Topic 1: CAPM to FF3

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CAPM

- Sharpe-Lintner CAPM:

$$E[R_i] = R_f + \beta_i\left(E[R_m] - R_f\right)$$

- Black (zero-beta) CAPM:

$$E[R_i] = E[R_{ZB}] + \beta_i(E[R_m] - E[R_{ZB}])$$

- CAPM is a single period model for expected returns. Its implications are that:
  - The intercept is zero
  - Beta fully captures cross-sectional variation in expected returns

- Testing CAPM $\Leftrightarrow$ checking whether the market portfolio is mean-variance efficient

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Testing Sharpe-Lintner CAPM

- Testable equation is:
$$R_{i,t} - r_{f,t} = \alpha_i + \beta_i(R_{m,t} - r_{f,t}) + \varepsilon_{i,t}$$
  - $R_{i,t}$ is the return on asset i at time t
  - $r_{f,t}$ is the risk-free rate at time t
  - $R_{m,t}$ is the return on the "market" portfolio at time t
    - $E[(R_{m,t} - r_{f,t})]$ is the market risk premium

- Standard assumptions for testing CAPM
  - Rational expectations for $R_{i,t}$, $R_{m,t}$, $Z_{i,t}$(any other variable)
    - Ex-ante ~ ex-post (i.e., "realized" proxy for "expected")
    - $R_{i,t} = E[R_{i,t}] + \varepsilon_t$, where ε_t is white noise
  - Constant beta
  - Holding period is known, usually a month

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Testing CAPM: Early tests overview

- Early tests focused on the Security Market Line (SML):

$$E[R_i] = \gamma + \beta_i \lambda + \epsilon_i \text{ (SML)}$$

  - $H_0: \lambda > 0$
  - Examples: Black, Jensen, Scholes (1972), Fama and MacBeth (1973), Blume and Friend (1973)

- Issues
  - We don't know β_i, so we need to estimate it. Anytime we use an estimate, we have measurement error that may lead to attenuation bias
  - We need to worry about the value of γ too

- Findings: mostly positive λ, though γ was too high

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Early tests: Two-pass technique

- First pass: time-series regression of security returns on a market index:

$$R_{i,t} - r_{f,t} = \alpha_i + \beta_i\left(R_{m,t} - r_{f,t}\right) + \varepsilon_{i,t}$$

- Second pass: cross-sectional estimation where the estimated CAPM beta from the first pass is related to average return:

$$R_i = \gamma + \lambda\beta_i + \epsilon_i \quad \text{(SML for security } i\text{)}$$

  - γ is the r_f in the Sharpe-Lintner CAPM and $E[R_{0m}]$ in the Black CAPM

- Measurement error in β_i is a big issue
  - Use portfolios where due to aggregation it should be less of an issue

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Early tests: Black-Jensen-Scholes (1972)

- Data: 1926 – 1965 NYSE stocks
- Use equal-weighted NYSE stock index
- Test procedure
  1. Start with monthly returns for 1926-1930 (60 months). Estimate a beta for each stock available at end of 1930 with at least 24 monthly observations in this period
  2. Rank securities by beta, group into 10 portfolios
  3. Calculate monthly equal-weighted returns for each portfolio during 1931
  4. Move to next period (1927-1931), go back to step 1.
     - Use rolling regressions
  5. When done with the entire sample:
     a) Calculate mean portfolio excess returns for each portfolio
     b) Calculate betas for each portfolio
  6. Plot average returns against betas

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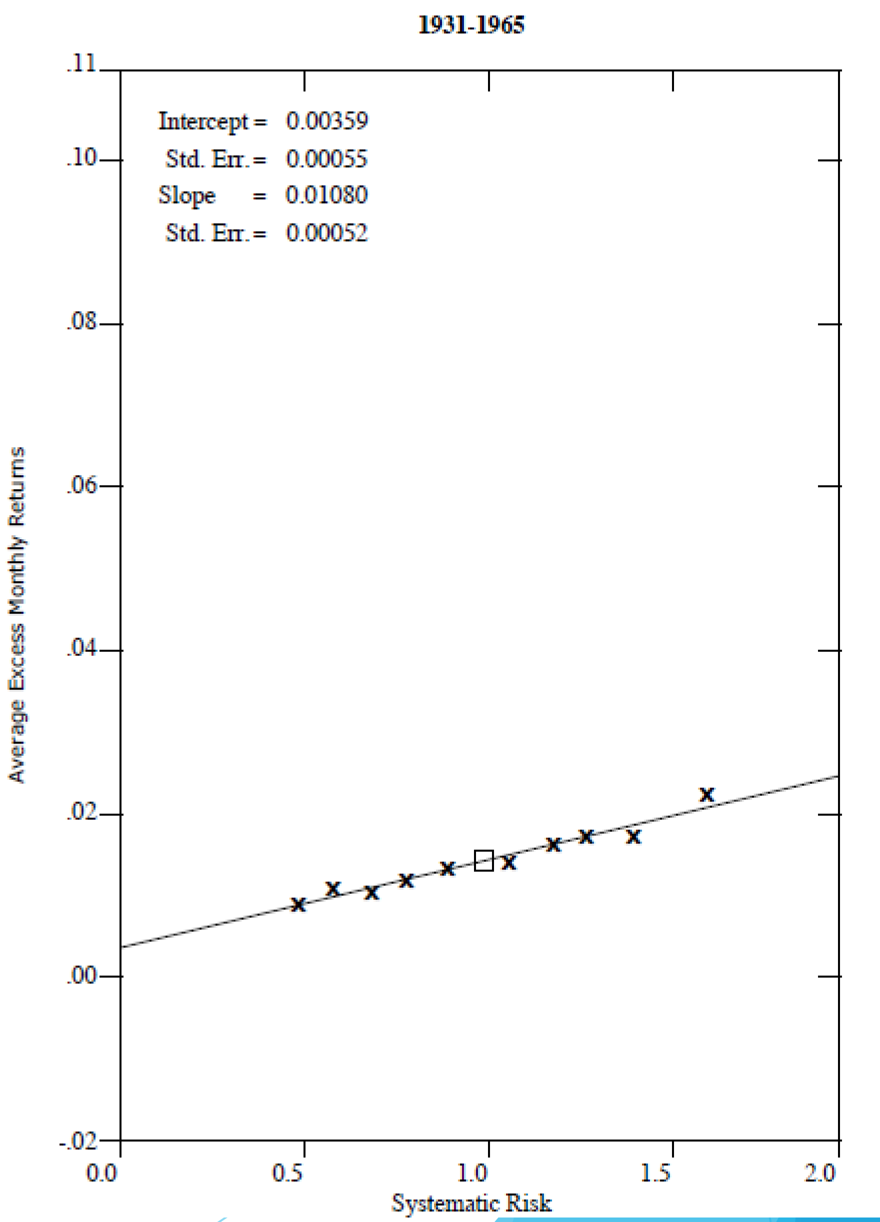
Early tests: Black-Jensen-Scholes (1972)

- Positive and significant slope for the SML.
  - Market risk-premium smaller than predicted
  - Risk-free rate too high



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Early tests: Fama and MacBeth (1973)

- Data: 1926 – 1968 NYSE stocks, NYSE EW index
- Test procedure

1. Divide sample into 9 subsets:

	1	2	3	4	5	6	7	8	9
Portfolio formation period	1926-1929	1927-1933	1931-1937	1935-1941	1939-1945	1943-1949	1947-1953	1951-1957	1955-1961
Initial estimation period	1930-1934	1934-1938	1938-1942	1942-1946	1946-1950	1950-1954	1954-1958	1958-1962	1962-1966
Testing period	1935-1938	1939-1942	1943-1946	1947-1950	1951-1954	1955-1958	1959-1962	1963-1966	1967-1968

2. Rank stocks by portfolio estimation period beta (e.g., 1926-29) and form 20 portfolios
3. Recalculate betas in the initial estimation period (e.g., 1930-34), average betas and returns across the 20 portfolios based on rankings in 2.
4. Update betas each year in testing period (e.g., 1935-38) & get 48 SMLs:

$$R_{pt} = \hat{\gamma}_{0t} + \hat{\gamma}_{1t}\hat{\beta}_{p,t-1} + \hat{\eta}_{pt}$$

5. Repeat above steps for all subsets & get 390 estimated SMLs
6. Do t-tests for $\hat{\gamma}_{0t}$ and $\hat{\gamma}_{1t}$

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Early tests: Fama and MacBeth (1973)

- Findings: positive for CAPM
  - Significant $\hat{\gamma}_{1t}$, although $\hat{\gamma}_{0t}$ again too high
  - No evidence of misspecification

TABLE 3

SUMMARY RESULTS FOR THE REGRESSION

$$R_p = \hat{\gamma}_{0t} + \hat{\gamma}_{1t}\hat{\beta}_p + \hat{\gamma}_{2t}\hat{\beta}^2{}_p + \hat{\gamma}_{3t}\bar{s}_p(\hat{\epsilon}_i) + \hat{\eta}_{pt}$$

Period	$\bar{\hat{\gamma}}_0$	$\bar{\hat{\gamma}}_1$	$\bar{\hat{\gamma}}_2$	$\bar{\hat{\gamma}}_3$	$\overline{\hat{\gamma}_0 - R_f}$	$s(\hat{\gamma}_0)$	$s(\hat{\gamma}_1)$	$s(\hat{\gamma}_2)$	$s(\hat{\gamma}_3)$	$\rho_0(\hat{\gamma}_0 - R_f)$	$\rho_M(\hat{\gamma}_1)$	$\rho_0(\hat{\gamma}_2)$	$\rho_0(\hat{\gamma}_3)$	$t(\bar{\hat{\gamma}}_0)$	$t(\bar{\hat{\gamma}}_1)$	$t(\bar{\hat{\gamma}}_2)$	$t(\bar{\hat{\gamma}}_3)$	$t(\overline{\hat{\gamma}_0 - R_f})$	$\bar{r}^2$	$s(r^2)$
Panel A:																				
1935–6/68 ..	.0061	.0085	...	...	.0048	.038	.066	...	...	.15	.02	...	...	3.24	2.57	...	...	2.55	.29	.30
1935–45	.0039	.0163	...	...	.0037	.052	.098	...	...	.10	—.03	...	...	.86	1.92	...	...	.82	.29	.29
1946–55	.0087	.0027	...	...	.0078	.026	.041	...	...	.18	.07	...	...	3.71	.70	...	...	3.31	.31	.32
1956–6/68 ..	.0060	.0062	...	...	.0034	.030	.044	...	...	.27	.15	...	...	2.45	1.73	...	...	1.39	.28	.29
1935–40	.0024	.0109	...	...	.0023	.064	.116	...	...	.07	—.09	...	...	.32	.79	...	...	.31	.23	.30
1941–45	.0056	.0229	...	...	.0054	.034	.069	...	...	.23	.15	...	...	1.27	2.55	...	...	1.22	.37	.28
1946–50	.0050	.0029	...	...	.0044	.031	.047	...	...	.20	.04	...	...	1.27	.48	...	...	1.10	.39	.33
1951–55	.0123	.0024	...	...	.0111	.019	.035	...	...	.20	.08	...	...	5.06	.53	...	...	4.56	.24	.29
1956–60	.0148	—.0059	...	...	.0128	.020	.034	...	...	.37	.18	...	...	5.68	—1.37	...	...	4.89	.22	.31
1961–6/68 ..	.0001	.0143	...	...	—.0029	.034	.048	...	...	.22	.09	...	...	.03	2.81	...	...	—.80	.32	.27

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Fama-MacBeth more generally

- Widely adopted tool in empirical asset pricing
- We can use it both to test new factors (traded or non-traded) and characteristics
  - More on that later

General issues with CAPM testing

- Measurement error: each beta is estimated with error. If the estimation errors are uncorrelated across stocks, a portfolio reduces estimation error and improves second-pass regression. The estimators are biased, but consistent.

- Sorting: random portfolios have a beta close to 1. The sorting preserves some cross-sectional variation for the second pass.

- Rolling regressions: to reduce bias in estimation error, estimate a lot of betas

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General issues with CAPM testing

- Inference issues
  - Standard errors to be adjusted for estimation bias (Shanken, 1992)
  - Simultaneous estimation (pass 1 and 2) (Cochrane, 2001)
  - Serial correlation in residuals: Use Newey-West SE

- Roll (1977) critique: only testable implication of CAPM is whether R_m is mean-variance efficient
  - Proxy problem: “true” market portfolio is unobservable
  - Any *ex-post* MVE portfolio used as the index will exactly satisfy the SML by the sample β time-series mean returns (simple algebra)

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General issues with CAPM testing

- Counter points to Roll (1977)
  - Stambaugh (1982): adds bonds and real estate to stock index
  - Shanken (1987): if correlations between observable stock index and true global index exceeds .7, CAPM is rejected
  - Roll and Ross (1994): even when proxy is not far from frontier, CAPM can be rejected

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Testing CAPM: Later tests overview

- Later tests focused on the time-series behavior

$$R_{i,t} - r_{f,t} = \alpha_i + \beta_i\left(R_{m,t} - r_{f,t}\right) + \varepsilon_{i,t}$$

- Two popular approaches
  - Test $H_0: \alpha_i = 0$ (α_i is the pricing error; Jensen's alpha)
    - More efficient to run joint tests ($H_0: \alpha_1 = \alpha_2 = \cdots = \alpha_N = 0$)
  - Test CAPM's specification
    - Add more explanatory variables $\boldsymbol{Z_{i,t}}$ on the right-hand-side (i.e., test $H_0: \delta_i = 0$ in):

$$R_{i,t} - r_{f,t} = \alpha_i + \beta_i\left(R_{m,t} - r_{f,t}\right) + \boldsymbol{\delta_i Z_{i,t}} + \varepsilon_{i,t}$$

- Findings: negative for CAPM ($\alpha_i \neq 0, \delta_i \neq 0$)

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Time-series tests

- Sharpe-Lintner CAPM:
  $$R_{i,t} - r_{f,t} = \alpha_i + \beta_i\left(R_{m,t} - r_{f,t}\right) + \varepsilon_{i,t} \; (+\delta_i X_{i,t})$$

- Two hypothesis:
  - Test $H_0: \alpha_i = 0$ (α_i is the pricing error; Jensen's alpha)
    - Assume $R_{i,t}$~i.i.d. normal
      - Estimate by Maximum Likelihood (ML), which is equivalent to OLS
      - Wald test (Gibbons, Ross, Shanken, 1989), LR test (Jobson and Korkie, 1982), or LM test
    - Test by GMM
      - Non-normality
      - Heteroskedasticity
      - Autocorrelation in returns
  - Test $H_0: \delta_i = 0$
    - Test of CAPM's specification

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Time-series tests: ML approach

- We assume returns $R_{i,t}$~i.i.d. normal
- α_i and β_i are security-specific parameters.
- We can use Seemingly Unrelated Regressions (SUR) or estimate the individual regressions
- The distribution of α_i and β_i is given by (see derivation in CLM, Section 5.3.1):

$$\hat{\alpha} \sim N\left(\alpha, \frac{1}{T}\left[1 + \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}\right]\Sigma\right)$$

$$\hat{\beta} \sim N\left(\beta, \frac{1}{T}\left[1 + \frac{1}{\hat{\sigma}_m^2}\right]\Sigma\right)$$

$$\mathrm{T}\hat{\Sigma} \sim W_N(T - 2, \Sigma)$$

$$\text{where } \hat{\sigma}_m^2 = \frac{1}{T}\Sigma_{t=1}^{T}\left(R_{m,t} - \hat{\mu}_m\right)^2$$

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Time-series tests: Gibbons, Ross, Shanken (1989)

- For N securities, we can conduct a Wald test (using the unconstrained estimation), a Lagrange multiplier test (using the constrained estimation), or a likelihood ratio test (using both estimations)

- Gibbons, Ross, Shanken (1989) do the Wald test, but they work out the finite-sample distribution:

$$J_1 = \left(\frac{T-N-1}{N}\right)\left[1+\left(\frac{\hat{\mu}_m}{\hat{\sigma}_m}\right)^2\right]^{-1} \hat{\alpha}\hat{\Sigma}^{-1}\hat{\alpha} \;\sim F(N, T-N-1)$$

  - $\hat{\alpha}\hat{\Sigma}^{-1}\hat{\alpha} = \left(\frac{\hat{\mu}_q}{\hat{\sigma}_q}\right)^2 - \left(\frac{\hat{\mu}_m}{\hat{\sigma}_m}\right)^2$ - Squared Sharpe ratio difference between the ex-post tangency portfolio (mkt + test ptfs) and the market portfolio

- Leads to nice interpretation, i.e., how far inside the ex-post frontier the market

$$J_1 = \left(\frac{T-N-1}{N}\right)\frac{\left(\frac{\hat{\mu}_q}{\hat{\sigma}_q}\right)^2 - \left(\frac{\hat{\mu}_m}{\hat{\sigma}_m}\right)^2}{1+\left(\frac{\hat{\mu}_m}{\hat{\sigma}_m}\right)^2}$$

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Gibbons, Ross, and Shanken (1989)

- To see the equivalence, suppose we have N test assets and the market. Then,
  - The expected return of the $N+1$ assets is $\underbrace{\hat{\lambda}}_{(N+1)\times 1} = \begin{bmatrix} \underbrace{\widehat{\mu_M^e}}_{1\times 1} & \underbrace{\widehat{\boldsymbol{\mu}^e}}_{N\times 1} \end{bmatrix}$
  - The variance-covariance matrix of the $N+1$ assets is:

$$\underbrace{\widehat{\Omega}}_{(N+1)\times(N+1)} = \begin{bmatrix} \hat{\sigma}_M^2 & \hat{\beta}'\hat{\sigma}_M^2 \\ \hat{\beta}\,\hat{\sigma}_M^2 & \hat{V} \end{bmatrix}$$

where $\hat{\sigma}_M^2$ is the estimated variance of the market factors, $\hat{\beta}$ is $N\times 1$ vector of market loadings, and $\hat{V}$ is the covariance matrix of the N test assets

- However, $\hat{V} = \hat{\beta}\hat{\beta}'\hat{\sigma}_M^2 + \hat{\Sigma}$, where $\hat{\Sigma}$ is covariance matrix of regression residuals
- The squared slope of the tangency portfolio (Sharpe ratio) of the N + 1 assets is:

$$\widehat{SR}_{TP}^2 = \hat{\lambda}'\widehat{\Omega}^{-1}\hat{\lambda}$$

- The inverse of the covariance matrix is (check $\widehat{\Omega}\widehat{\Omega}^{-1} = I_{(N+1)\times(N+1)}$):

$$\widehat{\Omega}^{-1} = \begin{bmatrix} \left(\hat{\sigma}_M^2\right)^{-1} + \hat{\beta}'\hat{\Sigma}^{-1}\hat{\beta} & -\hat{\beta}'\hat{\Sigma}^{-1} \\ -\hat{\beta}'\hat{\Sigma}^{-1} & \hat{\Sigma}^{-1} \end{bmatrix}$$

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Gibbons, Ross, Shanken (1989)

- Thus, the squared Sharpe ratio of the tangency portfolio could be represented as

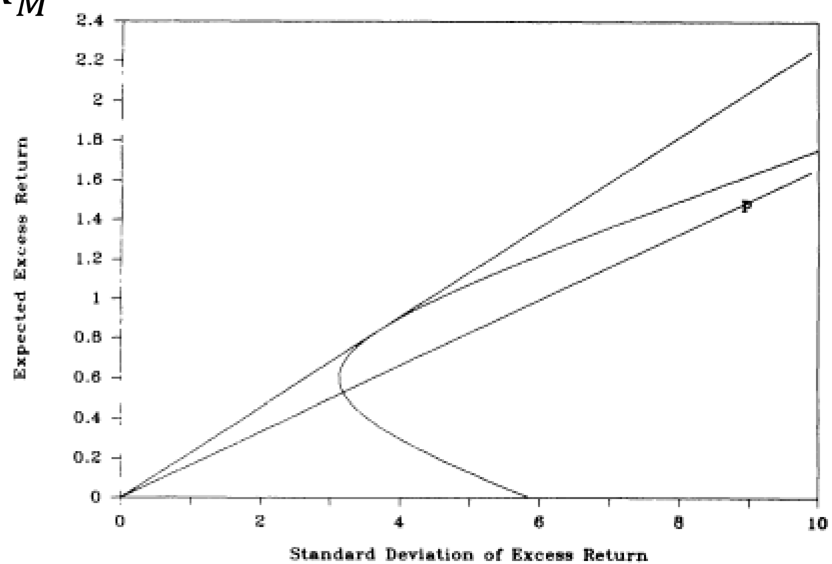
$$\widehat{SR}_{TP}^2 = \left(\frac{\hat{\mu}_M^e}{\hat{\sigma}_M}\right)^2 + [\hat{\mu}^e - \hat{\beta}\hat{\mu}_M^e]'\hat{\Sigma}^{-1}[\hat{\mu}^e - \hat{\beta}\hat{\mu}_M^e]$$

$$\Rightarrow \widehat{SR}_{TP}^2 = \widehat{SR}_M^2 + \hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha}$$

$$\Rightarrow \hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha} = \widehat{SR}_{TP}^2 - \widehat{SR}_M^2$$

- That is, $\hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha}$ is the difference between the squared Sharpe ratio based on the $N+1$ assets and the squared Sharpe ratio of the market portfolio

- Under the null (CAPM), the extra N assets do not add anything to improving the risk-return tradeoff



1b.) *Ex post* efficient frontier based on 10 beta-sorted portfolios and the CRSP Equal-Weighted Index using monthly data, 1931-1965. Point p represents the CRSP Equal-Weighted Index.

FIGURE 1.—Various plots of ex post mean variance efficient frontiers.

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More modern tests: Findings

- Negative for CAPM
- CLM Table 5.3:
  - Data: US, 1965-1994, monthly returns of 10 size portfolios, VW-CRSP market index
  - Joint test rejects CAPM, esp. in the earlier part of the sample period
  - GRS test stat, J_1, has a 0.020 p-value for their full sample

Time	$_1$	p-value	J_2	p-value	J_3	p-value	J_7	p-value
Five-year subperiods								
1/65–12/69	2.038	0.049	20.867	0.022	18.432	0.048	22.105	0.015
1/70–12/74	2.136	0.039	21.712	0.017	19.179	0.038	21.397	0.018
1/75–12/79	1.914	0.066	19.784	0.031	17.476	0.064	27.922	0.002
1/80–12/84	1.224	0.300	13.378	0.203	11.818	0.297	13.066	0.220
1/85–12/89	1.732	0.100	18.164	0.052	16.045	0.098	16.915	0.076
1/90–12/94	1.153	0.344	12.680	0.242	11.200	0.342	12.379	0.260
Overall	77.224	0.004	106.586	**	94.151	0.003	113.785	**
Ten-year subperiods								
1/65–12/74	2.400	0.013	23.883	0.008	22.490	0.013	24.649	0.006
1/75–12/84	2.248	0.020	22.503	0.013	21.190	0.020	27.192	0.002
1/85–12/94	1.900	0.053	19.281	0.037	18.157	0.052	16.373	0.089
Overall	57.690	0.001	65.667	**	61.837	0.001	68.215	**
Thirty-year period								
1/65–12/94	2.159	0.020	21.612	0.017	21.192	0.020	22.176	0.014

**Less than 0.0005.

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CAPM theoretical extensions

- Black (1972): no risk-free asset - zero-beta CAPM
- Mayers (1972): non-traded assets – human capital
- Merton (1973): intertemporal CAPM - factors
- Brennan (1970), Litzenberger and Ramaswamy (1979): dividends, taxes
- Solnik (1974): foreign exchange risk:
- Long (1974), Friend, Landskroner and Losq (1976): inflation
- Sercu (1980), Stulz (1981), Adler and Dumas (1983): international CAPM, PPP risk
- Stulz (1983): investment restrictions

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Early CAPM “anomalies”

- Monday dummy (-): French (1980)
- January dummy (+): Roll (1983), Reinganum (1983)
- E/P (+): Basu (1977), Ball (1978), Jaffe, Keim and Westerfield (1989)
- Firm Size (-): Banz (1981), Basu (1983)
- Long-Term Reversals (-): DeBondt and Thaler (1985)
- B/M (+): Stattman (1980), Rosenberg, Reid and Lanstein (1985)
- D/E (Leverage) (+): Bhandari (1988)
- Momentum (+): Jegadeesh (1990), Jegadeesh and Titman (1993)

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Fama and French (1992): “Beta is dead”

- Evaluate joint roles of market beta, size, E/P, leverage, and BE/ME in explaining cross-sectional variation in US stock returns

- Takeaway:

In a nutshell, market β seems to have no role in explaining the average returns on NYSE, AMEX, and NASDAQ stocks for 1963-1990, while size and book-to-market equity capture the cross-sectional variation in average stock returns that is related to leverage and E/P.

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Fama and French (1992): Data & Methodology

- All non-financial firms in NYSE, AMEX, and (after 1972) NASDAQ in 1963-1990
- 'Anomaly' variables:
  - Size: ln(ME), Book-to-market: ln(BE/ME), Leverage: ln(A/ME) or ln(A/BE), Earnings-to-price: E/P

- Monthly Fama-MacBeth cross-sectional regressions of individual stock excess returns on beta, size, etc.
  - Post-ranking portfolio betas assigned to each stock
    - In June of each year
      - Do a conditional 10x10 sort on size and beta with NYSE breakpoints
      - Measure post-ranking monthly returns for next year
      - Use full-sample betas of the 100 size-beta EW portfolios

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Fama and French (1992): Results

- Even alone, beta fails to explain returns
- Size has a reliable negative relation with returns
- Book-to-market has an even stronger (positive) relation
- Market (+) and book (-) leverage have significant, but opposite effect on returns
  - Manifestations of size & book-to-market effects
- Earnings-to-price: position effect, but significance subsumed by B/M

Table III

Average Slopes (t-Statistics) from Month-by-Month Regressions of Stock Returns on β, Size, Book-to-Market Equity, Leverage, and E/P: July 1963 to December 1990

Stocks are assigned the post-ranking β of the size-β portfolio they are in at the end of June of year t (Table I). BE is the book value of common equity plus balance-sheet deferred taxes, A is total book assets, and E is earnings (income before extraordinary items, plus income-statement deferred taxes, minus preferred dividends). BE, A, and E are for each firm's latest fiscal year ending in calendar year $t-1$. The accounting ratios are measured using market equity ME in December of year $t-1$. Firm size ln(ME) is measured in June of year t. In the regressions, these values of the explanatory variables for individual stocks are matched with CRSP returns for the months from July of year t to June of year $t+1$. The gap between the accounting data and the returns ensures that the accounting data are available prior to the returns. If earnings are positive, E(+)/P is the ratio of total earnings to market equity and E/P dummy is 0. If earnings are negative, E(+)/P is 0 and E/P dummy is 1.

The average slope is the time-series average of the monthly regression slopes for July 1963 to December 1990, and the t-statistic is the average slope divided by its time-series standard error.

On average, there are 2267 stocks in the monthly regressions. To avoid giving extreme observations heavy weight in the regressions, the smallest and largest 0.5% of the observations on E(+)/P, BE/ME, A/ME, and A/BE are set equal to the next largest or smallest values of the ratios (the 0.005 and 0.995 fractiles). This has no effect on inferences.

β	ln(ME)	ln(BE/ME)	ln(A/ME)	ln(A/BE)	E/P Dummy	E(+)/P
0.15 (0.46)						
	−0.15 (−2.58)					
−0.37 (−1.21)	−0.17 (−3.41)					
		0.50 (5.71)				
			0.50 (5.69)	−0.57 (−5.34)		
					0.57 (2.28)	4.72 (4.57)
	−0.11 (−1.99)	0.35 (4.44)				
	−0.11 (−2.06)		0.35 (4.32)	−0.50 (−4.56)		
	−0.16 (−3.06)				0.06 (0.38)	2.99 (3.04)
	−0.13 (−2.47)	0.33 (4.46)			−0.14 (−0.90)	0.87 (1.23)
	−0.13 (−2.47)		0.32 (4.28)	−0.46 (−4.45)	−0.08 (−0.56)	1.15 (1.57)

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Fama and French (1992): Conclusion

- “Beta is dead”: no relation between beta and average returns in 1963-1990
  - Noisy beta estimates?
    - Post-ranking betas have low s.e. (most below 0.05)
    - Close correspondence between pre- and post-ranking betas for the beta-sorted portfolios
    - Same results if use 5y pre-ranking or 5y post-ranking betas
- Robustness:
  - Similar results in subsamples & for NYSE stocks in 1941-1990
- Suggest a new model for average returns, with size and book-to-market equity
  - This combination explains well CS variation in returns and absorbs other anomalies

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Multifactor models: Theoretical basis

- Arbitrage pricing theory (APT) of Ross (1976)
  - K-factor return-generating model:
  $$R_i = a_i + \boldsymbol{b_i f} + \epsilon_i$$
  - Ross (1976) shows that in the absence of arbitrage:
  $$\mu \approx \iota\lambda_0 + \boldsymbol{B\lambda_K}$$

- Intertemporal Capital Asset Pricing Model (ICAPM) of Merton (1973)
  - Market + state variables describing shifts in the mean-variance frontier

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Summary of testing factor asset pricing models

- Two general approaches
  - Time-series regressions
    - Factors are traded portfolios (e.g., HML)
  - Cross-sectional regressions
    - Non-traded factors (e.g., Inflation)

- People rarely test for consistency between the two, although recommended (Lewellen, Shanken, and Nagel, 2010)

- Both can be combined in a unified GMM framework (Cochrane, 2005)

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Time-series approach: Overview

- Used when factors are traded portfolios (e.g., CAPM, Fama and French, 1993)

- We estimate unconstrained time-series regressions:

$$R_t = \alpha + BF_t + \epsilon_t$$

  - R_t is an $N \times 1$ vector of excess returns
  - F_t is an K × 1 vector of factors, assumed to be excess returns of zero-cost portfolios

- We can use SUR estimation to allow for cross-correlation of the residuals

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Time-series approach: Estimates

- We estimate unconstrained time-series regressions:

$$R_t = \alpha + BF_t + \epsilon_t$$

- Estimates & standard errors given by:

$$\hat{\alpha} = \hat{\mu} - \hat{B}\hat{\mu}_F,$$

$$\hat{\alpha} \sim N\left[\alpha, \frac{1}{T}\left(1 + \mu_f' \Sigma_f^{-1} \mu_F\right)\Sigma_\epsilon\right]$$

$$\hat{B} = \left[\sum_{t=1}^{T}(R_t - \hat{\mu})\,(F_t - \hat{\mu}_F)'\right]\left[\sum_{t=1}^{T}(F_t - \hat{\mu}_F)\,(F_t - \hat{\mu}_F)'\right]^{-1},$$

$$\hat{B} \sim N\left[B, \frac{1}{T}\Sigma_F^{-1} \otimes \Sigma_\epsilon\right]$$

$$\hat{\Sigma}_\epsilon = \frac{1}{T}\sum_{t=1}^{T}(R_t - \hat{\alpha} - \hat{B}F_t)(R_t - \hat{\alpha} - \hat{B}F_t)$$

$$T\hat{\Sigma}_\epsilon \sim W(T-2, \Sigma_\epsilon)$$

- where $\hat{\mu} = \frac{1}{T}\Sigma_t R_t$, $\hat{\mu}_F = \frac{1}{T}\Sigma_t F_t$, $\hat{\Sigma}_F = \frac{1}{T-K}\Sigma_t (F_t - \hat{\mu}_F)(F_t - \hat{\mu}_F)'$

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Time-series approach: Risk premium & pricing errors

- In the TSR approach, factor risk premium is simply the factor mean:

$$\hat{\lambda} = \frac{1}{T}\sum_{t=1}^{T} F_t \sim N\left[\lambda, \frac{1}{T}\Sigma_F\right]$$

- We can test jointly whether $\lambda = 0$, assuming normality of factor returns:

$$J_3 = \left(\frac{T-K}{TK}\right)\hat{\lambda} var\left(\hat{\lambda}\right)^{-1}\hat{\lambda} \sim F(K, T-K)$$

- Pricing errors are given by the $\hat{\alpha}$'s, leading to the GRS test:

$$J_1 = \left(\frac{T-N-K}{N}\right)\left[1 + \hat{\mu}_F'\hat{\Sigma}_F^{-1}\hat{\mu}_F\ \right]^{-1}\hat{\alpha}\hat{\Sigma}_\epsilon{}^{-1}\hat{\alpha}$$

$$\sim F(N, T-N-K)$$

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Cross-sectional approach: Overview

- Used when factors are non-traded portfolios (e.g., inflation)

- We estimate a two-pass regression:
  - In first stage we estimate betas:
  $$R_t = \alpha + BF_t + \epsilon_t$$
  - In second stage we estimate a cross-sectional regression:
  $$\bar{R}_T = \hat{B}\lambda + a$$
    - where $\bar{R}_T$ is the average sample return

- In second stage, now $\hat{B}$ are the independent variables & λ are the coefficient estimates

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Cross-sectional approach: Equations

- Estimates & standard errors given by (using OLS):

$$\hat{\lambda} = \left(\hat{B}'\hat{B}\right)^{-1}\hat{B}'\bar{R}_T,$$
$$var\left(\hat{\lambda}\right) = (B'B)^{-1}(B'\Sigma_a B)(B'B)^{-1}$$
$$\hat{a} = \bar{R}_T - \hat{B}\hat{\lambda}$$
$$var(\hat{a}) = \left(I_N - B(B'B)^{-1}B\right)\Sigma_a\left(I_N - B(B'B)^{-1}B\right)'$$

- Note: We estimate Σ_a $\left(= E(aa')\right)$ with its sample covariance matrix of residuals $\frac{1}{T}\Sigma_u$, but since $u = R - \hat{B}\hat{\lambda}$ and $\hat{B}\hat{\lambda}$ being constant across time, boils down to $\frac{1}{T}\Sigma_R$

- GLS estimation gives you more efficient estimates

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Cross-sectional approach: Price of risk & pricing errors

- In the cross-sectional approach, the price of risk is the estimate $\hat{\lambda}$ from the second stage

- Note that the $\hat{\alpha}$ in the first stage are not pricing errors, because $\lambda \neq E(F)$
    - Pricing errors are given by the second-stage cross-sectional residuals $\hat{a}$
    - Common to run second-stage regression with an intercept, in which case the intercept is the zero-beta rate in excess of the risk-free rate

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Cross-sectional approach: Errors-in-variables

- The betas in the second-pass of the CSR are estimated, we have measurement error

- Shanken (1992), Theorem 1, helps with that:

$$var_{EIV}(\hat{\lambda}) = \frac{1}{T}\Sigma_F + var(\hat{\lambda}) \times \left(1 + \lambda' \Sigma_f^{-1} \lambda\right)$$

  - First term, $\left(\frac{1}{T}\Sigma_F\right)$, is the variance of the factors (Σ_F^* in the paper)
  - Second term, $\left(1 + \lambda' \Sigma_f^{-1} \lambda\right)$, for traded factors includes the squared Sharpe ratio of the tangency portfolio constructed from the factors

- For CAPM, corrected (monthly) variance is:

$$var_{EIV}(\hat{\lambda}) = \frac{0.05^2}{T} + var(\hat{\lambda}) \times \underbrace{\left(1 + \left(\frac{0.0066}{0.05}\right)^2\right)}_{=1.0154}$$

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Cross-sectional approach: Fama-MacBeth second stage

- In the second stage, we estimate a cross-sectional regression for each period, t:

$$R_t = \lambda_0 + \hat{B}\lambda_t + \alpha_t$$

  - We can estimate using OLS, GLS (Shanken, 1985) or WLS using diagonal elements of Σ_ϵ (Litzenberger and Ramaswamy, 1979)

- Price of risk & pricing errors given by the time-series of coefficient estimates:

$$\hat{\lambda} = \frac{1}{T}\sum_{t=1}^{T} \widehat{\lambda_t}, \qquad \hat{\alpha} = \frac{1}{T}\sum_{t=1}^{T} \widehat{\alpha_t}$$

$$\text{var}(\hat{\lambda}) = \frac{1}{T^2}\sum_{t=1}^{T} (\widehat{\lambda_t} - \hat{\lambda})(\widehat{\lambda_t} - \hat{\lambda})'$$

$$\text{var}(\hat{\alpha}) = \frac{1}{T^2}\sum_{t=1}^{T} (\widehat{\alpha_t} - \hat{\alpha})(\widehat{\alpha_t} - \hat{\alpha})',$$

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Cross-sectional approach: Fama-MacBeth EIV

- Shanken (1992) has a separate correction for Fama-MacBeth second stage in Theorem 2:

$$T \times \mathrm{var}_{EIV}(\hat{\lambda}) = (1 + \mathrm{c})\left[T\,\mathrm{var}(\hat{\lambda}) - \Sigma_F\right] + \Sigma_F$$

  - where $\mathrm{c} = \lambda' \Sigma_F^{-1} \lambda$ and Σ_F is variance of the factors again

- Standard error correction tends to matter less for traded factors
  - However, Kan, Robotti, and Shanken (2013) show that it matters for non-traded factors

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Cross-sectional approach: Traded factors

- Can use traded factors in the cross-sectional approach

- However, does not need to lead to same result as the time-series approach
  - In time-series approach, factor risk premium is sample mean, and the zero-beta rate equals the risk-free rate

- Good diagnostic test for traded factors is to check whether the risk-premia from the cross-sectional model is the statistically indistinguishable from the time-series average of the traded factor (Lewellen, Shanken, and Nagel, 2010)

Three approaches to estimate factors

- Statistical factors (e.g., Connor and Korajczyk, 1988)
  - Extracted from returns
  - Estimate $\boldsymbol{B}$ and $\boldsymbol{\lambda_K}$ simultaneously

- Macroeconomic factors (e.g., Chen, Rolls, and Ross, 1988)
  - Estimate $\boldsymbol{B}$, then $\boldsymbol{\lambda_K}$

- Fundamental factors (e.g., Fama and French, 1993)
  - Estimate $\boldsymbol{\lambda_K}$ for given $\boldsymbol{B}$ (proxied by firm characteristics)

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Statistical factors: Overview

- Factor analysis (e.g., Lehmann and Modest, 1988)
  - Factors are linear combinations of returns to underlying assets
  - Procedure to create basis portfolios that have minimum idiosyncratic risk

- Principal component analysis (Connor and Korajczyk, 1986; 1988; 1993)
  - Methodology for extracting principal components for a large cross section of returns when the number of time-series observations is smaller than the cross-sectional dimension

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Connor and Korajczyk (1988): Methodology

- Asymptotic PCA

- First developed in Connor and Korajczyk (1986)

- Take K largest eigenvectors of $\mathrm{T} \times T$ matrix $r'r/N$
  - where r is $N \times T$ excess return matrix

- As $N \to \infty$, $K \times T$ matrix of eigenvectors $\equiv$ factor realizations
  - The factor estimates allow for time-varying risk-premiums!

- Refinement (like GLS): same for the scaled cross-product matrix $r'D^{-1}r/N$
  - Where D has variances of the residuals from the first-stage OLS on the diagonal, zeros off the diagonal
  - This increases the efficiency of the estimation

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Connor and Korajczyk (1988): Results

- The first 4 or so factors are enough
  - Based on explicit statistical tests and asset pricing tests

- Explain up to 40% of cross-sectional variation in stock returns
  - Better than CAPM

- Explain some (January), but not all (size, BE/ME) anomalies

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Connor and Korajczyk (1988): Issues and extension

- Issues
  - Statistical factors lack economic interpretation
  - The explanatory power out of sample is much lower than in-sample
  - Number of factors rises with N

- Extensions
  - Jones (2001): accounts for time-series heteroskedasticity
  - Stock and Watson (2002): allow both large time-series and large cross-sections, time-varying betas
  - Bai (2003): large-sample distributions of factor returns and factor beta matrix estimates

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Macroeconomic factors: Overview

- Basically anything motivated from a consumption capital-asset pricing model (Breeden, 1979)
  - CAPM is a specific case (Cochrane, 2005)
  - So is ICAPM (Merton, 1973)
  - Chen, Roll, and Ross (1986) is an early test of the obvious macro variables
  - Consumption-to-wealth-to-income ratio (*cay*) (Lettau and Ludvigson, 2001)
  - Intermediary capital (Adrian, Etula, and Muir, 2014; He, Kelly, and Manela, 2017)

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Chen, Roll, and Ross (1986): Overview

- "Economic forces and the stock market"

- Studies the relation between macroeconomic state variables and stock returns
  - Variables theoretically related to cash-flow/discount rate news

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Chen, Roll, and Ross (1986): Data

- Monthly returns on 20 EW size-sorted portfolios (1953-1983)

- Macro variables
  - Industrial production growth: $MP_t = \ln\left(\frac{IP_t}{IP_{t-1}}\right); YP_t = \ln\left(\frac{IP_t}{IP_{t-12}}\right)$
  - Unanticipated inflation: $UI_t = I_t - E_{t-1}[I_t]$
  - Change in expected inflation: $\Delta EI_t = E_t[I_{t+1}] - E_{t-1}[I_t]$
  - Default premium: $UPR_t = Baa_t - LGB_t$
  - Term premium: $UTS_t = LGB_t - TB_{t-1}$
  - Real interest rate: $RHO_t = TB_{t-1} - I_t$
  - Market return: $EWNY_t$ and $VWNY_t$ (NYSE)
  - Real consumption growth: CG
  - Change in oil prices: OG

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Chen, Roll, and Ross (1986): Methodology

- Two-step cross-sectional regression with Fama-MacBeth in the second step

- First step: Each year, estimate factor loadings for 20 EW size-sorted portfolios:

$$R_{i,t} = a_i + b_i' f_t + \epsilon_{i,t}$$

- Second step: estimate ex-post risk premia from a cross-sectional regression for each of the next 12 months:

$$R_{i,t} = \lambda_{0,t} + b_i' \lambda_{K,t} + \epsilon_{i,t}$$

  - Risk premia: average $\lambda_{K,t}$ from the time-series of coefficient estimates

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Chen, Roll, and Ross (1986): Results

- Table 4 reports results for the estimated risk premia
  - MP: positive
    - Insurance against real systematic production risks
  - UPR: positive
    - Hedging against unexpected increases in aggregate risk premium
  - UTS: negative in 1968-77
    - Hedge against falling LR%
  - IU and DEI: negative in 1968-77
    - Although series were very volatile at the time
  - YP, EWNY, VWNY
    - Insignificant

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Chen, Roll, and Ross (1986): Conclusions

- Stocks are “exposed to systematic economic news and priced in accordance with their exposures”

- Market betas fail to explain cross-section of stock returns
  - Though market index is the most significant factor in the time-series regression

- Not much support for consumption-based asset pricing

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Fundamental factors: Overview

- These are factors loosely related to theory and motivated primarily based on empirical grounds

- Theoretical justification could either come from consumption or production side:
  - Consumption: motivated by either APT or ICAPM
    - Fama and French (1993, 2015, …) – size, value, profitability, investment
    - Carhart (1997) – momentum
    - Pastor and Stambaugh (2003) – liquidity
    - Stambaugh and Yuan (2017) – size + 2 mispricing factors
  - Production side: motivated by *q*-theory
    - Chen, Novy-Marx, and Zhang (2010) – investment, ROE
    - Hou, Xue, and Zhang (2015) - + size
    - Hou, Mo, Xue, and Zhang (2020) + expected growth

Fama and French (1993): Methodology

- Identify risk factors in stock and bond markets
  - Factors for stocks are size and book-to-market
  - Factors for bonds are term-structure variables

- Stock factors: factor-mimicking portfolios
  - In June of each year t, break stocks into:
    - Two size groups: Small/Big (below/above median NYSE)
    - Three B/M groups: Low/Medium/High (bottom 30%/ middle 40%/top 30% NYSE)
    - Compute monthly VW returns of the 6 size x B/M portfolios for next 12 months
  - Factor-mimicking zero-investment portfolios:
    - Size: SMB = 1/3 (SL + SM + SH) – 1/3 (BL + BM +BH)
    - Value: HML = 1/2 (SH + BH) – 1/2 (SL + BL)

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Fama and French (1993): Methodology

- Bond factors
  - TERM=(Return on long-term Treasuries) – (T-bill rate)
  - DEF = (Return on Corp Bonds) – (Return on long-term Treasuries)

- Test assets
  - 25 stock portfolios
    - In June of each year t, stocks are sorted by size (current ME) and independently by B/M (as of December of t-1)
    - Using NYSE quintile breakpoints, all stocks are allocated one of 5 size portfolios and one of 5 B/M portfolios
    - From July of t to June of t+1, monthly VW returns of 25 size x B/M portfolios are computed
  - 7 bond portfolios
    - 2 government bond portfolios: 1-5y, 6-10y maturity
    - 5 corporate bond portfolios: Aaa, Aa, A, Baa, below Baa

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Fama and French (1993): Methodology

- Time-series regressions of excess asset returns on factor returns:

$$R_{i,t} = \alpha_i + \boldsymbol{\beta_i R_{f,t}} + \epsilon_{i,t}$$

- Look at common variation (slopes and R^2) and pricing (intercepts)

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Fama and French (1993): Results

- Table 2: summary statistics
  - MKT, SMB, and HML: high mean and standard deviation, (marginally) significant)
  - TERM, DEF: low mean, but high volatility
  - SMB & HML have low correlation (-0.08)

- Table 4: explanatory power of market factor
  - R^2s for stocks are much higher compared to bonds (up to 90% for large caps, low B/M portfolios)

- Table 6: explanatory power of MKT, SMB, and HML
  - Slopes for stocks are highly significant, R^2s over 90%
  - Market betas move toward one R^2s

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Fama and French (1993): Results

- Table 7: explanatory power of five-factor model
  - Stocks: stock factors remain significant, but make bond factors redundant
  - Bonds: bond factors remain significant, stock factors become much less important

- Table 9c: joint test for zero intercepts
  - Rejects the null for all models
  - The best model seems to be one with three stock factors.

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