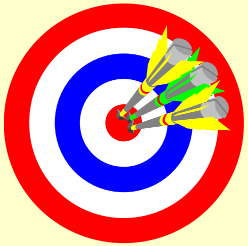
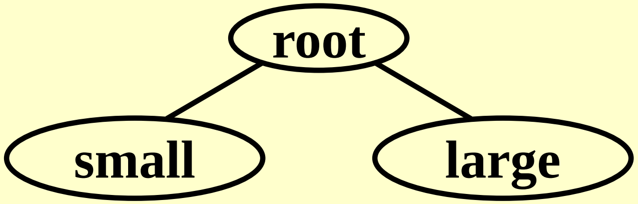
AVL Trees, Splay Trees, and Amortized Analysis

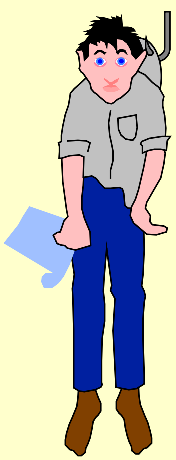
AVL Trees



Target : **Speed up searching (with insertion and deletion)**

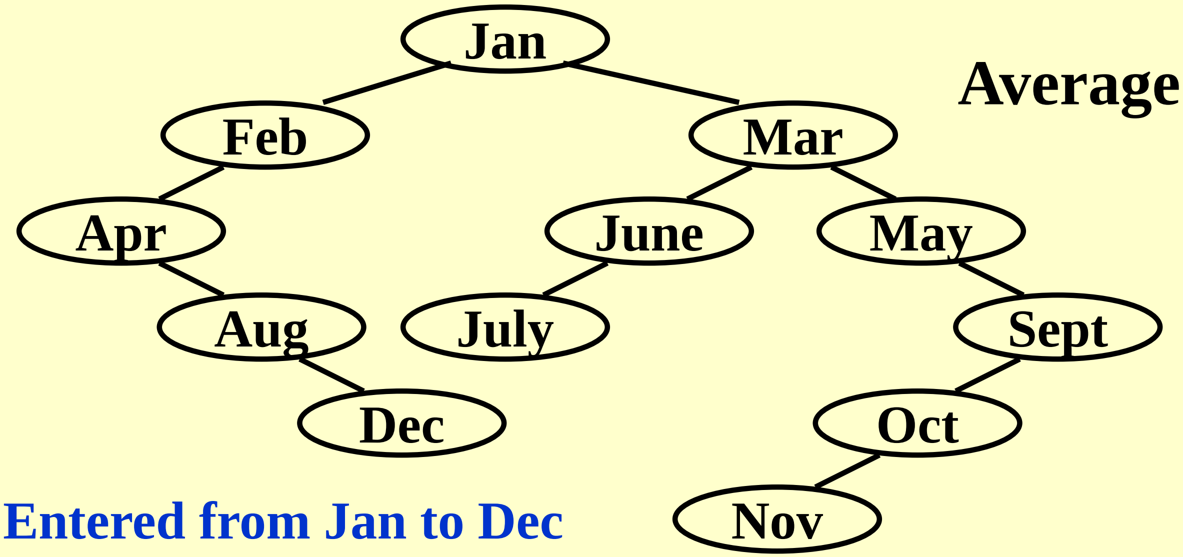
Tool : **Binary search trees**





Problem : **Although T_p = O(height), but the height can be as bad as O(N).**

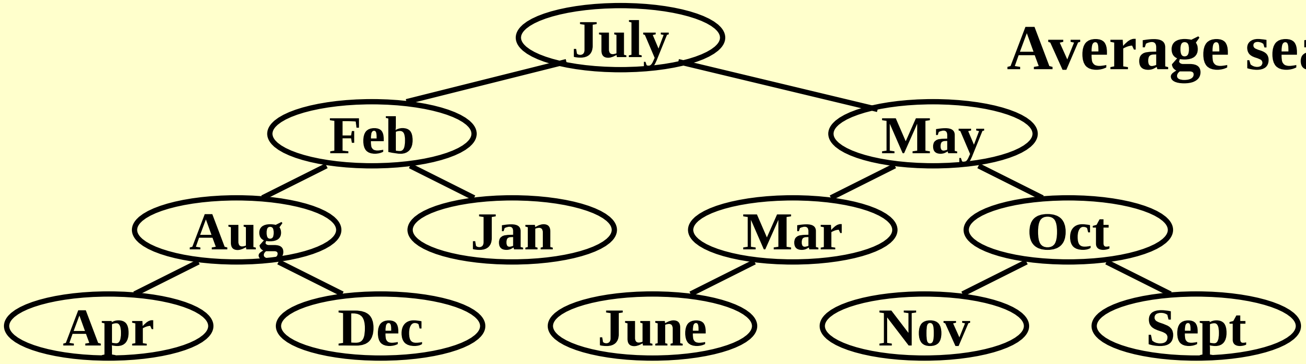
〖Example〗 2 binary search trees obtained for the months of the year



Average search time = 3.5

Average search time of the skew tree = 6.5

Entered from Jan to Dec



Average search time = 3.1

A balanced tree

Adelson-Velskii-Landis (AVL) Trees (1962)

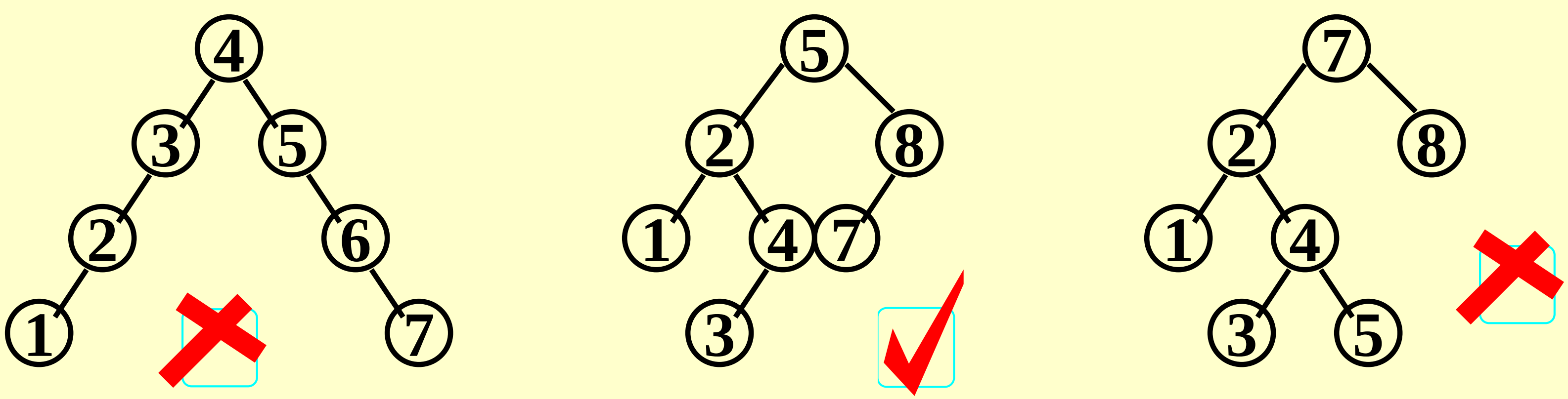
【**Definition**】 **An empty binary tree is height balanced. If T is a nonempty binary tree with T_L and T_R as its left and right subtrees, then T is height bala**

(1) T_L and T_R are height b

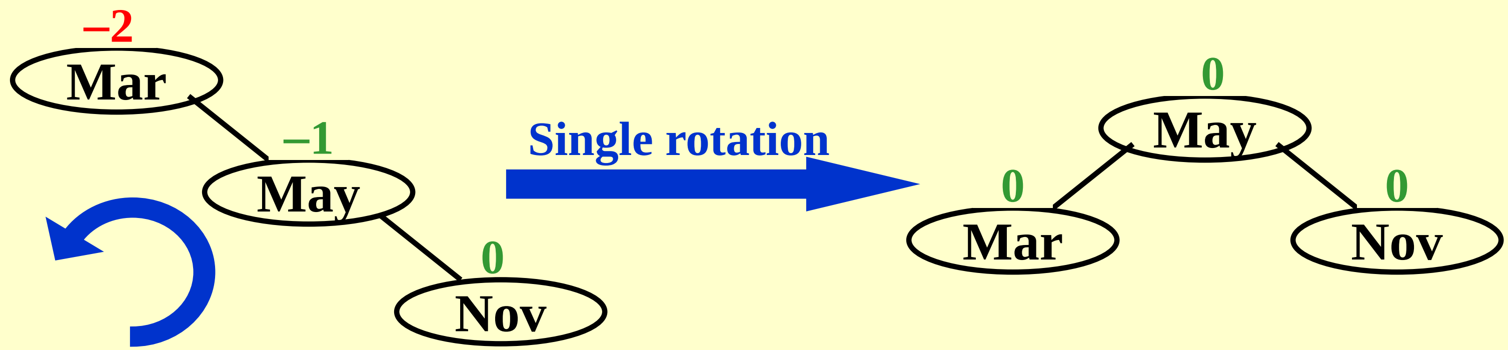
(2) $| h_L - h_R | \le 1$ where **and T_R, respectively.**

The height of an empty tree is defined to be –1.

【**Definition**】 **The balance factor BF(node) = $h_L - h_R$. In an AVL tree, BF(node) = –1, 0, or 1.**



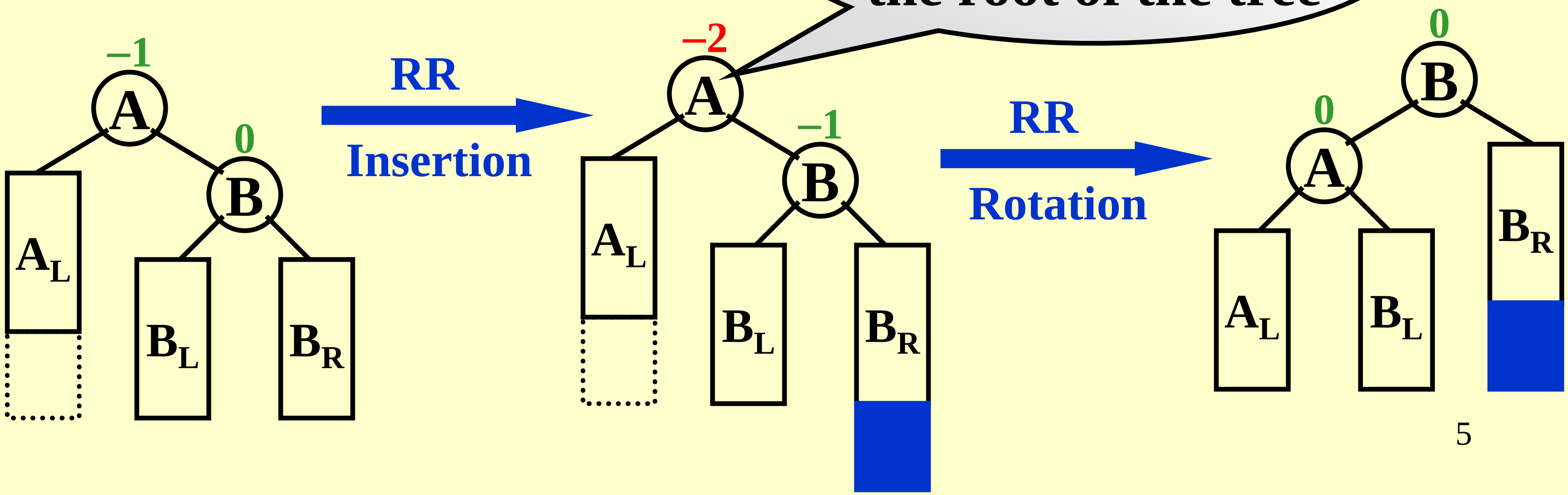
〖Example〗 Input the months Mar May Nov

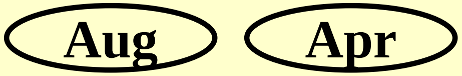


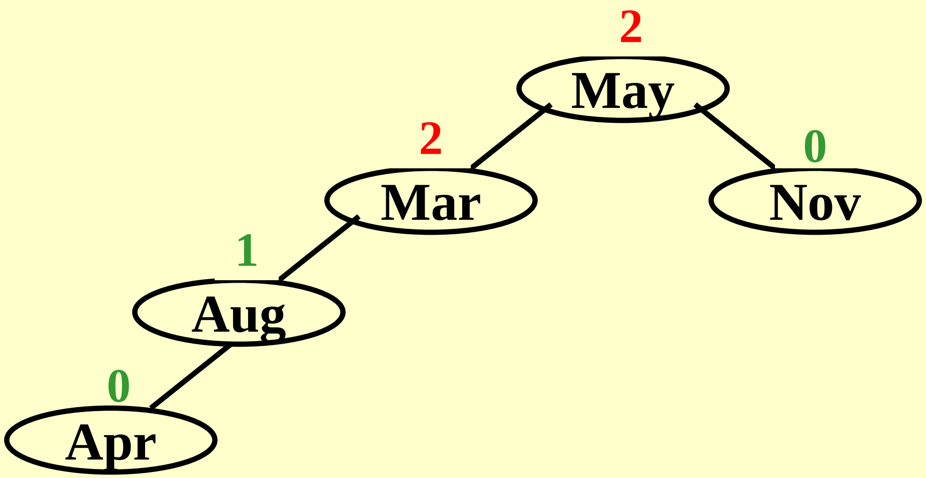
The trouble maker **Nov** is in the **r**ight subtree's **r**ight subtree of the trouble finder **Mar**. Hence it is called an **RR rotation**.

In general:

A is not necessarily the root of the tree

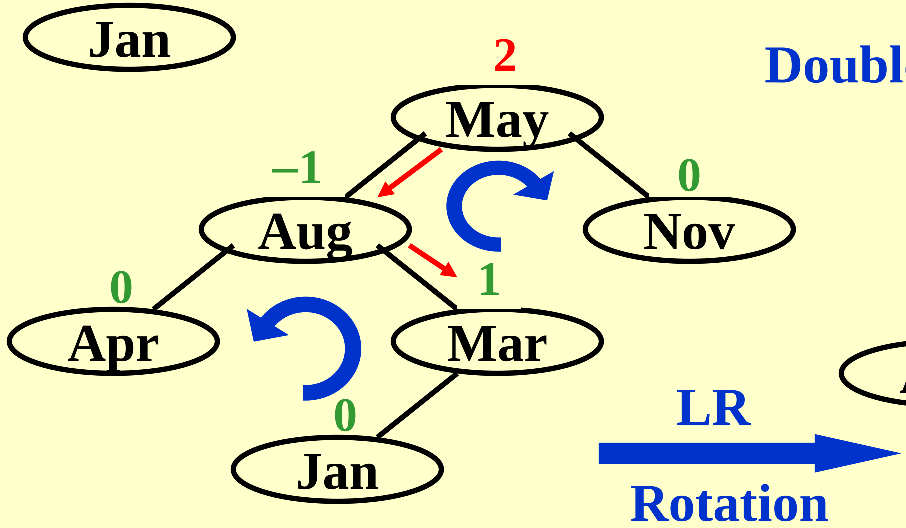


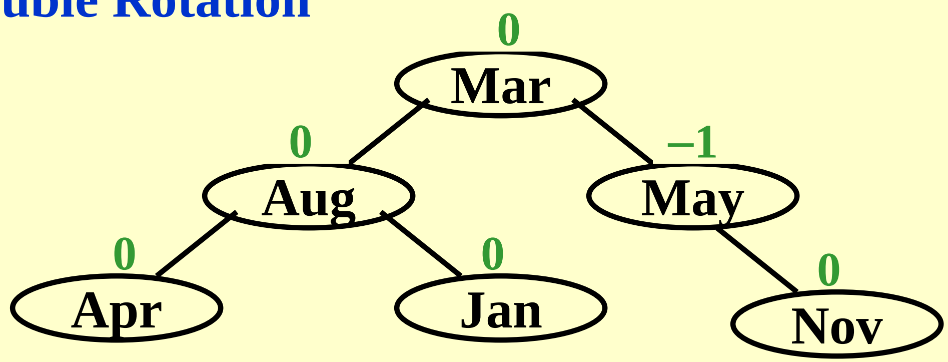




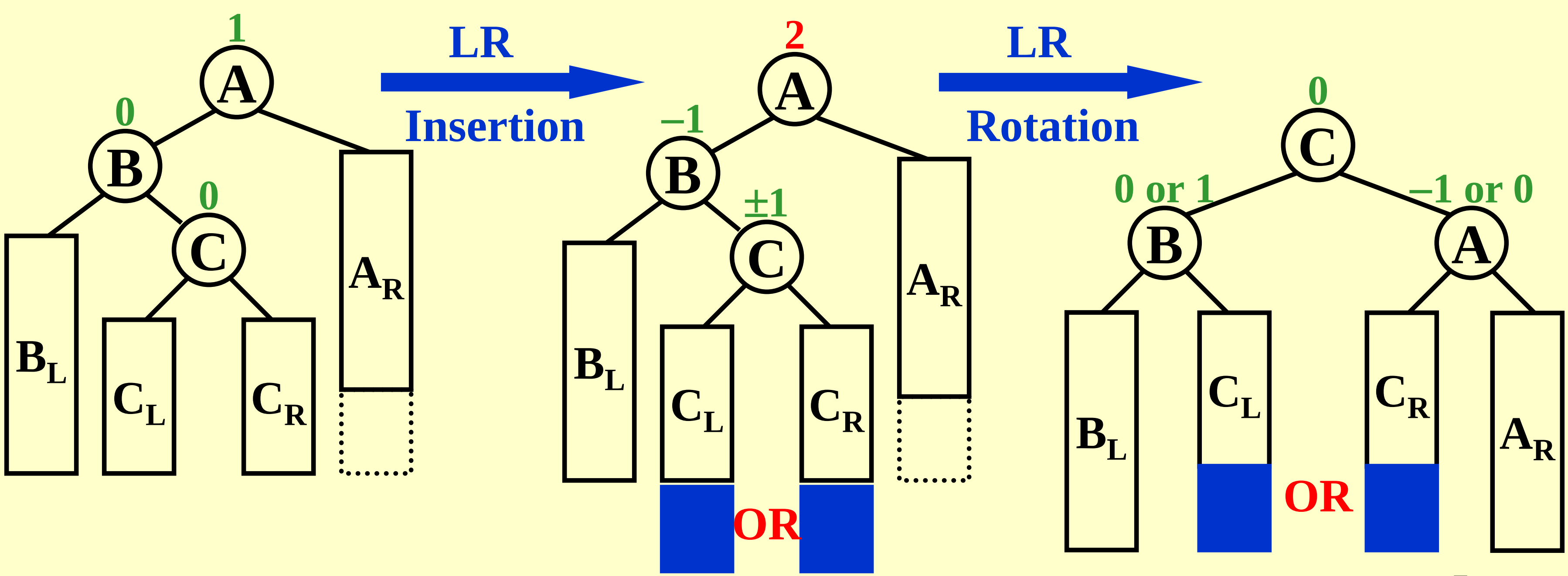
Discussion 1: What can we do now?

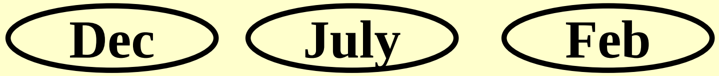
Double Rotation

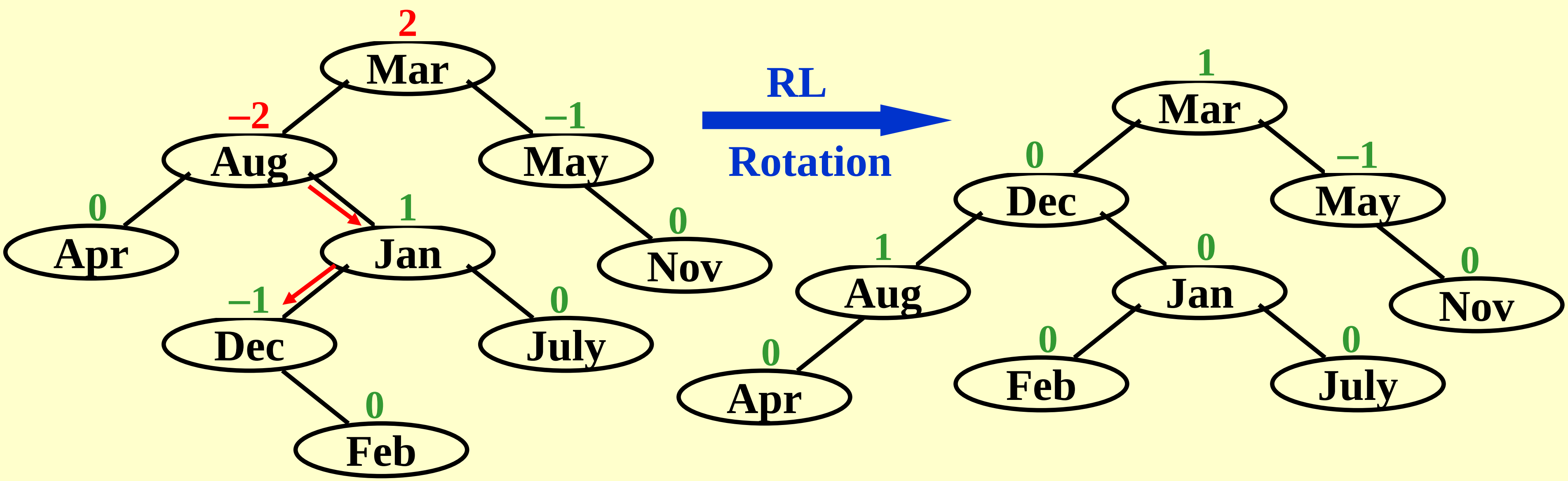




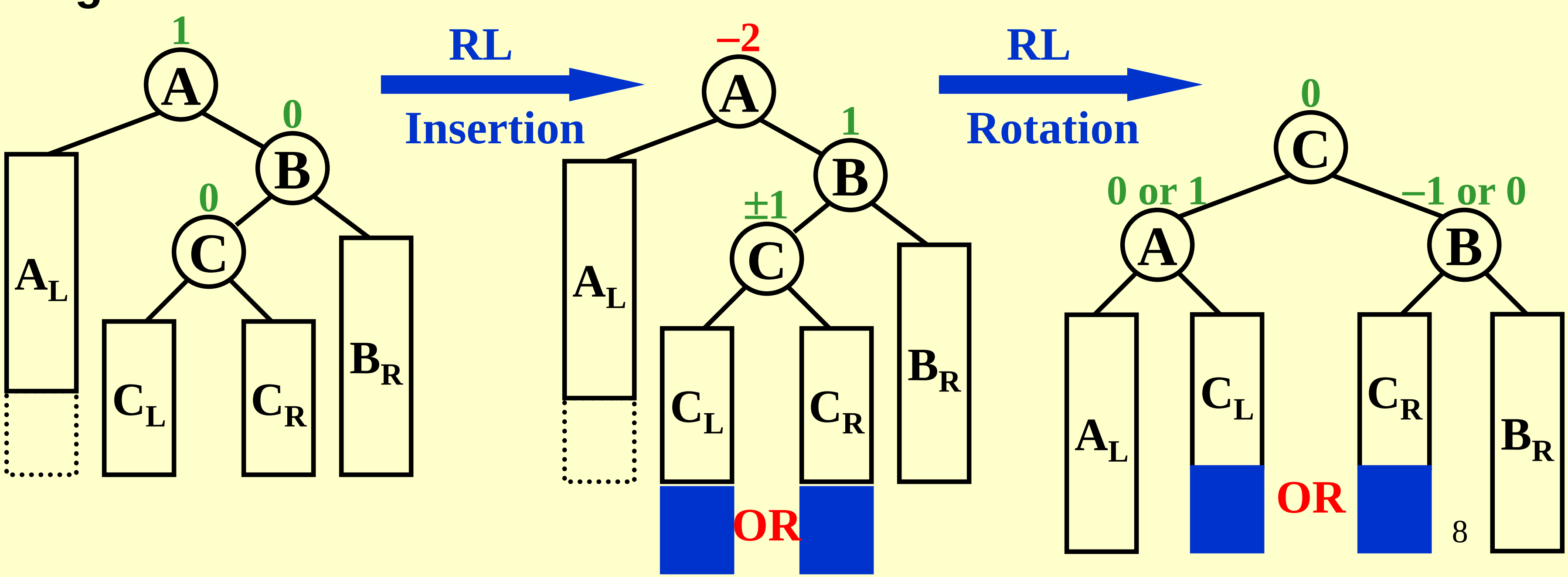
In general:

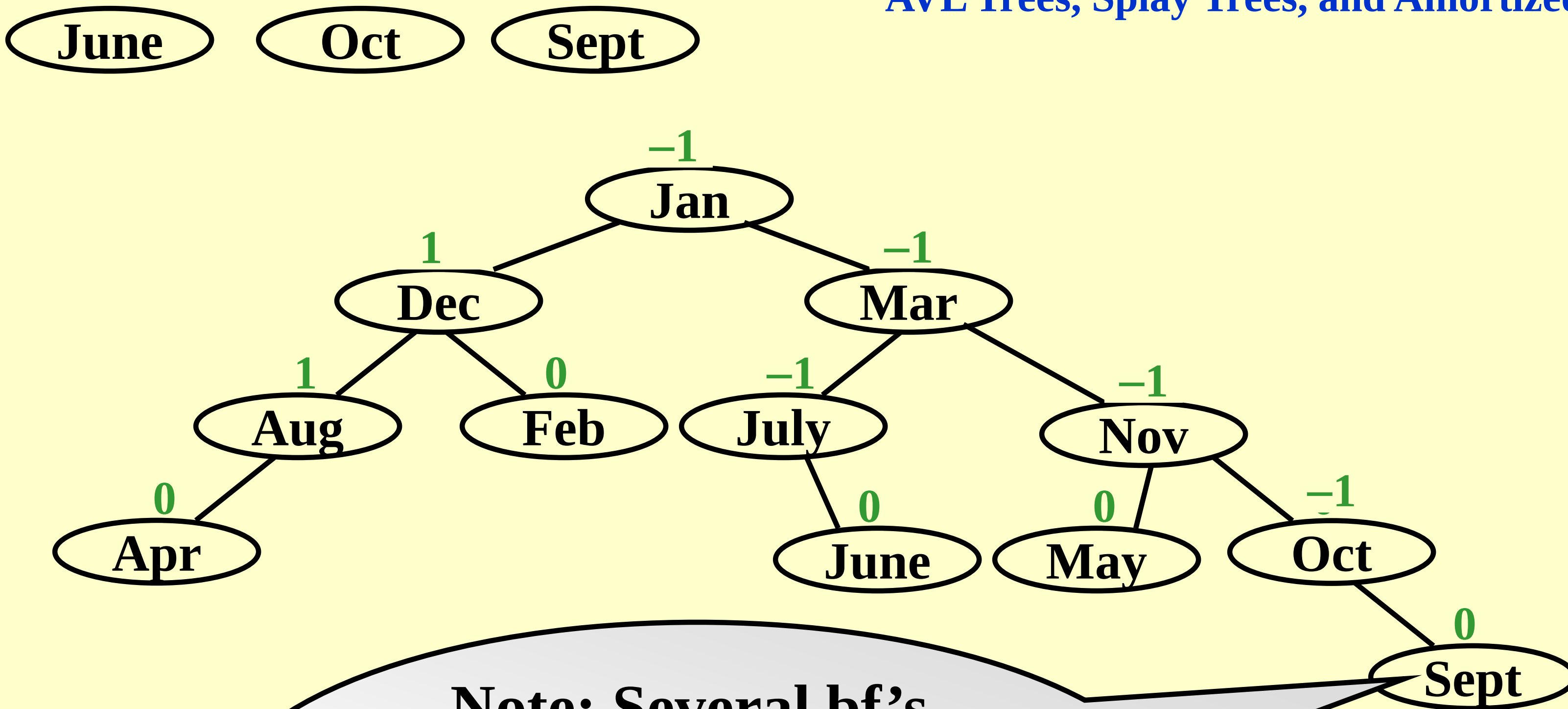



Dec
July
Feb


2
Mar
–2
Aug
–1
May
0
Apr
1
Jan
0
Nov
–1
Dec
0
July
0
Feb
RL
Rotation
1
Mar
0
Dec
–1
May
1
Aug
0
Jan
0
Nov
0
Apr
0
Feb
0
July

In general:


1
A
0
B
0
C
A_L
C_L
C_R
B_R
RL
Insertion
–2
A
1
B
±1
C
A_L
C_L
C_R
B_R
OR
RL
Rotation
0
C
0 or 1
A
–1 or 0
B
A_L
C_L
C_R
B_R
OR

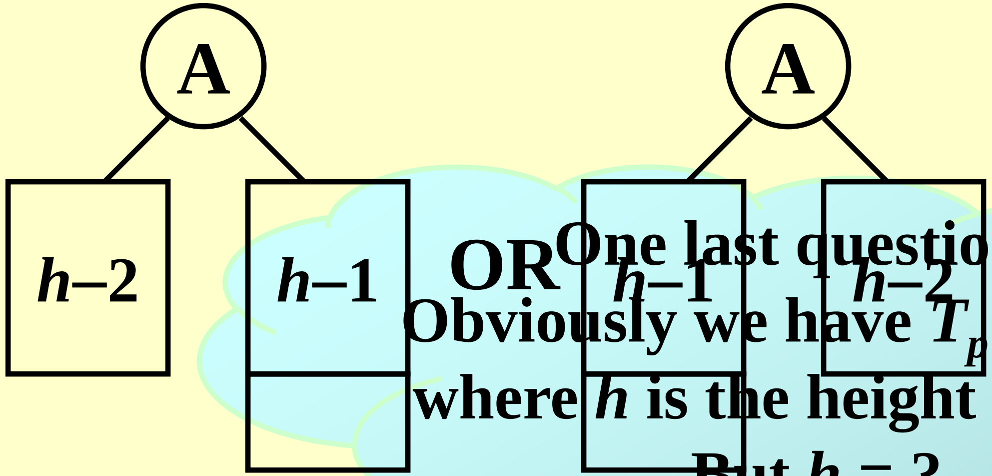


Note: Several bf's might be changed even if we don't need to reconstruct the tree.

Another option is to keep a *height* field for each node.

Read the declaration and functions in [1] Figures 4.42 – 4.48

Let n_h be the minimum number of nodes in a height balanced tree of height h. Then the tree must look like



$\Rightarrow$ $n_h = n_{h-1} + n_{h-2} + 1$

One last question:
Obviously we have $T_p = O(h)$ where h is the height of the tree.
But $h = ?$

Fibonacci numbers:

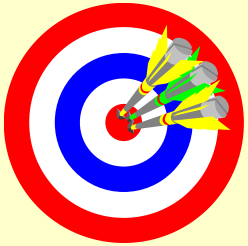
$F_0 = 0,\ F_1 = 1,\ F_i = F_{i-1} + F_{i-2}$ **for** $i > 1$

$\Rightarrow$ $n_h = F_{h+2} - 1$, **for** $h \geq 0$

Fibonacci number theory gives that $F_i \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^i$

$$\Rightarrow \quad n_h \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{h+2} - 1 \qquad \Rightarrow \quad h = O(\ln n)$$

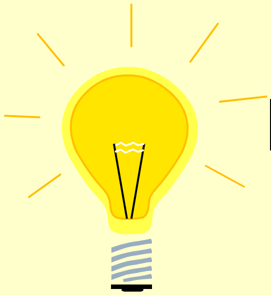
Splay Trees



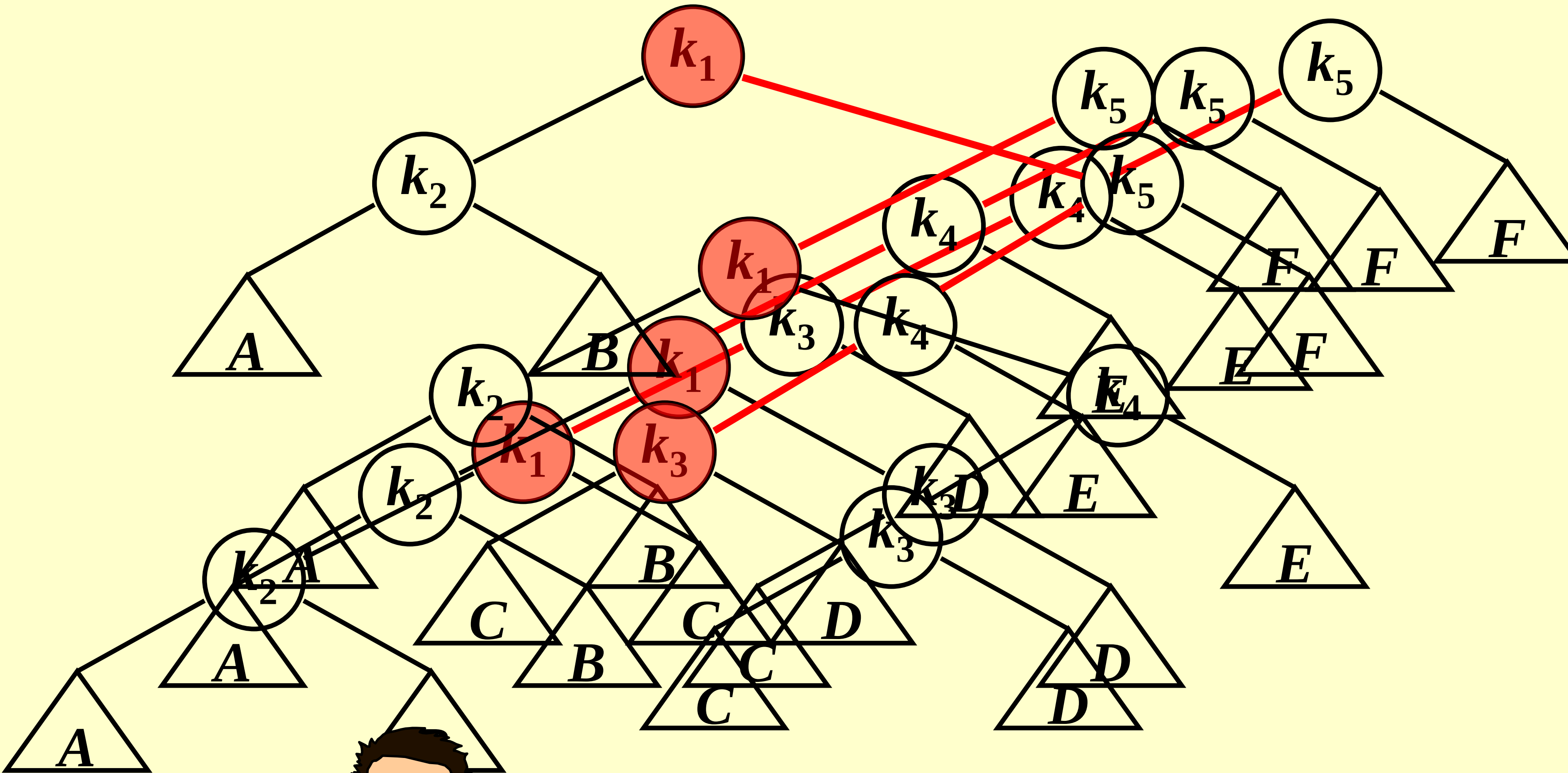
Target : **Any *M* consecutive tree operations starting from an empty tree take at most O(*M* log *N*) time.**

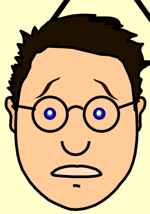
ne node takes O(*N*) time
we can keep accessing it
times, can't we?

Sure we can – that only means that whenever a node is accessed, it must be moved.



Idea : **After a node is accessed, it is pushed to the root by a series of AVL tree rotations.**

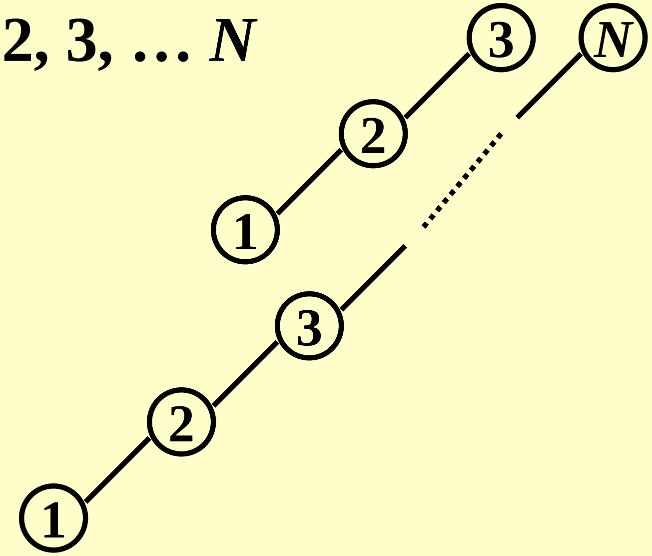




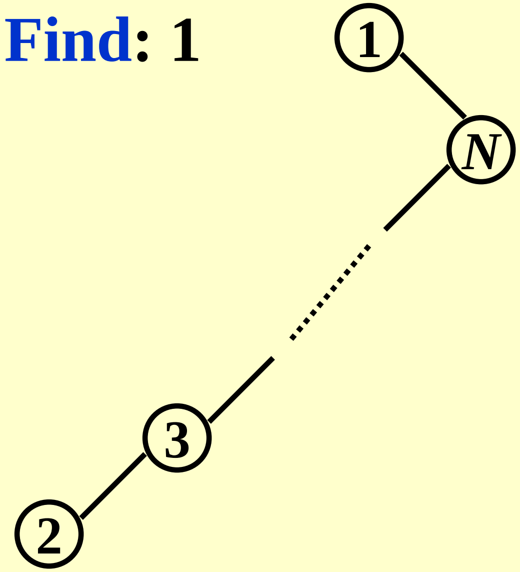
Does NOT work!

An even worse case:

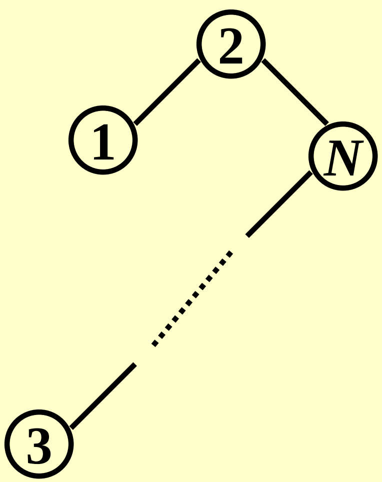
Insert: 1, 2, 3, … N



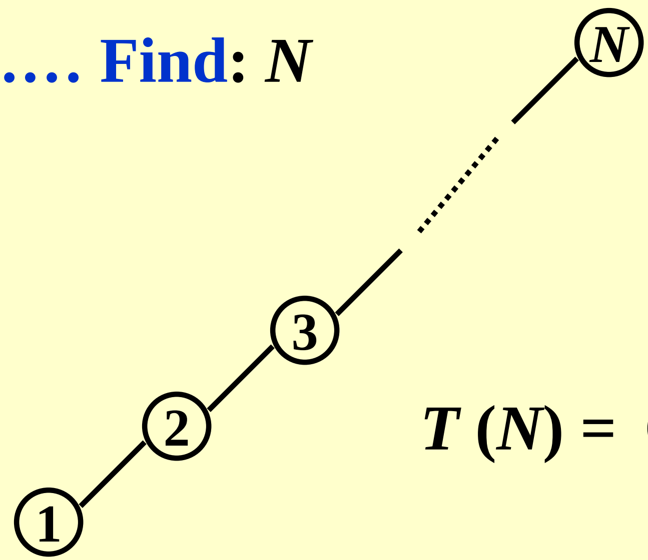
Find: 1



Find: 2



…… Find: N



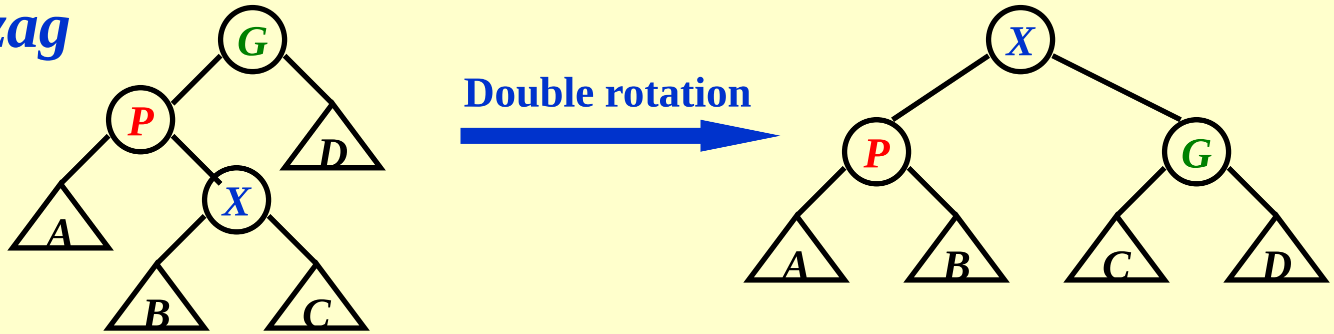
$$T(N) = O(N^2)$$

Try again -- For any nonroot node *X* , denote its parent by *P* and grandparent by *G* :

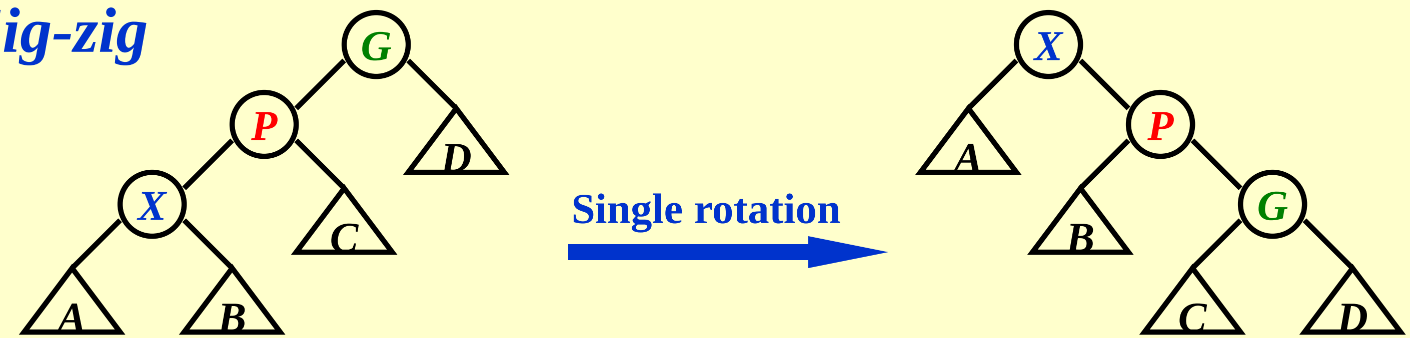
Case 1: *P* is the root → Rotate *X* and *P*

Case 2: *P* is not the root

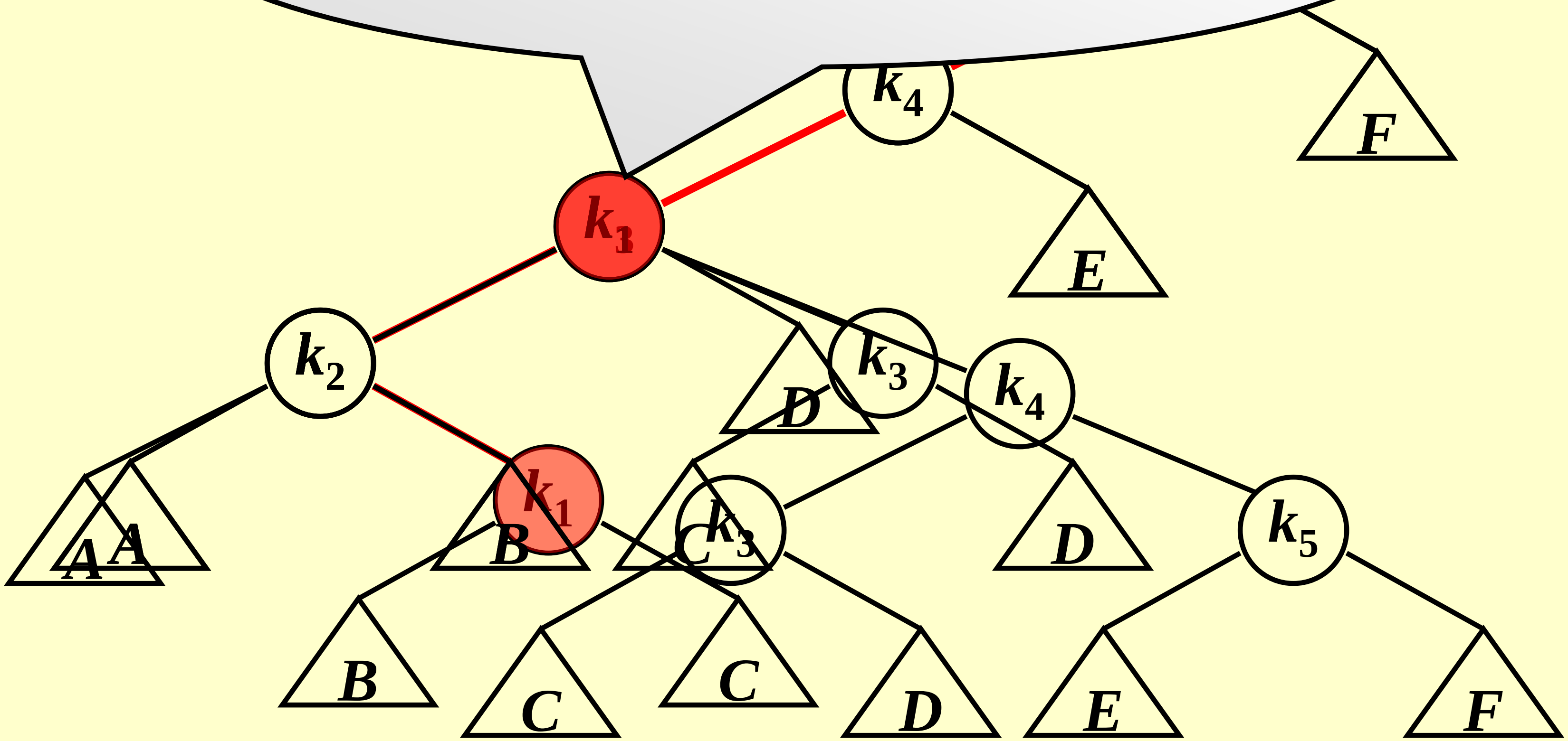
Zig-zag



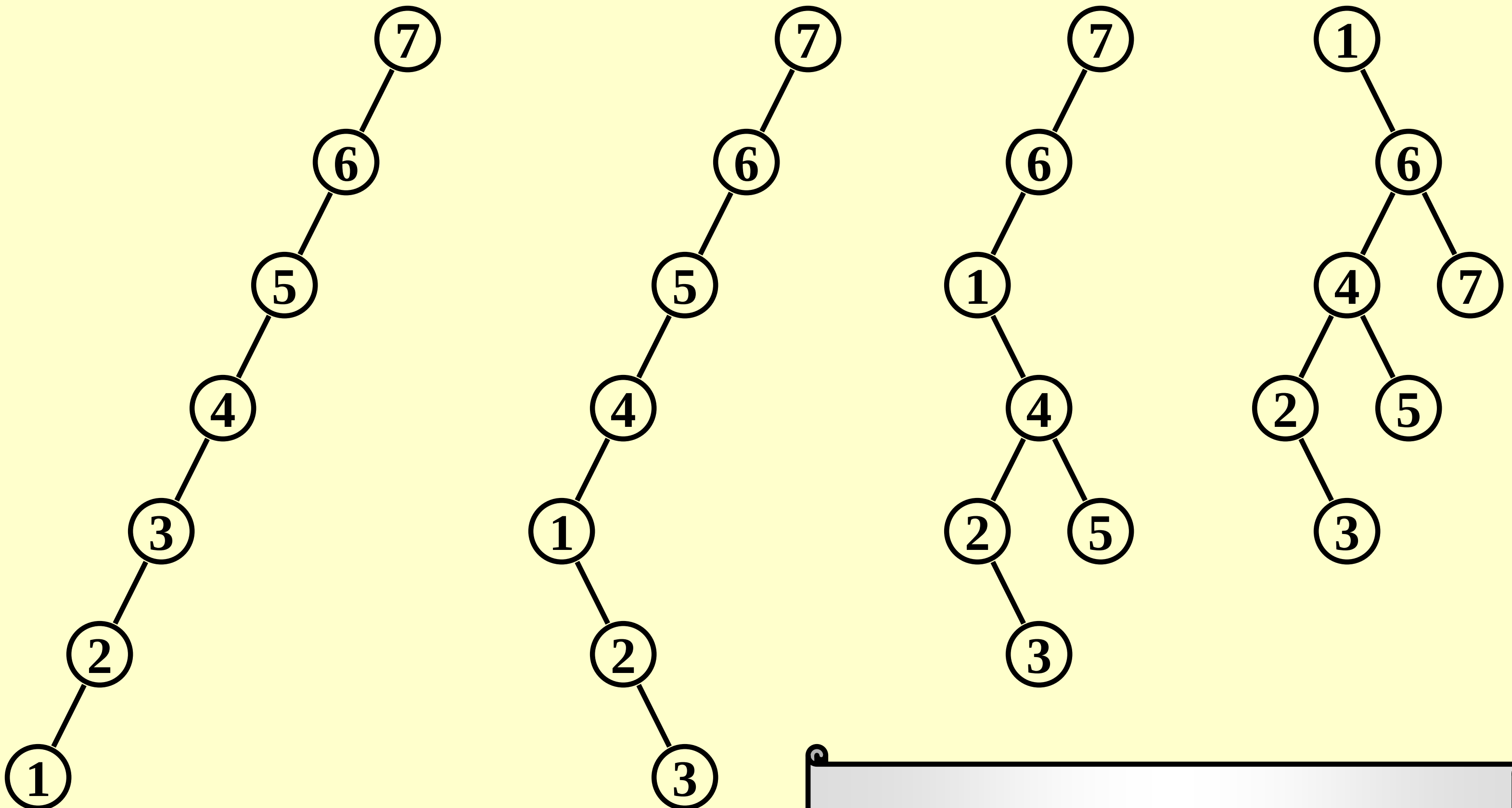
Zig-zig



Splaying not only moves the accessed node to the root, but also roughly halves the depth of most nodes on the path.



Insert: 1, 2, 3, 4, 5, 6, 7 **Find**: 1



Read the 32-node example given in Figures 4.52 – 4.60

Deletions:

- **Step 1:** **Find X ;**

  X **will be at the root.**

- **Step 2:** **Remove X ;**

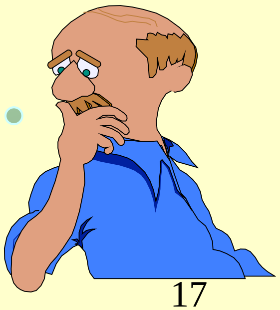
  There will be two subtrees T_L and T_R .

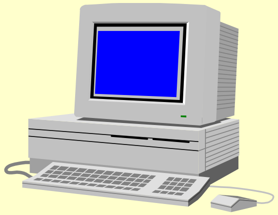
- **Step 3:** **FindMax (T_L) ;**

  The largest element will be the root of T_L , and *has no right child.*

- **Step 4:** **Make T_R the right child of the root of T_L .**

Are splay trees really better than AVL trees?





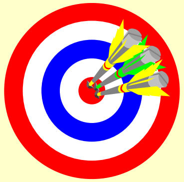
Research Project 1

Binary Search Trees (26)

This project requires you to implement operations on unbalanced binary search trees, AVL trees, and splay trees. You are to analyze and compare the performances of a sequence of insertions and deletions on these search tree structures.

Detailed requirements can be found at https://pintia.cn/ .

Amortized Analysis



Target : **Any *M* consecutive operations take at most O(*M* log *N*) time.**

-- *Amortized* time bound

worst-case bound $\geq$ amortized bound $\geq$ average-case bound

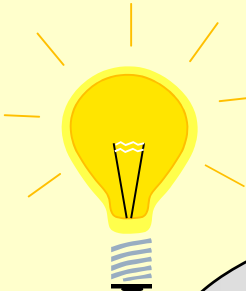
Probability is *not* involved

☞ **Aggregate analysis**

☞ **Accounting method**

☞ **Potential method**

☞ Aggregate analysis



Idea: **Show that for all n, a sequence of n operations takes *worst-case* time $T(n)$ in … ase, the average cost, … ration is**

We can **pop** each object from the stack ***at most once*** for each time we have **pushed** it onto the stack

Total $= O(n^2)$?

〖Example〗 Stack with … op(int k, Stack S)

```
Algorithm  {
   while ( !IsEmpty(S) && k>0 ) {
      Pop(S);
      k - -;
   } /* end while-loop */
}
```

$T = \min(\text{sizeof}(S), k)$

Consider a sequence of n Push, Pop, and MultiPop operations on an initially empty stack.

$\text{sizeof}(S) \le n$

$T_{amortized} = O(n)/n = O(1)$

☞ Accounting method

When an operation's *amortized cost* $\hat{c}_i$ exceeds its *actual cost* c_i , we assign the difference to specific objects in the data structure as *credit*. Credit can help *pay* for later operations whose amortized cost is less than their actual cost.

Note: For all sequences of *n* operations, we must have

$$\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i$$

$$T_{amortized} = \frac{\sum_{i=1}^{n} \hat{c}_i}{n}$$

〖Example〗 Stack with MultiPop(int k, Stack S)

c_i **for Push: 1 ; Pop: 1 ; and MultiPop: min (sizeof(*S*), *k*)**

$\hat{c}_i$ **for Push: 2 ; Pop: 0 ; and MultiPop: 0**

Starting f… an empt… *Credits* for

Push: +1 ; Po… …r each +1

sizeof(*S*) ≥ 0

The amortized costs of the operations may *differ* from each other

$$\Rightarrow O(n) = \sum_{i=1} c_i \geq \sum_{i=1} c_i$$

$$\Rightarrow T_{amortized} = O(n)/n = O(1)$$

☞ Potential method

Idea: **Take a closer look at the *credit* --**

$$\hat{c}_i - c_i = Credit_i = \Phi(D_i) - \Phi(D_{i-1})$$

Potential function

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} \left(c_i + \Phi(D_i) - \Phi(D_{i-1}) \right)$$

$$= \left(\sum_{i=1}^{n} c_i \right) + \underbrace{\Phi(D_n) - \Phi(D_0)}_{\geq 0}$$

In general, a good potential function should always assume its minimum at the start of the sequence.

〖Example〗 Stack with MultiPop(int k, Stack S)

D_i = the stack that results after the *i*-th operation

$\Phi(D_i)$ = the number of objects in the stack D_i

$$\Phi(D_i) \geq 0 = \Phi(D_0)$$

Push: $\Phi(D_i) - \Phi(D_{i-1}) = (sizeof(S)+1) - sizeof(S) = 1$

$\Rightarrow \hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 1 = 2$

Pop: $\Phi(D_i) - \Phi(D_{i-1}) = (sizeof(S)-1) - sizeof(S) = -1$

$\Rightarrow \hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 - 1 = 0$

MultiPop: $\Phi(D_i) - \Phi(D_{i-1}) = (sizeof(S)-k') - sizeof(S) = -k'$

$\Rightarrow \hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = k' - k' = 0$

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} O(1) = O(n) \geq \sum_{i=1}^{n} c_i \Rightarrow T_{amortized} = O(n)/n = O(1)$$

〖Example〗 Splay Trees: $T_{amortized} = O(\log N)$

D_i = the root of the resulting tree

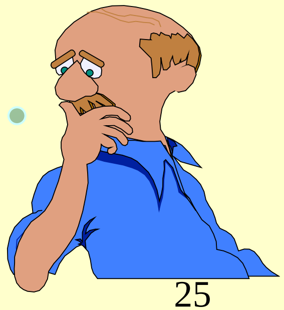
$\Phi(D_i)$ = must increase by at most $O(\log N)$ over n steps, AND will also cancel out the number of rotations (*zig*:1; *zig-zag*:2; *zig-zig*:2).

$$\Phi(T) = \sum_{i \in T} \log S(i)$$

where $S(i)$ is the number of descendants of i (i included).

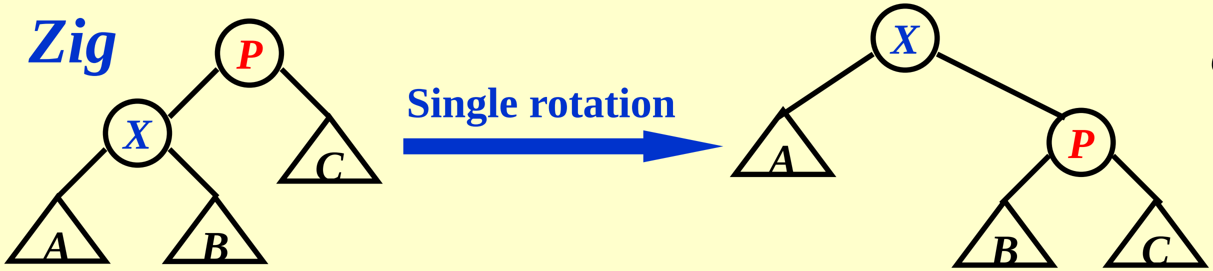
Rank of the subtree
≈ Height of the tree

Why not simply use the heights of the trees?



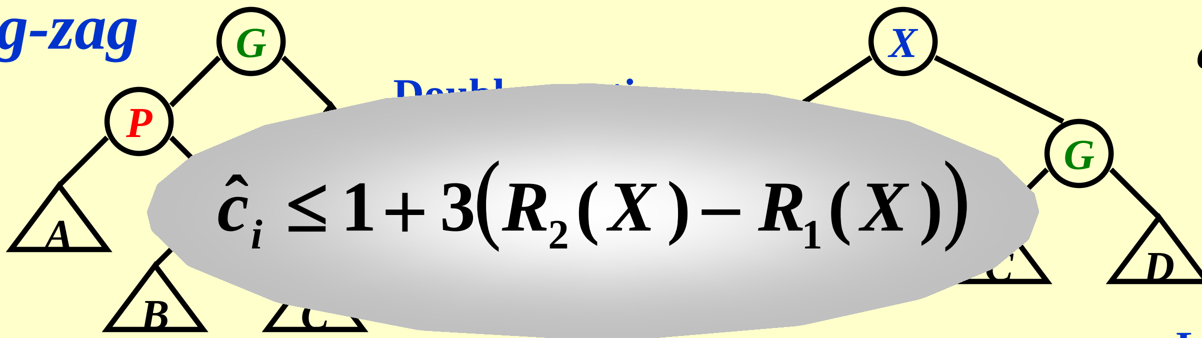
$$\Phi(T)=\sum_{i\in T} Rank(i)$$

Zig



$$\begin{aligned}\hat{c}_i &= 1+R_2(X)-R_1(X)\\ &\quad+\underline{R_2(P)-R_1(P)}\\ &\le 1+R_2(X)-R_1(X)\end{aligned}$$

Zig-zag

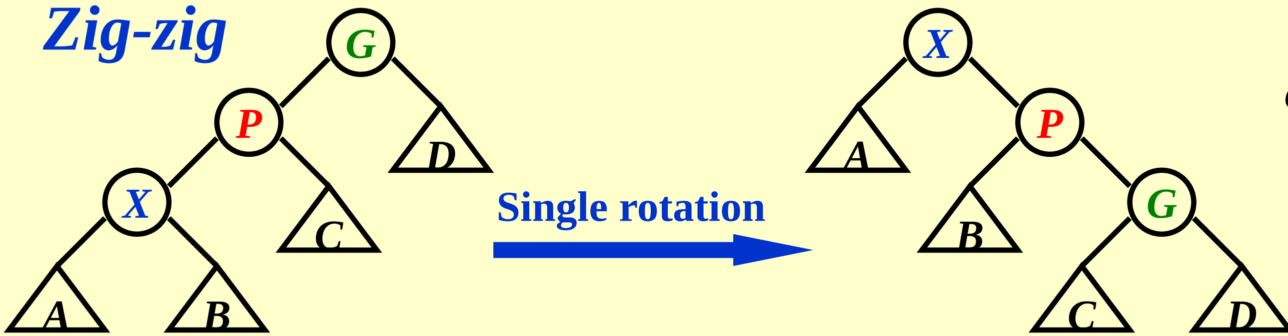


$$\hat{c}_i \le 1+3\big(R_2(X)-R_1(X)\big)$$

$$\begin{aligned}\hat{c}_i &= 2+\cancel{R_2(X)}-\underline{R_1(X)}\\ &\quad+R_2(P)-\underline{R_1(P)}\\ &\quad+R_2(G)-\cancel{R_1(G)}\\ &\le 2\big(R_2(X)-\underline{R_1(X)}\big)\end{aligned}$$

Lemma 11.4 on [1] p.448

Zig-zig



$$\begin{aligned}\hat{c}_i &= 2+R_2(X)-R_1(X)\\ &\quad+R_2(P)-R_1(P)\\ &\quad+R_2(G)-R_1(G)\\ &\le 3\big(R_2(X)-R_1(X)\big)\end{aligned}$$

【Theorem】 The amortized time to splay a tree with root T at node X is at most $3(R(T)-R(X))+1=O(\log N)$.

Reference:

Data Structure and Algorithm Analysis in C (2nd Edition)：Ch.4，p.106-128； Ch.11，p.447-451；*M.A.Weiss著、陈越改编，人民邮件出版社，2005*

Introduction to Algorithms, 3rd Edition: Ch.17，p. 451-478； *Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein. The MIT Press. 2009*