Thermal boundary conditions

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1 Model

1.1 CMB - heat flux pattern

We impose a heat flux balance between the inner and the outer core boundary. The total influx at the ICB Q_i equals the total outflux Q_o at the CMB:

$$-\mathrm{Q}_i = \mathrm{Q}_o \tag{1}$$

$$\Leftrightarrow \int\limits_{\Sigma_{\mathrm{icb}}} \kappa \nabla T\Big|_{\mathrm{icb}} \cdot \boldsymbol{e}_r dS = \int\limits_{\Sigma_{\mathrm{cmb}}} \kappa \nabla T\Big|_{\mathrm{cmb}} \cdot \boldsymbol{e}_r dS \tag{2}$$

$$\Leftrightarrow \int\limits_{\Sigma_{\mathrm{icb}}} \frac{\partial T}{\partial r}\bigg|_{\mathrm{icb}} dS = \int\limits_{\Sigma_{\mathrm{cmb}}} \frac{\partial T}{\partial r}\bigg|_{\mathrm{cmb}} dS \tag{3}$$

For the spectral decomposition it is important to know, that only the 0th order spherical harmonic $\mathcal{Y}_0^0$ yields values $\neq 0$ when integrated over a closed surface Σ:

$$\int\limits_{\Sigma} \frac{\partial T}{\partial r} dS = \int\limits_{\Sigma} \left(\frac{\partial T}{\partial r}\right)_0^0 \mathcal{Y}_0^0 dS \tag{4}$$

with $\left(\frac{\partial T}{\partial r}\right)_0^0$ being the spectral coefficient of degree and order 0.

$$\int\limits_{\Sigma_{\mathrm{icb}}} \left(\frac{\partial T}{\partial r}\right)_0^0 \bigg|_{\mathrm{icb}} \mathcal{Y}_0^0 dS = \int\limits_{\Sigma_{\mathrm{cmb}}} \left(\frac{\partial T}{\partial r}\right)_0^0 \bigg|_{\mathrm{cmb}} \mathcal{Y}_0^0 dS \tag{5}$$

$$\Leftrightarrow \left(\frac{\partial T}{\partial r}\right)_0^0 \bigg|_{\mathrm{icb}} r_i^2 = \left(\frac{\partial T}{\partial r}\right)_0^0 \bigg|_{\mathrm{cmb}} r_o^2 \tag{6}$$

This allows to formulate a simple relation between the mean radial temperature gradient at the inner and outer boundary.

$$\Leftrightarrow \left(\frac{\partial T}{\partial r}\right)_0^0 \bigg|_{\mathrm{icb}} = \left(\frac{\partial T}{\partial r}\right)_0^0 \bigg|_{\mathrm{cmb}} \frac{r_o^2}{r_i^2} = -\beta \frac{r_o^2}{r_i^2} = -\beta \frac{1}{a^2} \tag{7}$$

with $\beta := -\left(\frac{\partial T}{\partial r}\right)_0^0 \bigg|_{\mathrm{cmb}}$ as prescribed temperature gradient at the CMB.
This relation allows us to formulate Neumann boundary conditions for the stationary temperature equation that has the form of a Laplace equation,

$$\nabla^2 T = 0 \tag{8}$$

since we have no internal sources.

$$\text{ICB:} \quad \frac{\partial T}{\partial r}\bigg|_{\mathrm{icb}} = -\beta \frac{1}{a^2} \mathcal{Y}_0^0 \tag{9}$$

$$\text{CMB:} \quad \left.\frac{\partial T}{\partial r}\right|_{\text{cmb}} = -\beta \mathcal{Y}_0^0 + Amp_l^m \mathcal{Y}_l^m \tag{10}$$

where Amp_l^m is the amplitude of the heat flux heterogeneity.
The general solution of (8) in spherical coordinates reads

$$T = \sum_{l,m} \left[a_l^m r^l + b_l^m r^{-l-1} \right] \mathcal{Y}_l^m(\vartheta, \varphi) \tag{11}$$

and its radial derivative

$$\frac{\partial T}{\partial r} = \sum_{l,m} \left[l a_l^m r^{l-1} - (l+1) b_l^m r^{-l-2} \right] \mathcal{Y}_l^m(\vartheta, \varphi). \tag{12}$$

From (9) and (12) follows immediately for $l = m = 0$

$$-b_0^0 r_i^{-2} = -\beta \frac{1}{a^2} \quad \Rightarrow \quad b_0^0 = \beta r_o^2 \tag{13}$$

and the algebraic equation

$$l a_l^m r_i^{l-1} - (l+1) b_l^m r_i^{-l-2} = 0 \tag{14}$$

for all $l > 0$ and $m > 0$.
(10) and (11) together give another equation for the determination of a_l^m and b_l^m:

$$l a_l^m r_o^{l-1} - (l+1) b_l^m r_o^{-l-2} = Amp_l^m. \tag{15}$$

Since only a heat flux pattern with $l = m = 2$ will be used here, (14) and (15) simplify to

$$\begin{pmatrix} 2r_o & -3r_o^{-4} \\ 2r_i & -3r_i^{-4} \end{pmatrix} \begin{pmatrix} a_2^2 \\ b_2^2 \end{pmatrix} = \begin{pmatrix} Amp_2^2 \\ 0 \end{pmatrix}. \tag{16}$$

In a nondimensional form, the solution (8) is

$$\hat{T} = \frac{T}{d\beta} = \left(\frac{a_0^0}{d\beta} + \frac{1}{(a-1)^2 \hat{r}} \right) \mathcal{Y}_0^0 + \frac{Amp_2^2}{\beta} \cdot \frac{2a^5 \hat{r}^3 - 3(a-1)^5 \hat{r}^2}{6(a-1)^4 (1-a^5)} \mathcal{Y}_2^2 \tag{17}$$

with a_0^0 being an arbitrary integration constant that is chosen to be 0 in the following.