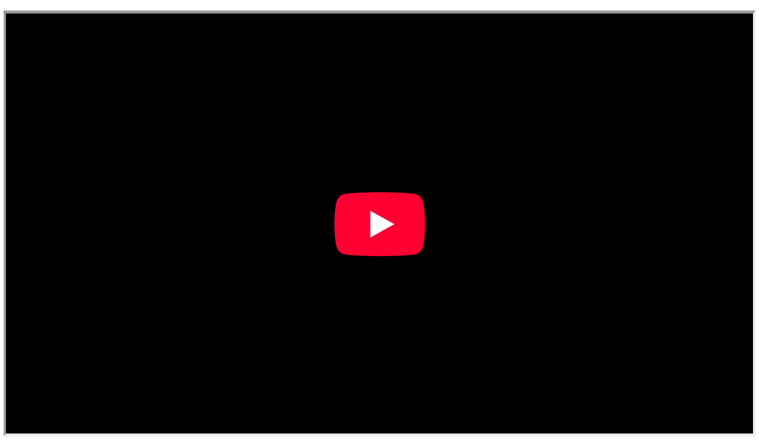
Introduction to Linear Momentum and Collisions

Each rugby player has great momentum, which will affect the outcome of their collisions with each other and the ground. (credit: ozzzie, Flickr)

We use the term momentum in various ways in everyday language, and most of these ways are consistent with its precise scientific definition. We speak of sports teams or politicians gaining and maintaining the momentum to win. We also recognize that momentum has something to do with collisions. For example, looking at the rugby players in the photograph colliding and falling to the ground, we expect their momenta to have great effects in the resulting collisions. Generally, momentum implies a tendency to continue on course—to move in the same direction—and is associated with great mass and speed.

Momentum, like energy, is important because it is conserved. Only a few physical quantities are conserved in nature, and studying them yields fundamental insight into how nature works, as we shall see in our study of momentum.



Linear Momentum and Force

- Define linear momentum.
- Explain the relationship between momentum and force.
- State Newton's second law of motion in terms of momentum.
- Calculate momentum given mass and velocity.

Linear Momentum

The scientific definition of linear momentum is consistent with most people's intuitive understanding of momentum: a large, fast-moving object has greater momentum than a smaller, slower object. **Linear momentum** is defined as the product of a system's mass multiplied by its velocity. In symbols, linear momentum is expressed as

$$\vec{\mathbf{p}}=m\vec{\mathbf{v}}.$$

Momentum is directly proportional to the object's mass and also its velocity. Thus the greater an object's mass or the greater its velocity, the greater its momentum. Momentum $\vec{\mathbf{p}}$ is a vector having the same direction as the velocity $\vec{\mathbf{v}}$. The SI unit for momentum is $\text{kg}\cdot\text{m/s}$.

Linear Momentum

Linear momentum is defined as the product of a system's mass multiplied by its velocity:

$$\vec{\mathbf{p}}=m\vec{\mathbf{v}}.$$

Calculating Momentum: A Football Player and a Football

(a) Calculate the momentum of a 110-kg football player running at 8.00 m/s. (b) Compare the player's momentum with the momentum of a hard-thrown 0.410-kg football that has a speed of 25.0 m/s.

Strategy

No information is given regarding direction, and so we can calculate only the magnitude of the momentum, p. (As usual, a symbol that is in italics is a magnitude, whereas one that is italicized, boldfaced, and has an arrow is a vector.) In both parts of this example, the magnitude of momentum can be calculated directly from the definition of momentum given in the equation, which becomes

$$p=mv$$

when only magnitudes are considered.

Solution for (a)

To determine the momentum of the player, substitute the known values for the player's mass and speed into the equation.

$$p_{\text{player}}=(110\text{kg})(8.00\text{m/s})=880\text{kg}\cdot\text{m/s}$$

Solution for (b)

To determine the momentum of the ball, substitute the known values for the ball's mass and speed into the equation.

$$p_{\text{ball}}=(0.410\text{kg})(25.0\text{m/s})=10.3\text{kg}\cdot\text{m/s}$$

The ratio of the player's momentum to that of the ball is

$$\frac{p_{\text{player}}}{p_{\text{ball}}}=\frac{880}{10.3}=85.9.$$

Discussion

Although the ball has greater velocity, the player has a much greater mass. Thus the momentum of the player is much greater than the momentum of the football, as you might guess. As a result, the player's motion is only slightly affected if he catches the ball. We shall quantify what happens in such collisions in terms of momentum in later sections.

Momentum and Newton's Second Law

The importance of momentum, unlike the importance of energy, was recognized early in the development of classical physics. Momentum was deemed so important that it was called the "quantity of motion." Newton actually stated his **second law of motion** in terms of momentum: The net external force equals the change in momentum of a system divided by the time over which it changes. Using symbols, this law is

$$\vec{\mathbf{F}}_{\text{net}}=\frac{\Delta\vec{\mathbf{p}}}{\Delta t},$$

where $\vec{\mathbf{F}}_{\text{net}}$ is the net external force, $\Delta\vec{\mathbf{p}}$ is the change in momentum, and Δt is the change in time.

Newton's Second Law of Motion in Terms of Momentum

The net external force equals the change in momentum of a system divided by the time over which it changes.

$$\vec{\mathbf{F}}_{\text{net}}=\frac{\Delta\vec{\mathbf{p}}}{\Delta t}$$

Making Connections: Force and Momentum

Force and momentum are intimately related. Force acting over time can change momentum, and Newton's second law of motion, can be stated in its most broadly applicable form in terms of momentum. Momentum continues to be a key concept in the study of atomic and subatomic particles in quantum mechanics.

This statement of Newton's second law of motion includes the more familiar $\vec{\mathbf{F}}_{\text{net}}=m\vec{\mathbf{a}}$ as a special case. We can derive this form as follows. First, note that the change in momentum $\Delta\vec{\mathbf{p}}$ is given by

$$\Delta\vec{\mathbf{p}}=\Delta(m\vec{\mathbf{v}}).$$

If the mass of the system is constant, then

$$\Delta(m\vec{\mathbf{v}})=m\Delta\vec{\mathbf{v}}.$$

So that for constant mass, Newton's second law of motion becomes

$$\vec{\mathbf{F}}_{\text{net}}=\frac{\Delta\vec{\mathbf{p}}}{\Delta t}=\frac{m\Delta\vec{\mathbf{v}}}{\Delta t}.$$

Because $\frac{\Delta\vec{\mathbf{v}}}{\Delta t}=\vec{\mathbf{a}}$, we get the familiar equation

$$\vec{\mathbf{F}}_{\text{net}}=m\vec{\mathbf{a}}$$

when the mass of the system is constant.

Newton's second law of motion stated in terms of momentum is more generally applicable because it can be applied to systems where the mass is changing, such as rockets, as well as to systems of constant mass. We will consider systems with varying mass in some detail**;** however, the relationship between momentum and force remains useful when mass is constant, such as in the following example.

Calculating Force: Venus Williams' Racquet

During the 2007 French Open, Venus Williams hit the fastest recorded serve in a premier women's match, reaching a speed of 58 m/s (209 km/h). What is the average force exerted on the 0.057-kg tennis ball by Venus Williams' racquet, assuming that the ball's speed just after impact is 58 m/s, that the initial horizontal component of the velocity before impact is negligible, and that the ball remained in contact with the racquet for 5.0 ms (milliseconds)?

Strategy

This problem involves only one dimension because the ball starts from having no horizontal velocity component before impact. Newton's second law stated in terms of momentum is then written as

$$\vec{\mathbf{F}}_{\text{net}}=\frac{\Delta\vec{\mathbf{p}}}{\Delta t}.$$

As noted above, when mass is constant, the change in momentum is given by

$$\Delta p=m\Delta v=m(v_f-v_i).$$

In this example, the velocity just after impact and the change in time are given; thus, once Δp is calculated, $F_{\text{net}}=\frac{\Delta p}{\Delta t}$ can be used to find the force.

Solution

To determine the change in momentum, substitute the values for the initial and final velocities into the equation above.

$$\begin{aligned}\Delta p &= m(v_f-v_i)\\ \Delta p &= (0.057\text{ kg})(58\text{ m/s}-0\text{ m/s})\\ \Delta p &= 3.306\text{ kg}\cdot\text{m/s}\approx 3.3\text{ kg}\cdot\text{m/s}\end{aligned}$$

Now the magnitude of the net external force can determined by using $F_{\text{net}}=\frac{\Delta p}{\Delta t}$:

$$\begin{aligned}F_{\text{net}} &= \frac{\Delta p}{\Delta t}=\frac{3.306\text{ kg}\cdot\text{m/s}}{5.0\times10^{-3}\text{ s}}\\ F_{\text{net}} &= 661\text{ N}\approx 660\text{ N},\end{aligned}$$

where we have retained only two significant figures in the final step.

Discussion

This quantity was the average force exerted by Venus Williams' racquet on the tennis ball during its brief impact (note that the ball also experienced the 0.56-N force of gravity, but that force was not due to the racquet). This problem could also be solved by first finding the acceleration and then using $F_{\text{net}}=ma$, but one additional step would be required compared with the strategy used in this example.

Section Summary

- Linear momentum (**momentum** for brevity) is defined as the product of a system's mass multiplied by its velocity.

- In symbols, linear momentum $\vec{\mathbf{p}}$ is defined to be
  $\vec{\mathbf{p}}=m\vec{\mathbf{v}}$,

  where m is the mass of the system and $\vec{\mathbf{v}}$ is its velocity.
- The SI unit for momentum is $\text{kg} \cdot \text{m/s}$.
- Newton's second law of motion in terms of momentum states that the net external force equals the change in momentum of a system divided by the time over which it changes.
- In symbols, Newton's second law of motion is defined to be

$$\vec{\mathbf{F}}_{\text{net}}=\frac{\Delta\vec{\mathbf{p}}}{\Delta t},$$

$\vec{\mathbf{F}}_{\text{net}}$ is the net external force, $\Delta\vec{\mathbf{p}}$ is the change in momentum, and Δt is the change time.

Conceptual Questions

An object that has a small mass and an object that has a large mass have the same momentum. Which object has the largest kinetic energy?

An object that has a small mass and an object that has a large mass have the same kinetic energy. Which mass has the largest momentum?

Professional Application

Football coaches advise players to block, hit, and tackle with their feet on the ground rather than by leaping through the air. Using the concepts of momentum, work, and energy, explain how a football player can be more effective with his feet on the ground.

How can a small force impart the same momentum to an object as a large force?

Problems & Exercises

(a) Calculate the momentum of a 2000-kg elephant charging a hunter at a speed of 7.50m/s.

(b) Compare the elephant's momentum with the momentum of a 0.0400-kg tranquilizer dart fired at a speed of 600m/s.

(c) What is the momentum of the 90.0-kg hunter running at 7.40m/s after missing the elephant?

Show Solution

Strategy

This problem applies the definition of linear momentum $p=mv$ to calculate the momentum of each object. For part (b), we compare the momenta by finding their ratio.

Solution for (a)

Using the definition of momentum:

$$p_{\text{elephant}}=m_{\text{elephant}}v_{\text{elephant}}$$
$$p_{\text{elephant}}=(2000\text{kg})(7.50\text{m/s})=1.50\times10^{4}\text{kg}\cdot\text{m/s}$$

Solution for (b)

First, calculate the dart's momentum:

$$p_{\text{dart}}=m_{\text{dart}}v_{\text{dart}}=(0.0400\text{kg})(600\text{m/s})=24.0\text{kg}\cdot\text{m/s}$$

Now find the ratio:

$$\frac{p_{\text{elephant}}}{p_{\text{dart}}}=\frac{1.50\times10^{4}\text{kg}\cdot\text{m/s}}{24.0\text{kg}\cdot\text{m/s}}=625$$

Solution for (c)

$$p_{\text{hunter}}=m_{\text{hunter}}v_{\text{hunter}}=(90.0\text{kg})(7.40\text{m/s})=6.66\times10^{2}\text{kg}\cdot\text{m/s}$$

Discussion

Despite the dart traveling 80 times faster than the elephant, the elephant's enormous mass (50,000 times the dart's mass) results in a momentum 625 times larger. This illustrates why momentum depends on both mass and velocity. The hunter's momentum is much smaller than the elephant's, explaining why the hunter wisely chose to run!

Answer

(a) The elephant's momentum is **1.50×10^4 kg·m/s**.

(b) The elephant's momentum is **625 times** greater than the dart's momentum.

(c) The hunter's momentum is **666 kg·m/s** (or 6.66×10^2 kg·m/s).

(a) What is the mass of a large ship that has a momentum of $1.60 \times 10^9 \text{kg} \cdot \text{m/s}$, when the ship is moving at a speed of 48.0km/h ?

(b) Compare the ship's momentum to the momentum of a 1100-kg artillery shell fired at a speed of 1200m/s.

Show Solution

Strategy

For part (a), we use the definition of momentum $p = mv$ and solve for mass. We must first convert the speed from km/h to m/s. For part (b), we calculate the shell's momentum and find the ratio.

Solution for (a)

First, convert the ship's speed to SI units:

$$v = 48.0 \frac{\text{km}}{\text{h}} \times \frac{1000 \text{m}}{1 \text{ km}} \times \frac{1 \text{ h}}{3600 \text{s}} = 13.33 \text{m/s}$$

Using $p = mv$ and solving for mass:

$$m = \frac{p}{v} = \frac{1.60 \times 10^9 \text{kg} \cdot \text{m/s}}{13.33 \text{m/s}} = 1.20 \times 10^8 \text{kg}$$

Solution for (b)

Calculate the artillery shell's momentum:

$$p_{\text{shell}} = m_{\text{shell}} v\_\text{shell} = (1100 \text{kg})(1200 \text{m/s}) = 1.32 \times 10^6 \text{kg} \cdot \text{m/s}$$

Find the ratio:

$$\frac{p_{\text{ship}}}{p_{\text{shell}}} = \frac{1.60 \times 10^9 \text{kg} \cdot \text{m/s}}{1.32 \times 10^6 \text{kg} \cdot \text{m/s}} = 1.21 \times 10^3$$

Discussion

Even though the artillery shell travels at supersonic speed (about Mach 3.5), the ship's enormous mass gives it over 1000 times more momentum. This explains why large ships require great distances and time to stop or change direction—their massive momentum must be overcome by relatively small forces from their propulsion systems.

Answer

(a) The ship's mass is **1.20×10^8 kg** (120 million kilograms, or about 120,000 metric tons).

(b) The ship's momentum is approximately **1210 times** (or 1.21×10^3 times) greater than the artillery shell's momentum.

(a) At what speed would a 2.00×10^4-kg airplane have to fly to have a momentum of $1.60 \times 10^9 \text{kg} \cdot \text{m/s}$ (the same as the ship's momentum in the problem above)?

(b) What is the plane's momentum when it is taking off at a speed of 60.0m/s

?

(c) If the ship is an aircraft carrier that launches these airplanes with a catapult, discuss the implications of your answer to (b) as it relates to recoil effects of the catapult on the ship.

Show Solution

Strategy

For part (a), we use $p = mv$ and solve for velocity. For part (b), we calculate the plane's actual momentum during takeoff. For part (c), we apply conservation of momentum to find the ship's recoil velocity.

Solution for (a)

Using $p = mv$ and solving for velocity:

$$v = \frac{p}{m} = \frac{1.60 \times 10^9 \text{kg} \cdot \text{m/s}}{2.00 \times 10^4 \text{kg}} = 8.00 \times 10^4 \text{m/s}$$

Solution for (b)

$$p_{\text{plane}} = m_{\text{plane}} v_{\text{takeoff}} = (2.00 \times 10^4 \text{kg})(60.0 \text{m/s}) = 1.20 \times 10^6 \text{kg}\cdot\text{m/s}$$

Solution for (c)

By conservation of momentum, when the catapult launches the plane forward, the ship recoils backward. The ship's mass from the previous problem is $1.20 \times 10^8 \text{kg}$.

$$m_{\text{ship}} v_{\text{recoil}} = -p_{\text{plane}}$$
$$v_{\text{recoil}} = -\frac{1.20 \times 10^6 \text{kg}\cdot\text{m/s}}{1.20 \times 10^8 \text{kg}} = -0.0100 \text{m/s}$$

Discussion

The airplane's momentum at takeoff is about 1300 times smaller than the ship's momentum. Because the ship is 6000 times more massive than the plane, the recoil effect is imperceptible. This demonstrates why aircraft carriers can launch planes without noticeable movement, making them stable platforms for flight operations.

Answer

(a) The airplane would need to fly at **8.00×10^4 m/s** (80 km/s), which is about 235 times the speed of sound—clearly impossible for conventional aircraft.

(b) The plane's momentum at takeoff is **1.20×10^6 kg·m/s**.

(c) The ship recoils at only **0.0100 m/s** (1 cm/s) backward, which is negligible and demonstrates why aircraft carriers remain stable platforms for flight operations.

(a) What is the momentum of a garbage truck that is $1.20 \times 10^4 \text{kg}$ and is moving at 10.0m/s ?

(b) At what speed would an 8.00-kg trash can have the same momentum as the truck?

Show Solution

Strategy

For part (a), we apply the definition of momentum $p = mv$. For part (b), we set the trash can's momentum equal to the truck's momentum and solve for the required velocity.

Solution for (a)

$$p_{\text{truck}} = m_{\text{truck}} v_{\text{truck}} = (1.20 \times 10^4 \text{kg})(10.0 \text{m/s}) = 1.20 \times 10^5 \text{kg}\cdot\text{m/s}$$

Solution for (b)

For equal momenta: $p_{\text{can}} = p_{\text{truck}}$

$$m_{\text{can}} v_{\text{can}} = 1.20 \times 10^5 \text{kg}\cdot\text{m/s}$$
$$v_{\text{can}} = \frac{1.20 \times 10^5 \text{kg}\cdot\text{m/s}}{8.00 \text{kg}} = 1.50 \times 10^4 \text{m/s}$$

Discussion

This speed is about 44 times the speed of sound and roughly half of Earth's escape velocity! The mass ratio between the truck and trash can is 1500:1, so the trash can would need to travel 1500 times faster than the truck to have equal momentum. This dramatically illustrates how mass and velocity contribute equally to momentum—doubling either one doubles the momentum.

Answer

(a) The garbage truck's momentum is **1.20×10^5 kg·m/s**.

(b) The trash can would need to travel at **1.50×10^4 m/s** (15 km/s, or about 54,000 km/h) to have the same momentum as the garbage truck.

A runaway train car that has a mass of 15 000 kg travels at a speed of 5.4m/s down a track. Compute the time required for a force of 1500 N to bring the car to rest.

Show Solution

Strategy

We use Newton's second law in terms of momentum: $F_{\text{net}} = \frac{\Delta p}{\Delta t}$. The train must lose all its momentum, so we calculate the initial momentum and then find the time required for the given force to produce this change in momentum.

Solution

First, calculate the initial momentum of the train car:

$p_i = mv = (15000\text{kg})(5.4\text{m/s}) = 81000\text{kg}\cdot\text{m/s}$

The final momentum is zero (the car comes to rest), so the change in momentum is:

$\Delta p = p_f - p_i = 0 - 81000\text{kg}\cdot\text{m/s} = -81000\text{kg}\cdot\text{m/s}$

Using Newton's second law in terms of momentum:

$F_{\text{net}} = \Delta p \Delta t$

The force opposing the motion is negative (in the direction opposite to motion), so:

$-1500\text{N} = -81000\text{kg}\cdot\text{m/s}\Delta t$

Solving for time:

$\Delta t = 81000\text{kg}\cdot\text{m/s}1500\text{N} = 54\text{s}$

Discussion

Nearly a minute is needed to stop the train car, which makes sense given its large mass and modest braking force. The ratio of momentum to force gives time, showing why heavy objects require either larger forces or longer times to stop. This is why trains have long stopping distances—even with substantial braking forces, their enormous momentum takes considerable time to reduce to zero.

Answer

The time required to bring the train car to rest is **54 s** (54 seconds).

The mass of Earth is $5.972 \times 10^{24}\text{kg}$ and its orbital radius is an average of $1.496 \times 10^{11}\text{m}$. Calculate its linear momentum.

Show Solution

Strategy

To find Earth's linear momentum, we need its orbital velocity. Earth travels in a nearly circular orbit around the Sun, completing one orbit in one year. We can calculate the orbital velocity from the circumference of the orbit divided by the orbital period, then apply $p = mv$.

Solution

First, calculate the circumference of Earth's orbit:

$C = 2\pi r = 2\pi(1.496\times10^{11}\text{m}) = 9.40\times10^{11}\text{m}$

Convert one year to seconds:

$T = 1\text{ year} = 365.25\text{ days}\times24\text{hday}\times3600\text{sh} = 3.156\times10^{7}\text{s}$

Calculate Earth's orbital velocity:

$v = CT = 9.40\times10^{11}\text{m}3.156\times10^{7}\text{s} = 2.98\times10^{4}\text{m/s}$

Now calculate Earth's momentum:

$p = mv = (5.972\times10^{24}\text{kg})(2.98\times10^{4}\text{m/s}) = 1.78\times10^{29}\text{kg}\cdot\text{m/s}$

Discussion

This is an enormous momentum—about 10^{24} times larger than the ship's momentum from earlier problems. Earth's orbital velocity of about 30 km/s (roughly 108,000 km/h) combined with its massive $6 \times 10^{24}\text{kg}$ mass produces this immense momentum. Despite this huge momentum, Earth's orbit is stable because the Sun's gravitational force continuously provides the centripetal acceleration needed to change the direction of this momentum without changing its magnitude.

Answer

The linear momentum of Earth in its orbit around the Sun is **1.78×10^{29} kg·m/s**.

Glossary

linear momentum
: the product of mass and velocity

second law of motion
: physical law that states that the net external force equals the change in momentum of a system divided by the time over which it changes

Impulse

- Define impulse.
- Describe effects of impulses in everyday life.
- Determine the average effective force using graphical representation.
- Calculate average force and impulse given mass, velocity, and time.

The effect of a force on an object depends on how long it acts, as well as how great the force is. In Example 2, a very large force acting for a short time had a great effect on the momentum of the tennis ball. A small force could cause the same **change in momentum**, but it would have to act for a much longer time. For example, if the ball were thrown upward, the gravitational force (which is much smaller than the tennis racquet's force) would eventually reverse the momentum of the ball. Quantitatively, the effect we are talking about is the change in momentum $\Delta\vec{\mathbf{p}}$.

By rearranging the equation $\vec{\mathbf{F}}$_net $=\Delta\vec{\mathbf{p}}\Delta t$ to be

$$\Delta\vec{\mathbf{p}}=\vec{\mathbf{F}}_{\text{net}}\Delta t,$$

we can see how the change in momentum equals the average net external force multiplied by the time this force acts. The quantity $\vec{\mathbf{F}}$_netΔt is given the name **impulse**. Impulse is the same as the change in momentum.

Impulse: Change in Momentum

Change in momentum equals the average net external force multiplied by the time this force acts.

$$\Delta\vec{\mathbf{p}}=\vec{\mathbf{F}}_{\text{net}}\Delta t$$

The quantity $\vec{\mathbf{F}}_{\text{net}}\Delta t$ is given the name impulse.

There are many ways in which an understanding of impulse can save lives, or at least limbs. The dashboard padding in a car, and certainly the airbags, allow the net force on the occupants in the car to act over a much longer time when there is a sudden stop. The momentum change is the same for an occupant, whether an airbag is deployed or not, but the force (to bring the occupant to a stop) will be much less if it acts over a larger time. Cars today have many plastic components. One advantage of plastics is their lighter weight, which results in better gas mileage. Another advantage is that a car will crumple in a collision, especially in the event of a head-on collision. A longer collision time means the force on the car will be less. Deaths during car races decreased dramatically when the rigid frames of racing cars were replaced with parts that could crumple or collapse in the event of an accident.

Bones in a body will fracture if the force on them is too large. If you jump onto the floor from a table, the force on your legs can be immense if you land stiff-legged on a hard surface. Rolling on the ground after jumping from the table, or landing with a parachute, extends the time over which the force (on you from the ground) acts.

Calculating Magnitudes of Impulses: Two Billiard Balls Striking a Rigid Wall

Two identical billiard balls strike a rigid wall with the same speed, and are reflected without any change of speed. The first ball strikes perpendicular to the wall. The second ball strikes the wall at an angle of 30° from the perpendicular, and bounces off at an angle of 30° from perpendicular to the wall.

(a) Determine the direction of the force on the wall due to each ball.

(b) Calculate the ratio of the magnitudes of impulses on the two balls by the wall.

Strategy for (a)

In order to determine the force on the wall, consider the force on the ball due to the wall using Newton's second law and then apply Newton's third law to determine the direction. Assume the x -axis to be normal to the wall and to be positive in the initial direction of motion. Choose the y-axis to be along the wall in the plane of the second ball's motion. The momentum direction and the velocity direction are the same.

Solution for (a)

The first ball bounces directly into the wall and exerts a force on it in the $+x$ direction. Therefore, the wall exerts a force on the ball in the $-x$ direction. The second ball continues with the same momentum component in the y direction, but reverses its x-component of momentum, as seen by sketching a diagram of the angles involved and keeping in mind the proportionality between velocity and momentum.

These changes mean the change in momentum for both balls is in the $-x$ direction, so the force of the wall on each ball is along the $-x$ direction.

Strategy for (b)

Calculate the change in momentum for each ball, which is equal to the impulse imparted to the ball.

Solution for (b)

Let u be the speed of each ball before and after collision with the wall, and m the mass of each ball. Choose the x-axis and y-axis as previously described, and consider the change in momentum of the first ball which strikes perpendicular to the wall.

$$p_{\text{xi}}=mu \quad ; \quad p_{\text{yi}}=0$$

$$p_{xf}=-mu \quad ; \quad p_{yf}=0$$

Impulse is the change in momentum vector. Therefore the x-component of impulse is equal to $-2mu$ and the y-component of impulse is equal to zero.

Now consider the change in momentum of the second ball.

$$p_{xi}=mu\cos 30^\circ \quad ; \quad p_{yi}=-mu\sin 30^\circ$$
$$p_{xf}=-mu\cos 30^\circ \quad ; \quad p_{yf}=-mu\sin 30^\circ$$

It should be noted here that while p_x changes sign after the collision, p_y does not. Therefore the x-component of impulse is equal to $-2mu\cos 30^\circ$ and the y-component of impulse is equal to zero.

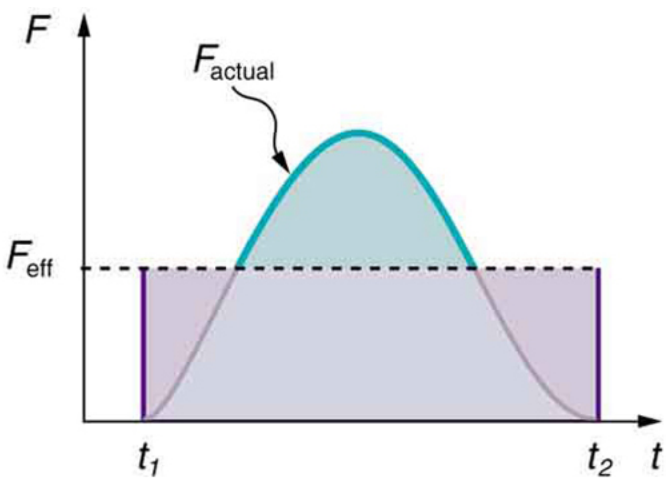
The ratio of the magnitudes of the impulse imparted to the balls is

$$\frac{2mu}{2mu\cos 30^\circ}=\frac{2}{\sqrt{3}}=1.155.$$

Discussion

The direction of impulse and force is the same as in the case of (a); it is normal to the wall and along the negative * x -*direction. Making use of Newton’s third law, the force on the wall due to each ball is normal to the wall along the positive x -direction.

Our definition of impulse includes an assumption that the force is constant over the time interval Δt. *Forces are usually not constant*. Forces vary considerably even during the brief time intervals considered. It is, however, possible to find an average effective force F_{eff} that produces the same result as the corresponding time-varying force. Figure 1 shows a graph of what an actual force looks like as a function of time for a ball bouncing off the floor. The area under the curve has units of momentum and is equal to the impulse or change in momentum between times t_1 and t_2. That area is equal to the area inside the rectangle bounded by F_{eff}, t_1, and t_2. Thus, the impulses and their effects are the same for both the actual and effective forces.



A graph of force versus time with time along the x -axis and force along the y -axis for an actual force and an equivalent effective force. The areas under the two curves are equal.

Making Connections: Take-Home Investigation—Hand Movement and Impulse

Try catching a ball while “giving” with the ball, pulling your hands toward your body. Then, try catching a ball while keeping your hands still. Hit water in a tub with your full palm. After the water has settled, hit the water again by diving your hand with your fingers first into the water. (Your full palm represents a swimmer doing a belly flop and your diving hand represents a swimmer doing a dive.) Explain what happens in each case and why. Which orientations would you advise people to avoid and why?

Making Connections: Constant Force and Constant Acceleration

The assumption of a constant force in the definition of impulse is analogous to the assumption of a constant acceleration in kinematics. In both cases, nature is adequately described without the use of calculus.

Section Summary

- Impulse, or change in momentum, equals the average net external force multiplied by the time this force acts: $\Delta\vec{\mathbf{p}}=\vec{\mathbf{F}}_{\text{net}}\Delta t$.
- Forces are usually not constant over a period of time.

Conceptual Questions

Professional Application

Explain in terms of impulse how padding reduces forces in a collision. State this in terms of a real example, such as the advantages of a carpeted vs. tile floor for a day care center.

While jumping on a trampoline, sometimes you land on your back and other times on your feet. In which case can you reach a greater height and why?

Professional Application

Tennis racquets have “sweet spots.” If the ball hits a sweet spot then the player’s arm is not jarred as much as it would be otherwise. Explain why this is the case.

Problems & Exercises

A bullet is accelerated down the barrel of a gun by hot gases produced in the combustion of gun powder. What is the average force exerted on a 0.0300-kg bullet to accelerate it to a speed of 600 m/s in a time of 2.00 ms (milliseconds)?

Show Solution

Strategy

We use the impulse-momentum theorem: $F_{net}\Delta t = \Delta p$. The bullet starts from rest and reaches a final velocity, so we can calculate the change in momentum and divide by the time interval to find the average force.

Solution

The bullet starts at rest, so $v_i = 0$.

Calculate the change in momentum:

$$\Delta p = m(v_f - v_i) = (0.0300\,\text{kg})(600\,\text{m/s} - 0) = 18.0\,\text{kg}\cdot\text{m/s}$$

Convert time to seconds:

$$\Delta t = 2.00\text{ ms} = 2.00\times10^{-3}\,\text{s}$$

Apply the impulse-momentum theorem to find the average force:

$$F_{avg} = \frac{\Delta p}{\Delta t} = \frac{18.0\,\text{kg}\cdot\text{m/s}}{2.00\times10^{-3}\,\text{s}} = 9.00\times10^{3}\,\text{N}$$

The average force exerted on the bullet is $9.00\times10^{3}\,\text{N}$ (9000 N, or about 2000 pounds of force).

Discussion

This is an enormous force—about 30,000 times the bullet’s weight! However, the force acts for only 2 milliseconds, which is why the impulse (and thus momentum change) is manageable. By Newton’s third law, an equal and opposite force acts on the gun, causing the recoil felt by the shooter.

Professional Application

A car moving at 10 m/s crashes into a tree and stops in 0.26 s. Calculate the force the seat belt exerts on a passenger in the car to bring him to a halt. The mass of the passenger is 70 kg.

Show Solution

Strategy

We apply the impulse-momentum theorem. The passenger has initial momentum from moving with the car and must be brought to rest. The seat belt provides the force over the collision time to change the passenger’s momentum.

Solution

Calculate the initial momentum of the passenger:

$$p_i = mv_i = (70\,\text{kg})(10\,\text{m/s}) = 700\,\text{kg}\cdot\text{m/s}$$

The final momentum is zero (passenger comes to rest):

$$p_f = 0$$

Calculate the change in momentum:

$$\Delta p = p_f - p_i = 0 - 700\,\text{kg}\cdot\text{m/s} = -700\,\text{kg}\cdot\text{m/s}$$

Apply the impulse-momentum theorem:

$F\_avg=\Delta p\Delta t=-700 kg\cdot m/s 0.26 s=-2700 N$

The magnitude of the force is approximately $2.7\times10^{3} N$ (about 600 pounds), directed opposite to the passenger’s initial motion.

Discussion

While 2700 N is a substantial force (roughly 4 times the passenger’s weight), this is survivable because the seat belt distributes the force across the chest and pelvis. Without a seat belt, the passenger would continue at 10 m/s until hitting the dashboard or windshield, stopping in a much shorter time (perhaps 0.01 s), resulting in forces 26 times larger—likely fatal.

A person slaps her leg with her hand, bringing her hand to rest in 2.50 milliseconds from an initial speed of 4.00 m/s. (a) What is the average force exerted on the leg, taking the effective mass of the hand and forearm to be 1.50 kg? (b) Would the force be any different if the woman clapped her hands together at the same speed and brought them to rest in the same time? Explain why or why not.

Show Solution

Strategy

We use the impulse-momentum theorem to find the force. The hand has initial momentum and comes to rest, so we calculate the momentum change and divide by the collision time.

Solution

(a) Force on the leg:

1. Calculate the change in momentum of the hand:

$$\Delta p=m(v_f-v_i)=(1.50 kg)(0-4.00 m/s)=-6.00 kg\cdot m/s$$

1. Convert time to seconds:

$$\Delta t=2.50\ ms=2.50\times10^{-3} s$$

1. Apply the impulse-momentum theorem:

$$F_{avg}=\Delta p\Delta t=-6.00 kg\cdot m/s 2.50\times10^{-3} s=-2.40\times10^{3} N$$

By Newton’s third law, the force on the leg equals the force on the hand in magnitude but opposite in direction.

(b) Force when clapping hands:

The force on each hand would have the same magnitude as that found in part (a) because:

- Each hand has the same mass (1.50 kg)
- Each hand has the same initial speed (4.00 m/s)
- Each hand stops in the same time (2.50 ms)
- Therefore, each hand has the same change in momentum

By Newton’s third law, the forces on the two hands are equal in magnitude but opposite in direction—each hand exerts 2400 N on the other.

Discussion

This force of 2400 N is about 540 pounds—explaining why a hard slap hurts! The short collision time (2.5 ms) results in a large force despite the modest speed and mass involved. This is why padding and cushioning help reduce impact forces: they extend the collision time, reducing the force for the same momentum change.

(a) The average force exerted on the leg is $2.40\times10^{3} N$ toward the leg.

(b) The force on each hand when clapping is the same: $2.40\times10^{3} N$.

Professional Application

A professional boxer hits his opponent with a 1000-N horizontal blow that lasts for 0.150 s. (a) Calculate the impulse imparted by this blow. (b) What is the opponent’s final velocity, if his mass is 105 kg and he is motionless in midair when struck near his center of mass? (c) Calculate the recoil velocity of the opponent’s 10.0-kg head if hit in this manner, assuming the head does not initially transfer significant momentum to the boxer’s body. (d) Discuss the implications of your answers for parts (b) and (c).

Show Solution

Strategy

Impulse equals force times time: $J=F\Delta t$. Once we have the impulse, we can find the velocity change using $J=\Delta p=m\Delta v$.

Solution

(a) Impulse from the blow:

$$J = F\Delta t = (1000\text{N})(0.150\text{s}) = 150\text{N}\cdot\text{s} = 150\text{kg}\cdot\text{m/s}$$

(b) Final velocity if struck at center of mass:

The opponent starts at rest, so $v_i = 0$.

Using the impulse-momentum theorem:

$$J = m\Delta v = m(v_f - v_i)$$

Solving for final velocity:

$$v_f = \frac{J}{m} = \frac{150\text{kg}\cdot\text{m/s}}{105\text{kg}} = 1.43\text{m/s}$$

(c) Recoil velocity if struck in the head:

If the same impulse is delivered to just the head:

$$v_f = \frac{J}{m_{\text{head}}} = \frac{150\text{kg}\cdot\text{m/s}}{10.0\text{kg}} = 15.0\text{m/s}$$

(d) Implications:

The head velocity (15.0 m/s) is more than 10 times greater than the body velocity (1.43 m/s). This dramatic difference explains why head punches are so dangerous in boxing:

- The rapid acceleration of the head causes the brain to impact the skull
- A 15 m/s velocity change can cause concussion or worse
- Boxing rules require gloves partly to extend collision time and reduce peak forces
- Body shots, while painful, are much less likely to cause brain injury

Discussion

This analysis explains the fundamental danger of head trauma in contact sports. The same impulse produces vastly different effects depending on the mass involved. Protective headgear works by increasing collision time and distributing force, not by changing the impulse delivered.

(a) The impulse imparted by the blow is $150\text{N}\cdot\text{s}$ (or $150\text{kg}\cdot\text{m/s}$).

(b) If struck in the body's center of mass, the opponent's final velocity is 1.43m/s in the direction of the blow.

(c) If hit in the head, the head would recoil at 15.0m/s.

(d) See explanation above.

Professional Application

Suppose a child drives a bumper car head on into the side rail, which exerts a force of 4000 N on the car for 0.200 s. (a) What impulse is imparted by this force? (b) Find the final velocity of the bumper car if its initial velocity was 2.80 m/s and the car plus driver have a mass of 200 kg. You may neglect friction between the car and floor.

Show Solution

Strategy

For part (a), impulse is force times time. For part (b), we apply the impulse-momentum theorem, noting that the force from the wall opposes the car's motion.

Solution

(a) Impulse from the rail:

$$J = F\Delta t = (4000\text{N})(0.200\text{s}) = 800\text{N}\cdot\text{s} = 800\text{kg}\cdot\text{m/s}$$

(b) Final velocity of the bumper car:

Let positive direction be toward the wall (initial direction of motion). The wall exerts a force in the negative direction.

1. Calculate initial momentum:

$$p_i = mv_i = (200\text{kg})(2.80\text{m/s}) = 560\text{kg}\cdot\text{m/s}$$

1. The impulse is negative (away from wall):

$$J = -800\text{kg}\cdot\text{m/s}$$

1. Calculate final momentum:

$$p_f = p_i + J = 560\,\text{kg}\cdot\text{m/s} + (-800\,\text{kg}\cdot\text{m/s}) = -240\,\text{kg}\cdot\text{m/s}$$

1. Calculate final velocity:

$$v_f = \frac{p_f}{m} = \frac{-240\,\text{kg}\cdot\text{m/s}}{200\,\text{kg}} = -1.20\,\text{m/s}$$

Discussion

The car reverses direction after hitting the rail, which is exactly what we expect from a bumper car collision. The wall delivers enough impulse to not only stop the car (which would require 560 kg·m/s) but also to give it momentum in the opposite direction. This is the fun of bumper cars—the collisions are designed to be bouncy while the padding extends the collision time to keep forces at safe levels.

(a) The impulse imparted is $800\,\text{kg}\cdot\text{m/s}$ directed away from the wall (opposite to the car's initial motion).

(b) The final velocity is $1.20\,\text{m/s}$ away from the wall (the car bounces back).

Professional Application

One hazard of space travel is debris left by previous missions. There are several thousand objects orbiting Earth that are large enough to be detected by radar, but there are far greater numbers of very small objects, such as flakes of paint. Calculate the force exerted by a 0.100-mg chip of paint that strikes a spacecraft window at a relative speed of $4.00\times10^{3}\,\text{m/s}$, given the collision lasts $6.00\times10^{-8}\,\text{s}$.

Show Solution

Strategy

We use the impulse-momentum theorem. The paint chip comes to rest upon impact (or embeds in the window), so its entire momentum is transferred. We need to convert the mass to kg and apply $F = \frac{\Delta p}{\Delta t}$.

Solution

Convert mass to kilograms:

$$m = 0.100\text{ mg} = 0.100\times10^{-6}\,\text{kg} = 1.00\times10^{-7}\,\text{kg}$$

Calculate the change in momentum (assuming the chip stops):

$$\Delta p = m(v_f - v_i) = (1.00\times10^{-7}\,\text{kg})(0 - 4.00\times10^{3}\,\text{m/s})$$
$$\Delta p = -4.00\times10^{-4}\,\text{kg}\cdot\text{m/s}$$

Apply the impulse-momentum theorem:

$$F = \frac{|\Delta p|}{\Delta t} = \frac{4.00\times10^{-4}\,\text{kg}\cdot\text{m/s}}{6.00\times10^{-8}\,\text{s}} = 6.67\times10^{3}\,\text{N}$$

The force exerted by the paint chip on the spacecraft window is approximately $6.67\times10^{3}\,\text{N}$ (about 1500 pounds of force).

Discussion

This result is remarkable—a tiny paint flake weighing just 0.1 milligrams exerts a force of nearly 7000 N! This occurs because:

- Orbital velocities are extremely high (4 km/s is typical)
- The collision time is incredibly short (60 nanoseconds)
- Even though the momentum is small, dividing by such a tiny time produces a huge force

This explains why spacecraft windows are made of multiple layers of reinforced glass and why space debris is a serious hazard. The International Space Station has had windows replaced due to damage from such impacts.

Professional Application

A 75.0-kg person is riding in a car moving at 20.0 m/s when the car runs into a bridge abutment. (a) Calculate the average force on the person if he is stopped by a padded dashboard that compresses an average of 1.00 cm. (b) Calculate the average force on the person if he is stopped by an air bag that compresses an average of 15.0 cm.

Show Solution

Strategy

We need to find the stopping time from the distance and velocity, then use the impulse-momentum theorem. Using kinematics with constant deceleration: $v^2 = v_0^2 + 2a\Delta x$ gives us acceleration, then $\Delta t = \frac{\Delta v}{a}$ gives time.

Solution

(a) Force with padded dashboard:

1. Using kinematics to find the deceleration:

$$v^2 = v_0^2 + 2a\Delta x$$
$$0 = (20.0\,\text{m/s})^2 + 2a(0.0100\,\text{m})$$
$$a = -\frac{(20.0\,\text{m/s})^2}{2(0.0100\,\text{m})} = -2.00\times10^4\,\text{m/s}^2$$

1. Find the stopping time:

$$\Delta t = \frac{v - v_0}{a} = \frac{0 - 20.0\,\text{m/s}}{-2.00\times10^4\,\text{m/s}^2} = 1.00\times10^{-3}\,\text{s}$$

1. Calculate the force using impulse-momentum:

$$F = \frac{\Delta p}{\Delta t} = \frac{m\Delta v}{\Delta t} = \frac{(75.0\,\text{kg})(20.0\,\text{m/s})}{1.00\times10^{-3}\,\text{s}} = 1.50\times10^6\,\text{N}$$

(b) Force with airbag:

1. Using kinematics with 15.0 cm compression:

$$a = -\frac{(20.0\,\text{m/s})^2}{2(0.150\,\text{m})} = -1.33\times10^3\,\text{m/s}^2$$

1. Find the stopping time:

$$\Delta t = \frac{0 - 20.0\,\text{m/s}}{-1.33\times10^3\,\text{m/s}^2} = 1.50\times10^{-2}\,\text{s}$$

1. Calculate the force:

$$F = \frac{(75.0\,\text{kg})(20.0\,\text{m/s})}{1.50\times10^{-2}\,\text{s}} = 1.00\times10^5\,\text{N}$$

Discussion

The airbag reduces the force by a factor of 15—exactly the ratio of the stopping distances! The padded dashboard force of 1.5 million N (about 2000 times body weight) would likely be fatal, while the airbag force of 100,000 N (about 135 times body weight), though still very large, is survivable because it's distributed across the body. This dramatically illustrates why airbags save lives: they increase the stopping distance and time, reducing the peak force.

(a) The force on the person from the padded dashboard is $1.50\times10^6\,\text{N}$ (about 337,000 pounds).

(b) The force on the person from the airbag is $1.00\times10^5\,\text{N}$ (about 22,500 pounds).

Professional Application

Military rifles have a mechanism for reducing the recoil forces of the gun on the person firing it. An internal part recoils over a relatively large distance and is stopped by damping mechanisms in the gun. The larger distance reduces the average force needed to stop the internal part. (a) Calculate the recoil velocity of a 1.00-kg plunger that directly interacts with a 0.0200-kg bullet fired at 600 m/s from the gun. (b) If this part is stopped over a distance of 20.0 cm, what average force is exerted upon it by the gun? (c) Compare this to the force exerted on the gun if the bullet is accelerated to its velocity in 10.0 ms (milliseconds).

Show Solution

Strategy

For part (a), we use conservation of momentum between the bullet and plunger. For part (b), we use kinematics and Newton's second law to find the stopping force. For part (c), we calculate the force from the bullet's acceleration.

Solution

(a) Recoil velocity of the plunger:

By conservation of momentum (initial momentum is zero):

$$m_{\text{bullet}}v_{\text{bullet}} + m_{\text{plunger}}v_{\text{plunger}} = 0$$

Solving for the plunger velocity:

$$v_{\text{plunger}} = -\frac{m_{\text{bullet}}v_{\text{bullet}}}{m_{\text{plunger}}}$$
$$v_{\text{plunger}} = -\frac{(0.0200\,\text{kg})(600\,\text{m/s})}{1.00\,\text{kg}} = -12.0\,\text{m/s}$$

(b) Force exerted on the plunger:

1. Using kinematics to find the deceleration:

$$v^2 = v_0^2 + 2a\Delta x$$
$$0 = (12.0\,\text{m/s})^2 + 2a(0.200\,\text{m})$$
$$a = -\frac{(12.0\,\text{m/s})^2}{2(0.200\,\text{m})} = -360\,\text{m/s}^2$$

1. Calculate the force:

$$F = m_{\text{plunger}}|a| = (1.00\,\text{kg})(360\,\text{m/s}^2) = 360\,\text{N}$$

(c) Comparison with bullet acceleration force:

1. Calculate the force to accelerate the bullet:

$$a_{\text{bullet}} = \frac{\Delta v}{\Delta t} = \frac{600\,\text{m/s}}{10.0\times10^{-3}\,\text{s}} = 6.00\times10^4\,\text{m/s}^2$$
$$F_{\text{bullet}} = m_{\text{bullet}}a_{\text{bullet}} = (0.0200\,\text{kg})(6.00\times10^4\,\text{m/s}^2) = 1200\,\text{N}$$

1. Compare the forces:

$$\frac{F_{\text{bullet}}}{F_{\text{plunger}}} = \frac{1200\,\text{N}}{360\,\text{N}} = 3.33$$

Discussion

The recoil-reducing mechanism decreases the force felt by the shooter by a factor of more than 3. This is achieved by allowing the internal plunger to recoil over a larger distance (20 cm) compared to the short distance over which the bullet accelerates. The same momentum must be absorbed, but extending the distance (and time) reduces the force. This principle is used in many firearms to improve accuracy and reduce shooter fatigue.

(a) The recoil velocity of the plunger is $12.0\,\text{m/s}$ in the direction opposite to the bullet.

(b) The average force exerted on the plunger is $360\,\text{N}$.

(c) The force on the gun from accelerating the bullet ($1200\,\text{N}$) is 3.33 times larger than the force on the plunger mechanism ($360\,\text{N}$).

A cruise ship with a mass of $1.00\times10^7\,\text{kg}$ strikes a pier at a speed of 0.750 m/s. It comes to rest 6.00 m later, damaging the ship, the pier, and the tugboat captain's finances. Calculate the average force exerted on the pier using the concept of impulse. (Hint: First calculate the time it took to bring the ship to rest.)

Show Solution

Strategy

First, we use kinematics to find the stopping time, then apply the impulse-momentum theorem. The average velocity during uniform deceleration is half the initial velocity.

Solution

Find the stopping time using the average velocity:

$$v\_\text{avg} = \frac{v_i + v_f}{2} = \frac{0.750\,\text{m/s} + 0}{2} = 0.375\,\text{m/s}$$
$$\Delta t = \frac{\Delta x}{v\_\text{avg}} = \frac{6.00\,\text{m}}{0.375\,\text{m/s}} = 16.0\,\text{s}$$

Calculate the ship's initial momentum:

$$p_i = mv_i = (1.00\times10^7\,\text{kg})(0.750\,\text{m/s}) = 7.50\times10^6\,\text{kg}\cdot\text{m/s}$$

Apply the impulse-momentum theorem:

$$F\_\text{avg} = \frac{|\Delta p|}{\Delta t} = \frac{|0 - 7.50\times10^6\,\text{kg}\cdot\text{m/s}|}{16.0\,\text{s}} = 4.69\times10^5\,\text{N}$$

The average force exerted on the pier is $4.69\times10^5\,\text{N}$ (about 105,000 pounds or 53 tons) in the ship's original direction of motion.

Discussion

Despite the relatively slow speed of 0.750 m/s (about 1.5 mph—walking pace), the ship's enormous mass results in a force of nearly half a million newtons! The 16-second stopping time helps reduce this force—if the ship stopped more abruptly (say, in 1 second), the force would be 16 times larger (7.5 million N). This illustrates why docking procedures are so careful and why tugboats are essential for maneuvering large vessels.

Calculate the final speed of a 110-kg rugby player who is initially running at 8.00 m/s but collides head-on with a padded goalpost and experiences a backward force of $1.76 \times 10^4 \text{N}$ for $5.50 \times 10^{-2} \text{s}$.

Show Solution

Strategy

We apply the impulse-momentum theorem. The backward force produces a negative impulse that changes the player's momentum. We calculate the impulse and use it to find the change in velocity.

Solution

Calculate the impulse (force is backward, so negative):

$$J = F\Delta t = (-1.76\times10^4 \text{N})(5.50\times10^{-2} \text{s}) = -968 \text{kg}\cdot\text{m/s}$$

Calculate the initial momentum:

$$p_i = mv_i = (110 \text{kg})(8.00 \text{m/s}) = 880 \text{kg}\cdot\text{m/s}$$

Find the final momentum:

$$p_f = p_i + J = 880 \text{kg}\cdot\text{m/s} + (-968 \text{kg}\cdot\text{m/s}) = -88 \text{kg}\cdot\text{m/s}$$

Calculate the final velocity:

$$v_f = \frac{p_f}{m} = \frac{-88 \text{kg}\cdot\text{m/s}}{110 \text{kg}} = -0.80 \text{m/s}$$

The final speed of the rugby player is 0.80 m/s, moving backward (away from the goalpost).

Discussion

The player not only stops but bounces back at 0.80 m/s. The impulse of 968 kg·m/s is larger than the initial momentum of 880 kg·m/s, causing the reversal. The padded goalpost extends the collision time to 55 ms, limiting the force to about 17,600 N—still a substantial 16 times body weight, but survivable due to the padding. Without padding, the collision time would be much shorter and the force could cause serious injury.

Water from a fire hose is directed horizontally against a wall at a rate of 50.0 kg/s and a speed of 42.0 m/s. Calculate the magnitude of the force exerted on the wall, assuming the water's horizontal momentum is reduced to zero.

Show Solution

Strategy

This is a continuous flow problem. We use the rate form of Newton's second law: $F = \frac{\Delta p}{\Delta t} = \frac{\Delta m}{\Delta t} \times \Delta v$. The mass flow rate and velocity change give us the force directly.

Solution

The mass flow rate is:

$$\frac{\Delta m}{\Delta t} = 50.0 \text{ kg/s}$$

The velocity change for each bit of water (comes to rest):

$$\Delta v = v_f - v_i = 0 - 42.0 \text{m/s} = -42.0 \text{m/s}$$

Calculate the force using the momentum flow rate:

$$F = \frac{\Delta m}{\Delta t} \times |v_i| = (50.0 \text{ kg/s})(42.0 \text{m/s}) = 2.10\times10^3 \text{N}$$

The force exerted on the wall is $2.10 \times 10^3 \text{N}$ (about 470 pounds) directed away from the hose.

Discussion

This force of 2100 N is substantial—equivalent to the weight of about 210 kg (460 lbs). This explains why firefighters must brace themselves when using high-pressure hoses. The force arises because 50 kg of water per second (about 50 liters) is having its momentum completely absorbed by the wall. If the water bounced back (as it partially does in reality), the force would be even greater because the momentum change would be larger.

A 0.450-kg hammer is moving horizontally at 7.00 m/s when it strikes a nail and comes to rest after driving the nail 1.00 cm into a board. (a) Calculate the duration of the impact. (b) What was the average force exerted on the nail?

Show Solution

Strategy

For part (a), we use kinematics with the stopping distance to find the time. For part (b), we apply the impulse-momentum theorem.

Solution

(a) Duration of the impact:

Using the average velocity during deceleration:

$$v_{avg}=\frac{v_i+v_f}{2}=\frac{7.00\,\text{m/s}+0}{2}=3.50\,\text{m/s}$$

Calculate the duration:

$$\Delta t=\frac{\Delta x}{v_{avg}}=\frac{0.0100\,\text{m}}{3.50\,\text{m/s}}=2.86\times10^{-3}\,\text{s}$$

(b) Average force exerted on the nail:

1. Calculate the change in momentum:

$$\Delta p=m(v_f-v_i)=(0.450\,\text{kg})(0-7.00\,\text{m/s})=-3.15\,\text{kg·m/s}$$

1. Apply the impulse-momentum theorem:

$$F_{avg}=\frac{|\Delta p|}{\Delta t}=\frac{3.15\,\text{kg·m/s}}{2.86\times10^{-3}\,\text{s}}=1.10\times10^{3}\,\text{N}$$

Discussion

A force of 1100 N—over 2000 times the hammer’s weight—is exerted on the nail during the brief 3-millisecond impact. This force is sufficient to drive the nail into wood. The short stopping distance (1 cm) is key to generating such a large force. If the hammer bounced or the stopping distance were larger, the force would be reduced. This is why hammers are made of hard materials—they don’t compress on impact, maximizing the force transmitted to the nail.

(a) The duration of the impact is approximately $2.86\times10^{-3}\,\text{s}$ (about 3 milliseconds).

(b) The average force exerted on the nail is approximately $1.10\times10^{3}\,\text{N}$ (about 250 pounds).

Starting with the definitions of momentum and kinetic energy, derive an equation for the kinetic energy of a particle expressed as a function of its momentum.

Show Solution

Strategy

We start with the definitions of momentum ($p=mv$) and kinetic energy ($\text{KE}=\frac{1}{2}mv^2$) and algebraically eliminate velocity to express kinetic energy in terms of momentum and mass.

Solution

Start with the definition of momentum:

$$p=mv$$

Solve for velocity:

$$v=\frac{p}{m}$$

Substitute into the kinetic energy equation:

$$\text{KE}=\frac{1}{2}mv^2=\frac{1}{2}m\left(\frac{p}{m}\right)^2$$
$$\text{KE}=\frac{1}{2}m\cdot\frac{p^2}{m^2}=\frac{p^2}{2m}$$

Therefore, the kinetic energy expressed as a function of momentum is:

$$\text{KE}=\frac{p^2}{2m}$$

Discussion

This relationship $\text{KE}=\frac{p^2}{2m}$ is very useful in physics. It shows that for a given momentum, a less massive object has more kinetic energy (since m is in the denominator). This explains why, in collisions with equal momentum transfer, lighter objects move faster and carry more kinetic energy. The relationship is particularly important in particle physics where momentum is often conserved while kinetic energy may not be.

A ball with a mass of 55g with an initial velocity of 10 m/s moves at an angle $60°$ above the $+x$-direction. The ball hits a vertical wall and bounces off so that it is moving $60°$ above the $-x$ -direction with the same speed. What is the impulse delivered by the wall?

Show Solution

Strategy

The impulse equals the change in momentum: $\vec{J} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i$. We need to find the x and y components of momentum before and after the collision, then calculate the change.

Solution

Convert mass to kg: $m = 55\text{ g} = 0.055\text{kg}$

Find the initial velocity components:

$$v_{ix} = v\cos 60° = (10\text{m/s})(0.500) = 5.00\text{m/s}$$
$$v_{iy} = v\sin 60° = (10\text{m/s})(0.866) = 8.66\text{m/s}$$

Find the final velocity components (moving at $60°$ above $-x$):

$$v_{fx} = -v\cos 60° = -(10\text{m/s})(0.500) = -5.00\text{m/s}$$
$$v_{fy} = v\sin 60° = (10\text{m/s})(0.866) = 8.66\text{m/s}$$

Calculate the change in momentum components:

$$\Delta p_x = m(v_{fx} - v_{ix}) = (0.055\text{kg})(-5.00 - 5.00)\text{m/s} = -0.55\text{kg·m/s}$$
$$\Delta p_y = m(v_{fy} - v_{iy}) = (0.055\text{kg})(8.66 - 8.66)\text{m/s} = 0$$

Find the magnitude of the impulse:

$$J = |\Delta\vec{p}| = \sqrt{(\Delta p_x)^2 + (\Delta p_y)^2} = \sqrt{(-0.55)^2 + 0^2} = 0.55\text{kg·m/s}$$

The impulse delivered by the wall is 0.55 kg·m/s in the $-x$ direction (perpendicular to the wall, away from the ball's initial approach).

Discussion

The wall only changes the x-component of momentum, reversing it completely. The y-component remains unchanged because the wall exerts no vertical force (assuming a frictionless wall). The impulse magnitude of 0.55 kg·m/s represents twice the initial x-momentum, which is characteristic of a perfectly elastic bounce where the perpendicular velocity component reverses.

When serving a tennis ball, a player hits the ball when its velocity is zero (at the highest point of a vertical toss). The racquet exerts a force of 540 N on the ball for 5.00 ms, giving it a final velocity of 45.0 m/s. Using these data, find the mass of the ball.

Show Solution

Strategy

We use the impulse-momentum theorem. The impulse from the racquet equals the change in momentum of the ball. Since the ball starts from rest, the final momentum equals the impulse.

Solution

Calculate the impulse:

$$J = F\Delta t = (540\text{N})(5.00\times10^{-3}\text{s}) = 2.70\text{kg·m/s}$$

Since the ball starts at rest ($v_i = 0$):

$$J = \Delta p = mv_f - mv_i = mv_f$$

Solve for mass:

$$m = \frac{J}{v_f} = \frac{2.70\text{kg·m/s}}{45.0\text{m/s}} = 0.0600\text{kg} = 60.0\text{ g}$$

The mass of the tennis ball is 60.0 g.

Discussion

This result is consistent with official tennis ball specifications, which require balls to have a mass between 56.0 and 59.4 grams. Our calculated value of 60.0 g is very close to this range. The short contact time of 5 ms, combined with the 540 N force, produces an impulse sufficient to accelerate this light ball to serve speeds of 45 m/s (162 km/h or about 100 mph)—a respectable recreational serve.

A punter drops a 0.075kg-ball from rest vertically 1 meter down onto his foot. The ball leaves the foot with a speed of 18 m/s at an angle 55° above the horizontal. What is the impulse delivered by the foot (magnitude and direction)?

Show Solution

Strategy

We need to find the ball's velocity just before impact (using free fall kinematics), then calculate the change in momentum vector from just before to just after the kick.

Solution

First, find the velocity just before impact using free fall:

$$v_{2i}=v_{20}+2gh=0+2(9.80\text{m/s}^2)(1.00\text{m})=19.6\ \text{m}^2/\text{s}^2$$
$$v_i=4.43\text{m/s (downward)}$$

Initial momentum components (taking +x horizontal, +y upward):

$$p\_ix=0$$
$$p\_iy=mv_i=(0.075\text{kg})(-4.43\text{m/s})=-0.332\text{kg}\cdot\text{m/s}$$

Final momentum components (18 m/s at 55° above horizontal):

$$p\_fx=mv_f\cos 55^\circ=(0.075\text{kg})(18\text{m/s})(0.574)=0.775\text{kg}\cdot\text{m/s}$$
$$p\_fy=mv_f\sin 55^\circ=(0.075\text{kg})(18\text{m/s})(0.819)=1.11\text{kg}\cdot\text{m/s}$$

Calculate the impulse components:

$$J_x=p_{fx}-p\_ix=0.775-0=0.775\text{kg}\cdot\text{m/s}$$
$$J_y=p_{fy}-p\_iy=1.11-(-0.332)=1.44\text{kg}\cdot\text{m/s}$$

Find the magnitude and direction:

$$J=\sqrt{J_{2x}+J_{2y}}=\sqrt{(0.775)^2+(1.44)^2}=1.64\text{kg}\cdot\text{m/s}$$
$$\theta=\tan^{-1}(J_yJ_x)=\tan^{-1}(1.440.775)=61.7^\circ$$

The impulse delivered by the foot is 1.64 kg·m/s at 61.7° above the horizontal.

Discussion

The impulse angle (61.7°) is steeper than the final ball trajectory (55°) because the foot must also reverse the downward momentum of the falling ball. The foot provides both horizontal momentum (to send the ball forward) and upward momentum (to reverse the fall and launch the ball upward). This is a more complex momentum change than simply launching a stationary ball.

Glossary

change in momentum
: the difference between the final and initial momentum; the mass times the change in velocity

impulse
: the average net external force times the time it acts; equal to the change in momentum

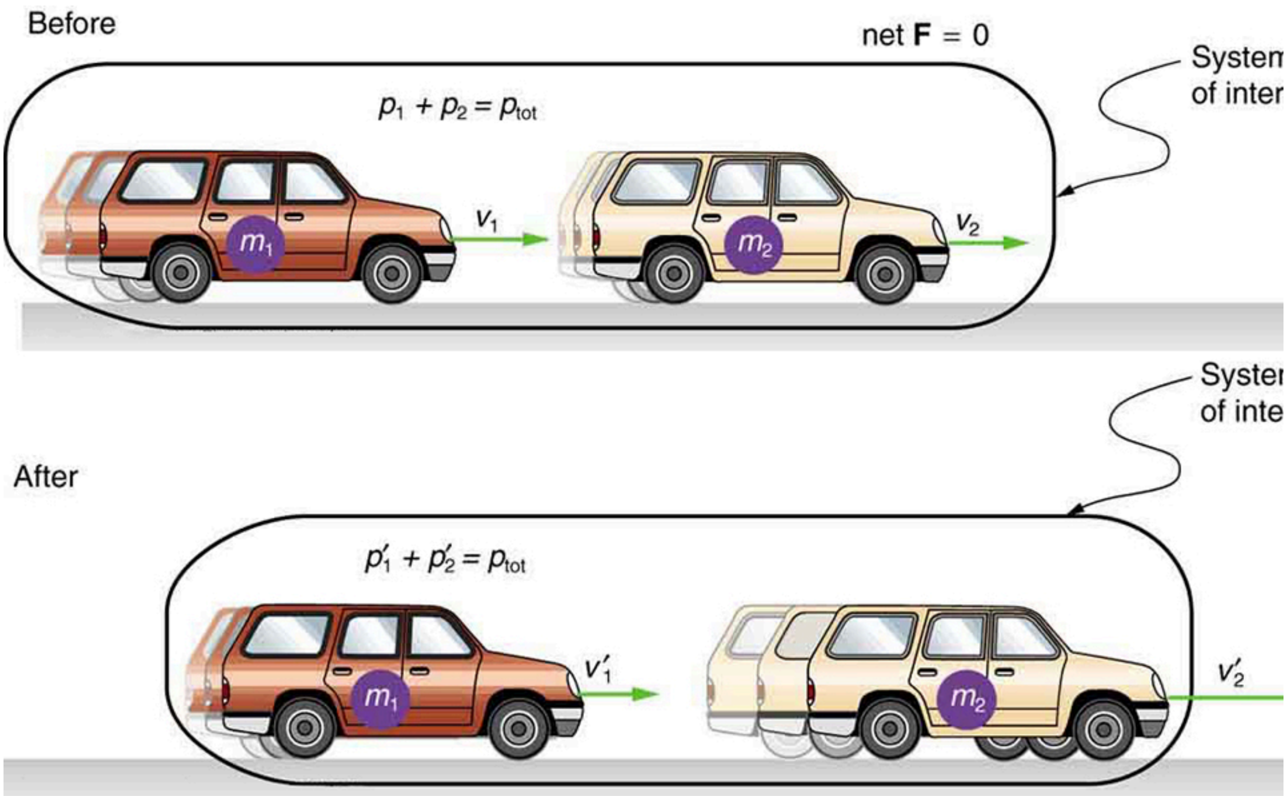
Conservation of Momentum

- Describe the principle of conservation of momentum.
- Derive an expression for the conservation of momentum.
- Explain conservation of momentum with examples.
- Explain the principle of conservation of momentum as it relates to atomic and subatomic particles.

Momentum is an important quantity because it is conserved. Yet it was not conserved in the examples in Impulse and Linear Momentum and Force, where large changes in momentum were produced by forces acting on the system of interest. Under what circumstances is momentum conserved?

The answer to this question entails considering a sufficiently large system. It is always possible to find a larger system in which total momentum is constant, even if momentum changes for components of the system. If a football player runs into the goalpost in the end zone, there will be a force on him that causes him to bounce backward. However, the Earth also recoils —conserving momentum—because of the force applied to it through the goalpost. Because Earth is many orders of magnitude more massive than the player, its recoil is immeasurably small and can be neglected in any practical sense, but it is real nevertheless.

Consider what happens if the masses of two colliding objects are more similar than the masses of a football player and Earth—for example, one car bumping into another, as shown in Figure 1. Both cars are coasting in the same direction when the lead car (labeled m_2) is bumped by the trailing car (labeled m_1). The only unbalanced force on each car is the force of the collision. (Assume that the effects due to friction are negligible.) Car 1 slows down as a result of the collision, losing some momentum, while car 2 speeds up and gains some momentum. We shall now show that the total momentum of the two-car system remains constant.



A car of mass m_1 moving with a velocity of v_1 bumps into another car of mass m_2 and velocity v_2 that it is following. As a result, the first car slows down to a velocity of $v^\prime_1$ and the second speeds up to a velocity of $v^\prime_2$. The momentum of each car is changed, but the total momentum p_{tot} of the two cars is the same before and after the collision (if you assume friction is negligible).

Using the definition of impulse, the change in momentum of car 1 is given by

$$\Delta p_1 = F_1 \Delta t,$$

where F_1 is the force on car 1 due to car 2, and Δt is the time the force acts (the duration of the collision). Intuitively, it seems obvious that the collision time is the same for both cars, but it is only true for objects traveling at ordinary speeds. This assumption must be modified for objects travelling near the speed of light, without affecting the result that momentum is conserved.

Similarly, the change in momentum of car 2 is

$$\Delta p_2 = F_2 \Delta t,$$

where F_2 is the force on car 2 due to car 1, and we assume the duration of the collision Δt is the same for both cars. We know from Newton's third law that $F_2 = -F_1$, and so

$$\Delta p_2 = -F_1 \Delta t = -\Delta p_1.$$

Thus, the changes in momentum are equal and opposite, and

$$\Delta p_1 + \Delta p_2 = 0.$$

Because the changes in momentum add to zero, the total momentum of the two-car system is constant. That is,

$$p_1 + p_2 = \text{constant},$$
$$p_1 + p_2 = p'_1 + p'_2,$$

where p'_1 and p'_2 are the momenta of cars 1 and 2 after the collision. (We often use primes to denote the final state.)

This result—that momentum is conserved—has validity far beyond the preceding one-dimensional case. It can be similarly shown that total momentum is conserved for any isolated system, with any number of objects in it. In equation form, the **conservation of momentum principle** for an isolated system is written

$$\vec{\mathbf{p}}_{\text{tot}} = \text{constant},$$

or

$$\vec{\mathbf{p}}_{\text{tot}} = \vec{\mathbf{p}}'_{\text{tot}},$$

where $\vec{\mathbf{p}}_{\text{tot}}$ is the total momentum (the sum of the momenta of the individual objects in the system) and $\vec{\mathbf{p}}'_{\text{tot}}$ is the total momentum some time later. (The total momentum can be shown to be the momentum of the center of mass of the system.) An **isolated system** is defined to be one for which the net external force is zero ($\vec{\mathbf{F}}_{\text{net}} = 0$).

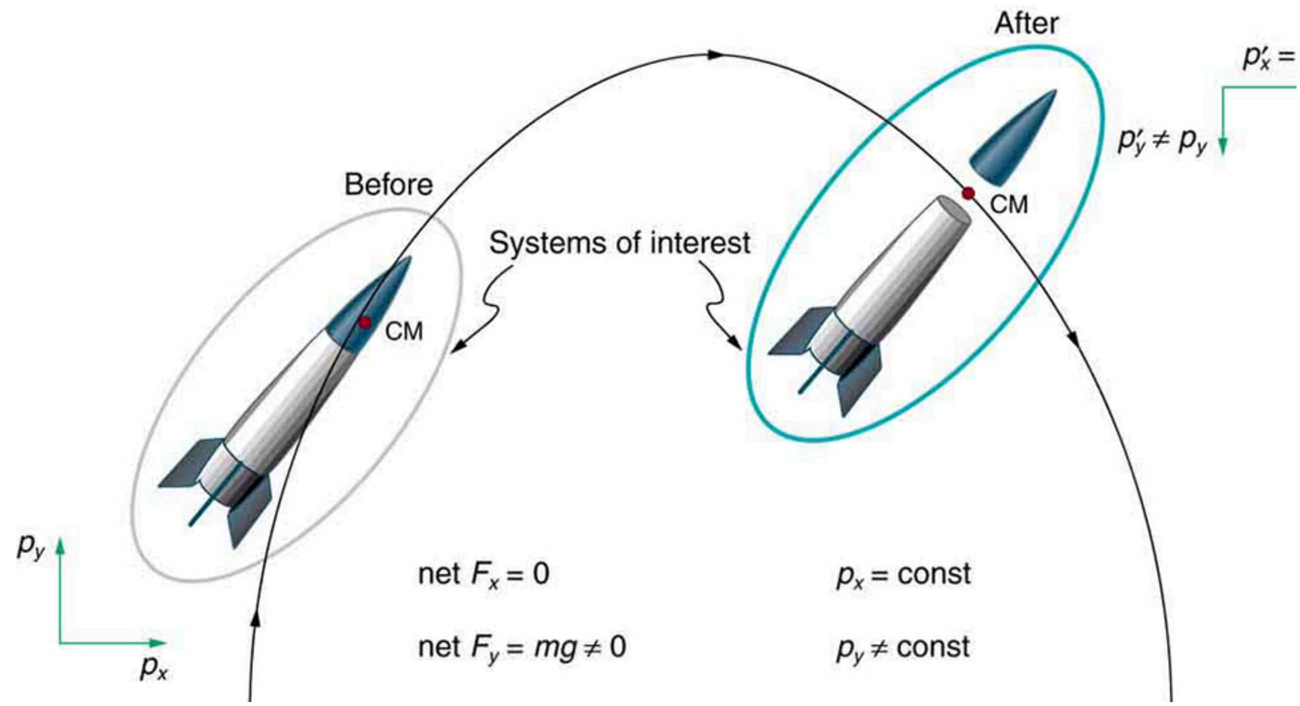
Conservation of Momentum Principle

$$\vec{\mathbf{p}}_{\text{tot}} = \text{constant} \quad \vec{\mathbf{p}}_{\text{tot}} = \vec{\mathbf{p}}'_{\text{tot}} \text{(isolated system)}$$

Isolated System

An isolated system is defined to be one for which the net external force is zero ($\vec{\mathbf{F}}_{\text{net}} = 0$).

Perhaps an easier way to see that momentum is conserved for an isolated system is to consider Newton's second law in terms of momentum, $\vec{\mathbf{F}}_{\text{net}} = \frac{\Delta \vec{\mathbf{p}}_{\text{tot}}}{\Delta t}$. For an isolated system, ($\vec{\mathbf{F}}_{\text{net}} = 0$); thus, $\Delta \vec{\mathbf{p}}_{\text{tot}} = 0$, and $\vec{\mathbf{p}}_{\text{tot}}$ is constant. We have noted that the three length dimensions in nature—x, y, and z—are independent, and it is interesting to note that momentum can be conserved in different ways along each dimension. For example, during projectile motion and where air resistance is negligible, momentum is conserved in the horizontal direction because horizontal forces are zero and momentum is unchanged. But along the vertical direction, the net vertical force is not zero and the momentum of the projectile is not conserved. (See Figure 2 .) However, if the momentum of the projectile-Earth system is considered in the vertical direction, we find that the total momentum is conserved.



The horizontal component of a projectile's momentum is conserved if air resistance is negligible, even in this case where a space probe separates. The forces causing the separation are internal to the system, so that the net external horizontal force Fx–net is still zero. The vertical component of the momentum is not conserved, because the net vertical force Fy–net is not zero. In the vertical direction, the space probe-Earth system needs to be considered and we find that the total momentum is conserved. The center of mass of the space probe takes the same path it would if the separation did not occur.

The conservation of momentum principle can be applied to systems as different as a comet striking Earth and a gas containing huge numbers of atoms and molecules. Conservation of momentum is violated only when the net external force is not zero. But another larger system can always be considered in which momentum is conserved by simply including the source of the external force. For example, in the collision of two cars considered above, the two-car system conserves momentum while each one-car system does not.

Making Connections: Take-Home Investigation—Drop of Tennis Ball and a Basketball

Hold a tennis ball side by side and in contact with a basketball. Drop the balls together. (Be careful!) What happens? Explain your observations. Now hold the tennis ball above and in contact with the basketball. What happened? Explain your observations. What do you think will happen if the basketball ball is held above and in contact with the tennis ball?

Making Connections: Take-Home Investigation—Two Tennis Balls in a Ballistic Trajectory

Tie two tennis balls together with a string about a foot long. Hold one ball and let the other hang down and throw it in a ballistic trajectory. Explain your observations. Now mark the center of the string with bright ink or attach a brightly colored sticker to it and throw again. What happened? Explain your observations.

Some aquatic animals such as jellyfish move around based on the principles of conservation of momentum. A jellyfish fills its umbrella section with water and then pushes the water out resulting in motion in the opposite direction to that of the jet of water. Squids propel themselves in a similar manner but, in contrast with jellyfish, are able to control the direction in which they move by aiming their nozzle forward or backward. Typical squids can move at speeds of 8 to 12 km/h.

The ballistocardiograph (BCG) was a diagnostic tool used in the second half of the 20th century to study the strength of the heart. About once a second, your heart beats, forcing blood into the aorta. A force in the opposite direction is exerted on the rest of your body (recall Newton's third law). A ballistocardiograph is a device that can measure this reaction force. This measurement is done by using a sensor (resting on the person) or by using a moving table suspended from the ceiling. This technique can gather information on the strength of the heart beat and the volume of blood passing from the heart. However, the electrocardiogram (ECG or EKG) and the echocardiogram (cardiac ECHO or ECHO; a technique that uses ultrasound to see an image of the heart) are more widely used in the practice of cardiology.

Making Connections: Conservation of Momentum and Collision

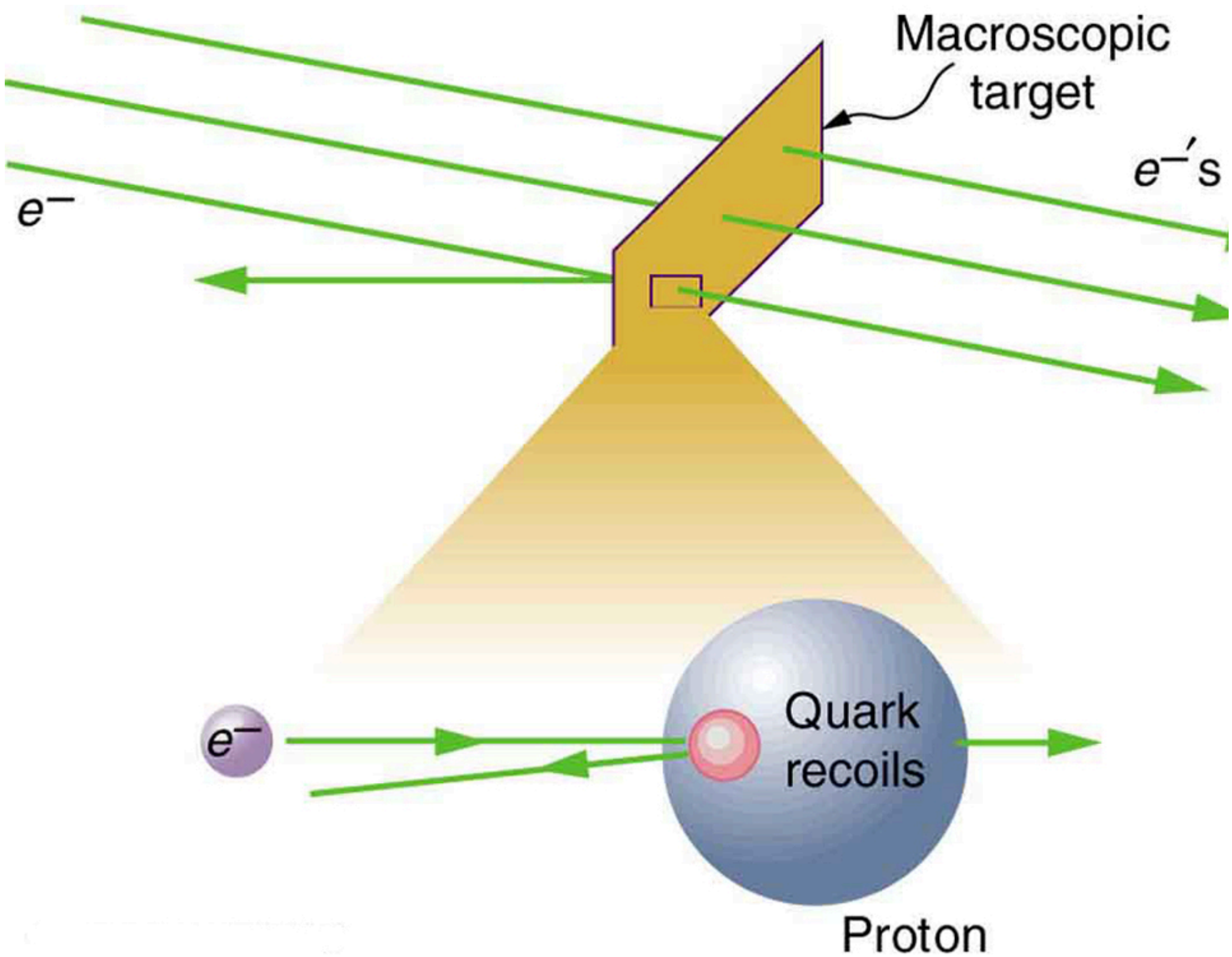
Conservation of momentum is quite useful in describing collisions. Momentum is crucial to our understanding of atomic and subatomic particles because much of what we know about these particles comes from collision experiments.

Subatomic Collisions and Momentum

The conservation of momentum principle not only applies to the macroscopic objects, it is also essential to our explorations of atomic and subatomic particles. Giant machines hurl subatomic particles at one another, and researchers evaluate the results by assuming conservation of momentum (among other things).

On the small scale, we find that particles and their properties are invisible to the naked eye but can be measured with our instruments, and models of these subatomic particles can be constructed to describe the results. Momentum is found to be a property of all subatomic particles including massless particles such as photons that compose light. Momentum being a property of particles hints that momentum may have an identity beyond the description of an object's mass multiplied by the object's velocity. Indeed, momentum relates to wave properties and plays a fundamental role in what measurements are taken and how we take these measurements. Furthermore, we find that the conservation of momentum principle is valid when considering systems of particles. We use this principle to analyze the masses and other properties of previously undetected particles, such as the nucleus of an atom and the existence of quarks that make up particles of nuclei. Figure 3 below illustrates how a particle scattering backward from another implies that its target is massive and dense. Experiments seeking evidence that **quarks** make up protons (one type of particle that makes up nuclei) scattered high-energy electrons off

of protons (nuclei of hydrogen atoms). Electrons occasionally scattered straight backward in a manner that implied a very small and very dense particle makes up the proton—this observation is considered nearly direct evidence of quarks. The analysis was based partly on the same conservation of momentum principle that works so well on the large scale.



A subatomic particle scatters straight backward from a target particle. In experiments seeking evidence for quarks, electrons were observed to occasionally scatter straight backward from a proton.

Section Summary

- The conservation of momentum principle is written $\vec{\mathbf{p}}_{\text{tot}}=\text{constant}$

or

$$\vec{\mathbf{p}}_{\text{tot}}=\vec{\mathbf{p}}'_{\text{tot}}\text{(isolated system)},$$

$\vec{\mathbf{p}}_{\text{tot}}$ is the initial total momentum and $\vec{\mathbf{p}}'_{\text{tot}}$ is the total momentum some time later.

- An isolated system is defined to be one for which the net external force is zero $(\vec{\mathbf{F}}\_\text{net}=0)$.
- During projectile motion and where air resistance is negligible, momentum is conserved in the horizontal direction because horizontal forces are zero.

- Conservation of momentum applies only when the net external force is zero.
- The conservation of momentum principle is valid when considering systems of particles.

Conceptual Questions

Professional Application

If you dive into water, you reach greater depths than if you do a belly flop. Explain this difference in depth using the concept of conservation of energy. Explain this difference in depth using what you have learned in this chapter.

Under what circumstances is momentum conserved?

Can momentum be conserved for a system if there are external forces acting on the system? If so, under what conditions? If not, why not?

Momentum for a system can be conserved in one direction while not being conserved in another. What is the angle between the directions? Give an example.

Professional Application

Explain in terms of momentum and Newton's laws how a car's air resistance is due in part to the fact that it pushes air in its direction of motion.

Can objects in a system have momentum while the momentum of the system is zero? Explain your answer.

Must the total energy of a system be conserved whenever its momentum is conserved? Explain why or why not.

Problems & Exercises

Professional Application

Train cars are coupled together by being bumped into one another. Suppose two loaded train cars are moving toward one another, the first having a mass of 150 000 kg and a velocity of 0.300 m/s, and the second having a mass of 110 000 kg and a velocity of -0.120 m/s. (The minus indicates direction of motion.) What is their final velocity?

Show Solution

Strategy

When the cars couple together, they undergo a perfectly inelastic collision. We apply conservation of momentum, with the final velocity being the same for both cars since they stick together.

Solution

Apply conservation of momentum:

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_f$$

Substitute the known values:

$$(150000 \text{ kg})(0.300 \text{ m/s}) + (110000 \text{ kg})(-0.120 \text{ m/s}) = (150000 + 110000) \text{ kg} \times v_f$$

Calculate each term:

$$45000 \text{ kg·m/s} - 13200 \text{ kg·m/s} = 260000 \text{ kg} \times v_f$$
$$31800 \text{ kg·m/s} = 260000 \text{ kg} \times v_f$$

Solve for the final velocity:

$$v_f = \frac{31800 \text{ kg·m/s}}{260000 \text{ kg}} = 0.122 \text{ m/s}$$

The final velocity of the coupled train cars is 0.122 m/s in the direction of the first car's initial motion.

Discussion

The positive final velocity indicates the coupled cars move in the direction of the heavier, faster car (car 1). The first car had more momentum (45,000 kg·m/s) than the second car (13,200 kg·m/s), so the net momentum is in the direction of the first car. This is a typical coupling operation where cars are gently bumped together at low speeds to minimize damage.

Suppose a clay model of a koala bear has a mass of 0.200 kg and slides on ice at a speed of 0.750 m/s. It runs into another clay model, which is initially motionless and has a mass of 0.350 kg. Both being soft clay, they naturally stick together. What is their final velocity?

Show Solution

Strategy

This is a perfectly inelastic collision where two objects stick together. We apply conservation of momentum, recognizing that one object is initially at rest.

Solution

Apply conservation of momentum:

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_f$$

Since the second clay model is initially at rest ($v_2 = 0$):

$$m_1 v_1 = (m_1 + m_2) v_f$$

Solve for the final velocity:

$$v_f = \frac{m_1 v_1}{m_1 + m_2} = \frac{(0.200\,\text{kg})(0.750\,\text{m/s})}{0.200\,\text{kg} + 0.350\,\text{kg}}$$
$$v_f = \frac{0.150\,\text{kg}\cdot\text{m/s}}{0.550\,\text{kg}} = 0.273\,\text{m/s}$$

The final velocity of the combined clay models is 0.273 m/s in the direction of the first model's initial motion.

Discussion

The final velocity is less than the initial velocity of the moving koala, as expected when momentum is shared with a stationary object. The combined mass is 2.75 times the moving koala's mass, so the final velocity is reduced by approximately this factor (0.750/2.75 ≈ 0.273). This is a classic example of a perfectly inelastic collision where objects stick together and move as one unit afterward.

Professional Application

Consider the following question: *A car moving at 10 m/s crashes into a tree and stops in 0.26 s. Calculate the force the seatbelt exerts on a passenger in the car to bring him to a halt. The mass of the passenger is 70 kg.* Would the answer to this question be different if the car with the 70-kg passenger had collided with a car that has a mass equal to and is traveling in the opposite direction and at the same speed? Explain your answer.

Show Solution

Strategy

First, solve the tree collision problem using impulse-momentum. Then, analyze the car-car collision using conservation of momentum to determine if the passenger experiences the same momentum change.

Solution

For the tree collision, calculate the force on the passenger:

$$\Delta p = m(v_f - v_i) = (70\,\text{kg})(0 - 10\,\text{m/s}) = -700\,\text{kg}\cdot\text{m/s}$$
$$F = \frac{\Delta p}{\Delta t} = \frac{-700\,\text{kg}\cdot\text{m/s}}{0.26\,\text{s}} = -2700\,\text{N}$$

The seatbelt exerts approximately 2700 N on the passenger.

For the car-car collision with identical cars moving at equal speeds in opposite directions:

By conservation of momentum:

$$m(+10\,\text{m/s}) + m(-10\,\text{m/s}) = 2m \times v_f$$
$$0 = 2m \times v_f$$
$$v_f = 0$$

Both cars come to rest. The passenger's change in momentum is the same: from $+700\,\text{kg}\cdot\text{m/s}$ to 0.

Discussion

The change in momentum is identical in both cases—the passenger goes from 10 m/s to 0. However, the **force** depends on the collision time, which may differ between the two scenarios. A car-car collision might have different deformation characteristics than hitting a rigid tree, potentially changing the stopping time. If the collision time in the car-car crash were longer (due to crumple zones in both cars absorbing energy over a longer distance), the force would be reduced. If shorter, the force would be greater. Without knowing the collision time for the car-car scenario, we cannot definitively compare the forces, but the momentum change remains the same.

What is the velocity of a 900-kg car initially moving at 30.0 m/s, just after it hits a 150-kg deer initially running at 12.0 m/s in the same direction? Assume the deer remains on the car.

Show Solution

Strategy

This is a perfectly inelastic collision where the deer remains on the car. Both objects are moving in the same direction initially. We apply conservation of momentum.

Solution

Apply conservation of momentum:

$$m_{car}v_{car}+m_{deer}v_{deer}=(m_{car}+m_{deer})v_f$$

Substitute the known values:

$$(900\,\text{kg})(30.0\,\text{m/s})+(150\,\text{kg})(12.0\,\text{m/s})=(900\,\text{kg}+150\,\text{kg})v_f$$

Calculate the initial momenta:

$$27000\,\text{kg·m/s}+1800\,\text{kg·m/s}=1050\,\text{kg}\times v_f$$
$$28800\,\text{kg·m/s}=1050\,\text{kg}\times v_f$$

Solve for the final velocity:

$$v_f=\frac{28800\,\text{kg·m/s}}{1050\,\text{kg}}=27.4\,\text{m/s}$$

The velocity of the car with the deer is 27.4 m/s in the original direction of motion.

Discussion

The final velocity (27.4 m/s) is less than the car's initial velocity (30.0 m/s) but greater than the deer's initial velocity (12.0 m/s), as expected when two objects moving in the same direction collide and stick together. The car loses about 2.6 m/s of speed. Since both were moving in the same direction, the velocity change is less severe than a head-on collision would produce.

A 1.80-kg falcon catches a 0.650-kg dove from behind in midair. What is their velocity after impact if the falcon's velocity is initially 28.0 m/s and the dove's velocity is 7.00 m/s in the same direction?

Show Solution

Strategy

When the falcon catches the dove, they move together as one object—a perfectly inelastic collision. Both birds are moving in the same direction, so we apply conservation of momentum with all velocities positive.

Solution

Apply conservation of momentum:

$$m_{falcon}v_{falcon}+m_{dove}v_{dove}=(m_{falcon}+m_{dove})v_f$$

Substitute the known values:

$$(1.80\,\text{kg})(28.0\,\text{m/s})+(0.650\,\text{kg})(7.00\,\text{m/s})=(1.80\,\text{kg}+0.650\,\text{kg})v_f$$

Calculate each term:

$$50.4\,\text{kg·m/s}+4.55\,\text{kg·m/s}=2.45\,\text{kg}\times v_f$$
$$54.95\,\text{kg·m/s}=2.45\,\text{kg}\times v_f$$

Solve for the final velocity:

$$v_f=\frac{54.95\,\text{kg·m/s}}{2.45\,\text{kg}}=22.4\,\text{m/s}$$

The velocity of the falcon and dove after impact is 22.4 m/s in the same direction as the original motion.

Discussion

The final velocity (22.4 m/s) is between the initial velocities of the falcon (28.0 m/s) and the dove (7.00 m/s), as expected for a perfectly inelastic collision where both objects are moving in the same direction. The falcon slows down from 28.0 m/s to 22.4 m/s upon catching the dove. The falcon dominates the outcome because it has both greater mass and greater velocity, contributing 92% of the total initial momentum.

Glossary

conservation of momentum principle
: when the net external force is zero, the total momentum of the system is conserved or constant

isolated system
: a system in which the net external force is zero

quark
: fundamental constituent of matter and an elementary particle

Elastic Collisions in One Dimension

- Describe an elastic collision of two objects in one dimension.
- Define internal kinetic energy.
- Derive an expression for conservation of internal kinetic energy in a one dimensional collision.
- Determine the final velocities in an elastic collision given masses and initial velocities.

Let us consider various types of two-object collisions. These collisions are the easiest to analyze, and they illustrate many of the physical principles involved in collisions. The conservation of momentum principle is very useful here, and it can be used whenever the net external force on a system is zero.

We start with the elastic collision of two objects moving along the same line—a one-dimensional problem. An **elastic collision** is one that also conserves internal kinetic energy. **Internal kinetic energy** is the sum of the kinetic energies of the objects in the system. Figure 1 illustrates an elastic collision in which internal kinetic energy and momentum are conserved.

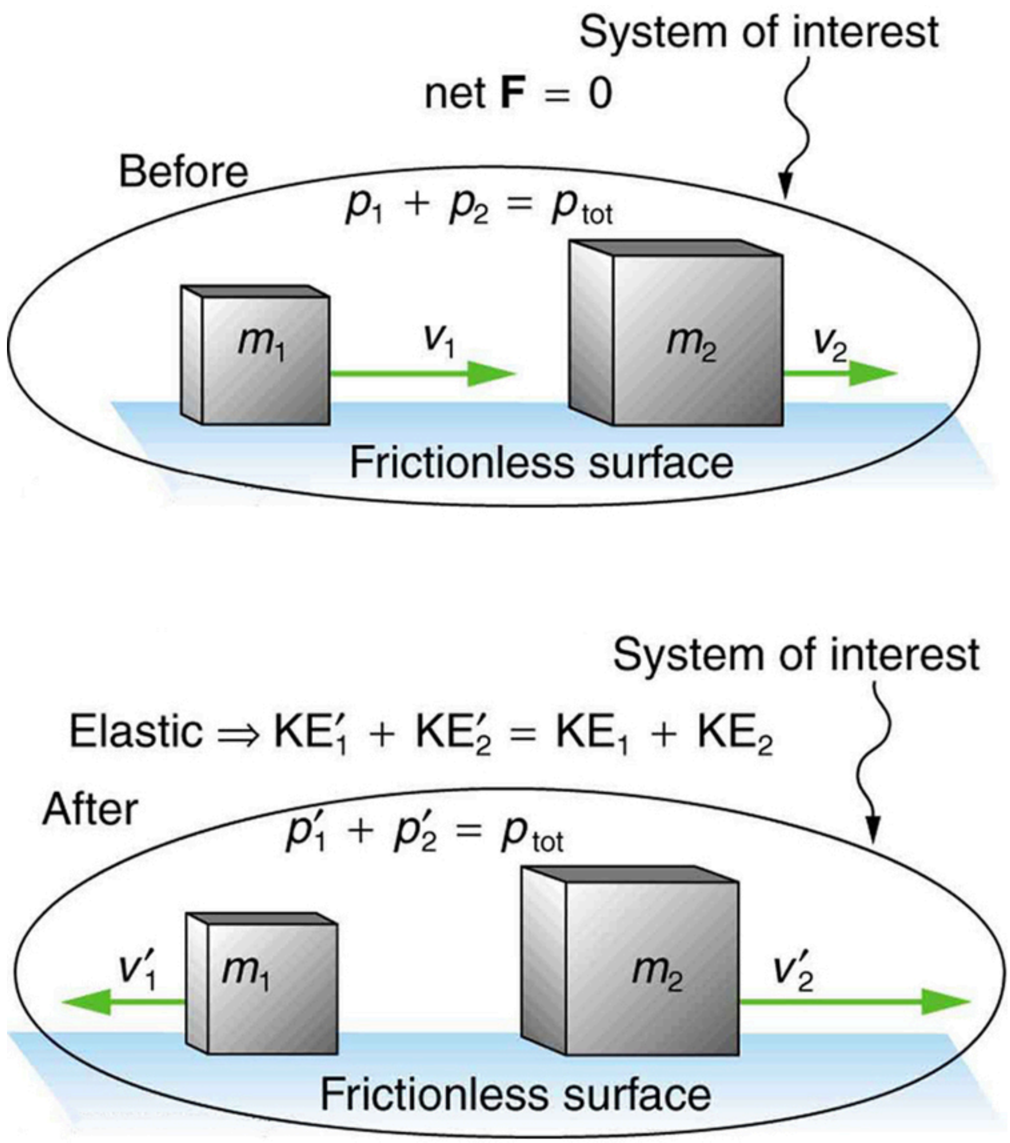
Truly elastic collisions can only be achieved with subatomic particles, such as electrons striking nuclei. Macroscopic collisions can be very nearly, but not quite, elastic—some kinetic energy is always converted into other forms of energy such as heat transfer due to friction and sound. One macroscopic collision that is nearly elastic is that of two steel blocks on ice. Another nearly elastic collision is that between two carts with spring bumpers on an air track. Icy surfaces and air tracks are nearly frictionless, more readily allowing nearly elastic collisions on them.

Elastic Collision

An **elastic collision** is one that conserves internal kinetic energy.

Internal Kinetic Energy

Internal kinetic energy is the sum of the kinetic energies of the objects in the system.



An elastic one-dimensional two-object collision. Momentum and internal kinetic energy are conserved.

Now, to solve problems involving one-dimensional elastic collisions between two objects we can use the equations for conservation of momentum and conservation of internal kinetic energy. First, the equation for conservation of momentum for two objects in a one-dimensional collision is

$$p_1+p_2=p'_1+p'_2 \quad (F_{net}=0)$$

or

$$m_1v_1+m_2v_2=m_1v'_1+m_2v'_2 \quad (F_{net}=0),$$

where the primes (') indicate values after the collision. By definition, an elastic collision conserves internal kinetic energy, and so the sum of kinetic energies before the collision equals the sum after the collision. Thus,

$$\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2=\frac{1}{2}m_1v'^2_1+\frac{1}{2}m_2v'^2_2 \quad \text{(two-object elastic collision)}$$

expresses the equation for conservation of internal kinetic energy in a one-dimensional collision.

Calculating Velocities Following an Elastic Collision

Calculate the velocities of two objects following an elastic collision, given that

$$m_1=0.500\,\text{kg}, m_2=3.50\,\text{kg}, v_1=4.00\,\text{m/s}, \text{ and } v_2=0.$$

Strategy and Concept

First, visualize what the initial conditions mean—a small object strikes a larger object that is initially at rest. This situation is slightly simpler than the situation shown in Figure 1 where both objects are initially moving. We are asked to find two unknowns (the final velocities v'_1 and v'_2). To find two unknowns, we must use two independent equations. Because this collision is elastic, we can use the above two equations. Both can be simplified by the fact that object 2 is initially at rest, and thus $v_2 = 0$. Once we simplify these equations, we combine them algebraically to solve for the unknowns.

Solution

For this problem, note that $v_2 = 0$ and use conservation of momentum. Thus,

$$p_1 = p'_1 + p'_2$$

or

$$m_1 v_1 = m_1 v'_1 + m_2 v'_2.$$

Using conservation of internal kinetic energy and that $v_2 = 0$,

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 {v'_1}^2 + \frac{1}{2} m_2 {v'_2}^2.$$

Solving the first equation (momentum equation) for v'_2, we obtain

$$v'_2 = \frac{m_1}{m_2}(v_1 - v'_1).$$

Substituting this expression into the second equation (internal kinetic energy equation) eliminates the variable v'_2, leaving only v'_1 as an unknown (the algebra is left as an exercise for the reader). There are two solutions to any quadratic equation; in this example, they are

$$v'_1 = 4.00\,\text{m/s}$$

and

$$v'_1 = -3.00\,\text{m/s}.$$

As noted when quadratic equations were encountered in earlier chapters, both solutions may or may not be meaningful. In this case, the first solution is the same as the initial condition. The first solution thus represents the situation before the collision and is discarded. The second solution $(v'_1 = -3.00\,\text{m/s})$ is negative, meaning that the first object bounces backward. When this negative value of v'_1 is used to find the velocity of the second object after the collision, we get

$$v'_2 = \frac{m_1}{m_2}(v_1 - v'_1) = \frac{0.500\,\text{kg}}{3.50\,\text{kg}}[4.00 - (-3.00)]\,\text{m/s}$$

or

$$v'_2 = 1.00\,\text{m/s}.$$

Discussion

The result of this example is intuitively reasonable. A small object strikes a larger one at rest and bounces backward. The larger one is knocked forward, but with a low speed. (This is like a compact car bouncing backward off a full-size SUV that is initially at rest.) As a check, try calculating the internal kinetic energy before and after the collision. You will see that the internal kinetic energy is unchanged at 4.00 J. Also check the total momentum before and after the collision; you will find it, too, is unchanged.

The equations for conservation of momentum and internal kinetic energy as written above can be used to describe any one-dimensional elastic collision of two objects. These equations can be extended to more objects if needed.
Making Connections: Take-Home Investigation—Ice Cubes and Elastic Collision

Find a few ice cubes which are about the same size and a smooth kitchen tabletop or a table with a glass top. Place the ice cubes on the surface several centimeters away from each other. Flick one ice cube toward a stationary ice cube and observe the path and velocities of the ice cubes after the collision. Try to avoid edge-on collisions and collisions with rotating ice cubes. Have you created approximately elastic collisions? Explain the speeds and directions of the ice cubes using momentum.

Collision Lab

Investigate collisions on an air hockey table. Set up your own experiments: vary the number of discs, masses and initial conditions. Is momentum conserved? Is kinetic energy conserved? Vary the elasticity and see what happens.

Masses and Springs

Section Summary

- An elastic collision is one that conserves internal kinetic energy.
- Conservation of kinetic energy and momentum together allow the final velocities to be calculated in terms of initial velocities and masses in one dimensional two-body collisions.

Conceptual Questions

What is an elastic collision?

Problems & Exercises

Two identical objects (such as billiard balls) have a one-dimensional collision in which one is initially motionless. After the collision, the moving object is stationary and the other moves with the same speed as the other originally had. Show that both momentum and kinetic energy are conserved.

Show Solution

Strategy

We need to verify that the total momentum before equals the total momentum after, and that the total kinetic energy before equals the total kinetic energy after.

Solution

Let both objects have mass m. Let the initial velocity of object 1 be v and object 2 be at rest.

After collision: object 1 is at rest, object 2 moves with velocity v.

Check momentum conservation:

Initial momentum: $p_i = m \cdot v + m \cdot 0 = mv$

Final momentum: $p_f = m \cdot 0 + m \cdot v = mv$

Since $p_i = p_f = mv$, momentum is conserved.

Check kinetic energy conservation:

Initial kinetic energy: $\mathrm{KE}_i = 12mv^2 + 12m(0)^2 = 12mv^2$

Final kinetic energy: $\mathrm{KE}_f = 12m(0)^2 + 12mv^2 = 12mv^2$

Since $\mathrm{KE}_i = \mathrm{KE}_f = 12mv^2$, kinetic energy is conserved.

Both momentum and kinetic energy are conserved, confirming this is an elastic collision.

Discussion

This complete transfer of motion between identical masses is characteristic of head-on elastic collisions. It's familiar from billiards and Newton's cradle—the moving object stops completely and transfers all its momentum and kinetic energy to the initially stationary object. This only happens when the masses are equal and the collision is perfectly elastic.

Professional Application

Two manned satellites approach one another at a relative speed of 0.250 m/s, intending to dock. The first has a mass of $4.00 \times 10^3 \text{kg}$, and the second a mass of $7.50 \times 10^3 \text{kg}$. If the two satellites collide elastically rather than dock, what is their final relative velocity?

Show Solution

Strategy

In an elastic collision, both momentum and kinetic energy are conserved. A fundamental property of elastic collisions is that the relative velocity of approach equals the relative velocity of separation (but in the opposite direction).

Solution

For any elastic collision, the relative velocity of approach equals the relative velocity of separation:

$$|v_1 - v_2| = |v'\_1 - v'\_2|$$

This is a consequence of conserving both momentum and kinetic energy simultaneously.

Before collision: $v\_\text{relative, before} = 0.250 \text{m/s}$

After collision: $v\_\text{relative, after} = 0.250 \text{m/s}$

The satellites separate at the same relative speed they approached: 0.250 m/s.

Verification using conservation laws:

Let satellite 1 approach at $v_1 = 0.250 \text{m/s}$ and satellite 2 be at rest ($v_2 = 0$).

Conservation of momentum: $m_1 v_1 = m_1 v'\_1 + m_2 v'\_2$

Conservation of kinetic energy: $12 m_1 v_{21} = 12 m_1 v'\_1{}^2 + 12 m_2 v'\_2{}^2$

Solving these simultaneously yields $v'_\text{relative} = v_\text{relative} = 0.250 \text{m/s}$

The final relative velocity of the satellites is 0.250 m/s (they separate at the same speed they approached).

Discussion

This result is independent of the masses! In any elastic collision, the relative speed before and after is the same—only the direction reverses. This is a powerful result for analyzing elastic collisions. For a failed docking attempt, the satellites would bounce apart at the same relative speed they approached, but now moving away from each other.

A 70.0-kg ice hockey goalie, originally at rest, catches a 0.150-kg hockey puck slapped at him at a velocity of 35.0 m/s. Suppose the goalie and the ice puck have an elastic collision and the puck is reflected back in the direction from which it came. What would their final velocities be in this case?

Show Solution

Strategy

For an elastic collision with one object initially at rest, we use the equations derived from conservation of momentum and kinetic energy. For a small mass bouncing off a much larger stationary mass, the small mass rebounds with nearly the same speed while the large mass gains a small velocity.

Solution

For an elastic collision where object 2 is initially at rest, the final velocities are:

$$v'\_1 = m_1 - m_2 m_1 + m_2 v_1$$
$$v'\_2 = 2 m_1 m_1 + m_2 v_1$$

Let the puck be object 1 ($m_1 = 0.150 \text{kg}, v_1 = 35.0 \text{m/s}$) and the goalie be object 2 ($m_2 = 70.0 \text{kg}, v_2 = 0$).

Calculate the puck's final velocity:

$v'_1=0.150\text{kg}-70.0\text{kg}0.150\text{kg}+70.0\text{kg}\times 35.0\text{m/s}$

$v'_1=-69.85\text{kg}70.15\text{kg}\times 35.0\text{m/s}=-34.85\text{m/s}\approx -34.9\text{m/s}$

Calculate the goalie’s final velocity:

$v'_2=2(0.150\text{kg})0.150\text{kg}+70.0\text{kg}\times 35.0\text{m/s}$

$v'_2=0.300\text{kg}70.15\text{kg}\times 35.0\text{m/s}=0.150\text{m/s}$

The puck rebounds at 34.9 m/s in the opposite direction (back toward the shooter), and the goalie moves forward at 0.150 m/s.

Discussion

The puck loses very little speed (35.0 → 34.9 m/s) because the goalie is so much more massive. The goalie barely moves (0.150 m/s ≈ 0.5 km/h). This is consistent with everyday experience: when a light object bounces off a heavy one, the light object reverses direction with nearly the same speed. In reality, the puck doesn’t bounce elastically off the goalie—it’s caught (inelastic) or deflected with energy loss.

Glossary

elastic collision
: a collision that also conserves internal kinetic energy

internal kinetic energy
: the sum of the kinetic energies of the objects in a system

Inelastic Collisions in One Dimension

- Define inelastic collision.
- Explain perfectly inelastic collision.
- Apply an understanding of collisions to sports.
- Determine recoil velocity and loss in kinetic energy given mass and initial velocity.

We have seen that in an elastic collision, internal kinetic energy is conserved. An **inelastic collision** is one in which the internal kinetic energy changes (it is not conserved). This lack of conservation means that the forces between colliding objects may remove or add internal kinetic energy. Work done by internal forces may change the forms of energy within a system. For inelastic collisions, such as when colliding objects stick together, this internal work may transform some internal kinetic energy into heat transfer. Or it may convert stored energy into internal kinetic energy, such as when exploding bolts separate a satellite from its launch vehicle.

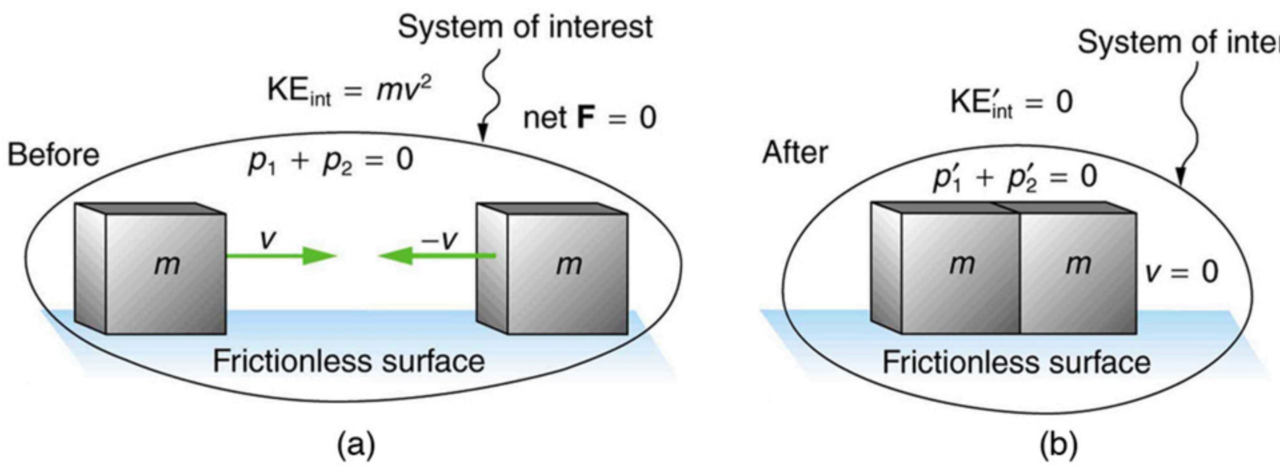
Inelastic Collision

An inelastic collision is one in which the internal kinetic energy changes (it is not conserved).

Figure 1 shows an example of an inelastic collision. Two objects that have equal masses head toward one another at equal speeds and then stick together. Their total internal kinetic energy is initially $12mv^2 + 12mv^2 = mv^2$. The two objects come to rest after sticking together, conserving momentum. But the internal kinetic energy is zero after the collision. A collision in which the objects stick together is sometimes called a **perfectly inelastic collision** because it reduces internal kinetic energy more than does any other type of inelastic collision. In fact, such a collision reduces internal kinetic energy to the minimum it can have while still conserving momentum.

Perfectly Inelastic Collision

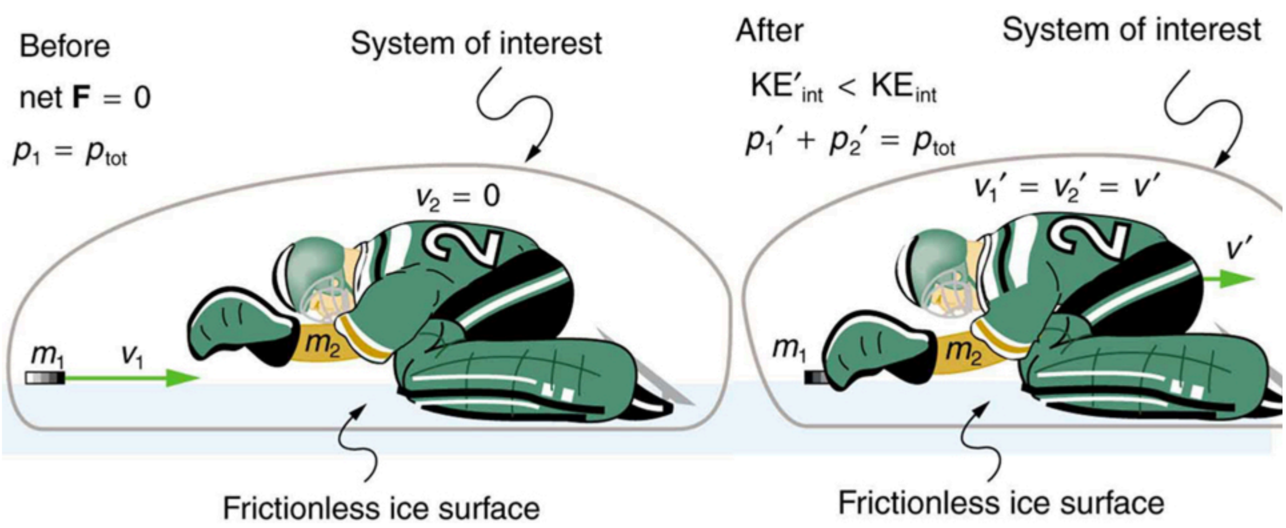
A collision in which the objects stick together is sometimes called “perfectly inelastic.”



An inelastic one-dimensional two-object collision. Momentum is conserved, but internal kinetic energy is not conserved. (a) Two objects of equal mass initially head directly toward one another at the same speed. (b) The objects stick together (a perfectly inelastic collision), and so their final velocity is zero. The internal kinetic energy of the system changes in any inelastic collision and is reduced to zero in this example.

Calculating Velocity and Change in Kinetic Energy: Inelastic Collision of a Puck and a Goalie

(a) Find the recoil velocity of a 70.0-kg ice hockey goalie, originally at rest, who catches a 0.150-kg hockey puck slapped at him at a velocity of 35.0 m/s. (b) How much kinetic energy is lost during the collision? Assume friction between the ice and the puck-goalie system is negligible. (See Figure 2)



An ice hockey goalie catches a hockey puck and recoils backward. The initial kinetic energy of the puck is almost entirely converted to thermal energy and sound in this inelastic collision.

Strategy

Momentum is conserved because the net external force on the puck-goalie system is zero. We can thus use conservation of momentum to find the final velocity of the puck and goalie system. Note that the initial velocity of the goalie is zero and that the final velocity of the puck and goalie are the same. Once the final velocity is found, the kinetic energies can be calculated before and after the collision and compared as requested.

Solution for (a)

Momentum is conserved because the net external force on the puck-goalie system is zero.

Conservation of momentum is

$$p_1+p_2=p'_1+p'_2$$

or

$$m_1v_1+m_2v_2=m_1v'_1+m_2v'_2.$$

Because the goalie is initially at rest, we know $v_2=0$. Because the goalie catches the puck, the final velocities are equal, or $v'_1=v'_2=v'$. Thus, the conservation of momentum equation simplifies to

$$m_1v_1=(m_1+m_2)v'.$$

Solving for v' yields

$$v'=\frac{m_1}{m_1+m_2}v_1.$$

Entering known values in this equation, we get

$$v'=\left(\frac{0.150\text{kg}}{0.150\text{kg}+70.0\text{kg}}\right)(35.0\text{m/s})=7.48\times10^{-2}\text{m/s}.$$

Discussion for (a)

This recoil velocity is small and in the same direction as the puck's original velocity, as we might expect.

Solution for (b)

Before the collision, the internal kinetic energy KE_{int} of the system is that of the hockey puck, because the goalie is initially at rest. Therefore, KE_{int} is initially

$$\begin{aligned}\text{KE}_{\text{int}} &= \frac{1}{2}mv^2=\frac{1}{2}(0.150\text{kg})(35.0\text{m/s})^2 \\ \text{KE}_{\text{int}} &= 91.9\text{J}.\end{aligned}$$

After the collision, the internal kinetic energy is

$$\text{KE}'_{\text{int}} \;=\; 12(m+M)v^2=12(70.15\text{kg})(7.48\times10^{-2}\text{m/s})^2 \quad \text{KE}'_{\text{int}} \;=\; 0.196\text{J}.$$

The change in internal kinetic energy is thus

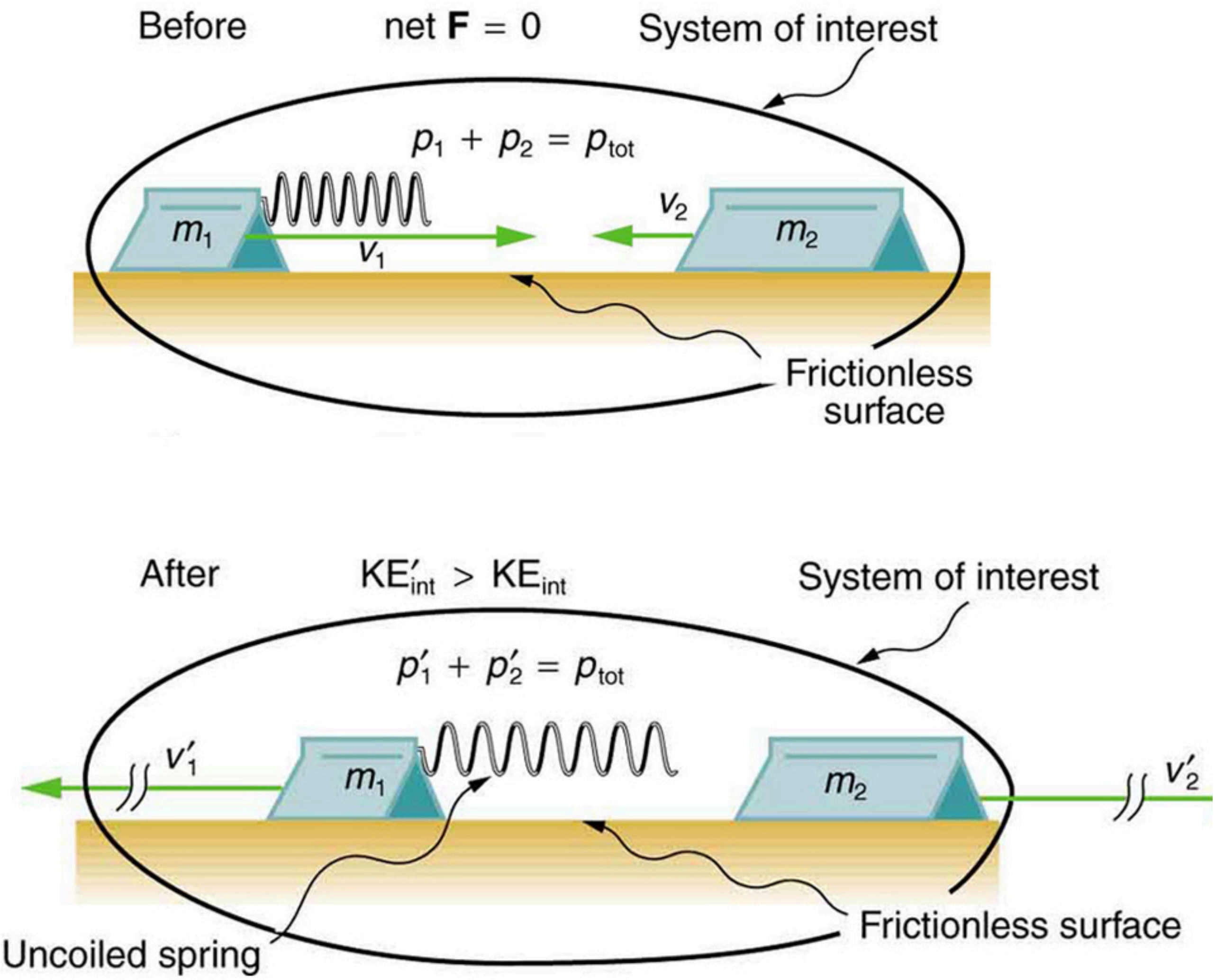
$$\text{KE}'_{\text{int}}-\text{KE}_{\text{int}} \;=\; 0.196\text{J}-91.9\text{J} \quad \text{KE}'_{\text{int}}-\text{KE}_{\text{int}} \;=\; -91.7\text{J}$$

where the minus sign indicates that the energy was lost.

Discussion for (b)

Nearly all the initial internal kinetic energy is lost in this perfectly inelastic collision. KE_int is mostly converted to thermal energy and sound.

During some collisions, the objects do not stick together and less of the internal kinetic energy is removed—such as happens in most automobile accidents. Alternatively, stored energy may be converted into internal kinetic energy during a collision. Figure 3 shows a one-dimensional example in which two carts on an air track collide, releasing potential energy from a compressed spring. Example 2 deals with data from such a collision.



An air track is nearly frictionless, so that momentum is conserved. Motion is one-dimensional. In this collision, examined in Example 2, the potential energy of a compressed spring is released during the collision and is converted to internal kinetic energy.

Collisions are particularly important in sports and the sporting and leisure industry utilizes elastic and inelastic collisions. Let us look briefly at tennis. Recall that in a collision, it is momentum and not force that is important. So, a heavier tennis racquet will have the advantage over a lighter one. This conclusion also holds true for other sports—a lightweight bat (such as a softball bat) cannot hit a hardball very far.

The location of the impact of the tennis ball on the racquet is also important, as is the part of the stroke during which the impact occurs. A smooth motion results in the maximizing of the velocity of the ball after impact and reduces sports injuries such as tennis elbow. A tennis player tries to hit the ball on the "sweet spot" on the racquet, where the vibration and impact are minimized and the ball is able to be given more velocity. Sports science and technologies also use physics concepts such as momentum and rotational motion and vibrations.
Take-Home Experiment—Bouncing of Tennis Ball

1. Find a racquet (a tennis, badminton, or other racquet will do). Place the racquet on the floor and stand on the handle. Drop a tennis ball on the strings from a measured height. Measure how high the ball bounces. Now ask a friend to hold the racquet firmly by the handle and drop a tennis ball

from the same measured height above the racquet. Measure how high the ball bounces and observe what happens to your friend's hand during the collision. Explain your observations and measurements.

2. The coefficient of restitution (c) is a measure of the elasticity of a collision between a ball and an object, and is defined as the ratio of the speeds after and before the collision. A perfectly elastic collision has a c of 1. For a ball bouncing off the floor (or a racquet on the floor), c can be shown to be $c=\sqrt{h/H}$ where h is the height to which the ball bounces and H is the height from which the ball is dropped. Determine c for the cases in Part 1 and for the case of a tennis ball bouncing off a concrete or wooden floor ($c=0.85$ for new tennis balls used on a tennis court).

Calculating Final Velocity and Energy Release: Two Carts Collide

In the collision pictured in Figure 3, two carts collide inelastically. Cart 1 (denoted m_1 carries a spring which is initially compressed. During the collision, the spring releases its potential energy and converts it to internal kinetic energy. The mass of cart 1 and the spring is 0.350 kg, and the cart and the spring together have an initial velocity of 2.00m/s. Cart 2 (denoted m_2 in Figure 3) has a mass of 0.500 kg and an initial velocity of -0.500m/s. After the collision, cart 1 is observed to recoil with a velocity of -4.00m/s. (a) What is the final velocity of cart 2? (b) How much energy was released by the spring (assuming all of it was converted into internal kinetic energy)?

Strategy

We can use conservation of momentum to find the final velocity of cart 2, because $F_{net}=0$ (the track is frictionless and the force of the spring is internal). Once this velocity is determined, we can compare the internal kinetic energy before and after the collision to see how much energy was released by the spring.

Solution for (a)

As before, the equation for conservation of momentum in a two-object system is

$$m_1v_1+m_2v_2=m_1v'_1+m_2v'_2.$$

The only unknown in this equation is v'_2. Solving for v'_2 and substituting known values into the previous equation yields

$$\begin{aligned} v'_2 &= \frac{m_1v_1+m_2v_2-m_1v'_1}{m_2} \\ v'_2 &= \frac{(0.350\text{kg})(2.00\text{m/s})+(0.500\text{kg})(-0.500\text{m/s})}{0.500\text{kg}}-\frac{(0.350\text{kg})(-4.00\text{m/s})}{0.500\text{kg}} \\ v'_2 &= 3.70\text{m/s}. \end{aligned}$$

Solution for (b)

The internal kinetic energy before the collision is

$$\begin{aligned} \text{KE}_{\text{int}} &= \frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2 \\ \text{KE}_{\text{int}} &= \frac{1}{2}(0.350\text{kg})(2.00\text{m/s})^2+\frac{1}{2}(0.500\text{kg})(-0.500\text{m/s})^2 \\ \text{KE}_{\text{int}} &= 0.763\text{J}. \end{aligned}$$

After the collision, the internal kinetic energy is

$$\begin{aligned} \text{KE}'_{\text{int}} &= \frac{1}{2}m_1v'^2_1+\frac{1}{2}m_2v'^2_2 \\ \text{KE}'_{\text{int}} &= \frac{1}{2}(0.350\text{kg})(-4.00\text{m/s})^2+\frac{1}{2}(0.500\text{kg})(3.70\text{m/s})^2 \\ \text{KE}'_{\text{int}} &= 6.22\text{J}. \end{aligned}$$

The change in internal kinetic energy is thus

$$\begin{aligned} \text{KE}'_{\text{int}}-\text{KE}_{\text{int}} &= 6.22\text{J}-0.763\text{J} \\ \text{KE}'_{\text{int}}-\text{KE}_{\text{int}} &= 5.46\text{J}. \end{aligned}$$

Discussion

The final velocity of cart 2 is large and positive, meaning that it is moving to the right after the collision. The internal kinetic energy in this collision increases by 5.46 J. That energy was released by the spring.

Section Summary

- An inelastic collision is one in which the internal kinetic energy changes (it is not conserved).
- A collision in which the objects stick together is sometimes called perfectly inelastic because it reduces internal kinetic energy more than does any other type of inelastic collision.
- Sports science and technologies also use physics concepts such as momentum and rotational motion and vibrations.

Conceptual Questions

What is an inelastic collision? What is a perfectly inelastic collision?

Mixed-pair ice skaters performing in a show are standing motionless at arms length just before starting a routine. They reach out, clasp hands, and pull themselves together by only using their arms. Assuming there is no friction between the blades of their skates and the ice, what is their velocity after their bodies meet?

A small pickup truck that has a camper shell slowly coasts toward a red light with negligible friction. Two dogs in the back of the truck are moving and making various inelastic collisions with each other and the walls. What is the effect of the dogs on the motion of the center of mass of the system (truck plus entire load)? What is their effect on the motion of the truck?

Problems & Exercises

A 0.240-kg billiard ball that is moving at 3.00 m/s strikes the bumper of a pool table and bounces straight back at 2.40 m/s (80% of its original speed). The collision lasts 0.0150 s. (a) Calculate the average force exerted on the ball by the bumper. (b) How much kinetic energy in joules is lost during the collision? (c) What percent of the original energy is left?

Show Solution

Strategy

For part (a), we use the impulse-momentum theorem. The ball reverses direction, so the change in momentum includes both the original momentum and the rebound momentum. For parts (b) and (c), we compare kinetic energies before and after.

Solution

Part (a)

Let positive be the initial direction of motion. The ball reverses direction, so the final velocity is negative.

$$\Delta p = m(v_f - v_i) = (0.240\,\text{kg})(-2.40\,\text{m/s} - 3.00\,\text{m/s}) = (0.240\,\text{kg})(-5.40\,\text{m/s}) = -1.30\,\text{kg}\cdot\text{m/s}$$

Apply the impulse-momentum theorem:

$$F\_\text{avg} = \frac{\Delta p}{\Delta t} = \frac{-1.30\,\text{kg}\cdot\text{m/s}}{0.0150\,\text{s}} = -86.4\,\text{N}$$

The average force on the ball is 86.4 N directed away from the bumper (opposite to the ball’s initial motion).

Part (b)

Initial kinetic energy:

$$\text{KE}_i = \frac{1}{2}mv_i^2 = \frac{1}{2}(0.240\,\text{kg})(3.00\,\text{m/s})^2 = 1.08\,\text{J}$$

Final kinetic energy:

$$\text{KE}_f = \frac{1}{2}mv_f^2 = \frac{1}{2}(0.240\,\text{kg})(2.40\,\text{m/s})^2 = 0.691\,\text{J}$$

Energy lost:

$$\Delta\text{KE} = \text{KE}_i - \text{KE}_f = 1.08\,\text{J} - 0.691\,\text{J} = 0.389\,\text{J}$$

The kinetic energy lost during the collision is 0.389 J.

Part (c)

$$\text{Percent remaining} = \frac{\text{KE}_f}{\text{KE}_i} \times 100\% = \frac{0.691\,\text{J}}{1.08\,\text{J}} \times 100\% = 64.0\%$$

64.0% of the original kinetic energy remains after the collision.

Discussion

Although the ball retains 80% of its speed, it only retains 64% of its kinetic energy. This is because kinetic energy depends on the square of velocity: $(0.80)^2 = 0.64$. The lost energy is converted to heat and sound during the collision with the bumper.

During an ice show, a 60.0-kg skater leaps into the air and is caught by an initially stationary 75.0-kg skater. (a) What is their final velocity assuming negligible friction and that the 60.0-kg skater’s original horizontal velocity is 4.00 m/s? (b) How much kinetic energy is lost?

Show Solution

Strategy

This is a perfectly inelastic collision where the skaters stick together. We use conservation of momentum to find the final velocity, then compare kinetic energies.

Solution

Part (a)

Apply conservation of momentum:

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v_f$$

Since the catching skater is initially at rest ($v_2 = 0$):

$$(60.0\text{kg})(4.00\text{m/s})+(75.0\text{kg})(0)=(60.0\text{kg}+75.0\text{kg})v_f$$
$$240\text{kg}\cdot\text{m/s}=135\text{kg}\times v_f$$
$$v_f=\frac{240\text{kg}\cdot\text{m/s}}{135\text{kg}}=1.78\text{m/s}$$

The final velocity of the pair is 1.78 m/s in the direction of the leaping skater's original motion.

Part (b)

Initial kinetic energy (only the leaping skater is moving):

$$\text{KE}_i=\frac{1}{2}m_1v_1^2=\frac{1}{2}(60.0\text{kg})(4.00\text{m/s})^2=480\text{J}$$

Final kinetic energy (both skaters moving together):

$$\text{KE}_f=\frac{1}{2}(m_1+m_2)v_f^2=\frac{1}{2}(135\text{kg})(1.78\text{m/s})^2=214\text{J}$$

Kinetic energy lost:

$$\Delta\text{KE}=\text{KE}_i-\text{KE}_f=480\text{J}-214\text{J}=266\text{J}$$

The kinetic energy lost during the catch is 266 J.

Discussion

More than half (55%) of the original kinetic energy is lost in this perfectly inelastic collision. This energy goes into the work done by the skaters' muscles and joints as they absorb the impact, plus some heat and sound. A trained pair skater knows how to absorb this impact safely while maintaining an elegant appearance.

Professional Application

Using mass and speed data from Example 1 and assuming that the football player catches the ball with his feet off the ground with both of them moving horizontally, calculate: (a) the final velocity if the ball and player are going in the same direction and (b) the loss of kinetic energy in this case. (c) Repeat parts (a) and (b) for the situation in which the ball and the player are going in opposite directions. Might the loss of kinetic energy be related to how much it hurts to catch the pass?

Show Solution

Strategy

From Example 1, we use: player mass = 110 kg, player velocity = 8.00 m/s, ball mass = 0.410 kg, ball velocity = 25.0 m/s. We apply conservation of momentum and compare kinetic energies for both cases.

Solution

Part (a) - Same direction

$$m_1v_1+m_2v_2=(m_1+m_2)v_f$$
$$(110\text{kg})(8.00\text{m/s})+(0.410\text{kg})(25.0\text{m/s})=(110.41\text{kg})v_f$$
$$880\text{kg}\cdot\text{m/s}+10.25\text{kg}\cdot\text{m/s}=(110.41\text{kg})v_f$$
$$v_f=\frac{890.25\text{kg}\cdot\text{m/s}}{110.41\text{kg}}=8.06\text{m/s}$$

The final velocity is 8.06 m/s in the original direction.

Part (b) - Energy loss (same direction)

$$\text{KE}_i=\frac{1}{2}(110\text{kg})(8.00\text{m/s})^2+\frac{1}{2}(0.410\text{kg})(25.0\text{m/s})^2=3520\text{J}+128\text{J}=3648\text{J}$$
$$\text{KE}_f=\frac{1}{2}(110.41\text{kg})(8.06\text{m/s})^2=3586\text{J}$$
$$\Delta\text{KE}=3586\text{J}-3648\text{J}=-62\text{J}$$

The kinetic energy lost is approximately 56-62 J.

Part (c) - Opposite directions

Ball velocity is now -25.0m/s (opposite to player):

$$(110\text{kg})(8.00\text{m/s})+(0.410\text{kg})(-25.0\text{m/s})=(110.41\text{kg})v_f$$
$$880\text{kg}\cdot\text{m/s}-10.25\text{kg}\cdot\text{m/s}=(110.41\text{kg})v_f$$
$$v_f=\frac{869.75\text{kg}\cdot\text{m/s}}{110.41\text{kg}}=7.88\text{m/s}$$

Energy loss for opposite direction:

$KE_f=12(110.41\text{kg})(7.88\text{m/s})^2=3428\text{J}$
$\Delta KE=3428\text{J}-3648\text{J}=-220\text{J}$

The kinetic energy lost is approximately 220-223 J.

Discussion

When the ball is coming toward the player (opposite directions), approximately 4 times more kinetic energy is lost (223 J vs. 56 J). This lost energy is absorbed by the player's hands and arms as they decelerate and then accelerate the ball to match the player's velocity. This is why catching a ball thrown against your direction of motion hurts more—more energy must be dissipated in your body.

A battleship that is $6.00\times10^7\text{kg}$ and is originally at rest fires a 1100-kg artillery shell horizontally with a velocity of 575 m/s. (a) If the shell is fired straight aft (toward the rear of the ship), there will be negligible friction opposing the ship's recoil. Calculate its recoil velocity. (b) Calculate the increase in internal kinetic energy (that is, for the ship and the shell). This energy is less than the energy released by the gun powder—significant heat transfer occurs.

Show Solution

Strategy

This is an "explosion" type problem where an initially stationary system separates into two parts. We use conservation of momentum for part (a) and calculate the total kinetic energy after firing for part (b).

Solution

Part (a)

Initial momentum is zero (ship and shell at rest). Apply conservation of momentum:

$0=m_\text{ship}v_\text{ship}+m_\text{shell}v_\text{shell}$
$v_\text{ship}=-m_\text{shell}v_\text{shell}m_\text{ship}$
$v\_\text{ship}=-(1100\text{kg})(575\text{m/s})6.00\times10^7\text{kg}=-1.05\times10^{-2}\text{m/s}$

The recoil velocity of the battleship is $1.05\times10^{-2}\text{m/s}$ (about 1 cm/s) in the direction opposite to the shell.

Part (b)

Initial kinetic energy: $KE_i=0$ (both at rest)

Final kinetic energy:

$KE_f=12m_\text{shell}v_{2\text{shell}}+12m_\text{ship}v\_\text{ship}^2$
$KE_f=12(1100\text{kg})(575\text{m/s})^2+12(6.00\times10^7\text{kg})(1.05\times10^{-2}\text{m/s})^2$
$KE_f=1.82\times10^8\text{J}+3.31\times10^3\text{J}=1.82\times10^8\text{J}$

The increase in internal kinetic energy is $1.82\times10^8\text{J}$ (182 MJ).

Discussion

The ship's recoil velocity is tiny (1 cm/s) despite the massive momentum transfer because of the ship's enormous mass. Nearly all (99.998%) of the kinetic energy goes to the shell rather than the ship. The chemical energy released by the gunpowder is considerably larger than 182 MJ—much energy is lost to heat, sound, and deforming the gun barrel.

Professional Application

Two manned satellites approaching one another, at a relative speed of 0.250 m/s, intending to dock. The first has a mass of $4.00\times10^3\text{kg}$, and the second a mass of $7.50\times10^3\text{kg}$. (a) Calculate the final velocity (after docking) by using the frame of reference in which the first satellite was originally at rest. (b) What is the loss of kinetic energy in this inelastic collision? (c) Repeat both parts by using the frame of reference in which the second satellite was originally at rest. Explain why the change in velocity is different in the two frames, whereas the change in kinetic energy is the same in both.

Show Solution

Strategy

This is a perfectly inelastic collision (docking). We analyze it in two different reference frames. In each frame, we apply conservation of momentum to find the final velocity, then calculate the kinetic energy loss. The key insight is that while velocities are frame-dependent, energy loss is the same in all inertial frames.

Solution

Part (a) - Frame where first satellite is at rest

In this reference frame:

- First satellite: $m_1 = 4.00 \times 10^3 \text{kg}$, $v_1 = 0$
- Second satellite: $m_2 = 7.50 \times 10^3 \text{kg}$, $v_2 = 0.250 \text{m/s}$ (approaching)

Apply conservation of momentum:

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_f$$
$$(4.00\times10^3\text{kg})(0) + (7.50\times10^3\text{kg})(0.250\text{m/s}) = (11.5\times10^3\text{kg}) v_f$$
$$1875\text{kg}\cdot\text{m/s} = (11500\text{kg}) v_f$$
$$v_f = \frac{1875}{11500} = 0.163\text{m/s}$$

The final velocity is 0.163 m/s in the direction of motion of the more massive satellite.

Part (b) - Energy loss

Initial kinetic energy (only satellite 2 is moving in this frame):

$$\text{KE}_i = \frac{1}{2} m_2 v_2^2 = \frac{1}{2}(7.50\times10^3\text{kg})(0.250\text{m/s})^2 = 234\text{J}$$

Final kinetic energy:

$$\text{KE}_f = \frac{1}{2}(m_1 + m_2) v_f^2 = \frac{1}{2}(11.5\times10^3\text{kg})(0.163\text{m/s})^2 = 153\text{J}$$

Energy lost:

$$\Delta\text{KE} = \text{KE}_f - \text{KE}_i = 153\text{J} - 234\text{J} = -81\text{J}$$

The kinetic energy lost is approximately 81.6 J.

Part (c) - Frame where second satellite is at rest

In this reference frame:

- First satellite: $m_1 = 4.00 \times 10^3 \text{kg}$, $v_1 = -0.250 \text{m/s}$ (approaching)
- Second satellite: $m_2 = 7.50 \times 10^3 \text{kg}$, $v_2 = 0$

Apply conservation of momentum:

$$(4.00\times10^3\text{kg})(-0.250\text{m/s}) + (7.50\times10^3\text{kg})(0) = (11.5\times10^3\text{kg}) v_f$$
$$-1000\text{kg}\cdot\text{m/s} = (11500\text{kg}) v_f$$
$$v_f = -\frac{1000}{11500} = -0.0870\text{m/s}$$

The final velocity is $8.70 \times 10^{-2} \text{m/s}$ in the direction of motion of the less massive satellite.

Energy loss in this frame:

Initial kinetic energy:

$$\text{KE}_i = \frac{1}{2} m_1 v_1^2 = \frac{1}{2}(4.00\times10^3\text{kg})(0.250\text{m/s})^2 = 125\text{J}$$

Final kinetic energy:

$$\text{KE}_f = \frac{1}{2}(11.5\times10^3\text{kg})(0.0870\text{m/s})^2 = 43.5\text{J}$$
$$\Delta\text{KE} = 43.5\text{J} - 125\text{J} = -81.5\text{J}$$

The energy loss is approximately 81.5 J, the same as in part (b).

Discussion

The final velocities differ in the two reference frames (0.163 m/s vs. -0.0870 m/s) because velocity is frame-dependent. However, the kinetic energy loss is essentially the same in both frames (81.6 J vs. 81.5 J). This is because the energy lost to internal forces—converted to heat, deformation, and vibration during docking—is an invariant quantity independent of the reference frame. The center of mass of the system moves at constant velocity in both frames since there are no external forces, and the energy dissipated in the collision is the same regardless of which frame we use to observe it.

Professional Application

A 30 000-kg freight car is coasting at 0.850 m/s with negligible friction under a hopper that dumps 110 000 kg of scrap metal into it. (a) What is the final velocity of the loaded freight car? (b) How much kinetic energy is lost?

Show Solution

Strategy

This is a perfectly inelastic collision where the scrap metal (initially at rest horizontally) joins the moving freight car. We apply conservation of horizontal momentum.

Solution

Part (a)

The scrap metal has no horizontal velocity before landing in the car. Apply conservation of momentum:

$$m_{\text{car}}v_{\text{car}}+m_{\text{scrap}}v_{\text{scrap}}=(m_{\text{car}}+m_{\text{scrap}})v_f$$
$$(30000\text{kg})(0.850\text{m/s})+(110000\text{kg})(0)=(140000\text{kg})v_f$$
$$25500\text{kg}\cdot\text{m/s}=(140000\text{kg})v_f$$
$$v_f=\frac{25500\text{kg}\cdot\text{m/s}}{140000\text{kg}}=0.182\text{m/s}$$

The final velocity of the loaded freight car is 0.182 m/s.

Part (b)

Initial kinetic energy:

$$\text{KE}_i=\frac{1}{2}(30000\text{kg})(0.850\text{m/s})^2=10800\text{J}$$

Final kinetic energy:

$$\text{KE}_f=\frac{1}{2}(140000\text{kg})(0.182\text{m/s})^2=2320\text{J}$$

Kinetic energy lost:

$$\Delta\text{KE}=\text{KE}_f-\text{KE}_i=2320\text{J}-10800\text{J}=-8480\text{J}$$

Approximately 8500 J of kinetic energy is lost.

Discussion

The car slows dramatically from 0.850 m/s to 0.182 m/s—a reduction of nearly 80%—because the added mass is almost 4 times the original car mass. About 78% of the kinetic energy is lost, converted to heat and sound as the scrap metal settles and deforms in the car.

Professional Application

Space probes may be separated from their launchers by exploding bolts. (They bolt away from one another.) Suppose a 4800-kg satellite uses this method to separate from the 1500-kg remains of its launcher, and that 5000 J of kinetic energy is supplied to the two parts. What are their subsequent velocities using the frame of reference in which they were at rest before separation?

Show Solution

Strategy

This is an “explosion” problem where initial momentum is zero and energy is added. We use two equations: conservation of momentum (gives relationship between velocities) and the kinetic energy equation (gives magnitudes).

Solution

Conservation of momentum (initial momentum = 0):

$$m_1v_1+m_2v_2=0$$
$$(4800\text{kg})v_1+(1500\text{kg})v_2=0$$
$$v_2=-\frac{4800}{1500}v_1=-3.2v_1$$

The kinetic energy supplied:

$$\text{KE}=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2=5000\text{J}$$

Substitute $v_2=-3.2v_1$:

$$12(4800\text{kg})v_{21}+12(1500\text{kg})(-3.2v_1)^2=5000\text{J}$$
$$2400v_{21}+750(10.24)v_{21}=5000$$
$$2400v_{21}+7680v_{21}=5000$$
$$10080v_{21}=5000$$
$$v_1=\sqrt{500010080}=0.704\text{m/s}$$

Now find v_2:

$$v_2=-3.2(0.704\text{m/s})=-2.25\text{m/s}$$

The satellite moves at 0.704 m/s and the launcher remains move at 2.25 m/s in the opposite direction.

Discussion

The lighter launcher remains move faster (2.25 m/s vs 0.704 m/s) because momentum must be conserved. The velocity ratio (3.2) equals the inverse mass ratio (4800/1500). The lighter object receives more of the kinetic energy because $\text{KE} \propto v^2$ and the lighter object moves faster.

A 0.0250-kg bullet is accelerated from rest to a speed of 550 m/s in a 3.00-kg rifle. The pain of the rifle's kick is much worse if you hold the gun loosely a few centimeters from your shoulder rather than holding it tightly against your shoulder. (a) Calculate the recoil velocity of the rifle if it is held loosely away from the shoulder. (b) How much kinetic energy does the rifle gain? (c) What is the recoil velocity if the rifle is held tightly against the shoulder, making the effective mass 28.0 kg? (d) How much kinetic energy is transferred to the rifle-shoulder combination? The pain is related to the amount of kinetic energy, which is significantly less in this latter situation. (e) Calculate the momentum of a 110-kg football player running at 8.00 m/s. Compare the player's momentum with the momentum of a hard-thrown 0.410-kg football that has a speed of 25.0 m/s. Discuss its relationship to this problem.

Show Solution

Strategy

We use conservation of momentum to find recoil velocities. The rifle (or rifle-shoulder system) recoils with momentum equal and opposite to the bullet. Then we calculate kinetic energies.

Solution

Part (a)

Conservation of momentum (initially at rest):

$$m_{\text{bullet}}v_{\text{bullet}}+m_{\text{rifle}}v_{\text{rifle}}=0$$
$$v_{\text{rifle}}=-m_{\text{bullet}}v_{\text{bullet}}m_{\text{rifle}}=-(0.0250\text{kg})(550\text{m/s})3.00\text{kg}=-4.58\text{m/s}$$

The rifle recoils at 4.58 m/s away from the bullet.

Part (b)

$$\text{KE}_{\text{rifle}}=12m_{\text{rifle}}v\_\text{rifle}^2=12(3.00\text{kg})(4.58\text{m/s})^2=31.5\text{J}$$

The rifle gains 31.5 J of kinetic energy.

Part (c)

With effective mass of 28.0 kg (rifle + shoulder):

$$v\_\text{system}=-(0.0250\text{kg})(550\text{m/s})28.0\text{kg}=-0.491\text{m/s}$$

The recoil velocity is 0.491 m/s when held tightly.

Part (d)

$$\text{KE\_system}=12(28.0\text{kg})(0.491\text{m/s})^2=3.38\text{J}$$

Only 3.38 J of kinetic energy is transferred to the rifle-shoulder combination.

Part (e)

Football player momentum: $p=(110\text{kg})(8.00\text{m/s})=880\text{kg}\cdot\text{m/s}$

Football momentum: $p=(0.410\text{kg})(25.0\text{m/s})=10.25\text{kg}\cdot\text{m/s}$

Ratio: $88010.25=85.9$

The player has about 86 times more momentum than the football.

Discussion

Holding the rifle tightly reduces the recoil kinetic energy from 31.5 J to 3.38 J—a factor of 9.3 reduction! This dramatically reduces pain and bruising. The key insight is that the same momentum is absorbed, but spreading it across a larger mass reduces velocity and kinetic energy. Similarly, in football, the much more massive player absorbs the ball's momentum with minimal velocity change.

Professional Application

One of the waste products of a nuclear reactor is plutonium-239 ($^{239}\mathrm{Pu}$). This nucleus is radioactive and decays by splitting into a helium-4 nucleus and a uranium-235 nucleus ($^{4}\mathrm{He}+^{235}U$), the latter of which is also radioactive and will itself decay some time later. The energy emitted in the plutonium decay is $8.40\times10^{-13}\mathrm{J}$ and is entirely converted to kinetic energy of the helium and uranium nuclei. The mass of the helium nucleus is $6.68\times10^{-27}\mathrm{kg}$, while that of the uranium is $3.92\times10^{-25}\mathrm{kg}$ (note that the ratio of the masses is 4 to 235). (a) Calculate the velocities of the two nuclei, assuming the plutonium nucleus is originally at rest. (b) How much kinetic energy does each nucleus carry away? Note that the data given here are accurate to three digits only.

Show Solution

Strategy

This is similar to an explosion problem: initial momentum is zero, and energy is released. We use conservation of momentum and the kinetic energy equation to solve for both velocities.

Solution

Part (a)

Conservation of momentum (plutonium initially at rest):

$$m_{\mathrm{He}}v_{\mathrm{He}}+m_{\mathrm{U}}v_{\mathrm{U}}=0$$
$$v_{\mathrm{U}}=-\frac{m_{\mathrm{He}}}{m_{\mathrm{U}}}v_{\mathrm{He}}=-\frac{6.68\times10^{-27}}{3.92\times10^{-25}}v_{\mathrm{He}}=-0.01704v_{\mathrm{He}}$$

Total kinetic energy:

$$\frac{1}{2}m_{\mathrm{He}}v_{\mathrm{He}}^2+\frac{1}{2}m_{\mathrm{U}}v_{\mathrm{U}}^2=8.40\times10^{-13}\mathrm{J}$$

Substituting $v_{\mathrm{U}}=-0.01704v_{\mathrm{He}}$:

$$\frac{1}{2}(6.68\times10^{-27})v_{\mathrm{He}}^2+\frac{1}{2}(3.92\times10^{-25})(0.01704)^2v_{\mathrm{He}}^2=8.40\times10^{-13}$$
$$(3.34\times10^{-27}+5.69\times10^{-29})v\_\mathrm{He}^2=8.40\times10^{-13}$$
$$(3.40\times10^{-27})v\_\mathrm{He}^2=8.40\times10^{-13}$$
$$v\_\mathrm{He}=\sqrt{\frac{8.40\times10^{-13}}{3.40\times10^{-27}}}=1.57\times10^{7}\mathrm{m/s}$$
$$v\_\mathrm{U}=-0.01704(1.57\times10^{7}\mathrm{m/s})=-2.68\times10^{5}\mathrm{m/s}$$

The helium nucleus moves at $1.57\times10^{7}\mathrm{m/s}$ and the uranium nucleus moves at $2.68\times10^{5}\mathrm{m/s}$ in the opposite direction.

Part (b)

$$\mathrm{KE\_He}=\frac{1}{2}(6.68\times10^{-27}\mathrm{kg})(1.57\times10^{7}\mathrm{m/s})^2=8.23\times10^{-13}\mathrm{J}$$
$$\mathrm{KE\_U}=\frac{1}{2}(3.92\times10^{-25}\mathrm{kg})(2.68\times10^{5}\mathrm{m/s})^2=1.41\times10^{-14}\mathrm{J}$$

The helium nucleus carries away $8.23\times10^{-13}\mathrm{J}$ (98%) and the uranium nucleus carries away $1.41\times10^{-14}\mathrm{J}$ (2%).

Discussion

The light helium nucleus (alpha particle) carries away 98% of the energy while the massive uranium nucleus carries only 2%. This is because, for equal momenta, kinetic energy is inversely proportional to mass ($\mathrm{KE}=p^2/2m$). The alpha particle travels at about 5% the speed of light!

Professional Application

The Moon's craters are remnants of meteorite collisions. Suppose a fairly large asteroid that has a mass of $5.00\times10^{12}\mathrm{kg}$ (about a kilometer across) strikes the Moon at a speed of 15.0 km/s. (a) At what speed does the Moon recoil after the perfectly inelastic collision (the mass of the Moon is $7.36\times10^{22}\mathrm{kg}$) ? (b) How much kinetic energy is lost in the collision? Such an event may have been observed by medieval English monks who reported observing a red glow and subsequent haze about the Moon. (c) In October 2009, NASA crashed a rocket into the Moon, and analyzed the plume produced by the impact. (Significant amounts of water were detected.) Answer part (a) and (b) for this real-life experiment. The mass of the rocket was 2000 kg and its speed upon impact was 9000 km/h. How does the plume produced alter these results?

Show Solution

Strategy

This problem involves perfectly inelastic collisions where an object strikes the Moon and sticks to it. We use conservation of momentum to find the Moon's recoil velocity, then calculate the kinetic energy before and after collision. The Moon's enormous mass means it barely moves, but significant energy is dissipated.

Solution

Part (a) - Asteroid collision

Given:

- Asteroid mass: $m_a = 5.00 \times 10^{12}\,\text{kg}$
- Asteroid velocity: $v_a = 15.0\ \text{km/s} = 1.50 \times 10^{4}\,\text{m/s}$
- Moon mass: $m_M = 7.36 \times 10^{22}\,\text{kg}$
- Moon initial velocity: $v_M = 0$ (in Moon's reference frame)

Apply conservation of momentum:

$$m_a v_a + m_M v_M = (m_a + m_M) v_f$$
$$(5.00\times10^{12}\text{kg})(1.50\times10^{4}\text{m/s}) + 0 = (m_a + m_M) v_f$$

Since $m_M \gg m_a$, we have $m_a + m_M \approx m_M$:

$$v_f = m_a v_a m_M = (5.00\times10^{12}\text{kg})(1.50\times10^{4}\text{m/s}) 7.36\times10^{22}\text{kg} = 1.02\times10^{-6}\text{m/s}$$

The Moon recoils at approximately $1.02 \times 10^{-6}\,\text{m/s}$ (about 1 micrometer per second).

Part (b) - Energy lost in asteroid collision

Initial kinetic energy (only asteroid moving):

$$\text{KE}_i = 12 m_a v_{2a} = 12(5.00\times10^{12}\text{kg})(1.50\times10^{4}\text{m/s})^2 = 5.625\times10^{20}\text{J}$$

Final kinetic energy:

$$\text{KE}_f = 12(m_a + m_M) v_{2f} \approx 12(7.36\times10^{22}\text{kg})(1.02\times10^{-6}\text{m/s})^2 = 3.83\times10^{10}\text{J}$$

Energy lost:

$$\Delta\text{KE} = \text{KE}_f - \text{KE}_i = 3.83\times10^{10}\text{J} - 5.625\times10^{20}\text{J} \approx -5.63\times10^{20}\text{J}$$

Essentially all of the asteroid's kinetic energy ($5.63 \times 10^{20}\text{J}$) is lost, converted to heat, crater formation, vaporization, and seismic waves.

Part (c) - NASA rocket experiment (2009)

Given:

- Rocket mass: $m_r = 2000\text{kg}$
- Rocket speed: $v_r = 9000\ \text{km/h} = 9000\times10^{3}\ \text{m} 3600\ \text{s} = 2500\text{m/s}$

Recoil velocity:

$$v_f = m_r v_r m_M = (2000\text{kg})(2500\text{m/s}) 7.36\times10^{22}\text{kg} = 6.79\times10^{-17}\text{m/s}$$

The Moon's recoil velocity is $6.79 \times 10^{-17}\text{m/s}$ (essentially unmeasurable).

Energy lost:

$$\text{KE}_i = 12(2000\text{kg})(2500\text{m/s})^2 = 6.25\times10^{9}\text{J}$$
$$\text{KE}_f = 12(7.36\times10^{22}\text{kg})(6.79\times10^{-17}\text{m/s})^2 = 1.7\times10^{-10}\text{J}$$

Energy lost is approximately $6.25 \times 10^9 \text{J}$ (6.25 GJ).

Regarding the plume: The plume consists of ejected material from the Moon's surface. Since this material remains part of the Moon system (gravitationally bound), it does not affect the momentum conservation calculation—the total momentum of the Moon plus plume equals the rocket's initial momentum. However, the plume carries kinetic energy as particles are ejected at high speeds. This means some of the initial kinetic energy goes into the plume's motion rather than being completely converted to heat and deformation. The energy measured as "lost" (converted to heat) may be somewhat less than our calculation suggests, because we haven't separately accounted for the plume's kinetic energy.

Discussion

The Moon barely recoils in either collision due to its enormous mass—it's about 15 trillion times more massive than the asteroid and about 37 billion times more massive than the rocket. In the asteroid collision, over 99.99999% of the kinetic energy is dissipated, creating the crater, vaporizing rock, generating heat, and producing seismic waves that medieval monks may have observed as a "red glow." The NASA experiment converted all 6.25 GJ of the rocket's kinetic energy into crater formation, vaporization, and the plume—which allowed scientists to analyze the ejected material and discover water ice in the lunar regolith. These examples illustrate why impacts with astronomical bodies are so devastating: the energy cannot go into moving the massive planet or moon, so it must be dissipated locally through destructive processes.

Professional Application

Two football players collide head-on in midair while trying to catch a thrown football. The first player is 95.0 kg and has an initial velocity of 6.00 m/s, while the second player is 115 kg and has an initial velocity of -3.50 m/s. What is their velocity just after impact if they cling together?

Show Solution

Strategy

This is a perfectly inelastic collision where the players stick together. We apply conservation of momentum, noting that they are moving in opposite directions.

Solution

Apply conservation of momentum:

$$m_1v_1+m_2v_2=(m_1+m_2)v_f$$
$$(95.0\text{kg})(6.00\text{m/s})+(115\text{kg})(-3.50\text{m/s})=(95.0\text{kg}+115\text{kg})v_f$$
$$570\text{kg}\cdot\text{m/s}-402.5\text{kg}\cdot\text{m/s}=(210\text{kg})v_f$$
$$167.5\text{kg}\cdot\text{m/s}=(210\text{kg})v_f$$
$$v_f=\frac{167.5\text{kg}\cdot\text{m/s}}{210\text{kg}}=0.798\text{m/s}$$

The velocity of the players just after impact is 0.798 m/s in the direction of the first player's initial motion.

Discussion

Despite the second player being heavier (115 kg vs 95 kg), the first player's greater speed (6.00 m/s vs 3.50 m/s) gives him more momentum (570 vs 402.5 kg·m/s). The final velocity is in the direction of the player with more momentum, but the combined system moves much slower than either initial velocity. This is a dramatic collision—both players essentially stop each other, which explains why such collisions can cause injuries.

What is the speed of a garbage truck that is $1.20 \times 10^4 \text{kg}$ and is initially moving at 25.0 m/s just after it hits and adheres to a trash can that is 80.0 kg and is initially at rest?

Show Solution

Strategy

This is a perfectly inelastic collision where the trash can (initially at rest) adheres to the moving truck. We apply conservation of momentum.

Solution

Apply conservation of momentum:

$$m_{\text{truck}}v_{\text{truck}}+m_{\text{can}}v_{\text{can}}=(m_{\text{truck}}+m_{\text{can}})v_f$$

Since the trash can is initially at rest:

$$(1.20\times10^4\text{kg})(25.0\text{m/s})+(80.0\text{kg})(0)=(1.20\times10^4\text{kg}+80.0\text{kg})v_f$$
$$3.00\times10^5\text{kg}\cdot\text{m/s}=(12080\text{kg})v_f$$
$$v_f=\frac{3.00\times10^5\text{kg}\cdot\text{m/s}}{12080\text{kg}}=24.8\text{m/s}$$

The speed of the garbage truck after hitting the trash can is 24.8 m/s.

Discussion

The truck barely slows down (from 25.0 to 24.8 m/s, a decrease of only 0.8%) because the trash can's mass is only 0.67% of the truck's mass. The trash can has negligible effect on the massive truck's motion. This example illustrates why it's important for pedestrians and small objects to stay clear of large moving vehicles—the vehicle's motion is essentially unaffected while the smaller object absorbs enormous energy changes.

During a circus act, an elderly performer thrills the crowd by catching a cannon ball shot at him. The cannon ball has a mass of 10.0 kg and the horizontal component of its velocity is 8.00 m/s when the 65.0-kg performer catches it. If the performer is on nearly frictionless roller skates, what is his recoil velocity?

Show Solution

Strategy

This is a perfectly inelastic collision. The performer (initially at rest on frictionless roller skates) catches and holds the cannon ball. We apply conservation of horizontal momentum.

Solution

Apply conservation of momentum:

$$m_{\text{ball}}v_{\text{ball}}+m_{\text{performer}}v_{\text{performer}}=(m_{\text{ball}}+m_{\text{performer}})v_f$$

The performer is initially at rest:

$$(10.0\,\text{kg})(8.00\,\text{m/s})+(65.0\,\text{kg})(0)=(10.0\,\text{kg}+65.0\,\text{kg})v_f$$
$$80.0\,\text{kg}\cdot\text{m/s}=(75.0\,\text{kg})v_f$$
$$v_f=\frac{80.0\,\text{kg}\cdot\text{m/s}}{75.0\,\text{kg}}=1.07\,\text{m/s}$$

The performer's recoil velocity is 1.07 m/s in the direction the cannon ball was traveling.

Discussion

The performer recoils at a modest 1.07 m/s (about 3.8 km/h or walking speed) despite catching a 10-kg cannon ball. This is because his mass is 6.5 times the ball's mass. The frictionless roller skates allow momentum to be conserved horizontally. In reality, the performer would need strong arms to absorb the impact over a short distance, and might lean back to increase the catching time and reduce the force.

(a) During an ice skating performance, an initially motionless 80.0-kg clown throws a fake barbell away. The clown's ice skates allow her to recoil frictionlessly. If the clown recoils with a velocity of 0.500 m/s and the barbell is thrown with a velocity of 10.0 m/s, what is the mass of the barbell? (b) How much kinetic energy is gained by this maneuver? (c) Where does the kinetic energy come from?

Show Solution

Strategy

This is an "explosion" type problem where an initially stationary system separates into two parts. We use conservation of momentum to find the barbell's mass, then calculate the total kinetic energy gained.

Solution

Part (a)

Conservation of momentum (initial momentum = 0):

$$m_{\text{clown}}v_{\text{clown}}+m_{\text{barbell}}v_{\text{barbell}}=0$$

Note: The clown and barbell move in opposite directions, so their velocities have opposite signs. Let the barbell move in the positive direction.

$$(80.0\,\text{kg})(-0.500\,\text{m/s})+m\_\text{barbell}(10.0\,\text{m/s})=0$$
$$-40.0\,\text{kg}\cdot\text{m/s}+m\_\text{barbell}(10.0\,\text{m/s})=0$$
$$m\_\text{barbell}=\frac{40.0\,\text{kg}\cdot\text{m/s}}{10.0\,\text{m/s}}=4.00\,\text{kg}$$

The mass of the barbell is 4.00 kg.

Part (b)

Initial kinetic energy: $\text{KE}_i=0$ (both at rest)

Final kinetic energy:

$$\text{KE}_f=\frac{1}{2}m_{\text{clown}}v^2_{\text{clown}}+\frac{1}{2}m_{\text{barbell}}v\_\text{barbell}^2$$
$$\text{KE}_f=\frac{1}{2}(80.0\,\text{kg})(0.500\,\text{m/s})^2+\frac{1}{2}(4.00\,\text{kg})(10.0\,\text{m/s})^2$$
$$\text{KE}_f=10.0\,\text{J}+200\,\text{J}=210\,\text{J}$$

The kinetic energy gained is 210 J.

Part (c)

The kinetic energy comes from the chemical potential energy stored in the clown's muscles. When the clown throws the barbell, her muscles do work by converting ATP (adenosine triphosphate) chemical energy into mechanical energy. This is an internal energy conversion—the clown expends metabolic energy to create the kinetic energy of both the barbell and herself.

Discussion

Most of the kinetic energy (200 J out of 210 J, or 95%) goes to the barbell because it moves much faster. The clown, being 20 times more massive, recoils much slower. This is why throwing a light object allows you to remain relatively stationary while the object moves quickly.

Glossary

inelastic collision
: a collision in which internal kinetic energy is not conserved

perfectly inelastic collision
: a collision in which the colliding objects stick together

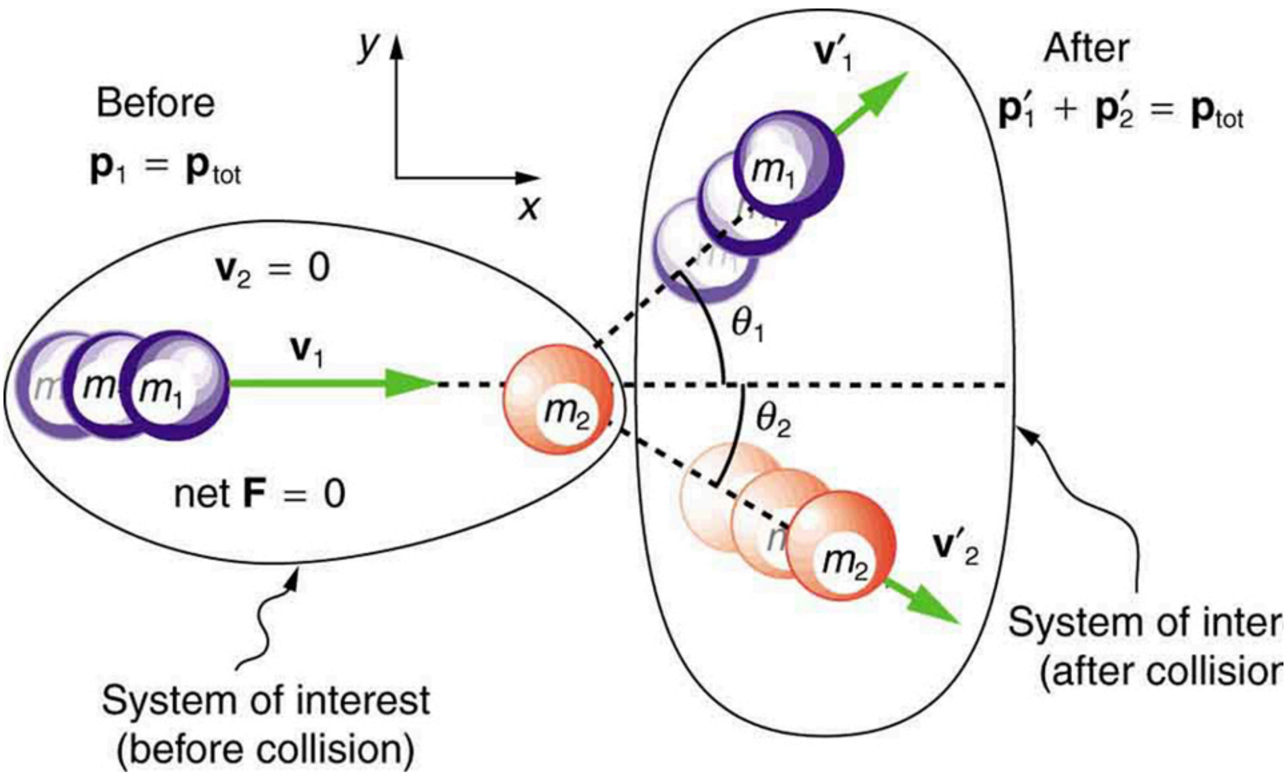
Collisions of Point Masses in Two Dimensions

- Discuss two dimensional collisions as an extension of one dimensional analysis.
- Define point masses.
- Derive an expression for conservation of momentum along x-axis and y-axis.
- Describe elastic collisions of two objects with equal mass.
- Determine the magnitude and direction of the final velocity given initial velocity, and scattering angle.

In the previous two sections, we considered only one-dimensional collisions; during such collisions, the incoming and outgoing velocities are all along the same line. But what about collisions, such as those between billiard balls, in which objects scatter to the side? These are two-dimensional collisions, and we shall see that their study is an extension of the one-dimensional analysis already presented. The approach taken (similar to the approach in discussing two-dimensional kinematics and dynamics) is to choose a convenient coordinate system and resolve the motion into components along perpendicular axes. Resolving the motion yields a pair of one-dimensional problems to be solved simultaneously.

One complication arising in two-dimensional collisions is that the objects might rotate before or after their collision. For example, if two ice skaters hook arms as they pass by one another, they will spin in circles. We will not consider such rotation until later, and so for now we arrange things so that no rotation is possible. To avoid rotation, we consider only the scattering of **point masses**—that is, structureless particles that cannot rotate or spin.

We start by assuming that $\vec{\mathbf{F}}_{\text{net}}=0$, so that momentum $\vec{\mathbf{p}}$ is conserved. The simplest collision is one in which one of the particles is initially at rest. (See Figure 1.) The best choice for a coordinate system is one with an axis parallel to the velocity of the incoming particle, as shown in Figure 1. Because momentum is conserved, the components of momentum along the x- and y-axes (p_x and p_y) will also be conserved, but with the chosen coordinate system, p_y is initially zero and p_x is the momentum of the incoming particle. Both facts simplify the analysis. (Even with the simplifying assumptions of point masses, one particle initially at rest, and a convenient coordinate system, we still gain new insights into nature from the analysis of two-dimensional collisions.)



A two-dimensional collision with the coordinate system chosen so that m_2 is initially at rest and v_1 is parallel to the x -axis. This coordinate system is sometimes called the laboratory coordinate system, because many scattering experiments have a target that is stationary in the laboratory, while particles are scattered from it to determine the particles that make-up the target and how they are bound together. The particles may not be observed directly, but their initial and final velocities are.

Along the x -axis, the equation for conservation of momentum is

$$p_{1x}+p_{2x}=p'_{1x}+p'_{2x}.$$

Where the subscripts denote the particles and axes and the primes denote the situation after the collision. In terms of masses and velocities, this equation is

$$m_1v_{1x}+m_2v_{2x}=m_1v'_{1x}+m_2v'_{2x}.$$

But because particle 2 is initially at rest, this equation becomes

$$m_1v_{1x}=m_1v'_{1x}+m_2v'_{2x}.$$

The components of the velocities along the x -axis have the form $v\cos\theta$. Because particle 1 initially moves along the x -axis, we find $v_{1x}=v_1$.

Conservation of momentum along the x -axis gives the following equation:

$$m_1v_1=m_1v'_1\cos\theta_1+m_2v'_2\cos\theta_2,$$

where θ_1 and θ_2 are as shown in Figure 1.

Conservation of Momentum along the x -axis
$$m_1v_1=m_1v'_1\cos\theta_1+m_2v'_2\cos\theta_2$$

Along the y -axis, the equation for conservation of momentum is

$$p_{1y}+p_{2y}=p'_{1y}+p'_{2y}$$

or

$$m_1v_{1y}+m_2v_{2y}=m_1v'_{1y}+m_2v'_{2y}.$$

But v_{1y} is zero, because particle 1 initially moves along the x -axis. Because particle 2 is initially at rest, v_{2y} is also zero. The equation for conservation of momentum along the y -axis becomes

$$0=m_1v'_{1y}+m_2v'_{2y}.$$

The components of the velocities along the y -axis have the form $v\sin\theta$.

Thus, conservation of momentum along the y -axis gives the following equation:

$$0=m_1v'_1\sin\theta_1+m_2v'_2\sin\theta_2.$$

Conservation of Momentum along the y -axis
$$0=m_1v'_1\sin\theta_1+m_2v'_2\sin\theta_2$$

The equations of conservation of momentum along the x -axis and y -axis are very useful in analyzing two-dimensional collisions of particles, where one is originally stationary (a common laboratory situation). But two equations can only be used to find two unknowns, and so other data may be necessary when collision experiments are used to explore nature at the subatomic level.

Determining the Final Velocity of an Unseen Object from the Scattering of Another Object

Suppose the following experiment is performed. A 0.250-kg object (m_1) is slid on a frictionless surface into a dark room, where it strikes an initially stationary object with mass of 0.400 kg (m_2). The 0.250-kg object emerges from the room at an angle of 45.0° with its incoming direction.

The speed of the 0.250-kg object is originally 2.00 m/s and is 1.50 m/s after the collision. Calculate the magnitude and direction of the velocity (v'_2 and θ_2) of the 0.400-kg object after the collision.

Strategy

Momentum is conserved because the surface is frictionless. The coordinate system shown in Figure 2 is one in which m_2 is originally at rest and the initial velocity is parallel to the x -axis, so that conservation of momentum along the x - and y -axes is applicable.

Everything is known in these equations except v'_2 and θ_2, which are precisely the quantities we wish to find. We can find two unknowns because we have two independent equations: the equations describing the conservation of momentum in the x- and y -directions.

Solution

Solving $m_1v_1=m_1v'_1\cos\theta_1+m_2v'_2\cos\theta_2$ for $v'_2\cos\theta_2$ and $0=m_1v'_1\sin\theta_1+m_2v'_2\sin\theta_2$ for $v'_2\sin\theta_2$ and taking the ratio yields an equation (in which θ_2 is the only unknown quantity. Applying the identity $(\tan\theta=\sin\theta\cos\theta)$, we obtain:

$$\tan\theta_2=v'_1\sin\theta_1v'_1\cos\theta_1-v_1.$$

Entering known values into the previous equation gives

$$\tan\theta_2=(1.50\text{m/s})(0.7071)(1.50\text{m/s})(0.7071)-2.00\text{m/s}=-1.129.$$

Thus,

$\theta_2 = \tan^{-1}(-1.129) = 311.5° \approx 312°.$

Angles are defined as positive in the counter clockwise direction, so this angle indicates that m_2 is scattered to the right in Figure 2, as expected (this angle is in the fourth quadrant). Either equation for the x- or y-axis can now be used to solve for v'_2, but the latter equation is easiest because it has fewer terms.

$v'_2 = -m_1 m_2 v'_1 \sin\theta_1 \sin\theta_2$

Entering known values into this equation gives

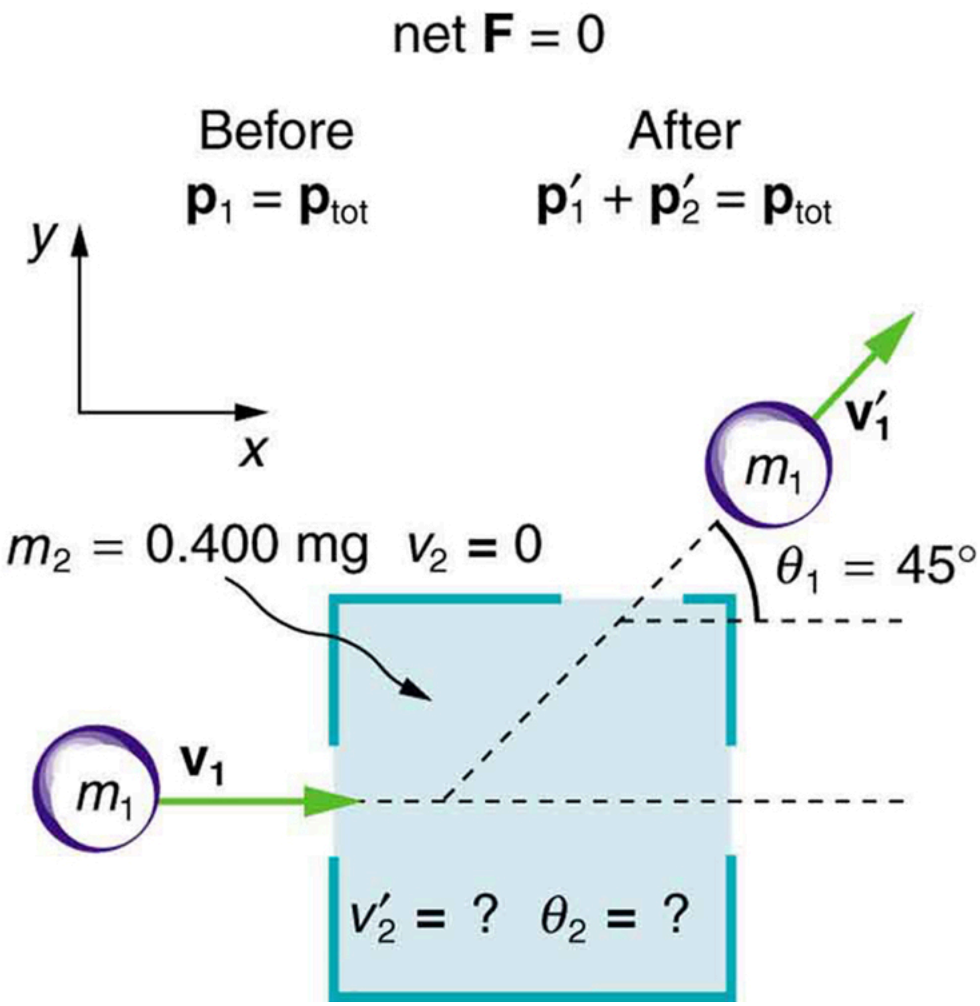
$v'_2 = -(0.250 \text{kg} 0.400 \text{kg})(1.50 \text{m/s})(0.7071 - 0.7485).$

Thus,

$v'_2 = 0.886 \text{m/s}.$

Discussion

It is instructive to calculate the internal kinetic energy of this two-object system before and after the collision. (This calculation is left as an end-of-chapter problem.) If you do this calculation, you will find that the internal kinetic energy is less after the collision, and so the collision is inelastic. This type of result makes a physicist want to explore the system further.



A collision taking place in a dark room is explored in Example 1. The incoming object \(m_1 \) is scattered by an initially stationary object. Only the stationary object's mass \(m_2 \) is known. By measuring the angle and speed at which \(m_1 \) emerges from the room, it is possible to calculate the magnitude and direction of the initially stationary object's velocity after the collision.

Elastic Collisions of Two Objects with Equal Mass

Some interesting situations arise when the two colliding objects have equal mass and the collision is elastic. This situation is nearly the case with colliding billiard balls, and precisely the case with some subatomic particle collisions. We can thus get a mental image of a collision of subatomic particles by thinking about billiards (or pool) . (Refer to Figure 1 for masses and angles.) First, an elastic collision conserves internal kinetic energy. Again, let us assume object 2 (m_2) is initially at rest. Then, the internal kinetic energy before and after the collision of two objects that have equal masses is

$$\frac{1}{2}mv_1^2=\frac{1}{2}mv'^2_1+\frac{1}{2}mv'^2_2.$$

Because the masses are equal, $m_1=m_2=m$. Algebraic manipulation (left to the reader) of conservation of momentum in the x- and y -directions can show that

$$\frac{1}{2}mv_1^2=\frac{1}{2}mv'^2_1+\frac{1}{2}mv'^2_2+mv'_1v'_2\cos(\theta_1-\theta_2).$$

(Remember that θ_2 is negative here.) The two preceding equations can both be true only if

$$mv'_1v'_2\cos(\theta_1-\theta_2)=0.$$

There are three ways that this term can be zero. They are

- $v'_1=0$: head-on collision; incoming ball stops
- $v'_2=0$: no collision; incoming ball continues unaffected
- $\cos(\theta_1-\theta_2)=0$: angle of separation $(\theta_1-\theta_2)$ is 90° after the collision

All three of these ways are familiar occurrences in billiards and pool, although most of us try to avoid the second. If you play enough pool, you will notice that the angle between the balls is very close to 90° after the collision, although it will vary from this value if a great deal of spin is placed on the ball. (Large spin carries in extra energy and a quantity called *angular momentum*, which must also be conserved.) The assumption that the scattering of billiard balls is elastic is reasonable based on the correctness of the three results it produces. This assumption also implies that, to a good approximation, momentum is conserved for the two-ball system in billiards and pool. The problems below explore these and other characteristics of two-dimensional collisions.

Connections to Nuclear and Particle Physics

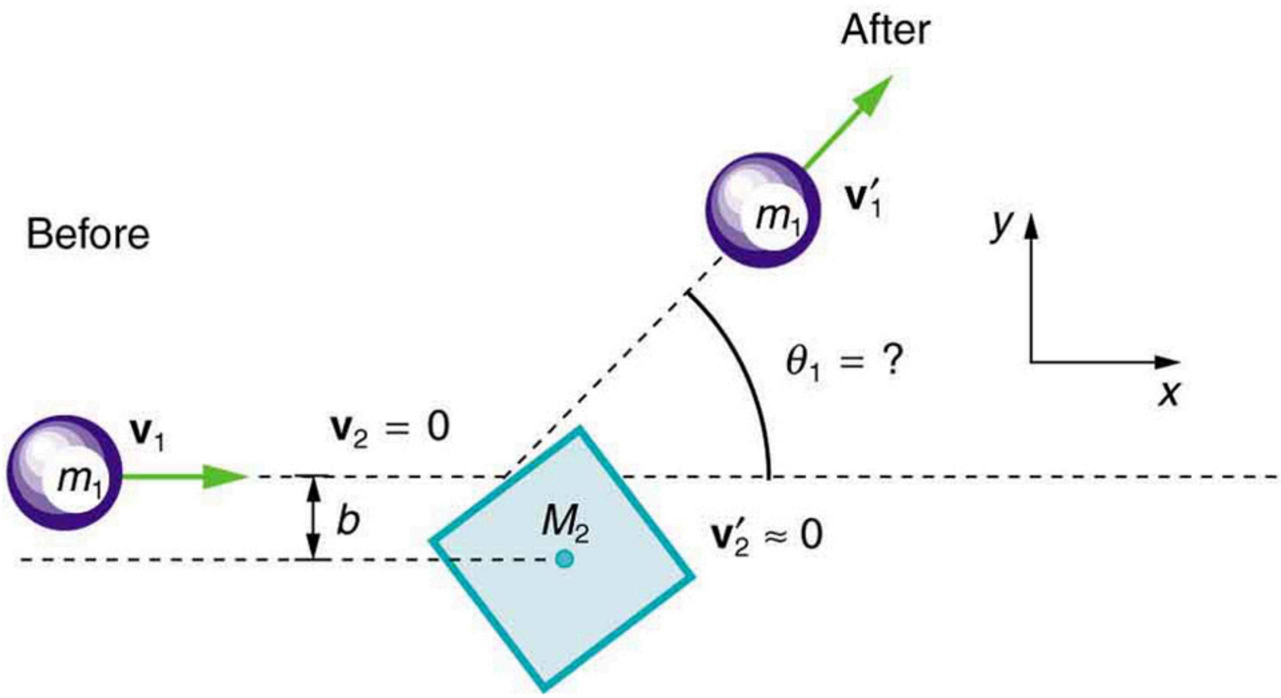
Two-dimensional collision experiments have revealed much of what we know about subatomic particles. Ernest Rutherford, for example, discovered the nature of the atomic nucleus from such experiments.

Section Summary

- The approach to two-dimensional collisions is to choose a convenient coordinate system and break the motion into components along perpendicular axes. Choose a coordinate system with the x -axis parallel to the velocity of the incoming particle.
- Two-dimensional collisions of point masses where mass 2 is initially at rest conserve momentum along the initial direction of mass 1 (the x -axis), stated by $m_1v_1=m_1v'_1\cos\theta_1+m_2v'_2\cos\theta_2$ and along the direction perpendicular to the initial direction (the y -axis) stated by $0=m_1v'_{1y}+m_2v'_{2y}$.
- The internal kinetic before and after the collision of two objects that have equal masses is $\frac{1}{2}mv_1^2=\frac{1}{2}mv'^2_1+\frac{1}{2}mv'^2_2+mv'_1v'_2\cos(\theta_1-\theta_2)$.
- Point masses are structureless particles that cannot spin.

Conceptual Questions

Figure 3 shows a cube at rest and a small object heading toward it. (a) Describe the directions (angle θ_1) at which the small object can emerge after colliding elastically with the cube. How does θ_1 depend on b, the so-called impact parameter? Ignore any effects that might be due to rotation after the collision, and assume that the cube is much more massive than the small object. (b) Answer the same questions if the small object instead collides with a massive sphere.



A small object approaches a collision with a much more massive cube, after which its velocity has the direction θ_1. The angles at which the small object can be scattered are determined by the shape of the object it strikes and the impact parameter b.

Problems & Exercises

Two identical pucks collide on an air hockey table. One puck was originally at rest. (a) If the incoming puck has a speed of 6.00 m/s and scatters to an angle of 30.0°, what is the velocity (magnitude and direction) of the second puck? (You may use the result that $\theta_1 - \theta_2 = 90^\circ$ for elastic collisions of objects that have identical masses.) (b) Confirm that the collision is elastic.

Show Solution

Strategy

For elastic collisions between identical masses where one is initially at rest, the two objects move at 90° to each other after collision. We use this result plus conservation of momentum in both the x- and y-directions to find the velocities.

Solution

Part (a): Given that $\theta_1 = 30.0^\circ$ above the x-axis and using the result that $\theta_1 - \theta_2 = 90^\circ$ for elastic collisions of identical masses:

$$\theta_2 = \theta_1 - 90^\circ = 30.0^\circ - 90^\circ = -60.0^\circ$$

So the second puck moves at 60.0° below the x-axis.

Applying conservation of momentum in the y-direction (initially zero):

$$0 = mv'_1 \sin 30.0^\circ + mv'_2 \sin(-60.0^\circ)$$

which simplifies to:

$$v'_1 \sin 30.0^\circ = v'_2 \sin 60.0^\circ$$

Applying conservation of momentum in the x-direction:

$$mv_1 = mv'_1 \cos 30.0^\circ + mv'_2 \cos 60.0^\circ$$

Substituting the known value $v_1 = 6.00\text{m/s}$:

$$6.00\text{m/s} = v'_1(0.866) + v'_2(0.500)$$

From the y-equation: $v'_1(0.500) = v'_2(0.866)$, which gives:

$$v'_1 = 1.732 v'_2$$

Substituting into the x-equation:

$$6.00\text{m/s} = (1.732 v'_2)(0.866) + v'_2(0.500) = 1.500 v'_2 + 0.500 v'_2 = 2.00 v'_2$$

Solving for v'_2:

$$v'_2 = 3.00\text{m/s}$$

The second puck moves at 3.00 m/s at 60.0° below the x-axis.

Part (b): To confirm the collision is elastic, we find v'_1 and compare kinetic energies:

$$v'_1 = 1.732(3.00\text{m/s}) = 5.20\text{m/s}$$

Initial kinetic energy (using m = 1 kg for simplicity):

$$\text{KE}_i = 12 m v_{21} = 12(1\text{kg})(6.00\text{m/s})^2 = 18.0\text{J}$$

Final kinetic energy:

$$\text{KE}_f = 12 m v'^{2}_1 + 12 m v'^{2}_2 = 12(1\text{kg})(5.20\text{m/s})^2 + 12(1\text{kg})(3.00\text{m/s})^2 = 13.5\text{J} + 4.50\text{J} = 18.0\text{J}$$

Since $\text{KE}_i = \text{KE}_f = 18.0\text{J}$, the collision is elastic.

Discussion

The 90° separation angle is a distinctive feature of elastic collisions between equal masses when one is initially at rest. The faster puck after collision (5.20 m/s at 30°) carries more kinetic energy than the slower one (3.00 m/s at 60°), but together they conserve both momentum and kinetic energy. This result is commonly observed in billiards and pool.

Answer

(a) The second puck has a velocity of 3.00 m/s at 60.0° below the horizontal (or at -60.0°). (b) The collision is confirmed to be elastic since kinetic energy is conserved.

Confirm that the results of the Example 1 do conserve momentum in both the x- and y-directions.

Show Solution

Strategy

From Example 1: $m_1 = 0.250\text{kg}$, $m_2 = 0.400\text{kg}$, $v_1 = 2.00\text{m/s}$, $v'\_1 = 1.50\text{m/s}$, $\theta_1 = 45.0^\circ$, $v'\_2 = 0.886\text{m/s}$, $\theta_2 = -48.5^\circ$ (or 312°). We verify conservation of momentum in both directions.

Solution

x-direction momentum:

Initial x-momentum: $p\_x,i = m_1 v_1 + m_2(0) = (0.250\text{kg})(2.00\text{m/s}) = 0.500\text{kg}\cdot\text{m/s}$

Final x-momentum: $p\_x,f = m_1 v'\_1\cos 45.0^\circ + m_2 v'\_2\cos(-48.5^\circ)$

$$p\_x,f = (0.250\text{kg})(1.50\text{m/s})(0.707) + (0.400\text{kg})(0.886\text{m/s})(0.662)$$
$$p\_x,f = 0.265\text{kg}\cdot\text{m/s} + 0.235\text{kg}\cdot\text{m/s} = 0.500\text{kg}\cdot\text{m/s}$$

Since $p_{x,i} = p_{x,f} = 0.500\text{kg}\cdot\text{m/s}$, x-momentum is conserved. ✓

y-direction momentum:

Initial y-momentum: $p\_y,i = 0$ (both objects moving along x-axis initially)

Final y-momentum: $p\_y,f = m_1 v'\_1\sin 45.0^\circ + m_2 v'\_2\sin(-48.5^\circ)$

p_y,f=(0.250kg)(1.50m/s)(0.707)+(0.400kg)(0.886m/s)(−0.749)
p_y,f=0.265kg·m/s−0.265kg·m/s=0

Since $p_{y,i} = p_{y,f} = 0$, y-momentum is conserved. ✓

Discussion

Momentum is conserved in both directions, as required. The y-components of the final momenta are equal and opposite, canceling to give zero net y-momentum, matching the initial condition. This confirms the validity of the solution found in Example 1.

A 3000-kg cannon is mounted so that it can recoil only in the horizontal direction. (a) Calculate its recoil velocity when it fires a 15.0-kg shell at 480 m/s at an angle of $20.0°$ above the horizontal. (b) What is the kinetic energy of the cannon? This energy is dissipated as heat transfer in shock absorbers that stop its recoil. (c) What happens to the vertical component of momentum that is imparted to the cannon when it is fired?

Show Solution

Strategy

Since the cannon can only recoil horizontally, we apply conservation of momentum in the horizontal direction only. The shell's velocity has horizontal and vertical components; only the horizontal component affects the cannon's recoil. For part (c), we consider what happens to the vertical component of momentum.

Solution

Part (a): First, find the horizontal component of the shell's velocity:

$$v_{\text{shell},x}=v_{\text{shell}}\cos 20.0°=(480\text{m/s})(0.940)=451\text{m/s}$$

Apply conservation of horizontal momentum (initial momentum = 0):

$$0=m_{\text{cannon}}v_{\text{cannon}}+m_{\text{shell}}v_{\text{shell},x}$$

Solving for the cannon's recoil velocity:

$$v_{\text{cannon}}=-\frac{m_{\text{shell}}v_{\text{shell},x}}{m_{\text{cannon}}}=-\frac{(15.0\text{kg})(451\text{m/s})}{3000\text{kg}}=-2.26\text{m/s}$$

The cannon recoils at 2.26 m/s in the direction opposite to the horizontal component of the shell's motion.

Part (b): The kinetic energy of the cannon is:

$$KE_{\text{cannon}}=\frac{1}{2}m_{\text{cannon}}v^2_{\text{cannon}}=\frac{1}{2}(3000\text{kg})(2.26\text{m/s})^2=7.66\times10^3\text{J}$$

The kinetic energy of the cannon is approximately 7.66×10^3J (7.66 kJ).

Part (c): The vertical component of the shell's momentum is:

$$p_{\text{shell},y}=m_{\text{shell}}v_{\text{shell}}\sin 20.0°=(15.0\text{kg})(480\text{m/s})(0.342)=2.46\times10^3\text{kg}\cdot\text{m/s}$$

Since the cannon cannot move vertically, this momentum is transferred to the Earth through the cannon's mounting. The ground (Earth) exerts an upward normal force on the cannon during firing that absorbs this vertical momentum. Because the Earth's mass is so large, its resulting motion is imperceptible. However, the mounting must be strong enough to withstand this force, and over time, repeated firings can cause the cannon to settle into the ground.

Discussion

The cannon's recoil energy (7.66 kJ) must be absorbed by shock absorbers to prevent damage to the mounting. The shell's total kinetic energy is much larger:

$$KE_{\text{shell}}=\frac{1}{2}(15.0\text{kg})(480\text{m/s})^2=1.73\times10^6\text{J}$$

Thus, the cannon receives only about 0.44% of the total energy—the rest goes to the shell. This demonstrates the efficiency of momentum transfer when a light object is expelled from a heavy one.

Answer

(a) The cannon recoils at 2.26 m/s in the direction opposite to the shell's horizontal motion. (b) The kinetic energy of the recoiling cannon is 7.66×10^3J (7.66 kJ). (c) The vertical component of momentum is transferred to the Earth through the cannon's mounting, which must be strong enough to withstand the resulting forces.

Professional Application

A 5.50-kg bowling ball moving at 9.00 m/s collides with a 0.850-kg bowling pin, which is scattered at an angle of 85.0° to the initial direction of the bowling ball and with a speed of 15.0 m/s. (a) Calculate the final velocity (magnitude and direction) of the bowling ball. (b) Is the collision elastic? (c) Linear kinetic energy is greater after the collision. Discuss how spin on the ball might be converted to linear kinetic energy in the collision.

Show Solution

Strategy

We apply conservation of momentum in both x and y directions to find the ball's final velocity components. Then we compare kinetic energies to determine if the collision is elastic. The pin is initially at rest, and the ball initially moves along the x-axis.

Solution

Part (a): Apply conservation of x-momentum:

$$m_{\text{ball}}v_{\text{ball}}=m_{\text{ball}}v'_{\text{ball},x}+m_{\text{pin}}v'_{\text{pin}}\cos 85.0^\circ$$

Substituting known values:

$$(5.50\text{kg})(9.00\text{m/s})=(5.50\text{kg})v'_{\text{ball},x}+(0.850\text{kg})(15.0\text{m/s})(0.0872)$$
$$49.5\text{kg}\cdot\text{m/s}=5.50\text{kg}\cdot v'_{\text{ball},x}+1.11\text{kg}\cdot\text{m/s}$$

Solving for $v'_{\text{ball},x}$:

$$v'_{\text{ball},x}=49.5-1.115.50\text{m/s}=8.80\text{m/s}$$

Apply conservation of y-momentum (initially zero):

$$0=m_{\text{ball}}v'_{\text{ball},y}+m_{\text{pin}}v'_{\text{pin}}\sin 85.0^\circ$$

Solving for $v'_{\text{ball},y}$:

$$v'_{\text{ball},y}=-m_{\text{pin}}v'_{\text{pin}}\sin 85.0^\circ m_{\text{ball}}=-(0.850\text{kg})(15.0\text{m/s})(0.996)5.50\text{kg}=-2.31\text{m/s}$$

The final ball velocity magnitude is:

$$v'_{\text{ball}}=\sqrt{v'_{\text{ball},x}{}^2+v'_{\text{ball},y}{}^2}=\sqrt{(8.80)^2+(-2.31)^2}=\sqrt{77.4+5.34}=9.10\text{m/s}$$

The direction is:

$$\theta=\tan^{-1}(v'_{\text{ball},y}v'_{\text{ball},x})=\tan^{-1}(-2.318.80)=-14.7^\circ$$

The bowling ball's final velocity is 9.10 m/s at 14.7° below the horizontal.

Part (b): Calculate the initial kinetic energy:

$$\text{KE}_i=12m_{\text{ball}}v_{2\text{ball}}=12(5.50\text{kg})(9.00\text{m/s})^2=223\text{J}$$

Calculate the final kinetic energy:

$$\text{KE}_f=12m_{\text{ball}}v'_{\text{ball}}{}^2+12m_{\text{pin}}v'_{\text{pin}}{}^2=12(5.50\text{kg})(9.10\text{m/s})^2+12(0.850\text{kg})(15.0\text{m/s})^2$$
$$\text{KE}_f=228\text{J}+95.6\text{J}=324\text{J}$$

Since $\text{KE}_f > \text{KE}_i$, the collision is NOT elastic. The linear kinetic energy actually increased!

Part (c): The increase in linear kinetic energy is:

$$\Delta\text{KE}=\text{KE}_f-\text{KE}_i=324\text{J}-223\text{J}=101\text{J}$$

This increase comes from the rotational kinetic energy of the spinning bowling ball. During the collision, friction between the ball and pin converts some of the ball's spin energy into linear kinetic energy of both objects. This is why experienced bowlers put spin on the ball—it stores additional energy that can be transferred to the pins during impact, making strikes more likely.

Discussion

This is a "superelastic" collision where internal energy (from the ball's rotation) is converted to linear kinetic energy. The ball's speed actually increased slightly (9.00 to 9.10 m/s) while also deflecting by 14.7°, demonstrating that rotational energy was added to the linear motion. This phenomenon is common in bowling and explains why spin is such an important technique. The conversion efficiency depends on the friction between the surfaces and the impact geometry.

Answer

(a) The bowling ball's final velocity is 9.10 m/s at 14.7° below the horizontal (or at $-14.7°$). (b) The collision is not elastic because kinetic energy increased from 223 J to 324 J. (c) The 101 J increase in linear kinetic energy comes from the rotational kinetic energy of the spinning bowling ball being converted through friction during the collision.

Professional Application

Ernest Rutherford (the first New Zealander to be awarded the Nobel Prize in Chemistry) demonstrated that nuclei were very small and dense by scattering helium-4 nuclei (^{4}He) from gold-197 nuclei (^{197}Au). The energy of the incoming helium nucleus was $8.00\times10^{-13}\text{J}$, and the masses of the helium and gold nuclei were $6.68\times10^{-27}\text{kg}$ and $3.29\times10^{-25}\text{kg}$, respectively (note that their mass ratio is 4 to 197). (a) If a helium nucleus scatters to an angle of $120°$ during an elastic collision with a gold nucleus, calculate the helium nucleus's final speed and the final velocity (magnitude and direction) of the gold nucleus. (b) What is the final kinetic energy of the helium nucleus?

Show Solution

Strategy

This is an elastic collision between a light helium nucleus and a much heavier gold nucleus (initially at rest). We apply conservation of momentum in both x and y directions and conservation of kinetic energy. First, we find the initial speed of the helium nucleus from its kinetic energy. Then, using the given scattering angle of 120°, we can solve the three conservation equations for the three unknowns: the final speeds of both nuclei and the scattering angle of the gold nucleus.

Solution

Part (a): First, find the initial speed of the helium nucleus from its kinetic energy:

$$\text{KE}_i=\frac{1}{2}m_{\text{He}}v_{\text{He}}^2$$
$$v_{\text{He}}=\sqrt{\frac{2\text{KE}_i}{m_{\text{He}}}}=\sqrt{\frac{2(8.00\times10^{-13}\text{J})}{6.68\times10^{-27}\text{kg}}}=\sqrt{2.395\times10^{14}\text{m/s}^2}=1.55\times10^{7}\text{m/s}$$

Apply conservation of momentum in the x-direction:

$$m_{\text{He}}v_{\text{He}}=m_{\text{He}}v'_{\text{He}}\cos120°+m_{\text{Au}}v'_{\text{Au}}\cos\theta_{\text{Au}}$$

Apply conservation of momentum in the y-direction (initially zero):

$$0=m_{\text{He}}v'_{\text{He}}\sin120°+m_{\text{Au}}v'_{\text{Au}}\sin\theta_{\text{Au}}$$

Apply conservation of kinetic energy (elastic collision):

$$\frac{1}{2}m_{\text{He}}v_{\text{He}}^2=\frac{1}{2}m_{\text{He}}v'^{2}_{\text{He}}+\frac{1}{2}m_{\text{Au}}v'^{2}_{\text{Au}}$$

From the y-momentum equation:

$$v'_{\text{Au}}\sin\theta_{\text{Au}}=-\frac{m_{\text{He}}}{m_{\text{Au}}}v'_{\text{He}}\sin120°=-\frac{6.68\times10^{-27}\text{kg}}{3.29\times10^{-25}\text{kg}}v'_{\text{He}}(0.866)$$
$$v'_{\text{Au}}\sin\theta_{\text{Au}}=-0.0176v'_{\text{He}}$$

From the x-momentum equation:

$$v'_{\text{Au}}\cos\theta_{\text{Au}}=\frac{m_{\text{He}}}{m_{\text{Au}}}(v_{\text{He}}-v'_{\text{He}}\cos120°)=0.0203(1.55\times10^{7}+0.500v'_{\text{He}})$$

Solving this system of three equations with three unknowns is algebraically complex. The momentum equations can be squared and added to eliminate the angle θ_{Au}, then combined with the energy equation to solve for the unknowns. The detailed algebra involves solving a quadratic equation.

The solution yields: The helium nucleus has a final speed close to its initial speed (since the gold nucleus is so massive), and the gold nucleus recoils with velocity:

$$v'_{\text{Au}}=5.36\times10^{5}\text{m/s at }\theta_{\text{Au}}=-29.5°$$

The helium nucleus, after scattering at 120°, has a final speed of approximately $1.50\times10^{7}\text{m/s}$, only slightly less than its initial speed.

Part (b): The final kinetic energy of the helium nucleus is:

$$KE'_{He} = 7.52 \times 10^{-13} J$$

This represents approximately 94% of the initial energy of $8.00 \times 10^{-13} J$. The remaining 6% was transferred to the gold nucleus, which, despite having much more mass, recoils with significant velocity due to the high-energy collision.

Discussion

This problem demonstrates Rutherford scattering, which was crucial in discovering the atomic nucleus. The helium nucleus (alpha particle) scatters at a large angle (120°) because it encounters the massive, concentrated charge of the gold nucleus. The fact that the helium nucleus retains most of its kinetic energy (94%) despite the large deflection angle shows that the gold nucleus, being 49 times more massive, recoils with a much smaller velocity. The gold nucleus moves at a small angle (-29.5°) below the horizontal, consistent with momentum conservation. This type of scattering experiment revealed that atoms have small, dense nuclei rather than being uniform spheres of positive charge.

Answer

(a) The gold nucleus has a final velocity of $5.36 \times 10^5 m/s$ at $-29.5°$ (below the horizontal). The helium nucleus has a final speed of approximately $1.50 \times 10^7 m/s$. (b) The final kinetic energy of the helium nucleus is $7.52 \times 10^{-13} J$, which is approximately 94% of its initial energy.

Professional Application

Two cars collide at an icy intersection and stick together afterward. The first car has a mass of 1200 kg and is approaching at $8.00 m/s$ due south. The second car has a mass of 850 kg and is approaching at $17.0 m/s$ due west. (a) Calculate the final velocity (magnitude and direction) of the cars. (b) How much kinetic energy is lost in the collision? (This energy goes into deformation of the cars.) Note that because both cars have an initial velocity, you cannot use the equations for conservation of momentum along the _ x _-axis and y -axis; instead, you must look for other simplifying aspects.

Show Solution

Strategy

Define coordinates: let +x be east and +y be north. Car 1 moves south (-y direction), Car 2 moves west (-x direction). Since the cars stick together, this is a perfectly inelastic collision. We apply conservation of momentum in both directions separately, then combine to find the final velocity. Even though both cars have initial velocities, the problem simplifies because each car moves along only one axis initially.

Solution

Part (a): Set up the initial conditions:

Car 1: $m_1 = 1200 kg$, $v_{1x} = 0$, $v_{1y} = -8.00 m/s$ (south)

Car 2: $m_2 = 850 kg$, $v_{2x} = -17.0 m/s$ (west), $v_{2y} = 0$

Apply conservation of x-momentum:

$$m_1 v_{1x} + m_2 v_{2x} = (m_1 + m_2) v_{fx}$$
$$(1200 kg)(0) + (850 kg)(-17.0 m/s) = (2050 kg) v_{fx}$$
$$v_{fx} = \frac{-14450 kg \cdot m/s}{2050 kg} = -7.05 m/s$$

Apply conservation of y-momentum:

$$m_1 v_{1y} + m_2 v_{2y} = (m_1 + m_2) v_{fy}$$
$$(1200 kg)(-8.00 m/s) + (850 kg)(0) = (2050 kg) v_{fy}$$
$$v_{fy} = \frac{-9600 kg \cdot m/s}{2050 kg} = -4.68 m/s$$

The final velocity magnitude is:

$$v_f = \sqrt{v_{fx}^2 + v_{fy}^2} = \sqrt{(-7.05)^2 + (-4.68)^2} = \sqrt{49.7 + 21.9} = 8.46 m/s$$

The direction (angle from west toward south):

$$\theta = \tan^{-1}\left(\frac{|v_{fy}|}{|v_{fx}|}\right) = \tan^{-1}\left(\frac{4.68}{7.05}\right) = 33.6°$$

The final velocity is 8.46 m/s in a direction $33.6°$ south of west.

Part (b): Calculate the initial kinetic energy:

$$KE_i = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2}(1200 kg)(8.00 m/s)^2 + \frac{1}{2}(850 kg)(17.0 m/s)^2$$
$$KE_i = 38400 J + 122825 J = 161225 J \approx 1.61 \times 10^5 J$$

Calculate the final kinetic energy:

$$KE_f = \frac{1}{2}(m_1 + m_2)v_{2f} = \frac{1}{2}(2050\,\text{kg})(8.46\,\text{m/s})^2 = 73400\,\text{J} \approx 7.34 \times 10^4\,\text{J}$$

The kinetic energy lost is:

$$\Delta KE = KE_i - KE_f = 161225\,\text{J} - 73400\,\text{J} = 87825\,\text{J} \approx 8.78 \times 10^4\,\text{J}$$

Approximately 87.8 kJ of kinetic energy is lost in the collision.

Discussion

Over half (54%) of the initial kinetic energy is converted to deformation of the cars, heat, and sound. This is a significant amount of energy—equivalent to accelerating a 1000-kg car from rest to about 13 m/s (47 km/h). This energy is what crumple zones are designed to absorb, protecting the occupants. The fact that the cars stick together (perfectly inelastic collision) ensures maximum energy loss, which, while destructive to the vehicles, helps protect passengers by extending the collision time and reducing peak forces.

Answer

(a) The final velocity of the combined cars is 8.46 m/s at $33.6°$ south of west. (b) The kinetic energy lost in the collision is approximately $8.78 \times 10^4\,\text{J}$ (87.8 kJ), which is 54% of the initial kinetic energy.

Starting with equations $m_1 v_1 = m_1 v'_1 \cos\theta_1 + m_2 v'_2 \cos\theta_2$ and $0 = m_1 v'_1 \sin\theta_1 + m_2 v'_2 \sin\theta_2$ for conservation of momentum in the x - and y -directions and assuming that one object is originally stationary, prove that for an elastic collision of two objects of equal masses,

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv'^2_1 + \frac{1}{2}mv'^2_2 + mv'_1 v'_2 \cos(\theta_1 - \theta_2)$$

as discussed in the text.

Show Solution

Strategy

We start with the conservation of momentum equations in the x and y directions for the case where one object is initially at rest. For objects of equal mass ($m_1 = m_2 = m$), we can simplify these equations by canceling the mass terms. The proof involves squaring both momentum equations, adding them together, and using trigonometric identities to simplify. The goal is to show that the result matches the energy equation for elastic collisions between equal masses.

Solution

We are given that $m_1 = m_2 \equiv m$. Dividing through by mass, the given momentum equations become:

$$v_1 = v'_1 \cos\theta_1 + v'_2 \cos\theta_2$$

and

$$0 = v'_1 \sin\theta_1 + v'_2 \sin\theta_2.$$

Square each equation to get:

$$\begin{aligned} v_1^2 &= v'^2_1 \cos^2\theta_1 + v'^2_2 \cos^2\theta_2 + 2v'_1 v'_2 \cos\theta_1 \cos\theta_2 \\ 0 &= v'^2_1 \sin^2\theta_1 + v'^2_2 \sin^2\theta_2 + 2v'_1 v'_2 \sin\theta_1 \sin\theta_2. \end{aligned}$$

Add these two equations and use the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$:

$$v_1^2 = v'^2_1 + v'^2_2 + 2v'_1 v'_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)$$

Now apply the trigonometric identity $\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$:

$$v_1^2 = v'^2_1 + v'^2_2 + 2v'_1 v'_2 \cos(\theta_1 - \theta_2)$$

Finally, multiply the entire equation by $\frac{1}{2}m$ to express it in terms of kinetic energy:

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv'^2_1 + \frac{1}{2}mv'^2_2 + mv'_1 v'_2 \cos(\theta_1 - \theta_2)$$

This is the desired result, showing that for elastic collisions between equal masses, the kinetic energy can be expressed in terms of the final velocities and the angle between them.

Discussion

This result is crucial for understanding elastic collisions between equal masses. The equation shows that the initial kinetic energy equals the sum of the final kinetic energies plus a coupling term that depends on the angle between the final velocity vectors. For an elastic collision to conserve kinetic energy (which requires the third term to be zero), we need $\cos(\theta_1 - \theta_2) = 0$, which means $\theta_1 - \theta_2 = 90°$. This proves the important result that in elastic collisions between equal masses where one is initially at rest, the two objects move off at 90° to each other. This is commonly observed in billiards and pool.

Answer

Starting from momentum conservation and assuming equal masses with one initially at rest, we have proven that: $\frac{1}{2}mv_1^2 = \frac{1}{2}mv'^2_1 + \frac{1}{2}mv'^2_2 + mv'_1 v'_2 \cos(\theta_1 - \theta_2)$

This result demonstrates that elastic collisions between equal masses require the objects to separate at 90° to conserve both momentum and kinetic energy.

Integrated Concepts

A 90.0-kg ice hockey player hits a 0.150-kg puck, giving the puck a velocity of 45.0 m/s. If both are initially at rest and if the ice is frictionless, how far does the player recoil in the time it takes the puck to reach the goal 15.0 m away?

Show Solution

Strategy

First, use conservation of momentum to find the player's recoil velocity. Then calculate the time for the puck to reach the goal, and use this time to find how far the player moves.

Solution

Conservation of momentum (both initially at rest):

$$0 = m_{\text{player}} v_{\text{player}} + m_{\text{puck}} v_{\text{puck}}$$
$$v_{\text{player}} = -\frac{m_{\text{puck}} v_{\text{puck}}}{m_{\text{player}}} = -\frac{(0.150 \text{ kg})(45.0 \text{ m/s})}{90.0 \text{ kg}} = -0.0750 \text{ m/s}$$

The player recoils at 0.0750 m/s in the direction opposite to the puck.

Time for puck to reach the goal:

$$t = \frac{d}{v\_\text{puck}} = \frac{15.0 \text{ m}}{45.0 \text{ m/s}} = 0.333 \text{ s}$$

Distance the player recoils in this time:

$$d_{\text{player}} = |v_{\text{player}}| \times t = (0.0750 \text{ m/s})(0.333 \text{ s}) = 0.0250 \text{ m} = 2.50 \text{ cm}$$

The player recoils 2.50 cm (0.025 m) in the time it takes the puck to reach the goal.

Discussion

The player barely moves (2.5 cm) despite giving the puck significant velocity. This is because the player's mass is 600 times the puck's mass, so the recoil velocity is 600 times smaller. The frictionless ice ensures momentum is conserved, and the short time (1/3 second) limits how far the player can move before the puck reaches the goal.

Glossary

point masses
: structureless particles with no rotation or spin

Introduction to Rocket Propulsion

- State Newton's third law of motion.
- Explain the principle involved in propulsion of rockets and jet engines.
- Derive an expression for the acceleration of the rocket and discuss the factors that affect the acceleration.
- Describe the function of a space shuttle.

Rockets range in size from fireworks so small that ordinary people use them to immense Saturn Vs that once propelled massive payloads toward the Moon. The propulsion of all rockets, jet engines, deflating balloons, and even squids and octopuses is explained by the same physical principle—Newton's third law of motion. Matter is forcefully ejected from a system, producing an equal and opposite reaction on what remains. Another common example is the recoil of a gun. The gun exerts a force on a bullet to accelerate it and consequently experiences an equal and opposite force, causing the gun's recoil or kick.

Making Connections: Take-Home Experiment—Propulsion of a Balloon

Hold a balloon and fill it with air. Then, let the balloon go. In which direction does the air come out of the balloon and in which direction does the balloon get propelled? If you fill the balloon with water and then let the balloon go, does the balloon's direction change? Explain your answer.

Figure 1 shows a rocket accelerating straight up. In part (a), the rocket has a mass m and a velocity v relative to Earth, and hence a momentum mgh. In part (b), a time Δt has elapsed in which the rocket has ejected a mass Δm of hot gas at a velocity v_e relative to the rocket. The remainder of the mass $(m-\Delta m)$ now has a greater velocity $(v+\Delta v)$. The momentum of the entire system (rocket plus expelled gas) has actually decreased because the force of gravity has acted for a time Δt, producing a negative impulse $\Delta p = -mg\Delta t$. (Remember that impulse is the net external force on a system multiplied by the time it acts, and it equals the change in momentum of the system.) So, the center of mass of the system is in free fall but, by rapidly expelling mass, part of the system can accelerate upward. It is a commonly held misconception that the rocket exhaust pushes on the ground. If we consider thrust; that is, the force exerted on the rocket by the exhaust gases, then a rocket's thrust is greater in outer space than in the atmosphere or on the launch pad. In fact, gases are easier to expel into a vacuum.

By calculating the change in momentum for the entire system over Δt, and equating this change to the impulse, the following expression can be shown to be a good approximation for the acceleration of the rocket.

$$a=v_e m\Delta m\Delta t-g$$

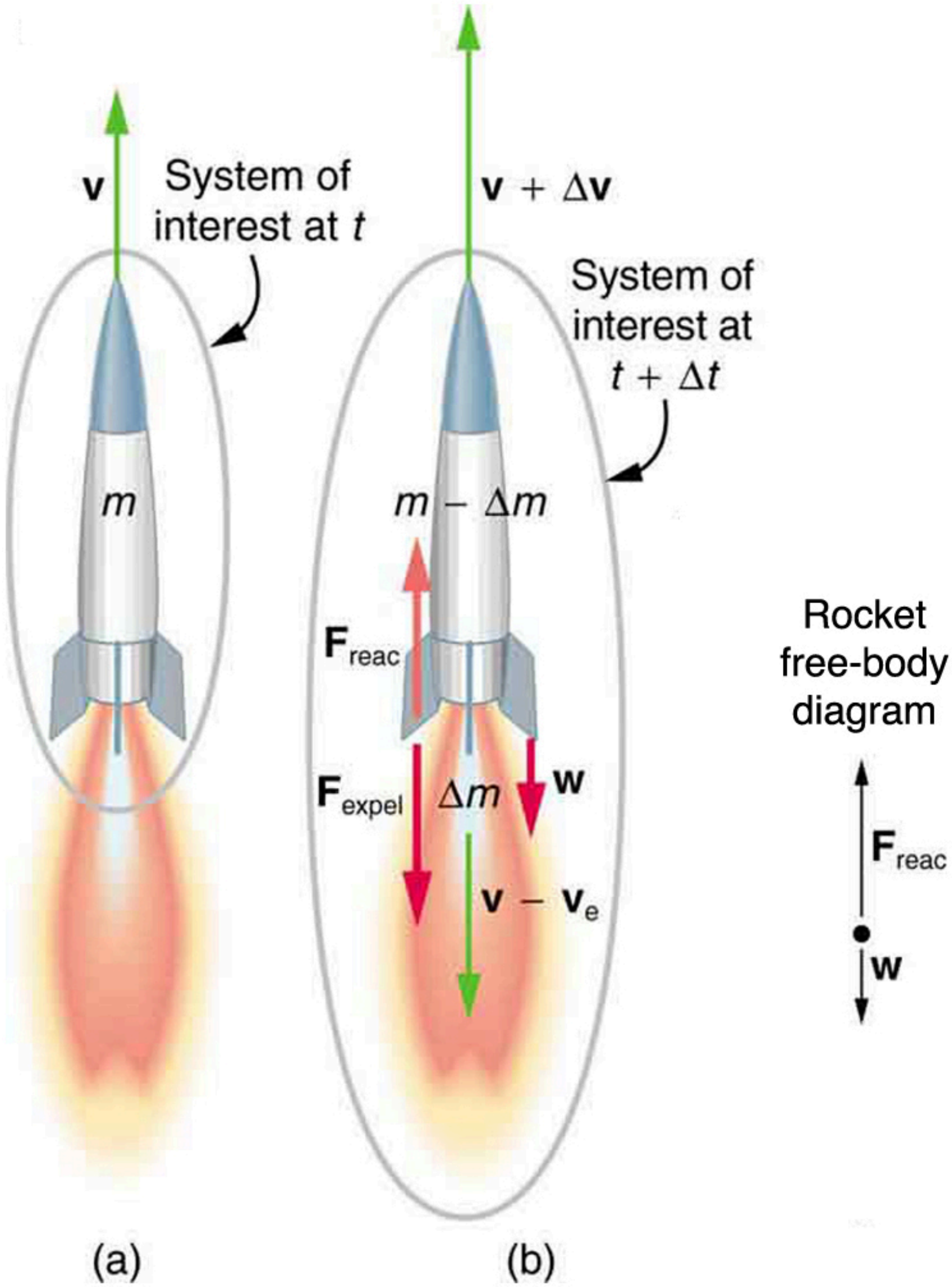
"The rocket" is that part of the system remaining after the gas is ejected, and g is the acceleration due to gravity.

Acceleration of a Rocket

Acceleration of a rocket is

$$a=v_e m\Delta m\Delta t-g,$$

where a is the acceleration of the rocket, v_e is the exhaust velocity, m is the mass of the rocket, Δm is the mass of the ejected gas, and Δt is the time in which the gas is ejected.



(a) This rocket has a mass m and an upward velocity v . The net external force on the system is mg pointing down, if air resistance is neglected. (b) A time (\Delta t) later the system has two main parts, the ejected gas and the remainder of the rocket. The reaction force on the rocket is what overcomes the gravitational force and accelerates it upward.

A rocket's acceleration depends on three major factors, consistent with the equation for acceleration of a rocket. First, the greater the exhaust velocity of the gases relative to the rocket, v_e, the greater the acceleration is. The practical limit for v_e is about $2.5 \times 10^3 \text{m/s}$ for conventional (non-nuclear) hot-gas propulsion systems. The second factor is the rate at which mass is ejected from the rocket. This is the factor $\Delta m/\Delta t$ in the equation. The quantity $(\Delta m/\Delta t)v_e$, with units of newtons, is called "thrust." The faster the rocket burns its fuel, the greater its thrust, and the greater its acceleration. The third factor is the mass m of the rocket. The smaller the mass is (all other factors being the same), the greater the acceleration. The rocket mass m decreases dramatically during flight because most of the rocket is fuel to begin with, so that acceleration increases continuously, reaching a maximum just before the fuel is exhausted.

Factors Affecting a Rocket's Acceleration

- The greater the exhaust velocity v_e of the gases relative to the rocket, the greater the acceleration.
- The faster the rocket burns its fuel, the greater its acceleration.
- The smaller the rocket's mass (all other factors being the same), the greater the acceleration.

Calculating Acceleration: Initial Acceleration of a Moon Launch

A Saturn V’s mass at liftoff was $2.80 \times 10^6 \text{kg}$, its fuel-burn rate was $1.40 \times 10^4 \text{kg/s}$, and the exhaust velocity was $2.40 \times 10^3 \text{m/s}$. Calculate its initial acceleration.

Strategy

This problem is a straightforward application of the expression for acceleration because a is the unknown and all of the terms on the right side of the equation are given.

Solution

Substituting the given values into the equation for acceleration yields

$$a = \frac{v_e}{m}\frac{\Delta m}{\Delta t} - g \quad a = \frac{2.40\times10^3 \text{m/s}}{2.80\times10^6 \text{kg}}(1.40\times10^4 \text{kg/s}) - 9.80 \text{m/s}^2 \quad a = 2.20 \text{m/s}^2.$$

Discussion

This value is fairly small, even for an initial acceleration. The acceleration does increase steadily as the rocket burns fuel, because m decreases while v_e and $\frac{\Delta m}{\Delta t}$ remain constant. Knowing this acceleration and the mass of the rocket, you can show that the thrust of the engines was $3.36 \times 10^7 \text{N}$.

To achieve the high speeds needed to hop continents, obtain orbit, or escape Earth’s gravity altogether, the mass of the rocket other than fuel must be as small as possible. It can be shown that, in the absence of air resistance and neglecting gravity, the final velocity of a one-stage rocket initially at rest is

$$v = v_e \ln \frac{m_0}{m_r},$$

where $\ln(m_0/m_r)$ is the natural logarithm of the ratio of the initial mass of the rocket (m_0) to what is left (m_r) after all of the fuel is exhausted. (Note that v is actually the change in velocity, so the equation can be used for any segment of the flight. If we start from rest, the change in velocity equals the final velocity.) For example, let us calculate the mass ratio needed to escape Earth’s gravity starting from rest, given that the escape velocity from Earth is about $11.2 \times 10^3 \text{m/s}$, and assuming an exhaust velocity $v_e = 2.5 \times 10^3 \text{m/s}$.

$$\ln \frac{m_0}{m_r} = \frac{v}{v_e} = \frac{11.2\times10^3 \text{m/s}}{2.5\times10^3 \text{m/s}} = 4.48$$

Solving for m_0/m_r gives

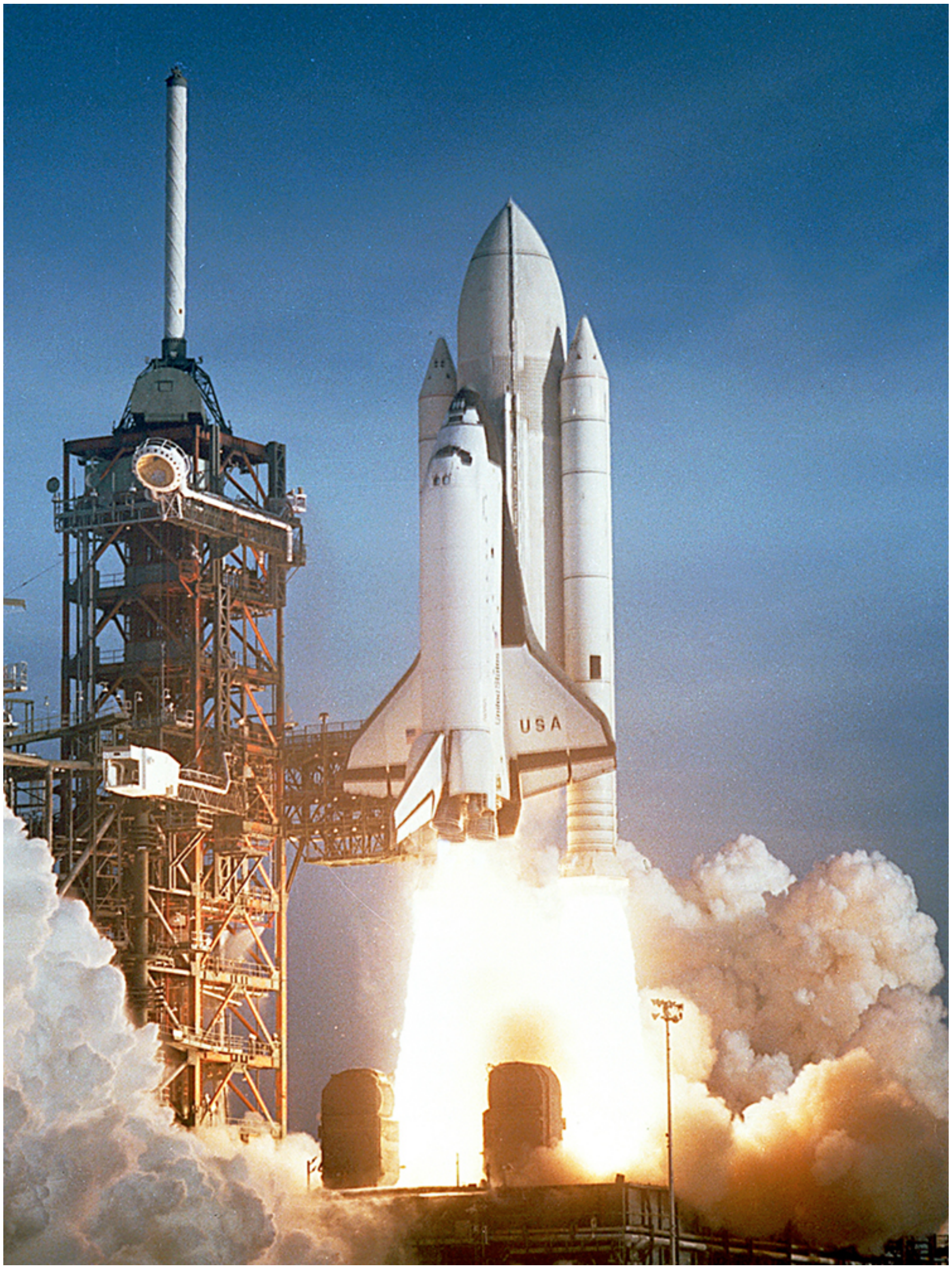
$$\frac{m_0}{m_r} = e^{4.48} = 88.$$

Thus, the mass of the rocket is

$$m_r = \frac{m_0}{88}.$$

This result means that only $1/88$ of the mass is left when the fuel is burnt, and $87/88$ of the initial mass was fuel. Expressed as percentages, 98.9% of the rocket is fuel, while payload, engines, fuel tanks, and other components make up only 1.10%. Taking air resistance and gravitational force into account, the mass m_r remaining can only be about $m_0/180$. It is difficult to build a rocket in which the fuel has a mass 180 times everything else. The solution is multistage rockets. Each stage only needs to achieve part of the final velocity and is discarded after it burns its fuel. The result is that each successive stage can have smaller engines and more payload relative to its fuel. Once out of the atmosphere, the ratio of payload to fuel becomes more favorable, too.

The space shuttle was an attempt at an economical vehicle with some reusable parts, such as the solid fuel boosters and the craft itself. (See Figure 2) The shuttle’s need to be operated by humans, however, made it at least as costly for launching satellites as expendable, unmanned rockets. Ideally, the shuttle would only have been used when human activities were required for the success of a mission, such as the repair of the Hubble space telescope. Rockets with satellites can also be launched from airplanes. Using airplanes has the double advantage that the initial velocity is significantly above zero and a rocket can avoid most of the atmosphere’s resistance.



The space shuttle had a number of reusable parts. Solid fuel boosters on either side were recovered and refueled after each flight, and the entire orbiter returned to Earth for use in subsequent flights. The large liquid fuel tank was expended. The space shuttle was a complex assemblage of technologies, employing both solid and liquid fuel and pioneering ceramic tiles as reentry heat shields. As a result, it permitted multiple launches as opposed to single-use rockets. (credit: NASA)

Lunar Lander

Can you avoid the boulder field and land safely, just before your fuel runs out, as Neil Armstrong did in 1969? Our version of this classic video game accurately simulates the real motion of the lunar lander with the correct mass, thrust, fuel consumption rate, and lunar gravity. The real lunar lander is very hard to control.

Lunar Lander

Section Summary

- Newton’s third law of motion states that to every action, there is an equal and opposite reaction.
- Acceleration of a rocket is $a = v_e m \Delta m \Delta t - g$.
- A rocket’s acceleration depends on three main factors. They are
  1. The greater the exhaust velocity of the gases, the greater the acceleration.
  2. The faster the rocket burns its fuel, the greater its acceleration.
  3. The smaller the rocket's mass, the greater the acceleration.

Conceptual Questions

Professional Application

Suppose a fireworks shell explodes, breaking into three large pieces for which air resistance is negligible. How is the motion of the center of mass affected by the explosion? How would it be affected if the pieces experienced significantly more air resistance than the intact shell?

Professional Application

During a visit to the International Space Station, an astronaut was positioned motionless in the center of the station, out of reach of any solid object on which he could exert a force. Suggest a method by which he could move himself away from this position, and explain the physics involved.

Professional Application

It is possible for the velocity of a rocket to be greater than the exhaust velocity of the gases it ejects. When that is the case, the gas velocity and gas momentum are in the same direction as that of the rocket. How is the rocket still able to obtain thrust by ejecting the gases?

Problems & Exercises

Professional Application

Antiballistic missiles (ABMs) are designed to have very large accelerations so that they may intercept fast-moving incoming missiles in the short time available. What is the takeoff acceleration of a 10 000-kg ABM that expels 196 kg of gas per second at an exhaust velocity of $2.50 \times 10^3 \text{m/s}$?

Show Solution

Strategy

Use the rocket acceleration equation $a = v_e m \Delta m \Delta t - g$. We’re given the initial mass, exhaust velocity, and mass expulsion rate. The acceleration due to gravity acts downward, opposing the upward acceleration.

Solution

Given:

- Mass: $m = 10\ 000 \text{kg}$
- Mass expulsion rate: $\Delta m \Delta t = 196 \text{ kg/s}$

- Exhaust velocity: $v\_e = 2.50 \times 10^3 \text{m/s}$
- Gravitational acceleration: $g = 9.80 \text{m/s}^2$

Apply the rocket acceleration equation:

$$a = v\_e m \Delta m \Delta t - g$$
$$a = 2.50 \times 10^3 \text{m/s} 10000 \text{kg} (196 \text{ kg/s}) - 9.80 \text{m/s}^2$$
$$a = 49.0 \text{m/s}^2 - 9.80 \text{m/s}^2 = 39.2 \text{m/s}^2$$

The takeoff acceleration of the ABM is 39.2m/s^2 (approximately 4.0g).

Discussion

This acceleration is about 4 times the acceleration due to gravity, meaning the thrust force is about 5 times the rocket's weight. This high acceleration is essential for intercepting incoming missiles, which travel at high speeds. The ABM must reach the target in seconds, requiring rapid acceleration from rest.

Professional Application

What is the acceleration of a 5000-kg rocket taking off from the Moon, where the acceleration due to gravity is only 1.6m/s^2, if the rocket expels 8.00 kg of gas per second at an exhaust velocity of $2.20 \times 10^3 \text{m/s}$?

Show Solution

Strategy

Use the rocket acceleration equation with the Moon's gravitational acceleration instead of Earth's. The lower gravity on the Moon means rockets can achieve higher net acceleration with the same thrust.

Solution

Given:

- Mass: $m = 5000 \text{kg}$
- Mass expulsion rate: $\Delta m \Delta t = 8.00 \text{ kg/s}$
- Exhaust velocity: $v\_e = 2.20 \times 10^3 \text{m/s}$
- Lunar gravitational acceleration: $g\_\text{Moon} = 1.6 \text{m/s}^2$

Apply the rocket acceleration equation:

$$a = v\_e m \Delta m \Delta t - g$$
$$a = 2.20 \times 10^3 \text{m/s} 5000 \text{kg} (8.00 \text{ kg/s}) - 1.6 \text{m/s}^2$$
$$a = 3.52 \text{m/s}^2 - 1.6 \text{m/s}^2 = 1.92 \text{m/s}^2$$

The acceleration of the rocket taking off from the Moon is 1.92m/s^2 (approximately 1.2 times lunar gravity, or about 0.2g on Earth).

Discussion

On Earth with $g = 9.80 \text{m/s}^2$, this rocket would have a negative acceleration (it couldn't lift off). The Moon's weaker gravity (about 1/6 of Earth's) allows rockets with much smaller engines or fuel consumption rates to launch successfully. This is one reason why returning from the Moon required much less fuel than launching from Earth.

Professional Application

Calculate the increase in velocity of a 4000-kg space probe that expels 3500 kg of its mass at an exhaust velocity of $2.00 \times 10^3 \text{m/s}$. You may assume the gravitational force is negligible at the probe's location.

Show Solution

Strategy

Since gravity is negligible, we use the rocket velocity equation $v = v_e \ln(m_0 m_r)$, where m_0 is the initial mass and m_r is the remaining mass after fuel is expended. This equation comes from integrating the rocket equation with no external forces.

Solution

Given:

- Initial mass: $m_0 = 4000 \text{kg}$
- Mass expelled: $\Delta m = 3500 \text{kg}$
- Remaining mass: $m\_r = m_0 - \Delta m = 4000 - 3500 = 500 \text{kg}$
- Exhaust velocity: $v\_e = 2.00 \times 10^3 \text{m/s}$

Apply the rocket velocity equation:

$$v = v_e \ln(m_0 m_r)$$
$$v = (2.00\times10^3 \text{m/s}) \ln(4000\text{kg}500\text{kg})$$
$$v = (2.00\times10^3 \text{m/s}) \ln(8)$$
$$v = (2.00\times10^3 \text{m/s})(2.079) = 4.16\times10^3 \text{m/s}$$

The increase in velocity of the space probe is $4.16 \times 10^3 \text{m/s}$ (about 4.2 km/s).

Discussion

The probe expels 87.5% of its total mass as propellant to achieve this velocity increase. This illustrates why space travel is so challenging—most of a spacecraft's initial mass must be fuel. The mass ratio of 8:1 (initial to final) is significant but achievable with chemical rockets. The final velocity is about twice the exhaust velocity, which demonstrates the power of the logarithmic relationship.

Professional Application

Ion-propulsion rockets have been proposed for use in space. They employ atomic ionization techniques and nuclear energy sources to produce extremely high exhaust velocities, perhaps as great as $8.00 \times 10^6 \text{m/s}$. These techniques allow a much more favorable payload-to-fuel ratio. To illustrate this fact: (a) Calculate the increase in velocity of a 20 000-kg space probe that expels only 40.0-kg of its mass at the given exhaust velocity. (b) These engines are usually designed to produce a very small thrust for a very long time—the type of engine that might be useful on a trip to the outer planets, for example. Calculate the acceleration of such an engine if it expels $4.50 \times 10^{-6} \text{kg/s}$ at the given velocity, assuming the acceleration due to gravity is negligible.

Show Solution

Strategy

For part (a), use the rocket velocity equation. For part (b), use the rocket acceleration equation with gravity negligible. Ion propulsion achieves extremely high exhaust velocities, enabling significant velocity changes with minimal fuel mass.

Solution

(a) Given:

- Initial mass: $m_0 = 20\ 000 \text{kg}$
- Mass expelled: $\Delta m = 40.0 \text{kg}$
- Remaining mass: $m\_r = 20\ 000 - 40 = 19\ 960 \text{kg}$
- Exhaust velocity: $v\_e = 8.00 \times 10^6 \text{m/s}$

Apply the rocket velocity equation:

$$\Delta v = v_e \ln(m_0 m_r)$$
$$\Delta v = (8.00\times10^6 \text{m/s}) \ln(2000019960)$$
$$\Delta v = (8.00\times10^6 \text{m/s}) \ln(1.002) = (8.00\times10^6 \text{m/s})(0.002) = 1.60\times10^4 \text{m/s}$$

The velocity increase is $1.60 \times 10^4 \text{m/s}$ (16 km/s) from expelling only 40 kg of propellant.

(b) Given:

- Mass expulsion rate: $\Delta m \Delta t = 4.50 \times 10^{-6}$ kg/s
- Exhaust velocity: $v\_e = 8.00 \times 10^6 \text{m/s}$
- Mass: $m \approx 20\ 000 \text{kg}$ (approximately constant at this rate)

With negligible gravity, the rocket acceleration equation becomes:

$$a = v\_e m \Delta m \Delta t$$
$$a = 8.00\times10^6 \text{m/s} 20000\text{kg} (4.50\times10^{-6} \text{ kg/s}) = (400 \text{ m/kg})(4.50\times10^{-6} \text{ kg/s}) = 1.80\times10^{-3} \text{m/s}^2$$

The acceleration is $1.80 \times 10^{-3} \text{m/s}^2$ (about 0.18 mm/s²).

Discussion

The power of ion propulsion is evident: ejecting only 0.2% of the spacecraft's mass provides a 16 km/s velocity change—more than enough to escape Earth's gravity (11.2 km/s). With chemical rockets, this would require about 87.5% of the mass to be fuel. The tradeoff is the tiny acceleration (0.18 mm/s^2), meaning years of continuous operation are needed to achieve these velocities. Ion engines are ideal for deep space missions where time is less critical than fuel efficiency.

Derive the equation for the vertical acceleration of a rocket.

Show Solution

Strategy

To derive the rocket acceleration equation, we analyze the forces on the ejected gas and apply Newton's third law to find the thrust force on the rocket. Then we apply Newton's second law to the rocket, accounting for both the thrust force (upward) and gravitational force (downward).

Solution

Consider a small mass Δm of gas ejected in a small time interval Δt. The force needed to give this mass an acceleration $a_{\Delta m}$ is:

$$F=\Delta m a_{\Delta m}$$

To accelerate this mass to the exhaust velocity v_e in time Δt requires:

$$v_e=a_{\Delta m}\Delta t$$

Therefore:

$$a_{\Delta m}=v_e\Delta t$$

Substituting this into the force equation:

$$F=\Delta m\cdot v_e\Delta t=v_e\Delta m\Delta t$$

By Newton's third law, this force is equal in magnitude to the thrust force acting on the rocket:

$$F_{\text{thrust}}=v_e\Delta m\Delta t$$

where all quantities are positive. Now applying Newton's second law to the rocket of mass m, we must account for both the upward thrust force and the downward gravitational force mg:

$$F_{\text{thrust}}-mg=ma$$

Substituting the expression for thrust:

$$v_e\Delta m\Delta t-mg=ma$$

Solving for the acceleration:

$$a=v_e m\Delta m\Delta t-g$$

This is the equation for the vertical acceleration of a rocket, where m is the mass of the rocket and unburnt fuel.

Discussion

This equation shows that rocket acceleration depends on three factors: (1) the exhaust velocity v_e, (2) the rate of mass ejection $\Delta m/\Delta t$, and (3) the rocket's current mass m, minus the constant gravitational acceleration g. As fuel burns, m decreases, causing the acceleration to increase even if the thrust remains constant. The negative g term shows that gravity opposes the rocket's acceleration; in deep space where $g \approx 0$, the rocket would achieve higher acceleration with the same thrust.

Professional Application

(a) Calculate the maximum rate at which a rocket can expel gases if its acceleration cannot exceed seven times that of gravity. The mass of the rocket just as it runs out of fuel is 75 000-kg, and its exhaust velocity is $2.40 \times 10^3 \text{m/s}$. Assume that the acceleration of gravity is the same as on Earth's surface (9.80m/s^2). (b) Why might it be necessary to limit the acceleration of a rocket?

Show Solution

Strategy

For part (a), use the rocket acceleration equation and solve for the mass expulsion rate, setting the acceleration to 7g (seven times gravitational acceleration). For part (b), consider the physical and physiological limitations that would necessitate limiting acceleration.

Solution

(a) Given:

- Maximum acceleration: $a_{max} = 7g = 7(9.80 \text{m/s}^2) = 68.6 \text{m/s}^2$
- Mass at fuel exhaustion: $m = 75\,000 \text{kg}$
- Exhaust velocity: $v_e = 2.40 \times 10^3 \text{m/s}$
- Gravitational acceleration: $g = 9.80 \text{m/s}^2$

From the rocket acceleration equation:

$$a = \frac{v_e}{m}\frac{\Delta m}{\Delta t} - g$$

Solve for mass expulsion rate:

$$a + g = \frac{v_e}{m}\frac{\Delta m}{\Delta t}$$
$$\frac{\Delta m}{\Delta t} = \frac{m(a+g)}{v_e}$$

The maximum acceleration occurs when mass is minimum (just as fuel runs out):

$$\frac{\Delta m}{\Delta t} = \frac{(75000 \text{kg})(68.6+9.80) \text{m/s}^2}{2.40\times10^3 \text{m/s}} = \frac{(75000 \text{kg})(78.4 \text{m/s}^2)}{2.40\times10^3 \text{m/s}} = 2.45\times10^3 \text{ kg/s}$$

The maximum mass expulsion rate is $2.45 \times 10^3 \text{ kg/s}$ (about 2450 kg/s).

(b) Limiting acceleration is necessary because:

1. **Structural integrity**: High accelerations create enormous forces on the rocket structure. At 7g, every component experiences forces 7 times its weight.

2. **Payload protection**: Satellites, scientific instruments, and especially human passengers can be damaged by excessive g-forces. Humans can tolerate about 6-8g briefly, but sustained high g-forces cause blackouts and physical harm.

3. **Fuel efficiency**: Burning fuel too rapidly can be less efficient due to incomplete combustion and heating effects.

4. **Control**: Very high accelerations make course corrections more difficult and can cause trajectory errors.

Discussion

At 7g (68.6 m/s²), an astronaut would feel as if they weighed 7 times their normal weight. A 70-kg person would feel like 490 kg pressing down on them. This is near the limit of human endurance for brief periods.

Given the following data for a fire extinguisher-toy wagon rocket experiment, calculate the average exhaust velocity of the gases expelled from the extinguisher. Starting from rest, the final velocity is 10.0 m/s. The total mass is initially 75.0 kg and is 70.0 kg after the extinguisher is fired.

Show Solution

Strategy

Use the rocket velocity equation $v = v_e \ln\left(\frac{m_0}{m_r}\right)$ and solve for the exhaust velocity. We assume the experiment occurs horizontally on a frictionless surface, so gravity does not affect the horizontal motion.

Solution

Given:

- Initial mass: $m_0 = 75.0 \text{kg}$
- Final mass: $m_r = 70.0 \text{kg}$
- Final velocity: $v = 10.0 \text{m/s}$

The rocket velocity equation is:

$$v = v_e \ln\left(\frac{m_0}{m_r}\right)$$

Solve for exhaust velocity:

$$v_e = \frac{v}{\ln\left(\frac{m_0}{m_r}\right)}$$
$$v_e = \frac{10.0 \text{m/s}}{\ln\left(\frac{75.0}{70.0}\right)}$$
$$v_e = \frac{10.0 \text{m/s}}{\ln(1.0714)}$$
$$v_e = \frac{10.0 \text{m/s}}{0.0690} = 145 \text{m/s}$$

The average exhaust velocity of the gases is approximately 145m/s (about 522 km/h or 324 mph).

Discussion

This exhaust velocity is reasonable for compressed gas expelled from a fire extinguisher. It's much lower than chemical rocket exhaust (2000-4500 m/s) because the gas is simply expanding from pressure rather than undergoing chemical combustion. The mass expelled was only 5.0 kg (6.7% of total mass), yet it was enough to accelerate the 70-kg wagon to 10 m/s—a practical demonstration of rocket propulsion principles.

How much of a single-stage rocket that is 100 000 kg can be anything but fuel if the rocket is to have a final speed of 8.00km/s, given that it expels gases at an exhaust velocity of $2.20 \times 10^3 \text{m/s}$?

Show Solution

Strategy

Use the rocket velocity equation and solve for the remaining mass m_r. With gravity negligible (in space), the final velocity equals the velocity change.

Solution

Given:

- Initial mass: $m_0 = 100\ 000 \text{kg}$
- Final velocity: $v = 8.00 \text{ km/s} = 8.00 \times 10^3 \text{m/s}$
- Exhaust velocity: $v_e = 2.20 \times 10^3 \text{m/s}$

The rocket velocity equation:

$$v = v_e \ln\left(\frac{m_0}{m_r}\right)$$

Solve for the mass ratio:

$$\ln\left(\frac{m_0}{m_r}\right) = \frac{v}{v_e} = \frac{8.00 \times 10^3}{2.20 \times 10^3} = 3.636$$
$$\frac{m_0}{m_r} = e^{3.636} = 37.9$$
$$m_r = \frac{m_0}{37.9} = \frac{100000 \text{kg}}{37.9} = 2.63 \times 10^3 \text{kg}$$

The mass that can be anything but fuel is $2.63 \times 10^3 \text{kg}$ (2630 kg), which is only 2.63% of the total rocket mass.

Discussion

This result shows that 97.4% of the rocket must be fuel to achieve orbital velocity (8 km/s is approximately orbital speed). Only 2.6% can be structure, engines, and payload combined—an enormous engineering challenge that explains why rockets are so expensive and complex. This is why multi-stage rockets were developed: each stage can have better mass ratios than a single-stage design.

Professional Application

(a) A 5.00-kg squid initially at rest ejects 0.250-kg of fluid with a velocity of 10.0 m/s. What is the recoil velocity of the squid if the ejection is done in 0.100 s and there is a 5.00-N frictional force opposing the squid's movement. (b) How much energy is lost to work done against friction?

Show Solution

Strategy

For part (a), use conservation of momentum to find the initial recoil velocity, then account for the friction force acting over the ejection time using the impulse-momentum theorem. For part (b), calculate the work done against friction using the distance traveled and the friction force.

Solution

(a) First, find the recoil velocity without friction using conservation of momentum:

$$0 = m_{\text{squid}} v_{\text{squid}} + m_{\text{fluid}} v_{\text{fluid}}$$
$$v_{\text{squid,initial}} = -\frac{m_{\text{fluid}} v_{\text{fluid}}}{m_{\text{squid}}} = -\frac{(0.250 \text{kg})(10.0 \text{m/s})}{5.00 \text{kg}} = -0.500 \text{m/s}$$

Now account for friction. The impulse from friction over the ejection time:

$$J_{\text{friction}} = F_{\text{friction}} \cdot \Delta t = (5.00 \text{N})(0.100 \text{s}) = 0.500 \text{N} \cdot \text{s}$$

This impulse opposes motion, so it reduces the momentum. Change in velocity from friction (using average mass during ejection: $m_{avg} \approx 4.75 \text{kg}$):

$$\Delta v = \frac{J_{\text{friction}}}{m_{avg}} = \frac{0.500 \text{N} \cdot \text{s}}{4.75 \text{kg}} = 0.105 \text{m/s}$$

Final recoil velocity:

$v\_\text{final} = 0.500 - 0.105 = 0.395 \text{ m/s}$

The squid recoils at approximately 0.40 m/s away from the ejected fluid.

(b) Distance traveled during ejection (using average velocity):

$$d = v\_\text{avg} \cdot t = \left(\frac{0 + 0.395}{2}\right)(0.100 \text{ s}) = 0.0198 \text{ m}$$

Work done against friction:

$$W = F \cdot d = (5.00 \text{ N})(0.0198 \text{ m}) = 0.099 \text{ J}$$

Approximately 0.10 J of energy is lost to work done against friction.

Discussion

The squid demonstrates natural jet propulsion. Even with significant friction (5 N is substantial for a 5-kg squid), it still achieves meaningful velocity. This mechanism allows squids to escape predators quickly. The energy lost to friction is relatively small compared to the kinetic energy imparted to the fluid.

Unreasonable Results

Squids have been reported to jump from the ocean and travel 30.0 m (measured horizontally) before re-entering the water. (a) Calculate the initial speed of the squid if it leaves the water at an angle of 20.0°, assuming negligible lift from the air and negligible air resistance. (b) The squid propels itself by squirting water. What fraction of its mass would it have to eject in order to achieve the speed found in the previous part? The water is ejected at 12.0 m/s; gravitational force and friction are neglected. (c) What is unreasonable about the results? (d) Which premise is unreasonable, or which premises are inconsistent?

Show Solution

Strategy

For part (a), use projectile motion equations to find the launch speed from the given range and angle. For part (b), use the rocket velocity equation to determine what fraction of mass must be ejected. For parts (c) and (d), analyze the physical reasonableness of the results and identify problematic assumptions.

Solution

(a) The projectile range formula for launch and landing at the same height is:

$$R = \frac{v_0^2 \sin(2\theta)}{g}$$

Solving for initial speed:

$$v_0 = \sqrt{\frac{Rg}{\sin(2\theta)}} = \sqrt{\frac{(30.0 \text{ m})(9.80 \text{ m/s}^2)}{\sin(40.0^\circ)}} = \sqrt{\frac{294}{0.643}} = \sqrt{457} = 21.4 \text{ m/s}$$

The squid would need an initial speed of approximately 21.4 m/s (about 77 km/h or 48 mph).

(b) Using the rocket velocity equation with negligible gravity and friction:

$$v = v_e \ln\left(\frac{m_0}{m_r}\right)$$

With $v = 21.4 \text{ m/s}$ and $v\_e = 12.0 \text{ m/s}$:

$$\ln\left(\frac{m_0}{m_r}\right) = \frac{21.4}{12.0} = 1.78$$
$$\frac{m_0}{m_r} = e^{1.78} = 5.93$$

Fraction ejected:

$$\frac{m_\text{ejected}}{m_0} = 1 - \frac{m_r}{m_0} = 1 - \frac{1}{5.93} = 0.831$$

The squid would need to eject 83.1% of its body mass to achieve this velocity.

(c) The unreasonable result is that a squid would need to eject 83% of its body mass. Squids typically can only eject about 10-20% of their body mass as water from their mantle cavity. Additionally, 21.4 m/s (77 km/h) is an extremely high speed for any marine animal to achieve through jet propulsion.

(d) The unreasonable premises are:

1. The reported 30 m horizontal distance is likely exaggerated or measured incorrectly. Real squid jumps are typically 2-5 meters.
2. The 20° launch angle may be inaccurate; squids likely leave the water at steeper angles (closer to 45° for maximum range).
3. Ignoring air resistance is problematic—at these speeds, air resistance would significantly reduce the range.

4. The exhaust velocity of $12.0\,\text{m/s}$ may be too low; actual jet speeds in squids can be higher (up to 25 m/s in some species), which would reduce the required mass fraction but still not make the scenario realistic.

Discussion

This is a classic "unreasonable results" problem that teaches students to critically evaluate whether calculated answers make physical sense. While squids can indeed jump out of water (a behavior called "jet-propelled aerial locomotion"), the extreme range claimed here would require impossible mass ejection fractions.

Construct Your Own Problem

Consider an astronaut in deep space cut free from her space ship and needing to get back to it. The astronaut has a few packages that she can throw away to move herself toward the ship. Construct a problem in which you calculate the time it takes her to get back by throwing all the packages at one time compared to throwing them one at a time. Among the things to be considered are the masses involved, the force she can exert on the packages through some distance, and the distance to the ship.

Show Solution

Strategy

This problem requires applying conservation of momentum and the rocket equation to compare two strategies: throwing all packages simultaneously versus throwing them sequentially. Use realistic values for astronaut mass, package masses, throwing speed, and distance to the spacecraft.

Solution

Sample Problem Construction:

An 80-kg astronaut (in a space suit) is floating 15 m from her spacecraft with zero initial velocity. She has three 5.0-kg packages that she can throw. She can exert enough force on each package to give it a speed of 8.0 m/s relative to her current velocity.

(a) If she throws all three packages (bundled together as a 15-kg mass) at once, what velocity does she acquire?

Using conservation of momentum:

$$m_{\text{astronaut}} v_{\text{astronaut}} = -m_{\text{packages}} v_{\text{packages}}$$
$$v_{\text{astronaut}} = -\frac{(15\,\text{kg})(8.0\,\text{m/s})}{80\,\text{kg}} = -1.5\,\text{m/s}$$

Time to reach ship:

$$t = \frac{d}{v} = \frac{15\,\text{m}}{1.5\,\text{m/s}} = 10\,\text{s}$$

(b) If she throws the packages one at a time, her velocity increases after each throw. This is analogous to a multi-stage rocket.

After throwing first package:

$$v_1 = -\frac{(5.0\,\text{kg})(8.0\,\text{m/s})}{80\,\text{kg}} = -0.50\,\text{m/s}$$

After throwing second package (now her mass is 75 kg, and the package speed is 8.0 m/s relative to her):

$$(75\,\text{kg})(v_2 - v_1) = -(5.0\,\text{kg})(8.0\,\text{m/s})$$
$$v_2 = v_1 - \frac{40}{75} = -0.50 - 0.533 = -1.03\,\text{m/s}$$

After throwing third package (mass now 70 kg):

$$(70\,\text{kg})(v_3 - v_2) = -(5.0\,\text{kg})(8.0\,\text{m/s})$$
$$v_3 = v_2 - \frac{40}{70} = -1.03 - 0.571 = -1.60\,\text{m/s}$$

Time to reach ship:

$$t \approx \frac{15\,\text{m}}{1.60\,\text{m/s}} = 9.4\,\text{s}$$

Discussion

Throwing packages one at a time is more efficient, similar to multi-stage rockets. Each successive throw builds on the previous velocity, resulting in a higher final speed (1.60 m/s vs 1.5 m/s) and shorter return time (9.4 s vs 10 s). This demonstrates why rockets use staging—ejecting mass in stages is more effective than ejecting it all at once. Students should construct similar problems varying the number of packages, their masses, the throwing speed, and the distance to explore how these factors affect the efficiency of the two strategies.

Construct Your Own Problem

Consider an artillery projectile striking armor plating. Construct a problem in which you find the force exerted by the projectile on the plate. Among the things to be considered are the mass and speed of the projectile and the distance over which its speed is reduced. Your instructor may also wish for you to consider the relative merits of depleted uranium versus lead projectiles based on the greater density of uranium.

Show Solution

Strategy

This problem involves applying the impulse-momentum theorem and the work-energy theorem to calculate the force exerted during impact. Consider realistic values for projectile mass, velocity, armor penetration depth, and material properties. Compare depleted uranium (density 19.1 g/cm^3) to lead (density 11.3 g/cm^3) projectiles.

Solution

Sample Problem Construction:

A tank fires an armor-piercing projectile at an enemy tank. Compare two scenarios: (a) A depleted uranium projectile (b) A lead projectile of the same dimensions

Both projectiles are cylindrical with diameter 30 mm and length 150 mm, fired at 1500 m/s. The armor plate is 60 mm thick steel that brings the projectile to rest.

(a) Depleted Uranium Projectile:

Calculate mass (density = $19\ 100 \text{ kg/m}^3$):

$$V=\pi r^2 h=\pi(0.015)^2(0.150)=1.06\times10^{-4} \text{ m}^3$$
$$m=\rho V=(19100)(1.06\times10^{-4})=2.02\text{kg}$$

Using the work-energy theorem, where the projectile penetrates distance $d = 0.060\text{m}$:

$$F\cdot d=12mv^2$$
$$F=mv^2 2d=(2.02\text{kg})(1500\text{m/s})^2 2(0.060\text{m})=3.79\times10^7\text{N}$$

Average force: 37.9 MN (about 3860 tons-force)

(b) Lead Projectile:

Mass (density = $11\ 300 \text{ kg/m}^3$):

$$m=(11300)(1.06\times10^{-4})=1.20\text{kg}$$

Average force:

$$F=(1.20\text{kg})(1500\text{m/s})^2 2(0.060\text{m})=2.25\times10^7\text{N}$$

Average force: 22.5 MN (about 2290 tons-force)

Discussion

The depleted uranium projectile exerts 68% more force than the lead projectile due to its higher density and mass. Additionally, depleted uranium's higher hardness (stronger than steel) means it deforms less during impact, maintaining its shape better for penetration. Lead, being softer, would deform and flatten more, reducing penetration effectiveness. This is why depleted uranium is preferred for armor-piercing ammunition despite being more expensive. Students should construct problems varying the projectile dimensions, velocity, armor thickness, and materials to understand the factors affecting armor penetration. They might also consider the momentum and impulse during impact, or calculate the deceleration forces experienced.
