Math 148 Notes

velo.x

Instructor: Nico Sprunk
Section: 002

Contents

1 INTEGRATION, SUMMATION

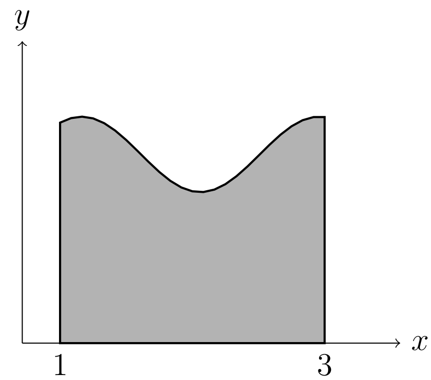
MOTIVATION: area, let $a < b$ in $\mathbb{R}$, and let $f : [a, b] \to [0, \infty]$, let

$$S_f = \{(x, y) : 0 \le y \le f(x), x \in [a, b]\}(\text{"}subgraph\text{"})$$

IDEA: area of rectangel = height * width

1.

Figure 1: The area under the function $\frac{1}{x}$ is $\log x$



2. approximate S_f by rectangle from below 13:43 JAN 6,

   $j = 1, \cdots, 4$, $m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$

$$\sum_{j=1}^{4} m_{j-1}(x_i - x_{j-1}) \le area(s_f)$$

3. approximate S_f by rectangle from above, $M_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\}$

$$area \le \sum_{j=1}^{4} M_j(x_j - x_{j-1})$$

4. if we can arrange lower sum $\approx$ upper sum, then we have some good approximation

1.1 Partition, Upper and Lower Sum

Let $a < b \in \mathbb{R}$, $f : [a, b] \in \mathbb{R}$,

Definition 1.1.1 (Riemann-Darboux).

*A **partition** of $[a, b]$ is any finite set of points including the endpoints.*

$$P : \{x_0, x_1, \cdots, x_n\} s.t. a = x_0 < x_1 < \cdots < x_n = b$$

often for convenience, we write $P = \{a = x_0 < \cdots < x_n = b\}$.

*A **Refinement** of P is any partition Q of $[a,b]$ s,t, $P \subseteq Q$.*

Now, fix a partition P of $[a,b]$ and let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$, i.e. $\sup_{x \in [a,b]} |f(x)| \leq M < \infty$. Write $P = \{a = x_0 < \cdots < x_n = b\}$. For $j = l, \cdots, n$,

$$\begin{aligned} m_j &= m_j(P) = \inf\{f(x) : x \in [x_{j-1}, x_j]\} \\ M_j &= M_j(P) = \sup\{f(x) : x \in [x_{j-1}, x_j]\} \end{aligned}$$

Notice that $-M \leq m_k \leq M_j \leq M$ for each j, and these "inf", "sup" exist. (Using that $\mathbb{R}$ is complete.)

Definition 1.1.2.

- ***Lower Sum:*** $L(f,P) = \sum_{j=1}^{n} m_j \underbrace{(x_j - x_{j-1})}_{width\ of\ [x_{j-1},x_j]}$
- ***Upper Sum:*** $U(f,P) = \sum_{j=1}^{n} M_j(x_j - x_{j-1})$

Remark:

1. if f is not bounded, then at least one of $L : (f,P)$ or $U(f,P)$ cannot be defined.
2. we have $L(f,P) \leq U(f,P)$, Indeed, for each $j = l, \cdots, n$, $m_j \leq M_j$. (exactly from definition),

$$L(f,P) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}) \leq \sum_{j=1}^{n} M_j(x_j - x_{j-1}) = U(f,P)$$

Lemma 1.1.1. *If P is a partition of $[a,b]$, $f : [a,b] \to \mathbb{R}$ is bounded, and Q is a refinement of P, then*

$$L(f,P) \leq L(f,Q) \qquad U(f,Q) \leq U(f,P)$$

Proof.

- Case 0: $Q = P$ obvious
- Case 1: $Q = P \cup \{q\}$ where $q \notin P$,

  write $P = \{a = x_0 < \cdots, x_n = b\}$ so $Q = \{a = x_0 < \cdots < x_{k-1} < q < x_k < \cdots < x_n = b\}$ Then,

$$\begin{aligned} m_k(P) &= \inf\{f(x) : x \in [x_{k-1}], x_k\} \qquad [x_{k-1}, x_k] = [x_{k-1}, q] \cup [q, x_k] \\ &= \min\{\inf\{f(x) : x \in [x_{k-1}, q] : x \in [x_{k-1}, q]\} \inf f(x) : x \in [q, x_k]\} \\ &= \min\{m_k(Q), m_k'(Q)\} \leq m_k(Q), m_k'(Q) \end{aligned}$$

Thus,

$$
\begin{aligned}
L(f,P) &= \sum_{j=1}^{m} m_j(P)(x_j - x_{j-1}) \\
&= \sum_{j=1}^{k-1} m_j(P)(x_j - x_{j-1}) + m_k(P)(x_k - x_{k-1}) + \sum_{j=k+1}^{n} m_j(P)(x_j - x_{j-1}) \\
&\leq \sum_{j=1}^{k-1} m_j(P)(x_j - x_{j-1}) + m_k
\end{aligned}
$$

- Case 2: $Q = P \cup \{q_1, \cdots, q_m\}, q_1, \cdots, q_m$ distinct, $q_u \notin P$, by case 1, we have

  $$L(f,P) \leq L(f, P \cup \{q_1\}) \leq L(f, P \cup \{q_1, q_2\}) \leq \cdots \leq L(f, P \cup \{q_1, \cdots, q_m\}) = L(f,Q)$$

  The case $U(f,Q) \leq U(f,P)$ is similar.

$\square$

Corollary 1.1.1. *let P, Q be any partition of $[a,b]$ and $f : [a,b] \to \mathbb{R}$ be bounded, then*

$$L(f,P) \leq U(f,Q)$$

Proof. We have $P, Q \subseteq P \cup Q$, i.e. $P \cup Q$ refines each of P and Q. Thus,

$$L(f,P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f,Q)$$

$\square$

1.2 Upper and Lower Sum

Definition 1.2.1. *Given a bounded $f : [a,b] \to \mathbb{R}$, define*

- ***Lower Integral*** *: $\underline{\int}_a^b f = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}$*
- ***Upper Integral:*** *$\overline{\int}_a^b f = \inf\{U(f,Q) : Q \text{ is a partition of } [a,b]\}$*

$\underline{\int}_a^b f = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\} \leq \inf\{U(f,Q) : Q \text{ is a partition of } [a,b]\} = \overline{\int}_a^b f$

*We say that f is **integrable** on $[a,b]$ provided that*

$$\underline{\int}_a^b f = \overline{\int}_a^b f$$

In this case, we write $\int_a^b f = \overline{\int}_a^b f = \underline{\int}_a^b f$

<u>Notation:</u> Write

$$\int_a^b f = \int_a^b f(x)d(x) = \int_a^b f(t)dt$$

Non-Example 1: not every bounded function is integrable.

Define: $\chi_{\mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$.

Let $P = \{0 = x_0 < \cdots < x_n = 1\}$ be any partition of $[0,1]$, We have that

- $\mathbb{Q}$ is dense in $\mathbb{R}$, so there is $q_j \in \mathbb{Q} \cap (x_{j-1}, x_j), j = l, \cdots, n$
- $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$, so there is $r_j \in (\mathbb{R}\backslash\mathbb{Q}) \cap (x_{j-1}, x_j), j = l, \cdots, n$

$$0 \leq L(\chi_{\mathbb{Q},P}) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}) \leq \sum_{j=1}^{n} \chi_{\mathbb{Q}}(r_j)(x_j - x_{j-1}) = 0 \Rightarrow \underline{\int}_0^1 = 0$$

Likewise,

$$1 \geq U(\chi_{\mathbb{Q}}, P) \geq \sum_{j=1}^{n} \chi_{\mathbb{Q}}(q_j)(x_j - x_{j-1}) = \sum_{j=1}^{n}(x_j - x_{j-1}) = 1 - 0 = 1 \Rightarrow \overline{\int}_0^1 = 1$$

hence,

$$\underline{\int}_0^1 \chi_{\mathbb{Q}} = 0 < 1 = \overline{\int}_0^1 \chi_{\mathbb{Q}}$$

so $\chi_{\mathbb{Q}}$ is not integrable on $[0,1]$.

Theorem 1.2.1 (**Cauchy Criterion For Integrability**). *Let $a < b \in \mathbb{R}$, $f : [a,b] \to \mathbb{R}$ be bounded, then TFAE,*

1. *f is integrable on $[a,b]$*
2. *given $\varepsilon > 0$, there exists a partition P_ε of $[a,b]$ s.t.*
$$U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$$
3. *given $\varepsilon > 0$, there exists a partition P_ε of $[a,b]$ so for every refinement P of P_ε*
$$U(f,P) - L(f,P) < \varepsilon$$

Proof. 1 to 2: we assume that

$$\sup\{L(f,P) : P\,\text{partition}\ \ of\ \ [a,b]\} = \underline{\int_a^b} f = \overline{\int_a^b} \inf\{U(f,P) : P\,\text{partition}\ \ of\ \ [a,b]\}$$

Let $\varepsilon > 0$, by first equality above, there is a partition P_1 of $[a,b]$ s.t.

$$\underline{\int_a^b} f - \frac{\varepsilon}{2} < L(f,P_1)$$

and by the third equality, there is a partition P_2 s.t.

$$\overline{\int_a^b} f < U(f,P_2) - \frac{\varepsilon}{2}$$

Let $P_\varepsilon = P_1 \cup P_2$, a refinement of P_1 and P_2, then since $\underline{\int_a^b} f = \overline{\int_a^b} = \int_a^b f$ we find

$$\int_a^b f - \frac{\varepsilon}{2} < L(f,P_1) \leq L(f,P_\varepsilon) \leq U(f,P_\varepsilon) \leq U_{f,P_2} < \int_a^b f + \frac{\varepsilon}{2}$$
$$\Rightarrow U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$$

2 to 3: we use the lemma.

If $P_\varepsilon \leq P$, we have

$$L(f,P_\varepsilon) \leq L(f,P) \leq U(f,P) \leq U(f,P_\varepsilon)$$

Hence,

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon \Rightarrow U(f,P) - L(f,P) < \varepsilon$$

3 to 2: $P_\varepsilon \subseteq P_\varepsilon$ i.e. P_ε self-defines itself

2 to 1: Given $\varepsilon > 0$, there is P_ε, a partition of $[a,b]$, so $U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$. We have

$$L(f,P_\varepsilon) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f,P_\varepsilon) \qquad \Rightarrow \qquad \underline{\int_a^b} f - \overline{\int_a^b} f < \varepsilon$$

$$\underline{\int_a^b} f = \overline{\int_a^b} f \qquad \Rightarrow \qquad f \text{ is integrable}$$

$\square$

1.3 Continuity and Inegrability

Definition 1.3.1 (Continuous). *$f : I \to \mathbb{R}$ is continuous if for every x in I, for every $\varepsilon > 0$, there exists $\delta > 0$ s.t. for all $|x - x'| < \delta$, $x' \in I$,*

$$|f(x) - f(x')| < \varepsilon$$

this definition is local. first we choose x, ε, then δ.

Definition 1.3.2 (uniform Continuity). *$f : I \to \mathbb{R}$ is uniformly continuous if for every $\varepsilon > 0$, there is $\delta > 0$ so $|f(x) - f(x')| < \varepsilon$ whenever $|x - x'| < \delta$ for $x, x' \in I$.*

Proposition 1.3.1 (Sequential Test of Continuity). *Let $f : I \to \mathbb{R}$, then f is uniformly continuous $\Rightarrow$ for any sequences $(x_n)_{n=1}^{\infty}$, $(x'_n)_{n=1}^{\infty} \subset I$, with $\lim_{n\to\infty} |x_n - x'_n| = 0$, we have $\lim\limits_{n\to\infty} |f(x_n) - f(x'_n)| = 0$.*

[Fact $\Leftarrow$ also true]

Proof. Given $\varepsilon > 0$, let δ be as in def'n of uniform continuity. Since $\lim\limits_{n\to\infty} |x_n - x'_n| = 0$, there is $N \in \mathbb{N}$, so for $n \geq N$, we have $|x_n - x'_n| < \delta$.

But then, for $n \geq N$, we also have that $|f(x_n) - f(x'_n)| < \varepsilon$. i.e. $\lim\limits_{n\to\infty} |f(x_n) - f(x'_n)| = 0$. $\square$

Example 1 $f : (0, 1] \to \mathbb{R}$, $f(x) = \frac{1}{x}$. Notice that f is continuous.

Let $x_n = \frac{1}{n}$, $x'_n = \frac{1}{2n}$, $|x_n - x'_n| = \frac{1}{2n} n \vec{\to} \infty 0$.

$$|f(x_n) - f(x'_n)| = |n - 2n| = n$$

Hence, not uniformly continuous.

Example 2: $g : (0, 1] \to \mathbb{R}$, $g(x) = \sin\frac{1}{x}$, then g is continuous.

$x_n = \frac{1}{\pi n}$, $x'_n = \frac{2}{(2n+1)\pi}$, $|x_n - x'_n| = \frac{1}{\pi n(2n+1)} n \vec{\to} \infty 0$,

$$|g(x_n) - g(x'_n)| = \left|\sin(n\pi) - \sin(\frac{2n+1}{2}\pi)\right| = 1$$

For $\varepsilon = 1$, uniform continuity fails.

Theorem 1.3.1. *Let $f : [a, b] \to \mathbb{R}$ be continuous, then f is uniformly continuous.*

Proof. Let us suppose that f is continuous, but not uniformly continuous, hence there exist $\varepsilon > 0$, such that for any $\delta > 0$, there are $x, x' \in [a, b]$ so

$$|f(x) - f(x')| \geq \varepsilon \text{ while } |x - x'| < \delta$$

Let us consider $\delta = \frac{1}{n}$, so there are x_n, x_n' in $[a,b]$ such that

$$|f(x_n) - f(x_n')| \geq \varepsilon \ while \ |x - x'| < \frac{1}{n}$$

By Bolzano Weierstrass Theorem, there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$, such that $x = \lim\limits_{k\to\infty} x_{n_k}$ exists in $[a,b]$.

Then, notice that

$$\left|x - x_{n_k}'\right| \leq |x_n - x_{n_k}| + \left|x_{n_k} - x_{n_k}'\right| < |x - x_{n_k}| + \frac{1}{n_k}$$

hence, by Squeeze Theorem, $\lim\limits_{k\to\infty} x_{n_k}' = x$. Since f is continuous, we have that

$$\lim_{k\to\infty} f(x_{n_k}) = f(x) = \lim_{k\to\infty} f(x_{n_k}')$$

$\Rightarrow$

$$\lim_{k\to\infty} \left|f(x_{n_k}) - f(x_{n_k}')\right| = 0$$

This contradicts that each $\left|f(x_{n_k}) - f(x_{n_k}')\right| \geq \varepsilon$. Thus by contradiction argument, f' must be uniformly continuous.

$\square$

Theorem 1.3.2 (**Continuous on a Closed Bounded Interval and Integrability**). *let* $f : [a,b] \to \mathbb{R}$ *be continuous, then* f *is integrable.*

Proof. Let $\varepsilon > 0$, then by uniform continuity of f, there exists a δ such that whenever $|x - x'| < \delta$, for $x, x' \in [a,b]$,

$$|f(x) - f(x')| < \frac{\varepsilon}{b-a}$$

Thus, we let $P = \{a = x_0 < \cdots < x_n = b\}$ be any partition with length $l(P) = \max\limits_{j=1,\cdots,n}(x_j - x_{j-1}) < \delta$.

Example: $P_n = \{a_i < a + \frac{b-a}{n} < a + 2\frac{b-a}{n} < \cdots < a + (n-1)\frac{b-1}{n} << b\}$, then $\lim_{n\to\infty} l(P_n) = 0$.

Now, by EVT, we have

$$x_j^* \in [x_{j-1}, x_j] \ s.t. \ f(x_j^*) = \inf\{f(x) : x \in [x_{j-1}, x_j]\} = m_j$$
$$x_j^{**} \in [x_{j-1}, x_j] \ s.t. \ f(x_j^{**}) = \sup\{f(x) : x \in [x_{j-1}, x_j]\} = M_j$$

Then

$$L(f,P) = \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) \qquad U(f,P) = \sum_{j=1}^{n} f(x_j^{**})(x_j - x_{j-1})$$

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{j=1}^{n}(f(x_j^{**}) - f(x_j^*))(x_j - x_{j-1}) \\ &= \sum_{j=1}^{n} \left|f(x_j^{**}) - f(x_j^*)\right|(x_j - x_{j-1}) < \sum_{j=1}^{n} \frac{\varepsilon}{b-a}(x_j - x_{j-1}) \\ &= \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon \end{aligned}$$

Hence, we have satisfied the Cauchy Criterion for integrability. $\square$

Corollary 1.3.1. *if $f : [a, b] \to \mathbb{R}$ is continuous, then*

$$\int_a^b f = \lim_{n\to\infty} \sum_{j=1}^{n} f(a + j\frac{b-a}{n})\frac{b-a}{n}$$

Proof. We have $a + j\frac{b-a}{n} \in [a + \frac{b-a}{n}(j-1), a + \frac{b-a}{n}(j-1)]$, $j = 1, \cdots, n$.

So,

$$m_j \leq f(a + j\frac{b-a}{n}) \leq M_j$$

and thus

$$L(f, P_n) \leq \sum_{j=1}^{n} f(a + j\frac{b-a}{n})\frac{b-a}{n} \leq U(f, P_n)$$

$\lim_{n\to\infty}(U(f, P_n) - L(f, P_n)) = 0$ as $\lim_{n\to\infty} l(P_n) = 0$.

where $P_n = \{a < a + \frac{b-a}{n} < a + 2\frac{b-a}{n} < \cdots < b\}$, then proof of theorem shows that $\lim_{n\to\infty} L(f, P_a) = \int_a^b f = \lim_{n\to\infty} U(f, P_n)$ as $\lim_{n\to\infty} l(P_n) = \lim_{n\to\infty} \frac{b-a}{n} = 0$.

and hence Cauchy Criterion is satisfied, hence $\int_a^b f$ exists and is $\lim_{n\to\infty} L(f, P_n)$, apply Squeeze Theorem. $\square$

1.4 Basic Properties of Integrals

Example 1: We will let $a > 0$ and compute $\int_0^a x^p dx$ for $p = 0, 1, 2$.

1. $p = 0$, $x^p = 1$, $P = \{0 = x_0 < x_1 = a\}$, $L(1, P) = a = U(1, P)$

   $[P'$ refines P, then $L(1, P) \leq L(l, P') \leq U(1, P') \leq U(1, P) = a]$

   It follows that $\int_0^a 1dx = a$.

2. From last corollary

$$\int_0^a xdx = \lim_{n\to\infty} \sum_{j=1}^{n} (j\frac{a}{n})\frac{a}{n} = \lim_{n\to\infty} \frac{a^2}{n^2} \sum_{j=1}^{n} j = \lim_{n\to\infty} \frac{a^2}{n^2} \frac{n(n+1)}{2} = \frac{a^2}{2}$$

3. We need a fomula for $\sum_{j=1}^{n} j^2$.

   Trick:

$$\begin{aligned}
(n+1)^3 - 1 &= \sum_{j=1}^{n} [(j-1)^3 - j^3] && \text{(telescope)} \\
&= \sum_{j=1}^{n} [\sum_{k=0}^{3} \binom{3}{k} j^k - j^3] && \text{(binomial theorem)} \\
&= \sum_{j=1}^{n} \sum_{k=0}^{2} \binom{3}{k} j^k \\
&= \sum_{k=0}^{3}
\end{aligned}$$

$$\begin{aligned}
\int_0^a x^2 dx &= \lim_{n\to\infty} \sum_{j=1}^{n} (j\frac{a}{n})^2 \frac{a}{n} \\
&= \lim_{n\to\infty} \frac{a^3}{n^3} \sum_{j=1}^{n} j^2 \\
&= \lim_{n\to\infty} \frac{a^3}{3n^3} a[(n+1)^3 - 1 - n - \frac{n(n+1)}{2}] \\
&= \frac{a^3}{3}
\end{aligned}$$

Algorithm 1.4.1 (Basic Properties Of Integrals).

Proposition 1.4.1 (Additivity over intervals). *Let $a < b < c \in \mathbb{R}$, and $f : [a, c] \to \mathbb{R}$ satisfies that f is integrable on each of $[a, b]$, $[b, c]$, then*

- *f is integrable on $[a, c]$ and $\int_a^c f = \int_a^b f + \int_b^c f$.*

Proof. Given $\varepsilon > 0$, the Cauchy Criterion provides that

- a partition P_1 of $[a,b]$ s.t. $U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}$
- a partition P_2 of $[b,c]$ s.t. $U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{2}$

Let P be any refinement of $P_1 \cup P_2$. Then

$$L(f,P) \geq L(f,P_1 \cup P_2) = L(f,P_1) + L(f,P_2)$$

$$U(f,P) \leq U(f,P_1 \cup P_2) = U(f,P_1) + U(f,P_2)$$

Then

$$U(f,P) - L(f,P) \leq U(f,P_1) - L(f,P_1) + U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

hence, f is integrable on $[a,c]$.

Let P as above, be written $P = \{a = x_0 < \cdots < x_n = c\}$.

Let $Q_1 = \{a = x_0 < \cdots < x_m = b\}$, $Q_2 = \{b = x_m < \cdots < x_n = c\}$.

We have

$$L(f,Q_1) \leq \int_a^b f \leq U(f,Q_1) \qquad L(f,Q_2) \leq \int_b^c f \leq U(f,Q_2)$$

from the proof above, we have

$$L(f,P) = L(f,Q_1) + L(f,Q_2) \leq \int_a^b f + \int_b^c f \leq U(f,Q_1) + U(f,Q_2) = U(f,P)$$

Since f is integrable on $[a,c]$, we have

$$\int_a^c = \sup\{L(f,P) : P\, partition\, of [a,c]\} \leq \int_a^b f + \int_a^c f \leq \inf\{U(f,P) : P\, partition\, of [a,c]\} = \int_a^c f$$

$\Rightarrow$

$$\int_a^c f = \int_a^b f + \int_b^c f$$

$\square$

1.5 Riemann Sum - Jan 13 Mon, Jan 15 Wed

Definition 1.5.1 (Riemann Sums). *Let $f : [a, b] \to \mathbb{R}$, $P = \{a = x_0 < \cdots < x_n = b\}$.*

*A **Riemann Sum** is any sum of the following form:*

$$S(f, P) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) \qquad t_j \in [x_{j-1}, x_j], j = 1, \cdots, n$$

Left Sum:

$$S_l(f, P) = \sum_{j=1}^{n} f(x_{j-1})(x_j - x_{j-1})$$

Right Sum:

$$S_r(f, P) = \sum_{j=1}^{n} f(x_j)(x_j - x_{j-1})$$

Mid-point Sum:

$$S_m(f, P) = \sum_{j=1}^{n} f(\frac{x_{j-1} + x_j}{2})(x_j - x_{j-1})$$

Trapezoid Sum

$$\begin{aligned} T(f, P) =& \frac{1}{2}[S_l(f) + S_r(f)] \\ =& \sum_{j=1}^{n} \frac{f(x_j) + f(x_j)}{2}(x_j - x_{j-1}) \\ =& \frac{1}{2} f(a)(x_1 - a) + \sum_{j=1}^{n-1} f(x_j)(x_j - x_{j-1}) + \frac{1}{2} f(b)(b - x_{n-1}) \end{aligned}$$

Theorem 1.5.1. *If $f : [a, b] \to \mathbb{R}$, then TFAE,*

1. *f is integrable and*
2. *there is a number I_f satisfying the following: given any $\varepsilon > 0$, there exists a partition P_ε of $[a, b]$ such that*

   for any refinement of P of P_ε, any Riemann Sum of $S(f, P)$ we have

   $$|S(f, P) - I(f)| < \varepsilon$$

   Furthermore, $I_f = \int_a^b f$.

Proof.

(i)⇒(ii) Given $\varepsilon > 0$, the Cauchy Criterion provides that P_ε so for any refinement P of P_ε,

$$U(f, P) - L(f, P) < \varepsilon \tag{1}$$

Write $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, and let for $j = 1, \cdots, n$, $t_j = [x_{j-1}, x_j]$.

We observe that

$$m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\} \leq \sup\{f(x) : x \in [x_{j-1}, x_j]\} = M_j$$

and hence

$$\sum_{j=1}^{n} m_j(x_j - x_{j-1}) \leq \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) \leq \sum_{j=1}^{n} M_j(x_j - x_{j-1})$$

i.e.

$$L(f,P) \leq S(f,P) \leq U(f,P \tag{2}$$

Also,

$$L(f,P) \leq \int_a^b f \leq U(f,P) \tag{3}$$

(1), (2) & (3) $\Rightarrow$

$$\left| S(f,P) - \int_a^b f \right| < \varepsilon$$

In particular, take $I_f = \int_a^b f$.

(ii)⇒(i) we let for $\varepsilon > 0$, given $P_{\varepsilon/4}$ be a partition s.t.

$$|S(f,P) - I_f| < \frac{\varepsilon}{4}$$

For P a refinement of $P_{\varepsilon/4}$, $S(f,P)$ a Riemann Sum. We fix such $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. For $j = 1, \cdots, n$, let m_j, M_j be as below, we then find for each j,

$$x_j^*, x_j^{**} \in [x_{j-1}, x_j] \qquad s.t. \qquad f(x_j^*) < m_j + \frac{\varepsilon}{4(b-a)} \qquad \& \qquad M_j - \frac{\varepsilon}{4(b-a)} < f(x_j^{**})$$

We then consider Riemann Sums

$$S^*(f,P) = \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}), \qquad S^{**}(f,P) = \sum_{j=1}^{n} f(x_j^{**})(x_j - x_{j-1})$$

Notice that

$$\begin{aligned} S^*(f,P) - L(f,P) &= \sum_{j=1}^{n} [f(x_j^*) - m_j](x_j - x_{j-1}) \\ &< \sum_{j=1}^{n} \frac{\varepsilon}{4(b-a)}(x_j - x_{j-1}) \\ &= \frac{\varepsilon}{4(b-a)}(b-a) = \frac{\varepsilon}{4} \end{aligned}$$

and likewise,

$$U(f,P) - S^{**}(f,P) < \frac{\varepsilon}{4}$$

thus

$$\begin{aligned}
&U(f,P)-L(f,P)\\
=&U(f,P)-S^{**}(f,P)+S^{**}(f,P)-I_f+I_f-S^{*}(f,P)+S^{*}(f,P)-L(f,P)\\
<&\frac{\varepsilon}{4}+|S^{**}(f,P)-I_f|+|I_f-S^{*}(f,P)|+\frac{\varepsilon}{4}<\varepsilon
\end{aligned}$$

hence, by Cauchy's Criterion, f is integrable.

□

Remark: If $f:[a,b]\to\mathbb{R}$ is continuous, then P a partition of $[a,b]$ then each of $L(f,P)$ and $U(f,P)$ are Riemann Sums, proof: See proof of integrability of continuous.

Proposition 1.5.1 (linearity of integration). *Let $f,g:[a,b]\to\mathbb{R}$ each be integrable and $\alpha,\beta\in\mathbb{R}$, then*

- $\alpha f+\beta g:[a,b]\to\mathbb{R}\qquad(\alpha f+\beta g)(x)=\alpha f(x)+\beta g(x)$
- $\int_a^b(\alpha f+\beta g)=\alpha\int_a^b f+\beta\int_a^b g$

Proof. Let $\varepsilon>0$, then find partitions of $[a,b]$.

- P_1 s.t. for any refinement P_p of P_1, and any Riemann Sum $S(f,P_p)$

$$\left|S(f,P_p)-\int_a^b f\right|<\frac{\varepsilon}{2|\alpha|+1}$$

- P_2 s.t. for any refinement of Q of P_2, and any Riemann Sum $S(g,Q)$,

$$\left|S(g,Q)-\int_a^b g\right|<\frac{\varepsilon}{2|\beta|+1}$$

Now we let $P=\{P_1\cup P_2\}$, a refinement of each of P_1 and P_2, write $P=\{a=x_0<x_1<\cdots<x_n=b\}$, and choose $t_j\in[x_{j-1},x_j]$ for each j. Then

$$\begin{aligned}
S(\alpha f+\beta g,P)&=\sum_{j=1}^{n}(\alpha f(t_j)+\beta g(t_j))(x_j-x_{j-1})\\
&=\alpha\sum_{j=1}^{n}f(t_j)(x_j-x_{j-1})+\beta\sum_{j=1}^{n}g(t_j)(x_j-x_{j-1})\\
&=\alpha S(f,P)+\beta S(g,P)
\end{aligned}$$

Then we have,

$$\begin{aligned}
\left|S(\alpha f+\beta g,P)-\left[\alpha\int_a^b f+\beta\int_a^b g\right]\right|&\le|\alpha|\left|S(f,P)-\int_a^b f\right|+|\beta|\left|S(g,P)-\int_a^b g\right|\\
&<|\alpha|\frac{\varepsilon}{2|\alpha|+1}+|\beta|\cdot\frac{\varepsilon}{2|\beta|+1}<\varepsilon
\end{aligned}$$

□

Proposition 1.5.2 (Order Properties of Integrals). *Let $f, g : [a, b] \to \mathbb{R}$ each be integrable, then*

1. $f \geq 0 \Rightarrow f \geq 0$
2. $f \geq g \Rightarrow \int_a^b f \geq 0$
3. $f \geq g$ *on* $[a, b] \Rightarrow \int_a^b f \geq \int_a^b g$
4. $|f| : [a, b] \to \mathbb{R}(|f|(x) = |f(x)|)$ *is integrable, with* $\left| \int_a^b f \right| \leq \int_a^b |f|$
5. $g \vee g,\ f \wedge g : [a, b] \to \mathbb{R}$ *(*$f \vee g(x) = \max\{f(x), g(x)\}, f \vee g(x) = \min\{f(x), g(x)\}$*) are each integrable*

Proof.

1. for any partition P, $L(f, P) > 0$.
2. $f - g$ is integrable with $f - g \geq 0$, so $\int_a^b f - \int_a^b g = \int_a^b (f - g) \geq 0$, by 1.
3. let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, and for each $j = 1, \cdots, n$

$\square$

2 ANTIDERIVATIVE

2.1 Fundamental Theorem Of Calculus I - Jan 17 Friday

Proposition 2.1.1. *Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, define*

$$F : [a, b] \to \mathbb{R}, \qquad F(x) = \int_a^x f = \int_a^x f(t)dt$$

<u>Note:</u> no $\int_a^x f(x)dx$.

We may call this "integral accumulation function".

1. *F is continuous on $(a, b]$*
2. *$\lim_{x \to a^+} F(x) = 0$*

hence, we define $F(a) = 0 = \int_a^a f$. Thus $F : [a, b] \to \mathbb{R}$, and is continuous on $[a, b]$.

Proof.

1. A1. Q5(c) assume that f is integrable on each $[a, x]$, $x \in [a, b]$, so $F(x) = \int_a^x f$ makes sense. Now, let $a < x < x' \leq b$, and we compute

$$\begin{aligned} F(x') - F(x) &= \int_a^{x'} f - \int_a^x f \\ &= \int_a^x f + \int_x^{x'} f - \int_a^x f && \text{(additivity)} \\ &= \int_x^{x'} f \end{aligned}$$

   Since f is integrable, it is bounded i.e. $\sup\limits_{x \in [a,b]} |f(x)| = M < \infty$. Thus, $|f(x)| \leq M$ on $[a, b]$. Hence, by order properties,

$$|F(x') - F(x)| = \left| \int_x^{x'} f \right| \leq \int_x^{x'} |f| \leq \int_x^{x'} M = M(x' - x) = M\,|x' - x|$$

   Thus, given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{M+1}$, we have

$$|x' - x| < \delta \Rightarrow |F(x') - F(x)| \leq M\delta = M\frac{\varepsilon}{M+1} < \varepsilon$$

   hence, F is uniformly continuous on $[a, b]$.

2. We use M as above

$$\left| \int_a^x f - 0 \right| = \left| \int_a^x f \right| \leq \int_a^b |f| \leq \int_a^x M = M(x - a)$$

   Porceed as above.

$\square$

Theorem 2.1.1 (**Mean Value For Integrals** or **Average Value for Integrals**)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous (integrability follows), then there exists $c \in [a, b]$, s.t.*

$$\int_a^b f = f(c)(b - a)$$

Proof. We use two important facts about continuous functions.

By **EVT**, there exists $x^*, x^{**} \in [a, b]$ s.t.

$$f(x^*) = m = min\{f(x) : x \in [a, b]\} \qquad and \qquad f(x^**) = M \max\{f(x) : x \in [a, b]\}$$

Then $m \le f \le M$, on $[a, b]$ so order properties provide

$$m(b - a) = \int_a^b m \le \int_a^b f \le \int_a^b M = M(b - a)$$

so

$$f(x^*) = m \le \frac{1}{b - a} \int_a^b f \le M = f(x^{**})$$

By **IVT**, Since $f(x^*) \le \frac{1}{b-a} \int_a^b f \le f(x^{**})$, there is c between x^* and x^{**}, and hence $c \in [a, b]$ s.t.

$$f(c) = \frac{1}{b - a} \int_a^b f$$

$\square$

Remark: f is integrable $\Rightarrow F(x) = \int_a^b f$ is a cts function. f cts $\Rightarrow F$ differentiable. (BELOW)

Theorem 2.1.2 (**Fundamental Theorem of Calculus (I)**)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous, then*

$$F : [a, b] \to \mathbb{R}, \qquad F(x) = \int_a^x f$$

satisfies that F is differentiable on $[a, b]$, with $F' = f$ on $[a, b]$.

Proof. Let $x \in [a, b]$, we want to examine the quotient

$$\frac{F(x + h) - F(x)}{h} \qquad when \qquad x + h \in [a, b]$$

$\underline{h > 0}$,

$$\frac{F(x + h) - F(x)}{h} = \frac{1}{h} \cdot \int_x^{x+h} f = \frac{1}{h} \cdot f(c_h^*)(x + h - x) = f(c_h^*)$$

by M.V.T for I, where $c_h^* \in [x, x + h]$,

$\underline{h < 0}$,

$$\frac{F(x + h) - F(x)}{h} = \frac{F(x) - F(x + h)}{-h} = \frac{1}{-h} \cdot \int_{x+h}^{x} f = \frac{1}{-h} \cdot f(c_h^{**})(x - (x + h)) = f(c_h^{**})$$

by M.V.T for I, where $c_h^{**} \in [x + h, x]$.

hence,

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\underbrace{\lim_{h\to 0}f(c_h^*)}_{continuity}=\underbrace{\lim_{h\to 0}f(c_h^{**})}_{squeeze}=f(x)$$

Thus, $F'(x)$ exists, and equals $f(x)$, for $x\in[a,b]$.

Remark: Notice that we really found

- left derivative at $x=b$
- right derivative at $x=a$

$\square$

Notation 2.1.1. *Let $-\infty\leq a<b\leq\infty\in\mathbb{R}$, $f:[a,b]\to\mathbb{R}$ be continuous, fix $c\in(a,b)$, dfeine*

$$F:(a,b)\to\mathbb{R}, F(x)=\begin{cases}\displaystyle\int_c^x f, & x\geq c\\ \displaystyle-\int_x^c f, & x<c\end{cases}$$

We know from FToCI, that $F'(x)=f(x)$ for $x>c$.

Proposition 2.1.2. *Let us compute $F'(x)$ for $x<c$, let $c'\in(a,c)$ and for $x\in(c',c)$ we have*

$$\begin{aligned} & \int_{c'}^c f=\int_{c'}^x f+\int_x^c f\\ \Rightarrow\quad & -\int_x^c f=\int_{c'}^x f-\int_{c'}^c f\\ \Rightarrow\quad & F'(x)=\frac{d}{dx}\int_{c'}^x f-\int_{c'}^c f=f(x)\end{aligned}$$

It will be convecient, hereafter, to let $\int_c^x f=-\int_x^c f$ if $x<c$, and we have FToCI

$$\frac{d}{dx}\int_c^x f=f(x),\qquad x\in(a,b).$$

2.2 Logrithm and Exponential Functions

Definition 2.2.1. *For* $x \in (0, \infty)$*,*

$$L(x) = \int_1^x \frac{1}{t} dt$$

we shall use only integral & differentiation rates to gain theory of L*.*

Proposition 2.2.1. *If* $a, b > 0$*, gthen* $L(ab) = L(a) + L(b)$*.*

Proof. Let $F(x) = L(ax)$, then chain rule provides

$$F'(x) = \frac{1}{ax}\frac{d}{dx}(ax) = \frac{1}{x} = L'(x)$$

hence, $F' - L' = 0 \Rightarrow F - L = C$ (constant), by MVT, $F = L + C(*)$. Then,

$$L(a) = F(1) = L(1) + C = C.$$

Also, $L(ab) = F(b) = L(b) + L(a)$.

□

Proposition 2.2.2. *For* $a > 0$*,* $q \in \mathbb{Q}$*,* $L(a^q) = qL(a)$*, (convention:* $a^0 = 1$*).*

Proof. First: $n \in \mathbb{N}$,

$$L(a^n) = L(a) + L(a^{n-1}) = \cdots = \underbrace{L(a) + L(a) + \cdots + L(a)}_{n} = nL(a) \tag{1}$$

Second:

$$L(a) = L((a^{\frac{1}{n}})^n) = nL(a^{\frac{1}{n}}) \Rightarrow L(a^{\frac{1}{n}}) = \frac{1}{n}L(a) \tag{2}$$

Third:

$$0 = L(1) = L(aa^{-1}) = L(a) + L(a^{-1}) \Rightarrow L(a^{-1}) - L(a) \tag{3}$$

Then, (1) & (2) $\Rightarrow L(a^m) = mL(a)$, for $m \in \mathbb{Z}$, for $q = \frac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$.

We combine (1), (2), &, (3) to get $L(a^q) = mL(a^{\frac{1}{n}}) = \frac{m}{n}L(a)$.

□

Proposition 2.2.3.

1. L *is inreasing:* $0 < x < x'$ *then* $L(x) < L(x')$
2. $\lim_{x \to 0^+} L(x) = -\infty$*,* $\lim_{x \to \infty} L(x) = \infty$

Proof.

1.
$$L(x') - L(x) = \int_x^{x'} \frac{1}{t} dt \geq \int_x^{x'} \frac{1}{x'} dt = \frac{1}{x'}(x' - x) > 0$$

   Alternatively: $L'(x) = \frac{1}{x} > 0$, MVT $\Rightarrow L$ is strictly increasing.

2. To see that $\lim_{x\to\infty} L(x) = \infty$, it suffices to find $(a_n)_{n=0}^{\infty} \subset (0,\infty)$ s.t. $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} L(x_n) = \infty$. Consider $(2^n)_{n=0}^{\infty}$ and we have $\lim_{n\to\infty} L(2^n) = \lim_{n\to\infty} nL(2) = \infty$. Likewise, $\lim_{n\to\infty} 2^{-n} = 0$, and $\lim_{n\to\infty}(2^{-n}) = \lim_{n\to\infty}(-n)L(2) = -\infty$.

□

Corollary 2.2.1. *$L : (0,\infty) \to \mathbb{R}$ is one-to-one and onto.*

Proof. Increasing $\Rightarrow$ one-to-one, since $\lim_{x\to 0^+} = -\infty$, $\lim_{x\to\infty} L(x) = \infty$, and IVT provides that L is onto.

□

Definition 2.2.2. *$E : \mathbb{R} \to (0,\infty)$ to be L^{-1}: inverse function. Hence,*

$$E(L(x)) = x, x \in (0,\infty) \qquad \text{and} \qquad L(E(y)) = y \qquad if\ y \in \mathbb{R}$$

Proposition 2.2.4. *If $y \in \mathbb{R}$, $L(E(y)) = y$, chain $\underset{\Rightarrow}{}$ rule$\frac{1}{E(y)}E'(y) = 1$*
$\Rightarrow E'(y) = E(y)$

Algorithm 2.2.1 (About E). *Let $c, d \in \mathbb{R}$,*

1. *$E(c+d) = E(c)E(d)$*
2. *$E(-c) = \frac{1}{E(c)}$*
3. *$E(0) = 1$*
4. *$E(qc) = E(c)^q$, $q \in \mathbb{Q}$*

Proof. 1. Let $c = L(a)$, $d = L(b)$ (L is onto) $E(c+d) = E(L(a)+L(b)) = E(L(ab)) = ab = E(a)E(b)$

2. $L(1) = 0$ so $E(0) = 1$
3. use (1) and (2)
4. $E(qc) = E(qL(a)) = E(L(a^q)) = a^q = E(c)^q$.

□

What is $E(1)$? We note that

$$\lim_{h\to 0} \frac{L(1+h)}{h} = L'(1) = \frac{1}{1} = 1$$

Hence,

$$1 = \lim_{n\to\infty} \frac{L(1+\frac{1}{n})}{\frac{1}{n}} = \lim_{n\to\infty} nL(1+\frac{1}{n}) = \lim_{n\to\infty} L((1+\frac{1}{n})^n)$$

Since E is continuous,

$$E(1) = E(\lim_{n\to\infty} L((1+\frac{1}{n})^n)) = \lim_{n\to\infty} E(L((1+\frac{1}{n})^n)) = \lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

From rule (iv), $E(q) = e^q$ for $q \in \mathbb{Q}$, if $x \in \mathbb{R}$, write $x = \lim_{n\to\infty} q_n$, each $q_n \in \mathbb{Q}$, and we define

$$e^x = E(x) = \lim_{n\to\infty} E(q_n) = \lim_{n\to\infty} e^{q_n}$$

Definition 2.2.3. *For $a > 0$, we have $a = E(L(a)) = e^{L(a)}$, and we let*

$$a^x = E(L(a)x) = e^{L(a)x}$$

Exercise With Chain Rule:

1. $\frac{d}{dx}(a^x) = L(a)a^x$,
2. $L(a^x) = L(a)x = xL(a)$,
3. $p \in \mathbb{R}$, $x > 0$, $x^p = e^{p(L(x))}$, $\frac{d}{dx}(x^p) = px^{p-1}$

2.3 Fundamental Theorem of Calculus II - Jan 22

Theorem 2.3.1 (Fundamental Theorem of Calculus II). *Let $f, F : [a, b] \to \mathbb{R}$ satisfy that*

- *f is integrable*
- *F is continuous on $[a, b]$*
- *F is differentiable on (a, b), with $F' = f$ on (a, b)*

Then,

$$F(b) - F(a) = \int_a^b f$$

Proof. Let $\varepsilon > 0$, find a partition P_ε on $[a, b]$, so

- for every refinement P of P_ε
- for every Riemann Sum $S(f, P)$, we have

$$\left| S(f, P) - \int_a^b f \right| < \varepsilon$$

Take P as above, write $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$.

Now let us consider F on each $[x_{j-1}, x_j]$

- F is continuous on $[x_{j-1}, x_j]$
- F is differentiable on $[x_{j-1}, x_j]$ [can be used in closed interval, except for $j = 0, n$]

Thus MVT tells us there exists $c_j \in (x_{j-1}, x_j) \subset [x_{j-1}, x_j]$ such that

$$F(x_j) - F(x_{j-1}) = F'(c_j)(x_j - x_{j-1}) = f(c_j)(x_j - x_{j-1}) \qquad (*)$$

Now we consider

$$\begin{aligned} F(b) - F(a) &= \sum_{j=1}^{n} [F(x_j) - F(x_{j-1})] && \text{(telescope)} \\ &= \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) && \text{(by *)} \\ &= S(f, P) && \text{(a Riemann Sum)} \end{aligned}$$

Hence,

$$\left| F(b) - F(a) - \int_a^b f \right| = \left| S(f, P) - \int_a^b f \right| < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we get desired result. $\square$

Remark:

- Suppose $F, G : [a, b] \to \mathbb{R}$, both satisfy $F' = f = G'$, for integrable f, then

$$(F - G)' = F' - G' = f - f = 0 \underset{\Rightarrow}{M.V.T} F - G = C(constant)$$

  hence, $F(x) = G(x) + C$ for any x in $[a, b]$.

- If $f : [a, b] \to \mathbb{R}$ is continuous, then f is integrable (theorem from earlier) & $F(x) = \int_a^b f$ defines on antiderivative.

  Moral: f continuous $\to$ an antiderivative exists.

Notation 2.3.1. *If f is continuous, (on same intervals), and F is an antiderivative of f, i.e. $F' = f$ (on interval of said intervals), write $\int f(x)dx = F(x) + C$.*

Antiderivatives of Basic Functions:

$$p \neq -1, \qquad \int x^p dx = \frac{x^{p+1}}{p+1} + C \qquad\qquad \int e^x dx = e^x + C$$

$$\int \cos x dx = \sin x + C \qquad\qquad \int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C \qquad\qquad \int \sec^2 x dx = \tan x + C$$

$$\int \frac{1}{x^2+1} dx = \arctan x + C[Tan = \tan|_{(\frac{\pi}{2}, \frac{-\pi}{2})]:(-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}} \qquad \text{one-to-one and onto}$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C[\text{Sin} = \sin|_{(\frac{\pi}{2}, \frac{-\pi}{2})]:(-\frac{\pi}{2}, \frac{\pi}{2}) \to [-1,1]} \qquad \text{one-to-one and onto}$$

Theorem 2.3.2 (Change of Variables/Substitution/Reverse Chain Rule). *Suppose*

- *$g : [a, b] \to \mathbb{R}$, differentiable with g' continuous*
- *f is defined on $g([a, b])$ with $f \circ g : [a, b] \to \mathbb{R}$ continuous*

Then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)d(u)$$

Anti Derivative Form:

$$\int f(g(x))g'(x)dx = \int f(u)du|_{u=g(x)}$$

Proof. Let F be any antiderivative of $f[g[(a, b)] = [c, d]$, let $F(x) = \int_x^c f]$.

Let $H : [a, b] \to \mathbb{R}$ be given by $H(x) = F(g(x))$. Then Chain Rule provides

$$H'(x) = F/(g(x))g'(x) = f(a(x))g'(x)$$

and F.T. of C II provides that

$$H(b) - H(a) = \int_a^b f(g(x))g'(x)dx$$

but F.T.of C provides that

$$\int_{g(a)}^{g(b)} f(u)d(u) = F(g(b)) - F(g(a)) = H(b) - H(a)$$

$\square$

Example:

1.

$$\begin{aligned}\int xe^{-x^2}dx &= -\frac{1}{2}\int e^{-x^2}(-2x)dx\\ &= -\frac{1}{2}\int e^u du\\ &= -\frac{1}{2}e^u + C\\ &= -\frac{1}{2}e^{-x^2} + C\end{aligned}$$

2.

$$\begin{aligned}\int_1^3 x(x^2+4)^{91}dx &= \frac{1}{2}\int_5^{13} u^{91}dx\\ &= \frac{1}{2}\frac{u^{92}}{92}|_5^{13}\\ &= \frac{1}{184}[(13)^{92} - 5^{92}]\end{aligned}$$

3.

$$\begin{aligned}\int \cos^m x \sin^n x dx &= \int \cos^m x \sin^{2k} x \sin x dx && (\text{n} \quad \text{odd})\\ &= \int \cos^m x(1-\cos^2 x)^k \sin x dx && (u = \cos x,\ du = -\sin x dx)\\ &= -\int u^m(1-u^2)^k du|_{u=\cos x}\end{aligned}$$

2.4 Integration and Trignometry - Jan 22 Wed, TUT

Definition 2.4.1. $\pi = 2\int_{-1}^{a} \sqrt{a - x^2}dx$

Definition 2.4.2. *Let for* $-1 \le x \le 1$,

$$\arccos x = x\sqrt{1-x^2} + 2\int_x^1 \sqrt{1-u^2}du$$

Then $\frac{1}{2}\arccos x$ *is the area of —–graph—–*

Note: $\frac{1}{2}\arccos x$ is proportional to the angle θ, hence it is reasonable to measure.

$$\theta = \arccos x \qquad \text{"}radians\text{"}$$

- $\arccos -1 = \pi$
- $\arccos 0 = 2\int_0^1 \sqrt{1-u^2}du \overset{symmetry}{=} \int_{-1}^1 \sqrt{1-u^2}du = \frac{\pi}{2}$
- $\arccos 1 = 0$

Derivatives:

$$\begin{aligned}\arccos' x &= \sqrt{1-x^2} + x\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) - 2\sqrt{1-x^2} \\ &= -\frac{x^2}{\sqrt{1-x^2}} - \sqrt{1-x^2}\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} = -\frac{1}{\sqrt{1-x^2}}\end{aligned}$$

hence,

- $\arccos' x < 0$ and by MVY, decreasing
- $\lim_{x\to -1^+} \arccos' x = -\infty = \lim_{x\to 1^-} \arccos' x$
- $\arccos' 0 = -1$
- $\arccos''(x) = -\frac{x}{(1-x^2)^{\frac{3}{2}}}$ hence,
  - $\arccos''(x) > 0$ if $x < 0 \Rightarrow$ concave up
  - $\arccos''(x) < 0$ if $x > 0 \Rightarrow$ concave down

Definition 2.4.3.

- $\operatorname{Cos} x = \arccos^{-1} : [0, \pi] \to [-1, 1]$
- $\sin\theta = \sqrt{1 - \cos^2\theta}$

Hence, $\sin : [0, \pi] \to [0, 1]$, *with*

- $\operatorname{Sin}(0) = \sqrt{1 - 1^2} = 0$
- $\operatorname{Sin}(\frac{\pi}{2}) = \sqrt{1 - 0^2} = 0$
- $\operatorname{Sin}(\pi) = \sqrt{1 - (-1)^2} = 0$

Derivatives of $\cos$, $\sin$

$\arccos(\cos\theta) = \theta$

$$\underset{\text{Chain Rule}}{\Rightarrow} \frac{-1}{\sqrt{1-\cos^2\theta}}\cos'\theta = 1 \Rightarrow \cos'\theta = -\sin\theta$$

$$\sin'\theta = \frac{d}{d\theta}\sqrt{1-\cos^2\theta} = \frac{1}{x}(1-\cos^2\theta)^{-\frac{1}{2}}(-2\cos\theta\cos'\theta) = \cos\theta$$

Hence, $\sin'(0) = 1$, $\sin'\frac{\pi}{2} = 0$, $\sin'(\pi) = -1$, and $\sin''(\theta) = -\sin\theta < 0$ if $0 < \theta < \pi \Rightarrow$ concave down

Extension to $\mathbb{R}$

(a) we define $\cos, \sin : [-\pi, \pi] \to [-1, 1]$

- $\cos$ is even: $\cos(-\theta) = \cos\theta$, $\theta \geq 0$
- $\sin$ is odd: $\sin(-\theta) = -\sin\theta$, $\sin\theta = \operatorname{Sin} x$, if $\theta \geq 0$

(b) we define $\cos, \sin : \mathbb{R} \to [-1, 1]$

$$\cos(\theta + 2\pi n) = \cos(\theta) \qquad \sin(\theta + 2\pi n) = \sin(\theta) \qquad \theta \in [-\pi, \pi], n \in \mathbb{Z}$$

Lemma 2.4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable, then*

- $f(0) = f'(0) = 0$ *and*
- $f'' + f = 0$

then $f = 0$.

Proof. Let $g = (f')^2 + f^2$ then

$$g(0) = 0 \qquad \text{and} \qquad g' = 2ff' + 2ff' = 2f[f'' + f] = 0$$

$\Rightarrow$ by MVT, g constant, hence, $g = 0$, then $0 \leq f^2 \leq g$. □

Lemma 2.4.2. *Double Angle Fomula for Cos*

Proof. Let $a, b \in \mathbb{R}$ be fixed, defined $f : \mathbb{R} \to \mathbb{R}$,

$$f(t) = \cos(s + t) - a\sin t + b\cos t$$

Then

$$\begin{aligned} f'(t) &= -\sin(s+t) + a\sin t + b\cos t \\ f''(t) &= -\cos(s+t) + a\cos t - b\sin t \\ \Rightarrow f'' + f &= 0 \end{aligned}$$

Now we wish to choose a, b to satisfy

$f(0) = 0$, hence $0 = f(0) = \cos s - a \Rightarrow a = \cos s$
$f(0) = 0$, hence $0 = f'(0) = -\sin s + b \Rightarrow b = \sin s$

With these choices of a, b, the lamma tells us that $f(t) = 0$, hence

$$0 = \cos(s + t) - [\cos s\cos t - \sin s\sin t)$$

□

Double Angle Fomula for $\cos$**:** Since $\cos^2 t + \sin^2 t = 1$, the angle sum fomula gives

$$\cos 2t = \cos^2 t - \sin^2 t = \begin{cases} 1 - 2\sin^2 t \Rightarrow \sin^2 t = \frac{1}{2}[1 - \cos^2 t] \\ 2\cos^2 t - 1 \Rightarrow \cos^2 t = \frac{1}{2}[1 - \cos^2 t] \end{cases}$$

Lemma 2.4.3. *Double Angle Fomula for* $\sin$*:* $\sin(s+t) = \cos s \sin t + \sin x \cos t$

Proof. Fix $s \in \mathbb{R}$, for t consider

$$\cos(s+t) = \cos s \cos t - \sin s \sin t$$

and take $\frac{d}{dt}$ to both sides. □

Double Angle Fomula for $\sin$**:**

$$\sin 2t = 2\cos t \sin t$$

Example 1:

1.

$$\begin{aligned} \int \sin^2 x dx =& \frac{1}{2}\int (1 - \cos 2x) dx \\ =& \frac{1}{2}[x - \frac{1}{2}\sin 2x] + C \\ =& \frac{1}{2}x - \frac{1}{4}\sin 2x + C \\ =& \frac{1}{2}x - \frac{1}{2}\sin x \cos x + C \end{aligned}$$

2.

$$\begin{aligned} \int \cos^4 x dx =& \int [\frac{1}{2}(1 + \cos 2x)]^2 dx \\ =& \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x) dx \\ =& \frac{1}{4}\int (1 + 2\cos 2x + \frac{1}{2}[1 + \cos 4x]) dx \end{aligned}$$

3.

$$\int \sin x \cos^4 x dx \qquad (u = \cos x, du = -\sin x dx)$$

$$= -\int u^4 du|_{u=\cos x}$$

$$= -\frac{\cos^5 x}{5} + C$$

4.

$$\begin{aligned}\int \sin^2 x \cos^4 x dx =& \int \sin^2 x \cos^2 x \cos^2 x dx \\ =& \int (\frac{1}{2}\sin 2x)^2 \frac{1}{2}[1+\cos 2x] dx \\ =& \frac{1}{8}\int [\sin^2 2x + \sin^2 2x \cos 2x] dx\end{aligned}$$

Change of Variables(Antiderivatives form)

$$\int f(g(x))g'(x)dx = \int f(u)du|_{u=g(x)}$$

f, g' continuous.

Inverse Form: Suppose we try $x = g(u)$,

$$\int f(x)dx = \int f(g(u))g'(u)du|_{x=g(u)}$$

Algorithm 2.4.1 (**Trig Substitution**).

Forms	*Substitution*	*Main Identity*	*dx*
$a^2 - x^2$	$x = a\sin\theta$	$a^2 - x^2 = a^2\cos^2\theta$	$dx = a\cos\theta d\theta$
$x^2 + a^2$	$x = a\tan\theta$	$x^2 + a^2 = a^2\sec\theta$	$dx = a\sec^2\theta d\theta$

Examples

1.

$$\begin{aligned}\int \frac{dx}{(9-x^2)^{3/2}} =& \int \frac{3\cos\theta}{(9\cos^2\theta)^{3/2}} dx \\ =& \frac{1}{9}\int \sec^2\theta d\theta = \frac{1}{9}\tan\theta + C \\ =& \frac{1}{9}\frac{\sin\theta}{\sqrt{1-\sin^2\theta}} + C \\ =& \frac{1}{9}\frac{\frac{1}{3}x}{\sqrt{1-(\frac{1}{3}x)^2}} + C = \frac{1}{9}\frac{x}{\sqrt{9-x^2}} + C\end{aligned}$$

2.

$$\begin{aligned}\int \frac{dx}{x^2+2x+5} =& \int \frac{dx}{(x+1)^2+4} \qquad (x+1 = 2\tan\theta, dx = 2\sec^2\theta d\theta) \\ =& \int \frac{2\sec^2\theta}{2^2\sec^2\theta} d\theta \\ =& \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C \\ =& \frac{1}{2}\arctan\frac{x+1}{2} + C\end{aligned}$$

3.

$$\begin{aligned}\int \sqrt{1-x^2}dx =& \int \cos^2\theta d\theta\\ =&\frac{1}{2}\int [1+\cos 2\theta]d\theta\\ =&\frac{1}{2}[\theta+\frac{1}{2}\sin 2\theta]+C\\ =&\frac{1}{2}[\arcsin x+\sin\theta\cos\theta]+C\\ =&\frac{1}{2}[\arcsin x]+x\sqrt{1-x^2}+C\\ \Rightarrow&\arcsin(x) = 2\int\sqrt{1-x^2}dx - x\sqrt{1-x^2}+C'\\ [\arcsin x = \frac{\pi}{2}-\arccos x]\checkmark\end{aligned}$$

4.

$$\begin{aligned}\int\frac{dx}{\sqrt{x^2+1}} =& \int\frac{\sec^2\theta d\theta}{\sec\theta} && (x=\tan\theta, dx=\sec^2\theta d\theta)\\ =&\int\sec\theta d\theta\\ =&\int\sec\theta\frac{\sec\theta\tan\theta}{\sec\theta+\tan\theta}d\theta\\ =&\int\frac{\sec^2\theta+\tan\theta\sec\theta}{\sec\theta+\tan\theta}d\theta\\ =&\log|\sec\theta+\tan\theta|+C\\ =&\log(\sqrt{x^2+1}+x)+C\end{aligned}$$

5.

$$\int\frac{dx}{\sqrt{x^2+1}} = \int\frac{\cosh t}{\cosh t}dt \qquad (x=\tan\theta,)$$

2.5 Integration by Partial Fraction - Jan 27

Warm Up:

$$\begin{aligned}
\int \sec\theta d\theta &= \int \frac{1}{\cos\theta} d\theta \\
&= \int \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} d\theta \\
&= \int \frac{\sin^2\theta}{\cos\theta} d\theta + \int \cos\theta d\theta \\
&= \int \frac{\sin^2\theta}{1-\sin^2\theta} \cos\theta d\theta + \int \cos\theta d\theta
\end{aligned}$$

Theorem 2.5.1.

1. *Let $q \neq 0$ be a polynomial with $\mathbb{R}$-coefficients, then we may write*

$$q(x) = a(x-r_1)^{m_1} \cdots (x-r_m)^{m_m} \cdot (x^2 + b_1 x + c_a)^{n_1} \cdots (x^2 + b_N x + C_N)^{n_N}$$

   where $a \neq 0$, $r_1, \cdots, r_M$ are the distinct $\mathbb{R}$-roots of q, and $b_1, \cdots, b_N, \cdots, c_N \in \mathbb{R}$..

   $b_j^2 - 4c_j < 0$ for $j = 1, \cdots, N$. Also, $m_1, \cdots, m_N, n_1, \cdots, n_N \in \mathbb{N}$.

2. *Let p be $\mathbb{R}-$polynomial with*

$$\deg p < \deg q$$

   Then there are unique $\mathbb{R}-$numbers $A_1, \cdots, B_N, C_N$. so

$$\frac{p(x)}{q(x)} = \sum_{j=1}^{M} \sum_{k=1}^{M} \frac{A_j, k}{(x-r_j)^k} = + \sum_{j=1}^{N} \sum_{k=1}^{n_j} \frac{B_{j,k}x + C_{j,k}}{x^2 + b_j x + c_j}^k$$

2.6 Integration by parts - Jan 29

Theorem 2.6.1 (Integration by Parts/"Reverse Product Rule"). *Let $f, gF : [a,b] \to \mathbb{R}$ satisfy*

- *f is integrable on $[a,b]$*
- *$F' = f$ on $[a,b]$*
- *g' is integrable on $[a,b]$*

Then

$$\int_a^b f(x)g(x)dx = F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)dx$$

Antiderivative Form:

$$\int f(x)g(x)dx = F(x)g(x) - \int F(x)g'(x)dx, \qquad F(x) = \int f(x)dx \qquad \textit{Can choose } c = 0$$

$$\int f'g = fg - \int fg'$$

Proof. Product Rule:

$$\frac{d}{dx}[F(x)g(x)] = F'(x)g(x) + F(x)g'(x) = f'(x)g(x) + F(x)g'(x)$$

FToCII:

$$F(b)g(b) - F(a)g(a) = \int_a^b [f(x)g(x) + F(x)g'(x)]dx$$

$$\Rightarrow F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)dx = \int_a^b f(x)g(x)dx$$

□

Example 1

$$\begin{aligned}\int \arctan(x)dx &= \int 1 \cdot \arctan(x)dx \\ &= x\arctan(x) - \int x\frac{1}{1+x^2}dx \\ &= x\arctan(x) - \frac{1}{2}\log(1+x^2) + C\end{aligned}$$

Example 2

$$\begin{aligned}\int x^2e^xdx &= x^2e^x - \int 2x \cdot e^xdx \\ &= x^2e^x - 2[xe^x - \int e^xdx] \\ &= x^2e^x - 2xe^x + 2e^x + C\end{aligned}$$

Example 3

$$\begin{aligned}
\int \cos^{2n}(x)dx \qquad n \geq 1 =& \int \cos x \cos^{2n-1} x dx && (I_n)\\
=& \sin x \cos^{2n-1} dx - \int \sin x (2n-1)\cos^{2n-2}(-\sin x) dx\\
=& \sin x \cos^{2n-1} x + (2n-1)\int (1-\cos^2 x)\cos^{2n-2} x dx\\
=& \sin x \cos^{2n-1} x + (2n-1)[\int \cos^{2n-2} x dx - \int \cos^{2n} x dx]\\
=& \sin x \cos^{2n-1} x + (2n-1)[I_{n-1}(x) - I_n(x)]\\
\Rightarrow 2nI_n(x) =& \sin x \cos^{2n-1} x + (2n-1)I_{n-1}(x)\\
I_n(x) =& \frac{1}{2n}\sin x \cos^{2n-1} x + \frac{2n-1}{2n} I_{n-1}(x) && (\text{"Reduction Fomula"})
\end{aligned}$$

Specific Example: $n = 0$, $I_0(x) = \int \cos^0 x dx = \int 1 dx = x + C$ Hence

$$\begin{aligned}
\int \cos^2 x dx = I_1(x) =& \frac{1}{2}\sin x \cos x + \frac{1}{2}[x + C]\\
=& \frac{1}{2}\sin x \cos x + \frac{1}{2}x + C'
\end{aligned}$$

$$\begin{aligned}
\int \cos^2 x dx =& \frac{1}{2}\int [1 + \cos 2x] dx\\
=& \frac{1}{2}x + \frac{1}{4}\sin 2x + C
\end{aligned}$$

$$\begin{aligned}
\int \cos^4 x dx =& I_2(x) = \frac{1}{4}\sin x \cos^3 x + \frac{3}{4}[\frac{1}{2}\sin x \cos x + \frac{1}{2}x] + C\\
=& \frac{1}{4}\sin x \cos^3 x + \frac{3}{8}\sin x \cos x + \frac{3}{8}x + C
\end{aligned}$$

Exmaple 3'

$$\begin{aligned}
\int \frac{dt}{(t^2+1)^3} =& \int \frac{\sec^2\theta}{(\sec^2\theta)^3} d\theta\\
=& \int \cos^4 \theta d\theta\\
=& \frac{1}{4}\sin\theta \cos^3\theta + \frac{3}{8}\sin\theta\cos\theta + \frac{3}{8}\theta + C\\
=& \frac{1}{4}\frac{t}{(1+t^2)^2} + \frac{3}{8}\frac{t}{1+t^2} + \frac{3}{8}\arctan(t) + C
\end{aligned}$$

2.7 Improper Integral - Jan 29

Recall: Integration involves upper and lower sums and hence requires

- bounded functions and
- bounded intervals

Definition 2.7.1. *let $a < b$ and $f : (a, b] \to \mathbb{R}$*

- *f is integrable on $[x, b]$ for each $x \in (a, b]$.*

*Then we define the **improper integral** by*

$$\int_a^b f = \lim_{x \to a^+} \int_x^b f, \qquad \textit{provided that limit exists}$$

Example 1:

$f(t) = \frac{1}{\sqrt{t}}$ on $(0, 2]$, notice that f is continuous, hence integrable on $[x, 2]$, $0 < x < 2$.

Compute

$$\int_x^2 \frac{dt}{\sqrt{t}} = \int_x^2 t^{-1/2} dt = 2t^{1/2}\bigg|_x^2 = 2\sqrt{2} - 2\sqrt{x}$$

Then

$$\int_0^2 \frac{dt}{\sqrt{t}} = \lim_{x \to 0^+} \int_x^2 \frac{dt}{\sqrt{t}} = 2\sqrt{2} - 2\sqrt{0} = 2\sqrt{2}$$

Example 2:

$g(t) = \frac{1}{t^2}$ on $[0, 2]$. g is cts, so integrable on each $[x, 2]$, $0 < x < 2$.

$$\int_x^2 \frac{dt}{t^2} = -\frac{1}{t}\bigg|_x 62 = \frac{1}{x} - \frac{1}{2}$$

$$\lim_{x \to 0^+} \int_x 62 \frac{dt}{t^2} = \lim_{x \to 0^+} [\frac{1}{x} - \frac{1}{2}] = \infty$$

We write $\int_0^2 \frac{dt}{t^2} = \infty$ or $\int_0^2 \frac{dt}{t^2}$ D.N.E..

Example 3:

$h(t) - \frac{\left|\sin \frac{1}{t}\right|}{\sqrt{t}}$, $t \in (0, 2]$, h is continuous on each $[x, 2]$, $0 < x < 2$.

How can we show if this is improperly integrable?

Comparison method

$$\begin{aligned} & 0 \leq \left|\sin \frac{1}{t}\right| \leq 1 \\ \Rightarrow \quad & 0 \leq \frac{\left|\sin \frac{1}{t}\right|}{\sqrt{t}} \leq \frac{1}{\sqrt{t}} \\ \Rightarrow \quad & 0 \leq \int_x^2 \frac{\left|\sin \frac{1}{t}\right|}{\sqrt{t}} dt \leq \int_x^2 \frac{dt}{\sqrt{t}} = 2\sqrt{2} - 2\sqrt{x} \leq 2\sqrt{2} \end{aligned}$$

$H(x) = \int_x^2 \frac{\left|\sin\frac{1}{t}\right|}{\sqrt{t}} dt$ is nonincreasing.

If $0 < x' < x < 2$, $H(x') - H(x) = \int_{x'}^2 h - \int_x^2 h = \int_{x'}^x h + \int_x^2 h - \int_x^2 h = \int_{x'}^x h \geq 0$.

$$H(x') \geq H(x)$$

$H'(x) - h(x) \leq 0$ by F.T. of C.I., $M.V.T. \Rightarrow H$ is non-increasing, $H(x)$ is bouneded on $(0, 2]$ and monotone.

$$\therefore \lim_{x \to 0^+} H(x) = \int_0^\infty \frac{\left|\sin(\frac{1}{t})\right|}{\sqrt{t}} dt \qquad \text{exists}$$

2.8 Jan 31

Facts from MATH 147:

1. $\lim_{x\to a} F(x) = L \Leftrightarrow$ for every sequence $(a_n)_{n=1}^{\infty}$ s.t. $\lim_{n\to\infty} a_n = a$, provides that $\lim_{n\to\infty} F(a_n) = L$.

2. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$, then $\lim_{n\to\infty} a_n$ exists $\Leftrightarrow$ for any $\varepsilon > 0$, there exists $n_\varepsilon \in N$ s.t. $|a_m - a_n| < \varepsilon$ whenever $m, n \geq n_\varepsilon$.

Cauchy criterion[Deep Fact: Bolzano Weierstrass Theorem]

Theorem 2.8.1 (Cauchy Criterion for limit of function). *Let $F : (a, b] \to \mathbb{R}$, then $\lim_{x\to a^+} F(x) \Leftrightarrow$ exists for any $\varepsilon > 0$, there is $\delta > 0$ s.t. $|F(u) - F(v)| < \varepsilon$ whenever $|u - a| < \delta$ and $|v - a| < \delta$ for $u, v \in (a, b]$ (*).*

Proof. $\Rightarrow$ Let $L = \lim_{x\to\infty^+} F(x)$, then, given $\varepsilon > 0$, there is $\delta > 0$ s.t.

$$|F(u) - L| < \frac{\varepsilon}{2}$$

where $|u - a| < \delta$ and $u \in (a, b]$.

Hence, if $u, v \in (a, b]$, $|u - a| < \delta$, $|v - a| < \delta$, then

$$|F(u) - F(v)| \leq |F(u) - L| + |L - F(v)| < \varepsilon$$

$\Leftarrow$ We will verify Fact 1. Hence, let $(a_n)_{n=1}^{\infty} \subset (a, b]$ be any sequence s.t. $\lim_{n\to\infty} a_n = a$, we wish to see that $(F(a_n))_{n=1}^{\infty}$ is Cauchy, hence, by fact 2, is convergent, let $\varepsilon > 0$ be given, find $\delta > 0$ as in $(*)$

$\lim_{n\to\infty} a_n = a \Rightarrow$ there exists $n_\delta \in \mathbb{N}$ s.t. $|a - a_n| < \delta$ whenever $n \geq n_\delta$.

Hence, if $m, n \geq n_\delta$, we have

$$\left.\begin{matrix} |a - a_m| < \delta \\ |a - a_n| < \delta \end{matrix}\right\} \Rightarrow |a_m - a_n| < \delta$$

note both a_n, a_m are to the right of a.

Thus $(*)$ provide that $|F(a_n) - F(a_m)| < \varepsilon$. Summary, we have $n_\varepsilon = n_\delta$ s.t. $|F(a_n) - F(a_m)| < \varepsilon$ whenever $n, m \geq n_\varepsilon$. $\square$

Last Time:

$$\int_0^2 \frac{\left|\sin(\frac{1}{t})\right|}{\sqrt{t}} dt = \lim_{x\to 0^+} \underbrace{\int_x^2 \frac{\left|\sin(\frac{1}{t})\right|}{\sqrt{t}} dt}_{H(x)}$$

H is monotone and bounded $\Rightarrow \lim_{x\to a^+} H(x)$ exists.

Example 1:

Consider

$$\begin{aligned}
&\int_0^1 \frac{\sin(\frac{1}{t})}{\sqrt{t}}dt = \lim_{x\to 0^+}\int_x^1 \frac{\sin(\frac{1}{t})}{\sqrt{t}}dt \\
&-1 \le \sin(\frac{1}{t}) \le 1 \\
\Rightarrow & -\frac{1}{\sqrt{y}} \le \frac{\sin(\frac{1}{t})}{\sqrt{t}} \le \frac{1}{\sqrt{t}} \overset{order\ properties}{\Rightarrow} -\int_x^1 \frac{dt}{\sqrt{t}} \le \int_x^1 \frac{\sin(\frac{1}{t})}{\sqrt{t}}dt \le \int_x^1 \frac{dt}{\sqrt{t}}
\end{aligned}$$

Now we consider $0 < u < v < 1$, again order properties give:

$$-\int_u^v \frac{dt}{\sqrt{t}} \le \int_u^v \frac{\sin(\frac{1}{t})}{\sqrt{t}}dt \le \int_u^v \frac{dt}{\sqrt{t}}$$

$$-2(\sqrt{v}-\sqrt{u}) \le F(v)-F(u) \le 2(\sqrt{v}-\sqrt{u})$$
$$|F(v)-F(u)| \le 2(\sqrt{v}-\sqrt{u}) \le 2\sqrt{v}$$

If $\delta = \frac{\varepsilon^2}{4}$ and if $0 < u < v < \delta$

$$|F(v)-F(u)| < 2\sqrt{\delta} = \varepsilon$$

hence, $\lim_{x\to 0^+} F(x) = \int_0^1 \frac{\sin(\frac{1}{t})}{\sqrt{t}}dt$ exists.

Example 2: $\int_0^\infty x^2 e^{-x}dx$ use integration by parts two times.

Other Types of Integrals:

$\int_a^b f$, f is integrable on each $[a,b]$, $a < x < b$, but unbounded.

Example:

$$\int_{-1}^1 \frac{1}{\sqrt{|t|}}dt = \int_{-1}^0 \frac{dt}{\sqrt{-t}} + \int_0^1 \frac{dt}{\sqrt{t}}$$

Definition 2.8.1. *Let $a \in \mathbb{R}$, $f : [a,\infty) \to \mathbb{R}$ satisfy that f is integrable on each $[a,x]$, $a < x$, let the improper integral be given by*

$$\int_a^\infty f = \lim_{x\to\infty}\int_a^x f, \qquad \textit{if the limit exists}$$

2.9 Convergence and Comparison Test- Feb 3

Notes on Comparison Test: Consider the improper integral $\int_a^b f$, either f is unbounded at a or at b, just one of a, b is $-\infty$, or ∞.

Case 1: $f \geq 0$ on (a, b),

- If we can find $0 \leq f \leq g$ on (a, b) and $\int_a^b g$ converges. $\Rightarrow \int_a^b f$ converges.

  [We use monotone convergence theorem, and boundedness]

- If we can find $0 \leq g \leq f$ on (a, b) and $\int_a^b g$ diverges, then $\Rightarrow \int_a^b f$ diverges.

$$\int_a^x f \geq \int_a^x g \xrightarrow{x\to\infty} \infty$$

Case 2: f is not (neccessarily) non-negative on (a, b),

- if we can find $g, h \geq 0$ with $-g \leq f \leq h$, and let both $\int_a^b g$, $\int_a^b h$ converge, then $\int_a^b f$ converges.

  [Need is Cauchy criterion]

Theorem 2.9.1 (**Cauchy Criterion for Limits at** ∞)**.** *If* $F : [0, \infty] \to \mathbb{R}$, *then* $\lim_{n\to\infty} F(x)$*exists* $\iff$ *given* $\varepsilon > 0$, *there is* $N >$ *as.t.* $|F(u) - F(v)| < \varepsilon$ *whenever* $u, v > N$.

Proof. ($\Leftarrow$) Let $(a_n)_{n=1}^\infty \subset [a, \infty)$ with $\lim_{n\to\infty} a_n = \infty$, then there is $n_0 \in \mathbb{N}$ so $m, n \geq n_0 \Rightarrow |F(a_n) - F(a_m)| < \varepsilon$.

Hence, $F(a_n)_{n=1}^\infty$ is Cauchy, hence admits limit L, check that for any $(b_n)_{n=1}^\infty \subset [a, \infty)$, $\lim_{n\to\infty} a_n \Rightarrow \lim_{n\to\infty} F(b_n) = L$.

Check that this implies that $\lim_{x\to\infty} F(x) = L$. $\square$

2.10 Integration and Area

let $f, g[a, b] \to \mathbb{R}$ be integrable with $f \leq g$.

Let $S = \{(x, y) : \text{y lies between} f(x) \text{and} g(x), a \leq x \leq b\}$.

Partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. $s_j, t_j \in [x_{j-1}, x_j]$.

$$\begin{aligned} area(S) &\approx \sum_{j=1}^{n} \underbrace{[g(t_j) - f(s_j)]}_{height} \underbrace{(x_j - x_{j-1})}_{width} \\ &= \sum_{j=1}^{n} g(t_j)(x_j - x_{j-1}) - \sum_{j=1}^{n} f(s_j)(x_j - x_{j-1}) \\ &\approx \int_a^b g - \int_a^b f = \int_a^b [g - f] \qquad \text{(Say } l(P) < \delta \text{, by A2Q1)} \end{aligned}$$

hence we define

$$area(S) := \int_a^b [g(x) - f(x)]dx$$

if S is a nice region,

$$area(S) = \int_a^b h_S(x)dx = \int_c^d W_S(y)dy$$

Warning Example:

$$S = \{(x, y) : 0 \leq y \leq 1, if\ x \in \mathbb{Q}, -1 \leq y \leq 0; \quad if\ x \in \mathbb{I}, 0 \leq x\ leq1\}$$

Notice "height funciton" $h_S(x) = 1$. But we should not imagine that

$$area(S) = \int_0^a h_S(x)dx = 1$$

Example 1: $S = \{(x, y) : y\ between\ x^3, y = x^2 - 2x, -1 \leq x \leq 1\}$.

$$area(S) = \int_{-1}^{1} \left| x^3 - (x^2 - 2x) \right| dx = \int_{-1}^{0} [x^2 - 2x - x^3]dx + \int_0^1 [x^3 - x^2 + 2x]dx$$

Example 2: Circle of radius $a > 0$: $x^2 + y^2 = a^2$.

$$area(C) = \int_{-a}^{a} [\sqrt{a^2 - x^2} - (-\sqrt{a^2 - x^2})]dx = 2 \int_{-a}^{a} \sqrt{a^2 - x^2}dx$$

Method 1: Substitute $x = au$, $dx = a \cdot du$,

$$= 2 \int_{-1}^{1} \sqrt{a^2 - (au)^2}du = a^2 \cdots 2 \int_{-1}^{1} \sqrt{1 - u^2}du = a^2\pi$$

Method 2: $x = a\sin\theta$, $dx = a\cos\theta d\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$$\begin{aligned}
area(C) =& 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2 - (a\sin\theta)^2} a\cos\theta d\theta \\
=& 2a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta \\
=& a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1 + \cos 2\theta] d\theta \\
=& a^2[\theta + \frac{1}{2}\sin 2\theta]\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = a^2\pi + 0 = a^2\pi
\end{aligned}$$

Example 3: let W be a circular wedge:

$$\begin{aligned}
area(W) =& \int_0^{a\cos\beta} (\tan\beta - \tan\alpha) x dx \\
=& \int_0^{a\cos\beta} (\tan\beta) \\
=& \frac{a^2}{2}\sin\beta\cos\beta - \frac{a^2}{2}\sin\alpha\cos\alpha - a^2\int_\beta^\alpha \sin^2\theta d\theta \\
=& \frac{a^2}{2}[\sin\beta\cos\beta - \sin\alpha\cos\alpha] + \frac{a^2}{2}\left[(\beta - \alpha) - \frac{1}{2}[\sin(2\beta) - sin 2\alpha]\right] \\
=& a^2\frac{\beta - \alpha}{2} = a^2\pi\frac{\beta - \alpha}{2\pi}
\end{aligned}$$

Therefore, the area is

$$area(W) = \frac{a^2}{2}(\beta - \alpha)$$

2.10.1 Average Value:

$$a = \{a_1, a_2, \cdots, a_n\} \subset \mathbb{R}$$
$$a_{ave} = \frac{a_1 + \cdots + a_n}{n}$$

Let $f : [a, b] \to \mathbb{R}$ be an integrable function, we wish to figure out the "average value" f_{ave}. Uniform partition

$$\{a < a + \frac{b-a}{n} < a + 2\frac{b-a}{n} < \cdots < b\} = P_n$$

Sample values

$$t_j \in [a + (j-1)\frac{b-a}{n}. a + j\frac{b-a}{n}], \qquad j = 1, \cdots, n$$

Expect:

$$f_{ave} \approx \frac{\sum_{j=1}^n f(t_j)}{=} \frac{1}{b-a}\sum_{j=1}^n f(t_j)\frac{b-a}{n} = \frac{1}{b-a}S(f, P_n)$$

A2Q1 $\Rightarrow \lim_{n\to\infty} \frac{1}{b-a}S(f, P_n) = \frac{1}{b-a}\int_a^b f$.

Definition 2.10.1. *The average height of f is:*

$$f_{ave} = \frac{1}{b-a}\int_a^b f$$

2.10.2 Weighted Average

$a = \{a_1, \cdots, a_n\} \subset \mathbb{R}$ weights $w_1, \cdots, w_n > 0$.

$$a_{w,ave} = \frac{a_1 w_1 + \cdots + a_n w_n}{w_1 + \cdots + w_n}$$

We have integrable $f : [a,b] \to \mathbb{R}$, $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, sample points $t_j \in [x_{j-1}, x_j]$.

$$\begin{aligned}
f_{ave} \approx & \frac{f(t_1)(x_1 - x_0) + f(t_2)(x_2 - x_1) + \cdots + f(t_n)(t_n - t_{n-1})}{(x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1})} \\
= & \frac{\sum_{j=1}^n f(t_j)(x_j - x_{j-1})}{b-a} \\
= & \frac{1}{b-a} S(f, P)
\end{aligned}$$

2.10.3 Centroid

S is a "nice region", $P = \{a = x_0 < \cdots < x_n = b\}$, tags: $t_j \in [x_{j-1}, x_j], j = 1, \cdots, n$.

x-center: $\bar{x}_s$,

$$S_j = \{(x,y) \in S : x_{j-1} \leq x \leq x_j\}$$

$$\bar{x}_S \approx \frac{\sum_{j=1}^n t_j area(S_j)}{\sum_{j=1}^n area(S_j)}$$

Then

$$\begin{aligned}
\bar{x}_S = & \frac{1}{area(S)} \cdot \int_a^b x h_S(x) dx \\
\bar{y}_S = & \frac{1}{area(S)} \cdot \int_c^d y w_S(x) dy
\end{aligned}$$

3 S M H

3.1 Polar Coordinates

Euclidean Coordinates: $(x, y) \in \mathbb{R}^2$.

$$r = \sqrt{x^2 + y^2} \qquad \text{distance from origin}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

Find θ:

- $x > 0$, $-\frac{-\pi}{2} < \theta < \frac{\pi}{2}$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \qquad \Rightarrow \qquad \arctan(\frac{y}{x}) = \theta$$

- $x < 0$: check that $\theta = \arctan(\frac{y}{x}) = \theta$
- $x = 0$:
  $y > 0$, $\theta = \frac{\pi}{2}$
  $y < 0$, $\theta = -\frac{\pi}{2}$, $\frac{3\pi}{2}$

Given $r > 0$, $0 \leq \theta < 2\pi$,

$$(x, y) = (r\cos\theta, r\sin\theta)\text{is a unique point in } \mathbb{R}^2\ \{(0,0)\}$$

Example 1: Circle $x^2 + y^2 = a^2$, $(a > 0)$.

$$\begin{aligned} x =& r\cos\theta, y = r\sin\theta \\ a^2 =& (r\cos\theta^2) + (r\sin\theta)^2 \\ =& r^2(\cos^2\theta + \sin^2\theta) = r^2 \\ \Rightarrow a =& \pm r \end{aligned}$$

But $a = r$ survives, as $r \geq 0$.

Circle, in polar coordinates, $r = a$.

Example 2: Vertical Lines:

- $x = a$, $a > 0$,

$$r\cos\theta = a \quad \Rightarrow \quad r = \frac{a}{\cos\theta} = a\sec\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

- $x = a$, $a < 0$,

$$r = \frac{a}{\cos\theta}, \qquad \frac{\pi}{2} < \theta < \frac{3\pi}{2}$$

- $x = 0$,

$$\theta = \frac{\pi}{2} \qquad \vee \qquad \theta = \frac{3\pi}{2}$$

3.2 Arclength - Feb 7

Definition 3.2.1. $f:[a,b]\to\mathbb{R}$ *continuous,*

$$\Gamma=\{(x,y):y=f(x),a\leq x\leq b\}$$

$P=\{a=x_0<\cdots<x_n=b\}$, *then*

$$\text{length}(L_i)=\sqrt{(x_i-x_{i-1})^2+[f(x_j)-f(x_{j-1})]^2}$$

$$\text{length}(\Gamma)\approx\sum_{j=1}^{n}length(L_j)=\sum_{j=1}^{n}\sqrt{(x_j-x_{j-1})^2+[f(x_j)-f(x_{j-1})]^2}$$

Add assumptions:

- *f' exists on* $[a,b]$ *and is continuous on* $[a,b]$

$$M.V.T.\Rightarrow f(x_j-x_{j-1})=f'(c_j)(x_j-x_{j-1}),\qquad c_j\in(x_{j-1},x_j)$$

$$\begin{aligned}&\Rightarrow\sqrt{(x_j-x_{j-1})^2+[f(x_j)-f(x_{j-1})]^2}\\&=\sqrt{(x_j-x_{j-1})^2+[f'(c_j)(x_j-x_{j-1})]^2}\\&=\sqrt{1+f'(c_j)^2}(x_j-x_{j-1})\end{aligned}$$

Then,

$$\text{length}(\Gamma)\approx\sum_{j=1}^{n}\sqrt{1+[f'(c_j)]^2}(x_j-x_{j-1})=S(\sqrt{1+(f')^2},P)$$

Definition 3.2.2. *If* f' *exists and is continuous on* $[a,b]$, $\Gamma=\{(x,y):y=f(x),a\leq x\leq b\}$

$$length(\Gamma)=\int_a^b\sqrt{1+[f'(x)]^2}dx$$

Example 1:

$0\leq\alpha<\beta\leq\pi$, $a>0$

$$\frac{dy}{dx}=\frac{-x}{\sqrt{a^2-x^2}}$$

$$\begin{aligned}\text{length}(\Gamma)&=\int_{a\cos\beta}^{a\cos\alpha}\sqrt{1+(\frac{-x}{\sqrt{a^2-x^2}})^2}dx\\&=\int_{a\cos\beta}^{a\cos\alpha}\frac{\alpha}{\sqrt{a^2-x^2}}dx\\&=\int_{\beta}^{\alpha}\frac{\alpha}{\sqrt{a^2-a^2\cos^2\theta}}(-a\sin\theta)d\theta\\&=\int_{\alpha}^{\beta}\alpha d\theta=\alpha(\beta-\alpha)\end{aligned}$$

Example 2: $\Gamma = \{(x,y) : y = x^2, 0 \le x \le 2\}$, $\frac{dy}{dx} = 2x$

$$\text{length}(\Gamma) = \int_0^2 \sqrt{1+(2x)^2}dx$$

$2x = \sinh t = \frac{e^t - e^{-t}}{2}$, $dx = \frac{1}{2}\cosh t dt$, $\cosh t = \frac{e^t+e^{-t}}{2}$.

$$\begin{aligned}
\text{length}(\Gamma) &= \int_0^{\log(4+\sqrt{17})} \sqrt{1+\sinh^2 t}\frac{1}{2}\cosh t dt && (t = \log(2x + \sqrt{(2x)^2+1})) \\
&= \frac{1}{2}\int_0^{\log(4+\sqrt{17})} \cosh^2 t dt && (\cosh^2 t = \tfrac{1}{2}[\cosh(2t)+1]) \\
&= \frac{1}{2}\int_0^{\log(4+\sqrt{17})} [\cosh(2t)+1]dt && (\sinh 2t = 2\sinh t \cosh t) \\
check &= \frac{1}{4}\sinh(2t) + t\Big|_0^{\log(4+\sqrt{17})}
\end{aligned}$$

Method 2 $2x = \tan t$, $dx = \frac{1}{2}\sec^2 t dt$,

$$length(\Gamma) = \int_0^{\arctan(4)} \sqrt{1+\tan^2 t}\frac{1}{2}\sec^2 t dt = \frac{1}{2}\int_0^{\arctan(4)} \sec^2 t dt$$

$$\begin{aligned}
\int \sec^3 t dt &= \int \sec^2 t \sec t dt \\
&= \tan t \sec t - \int \tan t \tan t \sec t dt \\
&= \tan \sec t - \int (\sec^3 t - \sec t)dt
\end{aligned}$$

$$\begin{aligned}
\Rightarrow 2\int \sec^3 t dt & \\
&= \tan t \sec t + \int \sec t dt \\
&= \tan t \sec + \log|\sec t + \tan t| + C \\
\sec(\arctan 4) &= \sqrt{1+\tan^2(\arctan 4)} = \sqrt{1+16} = \sqrt{17}
\end{aligned}$$

3.3 Parameterization

We regard $x, y \in [a,b] \to \mathbb{R}$ (coordinates are each functions)

$$\Gamma = \{y(t) : t \in [a,b]\}$$

Examples :

- Polar Curves: $x(\theta) = r(\theta)\cos\theta$, $y(\theta) = r(\theta)\sin\theta$.
- Hyperbolic Coordinates: $a, b > 0$, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

  $x(t) = a\cosh t$
  $y(t) = b\sinh t$

We wish to compute/define length(Γ),

assumption,

- $x'(t)$, $y'(t)$ always exist on $[a,b]$, $x', y' : [a,b] \to \mathbb{R}$ are each continuous,

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

$$\begin{aligned} length &\approx \sum_{j=1}^{n} length(L_j) \\ &= \sum_{j=1}^{n} \sqrt{[x(t_j) - x(t_{j-1})]^2 + [y(t_j) - y(t_{j-1})]^2} \end{aligned}$$

$$\begin{aligned} M.V.T. \Rightarrow x(t_j) - x(t_{j-1}) =& x'(c_j)(t_j - t_{j-1}), c_j \in (t_{j-1}, t_j) \\ y(t_j) - y(t_{j-1}) =& y'(c_j^*)(t_j - t_{j-1}), c_j^* \in (t_{j-1}, t_j) \end{aligned}$$

$$\begin{aligned} length(\Gamma) &\approx \sum_{j=1}^{n} \sqrt{[x'(c_j)]^2 + [y'(c_j^*)]^2}(t_j - t_{j-1}) \\ &\approx \sum_{j=1}^{n} \sqrt{[x'(c_j)]^2 + [y'(c_j)]^2}(t_j - t_{j-1}) \\ &= S(\sqrt{(x')^2 + (y')^2}, P) \end{aligned}$$

$$length(\Gamma) = \int_a^b \sqrt{(x'(t)^2 + y'(t)^2} dt$$

3.4 Volume and Integration

Volume: $S \subset \mathbb{R}^3$ "nice region", typically bounded by definable surfaces with definable cross-sections.

$$Partition = \{a = x_0 < x_1 < \cdots < x_n = b\} = Q$$
$$\mathrm{vol}(P) = \sum_{j=1}^{n} \mathrm{vol}(P_j) \approx \sum_{j=1}^{n} A(x_j)(x_j - x_{j-1})$$

We define $\mathrm{vol}(P) = \int_a^b A(x)dx$.

$$A(x) = \int_{c(x)}^{d(x)} [z_{top,x}(y) - z_{bot,x}(y)]dy$$

Hard Part: Figure out $z_{top,x}, z_{bot,x}, c(x), d(x)$.

Remark: we may interchange roles of x, y, z.

Circular Symmetry: circular symmetry about $x-$ axis, cross sections are circles.

Method of Disks

$$A(x) = \pi[r(x)]^2$$
$$\mathrm{vol}(S) = \pi \int_a^b [r(x)]^2 dx$$

Method of Cylindrical Shells

Suppose that $R \subset \mathbb{R}^3$ is circularly symmetric about $z - axis$.

$$P = \{0 = x_0 < x_1 < \cdots < x_n = b\}$$

$$\begin{aligned}\mathrm{vol}(R) \approx \sum_{j=1} \mathrm{vol}(S_j) &= \sum_{j=1}^{n} 2\pi t_j h(t_j)(x_j - x_{j-1}) \\ &= \sum_{j=1}^{n} S(H, P)(x_j - x_{j-1})\end{aligned}$$

$$\begin{aligned}\mathrm{vol}(S_i) =& \mathrm{vol}(cylinder, height\ h(t_j), radius\ x_j) - \mathrm{vol}(cylinder, height\ h(t_j), radius\ x_{j-1}) \\ =& \pi x_j^2 h(t_j) - \pi x_{j-1}^2 h(t_j) \\ =& \pi(x_j^2 - x_{j-1}^2)h(t_j) \\ =& 2\pi \frac{x_j + x_{j+1}}{2}(x_j - x_{j-1})h(t_j) \\ =& 2\pi t_j h(t_j)(x_j - x_{j-1})\end{aligned}$$

$$\text{vol}(R) = 2\pi \int_0^b x h(x) dx$$
$$\text{vol}(R) = 2\pi \int_0^b x[z_{top,0}(x) - z_{bot,0}(x)] dx$$

Example: S a sphere, radius $a > 0$, $x^2 + y^2 + x^2 = a^2$.

Compute volume (S),

Disks: Fix x, for the moment, $-a \leq x \leq a$. Set $y = 0$,

$$x^2 + z^2 = a^2 \Rightarrow z^2 = a^2 - x^2 \Rightarrow r(x) = \sqrt{a^2 - x^2}$$

$$\text{vol}(S) = \pi \int_0^a (\sqrt{a^2 - x^2})^2 dx = \frac{4}{3}\pi a^3$$

Cylindrical Shells:

$$h(x) = \sqrt{a^2 - x^2} - (-\sqrt{a^2 - x^2}) = 2\sqrt{a^2 - x^2}$$

$$\begin{aligned} \text{vol}(S) &= 2\pi \int_0^a x 2\sqrt{a^2 - x^2} dx \\ &= 4\pi \int_0^a x\sqrt{a^2 - x^2} dx \\ &= \frac{4}{3}\pi a^3 \end{aligned}$$

3.5 Application of Antiderivatives:

4 DIFFERENTIAL EQUATIONS

4.1 Differential Equations

1st order D.E., standard form:

$$y' = f(x, y) \qquad , \qquad \underbrace{y(x_0) = y_0}_{initial value}$$

Facts:

Theorem 4.1.1 (Caratheodory Existence Theorem:). *$f(x, y)$ is continuous near $(x_0, y_0) \Rightarrow$ solution to I.V.P. exists.*

Theorem 4.1.2 (Picard-Lindelof Theorem).

Nice assumption of 2nd variable of f near (x_0, y_0). Caratheodory Existence Theroem:

Example: $y' = x \cdot y^{\frac{1}{3}}$, $y(0) = x_0$

Solution #1: $y(x) = 0$

Solution #2: assume $y(0) \neq 0$, hence $y(x) \neq 0$ in neighborhood of x.

4.2 Feb 14

An object e.g. person with open parachute falls from a standstill to the earth from height H. (H large $H < R$.)

As the object falls, it experiences wind resistance proportional to velocity.

4.3 DE - Feb 24

4.3.1 First Order Linear Equation

Definition 4.3.1 (First Order Linear D.E.).

$$y' = p(x)y + q(x) \qquad p, q \text{ cts functions on some domain}$$

Facts: *Any I.V.P. with such a D.E. (i.e. $y(x_0) = y_0$) always admits a unique solution, assuming that p, q are continuous in the neighborhood of x_0.*

Algorithm 4.3.1.

1. ***Homogeneous Case:*** $y' = p(x)y$, *i.e.* $q(x) = 0$,

$$\begin{aligned}
\frac{y'}{y} =& p(x) \\
\Rightarrow \log|y| =& P(x) + C, P(X) = \int p(x)dx \\
\Rightarrow y =& ke^{P(x)}, k = e^C > 0
\end{aligned}$$

2. ***Non Homogeneous Case:*** *Let* $P(x) = \int p(x)dx$, *as above,* $y' = p(x)y + \underbrace{q(x)}_{forcing\ term}$

"Trick":

$$\begin{aligned}
(d^{-P(x)}y)' =& e^{-P(x)}y' + e^{-P(x)} \cdot (-p(x)) \\
=& e^{-P(x)}(y' - p(x)y) = e^{-P(x)}q(x) \\
\Rightarrow e^{-P(x)}y =& \int e^{-P(x)}q(x)dx \\
\Rightarrow y =& e^{P(x)} \int e^{-P(x)}q(x)dx
\end{aligned}$$

Dont forget the integration constant.

$e^{-P(x)} = e^{-\int p(x)dx}$ *"integrating factor"*

Example: Solve $xy' - ey = x^6$, $y' = \frac{3}{x}y + x^5$

$$\begin{aligned}
p(x) =& \frac{3}{x} && \text{(not defined at } x = 0) \\
P(x) =& \int \frac{3}{x}dx = 3\log|x| = \log(|x|^3) && \text{(did not worry about C)} \\
e^{-P(x)} =& \frac{1}{|x|^3} \\
e^{P(x)} =& |x|^3
\end{aligned}$$

$$y = |x|^3 \int \frac{x^5}{|x|^3}dx = \begin{cases} x^3[\frac{1}{3}x^3 + C], & x > 0 \\ -x^3[\int \frac{x^5}{-x^3}dx], & x < 0 \end{cases} = \begin{cases} \frac{1}{3}x^6 + Cx^3, & x > 0 \\ \frac{1}{3}x^6 - Cx^3, & x < 0 \end{cases}$$

Note: equation does not allow $x = 0$ in domain, we have for either $x > 0$ or $x < 0$.

4.3.2 Second Order Linear Equation

Definition 4.3.2 (**Second Order Linear D.E.**)**.**

$$y'' + p(x)y' + q(x)y = r(x)$$

Facts:

1. *if p, q, r are continuous on an open interval, then a "general solution" exist: $\varphi_1 y_1 + \varphi_2 y_2$, y_1, y_2 linearly independent, φ_1, φ_2 differentible functions, or constants*
2. *I.V.P $y(x_0) = y_0$, $y'(x_0) = y_0 \Rightarrow$ solution unique.*

Algorithm 4.3.2 (**Methods to Solve**)**.**

1. ***Homogeneous Case:***
$$y'' + p(x)y' + q(x)y = 0$$
   - *can be very difficult to compute solution unless p, q, are constant (A4)*
   - *general solution always exists: of form*
$$c_1 y_1 + c_2 y_2$$
     y_1, y_2 linearly independent solutions c_1, c_2 constants.

     In I.V.P. situation, use initial data to learn c_1, c_2.
2. ***Variation of Parameters - L. Euler:***
$$y'' + p(x)'y + q(x)y = r(x) \qquad \heartsuit$$
   Idea: replace c_1, c_2 from homogeneous case, with functions.

   We assume: we have φ_1, φ_2 differentiable with continuous φ_1', φ_2', and we consider
$$y = \varphi_1' y_1 + \varphi_2' y_2 \qquad (f)$$
   y_1, y_2 are linearly independent solutions to homogeneous case, and
$$\varphi_1' y_1 + \varphi_2' y_2 = 0 \qquad (*)$$
   Let's consider for y in (f).
$$\begin{aligned} y' &= (\varphi_1 y_1 + \varphi_2 y_2) \\ &= \varphi_1' y_1 + \varphi_2' y_2 + \varphi_1 y_1' + \varphi_2 y_2' \\ &= \varphi_1 y_1' + \varphi_2 y_2' && \text{(by (*))} \end{aligned}$$
   then
$$\begin{aligned} y'' + py' + qy &= \varphi_1' y_1' + \varphi_2' y_2' + \varphi_1 y_1'' + \varphi_2 y_2'' + p(\varphi_1 y_1' + \varphi_2 y_2') + q(\varphi_1 y_1 + \varphi_2 y_2) \\ &= \varphi_1' y_1' + \varphi_2' y_2' + \varphi_1(y_1'' + py_1' + qy_1) + \varphi_2(y_2'' + py_2' + qy_2) \\ &= \varphi_1' y_1' + \varphi_2' y_2' && (**) \end{aligned}$$

If we wish to solve ♡, then we have

$$\begin{cases}\varphi_1' y_1' + \varphi_2' y_2' = r, \qquad by\ \heartsuit\ and\ ** \qquad \varphi_1' y_1 + \varphi_2' y_2 = 0 \qquad by\ assumption*\end{cases}$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} \varphi_1' \\ \varphi_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix} \Rightarrow \begin{matrix} \varphi_1' \\ \varphi_0' \end{matrix} = \frac{1}{y_1 y_0' - y_1' y_2} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ r \end{bmatrix}$$

$$W = y_1 y_2' - y_1' y_2 \qquad Wronskian \qquad \Rightarrow \varphi_1' = -\frac{y_2 r}{W} \quad \varphi_2' = \frac{y_1 r}{W}$$

$$\varphi_1(x) = -\int \frac{y_2(x) r(x)}{W(x)} dx$$
$$\varphi_2(x) = \int \frac{y_1(x) r(x)}{W(x)} dx$$

dont forget integraiton constant General Solution:

$$y(x) = \varphi_1(x) y_1(x) + \varphi_2(x) y_2(x)$$

4.4 Feb 26

Theorem 4.4.1 (Taylor's Theorem). *Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$, be $(n+1)$-times differentiable, then for $a \in \mathbb{R}$, we have*

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(x)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{P_n(x)} + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)}$$

for all $x \in I$, $c = c_x$ is between a and x.

$$R_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-a)^{n+1}$$

Lagrange Remainder Theorem.

Proof. Let $C \in \mathbb{R}$ satisfy that $f(x) - P_n(x) = C(x-a)^{n+1}$, fix x, then for t between a and x

$$\varphi(t) = f(x) - [f(t) + f'(t)(x-t) + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n + C(x-t)^{n+1}]$$

$\varphi(x) = 0 = \varphi(a)$. Rolle's Theorem $\Rightarrow \varphi'(c) = 0$ for some c between a and x.

Calculate:

$$\varphi'(T) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)C(x-t)^n$$

Solve to get $C = \dfrac{f^{(n+1)}(c_x)}{(n+1)!}$.

□

Theorem 4.4.2 (Taylor's Theorem version 2). *Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$, be $(n+1)$-times differentiable with $f^{(n+1)}$ continuous, then for $a \in I$, we have*

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{P_n(x)} + \underbrace{\int_a^x \frac{f^{(n+1)}(t)}{n!}(t-a)^n dt}_{R_n(x), \textit{Cauchy Form of Remainder for } x \in I}$$

Proof. We have

$$\begin{aligned}
f(x) =& f(a) + \int_a^x f'(t)dt && \text{(F.T. of C)} \\
=& f(a) + \int_a^x f'(t)(x-t)^0 dt \\
=& f(a) - f'(t)(x-t)\Big|_{a=t}^{x=t} + \int_a^x f''(t)(x-t)dt && \text{(Integration by Parts)} \\
=& f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t)dt && (*)
\end{aligned}$$

Inductive Step:

$$\begin{aligned}\int_a^x f^{(m)}(t)(x-t)^{m-1}dt =& -\frac{1}{m}f^{(m)}(t)(x-t)^m\Big|_{t=a}^{t=x} + \frac{1}{m}\int_a^x f^{(m+1)}(t)(x-t)^m dt\\ =&\frac{1}{m}f^{(m)}(a)(x-a)^m + \frac{1}{m}\int_a^x f^{(m+1)}(t)(t-a)^m dt\end{aligned}$$

$$\begin{aligned}* =&f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\frac{1}{2}\int_a^x f^{(3)}(t)(x-t)^2dt\\ =&f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\frac{1}{2}[\frac{1}{3}f^{(3)}(a)(x-a)^3+\frac{1}{3}\int_a^x f^{(4)}(a)(x-t)^3dt]\\ &\vdots\\ =&\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k+\frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt\end{aligned}$$

□

Remark: we assumed $f^{(n+1)}$ is continuous, the M/AVT for integrals provides $c = c_x$ between a and x s.t.

$$\frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt = \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a)$$

above is the second version of Cauchy form of $R_n(x)$.

Compare: lagrange form

$$\frac{f^{(n+1)}(c_x)}{(n+1)!}(x-a)^{n+1} = R_n(x) = \frac{f^{(n+1)}(c_x^*)}{n!}(x-c_x^*)^n(x-a)$$

$c_x \neq c_x^*$ in general.

Proposition 4.4.1. *Given $f: I \to \mathbb{R}$, a as above, $P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$, we have that*

- *P_n is the unique polynomial with $\deg P_n \le n$ s.t.*

$$\lim_{x\to a}\frac{f(x)-P_n(x)}{(x-a)^n} = 0$$

Proof. Suppose Q is polynomial, $\deg Q \le n$ with

$$\lim_{x\to a}\frac{f(x)-Q(x)}{(x-a)^n} = 0$$

Then,

$$Q(x)+[f(x)-Q(x)] = f(x) = P_n(x)+R_n(x) \Rightarrow Q(x)-P_n(x) = R_n(x)-[f(x)-Q(x)]$$

$$
\begin{aligned}
x \neq a \qquad \frac{Q(x)-P_n(x)}{(x-a)^n} =& \frac{R_n(x)}{(x-a)^n} - \frac{f(x)-Q(x)}{(x-a)^n} \\
=& \frac{\frac{f^{(n+1)}(c_x)}{n!}(x-c_x)^n(x-a)}{(x-a)^n} - \frac{f(x)-Q(x)}{(x-a)^n} \\
=& \left[\frac{f^{(n+1)}(c_x)}{n!} \cdot \frac{(x-c_x)^n}{(x-a)^n}(x-a)\right] \\
=& 0
\end{aligned}
$$

and $\deg(Q-P_n) \leq n$, little effort $\Rightarrow Q = P_n$. □

Example:

$$
e^x = \sum_{k=0}^{n} \frac{1}{k!}x^k + \frac{e^c}{(n+1)!}x^{n+1}
$$

$f(x) = e^x$, $f'(x) = e^x$ centered at $a = 0$.

Wish to examine e^{-x^2}.

$$
\begin{aligned}
e^{-x^2} =& \sum_{k=0}^{n} \frac{1}{k!}(-x^2)^k + \frac{e^c}{(n+1)!}(-x^2)^{n+1} \\
=& \underbrace{\sum_{k=0}^{n} \frac{(-1)^k}{k!}x^{2k}}_{degree\ 2n} + \frac{e^c \cdot (-1)^{n+1}}{(n+1)!}x^{2n+2}
\end{aligned}
$$

Conclusion:

$$
\begin{aligned}
\frac{e^{-x^2} - \sum_{k=0}^{N} \frac{(-1)^k}{k!}x^{2k}}{x^{2n+1}} =& \frac{\frac{(-1)^{n+1}e^{c^k}}{(n+1)!}x^{2n+2}}{x^{2n+1}} \\
\lim_{x\to 0} \frac{\frac{(-1)^{n+1}e^{c^k}}{(n+1)!}x^{2n+2}}{x^{2n+1}} =& 0
\end{aligned}
$$

We know that $P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^n}{k!}x^{2k}$, for $f(t) = e^{-t^2}$ around $a = 0$.

We can learn $f^{(k)}(0)$ just from polynomial, for $k = 0, \cdots, n$.

4.5 Error Estimation - Feb 28

Example: Integral Functions:

$$E(x) = \int_0^x e^{-t^2} dt$$

Wish to estimate $E(1)$ with a polynomial in 1.

Wish to estimate $E(x)$ with a polynomial in x, $x \in [0, 1)$.

$a = 0$,

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^c}{(n+1)!} x^{n+1}$$

$$e^{-t^2} = \sum_{k=0}^{n} \frac{(-1)^k t^{2k}}{k!} + \frac{(-1)^{n+1} e^c}{(n+1)!} t^{2n+2}$$

$$E(x) = \int_0^x e^{-t^2} dt = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \int_0^x t^{2k} dt + \frac{(-1)^{n+1}}{(n+1)!} \int_0^x e^c t^{2n+2} dt$$

$$\begin{aligned}
\Rightarrow \qquad & \left| E(x) - \sum_{k=0}^{n} \frac{(-1)^k \cdot x^{2k+1}}{k! \cdot (2k+1)} \right| \\
= & \left| \frac{(-1)^{n+1}}{(n+1)!} \int_0^x e^c t^{2n+2} dt \right| \\
\leq & \frac{1}{(n+1)!} \int_0^x \left| e^c t^{2n+2} \right| dt \\
\leq & \frac{1}{(n+1)!} \int_0^x t^{2n+2} dt, \quad 0 \leq e^c \leq 1, \quad c \in [-1, 0] \\
= & \frac{x^{2n+3}}{(2n+3)(n+1)!} \leq \frac{1}{(2n+3)(n+1)!}, \text{ as } x \in [0, 1]
\end{aligned}$$

"Uniform Estimate": Estimate holds for any $x \in [0, 1]$.

Rate of Decay of Estimate:

Ratio of Estimates:

$$\frac{\frac{1}{(2(n+1)+3)((n+1)+1)!}}{\frac{1}{(2n+3)(n+1)!}} = \frac{(2n+3)}{(2n+5)(n+2)}$$

Better than exponential decay.

e_n error in n $e_n \sim r^n$ $(0 < r < 1)$, $\frac{e_{n+1}}{e_n} = r$ (fixed)

5 SERIES

5.1 Introduction to Series - Feb 28

Definition 5.1.1. *Let $(a_k)_{k=1}^{\infty} \subset \mathbb{R}$ be a sequence. We define the series*

$$\sum_{k=1}^{\infty} a_k := \lim_{n\to\infty} \sum_{k=0}^{n} a_k$$

provided the limit exists.

Series = Improper Sum.

Terminology: We say $\sum_{k=1}^{\infty} a_k$ converges provide $(\sum_{k=1}^{n} a_k)_{n=1}^{\infty}$ converges.

Essential Example: Geometric series

Let $a \in \mathbb{R}$, when does $\sum_{k=0}^{\infty} a^k$ converges?

Let $S_n = \sum_{k=0}^{n} a^k = 1 + a + a^2 + \ldots + a^n$,

$S_n(1-a) = 1 + + \ldots + a^n - [a + a^2 + \ldots + a^n + a^{n+1}] = 1 - a^{n+1}$,

$$S_n = \begin{cases} \frac{1-a^{n+1}}{1-a}, & if\ a \neq 1 \\ n+1, & if\ a = 1 \end{cases}$$

Fact:

$$\lim_{n\to\infty} a^{n+1} = \begin{cases} 0, & |a| < 1 \\ D.N.E., & |a| \geq 1, a \neq 1 \\ 1, & a = 1 \end{cases}$$

Hence, $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$ if $|a| < 1$.

Example: (Sometimes we get lucky)

Consider $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$,

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} [\frac{1}{j} - \frac{1}{k+1}] = 1 - \frac{1}{2} + \frac{1}{2} - \ldots + \frac{1}{n} - \frac{1}{n+1}$$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)} = \lim_{n\to\infty} [1 - \frac{1}{n+1}] = 1$$

Series converges to 1.

5.2 Series Convergence Test I - March 2

Recall:

$$\sum_{k=1}^{\infty} a_k := \lim_{n\to\infty} \sum_{k=1}^{n} a_k \quad \text{if limit eixsts.}$$

Fundmanetal Question of Series: Given $\sum_{k=1}^{\infty} a_k$, does it converge?

Tests For Convergence:

Proposition 5.2.1 (**Test #1: nth term test - weakest necessity result**).

$$\sum_{k=1}^{\infty} a_j \quad \textit{converges} \Rightarrow \lim_{k\to\infty} a_k = 0$$

Proof. Let $S_n = \sum_{k=1}^{N} a_k$. Then $a_n = S_n - S_{n-1}$. We assume $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n$ exists.

Hence $\lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} S_n$ exists. Hence by taking differences of limits of sequences, we get

$$0 = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} [S_n - S_{n-1}] = \lim_{n\to\infty} a_n$$

□

Example: $|a| \geq 1 \Rightarrow \sum_{k=0}^{\infty} a^k$ D.N.E. Indeed, $\lim_{k\to\infty} a^k \neq 0$. (or does not exist).

Theorem 5.2.1 (**Cauchy Criterion for Series Convergence**). $\sum_{k=1}^{\infty} a_k$ *converges* $\Leftrightarrow$ *given* $\varepsilon > 0$, *there is a* n_ε *in* $\mathbb{N}$ *s.t.* $|\sum_{k=m}^{n} a_k| < \varepsilon$. *whenever* $n > m \geq n_\varepsilon$. *da*

Proof. let $S_n = \sum_{k=1}^{n} a_k$. then $\sum_{k=1}^{\infty} a_k$ converges $\Leftrightarrow \lim_{n\to\infty} S_n$ exists, $\Leftrightarrow$ given $\varepsilon > 0$, there is n_ε in $\mathbb{N}$ so $|S_n - S_{m-1}| < \varepsilon$ whenever $n >, m \geq n_\varepsilon$.

Note that $S_n - S_{m-1} = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = \sum_{k=m}^{n} a_k$.

□

Example: For $\varepsilon = \frac{1}{2}$, then Cauchy Criterion fails for $\sum_{k=1}^{\infty} \frac{1}{k}$.

Proposition 5.2.2 (**Linearity of Converging series**). *Suppose* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *converges, then for* $\alpha, \beta \in \mathbb{R}$,

$$\sum_{k=1}^{\infty} (\alpha a_1 + \beta b_k) \quad \textit{converges}$$

with

$$\sum_{k=1}^{\infty} (\alpha a_K + \beta b_k) = \alpha \sum_{k=1}^{\infty} a_k + \beta \sum_{k=1}^{\infty} b_k$$

Proof. We use linearity of sums and of limits(when they exist).

$$\begin{aligned}
\sum_{k=1}^{\infty}(\alpha a_k+\beta b_k) &= \lim_{n\to\infty}\sum_{k=1}^{n}(\alpha a_k+\beta b_k)\\
&= \lim_{n\to\infty}(\alpha\sum_{k=1}^{n}a_k+\beta\sum_{k=1}^{n}b_k)\\
&=\alpha\lim_{n\to\infty}\sum_{k=1}^{n}a_k+\beta\lim_{n\to\infty}\sum_{k=1}^{n}b_k && \text{(some limit exist)}\\
&=\alpha\sum_{k=1}^{\infty}a_k+\beta\sum_{k=1}^{\infty}b_k
\end{aligned}$$

□

Theorem 5.2.2 (Comparison Test). *Suppose $0 \le a_k \le b_k$, $k \ge \mathbb{N}$, for some $N \in \mathbb{N}$, then*

1. *If $\sum_{k=1}^{\infty} b_k$ converges $\Rightarrow \sum_{k=1}^{\infty} a_k$ converges.*
2. *If $\sum_{k=1}^{\infty} a_k$ diverges $\Rightarrow \sum_{k=1}^{\infty} b_k$ diverges.*

Proof. 1. Assume $\sum_{k=1}^{\infty} b_k$ converges, then for $n \ge N$,

$$\begin{aligned}
\sum_{k=1}^{n}a_k &= \sum_{k=1}^{N-1}a_k+\sum_{k=N}^{n}a_k\\
&\le \sum_{k=1}^{n}a_k+\sum_{k=N}^{\infty}b_k\\
&\le \sum_{k=1}^{n}a_k+\underbrace{\lim_{n\to\infty}\sum_{k=1}^{n}b_k}_{nondecreasing\ in\ n}\\
&\le \underbrace{\sum_{k=1}^{N+1}a_k}_{finite}+\underbrace{\sum_{k=1}^{\infty}b_k}_{<\infty} && \text{(added in } \textstyle\sum_{k=1}^{\infty} b_k \ge 0)
\end{aligned}$$

Also $S_{n+1}-S_n=\sum_{k=1}^{n+1}a_k-\sum_{k=1}^{n}a_k=a_n\ge 0$. $\Rightarrow (S_n)_{n=1}^{\infty}$ is non-decreasing.

By monotone convergence theorem $\Rightarrow \sum_{k=1}^{\infty} a_n=\lim_{n\to\infty} S_n$ exists.

2. Assume $\sum_{k=1}^{\infty} a_k$ diverges, since $S_n=\sum_{k=1}^{\infty} a_k$ is non-decreasing, we must have that $\sum_{k=1}^{\infty} a_k=\infty$.

Now for $n \geq N$, we have

$$\begin{aligned}\sum_{k=1}^{n} b_k &= \sum_{k=1}^{N-1} b_k + \sum_{k=N}^{n} b_k \\ &\geq \sum_{k=1}^{N-1} b_k + \sum_{k=N}^{n} a_k \\ &= \underbrace{\sum_{k=1}^{N-1} b_k}_{independent\ \ of\ \ n} - \underbrace{\sum_{k=1}^{N-1} a_k}_{S_n} + \sum_{k=1}^{n} a_k\end{aligned}$$

$\lim_{n\to\infty} \to \infty \Rightarrow \sum_{k=1}^{\infty} b_k = \infty$. □

Example:

$$\sum_{k=2}^{\infty} \frac{1}{(\log k)^k}$$

$$\log k \geq 2 \Leftrightarrow k \geq e^2, i.e. k = \lfloor e^2 \rfloor + 1$$

$$\frac{1}{\log k}^{k} \leq \frac{1}{2^k}$$

By geometric series of $\frac{1}{2}$, the series converges.

5.3 Series Convergence Test II - March 4

Remark: Let $a_k \geq 0$, and $S_n = \sum_{k=1}^{n} a_k$

$$S_{n+1} - S_n = a_n \geq 0 \Rightarrow (S_n)_{n=1}^{\infty} is monotone increasing$$

$\sum_{k=1}^{\infty} S_n$ converges $\Leftrightarrow$ S_n is bounded.

Corollary 5.3.1 (Limit Comparison Test). *If $a_k \geq 0$ and $b_k > 0$, and $0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} = L$ exists, then,*

1. *If $L > 0$, $\sum_{k=1}^{\infty} b_k$ converges $\Leftrightarrow$ $\sum_{k=1}^{\infty} a_k$ converges.*
2. *If $L = 0$, $\sum_{k=1}^{\infty} b_k$ converges $\Rightarrow$ $\sum_{k=1}^{\infty} a_k$ converges.*
3. *If $L = 0$, $\sum_{k=1}^{\infty} a_k$ diverges $\Rightarrow$ $\sum_{k=1}^{\infty} b_k$ diverges. (Contrapositive of ii).*

Proof. 1) We suppose $L > 0$, thus there is $N \in \mathbb{N}$ such that

$$\begin{aligned}
& \left|\frac{a_k}{b_k} - L\right| < \frac{L}{2} && \text{if} \quad k \geq N \\
\Leftrightarrow \quad & -\frac{L}{2} < \frac{a_k}{b_k} - L < \frac{L}{2} && \text{if} \quad k \geq N \\
\Leftrightarrow \quad & \frac{L}{2} < \frac{a_k}{b_k} < \frac{3L}{2} && \text{if} \quad k \geq N \\
\Leftrightarrow \quad & \frac{L}{2} b_k < a_k < -\frac{3L}{2} b_k && \text{if} \quad k \geq N
\end{aligned}$$

We have $\sum_{k=1}^{\infty} b_k$ converges $\Leftrightarrow$ $\sum_{k=1}^{\infty} \frac{L}{2} b_k$ converges, and $\sum_{k=1}^{\infty} \frac{3L}{2} b_k$ converges.

We apply comparison test, twice.

□

Example: Let us consider $\sum_{k=1}^{\infty} \frac{1}{k^p}$, $p \geq 2$, recall that $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges (*).

$$\frac{\frac{1}{k^p}}{\frac{1}{k(k+1)}} = \frac{k^2 + k}{k^p} = \frac{1 + \frac{1}{k}}{k^{p-2}} \overset{k\to\infty}{\to} \begin{cases} 1, & p = 2 \\ 0, & p > 2 \end{cases}$$

Remark: The limit comparison test is typically easier to compute than comparison test, and hence useful (you should remember this)

Corollary 5.3.2 (Ratio Comparison Test). *If $a_k > 0$ and $b_k > 0$, and*

- $\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k}$ *for $k \geq N$, $N \in \mathbb{N}$.*

Then $\sum_{k=1}^{\infty} b_k$ converges $\Rightarrow$ $\sum_{k=1}^{\infty} a_k$ converges.

Remark: This is more difficult in practice than either comparison test or limit comparison test, we will see that it has strong theoretical value.

Proof. For $k \geq N$,

$$\begin{aligned}
\frac{a_{k+1}}{a_k} &\leq \frac{b_{k+1}}{b_k} \\
\Rightarrow \qquad \frac{a_{k+1}}{b_{k+1}} &\leq \frac{a_k}{b_k} \\
\Rightarrow \qquad \frac{a_k}{b_k} &\leq \frac{a_N}{b_N} = M \qquad for\ k \geq N \\
\Rightarrow \qquad a_k &\geq M b_k, \quad for\ k \geq N
\end{aligned}$$

Then $\sum_{k=1}^{\infty} b_k$ converges $\Rightarrow \sum_{k=1}^{\infty} M b_k$ converges $\Rightarrow$ comparison test $\sum_{k=1}^{\infty} a_k$ converges.

$\square$

Main Application of Ratio Comparison Test:

Theorem 5.3.1 (**Ratio test**)**.** *Suppose $a_k > 0$ and that*

$$\lim_{k\to\infty} a_k = r \qquad \textit{exists}$$

Then $r \geq 0$, and

1. *If $r < 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ converges,*
2. *If $r > 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ diverges.*

Remark:

- *test is easy to use, as no reference series are required*
- *case $r = 1$ is ambiguous e.g. $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges and $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges.*

Proof.

1. Say $r < 1$, pick any s so $r < s < 1$, then there is N in $\mathbb{N}$, so for $k \geq N$,

$$\frac{a_{k+1}}{a_k} < r - (r - s) = s = \frac{s^{k+1}}{s^k}$$

   We have that $\sum_{k=1}^{\infty} s_k$ converges $(0 < s < 1)$, and hence by R.L.T. $\sum_{k=1}^{\infty} a_k$ converges too.

2. Say $r > 1$, pick any s so $1 < s < r$, then there is N in $\mathbb{N}$, so for all $k \geq N$,

$$\frac{a_{k+1}}{a_k} > r - (r - s) = s = \frac{s^{k+1}}{s^k}$$

   However $\sum_{k=1}^{\infty} s_k$ diverges, If we have that $\sum_{k=1}^{\infty} a_k$ converges, then R.L.T. would imply $\sum_{k=1}^{\infty} S^k$ converges, contradiction.

$\square$

Example: Consider $\sum_{k=0}^{\infty} \frac{(1000)^k}{\sqrt{k!}}$

Ratio Test:

5.4 Series Convergence Test III - March 6

Theorem 5.4.1 (Integral Test). *Let $a_k > 0$, $k \in \mathbb{N}$, suppose there is a function $f : [1, \infty) \to \mathbb{R}$ s.t.*

- *$f(k) = a_k$ for $k \in \mathbb{N}$, and*
- *f is non-increasing*

then

$$\sum_{k=1}^{\infty} a_k \quad converges \Leftrightarrow \int_1^{infty} f(t)dt \quad converges$$

Remark: *f nonincreasing $\Rightarrow$ f is integrable on each $[1, x]$, $x \geq 1$, A_1*

Proof. f non-increasing, if $t \in [1, \infty]$, find $k \in \mathbb{N}$, so $t \leq k$, then $f(t) \geq f(k) = a_k > 0$, hence, $f(t) > 0$ for $t \in [1, \infty)$.

If $t \in [k, k+1]$, then

$$a_k = f(k) \geq f(t) \geq f(k+1) = a_{k+1}$$

and hence,

$$a_k \geq \int_k^{k+1} f(t)dt \geq a_{k+1} \qquad \text{since} \quad k + 1 - k = 1$$

$$\sum_{k=1}^{n+1} a_k \geq \int_1^{n+1} f(t)dt = \sum_{k=1}^{n} \int_k^{k+1} f(t)dt \geq \sum_{k=1}^{n} a_{k+1} = \sum_{k=2}^{n+1} a_k \tag{*}$$

If $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n+1} a_k$ converges, then, for $x > 1$,

$$0 \leq \int_1^x f(t)dt \leq \int_0^{\lceil x \rceil} f(t)dt \leq \sum_{k=1}^{\lceil x \rceil} a_k \overset{x\to\infty}{\longrightarrow} \sum_{k=1}^{\infty} a_k < \infty$$

hence $F(x) = \int_1^x f(t)dt$ is increasing, as $F'(x) = f(x) > 0$, and F is bounded.

Thus $\int_1^{\infty} f(t)dt = \lim_{x\to\infty} F(x)$ converges.

Conversely, if $\int_1^{\infty} f(t)dt$ converges, Then for $n \in \mathbb{N}$,

$$0 \leq \sum_{k=1}^{n+1} a_k = a_1 + \sum_{k=2}^{n+1} a_k \leq a_1 + \int_1^{n+1} f(t)dt \overset{x\to\infty}{\longrightarrow} a_1 + \int_1^{\infty} f(t)dt$$

and thus $S_{n+1} = \sum_{k=1}^{n+1} a_k$ is a bounded and non-decreasing sequence, hence, $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_{n+1}$ converges.

□

Remark: Variant: we may mildly weaken assumptions on f, above, If there is $M > 1$, so $f : [M, \infty] \to \mathbb{R}$ is nondecreasing,

- $f(k) = a_k$, for $k \in \mathbb{N}$, $k \geq M$,

then $\sum_{k=1}^{\infty} a_k$ converges $\Leftrightarrow$ $\int_1^{\infty} f(t)dt$ converges. [exercise]

Corollary 5.4.1. *If* $p > 0$, $\sum_{k=1}^{\infty} \frac{1}{k^p}$ *converges* $\Leftrightarrow p > 1$.

Proof. $f(t) = \frac{1}{t^p}$ which is decreasing on $[1, \infty)$.

$f(t) = \frac{1}{k^p}$, Integral Test: $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges $\Leftrightarrow \int_1^{\infty} \frac{dt}{t^p}$ converges.

$$\int_1^x \frac{dt}{t^p} = \int_1^x t^{-p} dt = \begin{cases} \frac{1}{1-p}(x^{1-p} - 1), & p \neq 1 \\ \log x, & p = 1 \end{cases} \quad \xrightarrow{x \to \infty} \quad \begin{cases} \infty, & p \leq 1 \\ \frac{1}{p-1}, & p > 1 \end{cases}$$

$\square$

Remark: Indecisive part of ratio test:

$$\frac{\frac{1}{(k+1)^p}}{\frac{1}{k^p}} = \frac{k^p}{(1+k)^p} = \frac{1}{(\frac{1}{k} + 1)^p} \xrightarrow{k \to \infty} 1$$

Example 1: $\sum_{k=1}^{\infty} \frac{k^3+1}{k^5+3k^3+1}$ converges? Use limit comparison test with $\sum_{k=1}^{\infty} \frac{1}{k^2}$. * ratio test fails.

Example 2: Does $\sum_{k=1}^{\infty} ke^{-k^2}$ converge?

1. integral test
$$\int_1^{\infty} te^{-t^2} = \frac{1}{2e} \Rightarrow \sum_{k=1}^{\infty} ke^{-k^2} \quad \text{converges}$$

2. ratio test
$$\frac{(k+1)e^{-(k+1)^2}}{ke^{-k^2}} = \frac{k+1}{k} e^{-2k-1} \xrightarrow{k \to \infty} 0 \Rightarrow \quad \text{series converges}$$

3. limit comparison test
$$\sum_{k=1}^{\infty} e^{-k} \text{converges by geometric series}$$

   Know that

5.5 Series Convergence Test IV - March 9

Example: Euler's Constant

$\gamma = \lim_{n\to\infty}[\sum_{k=1}^{n}\frac{1}{k} - \log n]$ exists.

Recall:
$\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$
$\lfloor(\rfloor t) \leq t \leq \lfloor t \rfloor + 1; t \geq 1$
$\frac{1}{\lfloor t \rfloor} \geq \frac{1}{t} \geq \frac{1}{\lfloor t \rfloor + 1}$
$\Rightarrow \qquad \frac{1}{\lfloor t \rfloor} - \frac{1}{t} = 0$

Consider

$$\begin{aligned} A_n &= \int_1^n (\frac{1}{\lfloor t \rfloor - \frac{1}{t}} dt) \\ &= \int_1^n \frac{1}{\lfloor t \rfloor} dt \\ &= \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{\lfloor t \rfloor} dt - \log n \\ &= \sum k = 1^{n-1} \frac{1}{k} - \log n \end{aligned}$$

$(A_n = \sum_{k=1}^{n-1}\frac{1}{k} - \log n = [\sum_{k=1}^{n}\frac{1}{k} - \log n] - \frac{1}{n} \overset{n\to\infty}{\longrightarrow} \lim_{n\to\infty}[\sum_{n=1}^{\infty}\frac{1}{k} - \log n])$

When $a_k > 0$, $\lim_{n\to\infty}\frac{a_{k+1}}{a_k} = 1$. (Indeterminate Case of Ratio Test)

Proposition 5.5.1 (Raabe's Test). *Suppose* $\lim_{n\to\infty} k(1 - \frac{a_{k+1}}{a_k}) = p \in \mathbb{R}$, *then*

1. *If* $p > 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ *converges*
2. *If* $p < 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ *diverges*
3. *If* $p = 1$ *and* $\left|k(1 - \frac{a_{k+1}}{a_k}) - 1\right| \leq \frac{m}{k}$ *for some* $M > 0$, *then* $\sum_{k=1}^{\infty} a_k$ *converges*

Remark: the case $p = \infty$ *also gives convergence, the proof is similar to* $p > 1$ *case.*

Proof.

1. Let $q > 0 \in \mathbb{R}$,
$$\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} = 1 - \frac{1}{k} + \frac{B_k}{k^2}$$
where $0 \leq B_k \leq (q+1)1$ (i.e. is bounded).

   Indeed,
$$\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} = \frac{(k+1)^q}{k^q} = \frac{1}{(1+\frac{1}{k})^q} = (1 + \frac{1}{k})^{-q}$$
Let $f = (1+x)^{-q}$, $f(x) = -q(1+x)^{-q-1}$, $f''(x) = q(q+1)(1+x)^{-q-2}$.

Taylor's Theorem about $a = 0$: $f(x) = 1 - qx\frac{q(q+1)}{(1+cx)^{q+2}}x^2$, cx between 0 and x.

$$\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} = (1+\frac{1}{k})^{-q} = 1 - \frac{q}{k} + \underbrace{\frac{(q+1)q}{(q+c_k)^{q+2}}}_{B_k, 0\le B_k \le q(q+1)} \frac{1}{k^2}$$

2. We write

$$\begin{aligned}\frac{a_{k+1}}{a_k} &= 1 - \frac{p}{k} + \frac{p}{k} - 1 + \frac{a_{k+1}}{a_k} \\ &= 1 - \frac{p}{k}\frac{1}{k}\underbrace{p - k(1 - \frac{a_{k+1}}{a_k}))}_{:=A_k} \\ &= 1 - \frac{p}{k} + \frac{A_k}{k}\end{aligned}$$

$$\lim_{k\to\infty} A_k = \lim_{k\to\infty}[p - k(1 - \frac{a_{k+1}}{a_k})] = p - \lim_{k\to\infty}(1 - \frac{a_{k+1}}{a_k}) = 0 \qquad (\textit{By Assumption})$$

3. Let we assume $p > 1$, find q with $p > q > 1$, then $\sum_{k=1}^{\infty} \frac{1}{k^q}$ converges, and (1, 2) shows that

$$\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} - \frac{a_{k+1}}{a_k} = (1 - \frac{q}{k} + \frac{B_k}{k^2}) - (1 - \frac{p}{k} + \frac{A_k}{k}) = \frac{p - q + \frac{B_k}{k} - A_k}{k}$$

where $lim_{k\to\infty}(\frac{B_k}{k} - A_k) = 0$.

Hence, $\exists N \in \mathbb{N}$ s.t. $-\frac{p-q}{2} < \frac{B_k}{k} - A_k < \frac{p-q}{2}$, so for $k \ge N$.

$$\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} - \frac{a_{k+1}}{a_k} > \frac{p-q}{2k} \Rightarrow \frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} > \frac{a_{k+1}}{a_k}$$

Thus by ratio comparison test, $\frac{a_{k+1}}{a_k}$ converges.

4. If $p < 1$, and find g so $p < g < 1$, thus $\sum_{k=1}^{\infty} \frac{1}{k^q}$ diverges. As in II.

$$\frac{a_{k+1}}{a_k} - \frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} == \frac{q - p + A_k - \frac{B_k}{k}}{k}$$

and as $\lim_{k\to\infty}(A_k - \frac{B_k}{k}) = 0$, $\exists N \in \mathbb{N}$, so for $k \ge N$, $\frac{q-p}{2} < A_k - \frac{B_k}{k} < \frac{q-p}{2}$, so

$$\frac{a_{k+1}}{a_k} - \frac{k^q}{(k+1)^q} > 0 \Rightarrow \frac{a_{k+1}}{a_k} > \frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} \Rightarrow \sum_{k=1}^{\infty} a_k \quad \text{diverges}$$

5. proof of 3, We suppose $p = 1$, then

$$\left| f(1 - \frac{a_{k+1}}{a_k}) - 1 \right| \le \frac{M}{k}, M > 0$$

$$\frac{a_{k+1}}{a_k} = 1 - \frac{1}{k} + \frac{1}{k}(1 - k(1 - \frac{a_{k+1}}{a_k})) \ge 1 - \frac{1}{k} - \frac{M}{k^2}$$

Now $\sum_{k=\lfloor M \rfloor + 2} \frac{1}{k-M+1}$ diverges.

$\square$

Example for Raabe's Test Find $a, b \geq 0$, s.t. $\sum_{k=1}^{\infty} \frac{(a+1)(a+2)\cdots(a+k)}{(b+1)(b+2)\cdots(b+k)}$ converges.

$$\frac{a_{k+1}}{a_k} = \sum_{k=1}^{\infty} \frac{\prod_{i=1}^{k+1} \frac{a+i}{b+i}}{\prod_{i=1}^{k} \frac{a+i}{b+i}} = \frac{a+k+1}{a+k} \xrightarrow{k\to\infty} 1$$

By Raabe's Test:

$$k(1-\frac{a_{k+1}}{a_k}) = k(1-\frac{a+k+1}{b+k+1}) = k(\frac{b-a}{b+k+1}) \xrightarrow{k\to\infty} b-a$$

$b-a>1 \Rightarrow$ converges, and $b-a<1 \Rightarrow$ diverges.

If $b-a=1$,

$$k(1-\frac{a_{k+1}}{a_k}) - 1 = \frac{k(b-a)}{b+k+1} = \frac{(b-a)k-(b+k+1)}{b+k+1} = -\frac{b+1}{k+b+1}$$

$$\left|k(1-\frac{a_{k+1}}{a_k}) - 1\right| = \frac{b+1}{k+b+1} = \frac{1}{k}[\frac{b+1}{1+\frac{b+1}{k}}] < \frac{b+1}{k}$$

$b-a=1 \Rightarrow$ converges.

5.6 Serires Convergence Test V - March 11

5.6.1 Leibnitz Alternating Series Test

Theorem 5.6.1 (leibnitz Alternating Series Test). *Suppose*

- $a_1 \geq a_2 \geq \cdots \geq 0$
- $\lim_{k\to\infty} a_k = 0$

then, $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ *converges.*

Furthermore, $\left|\sum_{k=1}^{\infty}(-1)^{k+1}a_k\right| \leq a_1$.

Proof. We let $S_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$, then

$$\begin{aligned} S_{2n} \leq& S_{2n} + a_{2n+1} - a_{2n+2} \\ =& S_{2n+2} \\ =& a_1 - a_2 + a_3 - a_4 + \cdots - a_{2n} + a_{2n+1} - a_{2n+2} \\ =& a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n} - a_{2n+1}) - a_{2n+2} \\ \leq& a_1 \end{aligned}$$

Hence, $0 \leq S_2 \leq S_{2n+2} \leq a_1$, i.e. $(S_{2n})_{n=1}^{\infty}$ is non-negative, non-decreasing, and bounded.

Monotone Convergence $\Rightarrow \alpha = \lim_{n\to\infty} S_{2n} \leq a_1$ exists.

$$|\alpha - S_{2k}| < \frac{\varepsilon}{2}, \quad a_{2k+1} < \frac{\varepsilon}{2}, \qquad \text{whenver } k \geq N.$$

If $n \geq 2N+1$, and with $k = \lfloor \frac{n}{2} \rfloor \geq N$, we have

$\square$