Chapter 3 Solution

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Problem 3.6

Take the model

$$
\begin{aligned}
C &= N_C \\
E &= \alpha C + N_E
\end{aligned}
$$

What is the distribution of $P(C|E=2)$

Solution

Using bayes rule we have that

$$
\begin{aligned}
P(C|E=2) &= \frac{P(E=2|C)P(C)}{P(E=2)} \\
&\propto P(E=2|C)P(C) \\
&\propto \exp\left(-\frac{1}{2}(2-\alpha C)^2\right)\exp\left(-\frac{1}{2}C^2\right) \\
&\propto \exp\left(-\frac{1}{2}(4-4\alpha C+(\alpha^2+1)C^2)\right) \\
&\propto \exp\left(-\frac{1}{2}(-4\alpha C+(\alpha^2+1)C^2)\right) \\
&\propto \exp\left(-\frac{\alpha^2+1}{2}(-4\frac{\alpha}{\alpha^2+1}C+C^2)\right) \\
&\propto \exp\left(-\frac{\alpha^2+1}{2}\left(C-\frac{2\alpha}{\alpha^2+1}\right)^2\right)
\end{aligned}
$$

Clearly, minus the normalizing constant, this is the pdf of a normal distribution with mean $\mu = \frac{2\alpha}{\alpha^2+1}$ and variance $\sigma^2 = \frac{1}{\alpha^2+1}$. Therefore, the distribution must be a normal with these parameters. And plugging back in $\alpha = 4$ we get the result in the book. Namely

$$P(C|E = 2) = \mathcal{N}(\mu = 8/17, \sigma^2 = 1/17)$$

Problem 3.7

See the book

Solution

We can intervene on X with the distribution $P(X) = \text{Bernoulli}(0.5)$ and then we can check the distribution of Y. If it is normal then we know that the model $Y \to X$ is correct. If it is not normal then we know that the model $X \to Y$ is correct.

Problem 3.8

See the book

Solution

a) We need to find $\alpha, \beta, \gamma, \delta$ such that for every n_x, n_y the equation holds. Plugging in $X = \alpha n_x + \beta n_y$ and $Y = \gamma n_x + \delta n_y$ we get that our solution must satisfy

$$\begin{aligned} \alpha n_x + \beta n_y &= 2\gamma n_x + 2\delta n_y + n_x \\ \gamma n_x + \delta n_y &= 2\alpha n_x + 2\beta n_y + n_y \end{aligned}$$

As this needs to hold for all n_x then we get the following system of equations

$$\begin{aligned} \alpha &= 2\gamma + 1 \\ \beta &= 2\delta \\ \gamma &= 2\alpha \\ \delta &= 2\beta + 1 \end{aligned}$$

Which we can solve as follows. Plugging in 3 into 1 we get

$$\alpha = 4\alpha + 1 \Rightarrow \alpha = -1/3$$

Plugging 2 in 4 we get

$$\delta = 4\delta + 1 \Rightarrow \delta = -1/3$$

and then using these two results in 2 and 3 we get

$$\begin{aligned} \beta &= 2\delta = -2/3 \\ \gamma &= 2\alpha = -2/3 \end{aligned}$$

So to summarize we have that

$$\alpha = -1/3 \tag{1}$$
$$\beta = -2/3 \tag{2}$$
$$\gamma = -2/3 \tag{3}$$
$$\delta = -1/3 \tag{4}$$