分支过程

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模型介绍

1. 设ξ是取值非负整数的随机变量，$P(\xi = k) = p_k, k = 0, 1, \cdots,$ 这里$p_0 < 1$.

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4. Z_1个个体独立繁衍, 方式与祖先一致. 用$\xi_{1,j}$表示第1代第j个个体的后代数, 则$\{\xi_{1,j}:j=1,2,\cdots\}$独立同分布, 与$\xi$同分布, 且与$Z_1$独立.

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$$Z_2 = \sum_{j=1}^{Z_1} \xi_{1,j}.$$

5. 令Z_n为第n代个体总数. 用$\xi_{n,j}$表示第n代第j个个体的后代数, 则$\{\xi_{n,j}: j=1,2,\cdots\}$独立同分布, 与$\xi$同分布, 且与$(Z_1,\cdots,Z_n)$独立.

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Example

设$P(\xi=1)=p$, $P(\xi=0)=1-p$, $0<p<1$.
则$P(Z_n=1)=p^n$, $P(Z_n=0)=1-p^n$.

Theorem

$\{Z_n; n \geq 0\}$ 是时齐$Markov$链, 状态空间为$\{0, 1, \cdots\}$,

$$p_{ij} = P(\sum_{l=1}^{i} \xi_l = j),\ i, j \geq 0,$$

其中$\xi_1, \xi_2, \cdots$独立同分布且与ξ分布相同.

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证明:

$$P(Z_{n+1} = j | Z_0 = 1, Z_1 = i_1, \cdots, Z_{n-1} = i_{n-1}, Z_n = i)$$

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$$\begin{aligned} & P(Z_{n+1} = j | Z_0 = 1, Z_1 = i_1, \cdots, Z_{n-1} = i_{n-1}, Z_n = i) \\ = \ & P(\sum_{l=1}^{i} \xi_{n,l} = j | Z_0 = 1, Z_1 = i_1, \cdots, Z_{n-1} = i_{n-1}, Z_n = i) \end{aligned}$$

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Theorem

设$E\xi = \mu$, $\mathrm{Var}(\xi) = \sigma^2$. 那么对$n \geq 1$,

(1) $E(Z_n) = \mu^n$;

(2) $\mathrm{Var}(Z_n) = \sigma^2 \mu^{n-1}(1 + \mu + \cdots + \mu^{n-1})$.

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由归纳法, (2)对所有$n\geq 1$成立.

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注： Z_n的分布律一般很难算.

Example

设$p_0 = 0.2$, $p_1 = 0.3$, $p_2 = 0.2$, $p_3 = 0.2$, $p_4 = 0.1$.
求$P(Z_2 = 0)$和$P(Z_2 = 1)$.

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P(Z_2 = 0) &= \sum_{k=0}^{4} P(Z_2 = 0 | Z_1 = k) P(Z_1 = k) = \sum_{k=0}^{4} p_0^k p_k \\
&= 0.2 + 0.2 \times 0.3 + 0.2^2 \times 0.2 \\
&\quad + 0.2^3 \times 0.2 + 0.2^4 \times 0.1 \\
&= 0.26976.
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$$P(Z_2 = 1) = \sum_{k=0}^{4} P(Z_2 = 1 | Z_1 = k) P(Z_1 = k)$$

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&= p_1^2 + 2p_0 p_1 p_2 + 3p_0^2 p_1 p_3 + 4p_0^3 p_1 p_4 \\
&= 0.12216.
\end{aligned}
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令Z_n的生成函数为$\phi_n(s) = E(s^{Z_n})$.

Theorem

$$\phi_0(s) = s,$$

$$\phi_1(s) = \phi(s),$$

对$n \geq 1$,

$$\phi_{n+1}(s) = \phi_n(\phi(s)) = \phi(\phi_n(s)).$$

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对$n \geq 1$,

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设$P(\xi = k) = \frac{1}{2^{k+1}}, k = 0, 1, \cdots,$ 对$n \geq 1$, 计算(1) $\phi_n(s)$, (2)Z_n的分布律.

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