DIGITAL COMMUNICATIONS

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1 Sum of Independent Random Variables

1. The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 1. The theoretical pmf obtained in is plotted for comparison.

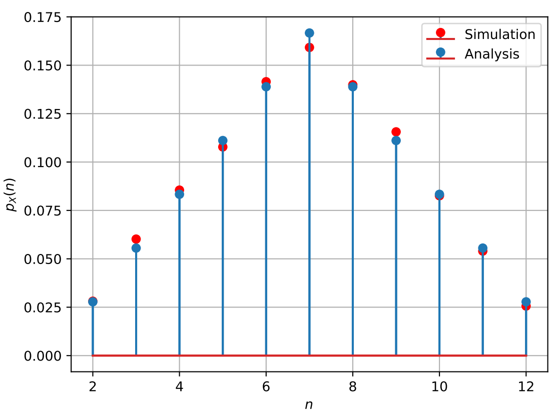

Figure 1: Plot of $p_X(n)$. Simulations are close to the analysis.

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-4/4.1.4/4.1.4.py
```

2 Random numbers

2.1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1. Generate 10^6 samples of U using a C program and save into a file called uni.dat .

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-5/5.1.1/uni.dat
```

2. Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1}$$

Solution: The following code plots Fig. 2

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-5/5.1.2/cdf_plot.py
```

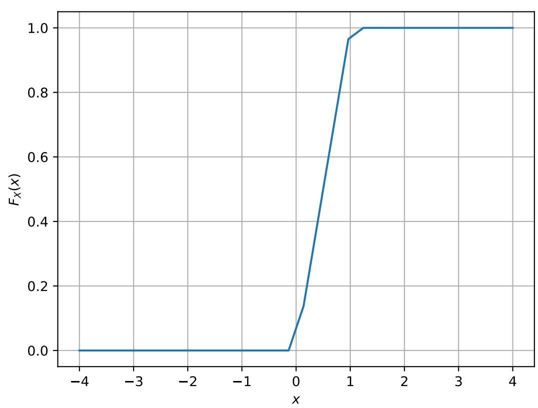

Figure 2: CDF of U

3. Find a theoretical expression for $F_U(x)$.
Uniform Random variable:

$$\left.\begin{aligned} f_U(x) &= \tfrac{1}{b-a} \quad && a \le x \le b \\ &= 0 && \text{elsewhere} \end{aligned}\right\} \begin{array}{l}\text{where a and}\\ b \text{ are real}\\ \text{constants}\\ -\alpha < a < \alpha\\ \& b > a\end{array}$$

$$F_U(x) = \int_{-\infty}^{x} f_x(x)dx$$

$$= \int_{-\infty}^{x} \frac{1}{b-a} \cdot dx$$

$$= \frac{1}{b-a} \cdot x\Big|_a^x$$

$$= \frac{x-a}{b-a}$$

$$F_U(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a) & a \le x < b \\ 1 & b \le x \end{cases}$$

4. The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{2}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{3}$$

Write a C program to find the mean and variance of U.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "coeffs.h"

int main(void) //main function begins
{

//Uniform random numbers
uniform("uni.dat", 1000000);

//Mean of uniform
printf("%lf\n",mean("uni.dat"));
//Variance of uniform
printf("%lf",variance("uni.dat"));
return 0;
}
```

```
double mean(char *str)
{
int i=0,c;
FILE *fp;
double x, temp=0.0;

fp = fopen(str,"r");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
//Count numbers in file
i=i+1;
//Add all numbers in file
temp = temp+x;
}
fclose(fp);
temp = temp/(i-1);
return temp;

}
```

```
double variance(char *str)
{
int i=0,j=0,c;
FILE *fp;
double x, temp=0.0,value,sumsqr=0,variance=0.0;

fp = fopen(str,"r");
//get numbers from file
while(fscanf(fp,"%lf",&x)!=EOF)
{
//Count numbers in file
i=i+1;
//Add all numbers in file
temp = temp+x;
}
fclose(fp);
temp = temp/(i-1);
fp = fopen(str,"r");
while(fscanf(fp,"%lf",&x)!=EOF)
{
  j=j+1;
  value=x-temp;
      sumsqr=sumsqr+value*value;
}
fclose(fp);
variance = sumsqr/(j-1);
return variance;
}
```

Result:

```
mean=0.500137
variance=0.83251
```

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-5/5.1.4/uniform.c

5. Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{4}$$

$$Mean : \mathrm{E}[U] = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{b^2 - a^2}{2} \cdot \frac{1}{b-a} \tag{5}$$

$$= \frac{(b-a)(b+a)}{2} \cdot \frac{1}{b-a} \tag{6}$$

$$= \frac{b+a}{2} \tag{7}$$

$$Here\ a = 0, b = 1 \tag{8}$$

$$\mu = \mathrm{E}[X] = \frac{1}{2} = 0.5 \tag{9}$$

$$\mathrm{E}[U^2] = \int_a^b x^2 \cdot \frac{1}{b-a} dx = \frac{b^3 - a^3}{3} \cdot \frac{1}{b-a} \tag{10}$$

$$= \frac{a^2 + ab + b^2}{3} \tag{11}$$

$$Variance : \sigma^2 = E\left(U^2\right) - [E(U)]^2 = \frac{(a-b)^2}{12} \tag{12}$$

$$\sigma^2 = 0.834 \tag{13}$$

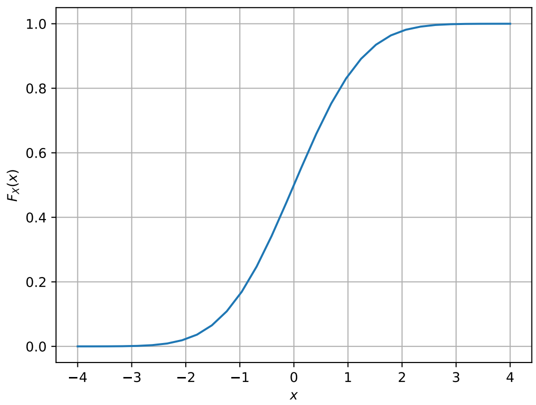

Figure 3: CDF of X

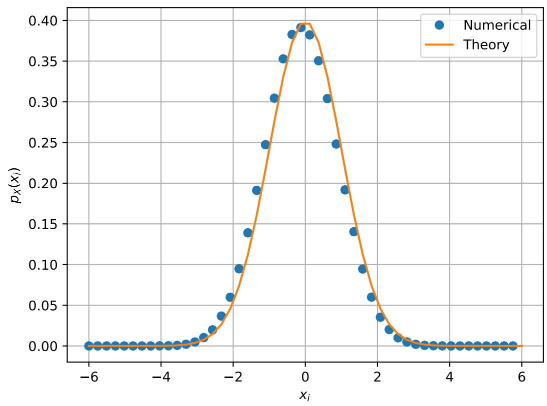

Figure 4: The PDF of X

2.2 Central Limit Theorem

1. Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{14}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-5/5.2.1/gau.dat

2. Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 3

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-5/5.2.2/cdf_plot.py

3. Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{15}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 4 using the code below

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-5/5.2.3/pdf_plot.py

4. Find the mean and variance of X by writing a C program.

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "coeffs.h"

int main(void) //main function begins
{

//gaussian random numbers
gaussian("gau.dat", 1000000);

//Mean ,variance of gaussian
printf("%lf\n",mean("gau.dat"));
printf("%lf",variance("gau.dat"));
return 0;
}
```

mean=−0.000283
variance=0.999702

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-5/5.2.4/gauss.c

5. Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{16}$$

repeat the above exercise theoretically.

$$E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx$$
$$= 0 \quad (\text{ odd function})$$
$$E\left(X^2\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx \quad (\text{evenfunction})$$
$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x^2 e^{-\frac{x^2}{2}} dx$$
$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \sqrt{2u} e^{-u} du \quad \left(\text{Let } \frac{x^2}{2} = u\right)$$
$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u} u^{\frac{3}{2}-1} du$$
$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right)$$
$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right)$$
$$= 1$$

where we have used the fact that

$$\because \Gamma(n) = (n-1)\Gamma(n-1); \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Thus, the variance is

$$\sigma^2 = E(X)^2 - E^2(X) = 1$$

2.3 From Uniform to Other

1. Generate samples of

$$V = -2\ln(1-U) \tag{17}$$

and plot its CDF.

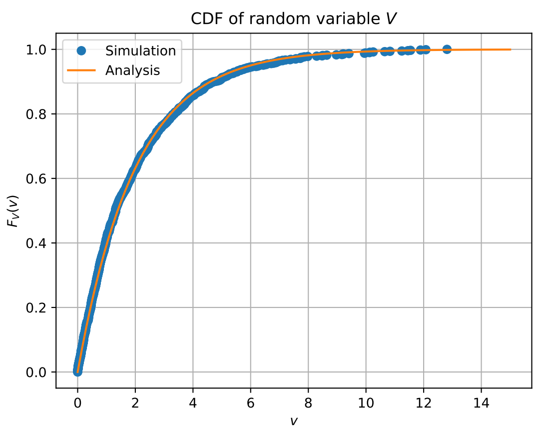

Figure 5: CDF of V

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-7/7.1.1/7.1.1_CDF.py
```

2. Find a theoretical expression for $F_V(x)$. The CDF of V is defined as

$$F_V(v) = \Pr(V \le v) \tag{18}$$
$$= \Pr(-2\ln(1-U) \le v) \tag{19}$$
$$= \Pr\left(\ln(1-U) \ge -\frac{v}{2}\right) \tag{20}$$
$$= \Pr\left(1-U \ge \exp\left(-\frac{v}{2}\right)\right) \tag{21}$$
$$= \Pr\left(U \le 1-\exp\left(-\frac{v}{2}\right)\right) \tag{22}$$
$$= F_U\left(1-\exp\left(-\frac{v}{2}\right)\right) \tag{23}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{24}$$

Substituting the above in (23),

$$F_U\left(1-\exp\left(-\frac{v}{2}\right)\right) = \begin{cases} 0 & 1-\exp\left(-\frac{v}{2}\right) < 0 \\ 1-\exp\left(-\frac{v}{2}\right) & 0 \le 1-\exp\left(-\frac{v}{2}\right) \le 1 \\ 1 & 1-\exp\left(-\frac{v}{2}\right) > 1 \end{cases} \tag{25}$$

After some algebra, the above conditions yield

$$F_V(v) = \begin{cases} 0 & v < 0 \\ 1-exp\left(-\frac{v}{2}\right) & v \ge 0 \end{cases} \tag{26}$$

which is the CDF of the exponential distribution with parameter $\frac{1}{2}$.

2.4 Triangular Distribution

2.1 Generate

$$Z = U_1 + U_2 \tag{27}$$

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-5/5.4/uniform_two.c
```

2.2 Find the CDF of T.

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-5/5.4/uniform_two.c
```

2.3 Find the PDF of T.

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-5/5.4/uniform_two.c
```

2.4 Find the theoretical expressions for the PDF and CDF of T.

$$f(z) = \begin{cases} z & 0 < z < 1 \\ 2-z & 1 \le z < 2 \\ 0 & otherwise \end{cases} \tag{28}$$

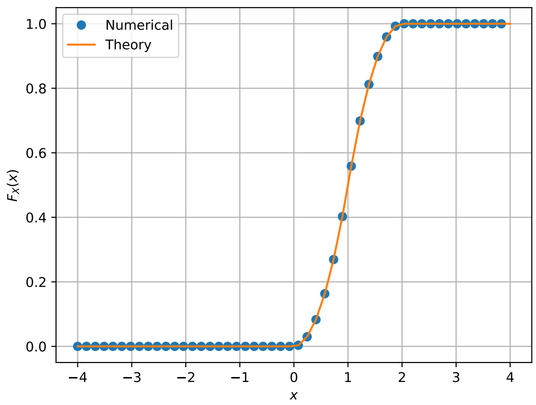

Figure 6: CDF of Z

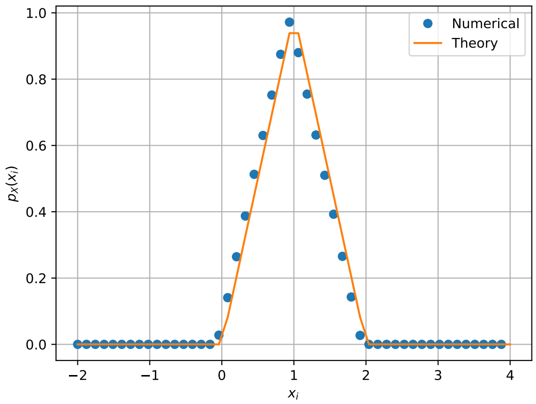

Figure 7: PDF of Z

$$F_Z(z) = \begin{cases} \frac{z^2}{2} & 0 < z < 1 \\ 2z - z^2 - 1 & 1 \le z < 2 \\ 1 & z > 0 \end{cases} \tag{29}$$

2.5 Verify your results through a plot.

3 transfromations of R.V

3.1 Gaussian to Other

1. Let $X_1 \sim (0,1)$ and $X_2 \sim (0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{30}$$

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-7/7.1.1/7.1.1_CDF.py

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-7/7.1.1/7.1.1_PDF.py

2. If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \ge 0 \\ 0 & x < 0, \end{cases} \tag{31}$$

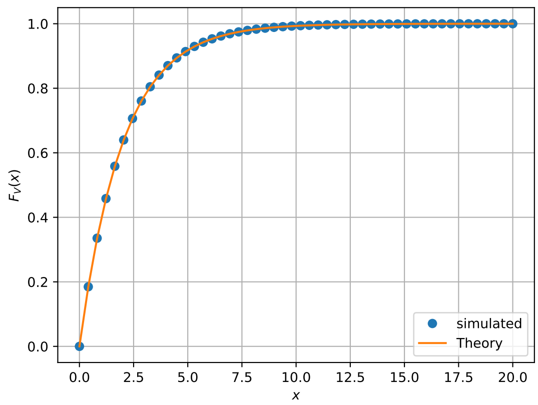

Figure 8: CDF of V

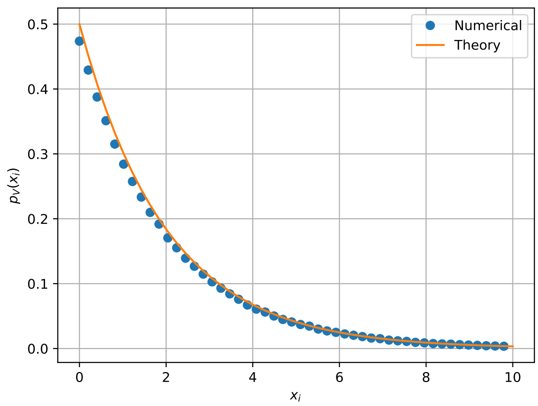

Figure 9: PDF of V

find α.

Solution: For the value $\alpha = 0.5$, the theory matches the simulation. The following code generates the CDF of V in Fig. 10

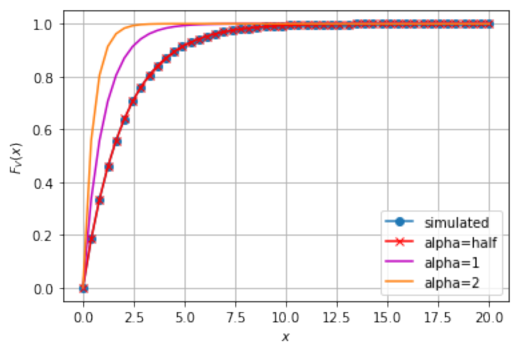

Figure 10: CDF of V

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-7/7.1.1/7.1.1_CDF.py

3. Plot the CDF and PDF of

$$A = \sqrt{V} \tag{32}$$

$$
\begin{aligned}
F_A(a) &= \Pr A < a \\
&= \Pr \sqrt{V} < a \\
&= \Pr \left(V < a^2\right) \\
&= F_V \left(a^2\right) \\
&= 1 - \exp\left(-\frac{a^2}{2}\right) \\
f(a) &= a \exp\left(-\frac{a^2}{2}\right)
\end{aligned}
$$

The CDF and PDF of A are plotted in Figs. 11 and 12 using the codes below.

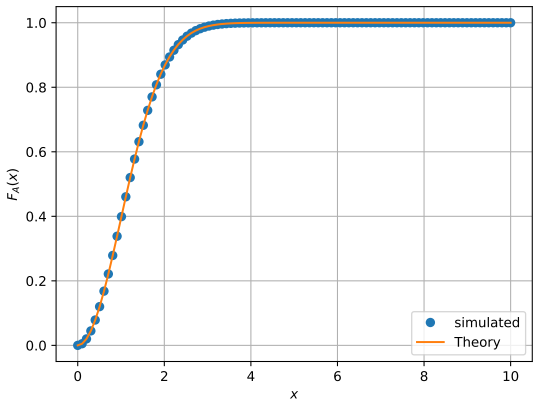

Figure 11: CDF of A

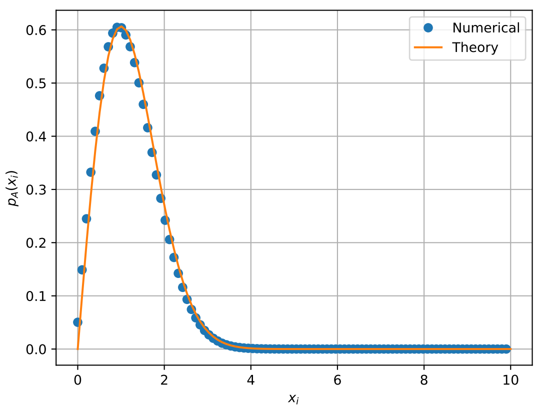

Figure 12: PDF of A

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-7/7.1.3/7.1.3_CDF.py
```

```
https://github.com/velicharlagokulkumar/
digital-communications/blob/main/codes/
chapter-7/7.1.3/7.1.3_PDF.py
```

3.2 Conditional Probability

1. Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{33}$$

for

$$Y = AX + N, \tag{34}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1) \in (-1,1)$ for $0 \le \gamma \le 10$ dB.

2. Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$.
Solution: The estimated value $\hat{X}$ is given by

$$\hat{X} = \begin{cases} +1 & Y > 0 \\ -1 & Y < 0 \end{cases} \tag{35}$$

For $X = 1$,

$$
\begin{aligned}
Y &= A + N && (36) \\
P_e &= \Pr\left(\hat{X} = -1 | X = 10\right) && (37) \\
&= \Pr\left(Y < 0 | X = 1\right) && (38) \\
&= \Pr\left(A < -N)\right) && (39) \\
&= F_A(-N) && (40) \\
&= \int_{-\infty}^{-N} f_A(x) dx && (41)
\end{aligned}
$$

By definition

$$f_A(x) = \begin{cases} \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) & x \ge 0 \\ 0 & otherwise \end{cases} \tag{42}$$

If $N > 0, f_A(x) = 0$. Then,

$$P_e = 0 \tag{43}$$

If $N < 0$. Then,

$$
\begin{aligned}
P_e(N) &= \int_{-\infty}^{-N} f_A(x) dx && (44) \\
&= \int_{-\infty}^{0} 0 dx + \int_{0}^{-N} f_A(x) dx && (45) \\
&= \int_{0}^{-N} \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx && (46) \\
&= 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) && (47)
\end{aligned}
$$

Therefore,

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{2\sigma^2}\right) & N < 0 \\ 0 & otherwise \end{cases} \tag{48}$$

3. For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{49}$$

Find $P_e = E\left[P_e(N)\right]$.
Solution: Since $N \sim \mathcal{N}(0,1)$,

$$p_N(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{50}$$

$$\tag{51}$$

And from (48)

$$P_e(x) = \begin{cases} 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) & x < 0 \\ 0 & otherwise \end{cases} \tag{52}$$

$$P_e = E\left[P_e(N)\right] = \int_{-\infty}^{\infty} P_e(x) p_N(x)\, dx \tag{53}$$

If $x < 0, P_e(x) = 0$ and using the fact that for an even function

$$\int_{-\infty}^{\infty} f(x) = 2\int_{-\infty}^{0} f(x) \tag{54}$$

we get

$$P_e = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(-\frac{x^2}{2}\right)\left(1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)\right) dx \tag{55}$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx - \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(1+\sigma^2)x^2}{2\sigma^2}\right) dx \tag{56}$$

$$= \frac{\sqrt{2\pi} - \sqrt{\frac{\pi(2\sigma^2)}{1+\sigma^2}}}{2\sqrt{2\pi}} \tag{57}$$

$$= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\sigma^2}{1+\sigma^2}} \tag{58}$$

For a Rayleigh Distribution with scale $= \sigma$,

$$E\left[A^2\right] = 2\sigma^2 \tag{59}$$

$$\gamma = 2\sigma^2 \tag{60}$$

$$\therefore P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{2+\gamma}} \tag{61}$$

4. Plot P_e in problems 1 and 3 on the same graph w.r.t γ. Comment.

https://github.com/velicharlagokulkumar/digital-communications/blob/main/codes/chapter-7/7.2/7.2.py