Divide and Conquer

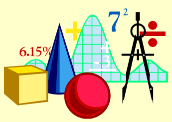
Recursively:

Divide the problem into a number of sub-problems

Conquer the sub-problems by solving them recursively

Combine the solutions to the sub-problems into the solution for the original problem

General recurrence: $T(N) = aT(N/b) + f(N)$



Cases solved by divide and conquer

- **The maximum subsequence sum – the O(N log N) solution**
- **Tree traversals – O(N)**
- **Mergesort and quicksort – O(N log N)**

Closest Points Problem

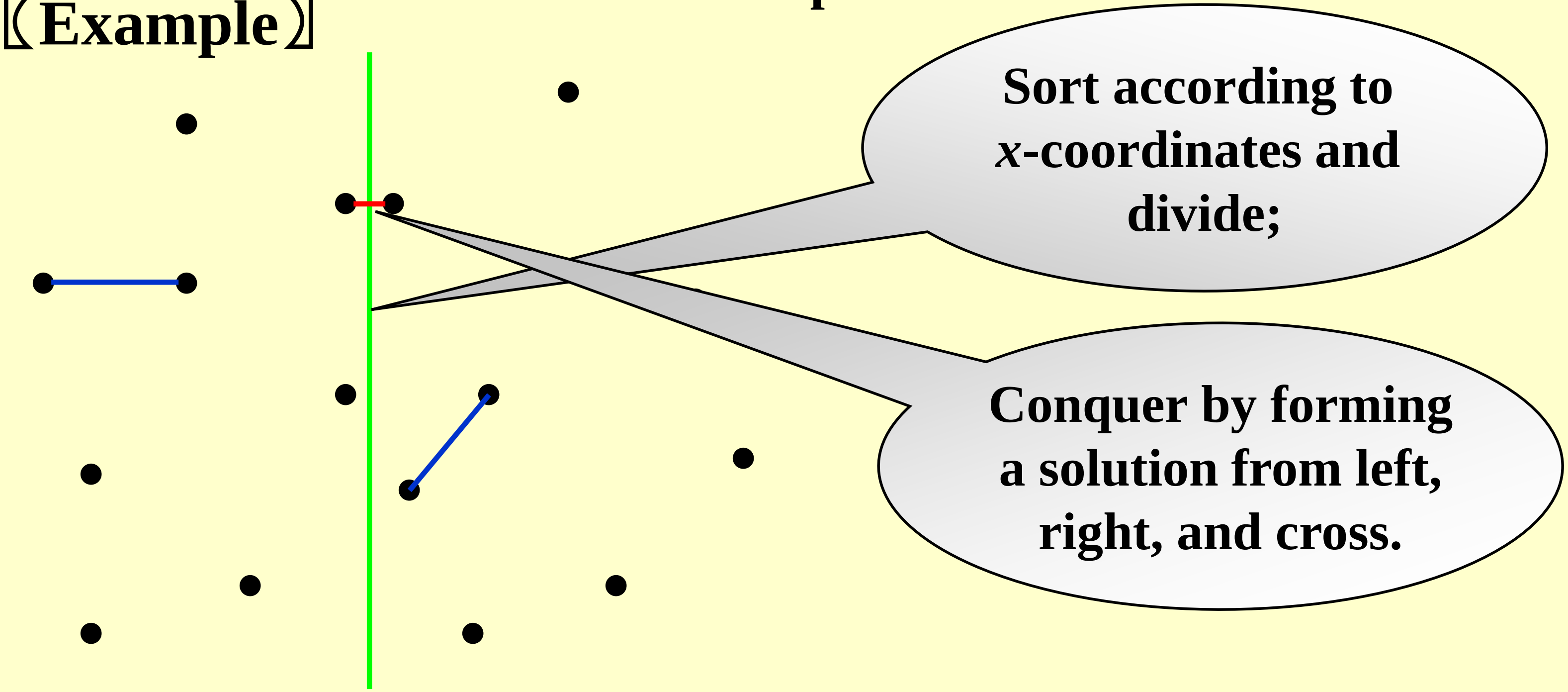
Given N points in a plane. Find the closest pair of points. (If two points have the same position, then that pair is the closest with distance 0.)

- ## Simple Exhaustive Search

Check $N(N-1)/2$ pairs of points. $T = O(N^2)$.

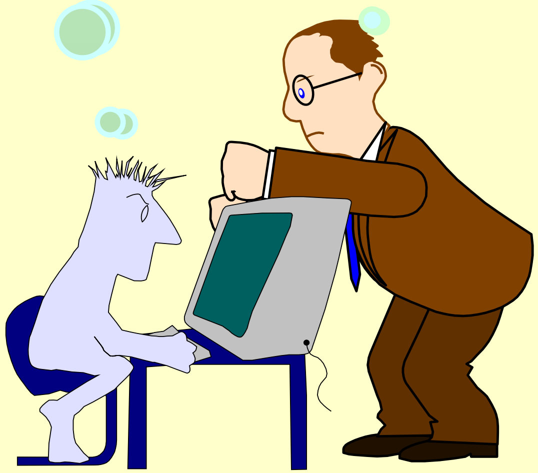
- ## Divide and Conquer – similar to the maximum subsequence sum problem

〖Example〗



st like finding
subsequence sum,
ve $a = b = 2$ …

How about $f(N)$?
Can you find the cross distance
in *linear* time?

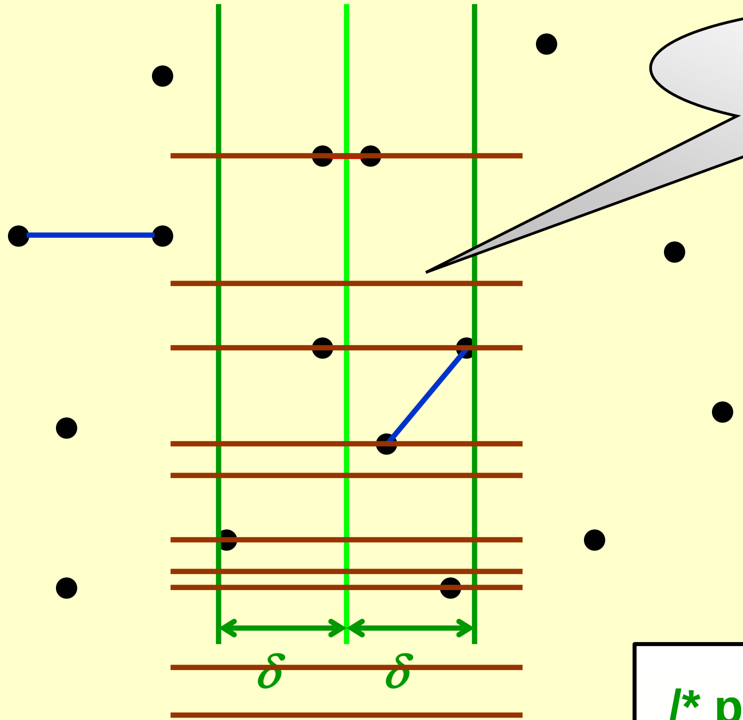


📖 **Recall:**

$$
\begin{aligned}
T(N) &= 2T(N/2) + cN \\
&= 2[\ 2T(N/2^2) + cN/2\ ] + cN \\
&= 2^2\, T(N/2^2) + 2cN \\
&= \ldots\ldots \\
&= 2^k\, T(N/2^k) + kcN \\
&= N + c\, N\log N \;=\; O(\, N\log N\,)
\end{aligned}
$$

if

$$
\begin{aligned}
T(N) &= 2T(N/2) + cN^2 \\
&= 2[\ 2T(N/2^2) + cN^2/2^2\ ] + cN^2 \\
&= 2^2\, T(N/2^2) + cN^2(1+1/2) \\
&= \ldots\ldots \\
&= 2^k\, T(N/2^k) + cN^2(1+1/2+\ldots+1/2^{k-1}\,) \\
&= O(\, N^2\,)
\end{aligned}
$$

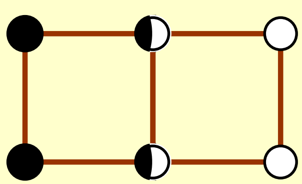


If NumPointInStrip = $O(\sqrt{N})$, we have

```
/* points are all in the strip */
for ( i=0; i<NumPointsInStrip; i++ )
    for ( j=i+1; j<NumPointsInStrip; j++ )
        if ( Dist( P_i , P_j ) < δ )
            δ = Dist( P_i , P_j );
```

The worst case: NumPointInStrip = N

The worst case:



For any p_i, at most 7 points are considered.

$f(N) = O(N)$

```
/* points are all in the strip */
/* and sorted by y coordinates */
for ( i = 0; i < NumPointsInStrip; i++ )
    for ( j = i + 1; j < NumPointsInStrip; j++ )
        if ( Dist_y( P_i , P_j ) > δ )
            break;
        else  if ( Dist( P_i , P_j ) < δ )
            δ = Dist( P_i , P_j );
```

Three methods for solving recurrences:

$$T(N) = a\,T(N/b) + f(N)$$

- **Substitution method**
- **Recursion-tree method**
- **Master method**

Details to be ignored:

- **if (N / b) is an integer or not**
- **always assume $T(n) = \Theta(1)$ for small n**

☞ **Substitution method — guess, then prove by induction**

〖Example〗 $T(N) = 2\,T(\lfloor N/2 \rfloor) + N$

Guess: $T(N) = O(N \log N)$

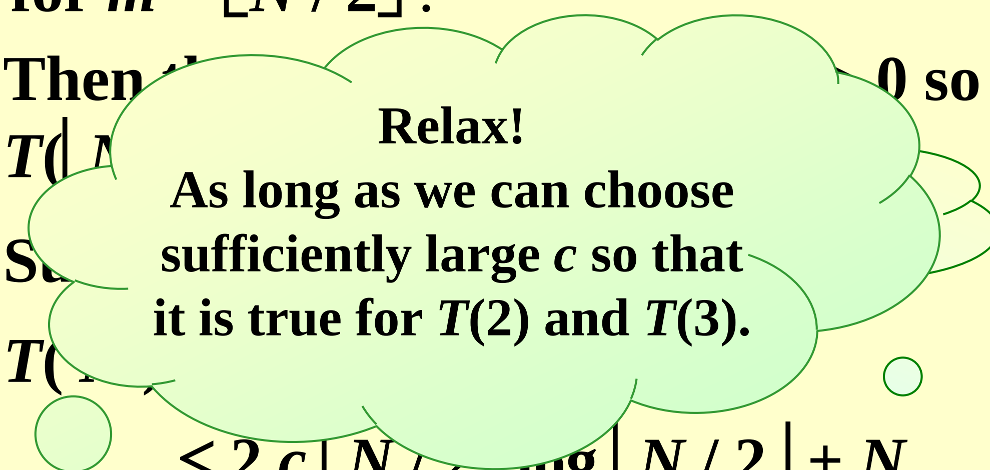
Proof: Assume it is true for all $m < N$, in particular for $m = \lfloor N/2 \rfloor$.

Then … 0 so that

$T(\lfloor N$ …

Su…

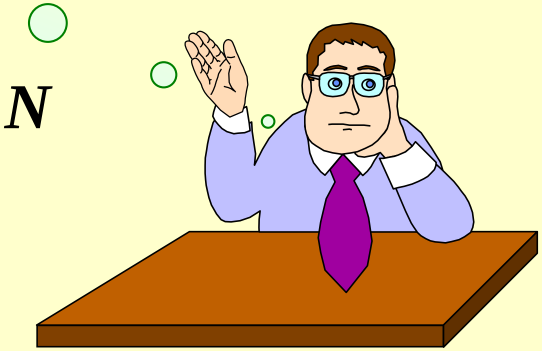
$T($ …



$\le 2\,c\,\lfloor N/2 \rfloor \log \lfloor N/2 \rfloor + N$

$\le c\,N(\log N - \log 2) + N$

$\le c\,N \log N \quad$ for $c \ge 1$



〖Example〗 $T(N) = 2\,T(\lfloor N/2 \rfloor) + N$

Wrong guess: $T(N) = O(N)$

Proof: Assume it is true for all $m < N$, in particular for $m = \lfloor N/2 \rfloor$.

$$T(\lfloor N/2 \rfloor) \le c \lfloor N/2 \rfloor$$

Substituting into the recurrence:

$$\begin{aligned} T(N) &= 2\,T(\lfloor N/2 \rfloor) + N \\ &\le 2c \lfloor N/2 \rfloor + N \\ &\le cN + N = O(N) \end{aligned}$$

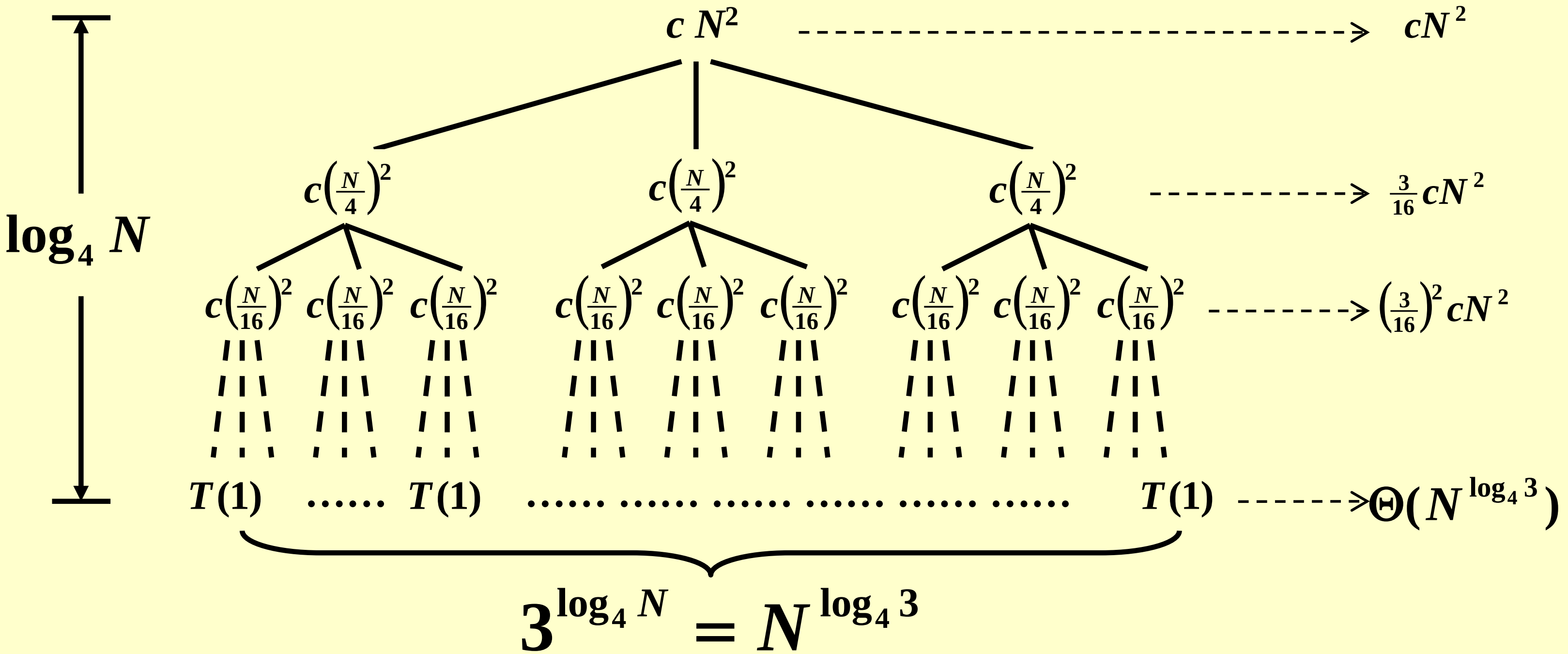
❌

Must prove the *exact form*

How to make a good guess?

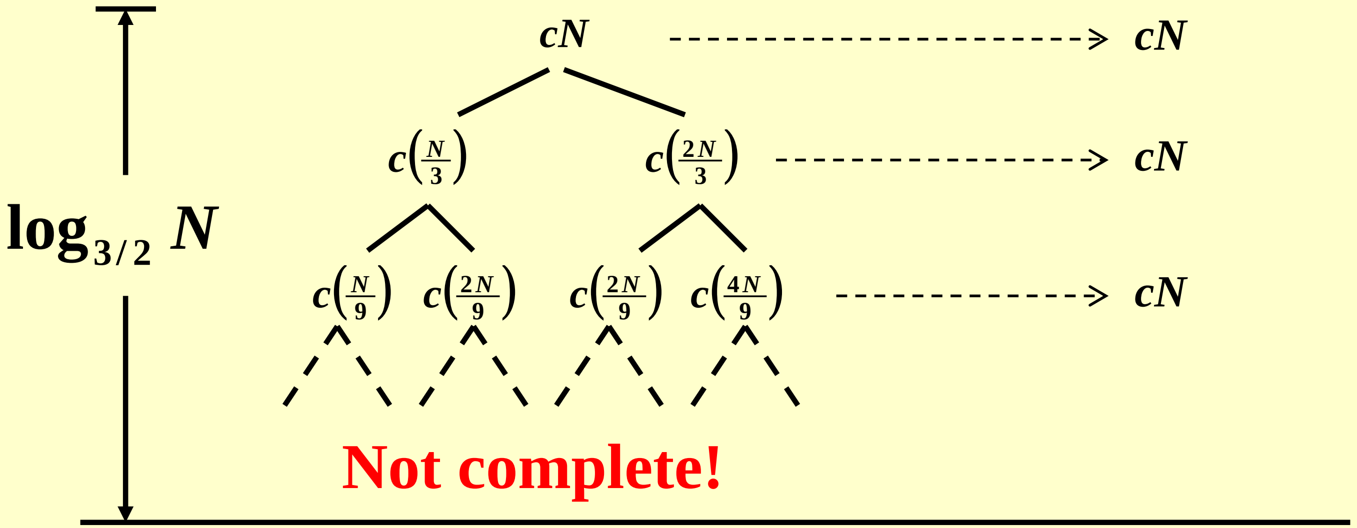
☞ Recursion-tree method

〖Example〗 $T(N) = 3\,T(N/4) + \Theta(N^2)$



$$T(N) = \sum_{i=0}^{\log_4 N - 1} \left(\frac{3}{16}\right)^i cN^2 + \Theta(N^{\log_4 3})$$

〖Example〗 $T(N) = T(N/3) + T(2N/3) + cN$



Guess: $O(N \log N)$

Proof by substitution:

$$T(N) = T(N/3) + T(2N/3) + cN \le d(N/3)\log(N/3) + d(2N/3)\log(2N/3) + cN$$

$$= dN \log N - dN(\log_2 3 - \tfrac{2}{3}) + cN \le dN \log N$$

for $d \ge c/(\log_2 3 - \frac{2}{3})$

Master method

【Master Theorem】 Let $a \geq 1$ and $b > 1$ be constants, let $f(N)$ be a function, and let $T(N)$ be defined on the nonnegative integers by the recurrence $T(N) = aT(N/b) + f(N)$. Then:

1. If $f(N) = O(\underline{N^{\log_b a - \varepsilon}})$ for some constant $\varepsilon > 0$, then $T(N) = \Theta(\underline{N^{\log_b a}})$
2. If $f(N) = \Theta(N^{\log_b a})$, then $T(N) = \Theta(N^{\log_b a} \log N)$

   regularity condition

3. If $\underline{f(N)} = \Omega(N^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af(N/b) < cf(N)$ for some constant $c < 1$ and all sufficiently large N, then $T(N) = \Theta(\underline{f(N)})$

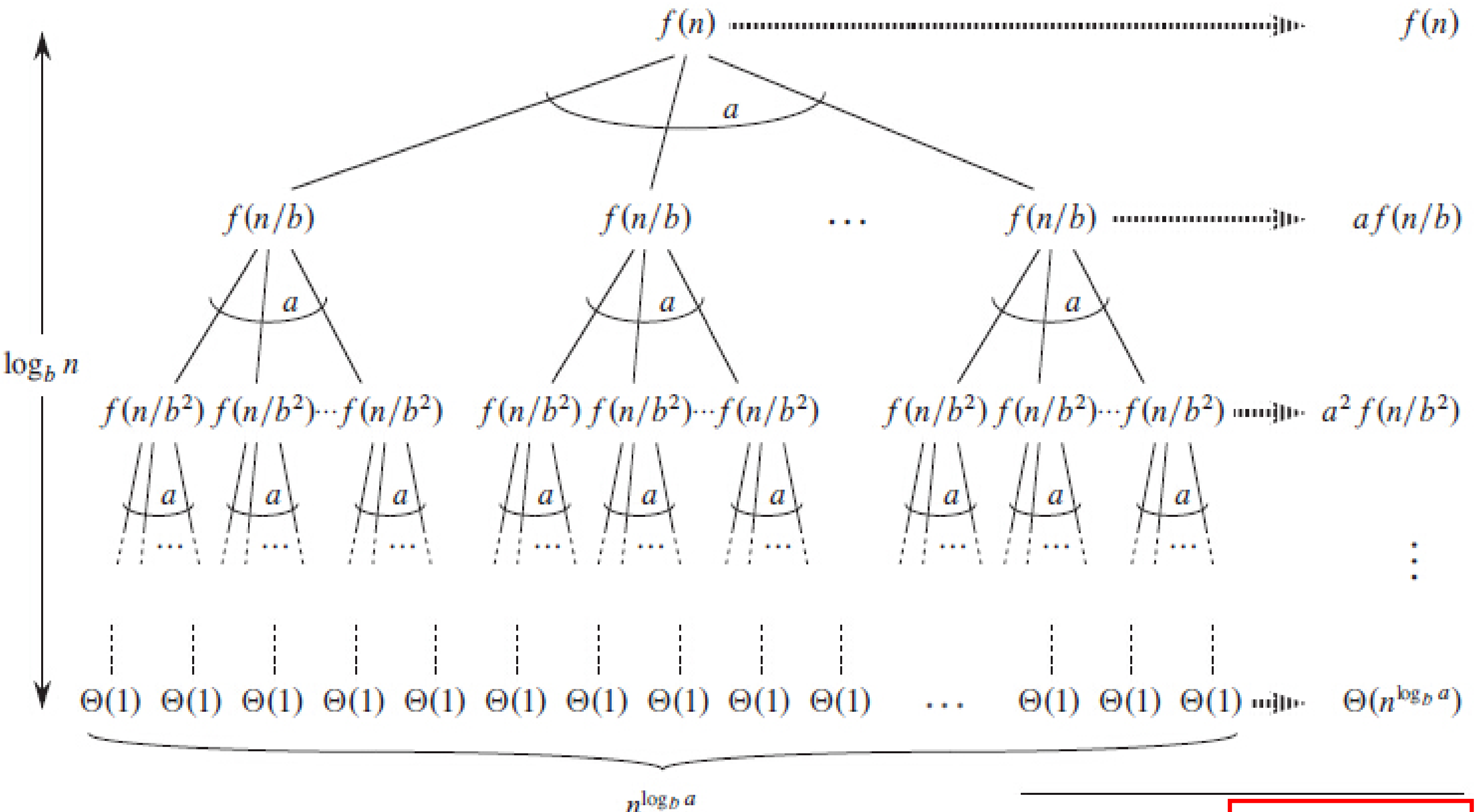
〖Example〗 Mergesort has $a = b = 2$, and case 2

→ $T = O(N \log N)$

〖Example〗 $a = b = 2$, $f(N) = N\log N$?

→ $T = O(N \cancel{\log} N)$

Proof by recursion tree: for $n = b^k$ for some integer k



$$\text{Total: } \Theta(n^{\log_b a}) + \boxed{\sum_{j=0}^{\log_b n - 1} a^j f(n/b^j)}$$

For case 1 where $f(N) = O(N^{\log_b a - \varepsilon})$

$$\sum_{j=0}^{\log_b N - 1} a^j f(N/b^j) =$$

$$= O\left(N^{\log_b a - \varepsilon} \sum_{j=0}^{\log_b N - 1} (b^{\varepsilon})^j\right) = O\left(N^{\log_b a - \varepsilon} \frac{b^{\varepsilon \log_b N} - 1}{b^{\varepsilon} - 1}\right)$$

$$= O(N^{\log_b a - \varepsilon} N^{\varepsilon}) = O(N^{\log_b a})$$

$$T(N) = \Theta(N^{\log_b a}) + O(N^{\log_b a}) = \Theta(N^{\log_b a})$$

Discussion 9:
Please prove case 2.

Read Ch.4 of "Introduction to Algorithms" for the rest of the proof.

☞ Master method – *another form*

【Master Theorem】The recurrence $T(N) = aT(N/b) + f(N)$ can be solved as follows:

1. If $af(N/b) = \kappa f(N)$ for some constant $\kappa < 1$, then $T(N) = \Theta(f(N))$
2. If $af(N/b) = K f(N)$ for some constant $K > 1$, then $T(N) = \Theta(N^{\log_b a})$
3. If $af(N/b) = f(N)$, then $T(N) = \Theta(f(N)\log_b N)$

〖Example〗 $a = 4$, $b = 2$, $f(N) = N\log N$

$$af(N/b) = 4(N/2)\log(N/2) = 2N\log N - 2N$$?

$$f(N) = N\log N \qquad O(N^{\log_b a-\varepsilon}) = O(N^{2-\varepsilon})$$

$\Longrightarrow T = O(N^2)$

【Theorem】 The solution to the equation

$$T(N) = a\,T(N/b) + \Theta(N^k \log^p N),$$

where $a \geq 1$, $b > 1$, and $p \geq 0$ is

$$T(N) = \begin{cases} O(N^{\log_b a}) & \text{if } a > b^k \\ O(N^k \log^{p+1} N) & \text{if } a = b^k \\ O(N^k \log^p N) & \text{if } a < b^k \end{cases}$$

〖Example〗 Mergesort has $a = b = 2$, $p = 0$ and $k = 1$.

→ $T = O(N \log N)$

〖Example〗 Divide with $a = 3$, and $b = 2$ for each recursion; Conquer with $O(N)$ – that is, $k = 1$ and $p = 0$.

→ $T = O(N^{1.59})$

If conquer takes $O(N^2)$ then $T = O(N^2)$.

〖Example〗 $a = b = 2$, $f(N) = N\log N$ → $T = O(N \log^2 N)$

Reference:

Introduction to Algorithms, 3rd Edition: Ch.4, p. 65-113; ***Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein. The MIT Press. 2009***

Data Structure and Algorithm Analysis in C (2nd Edition)：Ch.10，p.370-375；***M.A.Weiss著、陈越改编，人民邮件出版社，2005***

Lecture Notes of CS 373: Combinatorial Algorithms: Notes on Solving Recurrence Relations, p.10-13; ***Jeff Erickson, University of Illinois, Urbana-Champaign, 2003***