Statistical Rethinking: Lecture 02

with Python code

Tossing the globe example

The problem: to estimate the proportion of water covering the globe by "tossing it", i.e. sampling water and land observations. Imagine being away from the planet and being able only to send probes to get the information of whether it landed on land "L" or water "W". Alternatively, imagine a globe replica being tossed at a crowd and taking note of where in the globe, water or land, it first touched the hand of a person in the crowd.

In frequentist statistics, we could use the relative frequencies of "W" or "L" as the estimate for water and land, respectively. Here we deal with the Bayesian approach: given a sample consisiting on W and L observations of water and land, what value of p, the proportion of water covering the globe, is more likely to reproduce the sample?

Since p can range from 0 to 1, let us consider a 4-sided "globe", a tetrahedron or 4-sided die. This allows discretizing the problem. At the end we can let the limit of sides reach infinity, recovering a proper globe.

Recall the fundamental principle in Bayesian analysis

> For each possible explanation of the sample, count the ways each explanation can reproduce the sample. Explanations with the largest number of ways to reproduce the data are more likely.

And the basic workflow

1) Define a generative model of the sample (generate synthetic data?)
2) Define a specific estimand
3) Design a way to produce the estimate (estimator?)
4) Test 3) using 1) (use the estimator to get an estimate in controlled situation, where you know the answer)
5) Analyze sample (real data)

1) The generative model

Let us define a function to generate the sampling of water "W" or land "L" observations. If p is the proportion of water covering the globe, then the probability of observing "W" is p and "L" is $1 - p$.

```
def toss_globe(proportion, N):
    return np.random.choice(["W", "L"], size=N, p=[proportion, 1 - proportion])
```

We can toss a globe covered 70% by water ten times by calling `toss_globe(p=0.7, N=10)`, which gives

```
array(['L', 'W', 'W', 'W', 'L', 'W', 'W', 'W', 'W', 'L'], dtype='<U1')
```

2) The estimand

Our estimand will the number of ways each explanation can reproduce the sample. In this case, the number of ways each proportion of water p covering the globe can reproce a series of observations. That value of p with the largest number of ways to reproduce the data is more plausible to be the true value. Equivalently, instead of counting, we can normalize by the total number of ways of explaining the data and work with a probability. In this manner, our estimand will be the probability distribution for the prorportion of water p.

3) The estimator

For a total of $N = W + L$ tosses, the expected number of water and land observations are given by:

$$W \quad \text{or} \quad L = (4p)^W (4 - 4p)^L$$

which count the ways each explanation – the proportion of water p – can reproduce the sample. This is our estimator. Let us code it.

Using the sample observed in the previous toss, we can calculate the number of ways each proportion (explanation) can reproduce the sample as follows

```
W, L = np.sum(sample=='W'), np.sum(sample=='L')
proportions = [0, 0.25, 0.5, 0.75, 1]
ways = [(4 * p)**W * (4 - 4 * p)**L for p in proportions]
ways
```

```
[0, 27.0, 1024.0, 2187.0, 0]
```

Normalizing by the total number of ways we can calculate the probabilities each proportion p has to reproduce the data. The plot is shown in Figure 1.

```
probabilities = np.array(ways)/np.sum(ways)
probabilities
```

```
array([0.        , 0.00833848, 0.3162446 , 0.67541692, 0.        ])
```

```
fig, ax = mplt.subplots()
ax.bar([str(prop) for prop in proportions], probabilities)
ax.set_xlabel('water proportions')
ax.set_ylabel('proportion probability')

mplt.show()
```

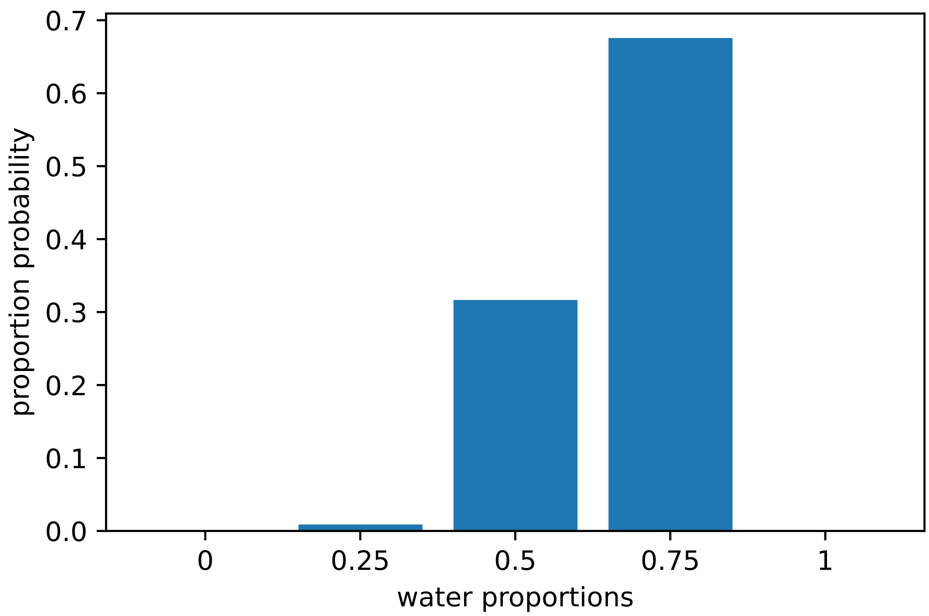


Figure 1: Bar plot for the plausability of each proportion of

This estimator calculates the estimate probability for each explanation p of the sample. In the Bayesian jargon, it computes the posterior distribution for p. We code it into a function for future convnience:

```
def compute_posterior(sample, proportions):
    W, L = np.sum(sample=='W'), np.sum(sample=='L')
    ways = [(4 * p)**W * (4 - 4 * p)**L for p in proportions]
    return np.array(ways)/np.sum(ways)
```

4) Testing

We have thus far succesfully designed the generative model, defined the estimand and designed the statistical estimator. We have also anticipated one estimate by applying the estimator to the first sample we got from the generative model. Now focus on testing if the estimand and estimator are valid.

Let us simulate extreme cases.

Water planet

Generate sample with water proportion equal to 1.

```
sample = toss_globe(proportion=1, N=10)
sample
```

```
array(['W', 'W', 'W', 'W', 'W', 'W', 'W', 'W', 'W', 'W'], dtype='<U1')
```

Compute the posterior

```
proportions = [0, 0.25, 0.5, 0.75, 1]
compute_posterior(sample, proportions)
```

```
array([0.00000000e+00, 9.01997925e-07, 9.23645876e-04, 5.32620755e-02,
       9.45813377e-01])
```

We see the most likely proportion is the correct $p = 1$.

Land planet

Generate sample with water proportion equal to 0.

```python
sample = toss_globe(proportion=0, N=10)
sample
```

```
array(['L', 'L', 'L', 'L', 'L', 'L', 'L', 'L', 'L', 'L'], dtype='<U1')
```

Compute the posterior

```python
proportions = [0, 0.25, 0.5, 0.75, 1]
compute_posterior(sample, proportions)
```

```
array([9.45813377e-01, 5.32620755e-02, 9.23645876e-04, 9.01997925e-07,
       0.00000000e+00])
```

We see the most likely proportion is the correct $p = 0$. Now lets play with large number of trials with $p = 0.5$ and see if we recover this frequency of water of observations

```python
sample = [toss_globe(proportion=0.5, N=10) for _ in range(1_000)]
sample = np.concatenate(sample)  # reshape it 1D array
np.sum(sample=='W')/np.size(sample)  # compute the frequency of water
```

```
0.5038
```

Abstracting: Bayes theorem

Let's turn the concrete example we have worked on into some more abstract reasoning before continuing.

The whole idea behind Bayesian analysis is that explanations with the largest number of ways of explaining the data are more likely. Bayesian statistics is a counting problem. Where's Bayes theorem in it?

This counting problem when normalized by the total number of possible outcomes becomes a probability problem. The probability for observing W water and L land instances when tossing the globe for a given proportion p of water is

$$\Pr(W, L|p) = \frac{(W+L)!}{W!L!} p^W (1-p)^L$$

saying that proportions with more ways to reproduce the sample are more plausible equals calculating the probability of p, given the observed sequence of W and L, $\Pr(p|W, L)$. If you are familiar with Bayes theorem, you know you can calculate this "inverted" conditional as

$$\Pr(p|W, L) = \frac{\Pr(W, L|p)\Pr(p)}{\Pr(W, L)}$$

The main theme in Bayesian statistics, however, is not Bayes theorem. Bayes theorem derives from the basic rules of probabilities and can be used in non-bayesian analysis. The main distinction of Bayesian reasoning is the philosophy of probabilities as measuring beliefs or creedances. $\Pr(W, L|p)$, is the likelihood function and $\Pr(p)$ is the prior distribution: the initial guess for the distribution of the explanations. Calculating $\Pr(p|W, L)$ can be seen as updating the initial belief on p given by the prior in face of the observed data (likelihood). If new observations come, the posterior is used as the new prior and the process is itrated, updating our belief about the distribution of p.

The $\Pr(W, L)$ term on the denominator is the normalization constant, formally calculated as

$$\Pr(W, L) = \mathbb{E}[\Pr(W, L|p)] = \int \Pr(W, L|p)\Pr(p)\mathrm{d}p$$

Grid approximation

What if we allow our 4-sided globe to have instead 11 sides? Well, now the possible proportions of water are

```
proportions11 = np.linspace(0, 1, 11, endpoint=True)
proportions11
```

```
array([0. , 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1. ])
```

Let's assume $p = 0.7$ and toss the globe 10 times. Let's calculate the posterior and repeat this process for a 21-sided globe.

```
sample11 = toss_globe(proportion=0.7, N=10)
sample21 = toss_globe(proportion=0.7, N=10)

post11 = compute_posterior(proportions=proportions11, sample=sample11)

proportions21 = np.linspace(0, 1, 21, endpoint=True)
post21 = compute_posterior(proportions=proportions21, sample=sample21)
```

We plot the results in Figure 2.

```
fig, (ax, ay) = mplt.subplots(1, 2, figsize=(10, 5))
data = [f'{prop:.1f}' for prop in proportions11]
ax.bar(data, post11)
ax.set_title('11-sides')
ax.set_xticklabels(labels=data, rotation=90)
ax.set_ylabel('proportion probability')

data = [f'{prop:.2f}' for prop in proportions21]
ay.set_xticklabels(labels=data, rotation=90)
ay.bar(data, post21)
ay.set_title('21-sides')
fig.supxlabel('water proportions')
fig.tight_layout()
mplt.show()
```

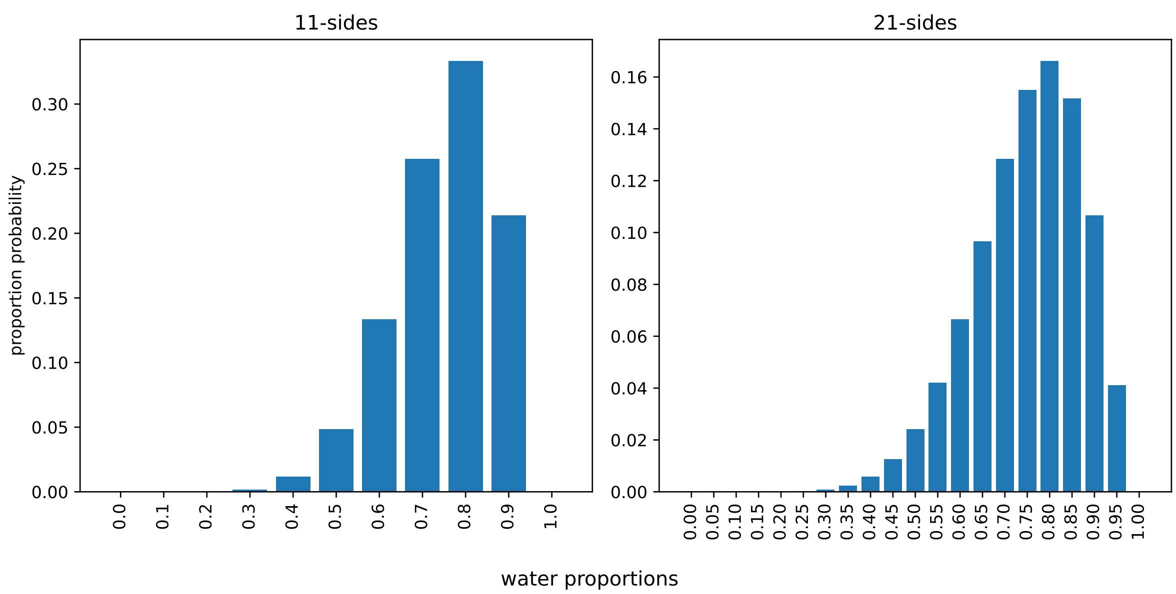


Figure 2: Distribution of p for 11- and 21-sided globes

Increasing the number of sides approaches our polygonal dies closer to becoming a proper globe. Approximating the posterior of the globe with increasing number of sides is a *grid approximation*. The grid is the number of sides.

Continuum limit

Recall that

$$\Pr(W, L|p) = \frac{1}{Z} p^W (1-p)^L$$

where the "partition function" reads $Z = \sum_p p^W (1-p)^L$. As the number of sides approach infinity, the sum approaches the integral

$$Z = \int p^W (1-p)^L \mathrm{d}p = \frac{W!L!}{(W+L+1)!}, \quad W, L \in \mathbb{Z}$$

so we recognize

$$\Pr(W, L|p) = \mathrm{Beta}(W+1, L+1) = \frac{(W+L+1)!}{W!L!} p^W (1-p)^L$$

Posterior: updating the prior

Before any observations, we assume any proportion p is equally plausible. So the probabiltiy for p is a constant normalized so it reads as probability:

$$\Pr(p)_0 = \frac{u}{Z_0},$$

where Z_0 is basically the `np.sum(ways)` term in our previous code snippets.

The first observation, say, water, allows us to calculate the posterior:

$$\Pr(p)_1 = \frac{p^1(1-p^0)\Pr(p)_0}{Z_1}$$

and so the second observation gives $\Pr(p)_2$ in the same manner, with Z_2 being whatever constant needed to normalize the probability to 1. For the beta distribution is easy to find analytical expressions for the posteriors as we iterate. Instead, let us generalize our function to update posteriors numerically. First we import `beta` from `scipy`

```
from scipy.stats import beta
```

And now we write our function

```
def compute_posterior(proportions, sample, prior):
    W, L = np.sum(sample=='W'), np.sum(sample=='L')
```

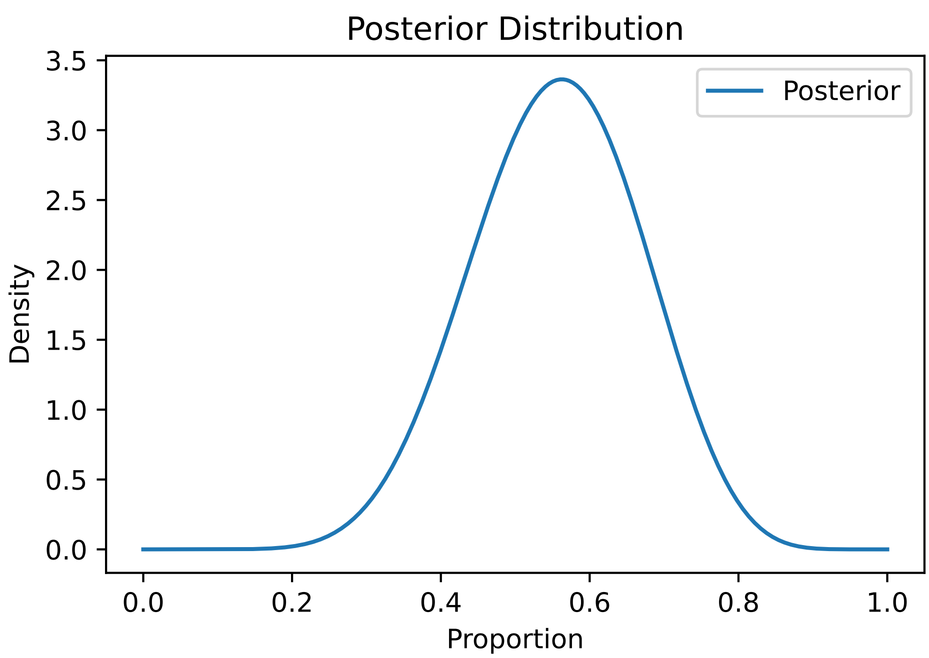
```
    Wprior = np.sum(prior=='W')
    Lprior = np.sum(prior=='L')
    return beta.pdf(proportions, W + Wprior, L + Lprior)
```

```
x_values = np.linspace(0, 1, 1000)

the_sample = np.array(['W', 'W', 'L', 'W', 'L', 'W', 'L', 'L', 'W'])
prior = np.array(['W', 'W', 'L', 'W', 'L', 'W', 'L', 'L', 'W'])

# Compute the posterior values for the given x values
posterior_values = compute_posterior(x_values, the_sample, prior)

# Plot the curve
mplt.figure()
mplt.plot(x_values, posterior_values, label='Posterior')
mplt.title('Posterior Distribution')
mplt.xlabel('Proportion')
mplt.ylabel('Density')
mplt.legend()
mplt.show()
```



Basically we count the ocurrences of water and land in the sample and in the prior, and use them as parameters to sample from the beta distribution. for a continuous .

Working with the posterior

Sampling the posterior

Suppose we constructed the posterior for the observation of 6 water and 3 land. We can then sample the posterior distribution $\text{Beta}(p, 6+1, 3+1)$. A thousand samples read

```
post_samples = np.random.beta(6+1, 3+1, 1000)
```

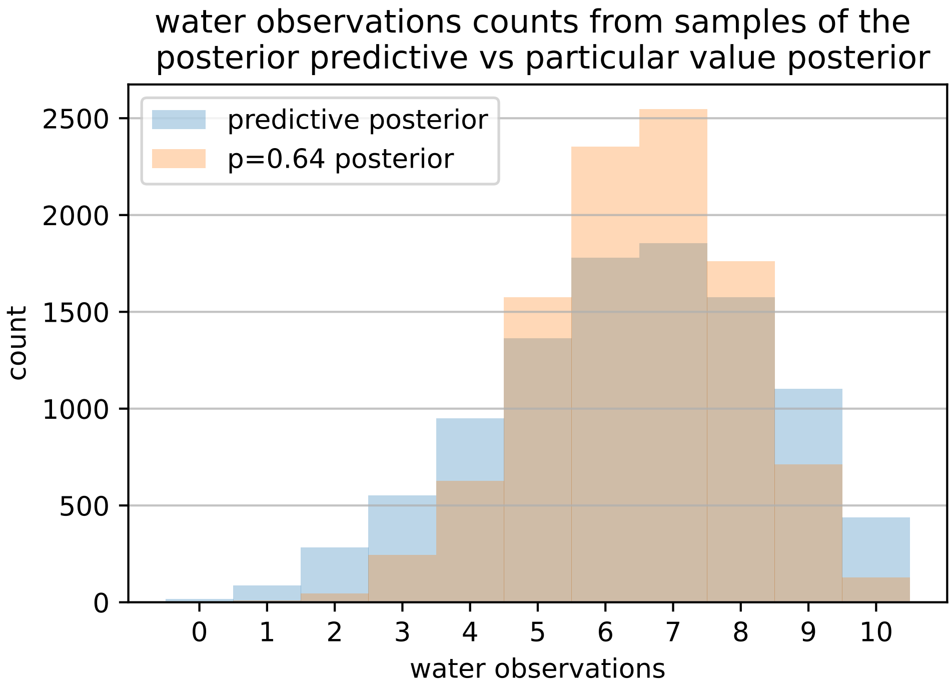
Posterior Predictive Distribution

Given the posterior distribution, how can we make predictions? Well, we could choose a particular value of p, maybe the mode, the peak of the distribution, the most probable value, and use it to make predictions about water and land observations with using the likelihood function or the generative model. But how do we incorporate the uncertainty in p in our estimates? How do we propagate the associated error of choosing a particular value and then doing a prediction? The solution is to sample values for p from the posterior, then calculate the likelihood for a given p or sample the generative model.

Below, we sample the water proportions from its posterior distribution and then sample the generative model 10-times for each value of proportion.

```
post_samples = np.random.beta(6+1, 3+1, 10_000)
pred_post = [np.sum(toss_globe(p, 10)=='W') for p in post_samples]

pred64 = [np.sum(toss_globe(0.65, 10)=='W') for _ in range(10_000)]
mplt.hist(pred_post, bins=np.arange(12)-0.5, alpha=0.3, label='predictive posterior')
mplt.hist(pred64, bins=np.arange(12)-0.5, alpha=0.3, label='p=0.64 posterior')
mplt.legend()
mplt.xlabel('water observations')
mplt.ylabel('count')
mplt.xticks(range(11))
mplt.grid(axis='y', alpha=0.75)
mplt.title('water observations counts from samples of the \n posterior predictive vs parti
mplt.show()
```



Notice the Predictive posterior is much more spread out, because it incorporates the uncertainty in the estimate for p.