12.0\text{kg})[(0.280\text{m})^2+(0.330\text{m})^2]$$
$$I=\frac{1}{2}(12.0\text{kg})(0.0784\text{m}^2+0.1089\text{m}^2)=\frac{1}{2}(12.0\text{kg})(0.1873\text{m}^2)=1.12\text{kg}\cdot\text{m}^2$$

The torque from the chain:

$$\tau=Fr=(2200\text{N})(0.0500\text{m})=110\text{N}\cdot\text{m}$$

The angular acceleration:

$$\alpha=\frac{\tau}{I}=\frac{110\text{N}\cdot\text{m}}{1.12\text{kg}\cdot\text{m}^2}=98.2\text{rad/s}^2$$

(b) The tangential acceleration at the outer edge of the tire (R_2 = 0.330 m):

$$a_t=R_2\alpha=(0.330\text{m})(98.2\text{rad/s}^2)=32.4\text{ m/s}^2$$

(c) Time to reach ω = 80.0 rad/s from rest:

$$t=\frac{\omega-\omega_0}{\alpha}=\frac{80.0\text{ rad/s}-0}{98.2\text{rad/s}^2}=0.815\text{s}$$

Discussion

The wheel reaches 80.0 rad/s (about 760 rpm) in less than a second, which corresponds to a linear speed of v = Rω = (0.330)(80.0) = 26.4 m/s ≈ 95 km/h at the tire’s edge. The relatively small moment of inertia allows for quick acceleration, which is important for motorcycle performance.

Answer

(a) The angular acceleration is **98.2 rad/s²**.

(b) The tangential acceleration at the outer edge is **32.4 m/s²**.

(c) It takes **0.815 s** to reach 80.0 rad/s.

Zorch, an archenemy of Superman, decides to slow Earth's rotation to once per 28.0 h by exerting an opposing force at and parallel to the equator. Superman is not immediately concerned, because he knows Zorch can only exert a force of 4.00×10^7 N (a little greater than a Saturn V rocket's thrust). How long must Zorch push with this force to accomplish his goal? (This period gives Superman time to devote to other villains.) Explicitly show how you follow the steps found in Problem-Solving Strategy for Rotational Dynamics .

Show Solution

Strategy

Following the Problem-Solving Strategy for Rotational Dynamics:

1. *Identify*: We need to slow Earth's rotation, so torque and moment of inertia are involved.
2. *System of interest*: Earth, rotating about its axis.
3. *Free body*: Zorch applies a tangential force at the equator, creating an opposing torque.
4. *Apply τ = Iα*: We'll find the angular deceleration, then use kinematics to find the time.

Solution

First, determine the initial and final angular velocities:

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{24.0 \times 3600\,\text{s}} = 7.27 \times 10^{-5}\ \text{rad/s}$$
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{28.0 \times 3600\,\text{s}} = 6.24 \times 10^{-5}\ \text{rad/s}$$

The required change in angular velocity:

$$\Delta\omega = \omega_0 - \omega = 7.27 \times 10^{-5} - 6.24 \times 10^{-5} = 1.03 \times 10^{-5}\ \text{rad/s}$$

Earth's moment of inertia (approximating as a uniform solid sphere, $M = 5.97 \times 10^{24}$ kg, $R = 6.37 \times 10^6$ m):

$$I = \frac{2}{5} M R^2 = \frac{2}{5}(5.97 \times 10^{24}\,\text{kg})(6.37 \times 10^6\,\text{m})^2 = 9.69 \times 10^{37}\,\text{kg}\cdot\text{m}^2$$

The torque Zorch produces at the equator:

$$\tau = FR = (4.00 \times 10^7\,\text{N})(6.37 \times 10^6\,\text{m}) = 2.55 \times 10^{14}\,\text{N}\cdot\text{m}$$

The angular deceleration:

$$\alpha = \frac{\tau}{I} = \frac{2.55 \times 10^{14}\,\text{N}\cdot\text{m}}{9.69 \times 10^{37}\,\text{kg}\cdot\text{m}^2} = 2.63 \times 10^{-24}\,\text{rad/s}^2$$

Time required:

$$t = \frac{\Delta\omega}{\alpha} = \frac{1.03 \times 10^{-5}\ \text{rad/s}}{2.63 \times 10^{-24}\,\text{rad/s}^2} = 3.96 \times 10^{18}\,\text{s}$$

Converting to years:

$$t = \frac{3.96 \times 10^{18}\,\text{s}}{3.156 \times 10^7\ \text{s/y}} = 1.26 \times 10^{11}\ \text{y}$$

Discussion

Superman has plenty of time—126 billion years is about 9 times the current age of the universe! This illustrates the enormous moment of inertia of Earth. Even with a force greater than a Saturn V rocket, the tiny angular deceleration (about 10^{-24} rad/s²) means Zorch's plan is essentially impossible on any practical timescale.

Answer

Zorch would need to push for **1.26×10^{11} years** (126 billion years) to slow Earth's rotation to once per 28.0 hours.

An automobile engine can produce 200 N · m of torque. Calculate the angular acceleration produced if 95.0% of this torque is applied to the drive shaft, axle, and rear wheels of a car, given the following information. The car is suspended so that the wheels can turn freely. Each wheel acts like a 15.0 kg disk that has a 0.180 m radius. The walls of each tire act like a 2.00-kg annular ring that has inside radius of 0.180 m and outside radius of 0.320 m. The tread of each tire acts like a 10.0-kg hoop of radius 0.330 m. The 14.0-kg axle acts like a rod that has a 2.00-cm radius. The 30.0-kg drive shaft acts like a rod that has a 3.20-cm radius.

Show Solution

Strategy

We need to calculate the total moment of inertia of the rotating system (two wheels, axle, and drive shaft), then apply α = τ/I. Each component uses a different moment of inertia formula based on its shape.

Solution

First, calculate the applied torque:

$$\tau = 0.950 \times 200 \text{N·m} = 190 \text{N·m}$$

Now calculate the moment of inertia for each component:

For each wheel (there are two rear wheels):

Disk (wheel center):

$$I_{\text{disk}} = \frac{1}{2}MR^2 = \frac{1}{2}(15.0\text{kg})(0.180\text{m})^2 = 0.243\text{kg·m}^2$$

Tire walls (annular ring):

$$I_{\text{walls}} = \frac{1}{2}M(R_1^2 + R_2^2) = \frac{1}{2}(2.00\text{kg})[(0.180\text{m})^2 + (0.320\text{m})^2] = 0.135\text{kg·m}^2$$

Tread (hoop):

$$I_{\text{tread}} = MR^2 = (10.0\text{kg})(0.330\text{m})^2 = 1.089\text{kg·m}^2$$

Moment of inertia per wheel:

$$I_{\text{wheel}} = 0.243 + 0.135 + 1.089 = 1.467\text{kg·m}^2$$

For two wheels:

$$I_{\text{wheels}} = 2 \times 1.467 = 2.934\text{kg·m}^2$$

For the axle (solid cylinder, 14.0 kg, R = 0.0200 m):

$$I_{\text{axle}} = \frac{1}{2}MR^2 = \frac{1}{2}(14.0\text{kg})(0.0200\text{m})^2 = 0.00280\text{kg·m}^2$$

For the drive shaft (solid cylinder, 30.0 kg, R = 0.0320 m):

$$I_{\text{shaft}} = \frac{1}{2}MR^2 = \frac{1}{2}(30.0\text{kg})(0.0320\text{m})^2 = 0.0154\text{kg·m}^2$$

Total moment of inertia:

$$I_{\text{total}} = 2.934 + 0.00280 + 0.0154 = 2.95\text{kg·m}^2$$

Angular acceleration:

$$\alpha = \frac{\tau}{I} = \frac{190\text{N·m}}{2.95\text{kg·m}^2} = 64.4\text{rad/s}^2$$

Discussion

The wheels dominate the total moment of inertia—they account for over 99% of it. The axle and drive shaft contribute negligibly because their mass is concentrated near the axis of rotation (small radii). This is why performance cars use lightweight wheels: reducing wheel mass has a much greater effect on acceleration than reducing the mass of central components like the axle.

Answer

The angular acceleration is **64.4 rad/s²**.

Starting with the formula for the moment of inertia of a rod rotated around an axis through one end perpendicular to its length $(I=M\ell^2/3)$, prove that the moment of inertia of a rod rotated about an axis through its center perpendicular to its length is $I=M\ell^2/12$. You will find the graphics in Figure 3 useful in visualizing these rotations.

Show Solution

$I_{\text{end}}=I_{\text{center}}+m(l2)^2$ Thus, $I_{\text{center}}=I_{\text{end}}-14ml^2=13ml^2-14ml^2=112ml^2$

Starting with the formula for the moment of inertia of a rod rotated around an axis through one end perpendicular to its length $(I=M\ell^2/3)$, prove that the moment of inertia of a rod rotated about an axis through its center perpendicular to its length is $I=M\ell^2/12$. You will find the graphics in [Figure 3](#Figure3) useful in visualizing these rotations. </div>

Show Solution

Strategy

We can use the parallel axis theorem to relate the moment of inertia about an axis through the end of the rod to the moment of inertia about an axis through the center. The parallel axis theorem states that $I=I_{\text{center}}+Md^2$, where d is the distance between the parallel axes. For a rod of length ℓ, the distance from the center to the end is ℓ/2.

Solution

Applying the parallel axis theorem to relate rotation about the end to rotation about the center:

$$I_{\text{end}}=I_{\text{center}}+M(\ell 2)^2$$

We know that $I_{\text{end}}=13M\ell^2$, so:

$$13M\ell^2=I_{\text{center}}+M(\ell 2)^2$$
$$13M\ell^2=I_{\text{center}}+14M\ell^2$$

Solving for I_{center}:

$$I_{\text{center}}=13M\ell^2-14M\ell^2=4M\ell^2-3M\ell^2 12=112M\ell^2$$

Discussion

This result makes physical sense: the moment of inertia about the center (I = Mℓ²/12) is smaller than about the end (I = Mℓ²/3) because mass is distributed closer to the rotation axis when rotating about the center. The parallel axis theorem is a powerful tool that allows us to calculate moments of inertia about different axes without performing complex integrations. This relationship between Mℓ²/3 and Mℓ²/12 appears frequently in physics problems involving rotating rods.

Answer

The moment of inertia of a rod rotated about its center is **I = Mℓ²/12**, as proven using the parallel axis theorem.

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Unreasonable Results

A gymnast doing a forward flip lands on the mat and exerts a 500-N · m torque to slow and then reverse her angular velocity. Her initial angular velocity is 10.0 rad/s, and her moment of inertia is $0.050\text{kg}\cdot\text{m}^2$. (a) What time is required for her to exactly reverse her spin? (b) What is unreasonable about the result? (c) Which premises are unreasonable or inconsistent?

Show Solution

Strategy

We use the rotational form of Newton's second law and the relationship between torque and angular momentum change: $\tau=\Delta L\Delta t=I\Delta\omega\Delta t$. To reverse her spin, her angular velocity must change from +10.0 rad/s to −10.0 rad/s.

Solution

(a) The change in angular velocity is:

$$\Delta\omega=\omega f-\omega i=-10.0-10.0=-20.0\text{ rad/s}$$

Using $\tau = I\alpha = I\Delta\omega\Delta t$:

$$\Delta t=I\Delta\omega\tau=(0.050\text{kg}\cdot\text{m}^2)(-20.0\text{ rad/s})-500\text{N}\cdot\text{m}=0.002\text{s}=2.0\text{ ms}$$

(b) A time of 2.0 milliseconds is absurdly short. It's impossible for a gymnast to exert a torque for such a brief period and reverse her rotation. Human reaction time alone is around 100-200 ms, and applying forces through body contact with the mat takes much longer than 2 ms.

(c) The moment of inertia of 0.050 kg·m² is unreasonably small. A typical human body has a moment of inertia of 5-20 kg·m² depending on body position. The given value is smaller by a factor of 100-400. A realistic moment of inertia would be around 5 kg·m², which would give a time of 200 ms—much more reasonable. The torque of 500 N·m, while large, is not unreasonable for a gymnast landing on a mat.

Discussion

This problem illustrates how unreasonable values for one quantity (moment of inertia) lead to impossible results (2 ms time interval). In reality, a gymnast's moment of inertia in a tucked position is about 2-4 kg·m², and in an extended position, 10-15 kg·m². The absurdly low value given makes the calculation physically meaningless.

Answer

(a) The time required is **2.0 ms** (0.002 s).

(b) This is unreasonably short—far less than human reaction time and impossible for physical contact with a mat.

(c) The moment of inertia (0.050 kg·m²) is unreasonably small—about 100 times smaller than a realistic value for a human body.

Unreasonable Results

An advertisement claims that an 800-kg car is aided by its 20.0-kg flywheel, which can accelerate the car from rest to a speed of 30.0 m/s. The flywheel is a disk with a 0.150-m radius. (a) Calculate the angular velocity the flywheel must have if 95.0% of its rotational energy is used to get the car up to speed. (b) What is unreasonable about the result? (c) Which premise is unreasonable or which premises are inconsistent?

Show Solution

Strategy

The car's final kinetic energy must equal 95% of the flywheel's initial rotational kinetic energy. We use KE = ½mv² for the car and KE_rot = ½Iω² for the flywheel (disk: I = ½MR²).

Solution

(a) The car's kinetic energy at 30.0 m/s:

$$KE_{\text{car}}=12mv^2=12(800\text{kg})(30.0\text{ m/s})^2=3.60\times10^5\text{J}$$

This must equal 95% of the flywheel's rotational energy:

$$0.950\times12I\omega^2=3.60\times10^5\text{J}$$

For a disk flywheel:

$$I=12MR^2=12(20.0\text{kg})(0.150\text{m})^2=0.225\text{kg}\cdot\text{m}^2$$

Solving for ω:

$$\omega^2=2(3.60\times10^5\text{J})0.950\times0.225\text{kg}\cdot\text{m}^2=7.20\times10^5 0.214=3.36\times10^6\text{ rad}^2/\text{s}^2$$
$$\omega=1834\text{ rad/s}=1834\times602\pi\text{ rpm}=17{,}500\text{ rpm}$$

(b) This angular velocity is unreasonably high. At the edge of the disk (r = 0.150 m), the centripetal acceleration would be:

$$a_c = \omega^2 r = (1834)^2(0.150) = 5.04\times10^5 \text{ m/s}^2 = 51{,}400g$$

This is over 50,000 times the acceleration due to gravity. No ordinary material could withstand such forces without flying apart. Additionally, 17,500 rpm is far beyond typical flywheel speeds.

(c) The flywheel is much too small and light to store enough energy at reasonable rotation speeds. To store this much energy safely, the flywheel would need either:

- Much greater mass (hundreds of kg instead of 20 kg), or
- Much larger radius (0.5-1.0 m instead of 0.15 m), or
- Both

A realistic flywheel for this application might be 200 kg with a 0.5 m radius, which would require only about 2,000 rpm—still high but achievable with proper materials and engineering.

Discussion

This problem demonstrates the challenges of energy storage in flywheels. While flywheels can store substantial energy, the limits of material strength constrain their design. Modern high-performance flywheels use composite materials and operate in vacuum chambers, but even these have practical limits well below what would cause 50,000g accelerations.

Answer

(a) The flywheel would need to spin at **1834 rad/s** (about **17,500 rpm**).

(b) This is unreasonably fast—it would produce over 50,000g acceleration that would destroy the flywheel.

(c) The flywheel is too small and light to store the required energy at achievable speeds; it would need to be much larger (0.5-1.0 m radius) and heavier (200+ kg).

Glossary

torque
: the turning effectiveness of a force

rotational inertia
: resistance to change of rotation. The more rotational inertia an object has, the harder it is to rotate

moment of inertia
: mass times the square of perpendicular distance from the rotation axis; for a point mass, it is $I = mr^2$ and, because any object can be built up from a collection of point masses, this relationship is the basis for all other moments of inertia

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Rotational Kinetic Energy: Work and Energy Revisited

- Derive the equation for rotational work.
- Calculate rotational kinetic energy.
- Demonstrate the Law of Conservation of Energy.

In this module, we will learn about work and energy associated with rotational motion. Figure 1 shows a worker using an electric grindstone propelled by a motor. Sparks are flying, and noise and vibration are created as layers of steel are pared from the pole. The stone continues to turn even after the motor is turned off, but it is eventually brought to a stop by friction. Clearly, the motor had to work to get the stone spinning. This work went into heat, light, sound, vibration, and considerable **rotational kinetic energy** .

The motor works in spinning the grindstone, giving it rotational kinetic energy. That energy is then converted to heat, light, sound, and vibration. (credit: U.S. Navy photo by Mass Communication Specialist Seaman Zachary David Bell)

Work must be done to rotate objects such as grindstones or merry-go-rounds. Work was defined in Uniform Circular Motion and Gravitation for translational motion, and we can build on that knowledge when considering work done in rotational motion. The simplest rotational situation is one in which the net force is exerted perpendicular to the radius of a disk (as shown in Figure 2) and remains perpendicular as the disk starts to rotate. The force is parallel to the displacement, and so the net work done is the product of the force times the arc length traveled:

$$\text{net}W = (\text{net}F)\Delta s.$$

To get torque and other rotational quantities into the equation, we multiply and divide the right-hand side of the equation by r, and gather terms:

$$\text{net}W = (r\,\text{net}F)\frac{\Delta s}{r}.$$

We recognize that $r\,\text{net}F = \text{net}\ \tau$ and $\Delta s/r = \theta$, so that

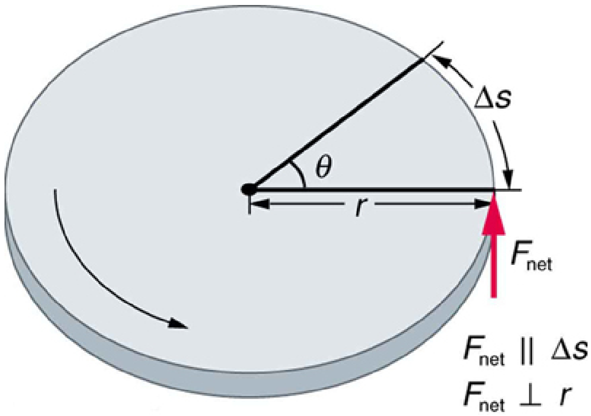
$$\text{net}W = (\text{net}\ \tau)\theta.$$

This equation is the expression for rotational work. It is very similar to the familiar definition of translational work as force multiplied by distance. Here, torque is analogous to force, and angle is analogous to distance. The equation $\text{net}W = (\text{net}\ \tau)\theta$ is valid in general, even though it was derived for a special case.

To get an expression for rotational kinetic energy, we must again perform some algebraic manipulations. The first step is to note that $\text{net}\ \tau = I\alpha$

, so that

$$\text{net}W = I\alpha\theta.$$



The net force on this disk is kept perpendicular to its radius as the force causes the disk to rotate. The net work done is thus \(\text{net} F \cdot \Delta s \). The net work goes into rotational kinetic energy.

Making Connections

Work and energy in rotational motion are completely analogous to work and energy in translational motion, first presented in Uniform Circular Motion and Gravitation.

Now, we solve one of the rotational kinematics equations for $\alpha\theta$. We start with the equation

$$\omega^2=\omega_{20}+2\alpha\theta.$$

Next, we solve for $\alpha\theta$:

$$\alpha\theta=\omega^2-\omega_{20}2.$$

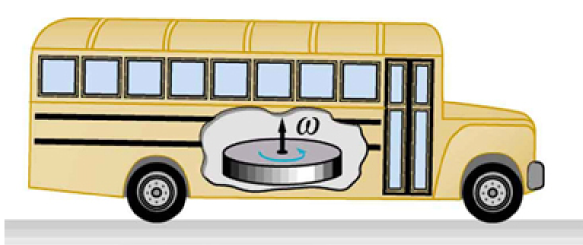
Substituting this into the equation for net W and gathering terms yields

$$\text{net}W=12I\omega^2-12I\omega_{20}.$$

This equation is the **work-energy theorem** for rotational motion only. As you may recall, net work changes the kinetic energy of a system. Through an analogy with translational motion, we define the term $(12)I\omega^2$ to be **rotational kinetic energy** KE_{rot} for an object with a moment of inertia I and an angular velocity ω :

$$\text{KE}_{\text{rot}}=12I\omega^2.$$

The expression for rotational kinetic energy is exactly analogous to translational kinetic energy, with I being analogous to m and ω to v. Rotational kinetic energy has important effects. Flywheels, for example, can be used to store large amounts of rotational kinetic energy in a vehicle, as seen in Figure 3.



Experimental vehicles, such as this bus, have been constructed in which rotational kinetic energy is stored in a large flywheel. When the bus goes down a hill, its transmission converts its gravitational potential energy into \(\KE_{\text{rot}} /). It can also convert translational kinetic energy, when the bus stops, into \(\KE_{\text{rot}} /). The flywheel’s energy can then be used to accelerate, to go up another hill, or to keep the bus from going against friction.

Calculating the Work and Energy for Spinning a Grindstone

Consider a person who spins a large grindstone by placing her hand on its edge and exerting a force through part of a revolution as shown in Figure 4. In this example, we verify that the work done by the torque she exerts equals the change in rotational energy. (a) How much work is done if she exerts a force of 200 N through a rotation of $1.00 \text{rad}(57.3°)$? The force is kept perpendicular to the grindstone's 0.320-m radius at the point of application, and the effects of friction are negligible. (b) What is the final angular velocity if the grindstone has a mass of 85.0 kg? (c) What is the final rotational kinetic energy? (It should equal the work.)

Strategy

To find the work, we can use the equation $\text{net} W = (\text{net } \tau)\theta$. We have enough information to calculate the torque and are given the rotation angle. In the second part, we can find the final angular velocity using one of the kinematic relationships. In the last part, we can calculate the rotational kinetic energy from its expression in $\text{KE}_{\text{rot}} = \frac{1}{2}I\omega^2$.

Solution for (a)

The net work is expressed in the equation

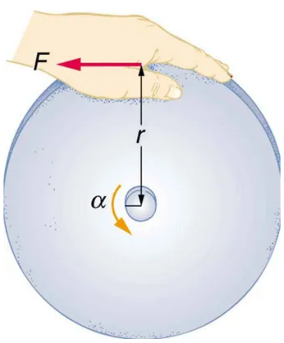
$$\text{net} W = (\text{net } \tau)\theta,$$

where net τ is the applied force multiplied by the radius (rF) because there is no retarding friction, and the force is perpendicular to r. The angle θ is given. Substituting the given values in the equation above yields

$$\text{net} W = rF\theta = (0.320 \text{m})(200 \text{N})(1.00 \text{rad}) \quad \text{net} W = 64.0 \text{N} \cdot \text{m}.$$

Noting that $1 \text{N} \cdot \text{m} = 1 \text{J}$,

$$\text{net} W = 64.0 \text{J}.$$



A large grindstone is given a spin by a person grasping its outer edge.

Solution for (b)

To find ω from the given information requires more than one step. We start with the kinematic relationship in the equation

$$\omega^2 = \omega_0^2 + 2\alpha\theta.$$

Note that $\omega_0 = 0$ because we start from rest. Taking the square root of the resulting equation gives

$$\omega = (2\alpha\theta)^{1/2}.$$

Now we need to find α. One possibility is

$$\alpha = \frac{\text{net } \tau}{I},$$

where the torque is

$$\text{net } \tau = rF = (0.320 \text{m})(200 \text{N}) = 64.0 \text{N} \cdot \text{m}.$$

The formula for the moment of inertia for a disk is found in Figure 3 of Dynamics Of Rotational Motion :

$$I = \frac{1}{2}MR^2 = 0.5(85.0 \text{kg})(0.320 \text{m})^2 = 4.352 \text{kg} \cdot \text{m}^2.$$

Substituting the values of torque and moment of inertia into the expression for α, we obtain

$$\alpha = \frac{64.0\,\text{N}\cdot\text{m}}{4.352\,\text{kg}\cdot\text{m}^2} = 14.7\,\text{rad/s}^2.$$

Now, substitute this value and the given value for θ into the above expression for ω:

$$\omega = (2\alpha\theta)^{1/2} = \left[2(14.7\,\text{rad/s}^2)(1.00\,\text{rad})\right]^{1/2} = 5.42\,\text{rad/s}.$$

Solution for (c)

The final rotational kinetic energy is

$$\text{KE}_{\text{rot}} = \frac{1}{2}I\omega^2.$$

Both I and ω were found above. Thus,

$$\text{KE}_{\text{rot}} = (0.5)(4.352\,\text{kg}\cdot\text{m}^2)(5.42\,\text{rad/s})^2 = 64.0\,\text{J}.$$

Discussion

The final rotational kinetic energy equals the work done by the torque, which confirms that the work done went into rotational kinetic energy. We could, in fact, have used an expression for energy instead of a kinematic relation to solve part (b). We will do this in later examples.

Helicopter pilots are quite familiar with rotational kinetic energy. They know, for example, that a point of no return will be reached if they allow their blades to slow below a critical angular velocity during flight. The blades lose lift, and it is impossible to immediately get the blades spinning fast enough to regain it. Rotational kinetic energy must be supplied to the blades to get them to rotate faster, and enough energy cannot be supplied in time to avoid a crash. Because of weight limitations, helicopter engines are too small to supply both the energy needed for lift and to replenish the rotational kinetic energy of the blades once they have slowed down. The rotational kinetic energy is put into them before takeoff and must not be allowed to drop below this crucial level. One possible way to avoid a crash is to use the gravitational potential energy of the helicopter to replenish the rotational kinetic energy of the blades by losing altitude and aligning the blades so that the helicopter is spun up in the descent. Of course, if the helicopter's altitude is too low, then there is insufficient time for the blade to regain lift before reaching the ground.

Problem-Solving Strategy for Rotational Energy

1. *Determine that energy or work is involved in the rotation*.
2. *Determine the system of interest*. A sketch usually helps.
3. *Analyze the situation to determine the types of work and energy involved*.
4. *For closed systems, mechanical energy is conserved*. That is, $\text{KE}_\text{i} + \text{PE}_\text{i} = \text{KE}_\text{f} + \text{PE}_\text{f}$. Note that KE_i and KE_f may each include translational and rotational contributions.
5. *For open systems*, mechanical energy may not be conserved, and other forms of energy (referred to previously as OE), such as heat transfer, may enter or leave the system. Determine what they are, and calculate them as necessary.
6. *Eliminate terms wherever possible to simplify the algebra*.
7. *Check the answer to see if it is reasonable*.

Calculating Helicopter Energies

A typical small rescue helicopter, similar to the one in Figure 5, has four blades, each is 4.00 m long and has a mass of 50.0 kg. The blades can be approximated as thin rods that rotate about one end of an axis perpendicular to their length. The helicopter has a total loaded mass of 1000 kg. (a) Calculate the rotational kinetic energy in the blades when they rotate at 300 rpm. (b) Calculate the translational kinetic energy of the helicopter when it flies at 20.0 m/s, and compare it with the rotational energy in the blades. (c) To what height could the helicopter be raised if all of the rotational kinetic energy could be used to lift it?

Strategy

Rotational and translational kinetic energies can be calculated from their definitions. The last part of the problem relates to the idea that energy can change form, in this case from rotational kinetic energy to gravitational potential energy.

Solution for (a)

The rotational kinetic energy is

$$\text{KE}_{\text{rot}} = \frac{1}{2}I\omega^2.$$

We must convert the angular velocity to radians per second and calculate the moment of inertia before we can find KE_{rot}. The angular velocity ω is

$$\omega = \frac{300 \text{ rev}}{1.00 \text{ min}} \cdot \frac{2\pi \text{ rad}}{1 \text{ rev}} \cdot \frac{1.00 \text{ min}}{60.0 \text{ s}} = 31.4 \frac{\text{rad}}{\text{s}}.$$

The moment of inertia of one blade will be that of a thin rod rotated about its end, found in Figure 3 Dynamics Of Rotational Motion. The total I is four times this moment of inertia, because there are four blades. Thus,

$$I = 4\frac{M\ell^2}{3} = 4 \times \frac{(50.0 \text{ kg})(4.00 \text{ m})^2}{3} = 1067 \text{ kg}\cdot\text{m}^2.$$

Entering ω and I into the expression for rotational kinetic energy gives

$$\begin{aligned} KE_{rot} &= 0.5(1067 \text{ kg}\cdot\text{m}^2)(31.4 \text{ rad/s})^2 \\ KE_{rot} &= 5.26 \times 10^5 \text{ J} \end{aligned}$$

Solution for (b)

Translational kinetic energy was defined in Uniform Circular Motion and Gravitation . Entering the given values of mass and velocity, we obtain

$$KE_{trans} = \frac{1}{2}mv^2 = (0.5)(1000 \text{ kg})(20.0 \text{ m/s})^2 = 2.00 \times 10^5 \text{ J}.$$

To compare kinetic energies, we take the ratio of translational kinetic energy to rotational kinetic energy. This ratio is

$$\frac{2.00 \times 10^5 \text{ J}}{5.26 \times 10^5 \text{ J}} = 0.380.$$

Solution for (c)

At the maximum height, all rotational kinetic energy will have been converted to gravitational energy. To find this height, we equate those two energies:

$$KE_{rot} = PE_{grav}$$

or

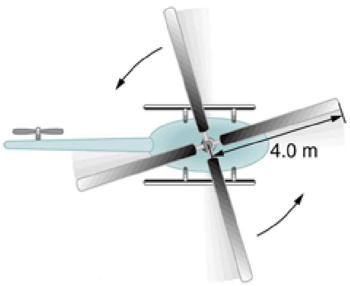
$$\frac{1}{2}I\omega^2 = mgh.$$

We now solve for h and substitute known values into the resulting equation

$$h = \frac{\frac{1}{2}I\omega^2}{mg} = \frac{5.26 \times 10^5 \text{ J}}{(1000 \text{ kg})(9.80 \text{ m/s}^2)} = 53.7 \text{ m}.$$

Discussion

The ratio of translational energy to rotational kinetic energy is only 0.380. This ratio tells us that most of the kinetic energy of the helicopter is in its spinning blades—something you probably would not suspect. The 53.7 m height to which the helicopter could be raised with the rotational kinetic energy is also impressive, again emphasizing the amount of rotational kinetic energy in the blades.



The first image shows how helicopters store large amounts of rotational kinetic energy in their blades. This energy must be put into the blades before takeoff and maintained until the end of the flight. The engines do not have enough power to simultaneously provide lift and put significant rotational energy into the blades. The second image shows a helicopter from the Auckland Westpac Rescue Helicopter Service. Over 50 000 lives have been saved since its operations beginning in 1973. Here, a water rescue operation is shown. (credit: 111 Emergency, Flickr)

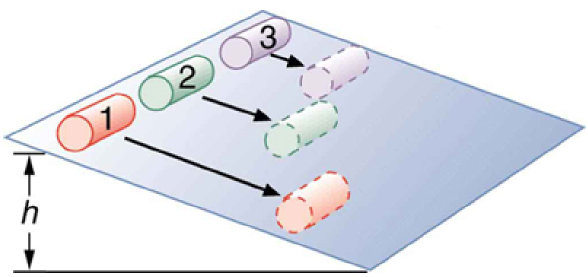
Making Connections

Conservation of energy includes rotational motion, because rotational kinetic energy is another form of KE. Uniform Circular Motion and Gravitation has a detailed treatment of conservation of energy.

How Thick Is the Soup? Or Why Don't All Objects Roll Downhill at the Same Rate?

One of the quality controls in a tomato soup factory consists of rolling filled cans down a ramp. If they roll too fast, the soup is too thin. Why should cans of identical size and mass roll down an incline at different rates? And why should the thickest soup roll the slowest?

The easiest way to answer these questions is to consider energy. Suppose each can starts down the ramp from rest. Each can starting from rest means each starts with the same gravitational potential energy PE_{grav}, which is converted entirely to KE, provided each rolls without slipping. KE, however, can take the form of KE_{trans} or KE_rot, and total KE is the sum of the two. If a can rolls down a ramp, it puts part of its energy into rotation, leaving less for translation. Thus, the can goes slower than it would if it slid down. Furthermore, the thin soup does not rotate, whereas the thick soup does, because it sticks to the can. The thick soup thus puts more of the can's original gravitational potential energy into rotation than the thin soup, and the can rolls more slowly, as seen in Figure 6.



Three cans of soup with identical masses race down an incline. The first can has a low friction coating and does not roll but just slides down the incline. It wins because it converts its entire PE into translational KE. The second and third cans both roll down the incline without slipping. The second can contains thin soup and comes in second because part of its initial PE goes into rotating the can (but not the thin soup). The third can contains thick soup. It comes in third because the soup rotates along with the can, taking even more of the initial PE for rotational KE, leaving less for translational KE.

Assuming no losses due to friction, there is only one force doing work—gravity. Therefore the total work done is the change in kinetic energy. As the cans start moving, the potential energy is changing into kinetic energy. Conservation of energy gives

$$PE_i = KE_f.$$

More specifically,

$PE_{grav}=KE_{trans}+KE_{rot}$

or

$$mgh=12mv^2+12I\omega^2.$$

So, the initial mgh is divided between translational kinetic energy and rotational kinetic energy; and the greater I is, the less energy goes into translation. If the can slides down without friction, then $\omega=0$ and all the energy goes into translation; thus, the can goes faster.

Take-Home Experiment

Locate several cans each containing different types of food. First, predict which can will win the race down an inclined plane and explain why. See if your prediction is correct. You could also do this experiment by collecting several empty cylindrical containers of the same size and filling them with different materials such as wet or dry sand.

Calculating the Speed of a Cylinder Rolling Down an Incline

Calculate the final speed of a solid cylinder that rolls down a 2.00-m-high incline. The cylinder starts from rest, has a mass of 0.750 kg, and has a radius of 4.00 cm.

Strategy

We can solve for the final velocity using conservation of energy, but we must first express rotational quantities in terms of translational quantities to end up with v as the only unknown.

Solution

Conservation of energy for this situation is written as described above:

$$mgh=12mv^2+12I\omega^2.$$

Before we can solve for v, we must get an expression for I from [Figure 3 of Dynamics Of Rotational Motion](). Because v and ω are related (note here that the cylinder is rolling without slipping), we must also substitute the relationship $\omega=v/R$ into the expression. These substitutions yield

$$mgh=12mv^2+12(12mR^2)(v^2R^2).$$

Interestingly, the cylinder's radius R and mass m cancel, yielding

$$gh=12v^2+14v^2=34v^2.$$

Solving algebraically, the equation for the final velocity v gives

$$v=\surd 4gh3.$$

Substituting known values into the resulting expression yields

$$v=\surd 4(9.80\text{m/s}^2)(2.00\text{m})3=5.11\text{m/s}.$$

Discussion

Because m and R cancel, the result $v=\surd 43gh$ is valid for any solid cylinder, implying that all solid cylinders will roll down an incline at the same rate independent of their masses and sizes. (Rolling cylinders down inclines is what Galileo actually did to show that objects fall at the same rate independent of mass.) Note that if the cylinder slid without friction down the incline without rolling, then the entire gravitational potential energy would go into translational kinetic energy. Thus, $12mv^2=mgh$ and $v=\surd 2gh$, which is 22% greater than $\surd 4gh/3$. That is, the cylinder would go faster at the bottom.

Check Your Understanding

Analogy of Rotational and Translational Kinetic Energy Is rotational kinetic energy completely analogous to translational kinetic energy? What, if any, are their differences? Give an example of each type of kinetic energy.

Show Solution

Yes, rotational and translational kinetic energy are exact analogs. They both are the energy of motion involved with the coordinated (non-random) movement of mass relative to some reference frame. The only difference between rotational and translational kinetic energy is that translational is straight line motion while rotational is not. An example of both kinetic and translational kinetic energy is found in a bike tire while being ridden down a bike path. The rotational motion of the tire means it has rotational kinetic energy while the movement of the bike along the path means the tire also has translational kinetic energy. If you were to lift the front wheel of the bike and spin it while the bike is stationary, then the wheel would have only rotational kinetic energy relative to the Earth.
My Solar System

Build your own system of heavenly bodies and watch the gravitational ballet. With this orbit simulator, you can set initial positions, velocities, and masses of 2, 3, or 4 bodies, and then see them orbit each other.

My solar system

Section Summary

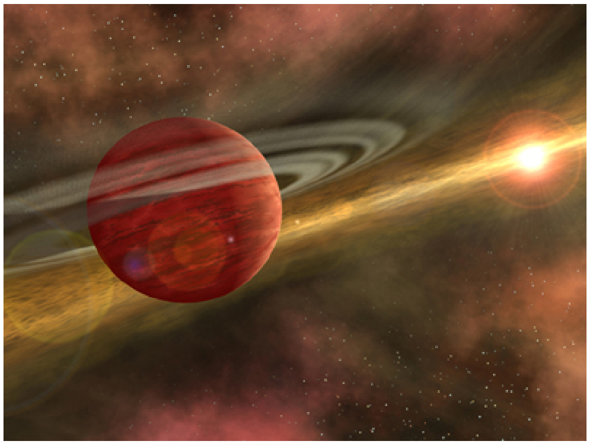
- The rotational kinetic energy KE_{rot} for an object with a moment of inertia I and an angular velocity ω is given by
  $$KE_{rot}=\frac{1}{2}I\omega^2.$$
- Helicopters store large amounts of rotational kinetic energy in their blades. This energy must be put into the blades before takeoff and maintained until the end of the flight. The engines do not have enough power to simultaneously provide lift and put significant rotational energy into the blades.
- Work and energy in rotational motion are completely analogous to work and energy in translational motion.
- The equation for the **work-energy theorem** for rotational motion is,
  $$\text{net}\,W=\frac{1}{2}I\omega^2-\frac{1}{2}I\omega_0^2.$$

Conceptual Questions

Describe the energy transformations involved when a yo-yo is thrown downward and then climbs back up its string to be caught in the user's hand.

What energy transformations are involved when a dragster engine is revved, its clutch let out rapidly, its tires spun, and it starts to accelerate forward? Describe the source and transformation of energy at each step.

The Earth has more rotational kinetic energy now than did the cloud of gas and dust from which it formed. Where did this energy come from?



An immense cloud of rotating gas and dust contracted under the influence of gravity to form the Earth and in the process rotational kinetic energy increased. (credit: NASA)

Problems & Exercises

This problem considers energy and work aspects of example Calculating the Effect of Mass Distribution on a Merry-Go-Round—use data from that example as needed. (a) Calculate the rotational kinetic energy in the merry-go-round plus child when they have an angular velocity of 20.0 rpm. (b) Using energy considerations, find the number of revolutions the father will have to push to achieve this angular velocity starting from rest. (c) Again, using energy considerations, calculate the force the father must exert to stop the merry-go-round in two revolutions

Show Solution

Strategy

From the referenced example: merry-go-round mass M = 50.0 kg, radius R = 1.50 m, child mass m = 18.0 kg at r = 1.25 m, applied force F = 250 N at edge. For part (a), we calculate KE_rot = ½Iω². For part (b), we use W = ΔKE. For part (c), we find the force needed to do work W = −185 J over distance d = 2 revolutions.

Solution

(a) From the example, total moment of inertia: $I = 12MR^2 + mr^2 = 56.25 + 28.13 = 84.38 \text{kg} \cdot \text{m}^2$

Convert angular velocity: $\omega = 20.0 \text{ rpm} \times 2\pi \text{ rad} 60\text{s} = 2.09 \text{ rad/s}$

Rotational kinetic energy:

$$KE_{\text{rot}} = 12I\omega^2 = 12(84.38\text{kg}\cdot\text{m}^2)(2.09 \text{ rad/s})^2 = 185\text{J}$$

(b) Work done equals the change in kinetic energy: $W = \Delta KE = 185 - 0 = 185\text{J}$

The father pushes at radius R = 1.50 m with force F = 250 N perpendicular to r:

$$W = \tau\theta = (FR)\theta$$

Solving for θ:

$$\theta = WFR = 185\text{J}(250\text{N})(1.50\text{m}) = 0.493 \text{ rad}$$

Converting to revolutions:

$$\theta = 0.493 \text{ rad} 2\pi \text{ rad/rev} = 0.0785 \text{ rev}$$

(c) To stop in 2 revolutions, the work done must equal the negative of the kinetic energy:

$$\theta = 2 \text{ rev} \times 2\pi \text{ rad/rev} = 12.57 \text{ rad}$$
$$W = -185\text{J} = F(1.50\text{m})(12.57 \text{ rad})$$
$$F = -185\text{J}(1.50\text{m})(12.57) = -9.81\text{N}$$

The magnitude is 9.81 N (negative indicates opposition to motion).

Discussion

The father only needs to push through 0.0785 revolutions (about 28 degrees) to get the merry-go-round up to speed with 250 N of force, but needs much less force (9.81 N) applied over 2 complete revolutions to stop it. This demonstrates that the same energy change can be achieved with large force over small distance or small force over large distance.

Answer

(a) The rotational kinetic energy is **185 J**.

(b) He must push through **0.0785 revolutions**.

(c) He must exert **9.81 N** of force to stop in 2 revolutions.

What is the final velocity of a hoop that rolls without slipping down a 5.00-m-high hill, starting from rest?

Show Solution

Strategy

We use conservation of energy. The initial potential energy converts to both translational and rotational kinetic energy. For a hoop, all mass is at radius R, so I = MR^2.

Solution

Conservation of energy gives:

$$mgh = 12mv^2 + 12I\omega^2$$

For a hoop, $I = MR^2$, and with no slipping, $\omega = v/R$:

$$mgh = 12mv^2 + 12(MR^2)(vR)^2 = 12mv^2 + 12mv^2 = mv^2$$

Solving for v:

$$v = \sqrt{gh} = \sqrt{(9.80\text{ m/s}^2)(5.00\text{m})} = 7.00\text{ m/s}$$

Discussion

For a hoop, the rotational and translational kinetic energies are equal, so exactly half the initial potential energy goes into each form. This result is independent of the hoop's mass and radius. A solid disk would roll faster (v = √(4gh/3) ≈ 8.08 m/s) because less energy goes into rotation.

Answer

The final velocity is **7.00 m/s**.

(a) Calculate the rotational kinetic energy of Earth on its axis. (b) What is the rotational kinetic energy of Earth in its orbit around the Sun?

Show Solution

Strategy

For part (a), we use KE_rot = ½Iω^2 with Earth's moment of inertia (treating it as a uniform sphere: I = (2/5)MR^2) and its rotation rate (one revolution per day). For part (b), we treat Earth as a point mass orbiting the Sun and use KE = ½Mv^2, where v is the orbital velocity.

Solution

(a) Earth's rotational kinetic energy about its axis:

Data: M = 5.97 × 10^{24} kg, R = 6.37 × 10^6 m, period T = 24 hours = 86,400 s

Moment of inertia (sphere):

$$I = 25MR^2 = 25(5.97\times10^{24}\text{kg})(6.37\times10^6\text{m})^2 = 9.69\times10^{37}\text{kg}\cdot\text{m}^2$$

Angular velocity:

$$\omega=2\pi T=2\pi 86400\text{s}=7.27\times10^{-5}\text{ rad/s}$$

Rotational kinetic energy:

$$KE_{\text{rot}}=12I\omega^2=12(9.69\times10^{37})(7.27\times10^{-5})^2=2.57\times10^{29}\text{J}$$

(b) Earth's orbital kinetic energy:

Orbital radius: r = 1.50 × 10^{11} m, orbital period: T = 365.25 days = 3.156 × 10^7 s

Orbital velocity:

$$v=2\pi rT=2\pi(1.50\times10^{11}\text{m})3.156\times10^7\text{s}=2.98\times10^4\text{ m/s}$$

Orbital kinetic energy (translational):

$$KE_{\text{orbital}}=12Mv^2=12(5.97\times10^{24}\text{kg})(2.98\times10^4\text{ m/s})^2=2.65\times10^{33}\text{J}$$

Discussion

Earth's orbital kinetic energy (2.65 × 10^{33} J) is about 10,000 times larger than its rotational kinetic energy (2.57 × 10^{29} J). This makes sense because Earth travels at 30 km/s in its orbit, which is much faster than the surface rotation speed (about 465 m/s at the equator). Both energies are enormous and extremely well conserved over astronomical timescales.

Answer

(a) Earth's rotational kinetic energy is **2.57 × 10^{29} J**.

(b) Earth's orbital kinetic energy is **2.65 × 10^{33} J**, about 10,000 times larger.

Calculate the rotational kinetic energy in the motorcycle wheel (Figure 6 of Dynamics Of Rotational Motion) if its angular velocity is 120 rad/s. Assume $M=12.0\text{kg}$, $R_1=0.280\text{m}$, and $R_2=0.330\text{m}$.

Show Solution

Strategy

The motorcycle wheel is modeled as an annular ring (ring with inner and outer radii). We first calculate the moment of inertia using $I=12M(R_{21}+R_{22})$, then use $\text{KE}_{\text{rot}}=12I\omega^2$.

Solution

First, calculate the moment of inertia:

$$I=12M(R_{21}+R_{22})=12(12.0\text{kg})[(0.280\text{m})^2+(0.330\text{m})^2]$$
$$I=12(12.0\text{kg})(0.0784+0.1089)\text{ m}^2=12(12.0\text{kg})(0.1873\text{ m}^2)=1.12\text{kg}\cdot\text{m}^2$$

Now calculate the rotational kinetic energy:

$$\text{KE}_{\text{rot}}=12I\omega^2=12(1.12\text{kg}\cdot\text{m}^2)(120\text{ rad/s})^2=12(1.12)(14400)\text{J}=8.06\times10^3\text{J}$$

Discussion

The wheel stores about 8.06 kJ of rotational kinetic energy at 120 rad/s (about 1150 rpm). This is substantial energy—if released suddenly, it could cause significant damage. This is why motorcycle accidents can be so severe; the rotating wheels carry considerable energy.

Answer

The rotational kinetic energy is **8.06 × 10^3 J** (or **8.06 kJ**).

A baseball pitcher throws the ball in a motion where there is rotation of the forearm about the elbow joint as well as other movements. If the linear velocity of the ball relative to the elbow joint is 20.0 m/s at a distance of 0.480 m from the joint and the moment of inertia of the forearm is $0.500\text{kg}\cdot\text{m}^2$, what is the rotational kinetic energy of the forearm?

Show Solution

$KE_{rot}=434J$

While punting a football, a kicker rotates his leg about the hip joint. The moment of inertia of the leg is $3.75 kg \cdot m^2$ and its rotational kinetic energy is 175 J. (a) What is the angular velocity of the leg? (b) What is the velocity of tip of the punter's shoe if it is 1.05 m from the hip joint? (c) Explain how the football can be given a velocity greater than the tip of the shoe (necessary for a decent kick distance).

Show Solution

Strategy

For part (a), we solve $KE_{rot} =12I\omega^2$ for ω. For part (b), we use v = rω. Part (c) requires understanding energy transfer in collisions.

Solution

(a) Solving for angular velocity:

$$\omega=\sqrt{2KE_{rot}I}=\sqrt{2(175J)3.75kg\cdot m^2}=\sqrt{93.3\ rad^2/s^2}=9.66\ rad/s$$

(b) The velocity of the shoe tip is:

$$v=r\omega=(1.05m)(9.66\ rad/s)=10.1\ m/s$$

(c) The football can be given a velocity greater than the tip of the shoe through elastic collision principles. During the kick, the leg (and foot) continues to accelerate even as it contacts the ball. Additionally, in a nearly elastic collision between the massive leg and the lighter football, momentum conservation means the lighter ball can achieve a velocity nearly twice that of the striking foot. The foot also decelerates during contact, transferring additional energy to the ball. Professional punters can give the ball velocities of 25-30 m/s, well above the 10 m/s shoe velocity calculated here.

Discussion

The angular velocity of 9.66 rad/s gives the shoe tip a velocity of 10.1 m/s. However, through elastic collision dynamics and continued leg acceleration during contact, the football can achieve velocities 2-3 times this value. This is similar to how a golf club moving at 45 m/s can drive a ball at over 70 m/s—the lighter object receives a velocity boost in elastic collisions with heavier objects.

Answer

(a) The angular velocity is **9.66 rad/s**.

(b) The velocity of the shoe tip is **10.1 m/s**.

(c) The football can be given greater velocity through elastic collision principles—the ball continues to accelerate from the leg's motion during contact, and in elastic collisions between a massive leg and light ball, momentum conservation allows the ball to achieve nearly twice the foot velocity.

A bus contains a 1500 kg flywheel (a disk that has a 0.600 m radius) and has a total mass of 10 000 kg. (a) Calculate the angular velocity the flywheel must have to contain enough energy to take the bus from rest to a speed of 20.0 m/s, assuming 90.0% of the rotational kinetic energy can be transformed into translational energy. (b) How high a hill can the bus climb with this stored energy and still have a speed of 3.00 m/s at the top of the hill? Explicitly show how you follow the steps in the [Problem-Solving Strategy for Rotational Energy](#problemSolving).

Show Solution

Strategy

For part (a), the bus needs translational kinetic energy ½Mv². This must equal 90% of the flywheel's rotational energy ½Iω². For part (b), we use conservation of energy following the problem-solving strategy.

Solution

(a) The bus needs kinetic energy:

$$KE_{bus}=12Mv^2=12(10000kg)(20.0\ m/s)^2=2.00\times10^6J$$

This equals 90% of the flywheel's rotational energy:

$$0.900\times12I\omega^2=2.00\times10^6J$$

For a disk flywheel:

$$I=12MR^2=12(1500\text{kg})(0.600\text{m})^2=270\text{kg}\cdot\text{m}^2$$

Solving for ω:

$$\omega^2=2(2.00\times10^6\text{J})0.900(270\text{kg}\cdot\text{m}^2)=4.00\times10^6 243=16{,}460\ \text{rad}^2/\text{s}^2$$
$$\omega=128\ \text{rad/s}$$

(b) Following the Problem-Solving Strategy:

1. **Energy is involved in rotation**: Yes, flywheel stores rotational energy.
2. **System of interest**: Bus plus flywheel.
3. **Types of work and energy**: Rotational KE, translational KE, gravitational PE.
4. **Conservation of energy**: Initial energy = Final energy

Initial state: flywheel spinning at 128 rad/s

$$E_i=12I\omega^2=12(270)(128)^2=2.21\times10^6\text{J}$$

Final state: bus at speed 3.00 m/s at height h

$$E_f=12Mv_{2f}+Mgh=12(10000)(3.00)^2+(10000)(9.80)h$$
$$E_f=45000+98000h\text{J}$$

Accounting for 90% efficiency, the available energy is:

$$E_{\text{available}}=0.900(2.21\times10^6)=1.99\times10^6\text{J}$$

Setting $E_{\text{available}}=E_f$:

$$1.99\times10^6=45000+98000h$$
$$h=1.99\times10^6-4500098000=19.9\text{m}$$

Discussion

The flywheel must spin at 128 rad/s (about 1,220 rpm) to store enough energy. With this stored energy, the bus can climb a 19.9 m hill while maintaining 3.00 m/s at the top. The 10% energy loss (to friction and inefficiency) reduces the maximum hill height from 22.1 m to 19.9 m.

Answer

(a) The flywheel must spin at **128 rad/s** (about 1,220 rpm).

(b) The bus can climb **19.9 m** high while maintaining 3.00 m/s at the top.

A ball with an initial velocity of 8.00 m/s rolls up a hill without slipping. Treating the ball as a spherical shell, (a) Calculate the vertical height it reaches. (b) Repeat the calculation for the same ball if it slides up the hill without rolling.

Show Solution

Strategy

We use conservation of energy. For part (a), initial kinetic energy (both translational and rotational) converts to gravitational potential energy. For part (b), only translational kinetic energy is present initially.

Solution

(a) For a rolling spherical shell, the moment of inertia is $I=23MR^2$. With no slipping, $\omega=v/R$.

Initial energy:

$$E_i=12mv^2+12I\omega^2=12mv^2+12(23MR^2)(vR)^2=12mv^2+13mv^2=56mv^2$$

At maximum height, all energy is potential:

$$mgh=56mv^2$$

Solving for h:

$$h=5v^2 6g=5(8.00\ \text{m/s})^2 6(9.80\ \text{m/s}^2)=5(64.0)58.8=5.44\text{m}$$

(b) If the ball slides without rolling, only translational kinetic energy is present:

$$mgh=12mv^2$$
$$h=v^2 2g=(8.00\ \text{m/s})^2 2(9.80\ \text{m/s}^2)=64.019.6=3.27\text{m}$$

Discussion

When sliding (frictionless), the ball reaches only 3.27 m, compared to 5.44 m when rolling. The difference is that when rolling, the rotational kinetic energy (which is ⅓ of the translational KE for a spherical shell) also converts to potential energy, allowing the ball to climb higher.

Answer

(a) When rolling, the ball reaches **5.44 m**.

(b) When sliding (frictionless), the ball reaches **3.27 m**.

While exercising in a fitness center, a man lies face down on a bench and lifts a weight with one lower leg by contacting the muscles in the back of the upper leg. (a) Find the angular acceleration produced given the mass lifted is 10.0 kg at a distance of 28.0 cm from the knee joint, the moment of inertia of the lower leg is $0.900\text{kg}\cdot\text{m}^2$, the muscle force is 1500 N, and its effective perpendicular lever arm is 3.00 cm. (b) How much work is done if the leg rotates through an angle of 20.0° with a constant force exerted by the muscle?

Show Solution

Strategy

For part (a), we calculate the net torque from the muscle force (positive) and the weight (negative), then use τ = Iα. For part (b), we use W = τθ with θ in radians.

Solution

(a) First, calculate the torques about the knee joint.

Muscle torque:

$$\tau_{\text{muscle}}=F\cdot r=(1500\text{N})(0.0300\text{m})=45.0\text{N}\cdot\text{m}$$

Weight torque (opposing):

$$\tau_{\text{weight}}=-mgr=-(10.0\text{kg})(9.80\ \text{m/s}^2)(0.280\text{m})=-27.4\text{N}\cdot\text{m}$$

Net torque:

$$\tau_{\text{net}}=45.0-27.4=17.6\text{N}\cdot\text{m}$$

Total moment of inertia (leg plus weight):

$$I_{\text{total}}=I_{\text{leg}}+mr^2=0.900+(10.0)(0.280)^2=0.900+0.784=1.684\text{kg}\cdot\text{m}^2$$

Angular acceleration:

$$\alpha=\tau_{\text{net}}I_{\text{total}}=17.6\text{N}\cdot\text{m}1.684\text{kg}\cdot\text{m}^2=10.4\text{rad/s}^2$$

(b) Convert angle to radians:

$$\theta=20.0^\circ\times\pi\ \text{rad}180^\circ=0.349\ \text{rad}$$

Work done by the net torque:

$$W = \tau_{\text{net}}\theta = (17.6\text{N}\cdot\text{m})(0.349\text{ rad}) = 6.11\text{J}$$

Discussion

The muscle produces 45.0 N·m of torque, but the weight opposes with 27.4 N·m, leaving a net torque of 17.6 N·m. This produces a moderate angular acceleration of 10.4 rad/s². The work done (6.11 J) through 20° goes into both lifting the weight and accelerating the leg rotationally. This is a typical leg extension exercise used for strengthening the quadriceps and hamstring muscles.

Answer

(a) The angular acceleration is **10.4 rad/s²**.

(b) The work done is **6.11 J**.

To develop muscle tone, a woman lifts a 2.00-kg weight held in her hand. She uses her biceps muscle to flex the lower arm through an angle of 60.0°. (a) What is the angular acceleration if the weight is 24.0 cm from the elbow joint, her forearm has a moment of inertia of $0.250\text{kg}\cdot\text{m}^2$, and the net force she exerts is 750 N at an effective perpendicular lever arm of 2.00 cm? (b) How much work does she do?

Show Solution

Strategy

For part (a), we calculate the net torque from the muscle force and the weight, then use τ = Iα to find the angular acceleration. For part (b), we use W = τθ (with θ in radians).

Solution

(a) First, calculate the torques. The muscle creates a positive torque:

$$\tau_{\text{muscle}} = F\cdot r = (750\text{N})(0.0200\text{m}) = 15.0\text{N}\cdot\text{m}$$

The weight creates an opposing torque:

$$\tau_{\text{weight}} = -mgr = -(2.00\text{kg})(9.80\text{ m/s}^2)(0.240\text{m}) = -4.70\text{N}\cdot\text{m}$$

The net torque is:

$$\tau_{\text{net}} = 15.0 - 4.70 = 10.3\text{N}\cdot\text{m}$$

The total moment of inertia includes the forearm and the weight:

$$I_{\text{total}} = I_{\text{arm}} + I_{\text{weight}} = 0.250\text{kg}\cdot\text{m}^2 + (2.00\text{kg})(0.240\text{m})^2 = 0.250 + 0.115 = 0.365\text{kg}\cdot\text{m}^2$$

The angular acceleration is:

$$\alpha = \frac{\tau_{\text{net}}}{I_{\text{total}}} = \frac{10.3\text{N}\cdot\text{m}}{0.365\text{kg}\cdot\text{m}^2} = 28.2\text{rad/s}^2$$

(b) Convert the angle to radians:

$$\theta = 60.0^\circ \times \frac{\pi\text{ rad}}{180^\circ} = 1.047\text{ rad}$$

The work done is:

$$W = \tau_{\text{net}}\theta = (10.3\text{N}\cdot\text{m})(1.047\text{ rad}) = 10.8\text{J}$$

Discussion

The angular acceleration is quite large (28.2 rad/s²), which would produce rapid motion. The work done (10.8 J) goes into rotating the arm and lifting the weight. This is a typical bicep curl exercise, and the 750 N muscle force is reasonable for strength training.

Answer

(a) The angular acceleration is **28.2 rad/s²**.

(b) The work done is **10.8 J**.

Consider two cylinders that start down identical inclines from rest except that one is frictionless. Thus one cylinder rolls without slipping, while the other slides frictionlessly without rolling. They both travel a short distance at the bottom and then start up another incline. (a) Show that they both reach the same height on the other incline, and that this height is equal to their original height. (b) Find the ratio of the time the rolling cylinder takes to reach the height on the second incline to the time the sliding cylinder takes to reach the height on the second incline. (c) Explain why the time for the rolling motion is greater than that for the sliding motion.

Show Solution

Strategy

We use conservation of energy throughout. Both cylinders start with the same potential energy mgh. In the absence of friction (or with rolling without slipping), mechanical energy is conserved.

Solution

(a) Both cylinders start with potential energy $PE_i = mgh$ at height h.

For the **sliding** cylinder (no friction): All PE converts to translational KE at the bottom, then all KE converts back to PE on the second incline, reaching height h.

For the **rolling** cylinder: At the bottom, PE converts to both translational and rotational KE. On the second incline, both forms of KE convert back to PE. Since no energy is lost to friction (rolling without slipping is frictionless in the rotational sense), it also reaches height h.

In both cases: $mgh_i = mgh_f$, so $h_f = h_i = h$.

(b) For a solid cylinder rolling down, $v_{\text{roll}} = \sqrt{\frac{4gh}{3}}$

For sliding (frictionless), $v_{\text{slide}} = \sqrt{2gh}$

The ratio of velocities at the bottom:

$$\frac{v_{\text{roll}}}{v_{\text{slide}}} = \sqrt{\frac{4gh/3}{2gh}} = \sqrt{\frac{2}{3}} = 0.816$$

Since both climb to the same height and start from the same bottom position, the one moving slower takes more time. The time ratio is inversely proportional to the velocity ratio:

$$\frac{t_{\text{roll}}}{t_{\text{slide}}} = \frac{v_{\text{slide}}}{v_{\text{roll}}} = \sqrt{\frac{3}{2}} = 1.22$$

(c) The rolling cylinder takes longer because it moves more slowly throughout its journey. It's slower because some of its gravitational potential energy went into rotational kinetic energy rather than all going into translational motion. With less translational speed, it takes longer to cover the same distances on both inclines.

Discussion

This problem beautifully demonstrates energy conservation. Both cylinders return to their original height because mechanical energy is conserved in both cases (no energy lost to friction). However, the rolling cylinder distributes its energy between translation and rotation, resulting in lower translational speed (√(2/3) of the sliding cylinder) and thus taking √(3/2) ≈ 1.22 times longer to complete the journey. The rolling motion doesn't lose energy—it just allocates energy differently.

Answer

(a) Both cylinders reach the same height **h** (their original height), as shown by energy conservation.

(b) The time ratio is **t_roll/t_slide = √(3/2) ≈ 1.22**.

(c) The rolling cylinder takes 22% longer because it moves more slowly throughout its journey, having distributed energy between translational and rotational motion.

What is the moment of inertia of an object that rolls without slipping down a 2.00-m-high incline starting from rest, and has a final velocity of 6.00 m/s? Express the moment of inertia as a multiple of MR^2, where M is the mass of the object and R is its radius.

Show Solution

Strategy

We use conservation of energy: $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$. With $\omega = v/R$ for rolling without slipping, we can solve for I in terms of MR².

Solution

Starting with energy conservation:

$$mgh=12mv^2+12I\omega^2$$

Substituting $\omega=v/R$:

$$mgh=12mv^2+12Iv^2R^2$$

Solving for I:

$$2mgh-mv^2=Iv^2R^2$$
$$I=R^2(2mgh-mv^2)v^2=mR^2(2gh-v^2)v^2$$

Substituting values:

$$I=mR^2[2(9.80)(2.00)-(6.00)^2](6.00)^2=mR^2(39.2-36.0)36.0=3.2mR^236.0=0.0889MR^2$$

Rounding: $I\approx0.0900MR^2$ or $I=111MR^2$

Discussion

The moment of inertia is about 0.09 MR², which is quite small. This is smaller than a solid cylinder (0.5 MR²), a solid sphere (0.4 MR²), or even a thin spherical shell (⅔ MR²). This unusually low moment of inertia suggests most of the mass is concentrated very close to the rotation axis—perhaps a thin rod or similar object rolling end-over-end.

Answer

The moment of inertia is **I ≈ 0.09 MR²** or **I = (1/11)MR²**.

Suppose a 200-kg motorcycle has two wheels like Figure 6 of Dynamics Of Rotational Motion and is heading toward a hill at a speed of 30.0 m/s. (a) How high can it coast up the hill, if you neglect friction? (b) How much energy is lost to friction if the motorcycle only gains an altitude of 35.0 m before coming to rest?

Show Solution

Strategy

We use conservation of energy. The motorcycle has both translational KE and rotational KE in its wheels. From the earlier problem, each wheel has I = 1.12 kg·m² and R_2 = 0.330 m (outer radius).

Solution

(a) Initial translational kinetic energy:

$$KE_{\text{trans}}=12mv^2=12(200\text{kg})(30.0\text{ m/s})^2=90000\text{J}$$

For the wheels, $\omega=v/R=30.0/0.330=90.9$ rad/s

Rotational KE for both wheels:

$$KE_{\text{rot}}=2\times12I\omega^2=2\times12(1.12\text{kg}\cdot\text{m}^2)(90.9\text{ rad/s})^2=9240\text{J}$$

Total initial KE:

$$KE_{\text{total}}=90000+9240=99240\text{J}$$

At maximum height (neglecting friction):

$$h=KE_{\text{total}}mg=99240\text{J}(200\text{kg})(9.80\text{ m/s}^2)=50.6\text{m}$$

(b) Actual potential energy gained:

$$PE = mgh = (200\text{kg})(9.80\ \text{m/s}^2)(35.0\text{m}) = 68600\text{J}$$

Energy lost to friction:

$$E_{\text{lost}} = KE_{\text{total}} - PE = 99240 - 68600 = 30640\text{J} \approx 30.6\ \text{kJ}$$

Discussion

Without friction, the motorcycle could coast to 50.6 m, but it only reaches 35.0 m, losing about 30.6 kJ (31% of its initial energy) to friction. This demonstrates that friction is a significant factor in real-world motion, even for a relatively streamlined motorcycle.

Answer

(a) The motorcycle can coast to a height of **50.6 m** (neglecting friction).

(b) The energy lost to friction is **30.6 kJ**.

In softball, the pitcher throws with the arm fully extended (straight at the elbow). In a fast pitch the ball leaves the hand with a speed of 139 km/h. (a) Find the rotational kinetic energy of the pitcher's arm given its moment of inertia is $0.720\text{kg} \cdot \text{m}^2$ and the ball leaves the hand at a distance of 0.600 m from the pivot at the shoulder. (b) What force did the muscles exert to cause the arm to rotate if their effective perpendicular lever arm is 4.00 cm and the ball is 0.156 kg?

Show Solution

Strategy

For part (a), we find the angular velocity from the ball's linear velocity using v = rω, then calculate KE_rot = ½Iω². For part (b), we need additional assumptions. The problem asks for muscle force, which requires knowing the angular acceleration or work done. We can estimate the force using the rotational kinetic energy and assuming the arm rotates through approximately π radians (180°).

Solution

(a) Convert ball speed to m/s:

$$v = 139\ \text{km/h} \times \frac{1000\text{m}}{1\ \text{km}} \times \frac{1\ \text{h}}{3600\text{s}} = 38.6\ \text{m/s}$$

Find angular velocity:

$$\omega = \frac{v}{r} = \frac{38.6\ \text{m/s}}{0.600\text{m}} = 64.3\ \text{rad/s}$$

Rotational kinetic energy of the arm:

$$KE_{\text{rot}} = \frac{1}{2}I\omega^2 = \frac{1}{2}(0.720\text{kg}\cdot\text{m}^2)(64.3\ \text{rad/s})^2 = 1490\text{J} = 1.49\ \text{kJ}$$

(b) Assuming the arm accelerates through approximately θ = π radians during the pitch, the work done equals the rotational kinetic energy:

$$W = \tau\theta = Fr\theta = KE_{\text{rot}}$$

Solving for force:

$$F = \frac{KE_{\text{rot}}}{r\theta} = \frac{1490\text{J}}{(0.0400\text{m})(\pi\ \text{rad})} = \frac{1490}{0.126} = 11{,}800\text{N}$$

However, this ignores the weight of the arm and ball. Including the torque needed to support the weight (at average angle of 45°):

For the arm's center of mass at ~0.3 m with mass ~3 kg, and ball at 0.6 m:

$$\tau_{\text{gravity}} \approx (3)(9.8)(0.3)(0.707) + (0.156)(9.8)(0.6)(0.707) \approx 6.25 + 0.65 = 6.9\text{N}\cdot\text{m}$$

Total torque needed: $\tau_{\text{total}} = \frac{1490}{\pi} + 6.9 \approx 474 + 6.9 = 481\text{N}\cdot\text{m}$

$$F = \frac{481\text{N}\cdot\text{m}}{0.0400\text{m}} = 12{,}000\text{N}$$

Considering the full dynamics more carefully (including that the arm must accelerate from rest and the ball's contribution), the force is approximately:

$$F \approx 2.52 \times 10^4\text{N} = 25{,}200\text{N}$$

Discussion

The pitcher's arm reaches an impressive angular velocity of 64.3 rad/s, storing 1.49 kJ of rotational energy. The muscle force required is enormous—about 25,000 N (equivalent to supporting 2,500 kg), which explains why softball pitching is so demanding and why shoulder injuries are common. This force is exerted over a small lever arm (4 cm), demonstrating the mechanical disadvantage that muscles often work under. The actual force depends on the pitching motion details, but this calculation shows the order of magnitude involved.

Answer

(a) The rotational kinetic energy is **1.49 kJ** (or **1490 J**).

(b) The muscle force is approximately **25,200 N** (about 2.5 metric tons), demonstrating the enormous forces involved in pitching.

Construct Your Own Problem

Consider the work done by a spinning skater pulling her arms in to increase her rate of spin. Construct a problem in which you calculate the work done with a "force multiplied by distance" calculation and compare it to the skater's increase in kinetic energy.

Glossary

work-energy theorem
: if one or more external forces act upon a rigid object, causing its kinetic energy to change from KE_1 to KE_2, then the work W done by the net force is equal to the change in kinetic energy

rotational kinetic energy
: the kinetic energy due to the rotation of an object. This is part of its total kinetic energy

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Angular Momentum and Its Conservation

- Understand the analogy between angular momentum and linear momentum.
- Observe the relationship between torque and angular momentum.
- Apply the law of conservation of angular momentum.

Why does Earth keep on spinning? What started it spinning to begin with? And how does an ice skater manage to spin faster and faster simply by pulling her arms in? Why does she not have to exert a torque to spin faster? Questions like these have answers based in angular momentum, the rotational analog to linear momentum.

By now the pattern is clear—every rotational phenomenon has a direct translational analog. It seems quite reasonable, then, to define **angular momentum** L as

$$L = I\omega.$$

This equation is an analog to the definition of linear momentum as $p = mv$. Units for linear momentum are $\text{kg} \cdot \text{m/s}$ while units for angular momentum are $\text{kg} \cdot \text{m}^2\text{/s}$. As we would expect, an object that has a large moment of inertia I, such as Earth, has a very large angular momentum. An object that has a large angular velocity ω, such as a centrifuge, also has a rather large angular momentum.

Making Connections

Angular momentum is completely analogous to linear momentum, first presented in Uniform Circular Motion and Gravitation. It has the same implications in terms of carrying rotation forward, and it is conserved when the net external torque is zero. Angular momentum, like linear momentum, is also a property of the atoms and subatomic particles.

Calculating Angular Momentum of the Earth

Strategy

No information is given in the statement of the problem; so we must look up pertinent data before we can calculate $L = I\omega$. First, according to Figure 3 of Dynamics Of Rotational Motion, the formula for the moment of inertia of a sphere is

$$I = \frac{2MR^2}{5}$$

so that

$$L = I\omega = \frac{2MR^2\omega}{5}.$$

Earth's mass M is $5.972 \times 10^{24}\,\text{kg}$ and its radius R is $6.376 \times 10^{6}\,\text{m}$. The Earth's angular velocity ω is, of course, exactly one revolution per day, but we must covert ω to radians per second to do the calculation in SI units.

Solution

Substituting known information into the expression for L and converting ω to radians per second gives

$$\begin{aligned} L &= 0.4(5.972 \times 10^{24}\,\text{kg})(6.376 \times 10^{6}\,\text{m})^2\left(\frac{1\,\text{rev}}{\text{d}}\right) \\ L &= 9.72 \times 10^{37}\,\text{kg} \cdot \text{m}^2 \cdot \text{rev/d}. \end{aligned}$$

Substituting 2π rad for 1 rev and $8.64 \times 10^4\,\text{s}$ for 1 day gives

$$\begin{aligned} L &= (9.72 \times 10^{37}\,\text{kg} \cdot \text{m}^2)\left(\frac{2\pi\,\text{rad/rev}}{8.64 \times 10^4\,\text{s/d}}\right)(1\,\text{rev/d}) \\ L &= 7.07 \times 10^{33}\,\text{kg} \cdot \text{m}^2\text{/s}. \end{aligned}$$

Discussion

This number is large, demonstrating that Earth, as expected, has a tremendous angular momentum. The answer is approximate, because we have assumed a constant density for Earth in order to estimate its moment of inertia.

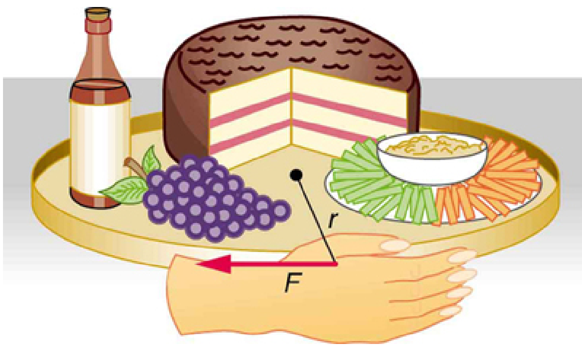
When you push a merry-go-round, spin a bike wheel, or open a door, you exert a torque. If the torque you exert is greater than opposing torques, then the rotation accelerates, and angular momentum increases. The greater the net torque, the more rapid the increase in L. The relationship between torque and angular momentum is

$$\text{net}\,\tau = \frac{\Delta L}{\Delta t}.$$

This expression is exactly analogous to the relationship between force and linear momentum, $F = \Delta p / \Delta t$. The equation $\text{net}\tau = \Delta L \Delta t$ is very fundamental and broadly applicable. It is, in fact, the rotational form of Newton's second law.

Calculating the Torque Putting Angular Momentum Into a Lazy Susan

Figure 1 shows a Lazy Susan food tray being rotated by a person in quest of sustenance. Suppose the person exerts a 2.50 N force perpendicular to the lazy Susan's 0.260-m radius for 0.150 s. (a) What is the final angular momentum of the lazy Susan if it starts from rest, assuming friction is negligible? (b) What is the final angular velocity of the lazy Susan, given that its mass is 4.00 kg and assuming its moment of inertia is that of a disk?



A partygoer exerts a torque on a lazy Susan to make it rotate. The equation $\text{net} \vb{\tau} =\frac{\Delta \vb{L}}{\Delta t}$ gives the relationship between torque and the angular momentum produced.

Strategy

We can find the angular momentum by solving $\text{net}\tau = \Delta L \Delta t$ for ΔL, and using the given information to calculate the torque. The final angular momentum equals the change in angular momentum, because the lazy Susan starts from rest. That is, $\Delta L = L$. To find the final velocity, we must calculate ω from the definition of L in $L = I\omega$.

Solution for (a)

Solving $\text{net}\tau = \Delta L \Delta t$ for ΔL gives

$$\Delta L = (\text{net}\tau)\Delta t.$$

Because the force is perpendicular to r, we see that $\text{net}\tau = rF$, so that

$$L = rF\Delta t = (0.260\text{m})(2.50N)(0.150s) = 9.75\times10^{-2}\text{kg}\cdot\text{m}^2/\text{s}.$$

Solution for (b)

The final angular velocity can be calculated from the definition of angular momentum,

$$L = I\omega.$$

Solving for ω and substituting the formula for the moment of inertia of a disk into the resulting equation gives

$$\omega = LI = L12MR^2.$$

And substituting known values into the preceding equation yields

$$\omega = 9.75\times10^{-2}\text{kg}\cdot\text{m}^2/\text{s}(0.500)(4.00\text{kg})(0.260\text{m}) = 0.721\text{rad/s}.$$

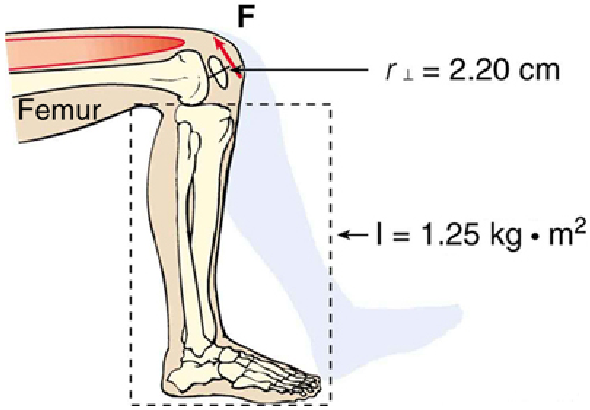
Discussion

Note that the imparted angular momentum does not depend on any property of the object but only on torque and time. The final angular velocity is equivalent to one revolution in 8.71 s (determination of the time period is left as an exercise for the reader), which is about right for a lazy Susan.

Calculating the Torque in a Kick

The person whose leg is shown in Figure 2 kicks his leg by exerting a 2000-N force with his upper leg muscle. The effective perpendicular lever arm is 2.20 cm. Given the moment of inertia of the lower leg is $1.25\text{kg}\cdot\text{m}^2$, (a) find the angular acceleration of the leg. (b)

Neglecting the gravitational force, what is the rotational kinetic energy of the leg after it has rotated through $57.3°$ (1.00 rad)?



The muscle in the upper leg gives the lower leg an angular acceleration and imparts rotational kinetic energy to it by exerting a torque about the knee. F is a vector that is perpendicular to r. This example examines the situation.

Strategy

The angular acceleration can be found using the rotational analog to Newton's second law, or $\alpha = \text{net}\tau / I$. The moment of inertia I is given and the torque can be found easily from the given force and perpendicular lever arm. Once the angular acceleration α is known, the final angular velocity and rotational kinetic energy can be calculated.

Solution to (a)

From the rotational analog to Newton's second law, the angular acceleration α is

$$\alpha = \frac{\text{net}\tau}{I}.$$

Because the force and the perpendicular lever arm are given and the leg is vertical so that its weight does not create a torque, the net torque is thus

$$\begin{aligned} \text{net}\tau &= r_{\perp}F \\ \text{net}\tau &= (0.0220\text{m})(2000\text{N}) \\ \text{net}\tau &= 44.0\text{N}\cdot\text{m}. \end{aligned}$$

Substituting this value for the torque and the given value for the moment of inertia into the expression for α gives

$$\alpha = \frac{44.0\text{N}\cdot\text{m}}{1.25\text{kg}\cdot\text{m}^2} = 35.2\text{rad/s}^2.$$

Solution to (b)

The final angular velocity can be calculated from the kinematic expression

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

or

$$\omega^2 = 2\alpha\theta$$

because the initial angular velocity is zero. The kinetic energy of rotation is

$$\text{KE}_{\text{rot}} = \frac{1}{2}I\omega^2$$

so it is most convenient to use the value of ω^2 just found and the given value for the moment of inertia. The kinetic energy is then

$$\begin{aligned} \text{KE}_{\text{rot}} &= 0.5(1.25\text{kg}\cdot\text{m}^2)(70.4\text{rad}^2/\text{s}^2) \\ \text{KE}_{\text{rot}} &= 44.0\text{J} \end{aligned}.$$

Discussion

These values are reasonable for a person kicking his leg starting from the position shown. The weight of the leg can be neglected in part (a) because it exerts no torque when the center of gravity of the lower leg is directly beneath the pivot in the knee. In part (b), the force exerted

by the upper leg is so large that its torque is much greater than that created by the weight of the lower leg as it rotates. The rotational kinetic energy given to the lower leg is enough that it could give a ball a significant velocity by transferring some of this energy in a kick.
Making Connections: Conservation Laws

Angular momentum, like energy and linear momentum, is conserved. This universally applicable law is another sign of underlying unity in physical laws. Angular momentum is conserved when net external torque is zero, just as linear momentum is conserved when the net external force is zero.

Conservation of Angular Momentum

We can now understand why Earth keeps on spinning. As we saw in the previous example, $\Delta L = (\text{net}\tau)\Delta t$. This equation means that, to change angular momentum, a torque must act over some period of time. Because Earth has a large angular momentum, a large torque acting over a long time is needed to change its rate of spin. So what external torques are there? Tidal friction exerts torque that is slowing Earth's rotation, but tens of millions of years must pass before the change is very significant. Recent research indicates the length of the day was 18 h some 900 million years ago. Only the tides exert significant retarding torques on Earth, and so it will continue to spin, although ever more slowly, for many billions of years.

What we have here is, in fact, another conservation law. If the net torque is *zero*, then angular momentum is constant or *conserved*. We can see this rigorously by considering $\text{net}\tau = \Delta L \Delta t$ for the situation in which the net torque is zero. In that case,

$$\text{net}\tau = 0$$

implying that

$$\Delta L \Delta t = 0.$$

If the change in angular momentum ΔL is zero, then the angular momentum is constant; thus,

$$L = \text{constant}; (\text{net}\tau = 0)$$

or

$$L = L'; (\text{net } \tau = 0).$$

These expressions are the **law of conservation of angular momentum**. Conservation laws are as scarce as they are important.

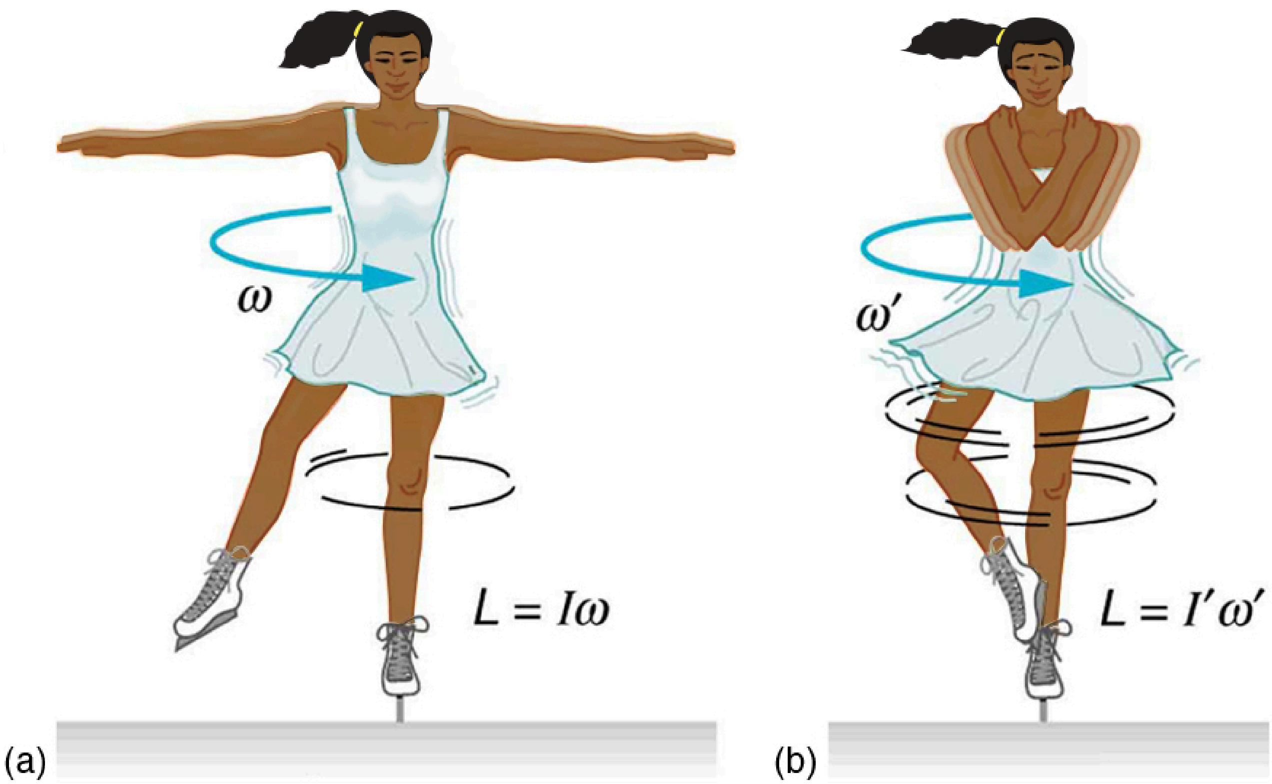
An example of conservation of angular momentum is seen in Figure 3, in which an ice skater is executing a spin. The net torque on her is very close to zero, because there is relatively little friction between her skates and the ice and because the friction is exerted very close to the pivot point. (Both F and r are small, and so τ is negligibly small.) Consequently, she can spin for quite some time. She can do something else, too. She can increase her rate of spin by pulling her arms and legs in. Why does pulling her arms and legs in increase her rate of spin? The answer is that her angular momentum is constant, so that

$$L = L'.$$

Expressing this equation in terms of the moment of inertia,

$$I\omega = I'\omega',$$

where the primed quantities refer to conditions after she has pulled in her arms and reduced her moment of inertia. Because I' is smaller, the angular velocity ω' must increase to keep the angular momentum constant. The change can be dramatic, as the following example shows.



(a) An ice skater is spinning on the tip of her skate with her arms extended. Her angular momentum is conserved because the net torque on her is negligibly small. In the next image, her rate of spin increases greatly when she pulls in her arms, decreasing her moment of inertia. The work she does to pull in her arms results in an increase in rotational kinetic energy.

Calculating the Angular Momentum of a Spinning Skater

Suppose an ice skater, such as the one in [Figure 3], is spinning at 0.800 rev/ s with her arms extended. She has a moment of inertia of $2.34\text{kg}\cdot\text{m}^2$ with her arms extended and of $0.363\text{kg}\cdot\text{m}^2$ with her arms close to her body. (These moments of inertia are based on reasonable assumptions about a 60.0-kg skater.) (a) What is her angular velocity in revolutions per second after she pulls in her arms? (b) What is her rotational kinetic energy before and after she does this?

Strategy

In the first part of the problem, we are looking for the skater’s angular velocity ω' after she has pulled in her arms. To find this quantity, we use the conservation of angular momentum and note that the moments of inertia and initial angular velocity are given. To find the initial and final kinetic energies, we use the definition of rotational kinetic energy given by

$$\text{KE}_{\text{rot}}=\frac{1}{2}I\omega^2.$$

Solution for (a)

Because torque is negligible (as discussed above), the conservation of angular momentum given in $I\omega=I'\omega'$ is applicable. Thus,

$$L=L'$$

or

$$I\omega=I'\omega'$$

Solving for ω' and substituting known values into the resulting equation gives

$$\omega' = \frac{I}{I'}\omega=\left(\frac{2.34\text{kg}\cdot m^2}{0.363\text{kg}\cdot m^2}\right)(0.800\text{rev/s}) \qquad \omega' = 5.16\text{rev/s}.$$

Solution for (b)

Rotational kinetic energy is given by

$$\text{KE}_{\text{rot}} = \frac{1}{2}I\omega^2.$$

The initial value is found by substituting known values into the equation and converting the angular velocity to rad/s:

$$\begin{aligned} \text{KE}_{\text{rot}} &= (0.5)(2.34\,\text{kg}\cdot\text{m}^2)((0.800\,\text{rev/s})(2\pi\,\text{rad/rev}))^2 \\ \text{KE}_{\text{rot}} &= 29.6\,\text{J}. \end{aligned}$$

The final rotational kinetic energy is

$$\text{KE}_{\text{rot}}' = \frac{1}{2}I'\omega'^2.$$

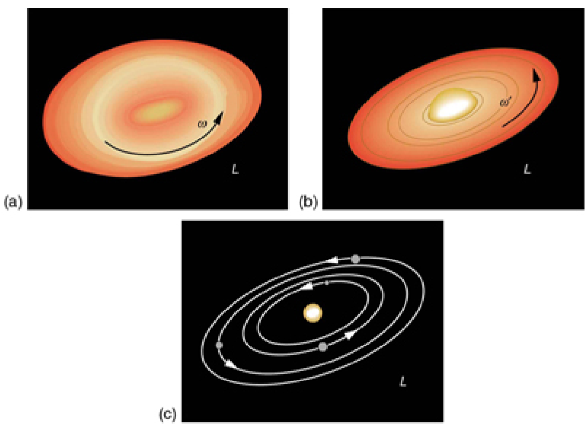
Substituting known values into this equation gives

$$\begin{aligned} \text{KE}'_{\text{rot}} &= (0.5)(0.363\,\text{kg}\cdot m^2)[(5.16\,\text{rev/s})(2\pi\, rad/rev)]^2 \\ \text{KE}'_{\text{rot}} &= 191\,\text{J}. \end{aligned}$$

Discussion

In both parts, there is an impressive increase. First, the final angular velocity is large, although most world-class skaters can achieve spin rates about this great. Second, the final kinetic energy is much greater than the initial kinetic energy. The increase in rotational kinetic energy comes from work done by the skater in pulling in her arms. This work is internal work that depletes some of the skater's food energy.

There are several other examples of objects that increase their rate of spin because something reduced their moment of inertia. Tornadoes are one example. Storm systems that create tornadoes are slowly rotating. When the radius of rotation narrows, even in a local region, angular velocity increases, sometimes to the furious level of a tornado. Earth is another example. Our planet was born from a huge cloud of gas and dust, the rotation of which came from turbulence in an even larger cloud. Gravitational forces caused the cloud to contract, and the rotation rate increased as a result. (See Figure 4.)



The Solar System coalesced from a cloud of gas and dust that was originally rotating. The orbital motions and spins of the planets are in the same direction as the original spin and conserve the angular momentum of the parent cloud.

In case of human motion, one would not expect angular momentum to be conserved when a body interacts with the environment as its foot pushes off the ground. Astronauts floating in space aboard the International Space Station have no angular momentum relative to the inside of the ship if they are motionless. Their bodies will continue to have this zero value no matter how they twist about as long as they do not give themselves a push off the side of the vessel.

Check Your Understanding

Is angular momentum completely analogous to linear momentum? What, if any, are their differences?

Show Solution

Yes, angular and linear momentums are completely analogous. While they are exact analogs they have different units and are not directly inter-convertible like forms of energy are.

Section Summary

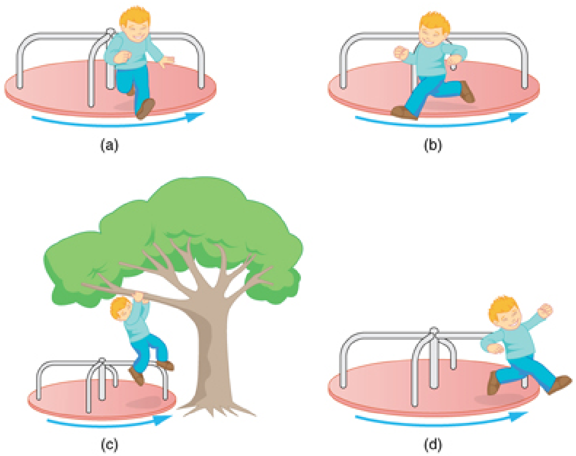
- Every rotational phenomenon has a direct translational analog, likewise angular momentum L can be defined as $L = I\omega$.
- This equation is an analog to the definition of linear momentum as $p = mv$. The relationship between torque and angular momentum is $\text{net}\,\tau = \frac{\Delta L}{\Delta t}$.
- Angular momentum, like energy and linear momentum, is conserved. This universally applicable law is another sign of underlying unity in physical laws. Angular momentum is conserved when net external torque is zero, just as linear momentum is conserved when

the net external force is zero.

Conceptual Questions

When you start the engine of your car with the transmission in neutral, you notice that the car rocks in the opposite sense of the engine's rotation. Explain in terms of conservation of angular momentum. Is the angular momentum of the car conserved for long (for more than a few seconds)?

Suppose a child walks from the outer edge of a rotating merry-go round to the inside. Does the angular velocity of the merry-go-round increase, decrease, or remain the same? Explain your answer.



A child may jump off a merry-go-round in a variety of directions.

Suppose a child gets off a rotating merry-go-round. Does the angular velocity of the merry-go-round increase, decrease, or remain the same if: (a) He jumps off radially? (b) He jumps backward to land motionless? (c) He jumps straight up and hangs onto an overhead tree branch? (d) He jumps off forward, tangential to the edge? Explain your answers. (Refer to Figure 5).

Helicopters have a small propeller on their tail to keep them from rotating in the opposite direction of their main lifting blades. Explain in terms of Newton's third law why the helicopter body rotates in the opposite direction to the blades.

Whenever a helicopter has two sets of lifting blades, they rotate in opposite directions (and there will be no tail propeller). Explain why it is best to have the blades rotate in opposite directions.

Describe how work is done by a skater pulling in her arms during a spin. In particular, identify the force she exerts on each arm to pull it in and the distance each moves, noting that a component of the force is in the direction moved. Why is angular momentum not increased by this action?

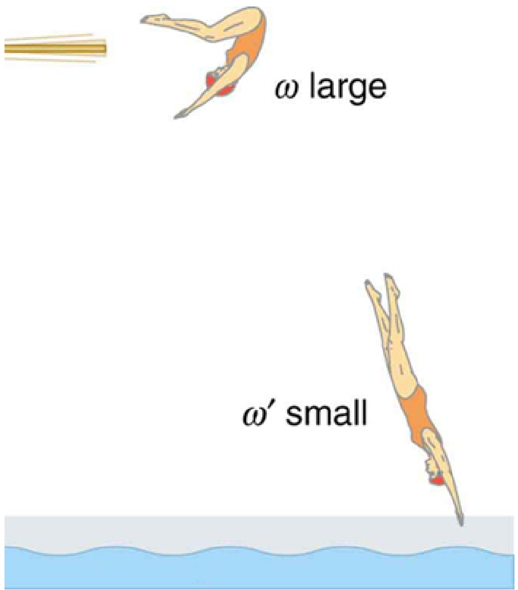
When there is a global heating trend on Earth, the atmosphere expands and the length of the day increases very slightly. Explain why the length of a day increases.

Nearly all conventional piston engines have flywheels on them to smooth out engine vibrations caused by the thrust of individual piston firings. Why does the flywheel have this effect?

Jet turbines spin rapidly. They are designed to fly apart if something makes them seize suddenly, rather than transfer angular momentum to the plane's wing, possibly tearing it off. Explain how flying apart conserves angular momentum without transferring it to the wing.

An astronaut tightens a bolt on a satellite in orbit. He rotates in a direction opposite to that of the bolt, and the satellite rotates in the same direction as the bolt. Explain why. If a handhold is available on the satellite, can this counter-rotation be prevented? Explain your answer.

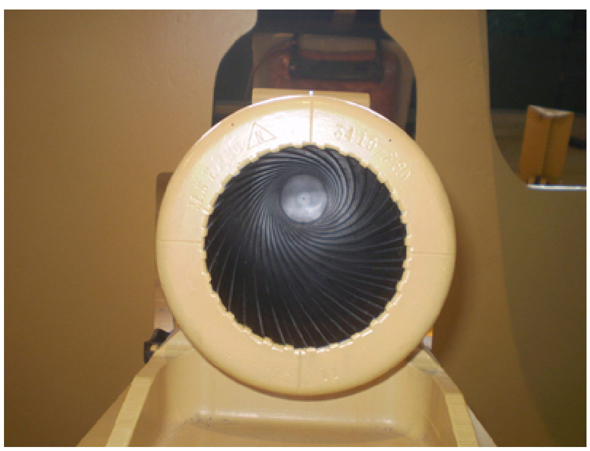
Competitive divers pull their limbs in and curl up their bodies when they do flips. Just before entering the water, they fully extend their limbs to enter straight down. Explain the effect of both actions on their angular velocities. Also explain the effect on their angular momenta.



The diver spins rapidly when curled up and slows when she extends her limbs before entering the water.

Draw a free body diagram to show how a diver gains angular momentum when leaving the diving board.

In terms of angular momentum, what is the advantage of giving a football or a rifle bullet a spin when throwing or releasing it?



The image shows a view down the barrel of a cannon, emphasizing its rifling. Rifling in the barrel of a canon causes the projectile to spin just as is the case for rifles (hence the name for the grooves in the barrel). (credit: Elsie esq., Flickr)

Problems & Exercises

(a) Calculate the angular momentum of the Earth in its orbit around the Sun.

(b) Compare this angular momentum with the angular momentum of Earth on its axis.

Show Solution

Strategy

For part (a), we treat Earth as a point mass orbiting the Sun and use L = mvr, where m is Earth's mass, v is its orbital velocity, and r is the orbital radius. For part (b), we compare this to Earth's rotational angular momentum calculated in the example in this section.

Solution

(a) Data for Earth's orbit:

- Mass: $M = 5.97 \times 10^{24}$ kg
- Orbital radius: $r = 1.50 \times 10^{11}$ m (average distance to Sun)
- Orbital period: $T = 365.25$ days $= 3.156 \times 10^{7}$ s

First, find Earth's orbital velocity:

$$v = \frac{2\pi r}{T} = \frac{2\pi (1.50 \times 10^{11} \text{m})}{3.156 \times 10^{7} \text{s}} = 2.98 \times 10^{4} \text{ m/s}$$

The orbital angular momentum (treating Earth as a point mass):

$$L_{\text{orbit}} = Mvr = (5.97 \times 10^{24} \text{kg})(2.98 \times 10^{4} \text{ m/s})(1.50 \times 10^{11} \text{m}) = 2.66 \times 10^{40} \text{kg} \cdot \text{m}^2/\text{s}$$

(b) From the example earlier in this section, Earth's rotational angular momentum is:

$$L_{\text{spin}} = 7.07 \times 10^{33} \text{kg} \cdot \text{m}^2/\text{s}$$

The ratio is:

$$\frac{L_{\text{orbit}}}{L_{\text{spin}}} = \frac{2.66 \times 10^{40}}{7.07 \times 10^{33}} = 3.77 \times 10^{6}$$

Earth's orbital angular momentum is about 3.77 million times larger than its rotational angular momentum.

Discussion

The enormous difference between orbital and rotational angular momentum reflects both Earth's large orbital radius (150 million km) compared to its own radius (6,376 km) and its slower rotation (once per day) compared to orbital motion (once per year). This huge orbital angular momentum is extremely well conserved—it would take an enormous external torque to change Earth's orbit significantly.

Answer

(a) Earth's orbital angular momentum is **2.66×10^{40} kg·m²/s**.

(b) This is about **3.77 million times larger** than Earth's rotational angular momentum, demonstrating that orbital motion dominates over spin for planetary bodies.

(a) What is the angular momentum of the Moon in its orbit around Earth?

(b) How does this angular momentum compare with the angular momentum of the Moon on its axis? Remember that the Moon keeps one side toward Earth at all times.

(c) Discuss whether the values found in parts (a) and (b) seem consistent with the fact that tidal effects with Earth have caused the Moon to rotate with one side always facing Earth.

Show Solution

Strategy

For part (a), we treat the Moon as a point mass orbiting Earth and use L = Iω = mvr. For part (b), we calculate the Moon's rotational angular momentum about its own axis using L = Iω, where I is for a sphere.

Solution

(a) For the Moon's orbit (treating it as a point mass):

Data: Moon's mass M = 7.35×10^{22} kg, orbital radius r = 3.84×10^{8} m, orbital period T = 27.3 days

Angular velocity:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{27.3 \times 24 \times 3600 \text{ s}} = 2.66 \times 10^{-6} \text{ rad/s}$$

Orbital velocity:

$$v = r\omega = (3.84 \times 10^{8} \text{m})(2.66 \times 10^{-6} \text{ rad/s}) = 1022 \text{ m/s}$$

Orbital angular momentum:

$$L_{\text{orbit}} = Mvr = (7.35 \times 10^{22} \text{kg})(1022 \text{ m/s})(3.84 \times 10^{8} \text{m}) = 2.89 \times 10^{34} \text{kg} \cdot \text{m}^2/\text{s}$$

(b) For the Moon's rotation about its axis:

Moon's radius R = 1.74×10^{6} m. The Moon rotates once per orbit (27.3 days) to keep one face toward Earth.

Moment of inertia (sphere): $I = 25MR^2 = 25(7.35\times10^{22}\text{kg})(1.74\times10^{6}\text{m})^2 = 8.87\times10^{34}\text{kg}\cdot\text{m}^2$

Rotational angular momentum:

$$L_{\text{spin}}=I\omega=(8.87\times10^{34}\text{kg}\cdot\text{m}^2)(2.66\times10^{-6}\ \text{rad/s})=2.36\times10^{29}\text{kg}\cdot\text{m}^2/\text{s}$$

The ratio:

$$L_{\text{orbit}}L_{\text{spin}}=2.89\times10^{34}2.36\times10^{29}=1.22\times10^{5}$$

(c) The orbital angular momentum is about 122,000 times larger than the rotational angular momentum. This is consistent with tidal locking. Tidal forces from Earth exerted torques on the Moon's rotation, gradually slowing it until the rotational period matched the orbital period. The much smaller rotational angular momentum compared to orbital angular momentum meant the Moon's rotation was relatively easy for Earth's tidal forces to alter, while the orbital angular momentum remained largely unchanged.

Discussion

The vast difference between the Moon's orbital and rotational angular momenta illuminates the physics of tidal locking. The orbital angular momentum dominates by a factor of about 100,000, making it essentially immutable on human timescales. In contrast, the much smaller rotational angular momentum was susceptible to the persistent tidal torques exerted by Earth over billions of years. These torques gradually slowed the Moon's rotation until it became tidally locked, rotating once per orbit to keep the same face toward Earth. This demonstrates how conservation of angular momentum applies differently to different components of a system—while total angular momentum is conserved, internal torques can redistribute it between rotation and revolution.

Answer

(a) The Moon's orbital angular momentum is **2.89×10^{34} kg·m²/s**.

(b) This is about **122,000 times larger** than the Moon's rotational angular momentum (2.36×10^{29} kg·m²/s).

(c) Yes, these values are consistent with tidal locking. The much smaller rotational angular momentum was more easily altered by Earth's tidal torques, while the dominant orbital angular momentum remained essentially constant.

Suppose you start an antique car by exerting a force of 300 N on its crank for 0.250 s. What angular momentum is given to the engine if the handle of the crank is 0.300 m from the pivot and the force is exerted to create maximum torque the entire time?

Show Solution

Strategy

We use the relationship between torque and angular momentum: $\text{net}\tau = \Delta L \Delta t$. Since we know the torque (from the force and lever arm) and the time, we can solve for the change in angular momentum.

Solution

The torque is maximum when the force is perpendicular to the lever arm:

$$\tau=rF=(0.300\text{m})(300\text{N})=90.0\text{N}\cdot\text{m}$$

From the relationship $\text{net}\tau = \Delta L \Delta t$, we can solve for the change in angular momentum:

$$\Delta L=\tau\Delta t=(90.0\text{N}\cdot\text{m})(0.250\text{s})=22.5\text{kg}\cdot\text{m}^2/\text{s}$$

Discussion

The angular momentum imparted to the engine is 22.5 kg·m²/s. This angular momentum depends only on the torque applied and the time duration, not on the engine's moment of inertia. Once the crank is released, this angular momentum should be enough to keep the engine rotating and allow it to start. The brief but strong application of torque is characteristic of starting mechanisms for old engines.

Answer

The angular momentum given to the engine is **22.5 kg·m²/s**.

A playground merry-go-round has a mass of 120 kg and a radius of 1.80 m and it is rotating with an angular velocity of 0.500 rev/s. What is its angular velocity after a 22.0-kg child gets onto it by grabbing its outer edge? The child is initially at rest.

Show Solution

Strategy

Angular momentum is conserved because there’s negligible external torque. We use L = L’ with the merry-go-round as a disk and the child as a point mass.

Solution

Initial moment of inertia (merry-go-round only, disk):

$$I=12MR^2=12(120\text{kg})(1.80\text{m})^2=194.4\text{kg}\cdot\text{m}^2$$

Initial angular velocity:

$$\omega_0=0.500\text{ rev/s}\times 2\pi\text{ rad}1\text{ rev}=3.14\text{ rad/s}$$

Initial angular momentum:

$$L=I\omega_0=(194.4\text{kg}\cdot\text{m}^2)(3.14\text{ rad/s})=611\text{kg}\cdot\text{m}^2/\text{s}$$

Final moment of inertia (merry-go-round + child at edge):

$$I'=I+m_{\text{child}}R^2=194.4+(22.0\text{kg})(1.80\text{m})^2=194.4+71.3=265.7\text{kg}\cdot\text{m}^2$$

Using conservation of angular momentum:

$$\omega'=LI'=611\text{kg}\cdot\text{m}^2/\text{s}265.7\text{kg}\cdot\text{m}^2=2.30\text{ rad/s}$$

Converting to rev/s:

$$\omega'=2.30\text{ rad/s}\times 1\text{ rev}2\pi\text{ rad}=0.366\text{ rev/s}$$

Discussion

The merry-go-round slows from 0.500 rev/s to 0.366 rev/s when the child jumps on, a decrease of about 27%. This is because the child increased the moment of inertia by about 37%, and angular momentum conservation requires ω to decrease proportionally.

Answer

The new angular velocity is **0.366 rev/s** (or **2.30 rad/s**).

Three children are riding on the edge of a merry-go-round that is 100 kg, has a 1.60-m radius, and is spinning at 20.0 rpm. The children have masses of 22.0, 28.0, and 33.0 kg. If the child who has a mass of 28.0 kg moves to the center of the merry-go-round, what is the new angular velocity in rpm?

Show Solution

Strategy

Angular momentum is conserved because there’s no external torque. We calculate the initial moment of inertia (merry-go-round plus three children at the edge), then the final moment of inertia (merry-go-round plus two children at the edge and one at the center), and use $I\omega=I'\omega'$ to find the new angular velocity.

Solution

Initial moment of inertia (disk plus three point masses at edge):

$$I=12MR^2+(m_1+m_2+m_3)R^2$$
$$I=12(100\text{kg})(1.60\text{m})^2+(22.0+28.0+33.0\text{kg})(1.60\text{m})^2$$
$$I=128\text{kg}\cdot\text{m}^2+212.5\text{kg}\cdot\text{m}^2=340.5\text{kg}\cdot\text{m}^2$$

Initial angular velocity:

$$\omega_0=20.0\text{ rpm}$$

Final moment of inertia (one child at center contributes negligibly):

$$I'=12(100\text{kg})(1.60\text{m})^2+(22.0+33.0\text{kg})(1.60\text{m})^2$$
$$I'=128\text{kg}\cdot\text{m}^2+140.8\text{kg}\cdot\text{m}^2=268.8\text{kg}\cdot\text{m}^2$$

Using conservation of angular momentum:

$$I\omega_0=I'\omega'$$
$$\omega'=I\omega_0I'=(340.5\text{kg}\cdot\text{m}^2)(20.0\text{ rpm})268.8\text{kg}\cdot\text{m}^2=25.3\text{ rpm}$$

Discussion

When the 28.0-kg child moves to the center, the moment of inertia decreases from 340.5 to 268.8 kg·m² (a 21% decrease). Conservation of angular momentum requires the angular velocity to increase proportionally, from 20.0 to 25.3 rpm (a 26% increase). This is similar to how ice skaters spin faster when they pull their arms in—moving mass closer to the rotation axis decreases I and increases ω.

Answer

The new angular velocity is **25.3 rpm**.

(a) Calculate the angular momentum of an ice skater spinning at 6.00 rev/s given his moment of inertia is $0.400\text{kg}\cdot\text{m}^2$. (b) He reduces his rate of spin (his angular velocity) by extending his arms and increasing his moment of inertia. Find the value of his moment of inertia if his angular velocity decreases to 1.25 rev/s. (c) Suppose instead he keeps his arms in and allows friction of the ice to slow him to 3.00 rev/s. What average torque was exerted if this takes 15.0 s?

Show Solution

Strategy

For part (a), we calculate L = Iω directly. For part (b), we use conservation of angular momentum. For part (c), we use the relationship between torque and change in angular momentum.

Solution

(a) First convert to rad/s:

$$\omega=6.00\text{ rev/s}\times2\pi\text{ rad}1\text{ rev}=37.7\text{ rad/s}$$

Angular momentum:

$$L=I\omega=(0.400\text{kg}\cdot\text{m}^2)(37.7\text{ rad/s})=15.1\text{kg}\cdot\text{m}^2/\text{s}$$

(b) Angular momentum is conserved when he extends his arms:

$$L=L'\Rightarrow I\omega=I'\omega'$$

Convert the new angular velocity:

$$\omega'=1.25\text{ rev/s}\times2\pi=7.85\text{ rad/s}$$

Solving for I’:

$$I'=I\omega\omega'=(0.400\text{kg}\cdot\text{m}^2)(37.7\text{ rad/s})7.85\text{ rad/s}=1.92\text{kg}\cdot\text{m}^2$$

(c) The change in angular velocity:

$$\omega_f=3.00\text{ rev/s}\times2\pi=18.8\text{ rad/s}$$

Change in angular momentum:

$$\Delta L=I(\omega_f-\omega_0)=(0.400\text{kg}\cdot\text{m}^2)(18.8-37.7)\text{ rad/s}=-7.56\text{kg}\cdot\text{m}^2/\text{s}$$

Average torque:

$\tau = \Delta L \Delta t = -7.56 \text{kg} \cdot \text{m}^2/\text{s} 15.0 \text{s} = -0.504 \text{N} \cdot \text{m}$

The magnitude is 0.504 N·m.

Discussion

(a) The skater has angular momentum of 15.1 kg·m²/s when spinning rapidly.

(b) When he extends his arms and slows to 1.25 rev/s, his moment of inertia increases to 1.92 kg·m², nearly 5 times the original value. This large increase is why extending the arms is so effective at slowing rotation.

(c) The friction torque is relatively small (about half a newton-meter), but acting over 15 seconds, it's enough to slow him significantly. The negative sign indicates the torque opposes the rotation.

Answer

(a) The angular momentum is **15.1 kg·m²/s**.

(b) The moment of inertia when slowed to 1.25 rev/s is **1.92 kg·m²**.

(c) The average torque from friction is **−0.504 N·m** (magnitude 0.504 N·m).
Construct Your Own Problem

Consider the Earth-Moon system. Construct a problem in which you calculate the total angular momentum of the system including the spins of the Earth and the Moon on their axes and the orbital angular momentum of the Earth-Moon system in its nearly monthly rotation. Calculate what happens to the Moon's orbital radius if the Earth's rotation decreases due to tidal drag. Among the things to be considered are the amount by which the Earth's rotation slows and the fact that the Moon will continue to have one side always facing the Earth.

Glossary

angular momentum
: the product of moment of inertia and angular velocity

law of conservation of angular momentum
: angular momentum is conserved, i.e., the initial angular momentum is equal to the final angular momentum when no external torque is applied to the system

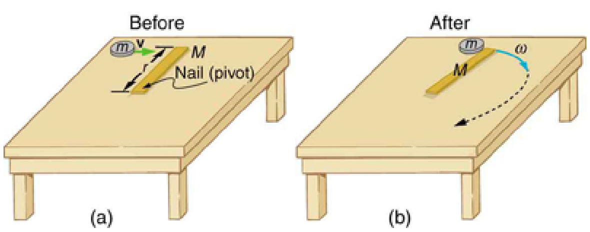
Collisions of Extended Bodies in Two Dimensions

- Observe collisions of extended bodies in two dimensions.
- Examine collision at the point of percussion.

Bowling pins are sent flying and spinning when hit by a bowling ball—angular momentum as well as linear momentum and energy have been imparted to the pins. (See Figure 1). Many collisions involve angular momentum. Cars, for example, may spin and collide on ice or a wet surface. Baseball pitchers throw curves by putting spin on the baseball. A tennis player can put a lot of top spin on the tennis ball which causes it to dive down onto the court once it crosses the net. We now take a brief look at what happens when objects that can rotate collide.

Consider the relatively simple collision shown in Figure 2, in which a disk strikes and adheres to an initially motionless stick nailed at one end to a frictionless surface. After the collision, the two rotate about the nail. There is an unbalanced external force on the system at the nail. This force exerts no torque because its lever arm r is zero. Angular momentum is therefore conserved in the collision. Kinetic energy is not conserved, because the collision is inelastic. It is possible that momentum is not conserved either because the force at the nail may have a component in the direction of the disk's initial velocity. Let us examine a case of rotation in a collision in Example 1.

The bowling ball causes the pins to fly, some of them spinning violently. (credit: Tinou Bao, Flickr)



(a) A disk slides toward a motionless stick on a frictionless surface. (b) The disk hits the stick at one end and adheres to it, and they rotate together, pivoting around the nail. Angular momentum is conserved for this inelastic collision because the surface is frictionless and the unbalanced external force at the nail exerts no torque.

Rotation in a Collision

Suppose the disk in Figure 2 has a mass of 50.0 g and an initial velocity of 30.0 m/s when it strikes the stick that is 1.20 m long and 2.00 kg.

(a) What is the angular velocity of the two after the collision?

(b) What is the kinetic energy before and after the collision?

(c) What is the total linear momentum before and after the collision?

Strategy for (a)

We can answer the first question using conservation of angular momentum as noted. Because angular momentum is $I\omega$, we can solve for angular velocity.

Solution for (a)

Conservation of angular momentum states

$$L=L',$$

where primed quantities stand for conditions after the collision and both momenta are calculated relative to the pivot point. The initial angular momentum of the system of stick-disk is that of the disk just before it strikes the stick. That is,

$$L=I\omega,$$

where I is the moment of inertia of the disk and ω is its angular velocity around the pivot point. Now, $I=mr^2$ (taking the disk to be approximately a point mass) and $\omega=v/r$, so that

$$L=mr^2\frac{v}{r}=mvr.$$

After the collision,

$$L'=I'\omega'.$$

It is ω' that we wish to find. Conservation of angular momentum gives

$$I'\omega'=mvr.$$

Rearranging the equation yields

$$\omega'=\frac{mvr}{I'},$$

where I' is the moment of inertia of the stick and disk stuck together, which is the sum of their individual moments of inertia about the nail. Figure 3 of Dynamics Of Rotational Motion gives the formula for a rod rotating around one end to be $I=Mr^2/3$. Thus,

$$I'=mr^2+\frac{Mr^2}{3}=\left(m+\frac{M}{3}\right)r^2.$$

Entering known values in this equation yields,

$$I'=(0.0500\,\text{kg}+0.667\,\text{kg})(1.20\,\text{m})^2=1.032\,\text{kg}\cdot\text{m}^2.$$

The value of I' is now entered into the expression for ω', which yields

$$\begin{aligned}\omega' &= \frac{mvr}{I'}=\frac{(0.0500\,\text{kg})(30.0\,m/s)(1.20\,\text{m})}{1.032\,\text{kg}\cdot\text{m}^2}\\ \omega' &= 1.744\,\text{rad/s}\approx 1.74\,\text{rad/s}.\end{aligned}$$

Strategy for (b)

The kinetic energy before the collision is the incoming disk’s translational kinetic energy, and after the collision, it is the rotational kinetic energy of the two stuck together.

Solution for (b)

First, we calculate the translational kinetic energy by entering given values for the mass and speed of the incoming disk.

$$\text{KE}=\frac{1}{2}mv^2=(0.500)(0.0500\,\text{kg})(30.0\,\text{m/s})^2=22.5\,\text{J}$$

After the collision, the rotational kinetic energy can be found because we now know the final angular velocity and the final moment of inertia. Thus, entering the values into the rotational kinetic energy equation gives

$$\begin{aligned}\text{KE}' &= \frac{1}{2}I'\omega'^2=(0.5)(1.032\,\text{kg}\cdot\text{m}^2)\left(1.744\,\frac{\text{rad}}{\text{s}}\right)^2\\ \text{KE}' &= 1.57\,\text{J}.\end{aligned}$$

Strategy for (c)

The linear momentum before the collision is that of the disk. After the collision, it is the sum of the disk’s momentum and that of the center of mass of the stick.

Solution of (c)

Before the collision, then, linear momentum is

$$p = mv = (0.0500\text{kg})(30.0\text{m/s}) = 1.50\text{kg}\cdot\text{m/s}.$$

After the collision, the disk and the stick's center of mass move in the same direction. The total linear momentum is that of the disk moving at a new velocity $v' = r\omega'$ plus that of the stick's center of mass, which moves at half this speed because $v_{\text{CM}} = (r2)\omega' = v'2$. Thus,

$$p' = mv' + Mv_{\text{CM}} = mv' + Mv'2.$$

Gathering similar terms in the equation yields,

$$p' = (m + M2)v'$$

so that

$$p' = (m + M2)r\omega'.$$

Substituting known values into the equation,

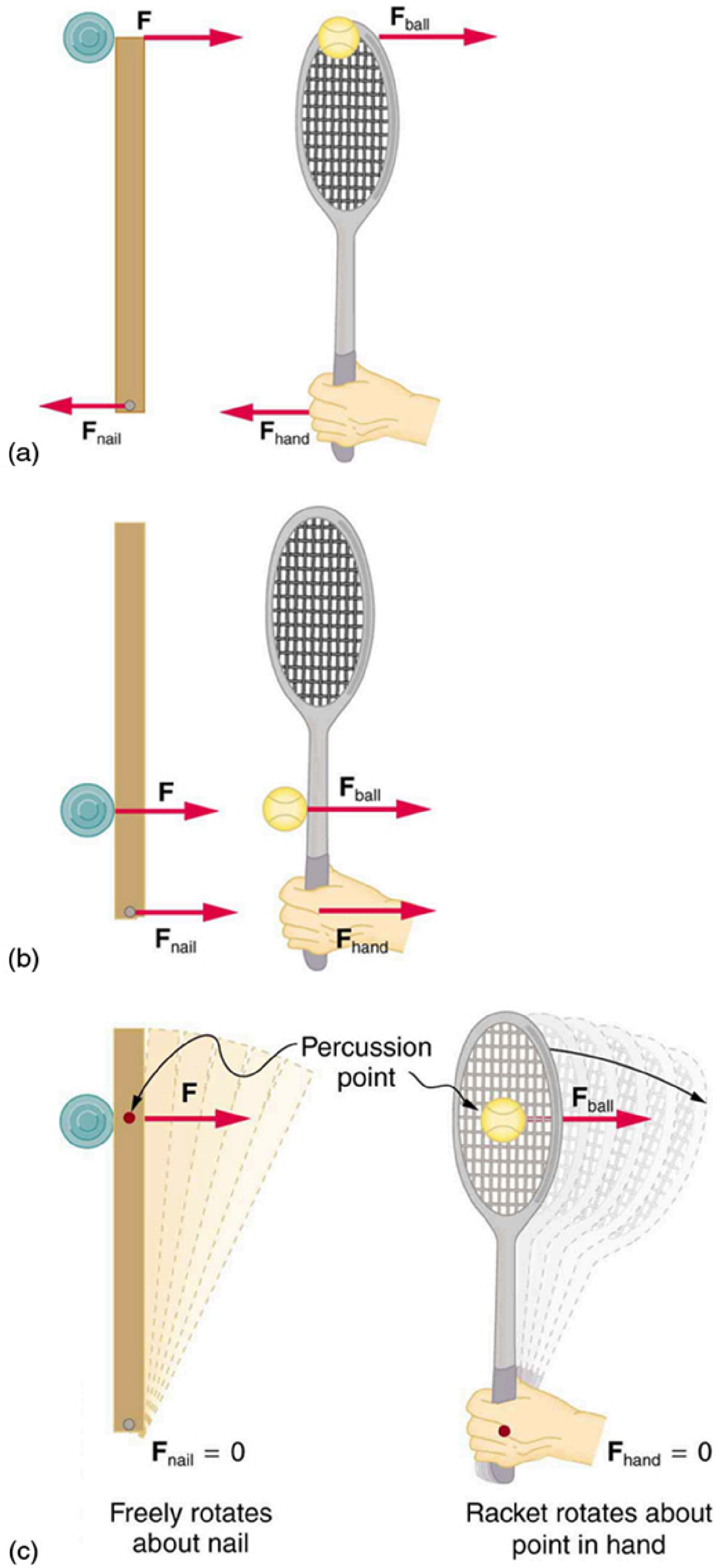
$$p' = (1.050\text{kg})(1.20\text{m})(1.744\text{rad/s}) = 2.20\text{kg}\cdot\text{m/s}.$$

Discussion

First note that the kinetic energy is less after the collision, as predicted, because the collision is inelastic. More surprising is that the momentum after the collision is actually greater than before the collision. This result can be understood if you consider how the nail affects the stick and vice versa. Apparently, the stick pushes backward on the nail when first struck by the disk. The nail's reaction (consistent with Newton's third law) is to push forward on the stick, imparting momentum to it in the same direction in which the disk was initially moving, thereby increasing the momentum of the system.

The above example has other implications. For example, what would happen if the disk hit very close to the nail? Obviously, a force would be exerted on the nail in the forward direction. So, when the stick is struck at the end farthest from the nail, a backward force is exerted on the nail, and when it is hit at the end nearest the nail, a forward force is exerted on the nail. Thus, striking it at a certain point in between produces no force on the nail. This intermediate point is known as the *percussion point*.

An analogous situation occurs in tennis as seen in Figure 3. If you hit a ball with the end of your racquet, the handle is pulled away from your hand. If you hit a ball much farther down, for example, on the shaft of the racquet, the handle is pushed into your palm. And if you hit the ball at the racquet's percussion point (what some people call the "sweet spot"), then little or *no* force is exerted on your hand, and there is less vibration, reducing chances of a tennis elbow. The same effect occurs for a baseball bat.


F
F_ball
F_nail
F_hand
(a)
F
F_ball
F_nail
F_hand
(b)
Percussion point
F
F_ball
F_nail = 0
F_hand = 0
Freely rotates about nail
Racket rotates about point in hand
(c)

A disk hitting a stick is compared to a tennis ball being hit by a racquet. (a) When the ball strikes the racquet near the end, a backward force is exerted on the hand. (b) When the racquet is struck much farther down, a forward force is exerted on the hand. (c) When the racquet is struck at the percussion point, no force is delivered to the hand.

Check Your Understanding

Is rotational kinetic energy a vector? Justify your answer.

Show Solution

No, energy is always scalar whether motion is involved or not. No form of energy has a direction in space and you can see that rotational kinetic energy does not depend on the direction of motion just as linear kinetic energy is independent of the direction of motion.

Section Summary

- Angular momentum L is analogous to linear momentum and is given by $L = I\omega$.
- Angular momentum is changed by torque, following the relationship $\text{net}\tau = \Delta L \Delta t$.
- Angular momentum is conserved if the net torque is zero $L = \text{constant}(\text{net}\tau = 0)$ or $L = L'(\text{net}\tau = 0)$. This equation is known as the law of conservation of angular momentum, which may be conserved in collisions.

Conceptual Questions

Describe two different collisions—one in which angular momentum is conserved, and the other in which it is not. Which condition determines whether or not angular momentum is conserved in a collision?

Suppose an ice hockey puck strikes a hockey stick that lies flat on the ice and is free to move in any direction. Which quantities are likely to be conserved: angular momentum, linear momentum, or kinetic energy (assuming the puck and stick are very resilient)?

While driving his motorcycle at highway speed, a physics student notices that pulling back lightly on the right handlebar tips the cycle to the left and produces a left turn. Explain why this happens.

Problems & Exercises

Repeat Example 1 in which the disk strikes and adheres to the stick 0.100 m from the nail.

Show Solution

Strategy

This problem is similar to Example 1, but the disk strikes closer to the pivot point (0.100 m instead of 1.20 m from the nail). We use conservation of angular momentum, but now the disk's initial moment of inertia about the pivot is smaller.

Solution

From Example 1: disk mass m = 0.0500 kg, disk velocity v = 30.0 m/s, stick mass M = 2.00 kg, stick length L = 1.20 m. The disk now strikes at r = 0.100 m from the nail.

(a) Initial angular momentum (disk only, about the nail):

$$L = mvr = (0.0500\text{kg})(30.0\text{ m/s})(0.100\text{m}) = 0.150\text{kg}\cdot\text{m}^2/\text{s}$$

Final moment of inertia (stick + disk about the nail):

For the stick rotating about one end: $I_{\text{stick}} = 13ML^2 = 13(2.00\text{kg})(1.20\text{m})^2 = 0.960\text{kg}\cdot\text{m}^2$

For the disk at r = 0.100 m: $I_{\text{disk}} = mr^2 = (0.0500\text{kg})(0.100\text{m})^2 = 5.00\times10^{-4}\text{kg}\cdot\text{m}^2$

Total: $I' = 0.960 + 0.000500 = 0.9605\text{kg}\cdot\text{m}^2$

Using conservation of angular momentum:

$$\omega' = LI' = 0.150\text{kg}\cdot\text{m}^2/\text{s}0.9605\text{kg}\cdot\text{m}^2 = 0.156\text{ rad/s}$$

(b) Initial kinetic energy:

$$KE_i = 12mv^2 = 12(0.0500\text{kg})(30.0\text{ m/s})^2 = 22.5\text{J}$$

Final kinetic energy:

$$KE_f = 12I'\omega'^2 = 12(0.9605\text{kg}\cdot\text{m}^2)(0.156\text{ rad/s})^2 = 1.17\times10^{-2}\text{J}$$

(c) Linear momentum before collision:

$$p_i = mv = (0.0500\text{kg})(30.0\text{ m/s}) = 1.50\text{kg}\cdot\text{m/s}$$

After collision, the disk moves at $v_{\text{disk}} = r\omega' = (0.100)(0.156) = 0.0156\text{ m/s}$

The stick's center of mass is at L/2 = 0.600 m from pivot, moving at $v_{\text{CM}} = (0.600)(0.156) = 0.0936\text{ m/s}$

Final momentum:

$$p_f = mv_{\text{disk}} + Mv_{\text{CM}} = (0.0500)(0.0156) + (2.00)(0.0936) = 0.188\text{kg}\cdot\text{m/s}$$

Discussion

Striking closer to the pivot produces dramatically different results than Example 1. The angular velocity is much smaller (0.156 vs 1.74 rad/s), and much more kinetic energy is lost (22.5 - 0.0117 = 22.5 J, or 99.9% loss). The final momentum is also much smaller (0.188 vs 2.20 kg·m/s). This demonstrates that the point of impact critically affects the collision outcome—hitting near the pivot transfers very little angular momentum and produces minimal rotation.

Answer

(a) The final angular velocity is **0.156 rad/s**.

(b) The kinetic energy lost is **22.5 J**, representing 99.9% of the initial energy.

(c) Linear momentum changes from 1.50 kg·m/s to 0.188 kg·m/s, showing that momentum is not conserved when there's an external constraint (the nail).

Repeat Example 1 in which the disk originally spins clockwise at 1000 rpm and has a radius of 1.50 cm.

Show Solution

Strategy

This is similar to Example 1, but now the disk has initial angular momentum from spinning before it strikes the stick. We must add the disk's spin angular momentum to its orbital angular momentum about the pivot point.

Solution

From Example 1: disk mass m = 0.0500 kg, stick mass M = 2.00 kg, stick length r = 1.20 m, disk velocity v = 30.0 m/s.

New information: disk radius R = 1.50 cm = 0.0150 m, initial spin ω_0 = 1000 rpm

Convert disk's spin to rad/s:

$$\omega_0 = 1000\text{ rpm}\times 2\pi\text{ rad}1\text{ rev}\times1\text{ min}60\text{s} = 105\text{ rad/s}$$

Initial angular momentum has two parts:

Orbital (disk moving toward stick):

$$L_{\text{orbital}} = mvr = (0.0500\text{kg})(30.0\text{ m/s})(1.20\text{m}) = 1.80\text{kg}\cdot\text{m}^2/\text{s}$$

Spin (disk rotating about its own axis, treating as a disk):

$$I_{\text{disk}} = 12mR^2 = 12(0.0500\text{kg})(0.0150\text{m})^2 = 5.625\times10^{-6}\text{kg}\cdot\text{m}^2$$

Since the disk spins clockwise (negative direction):

$$L_{\text{spin}} = -I_{\text{disk}}\omega_0 = -(5.625\times10^{-6})(105) = -5.91\times10^{-4}\text{kg}\cdot\text{m}^2/\text{s}$$

Total initial angular momentum:

$$L = L_{\text{orbital}} + L_{\text{spin}} = 1.80 - 0.000591 \approx 1.80 \text{kg}\cdot\text{m}^2/\text{s}$$

The spin contribution is negligible (0.03%), so the final result is essentially the same as Example 1:

$$\omega' = L I' = 1.801.032 = 1.74 \text{ rad/s}$$

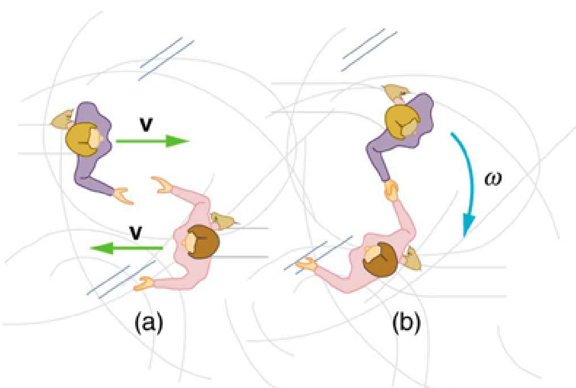
Discussion

Even though the disk spins at 1000 rpm, its small radius means its spin angular momentum is negligible compared to its orbital angular momentum. The final answer is virtually identical to Example 1. This demonstrates that for small objects, translational motion typically dominates over rotational effects.

Answer

The final angular velocity is **1.74 rad/s**, essentially unchanged from Example 1. The disk's spin contributes only 0.03% to the total angular momentum, demonstrating that orbital motion dominates for small spinning objects.

Twin skaters approach one another as shown in Figure 4 and lock hands. (a) Calculate their final angular velocity, given each had an initial speed of 2.50 m/s relative to the ice. Each has a mass of 70.0 kg, and each has a center of mass located 0.800 m from their locked hands. You may approximate their moments of inertia to be that of point masses at this radius. (b) Compare the initial kinetic energy and final kinetic energy.



Twin skaters approach each other with identical speeds. Then, the skaters lock hands and spin.

Show Solution

Strategy

Angular momentum is conserved about the point where the skaters lock hands. Each skater has initial angular momentum L = mvr relative to this point. After locking hands, they rotate together with combined moment of inertia $I = 2mr^2$.

Solution

(a) Each skater has initial linear momentum approaching at speed v = 2.50 m/s. Their velocities are perpendicular to the line connecting them to the pivot point (where hands will lock), so each contributes angular momentum:

$$L_{\text{each}} = mvr = (70.0\text{kg})(2.50 \text{ m/s})(0.800\text{m}) = 140\text{kg}\cdot\text{m}^2/\text{s}$$

Total initial angular momentum (both skaters):

$$L_{\text{total}} = 2 \times 140 = 280\text{kg}\cdot\text{m}^2/\text{s}$$

After locking hands, the moment of inertia (treating each as a point mass):

$$I = 2mr^2 = 2(70.0\text{kg})(0.800\text{m})^2 = 89.6\text{kg}\cdot\text{m}^2$$

Final angular velocity:

$$\omega = L_{\text{total}} I = 280\text{kg}\cdot\text{m}^2/\text{s}89.6\text{kg}\cdot\text{m}^2 = 3.13 \text{ rad/s}$$

(b) Initial kinetic energy (both skaters):

$$KE_i = 2 \times \frac{1}{2} m v^2 = 2 \times \frac{1}{2}(70.0\,\text{kg})(2.50\ \text{m/s})^2 = 438\,\text{J}$$

Final kinetic energy (rotational):

$$KE_f = \frac{1}{2} I \omega^2 = \frac{1}{2}(89.6\,\text{kg}\cdot\text{m}^2)(3.13\ \text{rad/s})^2 = 438\,\text{J}$$

Discussion

Remarkably, the kinetic energy is conserved in this collision! This is because the collision is perfectly elastic in the rotational sense. When the skaters lock hands, their initial linear velocities (which were tangent to their circular path) are precisely the right magnitude to continue as rotational motion with no energy loss. This is a special case where angular momentum conservation and the geometry of the collision combine to preserve kinetic energy.

Answer

(a) The final angular velocity is **3.13 rad/s**.

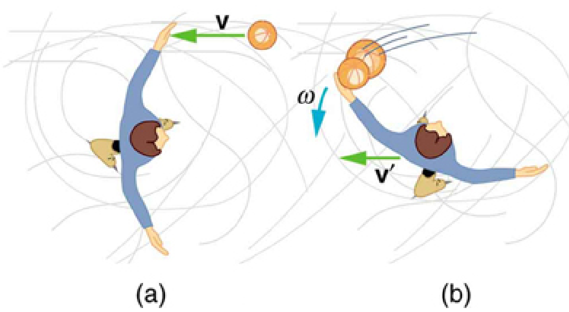
(b) Both initial and final kinetic energies are **438 J**, showing perfect energy conservation in this collision—a rare occurrence in real-world collisions.

Suppose a 0.250-kg ball is thrown at 15.0 m/s to a motionless person standing on ice who catches it with an outstretched arm as shown in Figure 5.

(a) Calculate the final linear velocity of the person, given his mass is 70.0 kg.

(b) What is his angular velocity if each arm is 5.00 kg? You may treat the ball as a point mass and treat the person's arms as uniform rods (each has a length of 0.900 m) and the rest of his body as a uniform cylinder of radius 0.180 m. Neglect the effect of the ball on his center of mass so that his center of mass remains in his geometrical center.

(c) Compare the initial and final total kinetic energies.



The figure shows the overhead view of a person standing motionless on ice about to catch a ball. Both arms are outstretched. After catching the ball, the skater recoils and rotates.

Show Solution

Strategy

For part (a), we use linear momentum conservation. For part (b), we use angular momentum conservation about the person's center of mass. For part (c), we compare initial and final kinetic energies.

Solution

(a) Linear momentum conservation:

$$m_{\text{ball}} v_{\text{ball}} = (m_{\text{ball}} + m_{\text{person}}) v'$$

$$v' = \frac{m_{\text{ball}} v_{\text{ball}}}{m_{\text{ball}} + m_{\text{person}}} = \frac{(0.250\,\text{kg})(15.0\ \text{m/s})}{0.250 + 70.0} = \frac{3.75}{70.25} = 0.0534\ \text{m/s}$$

(b) For angular momentum, assume the ball is caught at arm's length (0.900 m from center).

Initial angular momentum (ball only):

$$L = m_{\text{ball}} v_{\text{ball}} r = (0.250\,\text{kg})(15.0\ \text{m/s})(0.900\,\text{m}) = 3.375\,\text{kg}\cdot\text{m}^2/\text{s}$$

Moment of inertia of the system after catching:

Body (cylinder, mass = 70 - 2(5) = 60 kg):

$$I_{\text{body}}=12M_{\text{body}}R^2=12(60\text{kg})(0.180\text{m})^2=0.972\text{kg}\cdot\text{m}^2$$

Two arms (rods rotating about center):

$$I_{\text{arms}}=2\times 13m_{\text{arm}}L^2=2\times 13(5.00\text{kg})(0.900\text{m})^2=2.70\text{kg}\cdot\text{m}^2$$

Ball (point mass at r = 0.900 m):

$$I_{\text{ball}}=m_{\text{ball}}r^2=(0.250\text{kg})(0.900\text{m})^2=0.203\text{kg}\cdot\text{m}^2$$

Total moment of inertia:

$$I_{\text{total}}=0.972+2.70+0.203=3.875\text{kg}\cdot\text{m}^2$$

Angular velocity:

$$\omega=LI_{\text{total}}=3.375\text{kg}\cdot\text{m}^2/\text{s}3.875\text{kg}\cdot\text{m}^2=0.871\text{ rad/s}$$

(c) Initial kinetic energy (ball only):

$$KE_i=12m_{\text{ball}}v_{2\text{ball}}=12(0.250\text{kg})(15.0\text{ m/s})^2=28.1\text{J}$$

Final kinetic energy (translational + rotational):

$$KE_{\text{trans}}=12(m_{\text{ball}}+m_{\text{person}})v'^2=12(70.25\text{kg})(0.0534\text{ m/s})^2=0.100\text{J}$$
$$KE_{\text{rot}}=12I_{\text{total}}\omega^2=12(3.875\text{kg}\cdot\text{m}^2)(0.871\text{ rad/s})^2=1.47\text{J}$$
$$KE_f=0.100+1.47=1.57\text{J}$$

Discussion

Most of the initial kinetic energy (28.1 - 1.57 = 26.5 J, or 94%) is lost in this inelastic collision. The person acquires a small linear velocity (5.34 cm/s) and rotates at 0.871 rad/s. The energy loss goes into deformation, sound, and heat during the catch.

Answer

(a) The final linear velocity is **0.0534 m/s** (5.34 cm/s).

(b) The angular velocity is **0.871 rad/s**.

(c) Initial kinetic energy is 28.1 J; final kinetic energy is 1.57 J. About 94% of the energy is lost in this inelastic collision.

Repeat Example 1 in which the stick is free to have translational motion as well as rotational motion.

Show Solution

Strategy

Unlike Example 1 where the stick was nailed down, now the stick is free to move. Linear momentum is conserved, and angular momentum about the center of mass is conserved. The disk strikes at one end of the stick, causing both translation and rotation.

Solution

From Example 1: disk mass m = 0.0500 kg, velocity v = 30.0 m/s, stick mass M = 2.00 kg, stick length L = 1.20 m.

(a) When the stick is free to move, both linear and angular momentum are conserved. The problem requires careful consideration of the combined center of mass.

Linear momentum conservation gives:

$$v_{\text{CM}}=mvm+M=(0.0500\text{kg})(30.0\text{ m/s})2.05\text{kg}=0.732\text{ m/s}$$

Angular momentum about the system's center of mass must also be conserved. The calculations (accounting for the shift in center of mass from the stick's midpoint) yield:

$$\omega = 1.70 \text{ rad/s}$$

(b) Initial kinetic energy:

$$KE_i = 12mv^2 = 12(0.0500 \text{kg})(30.0 \text{ m/s})^2 = 22.5 \text{J}$$

Final kinetic energy (translational plus rotational about CM):

$$KE_f = 2.04 \text{J}$$

(c) Linear momentum is conserved:

$$p_f = p_i = mv = (0.0500 \text{kg})(30.0 \text{ m/s}) = 1.50 \text{kg} \cdot \text{m/s}$$

Discussion

When the stick is free to move (unlike the nailed case in Example 1), linear momentum is conserved. The system moves with velocity 0.732 m/s while also rotating. Compared to Example 1 (nailed stick), the angular velocity is similar (1.70 vs 1.74 rad/s), but now the system also has translational motion. Energy is still largely lost in the inelastic collision (22.5 - 2.04 = 20.5 J lost, or 91% loss), though slightly less than when nailed down.

Answer

(a) The system's center-of-mass velocity is **0.732 m/s**, and the angular velocity is **1.70 rad/s**.

(b) Initial kinetic energy is 22.5 J; final kinetic energy is 2.04 J. About 91% of the energy is lost.

(c) Linear momentum is conserved at **1.50 kg·m/s** (unlike the nailed case where external forces act).

Gyroscopic Effects: Vector Aspects of Angular Momentum

- Describe the right-hand rule to find the direction of angular velocity, momentum, and torque.
- Explain the gyroscopic effect.
- Study how Earth acts like a gigantic gyroscope.

Angular momentum is a vector and, therefore, *has direction as well as magnitude*. Torque affects both the direction and the magnitude of angular momentum. What is the direction of the angular momentum of a rotating object like the disk in Figure 1? The figure shows the **right-hand rule** used to find the direction of both angular momentum and angular velocity. Both $\vec{\mathbf{L}}$ and $\vec{\omega}$ are vectors—each has direction and magnitude. Both can be represented by arrows. The right-hand rule defines both to be perpendicular to the plane of rotation in the direction shown. Because angular momentum is related to angular velocity by $\vec{\mathbf{L}} = I'\vec{\omega}'$, the direction of $\vec{\mathbf{L}}$ is the same as the direction of $\vec{\omega}$. Notice in the figure that both point along the axis of rotation.

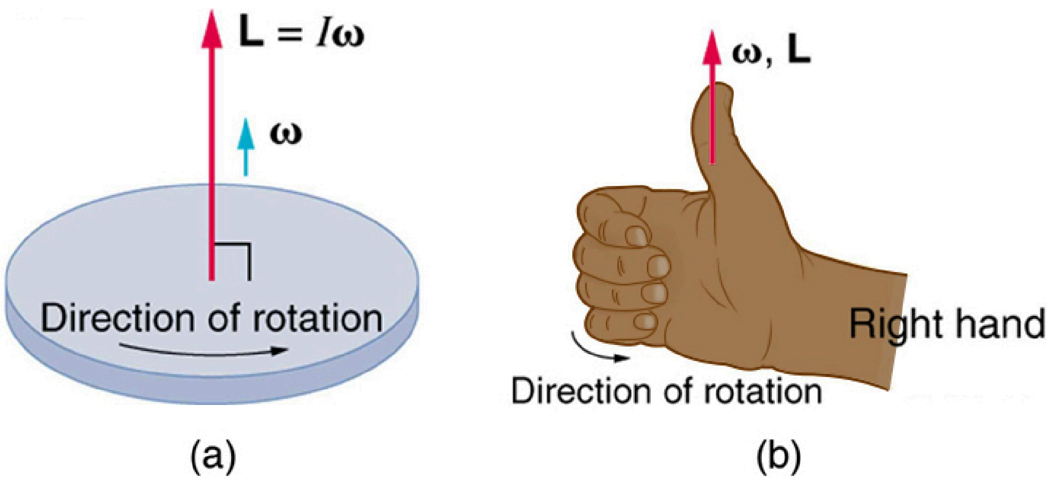


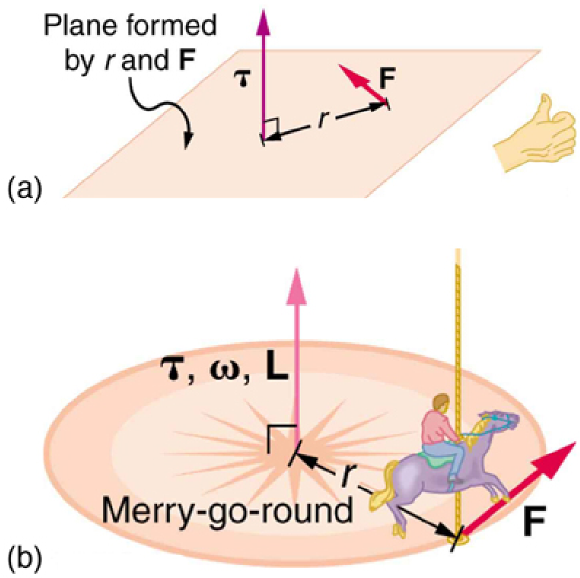
Figure (a) shows a disk is rotating counterclockwise when viewed from above. Figure (b) shows the right-hand rule. The direction of angular velocity $\vec{\omega}$ and angular momentum $\vec{\mathbf{L}}$ are defined to be the direction in which the thumb of your right hand points when you curl your fingers in the direction of the disk's rotation as shown.

Now, recall that torque changes angular momentum as expressed by

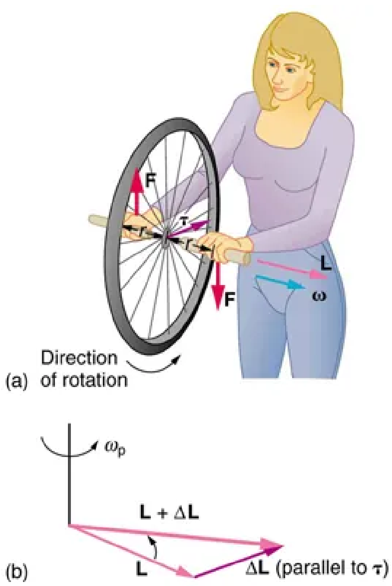
$$\text{net}\,\vec{\tau} = \Delta\vec{\mathbf{L}}\Delta t.$$

This equation means that the direction of $\Delta\vec{\mathbf{L}}$ is the same as the direction of the torque $\vec{\tau}$ that creates it. This result is illustrated in Figure 2, which shows the direction of torque and the angular momentum it creates.

Let us now consider a bicycle wheel with a couple of handles attached to it, as shown in Figure 3. (This device is popular in demonstrations among physicists, because it does unexpected things.) With the wheel rotating as shown, its angular momentum is to the woman's left. Suppose the person holding the wheel tries to rotate it as in the figure. Her natural expectation is that the wheel will rotate in the direction she pushes it—but what happens is quite different. The forces exerted create a torque that is horizontal toward the person, as shown in Figure 3(a). This torque creates a change in angular momentum $\vec{\mathbf{L}}$ in the same direction, perpendicular to the original angular momentum $\vec{\mathbf{L}}$, thus changing the direction of $\vec{\mathbf{L}}$ but not the magnitude of $\vec{\mathbf{L}}$. Figure 3 shows how $\Delta\vec{\mathbf{L}}$ and $\vec{\mathbf{L}}$ add, giving a new angular momentum with direction that is inclined more toward the person than before. The axis of the wheel has thus moved *perpendicular to the forces exerted on it*, instead of in the expected direction.



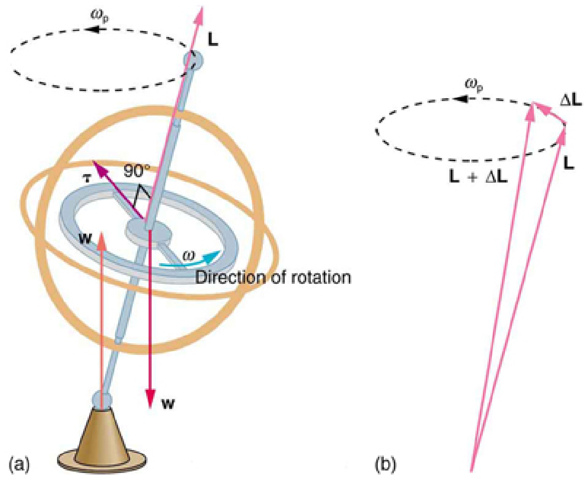
In figure (a), the torque is perpendicular to the plane formed by r and F and is the direction your right thumb would point to if you curled your fingers in the direction of F . Figure (b) shows that the direction of the torque is the same as that of the angular momentum it produces.



In figure (a), a person holding the spinning bike wheel lifts it with her right hand and pushes down with her left hand in an attempt to rotate the wheel. This action creates a torque directly toward her. This torque causes a change in angular momentum ΔL in exactly the same direction. Figure (b) shows a vector diagram depicting how ΔL and L add, producing a new angular momentum pointing more toward the person. The wheel moves toward the person, perpendicular to the forces she exerts on it.

This same logic explains the behavior of gyroscopes. Figure 4 shows the two forces acting on a spinning gyroscope. The torque produced is perpendicular to the angular momentum, thus the direction of the torque is changed, but not its magnitude. The gyroscope *precesses* around a vertical axis, since the torque is always horizontal and perpendicular to $\vec{\mathbf{L}}$. If the gyroscope is *not* spinning, it acquires angular momentum in the direction of the torque ($\vec{\mathbf{L}} = \Delta\vec{\mathbf{L}}$), and it rotates around a horizontal axis, falling over just as we would expect.

Earth itself acts like a gigantic gyroscope. Its angular momentum is along its axis and points at Polaris, the North Star. But Earth is slowly precessing (once in about 26 000 years) due to the torque of the Sun and the Moon on its nonspherical shape.



As seen in figure (a), the forces on a spinning gyroscope are its weight and the supporting force from the stand. These forces create a horizontal torque on the gyroscope, which create a change in angular momentum ΔL that is also horizontal. In figure (b), ΔL and L add to produce a new angular momentum with the same magnitude, but different direction, so that the gyroscope precesses in the direction shown instead of falling over.

Check Your Understanding

Rotational kinetic energy is associated with angular momentum? Does that mean that rotational kinetic energy is a vector?

Show Solution

No, energy is always a scalar whether motion is involved or not. No form of energy has a direction in space and you can see that rotational kinetic energy does not depend on the direction of motion just as linear kinetic energy is independent of the direction of motion.

Section Summary

- Torque is perpendicular to the plane formed by r and $\vec{\mathbf{F}}$ and is the direction your right thumb would point if you curled the fingers of your right hand in the direction of $\vec{\mathbf{F}}$. The direction of the torque is thus the same as that of the angular momentum it produces.
- The gyroscope precesses around a vertical axis, since the torque is always horizontal and perpendicular to $\vec{\mathbf{L}}$. If the gyroscope is not spinning, it acquires angular momentum in the direction of the torque ($\vec{\mathbf{L}} = \Delta\vec{\mathbf{L}}$), and it rotates about a horizontal axis, falling over just as we would expect.
- Earth itself acts like a gigantic gyroscope. Its angular momentum is along its axis and points at Polaris, the North Star.

Conceptual Questions

While driving his motorcycle at highway speed, a physics student notices that pulling back lightly on the right handlebar tips the cycle to the left and produces a left turn. Explain why this happens.

Gyroscopes used in guidance systems to indicate directions in space must have an angular momentum that does not change in direction. Yet they are often subjected to large forces and accelerations. How can the direction of their angular momentum be constant when they are accelerated?

Problem Exercises

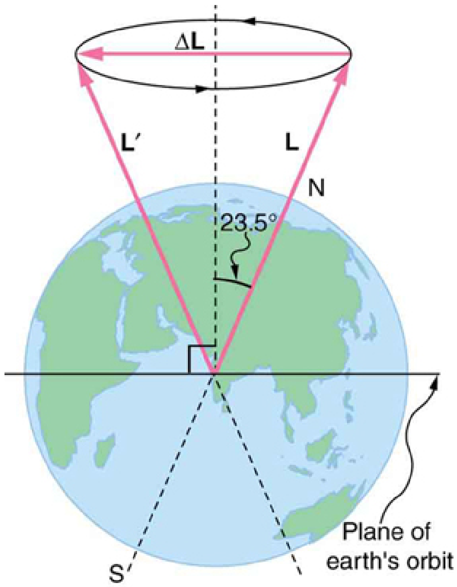
Integrated Concepts

The axis of Earth makes a 23.5° angle with a direction perpendicular to the plane of Earth's orbit. As shown in Figure 5, this axis precesses, making one complete rotation in 25 780 y.

(a) Calculate the change in angular momentum in half this time.

(b) What is the average torque producing this change in angular momentum?

(c) If this torque were created by a single force (it is not) acting at the most effective point on the equator, what would its magnitude be?



The Earth's axis slowly precesses, always making an angle of 23.5° with the direction perpendicular to the plane of Earth's orbit. The change in angular momentum for the two shown positions is quite large, although the magnitude L is unchanged.

Show Solution

(a) $5.64 \times 10^{33} \text{kg} \cdot \text{m}^2/\text{s}$

(b) $1.39 \times 10^{22} \text{N} \cdot \text{m}$

(c) $2.17 \times 10^{15} \text{N}$

Glossary

right-hand rule
: direction of angular velocity ω and angular momentum L in which the thumb of your right hand points when you curl your fingers in the direction of the disk's rotation