NP-Completeness

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📖 **Recall:**

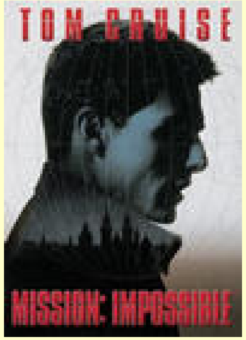
- ❖ **Euler circuit problem:** **Find a path that touches every edge exactly once.** ☺
- ❖ **Hamilton cycle problem:** **Find a single cycle that contains every vertex.** ☹
- ❖ **Single-source unweighted shortest-path problem** ☺
- ❖ **Single-source unweighted longest-path problem** ☹

☹ ***No*** **known algorithms are guaranteed to run in** ***polynomial*** **time.**

☞ Easy vs. Hard

The *easiest*: O(*N*) – since we have to read inputs at least once.

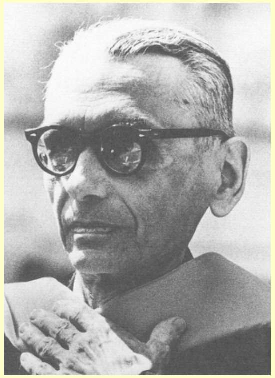
The *hardest*:



– undecidable problems.

The great mathematician David Hilbert at the 1900 International Congress of Mathematicians outlined 23 research problems to be investigated in the coming century. One of the problems is the question of Decidability — Could there exist, at least in principle, any definite method or process by which *all mathematical questions could be decided*?

Kurt Gödel proved in 1931 that not all true statements that evolve from an axiomatic system can be proven – *we can never know everything nor prove everything we discover*.



【Example】**Halting problem: Is it possible to have your C compiler detect all infinite loops?**

Answer: No.

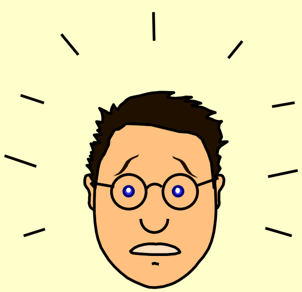
Proof: If there exists an infinite loop-checking program, then surely it could be used to **check itself**.

```
Loop( P )
{
/* 1 */ if ( P(P) loops )     print (YES);
/* 2 */ else infinite_loop();
}
```

Impossible to tell

What will happen to Loop(Loop) ?

- Terminates ➡ /* 2 */ is true ➡ Loops
- Loops ➡ /* 1 */ is true ➡ Terminates



☞ The Class NP

TURING MACHINE

Alan Mathison Turing (June 21, 1912 – June 7, 1954)

Founder of Computer Science.

More details can be found at http:\\www.turing.org.uk .

Task **To simulate any kind of computation which a mathematician can do by some arithmetical method (assuming that the mathematician has infinite time, energy, paper and pen, and is completely dedicated to the work).**

☞ The Class NP

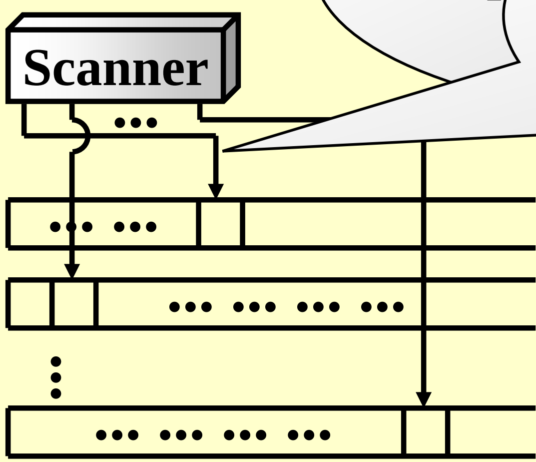
TURING MACHINE

Components — Infinite Memory and Scanner

Operation

Infini…
tap…

One head can point to
one and only one unit at each state,
and it may move at most one unit
to the left or right.



③ Head m… …storage …, or to the right (**R**), or stays at its current position (**S**).

A *Deterministic Turing Machine* executes one instruction at each point in time. Then depending on the instruction, it goes to the next *unique* instruction.

A *Nondeterministic Turing Machine* is *free to choose* its next step from a finite set. And if one of these steps leads to a solution, it will *always choose the correct one*.

NP: *Nondeterministic polynomial-time*

The problem is NP if we can prove any solution is true *in polynomial time.*

Undecidable problems are still undecidable.

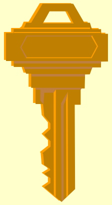
〖**Example**〗**Hamilton cycle problem: Find a single cycle that contains every vertex – *does this simple circuit include all the vertices?*   NP**

Note: **Not all decidable problems are in NP. For example, consider the problem of determining whether a graph does not have a Hamiltonian cycle.**

$P \subseteq NP$ ✔ $P \subset NP$?

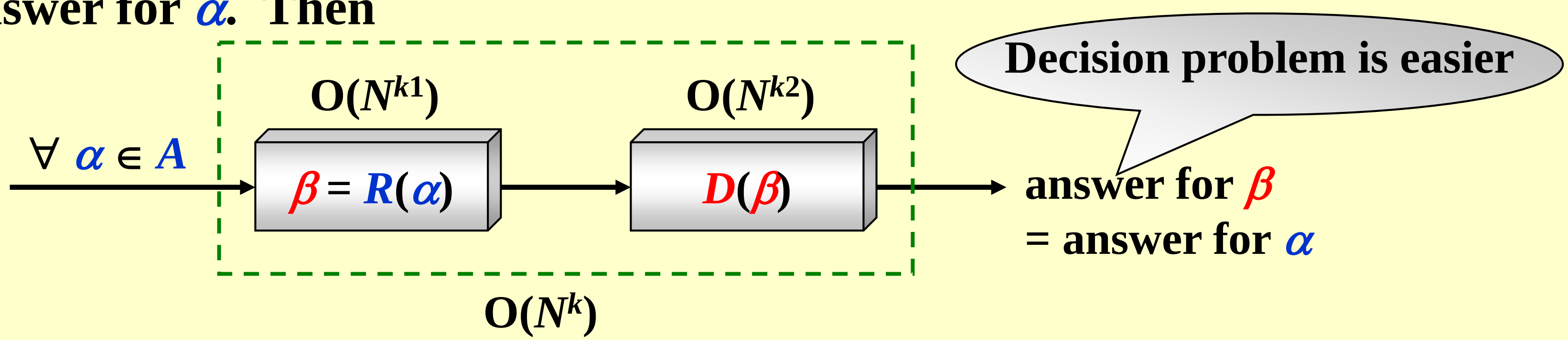
☞ **NP-Complete Problems -- the hardest**

An *NP-complete problem* has the property that any problem in NP can be *polynomially reduced* to it.



If we can solve any NP-complete problem in *polynomial* time, then we will be able to solve, in *polynomial* time, all the problems in NP!

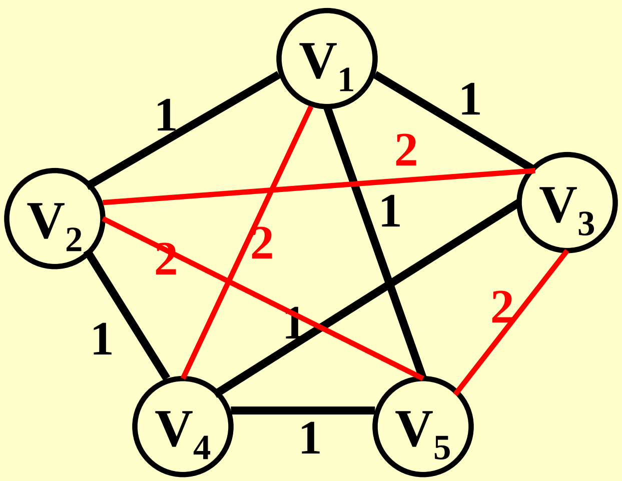
Given any instance $\alpha \in$ Problem A, if we can find a program $R(\alpha) \rightarrow \beta \in$ Problem B with $T_R(N) = O(N^{k1})$, and another program $D(\beta)$ to get an answer in time $O(N^{k2})$. And more, if the answer for β is the same as the answer for α. Then



〖Example〗 Suppose that we already know that the **Hamiltonian cycle problem** is NP-complete. Prove that the **traveling salesman problem** is NP-complete as well.

- ❖ **Hamiltonian cycle problem:** Given a graph G=(V, E), is there a simple cycle that visits all vertices?
- ❖ **Traveling salesman problem:** Given a complete graph G=(V, E), with edge costs, and an integer K, is there a simple cycle that visits all vertices and has total cost ≤ K?

Proof: TSP is obviously in NP, as its answer can be verified polynomially.



K = |V|

G has a Hamilton cycle iff G' has a traveling salesman tour of total weight |V|.

✔

The first problem that was proven to be NP-complete was the *Satisfiability* problem (Circuit-SAT): Input a boolean expression and ask if it has an assignment to the variables that gives the expression a value of 1.

Cook showed in 1971 that all the problems in NP could be polynomially transformed to Satisfiability. He proved it by solving this problem on a nondeterministic Turing machine in polynomial time.

A Formal-language Framework

☞ Abstract Problem

an *abstract problem Q* is a binary relation on a set I of problem *instances* and a set S of problem *solutions*.

〖Example〗 For SHORTEST-PATH problem

I = { <G, u, v>: G=(V, E) is an undirected graph; u, v $\in$ V };

S = { <u, w_1, w_2, …, w_k, v>: <u, w_1>, …, <w_k, v> $\in$ E }.

For every $i \in I$, SHORTEST-PATH(i) = $s \in S$.

For decision problem PATH:

I = { <G, u, v, k>: G=(V, E) is an undirected graph; u, v $\in$ V; $k \geq 0$ is an integer };

S = { 0, 1 }.

For every $i \in I$, PATH(i) = 1 or 0.

☞ Encodings

Map I into a binary string { 0, 1 }* $\rightarrow$ Q is a *concrete problem*.

☞ Formal-language Theory — *for decision problem*

- An *alphabet* Σ is a finite set of symbols { 0, 1 }
- A *language* L over Σ is any set of strings made up of symbols from Σ $L = \{ x \in \Sigma^*: Q(x) = 1 \}$
- Denote *empty string* by ε
- Denote *empty language* by $\emptyset$
- Language of all strings over Σ is denoted by Σ^*
- The *complement* of L is denoted by $\Sigma^*\text{-}L$
- The *concatenation* of two languages L_1 and L_2 is the language

  $L = \{ x_1x_2 : x_1 \in L_1 \text{ and } x_2 \in L_2 \}$.
- The *closure* or *Kleene star* of a language L is the language

  $L^* = \{\varepsilon\} \cup L \cup L^2 \cup L^3 \cup \cdots,$

  where L^k is the language obtained by concatenating L to itself k times

- **Algorithm A *accepts* a string $x \in \{0, 1\}^*$ if $A(x) = 1$**

- **Algorithm A *rejects* a string x if $A(x) = 0$**

- **A language L is *decided* by an algorithm A if every binary string *in L* is *accepted* by A and every binary string *not in L* is *rejected* by A**

- **To *accept* a language, an algorithm need only worry about strings in L, but to *decide* a language, it must correctly accept or reject every string in $\{0, 1\}^*$**

P = $\{ L \subseteq \{0, 1\}^*$: there exists an algorithm A that decides L in polynomial time $\}$

- A *verification algorithm* is a two-argument algorithm *A*, where one argument is an ordinary input string *x* and the other is a binary string *y* called a *certificate*.
- A two-argument algorithm *A* *verifies* an input string *x* if there exists a certificate *y* such that $A(x, y) = 1$.
- The *language* verified by a verification algorithm *A* is $L = \{ x \in \{0, 1\}^* : \text{there exists } y \in \{0, 1\}^* \text{ such that } A(x, y) = 1\}$.

〖Example〗 For SAT

$$x = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4)$$

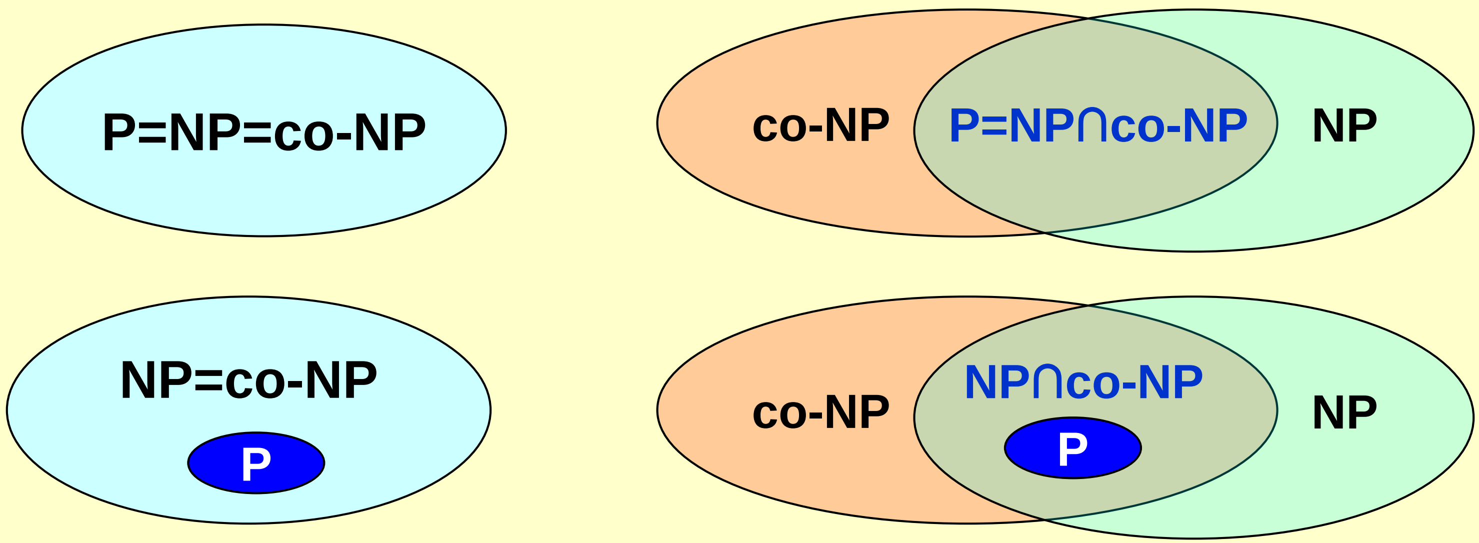
Certificate: $y = \{ x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 1 \}$

A language L belongs to NP iff there exist a two-input polynomial-time algorithm A and a constant c such that $L = \{ x \in \{0, 1\}^* : \text{there exists a certificate } y \text{ with } |y| = O(|x|^c) \text{ such that } A(x, y) = 1 \}$. We say that algorithm *A verifies* language L in *polynomial time*.

$$L \in NP \xrightarrow{?} \overline{L} \in NP$$

***complexity class* co-NP** = the set of languages L such that $\overline{L} \in NP$

Four possibilities:



no harder than

A language L_1 is *polynomial-time reducible* to a language L_2 ($L_1 \leq_P L_2$) if there exists a *polynomial-time computable* function $f : \{0, 1\}^* \rightarrow \{0,1\}^*$ such that for all x $\{0, 1\}^*$, $x \in L_1$ iff $f(x) \in L_2$.

We call the function f the *reduction function*, and a polynomial-time algorithm F that computes f is called a *reduction algorithm*.

A language $L \subseteq \{0, 1\}^*$ is *NP-complete* if

1. $L \in$ NP, and

2. $L' \leq_P L$ for every $L' \in$ NP.

〖Example〗 Suppose that we already know that the **clique problem** is NP-complete. Prove that the **vertex cover problem** is NP-complete as well.

- **Clique problem:** Given an undirected graph G = (V, E) and an integer K, does G contain a **complete subgraph** (***clique***) of (at least) K vertices?

  CLIQUE = { <G, K> : G is a graph with a clique of size K }.

- **Vertex cover problem:** Given an undirected graph G = (V, E) and an integer K, does G contain a subset V' $\subseteq$ V such that |V'| is (at most) K and every edge in G has a vertex in V' (***vertex cover***)?

  VERTEX-COVER = { <G, K> : G has a vertex cover of size K }.

Proof: ① VERTEX-COVER $\in$ NP

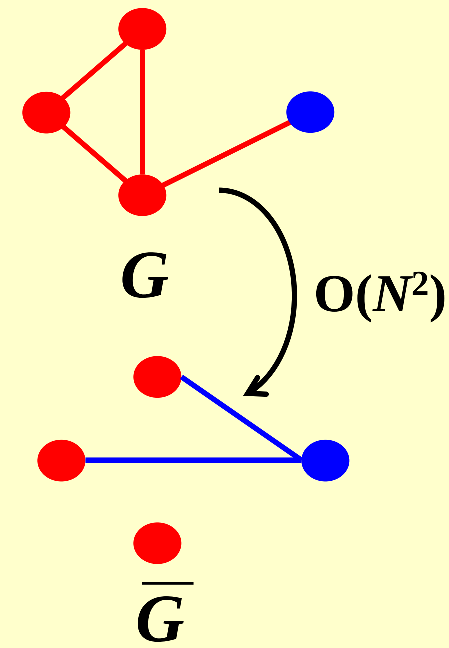
Given any x = <G, K>, take V' $\subseteq$ V as the certificate y.

Reduction algorithm: check if |V'| = K; check if for each edge $(u, v) \in$ E, that $u \in$ V' or $v \in$ V'.

$\boxed{O(N^3)}$

Proof (con.): ② CLIQUE $\leq_P$ VERTEX-COVER

G has a clique of size K iff $\overline{G}$ has a vertex cover of size |V| – K.



$\Rightarrow$ *G* has a clique V' $\subseteq$ V of size K

Let (u, v) be any edge in $\overline{E}$

At least one of u or v does not belong to V'

At least one of u or v does belong to V-V'

Every edge of $\overline{G}$ is covered by a vertex in V-V'

Hence, the set V-V', which has size |V| – K forms a vertex cover for $\overline{G}$

$\Leftarrow$ $\overline{G}$ has a vertex cover V' $\subseteq$ V of size |V| – K

For all $u, v \in$ V, if $(u, v) \notin$ E , then $u \in$ V' or $v \in$ V' or both.

For all $u, v \in$ V, if $u \notin$ V' AND $v \notin$ V', then $(u, v) \in$ E.

V-V' is a clique and it has size |V|–|V'| = K. ∎

Reference:

Introduction to Algorithms, 3rd Edition: Ch.34, p. 1048 - 1105; ***Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein. The MIT Press. 2009***