DIGITAL COMMUNICATIONS

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FWC22034 IITH Future Wireless Communication (FWC) ASSIGNMENT

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1 Sum of Independent Random Variables

1. The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 1. The theoretical pmf obtained in is plotted for comparison.

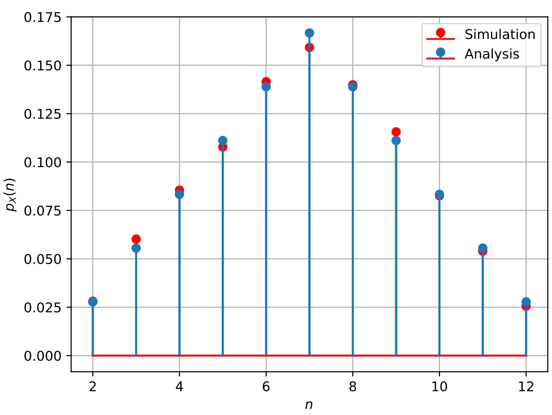


Figure 1: Plot of $p_X(n)$. Simulations are close to the analysis.

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-4/4.1.4/4.1.4.py
```

2 Random numbers

2.1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1. Generate 10^6 samples of U using a C program and save into a file called uni.dat .

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-5/5.1.1/uni.dat
```

2. Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1}$$

Solution: The following code plots Fig. 2

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-5/5.1.2/cdf_plot.py
```

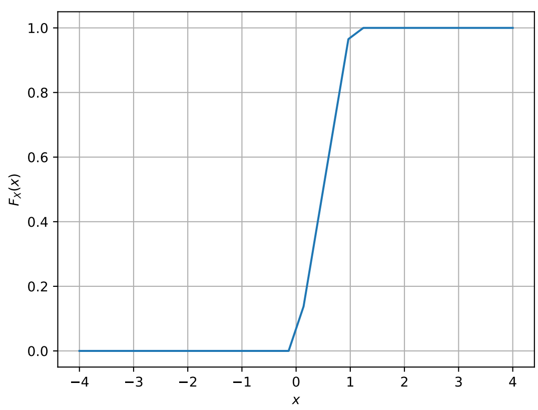


Figure 2: CDF of U

3. Find a theoretical expression for $F_U(x)$.
Uniform Random variable:

$$\left.\begin{aligned} f_U(x) &= \tfrac{1}{b-a} \quad a \leq x \leq b \\ &= 0 \quad \text{elsewhere} \end{aligned}\right\}$$

where a and b are real constants $-\alpha < a < \alpha$ $\&b > a$

$$F_U(x) = \int_{-\infty}^{x} f_x(x)dx$$
$$= \int_{-\infty}^{x} \frac{1}{b-a} \cdot dx$$
$$= \frac{1}{b-a} \cdot x\Big|_a^x$$
$$= \frac{x-a}{b-a}$$

$$F_U(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a) & a \leq x < b \\ 1 & b \leq x \end{cases}$$

4. The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{2}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{3}$$

Write a C program to find the mean and variance of U.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "coeffs.h"

int main(void) //main function begins
{

//Uniform random numbers
uniform("uni.dat", 1000000);

//Mean of uniform
printf("%lf\n",mean("uni.dat"));
//Variance of uniform
printf("%lf",variance("uni.dat"));
return 0;
}
```

```
double mean(char *str)
{
int i=0,c;
FILE *fp;
double x, temp=0.0;

fp = fopen(str,"r");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
//Count numbers in file
i=i+1;
//Add all numbers in file
temp = temp+x;
}
fclose(fp);
temp = temp/(i-1);
return temp;

}
```

```
double variance(char *str)
{
int i=0,j=0,c;
FILE *fp;
double x, temp=0.0,value,sumsqr=0,variance=0.0;

fp = fopen(str,"r");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
//Count numbers in file
i=i+1;
//Add all numbers in file
temp = temp+x;
}
fclose(fp);
temp = temp/(i-1);
fp = fopen(str,"r");
while(fscanf(fp,"%lf",&x)!=EOF)
{
  j=j+1;
  value=x-temp;
     sumsqr=sumsqr+value*value;
}
fclose(fp);
variance = sumsqr/(j-1);
return variance;
}
```

Result:

```
mean=0.500137
variance=0.83251
```

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-5/5.1.4/uniform.c

5. Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{4}$$

$$Mean : \mathrm{E}[U] = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{b^2-a^2}{2} \cdot \frac{1}{b-a} \tag{5}$$

$$= \frac{(b-a)(b+a)}{2} \cdot \frac{1}{b-a} \tag{6}$$

$$= \frac{b+a}{2} \tag{7}$$

$$Here\ a = 0, b = 1 \tag{8}$$

$$\mu = \mathrm{E}[X] = \frac{1}{2} = 0.5 \tag{9}$$

$$\mathrm{E}[U^2] = \int_a^b x^2 \cdot \frac{1}{b-a} dx = \frac{b^3-a^3}{3} \cdot \frac{1}{b-a} \tag{10}$$

$$= \frac{a^2+ab+b^2}{3} \tag{11}$$

$$Variance : \sigma^2 = E\left(U^2\right) - [E(U)]^2 = \frac{(a-b)^2}{12} \tag{12}$$

$$\sigma^2 = 0.834 \tag{13}$$

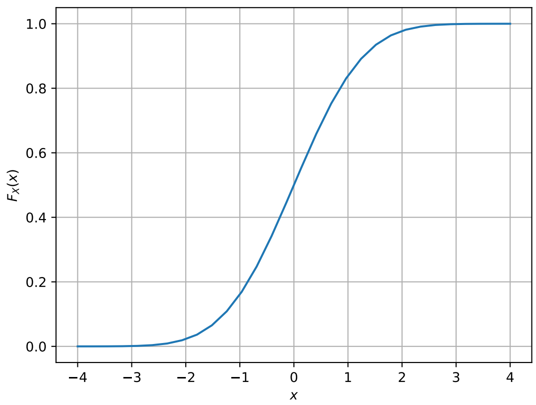


Figure 3: CDF of X

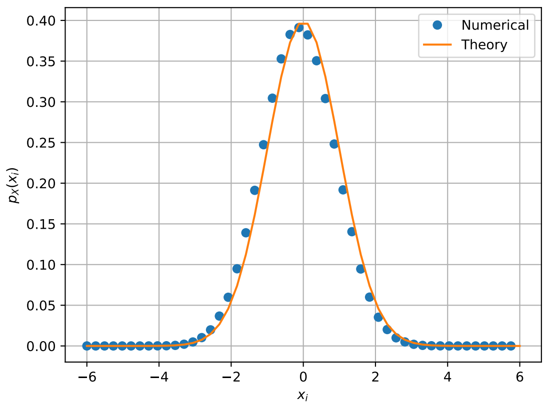


Figure 4: The PDF of X

2.2 Central Limit Theorem

1. Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{14}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-5/5.2.1/gau.dat
```

2. Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 3

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-5/5.2.2/cdf_plot.py
```

3. Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{15}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 4 using the code below

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-5/5.2.3/pdf_plot.py
```

4. Find the mean and variance of X by writing a C program.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "coeffs.h"
```

```
int main(void) //main function begins
{

//gaussian random numbers
gaussian("gau.dat", 1000000);

//Mean ,variance of gaussian
printf("%lf\n",mean("gau.dat"));
printf("%lf",variance("gau.dat"));
return 0;
}
```

```
double mean(char *str)
{
int i=0,c;
FILE *fp;
double x, temp=0.0;

fp = fopen(str,"r");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
//Count numbers in file
i=i+1;
//Add all numbers in file
temp = temp+x;
}
fclose(fp);
temp = temp/(i-1);
return temp;

}
```

```
double variance(char *str)
{
int i=0,j=0,c;
FILE *fp;
double x, temp=0.0,value,sumsqr=0,variance=0.0;

fp = fopen(str,"r");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
```

```
//Count numbers in file
i=i+1;
//Add all numbers in file
temp = temp+x;
}
fclose(fp);
temp = temp/(i-1);
fp = fopen(str,"r");
while(fscanf(fp,"%lf",&x)!=EOF)
{
  j=j+1;
  value=x-temp;
      sumsqr=sumsqr+value*value;
}
fclose(fp);
variance = sumsqr/(j-1);
return variance;
}
```

```
mean= 0.000283
variance=0.999702
```

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-5/5.2.4/gauss.c

5. Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (16)$$

repeat the above exercise theoretically.

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx$$

$$= 0 \quad \text{(odd function)}$$

$$E\left(X^2\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx \qquad \text{(evenfunction)}$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x^2 e^{-\frac{x^2}{2}} dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \sqrt{2u} e^{-u} du \qquad \left(\text{Let } \frac{x^2}{2} = u\right)$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u} u^{\frac{3}{2}-1} du$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right)$$

$$= 1$$

where we have used the fact that

$$\because \Gamma(n) = (n-1)\Gamma(n-1); \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Thus, the variance is

$$\sigma^2 = E(X)^2 - E^2(X) = 1$$

2.3 From Uniform to Other

1. Generate samples of

$$V = -2\ln(1-U) \quad (17)$$

and plot its CDF.

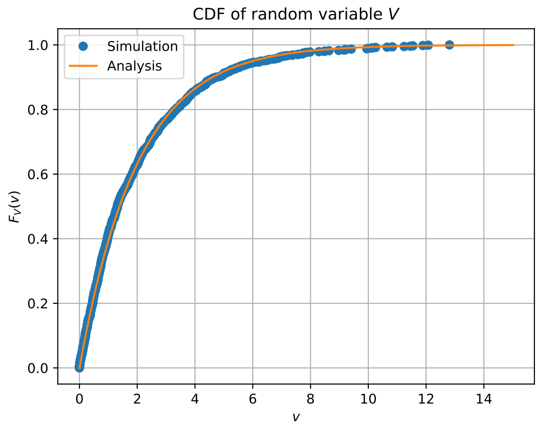


Figure 5: CDF of V

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-5/5.3/5.3.py

2. Find a theoretical expression for $F_V(x)$. The CDF of V is defined as

$$F_V(v) = \Pr(V \le v) \quad (18)$$

$$= \Pr(-2\ln(1-U) \le v) \quad (19)$$

$$= \Pr\left(\ln(1-U) \ge -\frac{v}{2}\right) \quad (20)$$

$$= \Pr\left(1-U \ge \exp\left(-\frac{v}{2}\right)\right) \quad (21)$$

$$= \Pr\left(U \le 1-\exp\left(-\frac{v}{2}\right)\right) \quad (22)$$

$$= F_U\left(1-\exp\left(-\frac{v}{2}\right)\right) \quad (23)$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \quad (24)$$

Substituting the above in (23),

$$F_U\left(1-\exp\left(-\frac{v}{2}\right)\right) =$$

$$\begin{cases} 0 & 1-\exp\left(-\frac{v}{2}\right) < 0 \\ 1-\exp\left(-\frac{v}{2}\right) & 0 \le 1-\exp\left(-\frac{v}{2}\right) \le 1 \\ 1 & 1-\exp\left(-\frac{v}{2}\right) > 1 \end{cases} \quad (25)$$

After some algebra, the above conditions yield

$$F_V(v) = \begin{cases} 0 & v < 0 \\ 1-exp\left(-\frac{v}{2}\right) & v \ge 0 \end{cases} \quad (26)$$

which is the CDF of the exponential distribution with parameter $\frac{1}{2}$.

2.4 Triangular Distribution

2.1 Generate

$$Z = U_1 + U_2 \tag{27}$$

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-5/5.4/uniform_two.c
```

2.2 Find the CDF of Z.

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-5/5.4/5.4_cdf.py
```

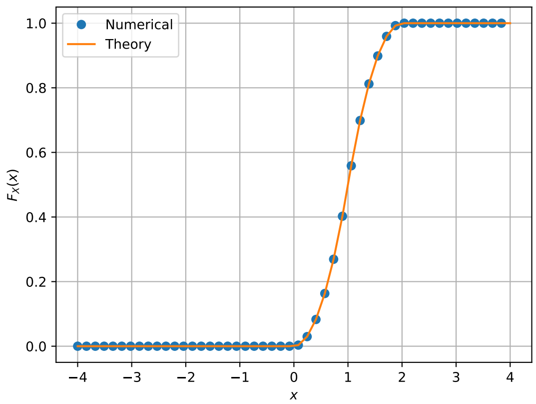


Figure 6: CDF of Z

2.3 Find the PDF of Z.

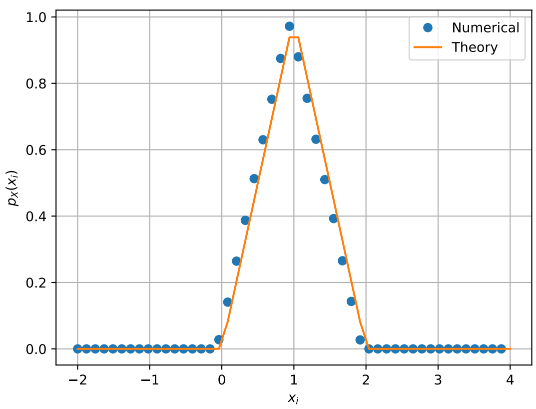


Figure 7: PDF of Z

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-5/5.4/5.4_pdf.py
```

2.4 Find the theoretical expressions for the PDF and CDF of Z.

$$f(z) = \begin{cases} z & 0 < z < 1 \\ 2 - z & 1 \le z < 2 \\ 0 & otherwise \end{cases} \tag{28}$$

$$F_Z(z) = \begin{cases} \frac{z^2}{2} & 0 < z < 1 \\ 2z - z^2 - 1 & 1 \le z < 2 \\ 1 & z > 0 \end{cases} \tag{29}$$

2.5 Verify your results through a plot.
Solution: From Figure:6,Figure:7

3 transfromations of R.V

3.1 Gaussian to Other

1. Let $X_1 \sim (0,1)$ and $X_2 \sim (0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{30}$$

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-8/8.1.1/8.1.1_CDF.py
```

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-8/8.1.1/8.1.1_PDF.py
```

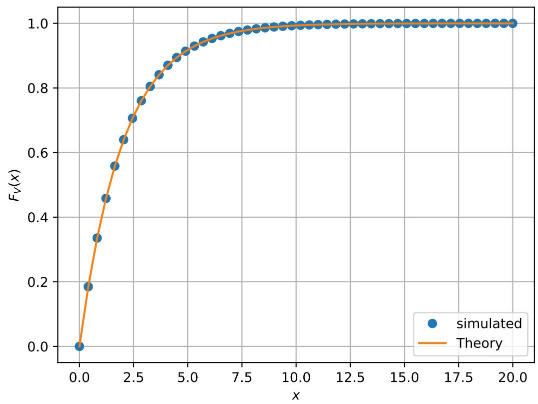


Figure 8: CDF of V

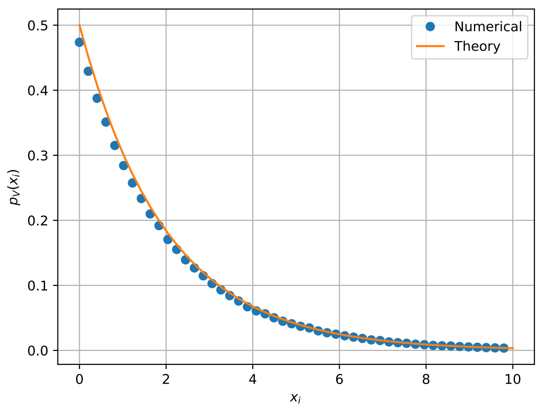


Figure 9: PDF of V

2. If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{31}$$

find α.
Solution: For the value $\alpha = 0.5$, the theory matches the simulation.

The following code generates the CDF of V in Fig. 10

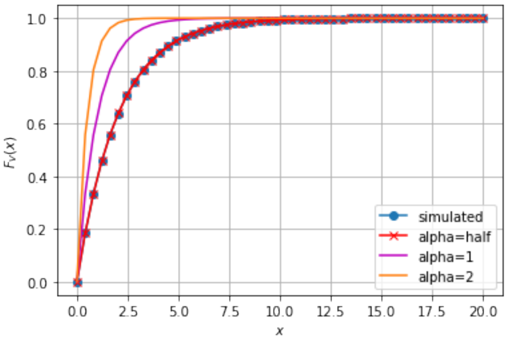


Figure 10: CDF of V

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-8/8.1.2/8.1.2.py
```

3. Plot the CDF and PDf of

$$A = \sqrt{V} \tag{32}$$

$$\begin{aligned} F_A(a) &= \Pr(A < a) \\ &= \Pr\left(\sqrt{V} < a\right) \\ &= \Pr\left(V < a^2\right) \\ &= F_V\left(a^2\right) \\ &= 1 - \exp\left(-\frac{a^2}{2}\right) \\ f(a) &= a \exp\left(-\frac{a^2}{2}\right) \end{aligned}$$

The CDF and PDF of A are plotted in Figs. 11 and 12 using the codes below.

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-8/8.1.3/8.1.3_CDF.py
```

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-8/8.1.3/8.1.3_PDF.py
```

3.2 Conditional Probability

1. Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{33}$$

for

$$Y = AX + N, \tag{34}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1) \in (-1,1)$ for $0 \leq \gamma \leq 10$ dB.

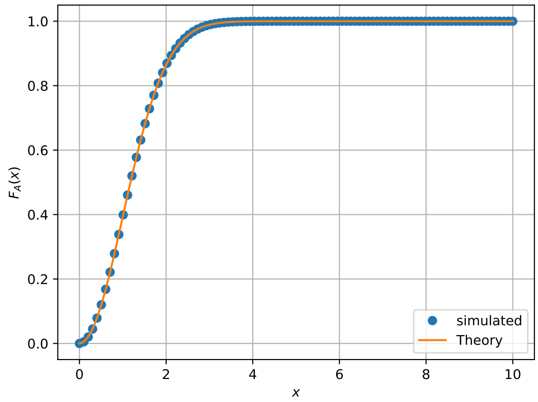


Figure 11: CDF of A

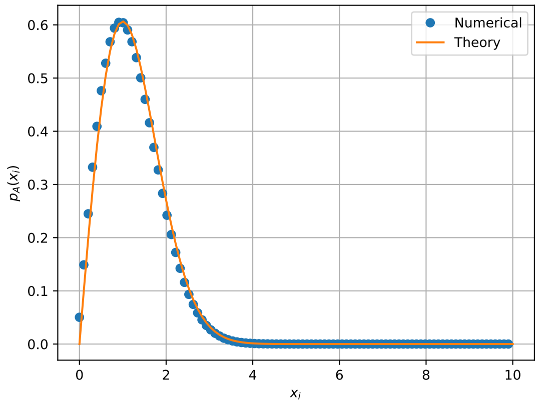


Figure 12: PDF of A

2. Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$.
Solution: The estimated value $\hat{X}$ is given by

$$\hat{X} = \begin{cases} +1 & Y > 0 \\ -1 & Y < 0 \end{cases} \tag{35}$$

For $X = 1$,

$$\begin{aligned} Y &= A + N & (36) \\ P_e &= \Pr\left(\hat{X} = -1 | X = 10\right) & (37) \\ &= \Pr(Y < 0 | X = 1) & (38) \\ &= \Pr(A < -N)) & (39) \\ &= F_A(-N) & (40) \\ &= \int_{-\infty}^{-N} f_A(x) dx & (41) \end{aligned}$$

By definition

$$f_A(x) = \begin{cases} \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) & x \geq 0 \\ 0 & otherwise \end{cases} \tag{42}$$

If $N > 0, f_A(x) = 0$. Then,

$$P_e = 0 \tag{43}$$

If $N < 0$. Then,

$$P_e(N) = \int_{-\infty}^{-N} f_A(x)dx \tag{44}$$

$$= \int_{-\infty}^{0} 0dx + \int_{0}^{-N} f_A(x)dx \tag{45}$$

$$= \int_{0}^{-N} \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \tag{46}$$

$$= 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) \tag{47}$$

Therefore,

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) & N < 0 \\ 0 & otherwise \end{cases} \tag{48}$$

3. For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)p_X(x)\,dx \tag{49}$$

Find $P_e = E\left[P_e(N)\right]$.
Solution: Since $N \sim \mathcal{N}(0, 1)$,

$$p_N(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{50}$$

$$\tag{51}$$

And from (48)

$$P_e(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) & x < 0 \\ 0 & otherwise \end{cases} \tag{52}$$

$$P_e = E\left[P_e(N)\right] = \int_{-\infty}^{\infty} P_e(x)p_N(x)\,dx \tag{53}$$

If $x < 0, P_e(x) = 0$ and using the fact that for an even function

$$\int_{-\infty}^{\infty} f(x) = 2\int_{-\infty}^{0} f(x) \tag{54}$$

we get

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{x^2}{2}\right)\left(1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)\right) dx \tag{55}$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx - \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(1+\sigma^2)x^2}{2\sigma^2}\right) dx \tag{56}$$

$$= \frac{\sqrt{2\pi} - \sqrt{\frac{\pi(2\sigma^2)}{1+\sigma^2}}}{2\sqrt{2\pi}} \tag{57}$$

$$= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\sigma^2}{1+\sigma^2}} \tag{58}$$

For a Rayleigh Distribution with scale $= \sigma$,

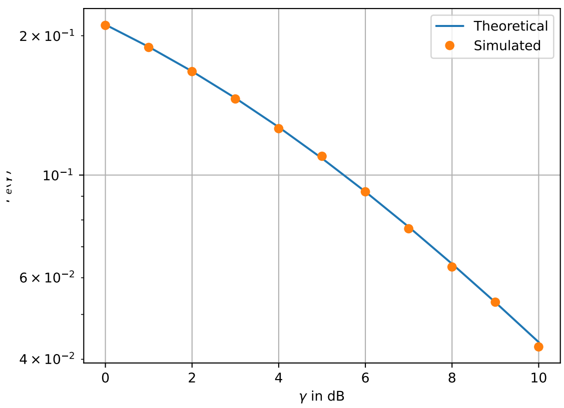
$$E\left[A^2\right] = 2\sigma^2 \tag{59}$$

$$\gamma = 2\sigma^2 \tag{60}$$

$$\therefore P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{2+\gamma}} \tag{61}$$

4. Plot P_e in problems 1 and 3 on the same graph w.r.t γ. Comment.

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-8/8.2/8.2.py



4 Maximum Likelihood Detection:BPSK

4.1 Maximum Likelihood

1. Generate equiprobable $X \in \{1, -1\}$,$Y = AX + N$
Plot Y using a scatter plot.
where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.
Solution:

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-6/scatter_plot.py

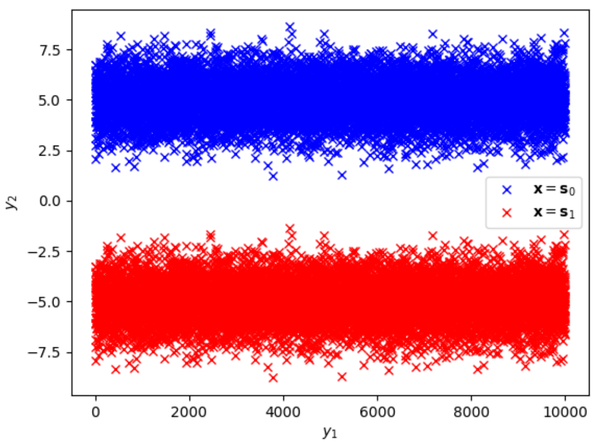


Figure 13: Scatter plot of Y

2. Guess how to estimate X from Y.
Solution: X and Y can be estimated by decision rule

$$y \underset{-1}{\overset{1}{\gtrless}} 0 \tag{62}$$

3. Find
$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{63}$$
and
$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{64}$$

Solution: using decision rule in

$$\begin{aligned}
\Pr\left(\hat{X} = -1 | X = 1\right) &= \Pr\left(Y < 0 | X = 1\right) && (65)\\
&= \Pr\left(AX + N < 0 | X = 1\right) && (66)\\
&= \Pr\left(A + N < 0\right) && (67)\\
&= \Pr\left(N < -A\right) && (68)\\
&= \Pr\left(N > A\right) && (69)
\end{aligned}$$

$$\begin{aligned}
\Pr\left(\hat{X} = 1 | X = -1\right) &= \Pr\left(Y > 0 | X = -1\right) && (70)\\
&= \Pr\left(N > A\right) && (71)
\end{aligned}$$

$$\begin{aligned}
\Pr\left(N > A\right) &= \Pr\left(\frac{N-0}{1} > \frac{A-0}{1}\right)\\
&= Q\left(\frac{A-0}{1}\right) = Q(A)
\end{aligned}$$
$$P_{e|0} = P_{e|1} = Q(A) \tag{72}$$

4. Find P_e assuming that X has equiprobable symbols.
Solution:

Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 1 was sent. This results in

$$\begin{aligned}
P_e &= \Pr\left(Y < 0 | X = 1\right) = \Pr\left(A + N < 0\right) && (73)\\
&= \Pr\left(N < -A\right) = \Pr\left(N > A\right) && (74)
\end{aligned}$$

since N has a symmetric pdf. Let $w \sim \mathcal{N}(0, 1)$. Then $N = \sqrt{1}w$. Substituting this in (74),

$$\begin{aligned}
P_e &= \Pr\left(w > A\right) && (75)\\
&= Q\left(A\right) && (76)
\end{aligned}$$

where

$$\begin{aligned}
Q\left(x\right) &\triangleq \Pr\left(w > x\right), x \geq 0. && (77)\\
&= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta && (78)
\end{aligned}$$

5. Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-6/ber_snr_plot.py
```

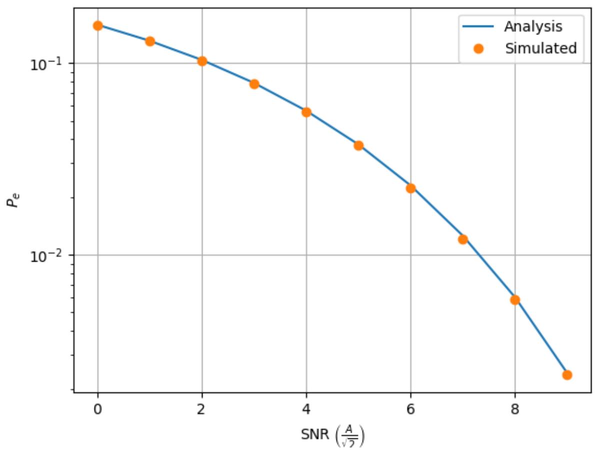


Figure 14: P_e versus A plot

6. Now, consider a threshold δ while estimating X from Y. Find the value of δ that minimize the theoretical P_e.
Solution:

Let $\Pr\left(0\right)$, $\Pr\left(1\right)$ is a probability of transmitting bit zero and bit one;
$P_{e|0}$,$P_{e|1}$is a probability of error when detecting bit zero and bit one.

$$P_e = P_{e|0}\Pr\left(0\right) + P_{e|1}\Pr\left(1\right) \tag{79}$$

Let V_0,V_1 be a nominal signal voltage of bit zero and one signal at the transmitter.

$$\begin{aligned}
P(e \mid 0) &= \int_\delta^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\left(\nu - V_0\right)^2/2\sigma^2\right) d\nu\\
P(e \mid 1) &= \int_{-\infty}^\delta \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\left(\nu - V_1\right)^2/2\sigma^2\right) d\nu
\end{aligned}$$

where δ is a detection threshold. Differentiating $P(e)$ of (79) w.r.t. T, we arrive at

$$\begin{aligned}
-\Pr\left(0\right)\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\left(\delta - V_0\right)^2/2\sigma^2\right) +\\
\Pr\left(1\right)\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\left(\delta - V_1\right)^2/2\sigma^2\right)
\end{aligned}$$

To find an optimal threshold, we equate the above expression to zero:

$$\begin{aligned}
\Pr\left(0\right)\exp\left(-\frac{\left(\delta - V_0\right)^2}{2\sigma^2}\right) &= \Pr\left(0\right)\exp\left(-\frac{\left(\delta - V_1\right)^2}{2\sigma^2}\right)\\
\delta &= \frac{V_0 + V_1}{2} + \sigma^2\ln\left(\frac{\Pr\left(1\right)}{\Pr\left(0\right)}\right)
\end{aligned}$$

$$\begin{aligned}
\implies \Pr(1) = \Pr(0) = \frac{1}{2}\\
\implies V_0 = 1, V_1 = -1\\
\therefore \delta = 0
\end{aligned}$$

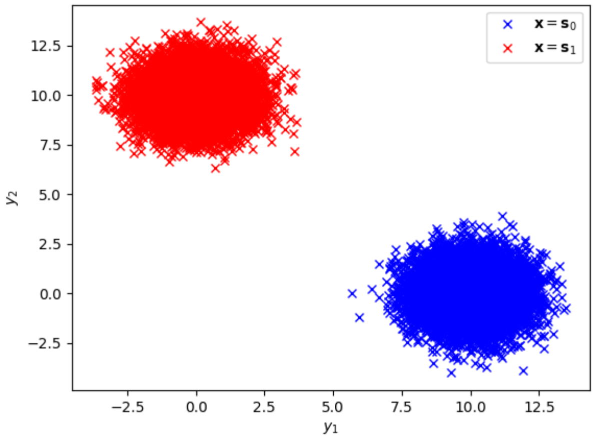


Figure 15: Scatter plot of $\mathbf{y}$ for $A = 10$

7. Repeat the above exercise when

$$p_X(0) = p \tag{80}$$

Solution:

$$P_e = (1-p)P_{e|1} + pP_{e|0} \tag{81}$$

From Above problem we know

$$\delta = \frac{V_0 + V_1}{2} + \sigma^2 \ln\left(\frac{P(1)}{P(0)}\right)$$
$$\delta = \ln\left(\frac{1-p}{p}\right)$$

5 Bivariate Random Variables: FSK

5.1 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{82}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix}0\\1\end{pmatrix} \tag{83}$$

$$\mathbf{n} = \begin{pmatrix}n_1\\n_2\end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0,1). \tag{84}$$

1. Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{85}$$

on the same graph using a scatter plot.
Solution: The following python code plots the scatter plot when $\mathbf{x} = \mathbf{s}_0$ and $\mathbf{x} = \mathbf{s}_1$ in Fig. 15

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-9/scatter_plot.py
```

2. For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

Solution: The multivariate Gaussian distribution is defined as

$$p_{\mathbf{y}}(y_1, \dots, y_k) = \frac{1}{\sqrt{(2\pi)^k |\boldsymbol{\sigma}|}} \exp\left\{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^T \boldsymbol{\sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})\right\} \tag{86}$$

where $\boldsymbol{\mu}$ is the mean vector, $\boldsymbol{\sigma} = E\left[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T\right]$ is the covariance matrix and $|\boldsymbol{\sigma}|$ is the determinant of $\boldsymbol{\Sigma}$.

$$p_{\mathbf{y}}(y_1, y_2) = \frac{1}{2\pi\sqrt{|\boldsymbol{\sigma}|}} \exp\left\{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^T \boldsymbol{\sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})\right\} \tag{87}$$

$$p(\mathbf{y}|s_0) = \frac{1}{2\pi\sqrt{|\boldsymbol{\sigma}|}} \exp\left\{-\frac{1}{2}(\mathbf{y}-\boldsymbol{s_0})^T \boldsymbol{\sigma}^{-1}(\mathbf{y}-\boldsymbol{s_0})\right\} \tag{88}$$

$$p(\mathbf{y}|s_1) = \frac{1}{2\pi\sqrt{|\boldsymbol{\sigma}|}} \exp\left\{-\frac{1}{2}(\mathbf{y}-\boldsymbol{s_1})^T \boldsymbol{\sigma}^{-1}(\mathbf{y}-\boldsymbol{s_1})\right\} \tag{89}$$

According to the MAP criterion, assuming equiprobably symbols,optimal decison criteria can be found by equating (88),(89)

$$p(\mathbf{y}|s_0) = p(\mathbf{y}|s_1) \tag{90}$$

$$\frac{1}{2\pi\sqrt{|\boldsymbol{\sigma}|}} \exp\left\{-\frac{1}{2}(\mathbf{y}-\boldsymbol{s_0})^T \boldsymbol{\sigma}^{-1}(\mathbf{y}-\boldsymbol{s_0})\right\} = \frac{1}{2\pi\sqrt{|\boldsymbol{\sigma}|}} \exp\left\{-\frac{1}{2}(\mathbf{y}-\boldsymbol{s_0})^T \boldsymbol{\sigma}^{-1}(\mathbf{y}-\boldsymbol{s_0})\right\}$$

$$(\mathbf{y}-\mathbf{s}_0)^\top(\mathbf{y}-\mathbf{s}_0) = (\mathbf{y}-\mathbf{s}_1)^\top(\mathbf{y}-\mathbf{s}_1)$$
$$\mathbf{y}^\top\mathbf{y} - 2\mathbf{s}_0^\top\mathbf{y} + \mathbf{s}_0^T\mathbf{s}_0 = \mathbf{y}^\top\mathbf{y} - 2\mathbf{s}_1^\top\mathbf{y} + \mathbf{s}_1^T\mathbf{s}_1$$
$$2(\mathbf{s}_1 - \mathbf{s}_0)^\top\mathbf{y} = \|\mathbf{s}_1\|^2 - \|\mathbf{s}_0\|^2$$
$$(\mathbf{s}_1 - \mathbf{s}_0)^\top\mathbf{y} = 0$$
$$\begin{pmatrix}-1\\1\end{pmatrix}^\top \mathbf{y} = 0$$

compare with the equation of line $n^\top$y=c

$$\implies y1 = y2$$

which is decsion making line as shown in below fig 16

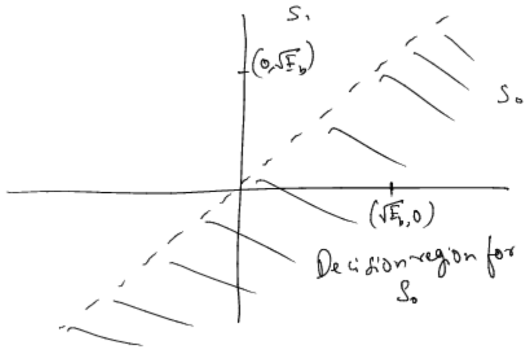


Figure 16:

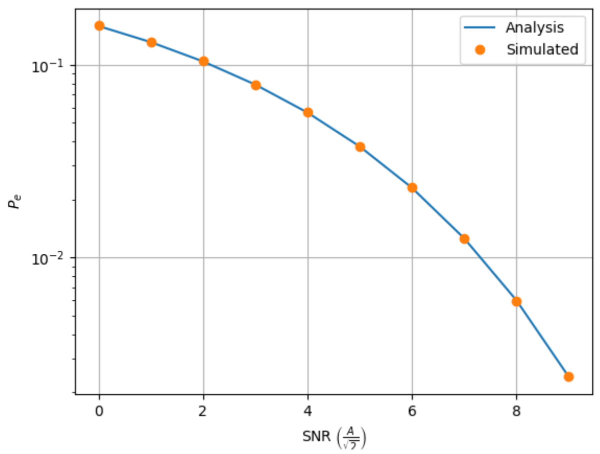


Figure 17: P_e with respect to SNR from 0 to 10 dB

3. Plot
$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{91}$$
with respect to the SNR from 0 to 10 dB.
Solution:

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-9/ber_snr_plot.py
```

4. Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.
Solution:
$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{92}$$
Given that $\mathbf{s}_0$ was transmitted, the received signal is
$$\mathbf{y}|\mathbf{s}_0 = \begin{pmatrix} A \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \tag{93}$$
From decision rule, the probability of error is given by
$$P_e = \Pr\left(y_1 < y_2 | \mathbf{s}_0\right) = \Pr\left(A + n_1 < n_2\right) \tag{94}$$
$$= \Pr\left(n_2 - n_1 > A\right) \tag{95}$$

Note that $n_2 - n_1 \sim \mathcal{N}(0, 2)$. Thus,
$$P_e = \Pr\left(\sqrt{2}w > A\right) \tag{96}$$
$$\Pr\left(w > \frac{A}{\sqrt{2}}\right) \tag{97}$$
$$\Rightarrow P_e = Q\left(\frac{A}{\sqrt{2}}\right) \tag{98}$$
where $w \sim \mathcal{N}(0, 1)$.Above code plots the P_e curve in Fig. (17).

6 Exercises

6.1 BPSK

1. The *signal constellation diagram* for BPSK is given by Fig. 18. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, The decision rule is
$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{99}$$
Repeat the previous exercise using the MAP criterion.

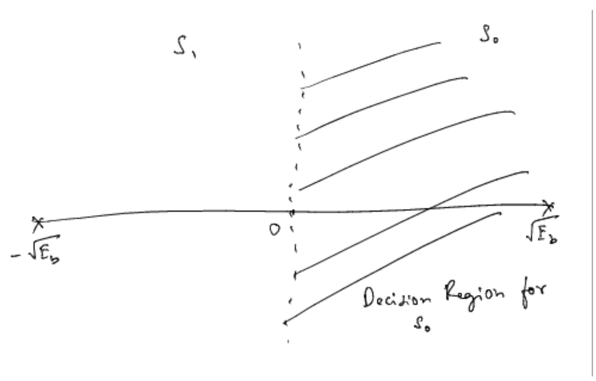


Figure 18:

Solution: According to MAP detection rule
$$\hat{s} = \max_{s \in \{s_0, s_1\}} p\left(s|y\right) \tag{100}$$
$$\implies p\left(s_0|y\right) \underset{s_1}{\overset{s_0}{\gtrless}} p\left(s_1|y\right) \tag{101}$$
Using Bayes rule,
$$p\left(s_0|y\right) = \frac{p\left(y|s_0\right)p\left(s_0\right)}{p\left(y\right)} \tag{102}$$
$$p\left(s_1|y\right) = \frac{p\left(y|s_1\right)p\left(s_1\right)}{p\left(y\right)} \tag{103}$$
Since symbols are equi probable, $p\left(s_0\right) = p\left(s_1\right)$. Hence the decision becomes
$$\frac{p\left(y|s_0\right)p\left(s_0\right)}{p\left(y\right)} \underset{s_1}{\overset{s_0}{\gtrless}} \frac{p\left(y|s_1\right)p\left(s_1\right)}{p\left(y\right)} \tag{104}$$
$$\implies p\left(y|s_0\right) \underset{s_1}{\overset{s_0}{\gtrless}} p\left(y|s_1\right) \tag{105}$$

The above condition is known as the maximum-likelihood (ML) criterion. (105) can be expressed as

$$\frac{1}{\sqrt{2\pi}}\exp -\frac{(y-\sqrt{E_b})^2}{\frac{N_o N_0}{2}} \underset{s_1}{\overset{s_0}{\gtrless}} \frac{1}{\sqrt{2\pi}}\exp -\frac{(y+\sqrt{E_b})^2}{\frac{N_o N_0}{2}} \tag{106}$$

$$\implies (y+\sqrt{E_b})^2 \underset{s_1}{\overset{s_0}{\gtrless}} (y-\sqrt{E_b})^2 \tag{107}$$

$$\implies y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{108}$$

2. The PDF of $w \sim \mathcal{N}(0,1)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{109}$$

and the complementary error function is defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt. \tag{110}$$

Show that

$$Q(x) = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{111}$$

Solution: we know that

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt. \tag{112}$$

$$Q(u) = \frac{1}{2\pi}\int_u^\infty e^{-\frac{t^2}{2}}\,dt \tag{113}$$

$$u = \frac{u'}{\sqrt{2}}, u' = \sqrt{2}u \tag{114}$$

$$du = \frac{du'}{\sqrt{2}} \tag{115}$$

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2/2}\frac{du'}{\sqrt{2}} \tag{116}$$

$$= \frac{2}{\sqrt{2\pi}}\int_{\sqrt{2}x}^\infty e^{-u^2/2}\,du' \tag{117}$$

$$\operatorname{erfc}(x) = 2Q(\sqrt{2}x) \tag{118}$$

$$\implies Q(\sqrt{2}x) = \frac{1}{2}\operatorname{erfc}(x) \tag{119}$$

$$\therefore Q(x) = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{120}$$

3. Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.
Solution: The following code yields Fig. 19

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-10/bpsk_ber.py
```

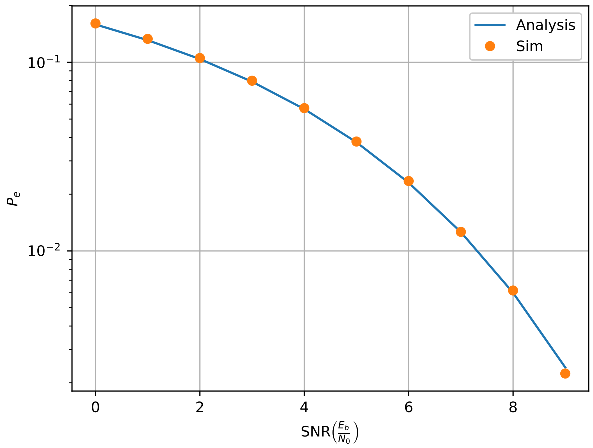


Figure 19:

4. Show that

$$Q(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\,d\theta \tag{121}$$

Solution: **Solution:** Consider the bivariate gaussian distribution of $X, Y \sim \mathcal{N}(0,1)$,

$$p_{X,Y}(x,y) = \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right) \tag{122}$$

Q-function can be expressed with (122) as

$$Q(z) = \int_z^\infty \int_{-\infty}^\infty p_{X,Y}(x,y)\,dx\,dy \tag{123}$$

$$= \frac{1}{2\pi}\int_z^\infty \int_{-\infty}^\infty \exp\left(-\frac{x^2+y^2}{2}\right)dx\,dy \tag{124}$$

Transforming the integral in (124) to polar coordinates (r,θ) for $z > 0$,

$$Q(z) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\frac{z}{\sin\theta}}^\infty \exp\left(-\frac{r^2}{2}\right) r\,dr\,d\theta \tag{125}$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left(-\frac{z^2}{2\sin^2\theta}\right) d\theta \tag{126}$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \exp\left(-\frac{z^2}{2\sin^2\theta}\right) d\theta \text{ , for } z > 0 \tag{127}$$

6.2 Coherent BFSK

1. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 20.The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix}\sqrt{E_b}\\0\end{pmatrix} + \begin{pmatrix}n_1\\n_2\end{pmatrix}, \tag{128}$$

and

$$\mathbf{y}|s_1 = \begin{pmatrix}0\\\sqrt{E_b}\end{pmatrix} + \begin{pmatrix}n_1\\n_2\end{pmatrix}, \tag{129}$$

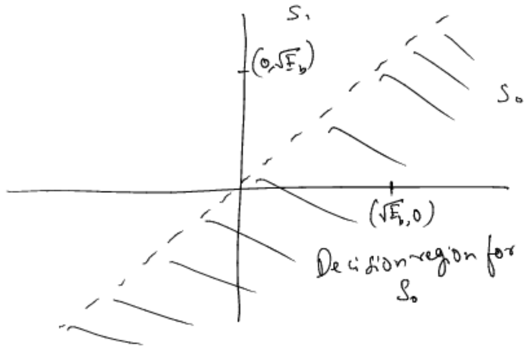


Figure 20:

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

decision rule for BFSK from Fig. 20. is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{130}$$

Repeat the above using the MAP criterion.
Solution: The multivariate Gaussian distribution is defined as

$$p_{\mathbf{x}}(x_1, \ldots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\boldsymbol{\sigma}|}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\} \tag{131}$$

where $\boldsymbol{\mu}$ is the mean vector, $\boldsymbol{\sigma} = E\left[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T\right]$ is the covariance matrix and $|\boldsymbol{\sigma}|$ is the determinant of $\boldsymbol{\Sigma}$. The PDF of the *bivariate* Gaussian is

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \times \left\{\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right\}\right] \tag{132}$$

where

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix} \tag{133}$$

According to the MAP criterion, assuming equiprobably symbols,

$$p(s_0|\mathbf{y}) \underset{s_1}{\overset{s_0}{\gtrless}} p(s_1|\mathbf{y}) \tag{134}$$

Use (132) in (134) to obtain (134).

Definition 1 *The joint PDF of X, Y is given by*

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \times \left\{\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right\}\right] \tag{135}$$

where

$$\mu_x = E[X], \sigma_x^2 = \mathit{var}(X), \rho = \frac{E[(X-\mu_x)(Y-\mu_y)]}{\sigma_x\sigma_y}. \tag{136}$$

2. Derive and plot the probability of error. Verify through simulation.

Solution: Given that s_0 was transmitted, the received symbols are

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{137}$$

From (134), the probability of error is given by

$$P_e = \Pr(y_1 < y_2 | s_0) = \Pr\left(\sqrt{E_b} + n_1 < n_2\right) \tag{138}$$

$$= \Pr\left(n_2 - n_1 > \sqrt{E_b}\right) \tag{139}$$

Note that $n_2 - n_1 \sim \mathcal{N}(0, N_0)$. Thus,

$$P_e = \Pr\left(\sqrt{N_0}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{E_b}{N_0}}\right) \tag{140}$$

$$\implies P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \tag{141}$$

where $w \sim \mathcal{N}(0,1)$. The following code plots the BER curves in Fig. 1

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-10/bfsk_ber.py
```

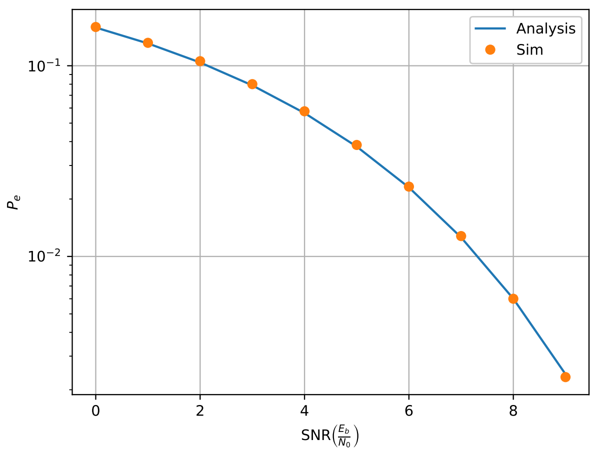


Figure 1: