State Space Averaging Technique:

Terms used:

$$x = \text{average variation of the variable over a cycle}$$

$$\dot{x} = \text{average rate of variation of the variable over a cycle}$$

$$T = switching\ time$$

$$\dot{x_{0n}} = \text{average rate of variation of the variable during on} - \text{state}$$

$$\dot{x_{off}} = \text{average rate of variation of the variable during off} - \text{state}$$

$$x = \dot{x}T = \dot{x_{0n}}DT + \dot{x_{off}}(1-D)T$$

$$\dot{x} = \dot{x_{0n}}D + \dot{x_{off}}(1-D) \ \ldots.(1)$$

Let

$$\dot{x}_{on} \dot{=} A_1x + B_1u \ \ldots (2)$$

$$\dot{x}_{off} \dot{=} A_2x + B_2u \ \ldots.(2)$$

Substitute eq. (2) in (1)

$$\dot{x} = (A_1x + B_1u)D \dot{+} (A_2x + B_2u)(1-D)$$

$$= [(A_1D + A_2(1-D)]x + [(B_1D + B_2(1-D)]u$$

Where

$$A = [(A_1D + A_2(1-D)] \qquad \ldots.(3)$$

$$B = [(B_1D + B_2(1-D)]$$

$$y = \text{average variation of the output variable over a cycle}$$

$$\dot{y_{on}} = \text{average variation of the output variable on} - \text{state}$$

$$\dot{y_{off}} = \text{average variation of the output variable off} - \text{state}$$

$$y = \dot{y_{0n}}D + \dot{y_{off}}(1-D)$$

$$y_{on} = C_1x + E_1u \qquad \ldots.(4)$$

$$y_{off} = C_2x + E_2u$$

Substitute eq. (3) in (4)

$$y = (C_1 x + E_1 u)D \dot{+} (C_2 x + E_2 u)(1 - D)$$

$$= [(C_1 D + E_2(1 - D)]x + [(C_1 D + E_2(1 - D)]u$$

Where

$$C = [(C_1 D + C_2(1 - D)] \qquad \ldots. (5)$$

$$E = [(E_1 D + E_2(1 - D)]$$

Under small signal consideration

$$\dot{x} = Ax + Bu = [A_1 D + A_2(1 - D)x + B_1 D + B_2(1 - D)u]$$

The variable is perturbed around the operating points as follows

$$x = (X + \hat{x}), d = \left(D + \hat{d}\right), u = (U + \hat{u})$$

Substitute the above perturbations in state equation

$$X \dot{+} \hat{x} = [A_1(D + \widehat{d}) + A_2\left(1 - D - \widehat{d}\right)](X + \hat{x}) + [B_1(D + \widehat{d}) + B_2\left(1 - D - \widehat{d}\right)](U + \hat{u})$$

The non-linear terms can be neglected as they are small. The above equations are written as

$$[X \dot{+} \hat{x}] = [AX + BU] + [(A_1 - A_2)X + (B_1 - B_2)U]\hat{d} + A\hat{x} + B\hat{u}$$

$$X = 0 = AX + B\dot{U}, \;\; X = -A^{-1}BU$$

$$[\dot{\hat{x}}] = [A\hat{x} + B\hat{u}] + [(A_1 - A_2)X + (B_1 - B_2)U]\hat{d}$$

$$[\dot{\hat{x}}] = \left[A\hat{x} + B\hat{u} + F\hat{d}\right]$$

Where $F = (A_1 - A_2)X + (B_1 - B_2)U$

For output equation

$$y = Cx + Eu$$

As output is not directly connected to input E=0

$$y = Cx$$

Under small signal consideration. Substitute the above perturbations in state equation

$$y \dot{+} \hat{x} = [C_1(D + \widehat{d}) + C_2\left(1 - D - \widehat{d}\right)](X + \hat{x})$$

The non-linear terms can be neglected as they are small. The above equations are written as

$$y = [C_1 D + C_2(1 - D)]X$$

$$\hat{y} = C\hat{x} + (C_1 - C_2)X\hat{d}$$

Control to output transfer function

$$\hat{X} = A\hat{x} + B\hat{u} + F\hat{d}$$

$$\hat{y} = C\hat{x} + (C_1 - C_2)X\hat{d}$$

Take LaPlace transform of above equations

$$S\hat{x}(s) = A\hat{x}(s) + B\hat{u}(s) + F\hat{d}(s)$$

For control to output transfer function $\hat{u}(s) = 0$

$$\hat{x}(s) = (sI - A)^{-1}F\hat{d}(s) \;\ldots.\; (6)$$

$$\text{Also}\,\hat{y}(s) = C\hat{x}(s) + (C_1 - C_2)X\hat{d}(s) \;\ldots..\; (7)$$

Substituting equation (6) in (7)

$$\hat{y}(s) = C(sI - A)^{-1}F\hat{d}(s) + (C_1 - C_2)X\hat{d}(s)$$

$$\frac{\hat{y}(s)}{\hat{d}(s)} = [C(sI - A)^{-1}F + (C_1 - C_2)X]$$