DIGITAL COMMUNICATIONS

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1 Random numbers

1.1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-5/5.1.1/uni.dat
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1}$$

Solution: The following code plots Fig. **??**

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-5/5.1.2/cdf_plot.py
```

images/uni_cdf.png

Figure 1: CDF

1.3 Find a theoretical expression for $F_U(x)$.
Uniform Random variable:

$$\left.\begin{array}{ll} f_U(x) = \frac{1}{b-a} & a \leq x \leq b \\ = 0 & \text{elsewhere} \end{array}\right\} \begin{array}{l} \text{where a and} \\ b \text{ are real} \\ \text{constants} \\ -\alpha < a < \alpha \\ \&b > a \end{array}$$

$$F_U(x) = \int_{-\infty}^{x} f_x(x)dx$$
$$= \int_{-\infty}^{x} \frac{1}{b-a} \cdot dx$$
$$= \frac{1}{b-a} \cdot x\Big|_a^x$$
$$= \frac{x-a}{b-a}$$

$$F_U(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a) & a \leq x < b \\ 1 & b \leq x \end{cases}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{2}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{3}$$

Write a C program to find the mean and variance of U.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "coeffs.h"

int main(void) //main function begins
{

//Uniform random numbers
uniform("uni.dat", 1000000);

//Mean of uniform
printf("%lf\n",mean("uni.dat"));
//Variance of uniform
printf("%lf",variance("uni.dat"));
return 0;
}
```

```
double mean(char *str)
{
int i=0,c;
FILE *fp;
double x, temp=0.0;

fp = fopen(str,"r");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
//Count numbers in file
i=i+1;
//Add all numbers in file
temp = temp+x;
}
fclose(fp);
temp = temp/(i-1);
return temp;

}
```

```
double variance(char *str)
{
int i=0,j=0,c;
FILE *fp;
double x, temp=0.0,value,sumsqr=0,variance=0.0;

fp = fopen(str,"r");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
//Count numbers in file
i=i+1;
//Add all numbers in file
temp = temp+x;
}
fclose(fp);
temp = temp/(i-1);
fp = fopen(str,"r");
while(fscanf(fp,"%lf",&x)!=EOF)
{
  j=j+1;
  value=x-temp;
     sumsqr=sumsqr+value*value;
}
fclose(fp);
variance = sumsqr/(j-1);
return variance;
}
```

Result:

```
mean=0.500137
variance=0.83251
```

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-5/5.1.4/uniform.c

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{4}$$

images/gau_cdf.pdf

Figure 2: CDF of X

$$Mean : \mathrm{E}[U] = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{b^2-a^2}{2} \cdot \frac{1}{b-a} \tag{5}$$

$$= \frac{(b-a)(b+a)}{2} \cdot \frac{1}{b-a} \tag{6}$$

$$= \frac{b+a}{2} \tag{7}$$

$$Here\ a = 0, b = 1 \tag{8}$$

$$\mu = \mathrm{E}[X] = \frac{1}{2} = 0.5 \tag{9}$$

$$\mathrm{E}[U^2] = \int_a^b x^2 \cdot \frac{1}{b-a} dx = \frac{b^3-a^3}{3} \cdot \frac{1}{b-a} \tag{10}$$

$$= \frac{a^2+ab+b^2}{3} \tag{11}$$

$$Variance : \sigma^2 = E\left(U^2\right) - [E(U)]^2 = \frac{(a-b)^2}{12} \tag{12}$$

$$\sigma^2 = 0.834 \tag{13}$$

1.2 Central Limit Theorem

1. Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{14}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-5/5.2.1/gau.dat

2. Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig.

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-5/5.2.2/cdf_plot.py

3. Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is

images/gau_pdf.pdf

Figure 3: The PDF of X

defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{15}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. **??** using the code below

```
https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-5/5.2.3/pdf_plot.py
```

4. Find the mean and variance of X by writing a C program.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "coeffs.h"

int main(void) //main function begins
{

//gaussian random numbers
gaussian("gau.dat", 1000000);

//Mean ,variance of gaussian
printf("%lf\n",mean("gau.dat"));
printf("%lf",variance("gau.dat"));
return 0;
}
```

```
mean=−0.000283
variance=0.999702
```

```
https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-5/5.2.4/gauss.c
```

5. Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{16}$$

repeat the above exercise theoretically.

$$\begin{aligned}
&E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx \\
&= 0 \quad (\text{ odd function}) \\
&E\left(X^2\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx \quad (\text{evenfunction}) \\
&= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x^2 e^{-\frac{x^2}{2}} dx \\
&= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \sqrt{2u} e^{-u} du \quad \left(\text{Let } \frac{x^2}{2} = u\right) \\
&= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u} u^{\frac{3}{2}-1} du \\
&= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \\
&= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) \\
&= 1
\end{aligned}$$

where we have used the fact that

$$\because \Gamma(n) = (n-1)\Gamma(n-1); \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Thus, the variance is

$$\sigma^2 = E(X)^2 - E^2(X) = 1$$

1.3 From Uniform to Other

1. Generate samples of

$$V = -2\ln\left(1-U\right) \tag{17}$$

and plot its CDF.

images/quantile.png

Figure 4: CDF

2. Find a theoretical expression for $F_V(x)$. The CDF of V is defined as

$$\begin{aligned}
F_V(v) &= pr(V \le v) && (18)\\
&= pr(-2\ln(1-U) \le v) && (19)\\
&= pr(\ln(1-U) \ge -\frac{v}{2}) && (20)\\
&= pr(1-U \ge \exp\left(-\frac{v}{2}\right)) && (21)\\
&= pr(U \le 1-\exp\left(-\frac{v}{2}\right)) && (22)\\
&= F_U\left(1-\exp\left(-\frac{v}{2}\right)\right) && (23)
\end{aligned}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{24}$$

Substituting the above in (**??**),

$$F_U\left(1-\exp\left(-\frac{v}{2}\right)\right) = \begin{cases} 0 & 1-\exp\left(-\frac{v}{2}\right) < 0 \\ 1-\exp\left(-\frac{v}{2}\right) & 0 \le 1-\exp\left(-\frac{v}{2}\right) \le 1 \\ 1 & 1-\exp\left(-\frac{v}{2}\right) > 1 \end{cases} \quad (25)$$

After some algebra, the above conditions yield

$$F_V(v) = \begin{cases} 0 & v < 0 \\ 1-exp\left(-\frac{v}{2}\right) & v \ge 0 \end{cases} \quad (26)$$

which is the CDF of the exponential distribution with parameter $\frac{1}{2}$.

1.4 Triangular Distribution

1.1 Generate

$$T = U_1 + U_2 \quad (27)$$

1.2 Find the CDF of T.

1.3 Find the PDF of T.

1.4 Find the theoretical expressions for the PDF and CDF of T.

1.5 Verify your results through a plot.

2 transfromations of R.V

2.1 Gaussian to Other

1. Let $X_1 \sim (0,1)$ and $X_2 \sim (0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (28)$$

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-7/7.1.1/7.1.1_CDF.py

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-7/7.1.1/7.1.1_PDF.py

2. If

$$F_V(x) = \begin{cases} 1-e^{-\alpha x} & x \ge 0 \\ 0 & x < 0, \end{cases} \quad (29)$$

find α.
For the value $\alpha = 0.5$, the theory matches the simulation. The following code generates the CDF of V in Fig. Fig.

3. Plot the CDF and PDf of

$$A = \sqrt{V} \quad (30)$$

The CDF and PDF of A are plotted in Figs. **??** and **??** using the codes below.

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-7/7.1.3/7.1.3_CDF.py

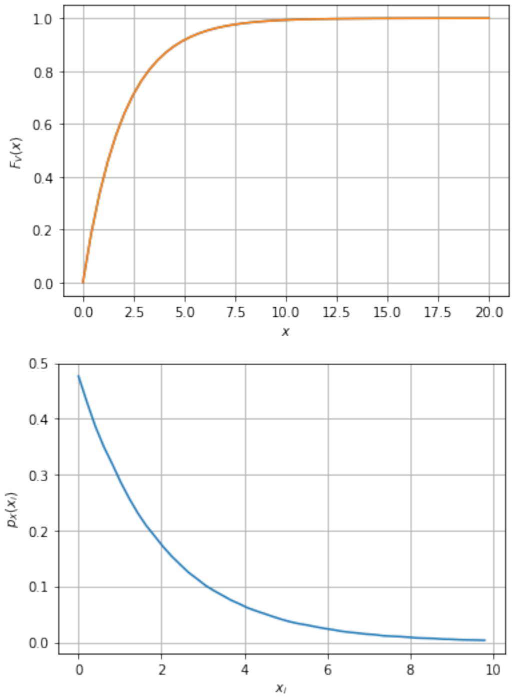


Figure 5: CDF,PDF of V

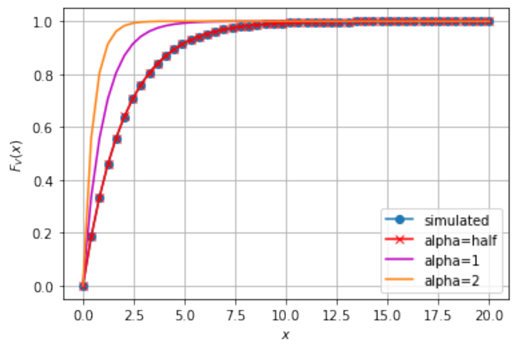


Figure 6: CDF of V

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-7/7.1.3/7.1.3_PDF.py

2.2 Conditional Probability

1. Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \quad (31)$$

for

$$Y = AX + N, \quad (32)$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1) \in (-1,1)$ for $0 \le \gamma \le 10$ dB.

2. Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$.
The estimated value $\hat{X}$ is given by

$$\hat{X} = \begin{cases} +1 & Y > 0 \\ -1 & Y < 0 \end{cases} \quad (33)$$

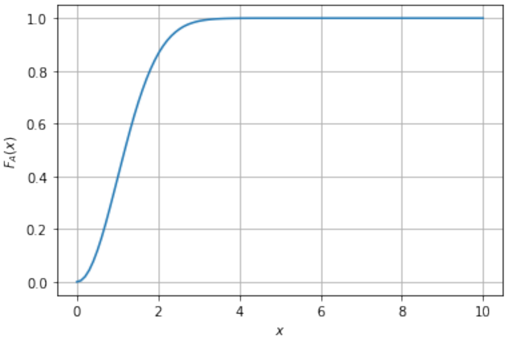


Figure 7: CDF

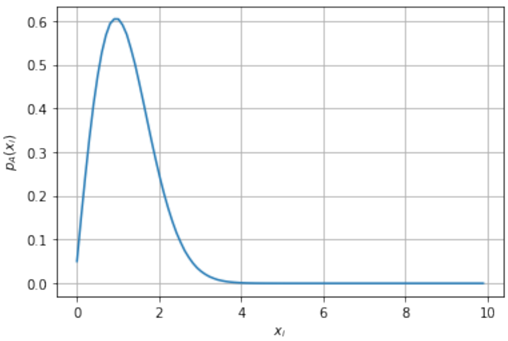


Figure 8: PDF

For $X = 1$,

$$Y = A + N \tag{34}$$
$$P_e = \Pr\left(\hat{X} = -1 | X = 10\right) \tag{35}$$
$$= \Pr\left(Y < 0 | X = 1\right) \tag{36}$$
$$= \Pr\left(A < -N)\right) \tag{37}$$
$$= F_A(-N) \tag{38}$$
$$= \int_{-\infty}^{-N} f_A(x) dx \tag{39}$$

By definition

$$f_A(x) = \begin{cases} \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) & x \geq 0 \\ 0 & otherwise \end{cases} \tag{40}$$

If $N > 0, f_A(x) = 0$. Then,

$$P_e = 0 \tag{41}$$

If $N < 0$. Then,

$$P_e(N) = \int_{-\infty}^{-N} f_A(x) dx \tag{42}$$
$$= \int_{-\infty}^{0} 0 dx + \int_{0}^{-N} f_A(x) dx \tag{43}$$
$$= \int_{0}^{-N} \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \tag{44}$$
$$= 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) \tag{45}$$

Therefore,

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) & N < 0 \\ 0 & otherwise \end{cases} \tag{46}$$

3. For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{47}$$

Find $P_e = E\left[P_e(N)\right]$.
Since $N \sim \mathcal{N}(0, 1)$,

$$p_N(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{48}$$
$$\tag{49}$$

And from (**??**)

$$P_e(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) & x < 0 \\ 0 & otherwise \end{cases} \tag{50}$$

$$P_e = E\left[P_e(N)\right] = \int_{-\infty}^{\infty} P_e(x) p_N(x)\, dx \tag{51}$$

If $x < 0, P_e(x) = 0$ and using the fact that for an even function

$$\int_{-\infty}^{\infty} f(x) = 2 \int_{-\infty}^{0} f(x) \tag{52}$$

we get

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{x^2}{2}\right) \left(1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)\right) dx \tag{53}$$

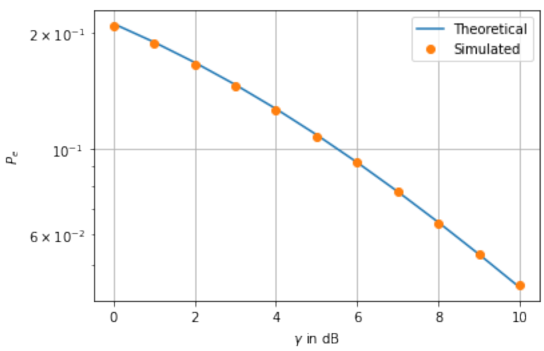
$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx - \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(1+\sigma^2)x^2}{2\sigma^2}\right) dx \tag{54}$$
$$= \frac{\sqrt{2\pi} - \sqrt{\frac{\pi(2\sigma^2)}{1+\sigma^2}}}{2\sqrt{2\pi}} \tag{55}$$
$$= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\sigma^2}{1+\sigma^2}} \tag{56}$$

For a Rayleigh Distribution with scale $= \sigma$,

$$E\left[A^2\right] = 2\sigma^2 \tag{57}$$
$$\gamma = 2\sigma^2 \tag{58}$$
$$\therefore P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{2+\gamma}} \tag{59}$$

4. Plot P_e in problems **??** and **??** on the same graph w.r.t γ. Comment.

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-7/7.2/7.2.py
```



Theoretical
Simulated
2×10^{-1}
10^{-1}
6×10^{-2}
P_e
0
2
4
6
8
10
γ in dB

Figure 9: PDF