Randomized Algorithms

What to Randomize?

The world behaves randomly – randomly generated input solved by traditional algorithm

Average-case Analysis

The algorithm behaves randomly – make random decisions as the algorithm processes the worst-case input

Randomized Algorithms

Why Randomize?

Efficient deterministic algorithms that always yield the correct answer are a special case of –

efficient randomized algorithms that only need to yield the correct answer with *high probability*

randomized algorithms that are always correct, and run efficiently *in expectation*

Symmetry-breaking* among processes in a *distributed system

Simpler

A Quick Review

Pr[*A*] := the *probability* of the even *A*

$\overline{A}$:= the *complementary* of the even *A* (*A* did not occur)

$$\Pr[A] + \Pr[\overline{A}] = 1$$

E[*X*] := the *expectation* (the "average value") of the random variable *X*

$$E[X] = \sum_{j=0}^{\infty} j \cdot \Pr[X = j]$$

〖Example〗 The Hiring Problem

- ☞ Hire an office assistant from headhunter
- ☞ Interview a different applicant per day for N days
- ☞ Interviewing Cost = C_i << Hiring Cost = C_h
- ☞ Analyze interview & hiring cost instead of running time

Assume M people are hired.

Total Cost: $O(NC_i + MC_h)$

Naïve Solution

```
int Hiring ( EventType C[ ], int N )
{   /* candidate 0 is a least-qualified dummy candidate */
    int Best = 0;
    int BestQ = the quality of candidate 0;
    for ( i=1; i<=N; i++ ) {
        Qi = interview( i ); /* $C_i$ */
        if ( Qi > BestQ ) {
            BestQ = Qi;
            Best = i;
            hire( i );  /* $C_h$ */
        }
    }
    return Best;
}
```

Worst case: The candidates come in increasing quality order

$$O(NC_h)$$

☞ **Assume candidates arrive in *random* order**

X = number of hires

$$E[X] = \sum_{j=1}^{N} j \cdot \Pr[X = j]$$?

Randomness assumption: any of first i candidates is equally likely to be best-qualified so far

$$X_i = \begin{cases} 1 & \text{if candidate } i \text{ is hired} \\ 0 & \text{if candidate } i \text{ is NOT hired} \end{cases}$$

➡ $X = \sum_{i=1}^{N} X_i$ $\qquad E[X_i] = \Pr[\textit{candidate i is hired}]$? $= 1/i$

➡ $E[X] = E[\sum_{i=1}^{N} X_i] = \sum_{i=1}^{N} E[X_i] = \sum_{i=1}^{N} 1/i = \ln N + O(1)$

➡ $O(C_h \ln N + N C_i)$

Radomized Algorithm

```
int RandomizedHiring ( EventType C[ ], int N )
{   /* candidate 0 is a least-qualified dummy candidate */
    int Best = 0;
    int BestQ = the quality of candidate 0;

    randomly permute the list of candidates;

    for ( i=1; i<=N; i++ ) {
        Qi = interview( i ); /* $C_i$ */
        if ( Qi > BestQ ) {
            BestQ = Qi;
            Best = i;
            hire( i );  /* $C_h$ */
        }
    }
```

👎 takes time

👍 no longer need to assume that candidates *are presented* in random order

Radomized Permutation Algorithm

Target : **Permute array A[ ]**

Assign each element A[i] a *random priority* P[i], and sort

```
void PermuteBySorting ( ElemType A[ ], int N )
{
   for ( i=1; i<=N; i++ )
      A[i].P = 1 + rand()%(N³);
      /* makes it more likely that all priorities are unique */
   Sort A, using P as the sort keys;
}
```

Claim: PermuteBySorting produces a *uniform random permutation* of the input, assuming all priorities are distinct.

Online Hiring Algorithm – hire only once

```
int OnlineHiring ( EventType C[ ], int N, int k )
{
    int Best = N;
    int BestQ = - ∞ ;
    for ( i=1; i<=k; i++ ) {
        Qi = interview( i );
        if ( Qi > BestQ )   BestQ = Qi;
    }
    for ( i=k+1; i<=N; i++ ) {
        Qi = interview( i );
        if ( Qi > BestQ ) {
            Best = i;
            break;
        }
    }
    return Best;
}
```

☞ What is the probability we hire the best qualified candidate for a given *k*?

☞ What is the best value of *k* to maximize above probability?

Online Hiring Algorithm

S_i := the *i*th applicant is the best

What needs to happen for S_i to be TRUE?

independent

{ A:= the best one is at position *i* }

$\cap$ { B:= no one at positions *k*+1 ~ *i*–1 are hired }

$$\Pr[S_i] = \Pr[A \cap B] = \underbrace{\Pr[A]}_{=1/N} \cdot \underbrace{\Pr[B]}_{=k/(i-1)} = \frac{k}{N(i-1)}$$

$$\Pr[S] = \sum_{i=k+1}^{N} \Pr[S_i] = \sum_{i=k+1}^{N} \frac{k}{N(i-1)} = \frac{k}{N} \sum_{i=k}^{N-1} \frac{1}{i}$$

Discussion 18: Prove $\int_k^N \frac{1}{x}dx \le \sum_{i=k}^{N-1} \frac{1}{i} \le \int_{k-1}^{N-1} \frac{1}{x}dx$

$$\Pr[S] = \sum_{i=k+1}^{N} \Pr[S_i] = \sum_{i=k+1}^{N} \frac{k}{N(i-1)} = \frac{k}{N}\sum_{i=k}^{N-1}\frac{1}{i}$$

☞ **What is the probability we hire the best qualified candidate for a given *k*?**

$$\frac{k}{N}\ln\left(\frac{N}{k}\right) \le \Pr[S] \le \frac{k}{N}\ln\left(\frac{N-1}{k-1}\right)$$

☞ **What is the best value of *k* to maximize the above probability?**

Discussion 19: **What is the maximum value of $f(k) = \frac{k}{N}\ln\left(\frac{N}{k}\right)$? And what is the best *k*?**

〖Example〗 Quicksort

📖 **Deterministic Quicksort**

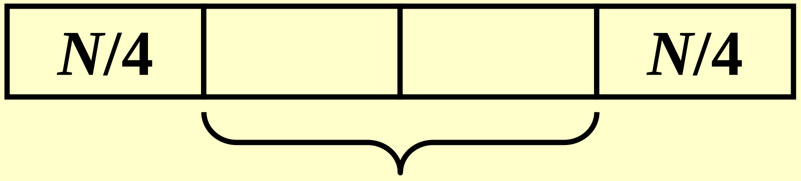
- ☞ **$\Theta(N^2)$ worst-case running time**
- ☞ **$\Theta(N \log N)$ average case running time, *assuming every input permutation is equally likely***

☝ **How about choosing the pivot *uniformly at random*?**

Central splitter := the pivot that divides the set so that each side contains at least $n/4$

Modified Quicksort := always select a central splitter before recursions

Claim: The expected number of iterations needed until we find a central splitter is at most 2.



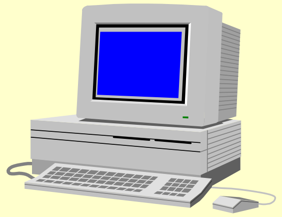
$N/2$ central splitters

Pr[*find a central splitter*] = 1/2 ✓

Type *j* : the subproblem S is of *type j* if $N\left(\frac{3}{4}\right)^{j+1} \le |S| \le N\left(\frac{3}{4}\right)^{j}$

Claim: There are at most $\left(\frac{4}{3}\right)^{j+1}$ subproblems of *type j*.

$$\left.\begin{array}{l} E[T_{type\ j}] = O(N\left(\frac{3}{4}\right)^{j}) \times \left(\frac{4}{3}\right)^{j+1} = O(N) \\ \text{Number of different types} = \log_{4/3} N = O(\log N) \end{array}\right\} O(N \log N)$$



Research Project 7

Skip Lists (26)

Skip list is a data structure that supports both searching and insertion in O(log*N*) expected time.

This project requires you to introduce the skip lists, and to implement insertion, deletion, and searching in skip lists.

Detailed requirements can be downloaded from https://pintia.cn/

Reference:

Introduction to Algorithms, 3rd Edition: Ch.5, p.114 - 145； ***Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein. The MIT Press. 2009***

Data Structure and Algorithm Analysis in C (2nd Edition)：Ch.10，p.399-401；*M.A.Weiss著、陈越改编，人民邮件出版社，2005*