PMATH 352 Complex Analysis

velo.x

Contents

1 Differentiation

1.1 Introduction to Complex Number - May 2

Definition 1.1.1 (Complex Number)**.**
Define i to be which $i^2 = -i$, define the set of complex numbers to be

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Note that there is no a priori distrinction between i and $-i$. Thus all behaviour in $\mathbb{C}$ should be invariant under a map $i \leftrightarrow -i$.

Operations:

- Addition:
$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
- Multiplication
$$(a + bi)(c + di) = ac + (bc + ad)i + bdi^2 = (ac - bd) + (bc + ad)i$$
- Division: assume $c \neq 0$ or $d \neq 0$,
$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\frac{c - di}{c - di} = \frac{ac + bd + (bc - ad)i}{c^2 + d^2}$$
- Conjugation:
$$\overline{a + bi} = a - bi$$
Note
$$(a + bi)(\overline{a + bi}) = (a + bi)(a - bi) = a^2 + b^2$$

Remark 1.1.2. There is a canonical bijection between $\mathbb{R}^2$ and $\mathbb{C}$. $a + bi \leftrightarrow (a, b)$. Therefore, every complex number $a + bi$ can be mapped as a point on the 2-dimensional axis.

Lemma 1.1.1. Every coordinate point (x, y) can be translated to polar coordinate (r, θ) where

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan\frac{y}{x}\ ;$$

every polar coordinate point (r, θ) can be translate to (x, y) where

$$x = r\cos\theta, \qquad y = r\sin\theta\ .$$

Definition 1.1.3 (Norm)**.**

$$|a + bi| = \sqrt{a^2 + b^2}$$

Lemma 1.1.2. Powers of i loop in a circle of 4.

$$i^1 = i \quad \Rightarrow \quad i^2 = -1 \quad \Rightarrow \quad i^3 = -i \quad \Rightarrow \quad i^4 = 1 \quad \Rightarrow \quad i^5 = i \quad \Rightarrow \quad \cdots$$

Lemma 1.1.3. Let $x \in \mathbb{R}$ and $z = a + bi, w = c + di \in \mathbb{C}$. We have $\frac{d}{dz}e^z = e^z$ and $e^{w+z} = e^w e^z$ and $\frac{d}{d(iy)}e^{iy} = e^{iy}$.

Proof. UNFINISHED $\square$

1.2 Limit, Continuity, Differentiability - May 4

Definition 1.2.1 (Distance)**.**
The distance between two points $w, z \in \mathbb{C}$ is $|w - z| = |z - w|$.

Remark 1.2.2. $\mathbb{C}$ and $\mathbb{R}^2$ are isomorphic as metric spaces.

Definition 1.2.3 (Open Set and Closed Set)**.**
An **open set** $S \subseteq \mathbb{C}$ is a set such that for every $z \in S$, there is ε such that $|z - w| < \varepsilon \Rightarrow w \in S$.

A **closed set** is if $\mathbb{C} \setminus S$ is open.

Definition 1.2.4 (Limit)**.**
Let $f : \mathbb{C} \to \mathbb{C}$. We say $\lim\limits_{z \to w} f(z) = L$ if for all $\varepsilon > 0 \in \mathbb{C}$, $\exists \delta > 0$ s.t. for all $z \in \mathbb{C}$, if $|z - w| < \delta$ then $|f(z) - L| < \varepsilon$.

Example 1.2.1. Consider $\lim\limits_{z \to 0} \dfrac{\bar{z}}{z}$.

Try approaching in different directions:

- Let $z = x, x \in \mathbb{R}$, then,
$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{x \to 0} \frac{\bar{x}}{x} = \lim_{x \to 0} \frac{x}{x} = 1 \ .$$
- Try $z = iy$, $y \in \mathbb{R}$,
$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{y \to 0} \frac{\overline{iy}}{iy} = \lim_{y \to 0} \frac{-iy}{iy} = -1 \ .$$

Therefore, the limit does not exist.

Definition 1.2.5 (Continuity at a Point)**.**
A function f is **continuous at the point** z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

Definition 1.2.6 (Continuity on a Set)**.**
A function is **continuous on a set** S if it is continuous at all point $z \in S$.

Example 1.2.2. Consider $f(z) = z^2$.

Let $\Delta z = z - z_0$. Then,

$$\begin{aligned} \lim_{z \to z_0} z^2 &= \lim_{\Delta z \to 0} (z_0 + \Delta z)^2 \\ &= \lim_{\Delta z \to 0} z_0^2 + 2\Delta z z_0 + \Delta z^2 = z_0^2 \end{aligned}$$

So f is continuous everywhere.

Definition 1.2.7 (Connectedness)**.**
A set S is **connected** if S cannot be written as $S = S_1 \cup S_2$, where S_1, S_2 are open and $S_1 \cup S_2 = \varnothing$.

Definition 1.2.8 (Path)**.**
A **path** is the image of $[0, 1]$ under a continuous function.

Definition 1.2.9 (Path-Connectedness)**.**
A set S is **path-connected** if $\forall z_1, z_2 \in S$, there exists a path from z_1 to z_2 lying in S.

Definition 1.2.10 (Domain)**.**
A **domain** is a path-connected open set.

Remark 1.2.11. path-connected $\Rightarrow$ connected; connected $\not\Rightarrow$ path-connected

Example 1.2.3. Consider the set $S \subseteq \mathbb{R}^2$, $S = \{(x, \sin\frac{1}{x}), x = 0\} \cup \{(0, y) : y \in \mathbb{R}\}$, this set S is connected but not path connected.

Definition 1.2.12 (Derivative)**.**
Let $f : \mathbb{C} \to \mathbb{C}$, if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, then f is **differentiable** at z_0 and that the limit is its derivative $f'(z_0)$.

Example 1.2.4. Consider $f(z) = z^2$.

$$\begin{aligned} f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{\Delta z_0^2 + 2z_0 \Delta z + \Delta z^2 - \Delta z_0^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} 2z_0 + \Delta z = 2z_0 = f'(z_0) \ . \end{aligned}$$

Example 1.2.5. Consider $f(z) = |z|$.

$$\lim_{\Delta z_0} \frac{f(z_0) + \Delta z - f(z_0)}{\Delta z} = \lim_{\Delta z_0} \frac{|z_0 + \Delta z| - |z_0|}{\Delta z} \quad .$$

Try $\Delta z = x \in \mathbb{R}$, let $z_0 = a + bi$, $a, b \in \mathbb{R}$, then, because let $g(x, y) = \sqrt{x^2 + y^2}$, then, $\frac{\partial g}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$. Therefore

$$\begin{aligned}\lim_{\Delta z_0 \to 0} \frac{|z_0 + \Delta z| - |z_0|}{\Delta z} &= \lim_{x \to 0} \frac{|a + x + bi| - |a + bi|}{x} \\ &= \lim_{x \to 0} \frac{\sqrt{(a + x)^2 + b^2} - \sqrt{a^2 + b^2}}{x} = \frac{a}{a^2 + b^2} \ .\end{aligned}$$

Similarly, try $\Delta z = yi, y \in \mathbb{R}$, then,

$$\begin{aligned}\lim_{\Delta z \to 0} \frac{|z_0 + \Delta z| - |z_0|}{\Delta z} &= \lim_{yi \to 0} \frac{\sqrt{a^2 + (b + y)^2} - \sqrt{a^2 + b^2}}{yi} \\ &= \frac{1}{i} \cdot \lim_{y \to 0} \frac{\partial g}{\partial y} = \frac{1}{i} \cdot \frac{b}{\sqrt{a^2 + b^2}} \ .\end{aligned}$$

Thus, $f(z) = |z|$ **is differentiable <u>nowhere</u>.**

1.3 Derivative Continued - May 6

Proposition 1.3.1 (Derivative Operations)**.**

- $(f + g)'(z) = f'(z) + g'(z)$
- $(cf)'(z) = cf'(z)$, where $c \in \mathbb{C}$
- $(fg)'(z) = f(z)g'(z) + f'(z)g(z)$, product rule
- $(f \circ g)'(z) = f'(g(z))g'(z)$, chain rule
- $(\frac{f}{g})'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{g^2(z)}$

Definition 1.3.1 (Real and Imaginary Parts)**.**
Let $z \in \mathbb{C}$, $z = a + bi, a, b \in \mathbb{R}$. Then a and b are called **the real and imagniary parts** of z respectively, denoted $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

Example 1.3.1. $f(z) = \mathrm{Re}(z)$,

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

Let $h = h_x \in \mathbb{R}$, then,

$$\lim_{h_x \to 0} \frac{\text{Re}(a + h_x + bi) - \text{Re}(a + bi)}{h_x} = \lim_{h_x \to 0} \frac{a + h_x - a}{h_x} = 1 \ .$$

Now let $h = ih_y$, $h_y \in \mathbb{R}$, then,

$$\lim_{h_y \to 0} \frac{\text{Re}(a + bi + ih_y) - \text{Re}(a + bi)}{ih_y} = \lim_{h_y \to 0} \frac{a - i}{ih_y} = 0 \ .$$

Therefore, Re(z) is differentiable nowhere.

And Im(z) is differentiable nowhere because Im(z) = Re$(-iz)$. Then, $\bar{z}$ = Re$(z) - i$Im(z) is differentiable nowhere.

Intuition: Differentiable functions are these that act on z and are blind to Re(z), Im(z).

Example 1.3.2. $f(z) = z^2$, let $z = a + bi$, $f(z) = a^2 + 2ab - b^2$.

Recall $z = a + bi = re^{i\theta}$, where $r = \sqrt{a^2 + b^2}$, $\theta = \arctan \frac{b}{a}$, therefore, $f(z) = r^2 e^{2i\theta}$.

Definition 1.3.2 (Modulus)**.**
The **modulus** (or magnitude, sbolute value) of z is $r = |z|$.

Definition 1.3.3 (Argument)**.**
The **argument** of z is $\theta = \arg(z)$. Note the argument of z is NOT unique to a diffrence of multiple of 2π.

Example 1.3.3. what is $i^{1/2}$?

$i = e^{i\frac{\pi}{2}}$, so

$$i^{1/2} = (e^{i\frac{\pi}{2}})^{1/2} = e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \ .$$

$i = e^{i\frac{5\pi}{2}}$, so

$$i^{1/2} = (e^{i\frac{5\pi}{2}})^{1/2} = e^{i\frac{5\pi}{4}} = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} = -e^{\frac{\pi}{4}} \ .$$

Therefore, $i^{1/2} = \pm e^{i\frac{\pi}{4}}$.

Proposition 1.3.2 (Number of Roots)**.** If $n > 0 \in \mathbb{Z}$, $z = re^{i\theta}$, then

$$\begin{aligned} z^{1/n} =& (r \exp(i\theta))^{1/n} \\ =& \left\{ r^{1/n} e^{i\frac{\theta}{n}}, r^{1/n} e^{i(\frac{\theta + 2\pi}{n})} \right\} \end{aligned}$$

So any nonzero $z \in \mathbb{C}$ has exactly n distinct n-th roots, which are $r^{1/n} e^{i\frac{\theta + 2\pi}{n}}$, $0 \le k < n$.

Example 1.3.4. $(-1)^{\frac{1}{4}} = \exp(i\frac{\pi}{4}), \exp(i\frac{3\pi}{4}), \exp(i\frac{5\pi}{4}), \exp(i\frac{7\pi}{4})$.

1.4 Holomorphic and Cauchy-Riemann Equation - May 9

DEFINITION 1.4.1 (Holomorphic)**.**
If $f : \mathbb{C} \to \mathbb{C}$ is differentiable on a domain D, we say f is **holomorphic** on D. Also called **(complex) analytic**, regular.

Remark 1.4.2. Sloppy terminology warning: A function is said to be **holomorphic at a point** z_0 if it is hlomorphic on some open set containing z_0.

Proposition 1.4.1 (Cauchy-Riemann Equation)**.** Let $f = u + iv : \mathbb{C} \to \mathbb{C}$ be holomorphic on a domain D, then f satisfy the following **Cauchy-Riemann(CR) Equations** on D,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\ .$$

Proof. Let $z_0 \in D$, since f is holomorphic on D, $\lim\limits_{h\to 0} \dfrac{f(z+h)-f(z)}{h}$ exists.

Consider $h = h_x \in \mathbb{R}$, then

$$f'(z) = \lim_{h_x \to 0} \frac{f(z+h_x)-f(z)}{h_x}\ .$$

Let $z = x + iy$, then

$$f'(z) = \lim_{h_x \to 0} \frac{f(x+h_x+iy)-f(x+iy)}{h_x}\ ,$$

let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$, then

$$\begin{aligned} f'(x+iy) &= \lim_{h_x \to 0} \frac{u(x+h_x,y)+iv(x+h_x,y)-u(x,y)-iv(x,y)}{h_x} \\ &= \lim_{h_x \to 0} \frac{u(x+h_x,y)-u(x,y)}{h_x} + i \lim_{h_x \to 0} \frac{v(x+h_x,y)-v(x,y)}{h_x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\ . \end{aligned}$$

Now consider $h = ih_y$, $h_y \in \mathbb{R}$, then

$$\begin{aligned} f'(x+iy) &= \lim_{h_y \to 0} \frac{u(x,hy+h_y)+iv(x,y+h_y)}{h_y} - fracu(x,y)+iv(x,y)ih_y \\ &= \frac{1}{i}\left(\lim_{h_y \to 0} \frac{u(x,y+h_y)-u(x,y)}{h_y} + \lim_{h_y \to 0} \frac{iv(x,y+h_y)-iv(x,y)}{h_y}\right) \\ &= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial v} - i\frac{\partial u}{\partial y}\ . \end{aligned}$$

Therefore,

$$\begin{aligned} f'(x+iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \\ \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\ . \end{aligned}$$

□

Example 1.4.1. Let f be holomorphic on a domain D and let $v(x, y) = \text{Im}(f) = xy$ on D. Find $u(x, y)$. Let $f = u + iv$, then,

$$\begin{aligned}\frac{\partial u}{\partial x} &= x = \frac{\partial v}{\partial y}\\ \frac{\partial v}{\partial x} &= y = -\frac{\partial u}{\partial y}\end{aligned}$$

Therefore, $\frac{\partial u}{\partial x} = x$, $\frac{\partial u}{\partial y} = -y$. Hence,

$$\begin{aligned}u &= \int \frac{\partial u}{\partial x} dx = \frac{1}{2}x^2 + C_1(y)\\ u &= \int \frac{\partial u}{\partial y} dy = \frac{1}{2}y^2 + C_2(x)\\ \Rightarrow \qquad u(x, y) &= \frac{1}{2}x^2 - \frac{1}{2}y^2 + C\ .\end{aligned}$$

Therefore,

$$f(x + iy) = (\frac{1}{2}x^2 - \frac{1}{2}y^2 + C) + xyi = \frac{z^2}{2} + C\ .$$

Example 1.4.2. Let f be holomorphic on a domain D and let $\text{Re}(f) = x^2 y$, prove such function D.N.E..

Let $z = x + iy$, $f(x + iy) = u(x, y) + iv(x, y)$, $u(x, y) = x^2 y$.

$$\begin{aligned}\frac{\partial u}{\partial x} &= 2xy = \frac{\partial v}{\partial y} \qquad \Rightarrow \qquad v = \int 2xy dy = xy^2 + C_1(x)\\ \frac{\partial v}{\partial x} &= -x^2 = -\frac{\partial u}{\partial y} \qquad \Rightarrow \qquad v = \int x^2 dx = \frac{1}{3}x^2 + C_2(y)\ .\end{aligned}$$

No such v exists because $C_2(y)$ cannot contain x.

Remark 1.4.3.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

Let $f = u + iv$ be holomorphic and let $v(x, y) = xy$. Then

1.5 Holomorphic and Cauchy-Riemann Equation (Continued) - May 11

Example 1.5.1. $f(z) = e^z$,

$$f(x+iy) = e^x e^{iy} = e^x(\cos y + i \sin y) = e^x \cos y + ie^x \sin y$$

Then,

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -e^x \sin y = \frac{\partial v}{\partial x}$$

Theorem 1.5.1 (Cauchy-Riemann Equation and Holomorphicity)**.**
Let $u, v : \mathbb{R}^2 \to \mathbb{R}$, have continuous partial derivatives at (x_0, y_0) which satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \ ,$$

then $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic at $z_0 = x_0 + iy_0$.

Proof. Let $D \subseteq \mathbb{C}$ be an open set with $z_0 \in D$, let $z = x + iy \in D$. Then,

$$u(x,y) = u(x_0,y_0) + (x-x_0)\left(\frac{\partial u}{\partial x} + \varepsilon_1(x,y)\right) + (y-y_0)\left(\frac{\partial u}{\partial y} + \varepsilon_2(x,y)\right),$$

where $\varepsilon_1, \varepsilon_2$ are continuous at (x_0, y_0) and $\varepsilon_1(x_0 y_0) = \varepsilon_2(x_0, y_0) = 0$. And,

$$v(x,y) = v(x_0,y_0) + (x-x_0)\left(\frac{\partial v}{\partial x} + \varepsilon_3(x,y)\right) + (y-y_0)\left(\frac{\partial v}{\partial y} + \varepsilon_4(x,y)\right),$$

So

$$f(x+iy) = u(x,y) + iv(x,y) = f(z_0) + (z-z_0)\left(\frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) + \varepsilon(z)\right) \ ,$$

where

$$\varepsilon(z) = \frac{x-x_0}{z-z_0}(\varepsilon_1 + i\varepsilon_3) + \frac{y-y_0}{z-z_0}(\varepsilon_2 + i\varepsilon_4) \ .$$

satisfying ε continuous at $z_0, \varepsilon(z_0) \to 0$. So

$$\begin{aligned}\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{z\to z_0} \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) + \varepsilon(z) \\ &= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) \\ &= f'(z_0)\end{aligned}$$

exists and thus. □

Remark 1.5.2. We now have a good test for holomorphicity.

Example 1.5.2. Consider $f(z) = \bar{z}$.

$f(x+iy) = x - iy$, $u = x$, $v = -y$. then

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1 \qquad\qquad \frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial y} .$$

Therefore, $f(z)$ is nowhere holomorphic at complex plane.

Example 1.5.3. Consider $f(z) = \frac{1}{z}$.

Method 1:

$$f(x+iy) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$

Therefore, $u = \frac{x}{x^2+y^2}$, $v = \frac{-y}{x^2+y^2}$.

$$\begin{aligned} \frac{\partial u}{\partial x} =& \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\ \frac{\partial v}{\partial y} =& \frac{y^2-x^2}{(x^2+y^2)^2} . \end{aligned}$$

And

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x} .$$

So f is holomorphic on $\mathbb{C} \setminus \{0\}$.

Method 2: $f(z) = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-r\theta}$ is well-behaved.

Lemma 1.5.1. If f, g are holomorphic at z_0 and $g(z_0) \neq 0$, then $\frac{f}{g}$ is holomorphic at z_0.

However, the converse is false, that is if $\frac{f}{g}$ is holomorphic at z_0, it does not imply that f, g are holomorphic at z_0.

Proposition 1.5.1 (Polar Form of Cauchy-Riemann Equation)**.** Let $z = re^{i\theta}$, $z_0 = r_0e^{i\theta_0}$, let $f(z)$ be holomorphic, and let $f(z) = u(r,\theta) + iv(r,\theta)$, we have

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} .$$

First, fix $\theta = \theta_0$ and let $r \to r_0$, then

$$\begin{aligned} f'(r_0e^{i\theta_0}) =& \lim_{r\to r_0} \frac{u(r,\theta_0) + iv(r,\theta_0) - (u(r_0,\theta_0) + iv(r_0,\theta_0))}{re^{i\theta_0} - r_0e^{i\theta_0}} \\ =& e^{-r\theta_0}\left(\frac{\partial u}{\partial r}(r_0,\theta_0) + i\frac{\partial v}{\partial r}(r_0,\theta_0)\right) \end{aligned}$$

Next let $r = r_0$, $\theta \to \theta_0$, then,

$$\begin{aligned}
f'(r, e^{i\theta_0}) =& \lim_{\theta\to\theta_0} \frac{u(r_0,\theta) + iv(r_0,\theta) - (u(r_0,\theta_0) + iv(r_0,\theta_0))}{r_0 e^{i\theta} - r_0 e^{i\theta_0}} \\
=& \frac{1}{r_0} \lim_{\theta\to\theta_0} \left[\frac{u(r_0,\theta) - u(r_0,\theta_0)}{\theta - \theta_0} + i \frac{v(r_0,\theta) - v(r_0,\theta_0)}{\theta - \theta_0} \right] \cdot \frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}} \\
=& \frac{1}{r} \frac{1}{ie^{\theta}} \left(\frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} \right) \\
=& e^{-i\theta} \left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial u}{\partial r} =& \frac{1}{r} \frac{\partial v}{\partial \theta} \\
\frac{\partial v}{\partial r} =& - \frac{1}{r} \frac{\partial u}{\partial \theta}
\end{aligned}$$

1.6 Branch - May 13

Remark 1.6.1. Assuming all partial derivatives are continuous and $z = re^{i\theta}$,

$$\begin{aligned}
& f = u + iv \text{ is holomorphic on a domain } D \\
\iff & \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ on D} \\
\iff & \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \text{ on D}
\end{aligned}$$

Remark 1.6.2. Recall $z^{1/n} = (re^{i\theta})^{1/n} = r^{1/n} e^{i\frac{\theta+2\pi m}{n}}$, $0 \le m < n$. Consider $z^{1/n} = r^{1/n} e^{i\frac{\theta}{n}} = r^{1/n} \cos \frac{\theta}{n} + i r^{1/n} \sin \frac{\theta}{n}$. Then let

$$u(r,\theta) = r^{1/n} \cos \frac{\theta}{n} \qquad v(r,\theta) = r^{1/n} \sin \frac{\theta}{n} \ ,$$

$$\begin{aligned}
\frac{\partial u}{\partial r} =& \frac{1}{n} r^{\frac{1}{n}-1} \cos \frac{\theta}{n} = \frac{1}{r} \frac{\partial v}{\partial \theta} \\
\frac{\partial v}{\partial r} =& \frac{1}{n} r^{\frac{1}{n}-1} \sin \frac{\theta}{n} = -\frac{1}{r} \frac{\partial u}{\partial v} \ .
\end{aligned}$$

This function is not continuous along $\text{Arg}(z) = \pi$. This function is holomorphic along $\mathbb{C} \setminus \mathbb{R}_{\le 0}$ (Note 0 is included). This is called the **branch cut**.

We can put the branch cut somewhere else, i.e.

$$f(re^{i\theta}) = \theta, 0 \le \theta < 2\pi$$

then, f is not continuous at $\text{Arg}(z) = 2\pi$.

DEFINITION 1.6.3.
Define $\arg(re^{i\theta}) = \theta + 2\pi m$, $m \in \mathbb{Z}$, and $\text{Arg}(re^{i\theta}) = \theta$, $-\pi < \theta \leq \pi$.

This is a **branch** of the multivalued function arg. In particular Arg is called the **principle branch**.

$\text{Arg}(z)$ is not continuous at $\text{Arg}(z) = \pi$.

Remark 1.6.4. Other branches:

- $f(re^{i\theta}) = \theta, \pi < \theta \leq 3\pi = \text{Arg}(re^{i\theta}) + 2\pi$
- $\text{Arg}(z) + 2\pi n$, $n \in \mathbb{Z}$ is a branch of $\arg(z)$.

DEFINITION 1.6.5 (Logarithm)**.**
For $z \neq 0 \in \mathbb{C}$ define

$$\begin{aligned}\log(z) = \log(re^{i\theta}) &= \log(r) + \log(e^{i\theta})\\ &= \log r + i\theta\\ &= \ln r + i\theta\\ &= \ln|z| + i\theta + 2\pi i m, m \in \mathbb{Z}.\end{aligned}$$

Note log is a multivalued function.

Notation 1.6.6. Use log for the multivalued function and Log for the principal branch:

$$\text{Log}(z) = \ln|z| + i\text{Arg}(z) \; .$$

$u = \ln r$, $v = \theta$,

$$\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial v}{\partial r} = 0 = -\frac{1}{r}\frac{\partial u}{\partial \theta} \; .$$

Example 1.6.1. $-\pi < \theta \leq \pi$

$$\begin{aligned}\text{Log}(z^{1/2}) =& \text{Log}((re^{i\theta})^{1/2})\\ =& \text{Log}(r^{1/2}e^{\frac{i\theta}{2}})\\ =& \ln|r^{1/2}| + i\frac{\theta}{2}\\ =& \frac{1}{2}(\ln|r| + i\theta) = \frac{1}{2}\text{Log}(z) \; .\end{aligned}$$

Remark 1.6.7. Note that $\text{Log}(wz) \neq \text{Log}(z) + \text{Log}(w)$ in general.

1.7 Trignometric Functions and Power - May 16

Definition 1.7.1 (Trignometric Functions on Complex Plane)**.**
Let $z \in \mathbb{C}$,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \ .$$

sin and cos are immediately holomorphic in the entire $\mathbb{C}$, since the exponential functions are.

Remark 1.7.2. If two holomorphic functions are equal on "enough" of a set, they must agree on their domains.

Lemma 1.7.1 (Derivatives of Trig Functions)**.**

$$\begin{aligned}
\frac{d}{dz}\sin(z) =& \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) \\
=& \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z \\
\frac{d}{dz}\cos(z) =& \frac{d}{dz}\left(\frac{e^{iz} + e^{-iz}}{2i}\right) \\
=& \frac{ie^{iz} - ie^{-iz}}{2} = \frac{-e^{iz} + e^{-iz}}{2i} = -\sin z \ .
\end{aligned}$$

Remark 1.7.3 (Trig Functions are unbounded in $\mathbb{C}$)**.** All the trig identities in $\mathbb{R}$ carry over in the obvious way to $\mathbb{C}$. However, $\cos, \sin$ have properties that are NOT true in $\mathbb{R}$.

Consider $\cos(iy)$,

$$|\cos(iy)| = \left|\frac{e^{i(iy)} + e^{-i(iy)}}{2}\right| = \left|\frac{e^{-y} + e^{y}}{2}\right| \ ,$$

then, $|\cos(iy)| \to \infty$ as $y \to \infty$ or $y \to -\infty$. Hence, $\sin z, \cos z$ are NOT bounded functions on $\mathbb{C}$.

Definition 1.7.4 (Hyperbolic Functions)**.**
Let $z \in \mathbb{C}$,

$$\cosh z = \frac{e^{z} - e^{-z}}{2} \qquad \text{and} \qquad \sinh z = \frac{e^{z} + e^{-z}}{2} \ .$$

Then,

$$\sinh(iz) = i \sin z \qquad \text{and} \qquad \cosh(iz) = \cos z$$

Example 1.7.1. Consider i^i.

$$i^i = (e^{\log i})^i = e^{i \log i} = e^{i(\frac{\pi i}{2} + 2\pi i k)} = e^{-\frac{\pi}{2} - 2\pi k}, k \in \mathbb{Z} \ .$$

Example 1.7.2.

$$\log i = \log e^{i\frac{\pi}{2}} = -\frac{i\pi}{2} + 2\pi ik, k \in \mathbb{Z}$$

Definition 1.7.5 (Derivative of Power)**.**
Define $z^w = e^{w \log z}$, then its derivative is

$$\frac{d}{dz}(z^w) = \frac{d}{dz}(e^{w \log z}) = e^{w \log z}\frac{d}{dz}(w \log z) = w\frac{1}{z}e^{w \log z} = \frac{w}{z}z^2 = wz^{w-1} .$$

Remark 1.7.6. How many values does z^w have?

$$z^w = e^{w \log z} = e^{w(\log z + 2\pi ik)} = e^{w \log z}e^{2\pi ikw}$$

When is $e^{2\pi ikw} = e^{2\pi inw}, n, w \in \mathbb{Z}, n \neq k$?

If $e^{2\pi ikw} = e^{2\pi inw}$, these are equal when

$$w\pi ikw = 2\pi inw + 2\pi im, m \in \mathbb{Z} .$$

$kw = nw + m$, $m \in \mathbb{Z}$, $w = \frac{m}{k-n}$, $m, n, k \in \mathbb{Z}$. Thus the powers z^w repeat if and only if $w \in \mathbb{Q}$. Then if $w = \frac{p}{q}$, $z^w = z^{p/q} = (z^p)^{1/q}$ has q distinct values.

Proposition 1.7.1. If $z \neq 0$,

$$z^w \text{ take on } \begin{cases} 1, & \text{if } w \in \mathbb{Z} \\ q, & \text{if } w = \frac{p}{q} \in \mathbb{Q} \\ \infty, & \text{otherwise} \end{cases} .$$

1.8 Rotalion Approximation - May 18

Algorithm 1.8.1 (Rotalion Approximation)**.** Let f be holomorphic at z_0, we have

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} .$$

The modulus and argument must converge individually:

- $$|f'(z_0)| = \lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right|$$

  So near z_0,

  $$|f(z) - f(z_0)| \approx |f'(z_0)| \cdot |z - z_0| .$$

- and
$$\arg(f'(z_0)) = \lim_{z\to z_0} \arg(\frac{f(z)-f(z_0)}{z-z_0}) \ .$$
for some branch of arg holomorphic near z_0, $f(z_0)$, $f'(z_0)$.
$$\begin{aligned} \arg(f'(z)) &= \arg(f(z)-f(z_0)) - \arg(z-z_0) \\ \Rightarrow \arg(f(z)-f(z_0)) &\approx \arg(f'(z_0)) + \arg(z-z_0) \ . \end{aligned}$$

So near z_0 we have
$$f(x) \approx f(z_0) + e^{i\arg(f'(z_0))}|f'(z_0)|(z-z_0) \ .$$
This is a rotalion of $z-z_0$ by $\arg(f'(z_0))$ and a scaling by $|f'(z_0)|$.

Example 1.8.1. Consider $f(z)=z^2$, $f(re^{i\theta}) = r^2e^{i2\theta}$.

Let $z_0 = 1+i = \sqrt{2}e^{i\frac{\pi}{4}}$, $f(z_0)=z_0^2=2i$, $f'(z)=2z$, $f'(z_0)=2(1+i)=2\sqrt{2}e^{i\frac{\pi}{4}}$.

So for small $h=z-z_0$,
$$f(z_0+h) \approx f(z_0) + e^{i\arg f'(z_0)}$$

Theorem 1.8.1.
If f is holomorphic on a domain D and $f'(z)=0$ for all $z\in D$, then f is constant on D.

Proof.
$$f'(z) = 0 = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$
Then, all partial derivatives are 0:
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \ .$$
Therefore, u, v are constant on any horizontal or vertical line segment in D. But D is domain, so D is open and path-connected. Then any two points in D can be connected by a path of horizontal and vertical segments. So u and v are constant on D so f. Therefore, $f=u+iv$ is constant on D. □

Example 1.8.2. Find a branch of $(z^2-1)^{1/2}$ holomorphic on $|z|>1$.

Note that the principal branch of $z^{1/2}$ does not work: $e^{1/2}\log(z^2-1)$.

Its branch cut is where $z^2-1\in\mathbb{R}$, $z^2-1\le 0$. But let $z=2i$, $z^2-1=-4-1=-5\le 0$.

Find $f(z)$ holomorphic on $|z|>1$ such that $f(z)^2=z^2-1$.

Consider the principal branch of $f(z) = z(1-\frac{1}{z^2})^{1/2}$.

Its branch cut lies wherever $1-\frac{1}{z^2}\le 0$ in $\mathbb{R}$, which is $\frac{1}{z^2}\ge 1$ in $\mathbb{R} \Rightarrow z^2\le 1$ in $\mathbb{R}$, so $|z|<1$.

2 Integration

2.1 Integrability - May 20

Definition 2.1.1 (Smooth Curve)**.**
A **smooth curve** in $\mathbb{C}$ is the image of a function $\gamma : [a, b] \to \mathbb{C}$ satisfying

- γ is continuously differentiable on $[a, b]$
- $\gamma' \neq 0$ on $[a, b]$
- γ is one to one

This definition rules out gaps, sharp corners, pausing(temporarily stopping at a point), retracing, self-intersection.

Definition 2.1.2 (Directed Smooth Curve)**.**
A **directed smooth curve** is a smooth curve with a fixed direction, $i.e.$ the points on the curve are ordered and any γ must trace them in order.

Definition 2.1.3 (Contour)**.**
A **contour** is a directed piecewise smooth curve. i.e. $\Gamma = C_1 \cup C_2 \cup \cdots \cup C_n$, where the C_j are directed small curves and the terminal point of C_j is the initial point of C_j.

Definition 2.1.4 (Simple Contour)**.**
A contour is **simple** if it has no self-intersections.

Definition 2.1.5 (Closed Contour)**.**
A contour is **closed** if its initial point conincides with its terminal point.

Definition 2.1.6 (Simple Closed Contour)**.**
A **simple closed contour** is a contour both simple and closed.

Example 2.1.1. $\Gamma : r_1(t) = z_0 t + z_1(1-t)$, $t \in [0, 1]$, $z_0, z_1 \in \mathbb{C}$.

Note parametrizations are NOT unique, e.g. $r_2(t) = z_0(2t) + z_1(1-2t), t \in [0, \frac{1}{2}]$, $r_3(t) = z_0 t^2 + z_1(1-t^2)$, $t \in [0, 1]$.

Definition 2.1.7 (Circular Contour)**.**
$C_r(z_0)$ is the circular contour with radius r and center z_0, traversed counterclockwise.

Example 2.1.2. $\Gamma = C_1 \cup C_2 \cup C_3$

$$\begin{aligned}
&C_1 : \gamma_1(t) = t, t \in [0, 1] \\
&C_2 : \gamma_2(t) = ti + (1-t), t \in [0, 1] \\
&C_3 : \gamma_3(t) = (1-t)i, t \in [0, 1] \\
&\Rightarrow \gamma(t) = \begin{cases} t, & t \in [0, 1] \end{cases}
\end{aligned}$$

THEOREM 2.1.8 (Jordan Curve Theorem)**.**
A simple closed contour divides $\mathbb{C}$ into two disjoint regoins, a bounded interior and an unhounded exterior.

DEFINITION 2.1.9 (Orientation)**.**
A simple closed contour is **positively oriented** if its interior is to the left when traverse, **negatively oriented** otherwise.

DEFINITION 2.1.10 (Partition)**.**
Let Γ be a directed smooth curve with initial point w_0 and terminal point w_1, a **partition** of Γ is a set of points $Z_0 = w_0, z_1, z_2, \dots, z_n = w_1$ such that $\forall j$, $0 \leq j < n$, z_{j+1} is farther along Γ than z_j.

DEFINITION 2.1.11 (Mesh)**.**
The **mesh** of a partition is the largest distance between two consecutive points z_j, z_{j+1} along Γ.

DEFINITION 2.1.12 (Riemann Sum)**.**
Let Γ lie in a domain D and let $f : D \to \mathbb{C}$. The **Riemann sum** of f with respect to P_n is

$$S_f(P_n) = f(z_1)(z_1 - z_0) + f(z_2)(z_2 - z_0) + \cdots + f_(z_n)(z_n - z_{n-1}) \ .$$

DEFINITION 2.1.13 (Integrable)**.**
f is integrable along Γ if

$$\lim_{\text{mesh}(P_n)\to 0} S(P_n) \text{ exists } .$$

DEFINITION 2.1.14.
If f is integrable along Γ, the integrable of f along Γ with partition $P_n = z_0, z_1, \dots, z_n$ is

$$\int_\Gamma f = \lim_{\text{mesh}(P_n)\to 0} S(P_n) = \lim_{\text{mesh}(P_n)\to 0} \sum_{j=0}^{n-1} f(z_j)(z_{j+1} - z_j)$$

Remark 2.1.15. Note in the above formula, we do not reference a parametrization of Γ and therefore, the integral of f is independent of any parametrization.

2.2 May 25

DEFINITION 2.2.1 (Integral).
Let Γ be a directed smooth curve, let $P_n = z_0, z_1, \ldots, z_n$ be a partition of Γ and let

$$\int_\Gamma f(z)dz = \lim_{mesh(P_n)\to 0} \sum_{j=0}^{n-1} f(z_j)(z_{j+1} - z_j) \ .$$

Now let Γ be parametrized by $\gamma : [a, b] \to \Gamma$ and let $t_0, t_1, \ldots, t_n$ be a partition of $[a, b]$ such that $\gamma(t_j) = z_j$, $0 \le j \le n$, then

$$\begin{aligned} \lim_{mesh(P_n)\to 0} \sum_{j=0}^{n-1} f(z_j)\,\Delta z_j &= \lim_{\Delta t_j \to 0, 0\le j<n} \sum_{j=0}^{n-1} f(\gamma(t_i))\,\Delta z_j \ , \\ &\qquad \text{where } \Delta t_j = t_{j+1} - t_j, \Delta z_j = z_{j+1} - z_j \\ &= \lim_{\Delta t_j \to 0} \sum_{j=0}^{n-1} f(\gamma(t_j))\,\gamma'(t_j)\,\Delta t_j \\ &= \int_a^b f(\gamma(t))\,\gamma'(t)dt \\ \Rightarrow \qquad \int_\Gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt \ . \end{aligned}$$

★ **Integral of parametrization**

DEFINITION 2.2.2.
Define the integral over a contour $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$, where Γ_j are smooth directed curves, to be

$$\int_\Gamma f = \int_{\Gamma_1} f + \cdots + \int_{\Gamma_n} f \ .$$

We immediately have

- $$\int_\Gamma f + g = \int_\Gamma f + \int_\Gamma g \ .$$
- $$\int_{\Gamma_1 + \Gamma_2} f = \int_{\Gamma_1} f + \int_{\Gamma_2} f$$
- For $c \in \mathbb{C}$,
$$\int_\Gamma cf = c \int_\Gamma f$$

Example 2.2.1. $\int_\Gamma z^2 dz$, $\Gamma = \gamma(t) = e^{it}$, $0 \le t \le \pi$.

$$\int_\Gamma z^2 dz = \int_0^\pi (e^{it})^2\,(ie^{it})dt = i\int_0^\pi e^{3it}dt = \frac{1}{3}(e^{3i\pi} - e^{3i(0)}) = -\frac{2}{3} \ .$$

Example 2.2.2. Let $C_1(0) = r(t) = e^{it}$, $0 \le t \le 2\pi$, then

$$\int_{C_1(0)} z dz = \int_0^{2\pi} e^{it}(ie^{it})dt = i\int_0^{2\pi} e^{2it}dt = \frac{e^{2it}}{2}\bigg|_0^{2\pi} = \frac{e^{4\pi i}-1}{2} = 0 \ .$$

and

$$\int_{C_1(0)} \frac{1}{z}dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}dt = 2\pi i \ .$$

Definition 2.2.3 (Length)**.**

The length of a contour Γ is parametrized by $\gamma : [a,b] \to \Gamma$, $\displaystyle\int_a^b |\gamma'(t)|dt$.

2.3 May 27

Theorem 2.3.1 (ML bound)**.**
Let f be integrable on Γ and let $|f(z)| \le M, \forall z \in \Gamma$,

$$\begin{aligned}\left|\int_\Gamma f(z)dz\right| &= \left|\int_a^b f(\gamma(r))\gamma'(t)dt\right| \\ &\le \int_a^b |f(\gamma(t))\gamma'(t)|dt \\ &= \int_a^b |f(\gamma(t))||\gamma'(t)|dt \\ &\le \int_a^b M|\gamma'(t)|dt \\ &= M \cdot \text{length}(\Gamma) \ .\end{aligned}$$

This is often called the ML bound.

Definition 2.3.2 (Primitive)**.**
A function F is a **primitive** (or **antiderivative**) for a function f on a domain D if F is holomorphic on D and $\forall z \in D$, $F'(z) = f(z)$.

Definition 2.3.3.
Let f have a primitive F on D and let Γ lie on D. Consider

$$\int_\Gamma f(z)dz = \int_\Gamma F'(z)dz \ .$$

Let $\gamma : [a,b] \to \Gamma$ parametrize Γ. Then

$$\int F'(z)dz = \int_a^b F'(\gamma(t))\gamma'(t)dt = \int_a^b \frac{dF}{d\gamma}(\gamma(t))\frac{d\gamma}{dt}dt = \int_{\gamma(a)}^{\gamma(b)} \frac{dF}{d\gamma}d\gamma = F(\gamma(b)) - F(\gamma(a)) \ .$$

By the Fundamental Theorem of Calculus on $\mathbb{R}$.

Theorem 2.3.4 (Fundamental Theorem of Calculus in $\mathbb{C}$)**.**
If f has a primitive F on a domain in D and Γ lies in D with initial point z_0 and terminal point z_1,

$$\int_\Gamma f(z)dz = F(z_1) - F(z_0) \ .$$

Example 2.3.1. $f(z) = z$ has a primitive $F(z) = \frac{1}{2}z^2$ on all of $\mathbb{C}$. So for Γ running from z_0 to z_1,

$$\int_\Gamma z dz = \frac{1}{2}z^2\Big|_{z_0}^{z_1} \ .$$

Example 2.3.2. $f(z) = \frac{1}{z}$ has primitive $\log z$. i.e. any branch of $\log z$ is a primitive of $\frac{1}{z}$ and its domain.

Corollary 2.3.1. If f has a primitive on a domain D and Γ is a closed contour lying in D,

$$\int_\Gamma f(z)dz = 0 \ .$$

Proof.

$$\int_\Gamma f = F(z_1) - F(z_0) = F(z_0) - F(z_0) = 0 \ .$$

□

Lemma 2.3.1. Let f be continuous on a domain D and let $\oint_\Gamma f = 0$ for any closed Γ lying in D. Then given Γ_1, Γ_2 in D with the same initial and terminal points,

$$\int_{\Gamma_1} f = \int_{\Gamma_2} f \ .$$

Proof. Note that $\Gamma_1 + (-\Gamma_2)$ is closed, so

$$\int_{\Gamma_1+(-\Gamma_2)} f = 0 = \int_{\Gamma_1} f - \int_{\Gamma_2} f = 0 \ .$$

□

Lemma 2.3.2. Let f be continuous on a domain D such that for Γ_1, Γ_2 in D sharing initial and terminal points,

$$\int_{\Gamma_1} f = \int_{\Gamma_2} f \ .$$

Then f has a primitive on D.

Remark 2.3.5. ★ important

Theorem 2.3.6.
Let f be continuous on a domain D TFAE

1. f has a primitive on D
2. for all closed contours Γ lying in D, $\int_\Gamma f = 0$.
3. for any two contours Γ_1, Γ_2 in D sharing initial and terminal points

$$\int_{\Gamma_1} f = \int_{\Gamma_2} f \ .$$

2.4 May 30

Definition 2.4.1.
A **Cauchy sequence** is a sequence $\{z_n\}_{n=1}^{\infty}$ s.t. $\forall \varepsilon > 0$, $\exists N > 0$ $\forall n_1, n_2 > N$, $|z_{n_1} - z_{n_2}| < \varepsilon$.

Lemma 2.4.1. Closed bounded subsets of $\mathbb{R}^n$ are compact.

Lemma 2.4.2 (Cauchy sequence convergence)**.** A Cauchy sequence in a compact set $S \subseteq \mathbb{R}$ converges to a point in S.

Lemma 2.4.3. Let f be holomorphic at z_0, then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \varepsilon(z)(z - z_0) \ ,$$

for some $\varepsilon(z)$ satisfying $\lim_{z \to z_0} \varepsilon(z) = 0$.

Proof. Let $\varepsilon(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$. Take $\lim_{z \to z_0}$. □

Theorem 2.4.2 (Goursat's Theorem)**.**
Let f be holomorphic on a domain D and let T be a triangle lying in D with interior in D. Then

$$\int_T f(z) dz = 0 \ .$$

Proof. Divide T into four triangles by connecting the midpoints of its side, now

$$\int_T f = \int_{T_{(1)}} f + \int_{T_2} f + \int_{T_{(3)}} f + \int_{T_{(4)}} f \ .$$

There exists a T_1 such that $|\int_T f| \leq 4 |\int_{T_1} f|$. Call it T_1. Note that $\text{length}(T_1) \leq \frac{1}{2}\text{length}(T)$, hence

$$\text{diam}(T_1) \leq \frac{1}{2}\text{diam}(T) \ .$$

Repeat this process, yielding $T = T_0, T_1, T_2, T_3, T_4, \ldots$,

$$|\int_T f| \le 4^n |\int_{T_n} f| \ ,$$

then, $\text{length}(T_n) \le \frac{1}{2^n}\text{length}(T)$ and $\text{diam}(T_n) \le \frac{1}{2^n}\text{diam}(T)$.

Let z_n be a point in the interior of T_n for each n. Then, $\{z_n\}$ is a Cauchy sequence and $\lim_{n\to\infty} z_n = w$, where w is in the interior of the triangle.

f is holomorphic at w, so

$$f(z) = f(w) + f'(w)(z-w) + \varepsilon(z)(z-w)$$

, where $\lim_{z\to w}\varepsilon(z) = 0$. Now consider

$$\int_{T_n} f(z)dz = \int_{T_n} f(w) + f'(w)(z-w) + \varepsilon(z)(z-w)dz \ .$$

$f(w)$ has primitive $zf(w)$, $f'(w)(z-w)$ has primitive $\frac{1}{2}f'(w)(z-w)^2$ so

$$\int_{T_n} f(w) + f'(w)(z-w)dz = 0 \ .$$

therefore,

$$\int_{T_n} f(z)dz = \int_{T_n} f(w) + f'(w)(z-w) + \varepsilon(z)(z-w)dz = \int_{T_n} \varepsilon(z)(z-w)dz \ .$$

Let $\varepsilon_n = \sup_{z\in T_n}|\varepsilon(z)|$, $|z-w| \le \text{diam}(T_n) \le \frac{1}{2^n}\text{diam}(T)$. $\text{length}(T_n) \le \frac{1}{2^n}\text{length}(T)$. So

$$\left|\int_{T_n]f(z)dz}\right| = \left|\int_{T_n}\varepsilon(z)(z-2)dz\right| \le \varepsilon\text{diam}(T_n)\text{length}(T_n) \le \varepsilon_n\frac{1}{4^n}\text{diam}(T)\text{length}(T) \ ,$$

thus

$$\left|\int_T f(z)dz\right| \le 4^n\left|\int_{T_n} f(z)dz\right| \le 4^n\varepsilon_n\frac{1}{4^n}\text{diam}(T)\text{length}(T) = \varepsilon_n\text{diam}(T)\text{length}(T).$$

Let $n\to\infty$ and $\varepsilon_n \to 0$,

$$\left|\int_{T_n} f(z)dz\right| = 0 \Rightarrow \int_{T_n} f(z)dz = 0 \ .$$

$\square$

Corollary 2.4.1. If f is holomorphic on an open disk, f has primitive on that disk.

Proof. Choose $z_0 \in D$, define $F(z) = \int_\Gamma f(z)dz$. Then

$$F(z+h) - F(z) = \int_{\Gamma_n} f(z)dz + \int_\Delta f + \int_\square f = \int_{\Gamma_n} f(z)dz$$

so that

$$\frac{d}{dz}\int_{\Gamma_n} f(z)dz = f(z)$$

as in last lecture. $\square$

2.5 June 1

THEOREM 2.5.1 (Goursat's Theorem)**.**
If f is holomorphic on and inside a polygon traingle T, then $\oint_T f = 0$.

THEOREM 2.5.2 (Cauchy's Theorem)**.**
On an open disk f holomorphic on an open disk D, let $\Gamma \subseteq$ be a closed contour, then

$$\oint_\Gamma f = 0 \ .$$

Example 2.5.1. $f(z) = \frac{1}{z}$ is holomorphic on $0 < |z|$ but has no primitive and

$$\int_{C_1(0)} \frac{1}{z} dz = 2\pi i \ .$$

DEFINITION 2.5.3 (Homotopic)**.**
Let Γ_1, Γ_2 be two contours in a domain in D with the same initial and terminal point. Γ_0 is **homotopic** (or **continuously deformable**) to Γ_1 if there exists $\gamma : [0,1]^2 \to \mathbb{C}$ satisfying

- γ is continuous on $[0,1]^2$
- for a fixed s, $\gamma(s,t)$ is a parametrization of a contour in D with initial and terminal point shared with Γ_1, Γ_2.
- $\gamma(0,t)$ parametrizes Γ_0
- $\gamma(1,t)$ parametrizes Γ_1

DEFINITION 2.5.4 (Simply Connected)**.**
A doamin D is **simply connected** if any two contours in D sharing initial and terminal points are homotopic to each other.

THEOREM 2.5.5 (Cauchy's Theorem)**.**
Let f be holomorphic on a simply connected domain D, and let Γ be a closed contour in D. Then $\int_\Gamma f = 0$.

Proof. Γ is homotopic on a triangle. □

Example 2.5.2. $\int_\Gamma z^2 dz$, then z^2 is entire, so by Cauchy's Theorem,

$$\int_\Gamma z^2 dz = \int_{z^2} dz = \int_1^{-1} x^2 dx = -\frac{2}{3} \ .$$

Example 2.5.3.

$$\int_{C_2(0)} \frac{1}{x^2-1} dz$$

$\frac{1}{z^2-1}$ is holomorphic on $\mathbb{C} \setminus \{1, -1\}$. Then

$$\int_{C_2(0)} \frac{1}{z^2-1} dz = \int_{C_\varepsilon(-1)} \frac{1}{z^2-1} dz + \int_{C_\varepsilon(1)} \frac{1}{z^2-1} dz, \text{ for } 0 < \varepsilon < 2 \ .$$

2.6 Cauchy Integral Formula - June 3

Theorem 2.6.1.
$f(z)$ holomorphic on a shaded region plus a bit, then

$$\int_{\Gamma_1} F(z) dz = \int_{\Gamma_2} F(z) dz$$

where Γ_1, Γ_2 have standard anticlockwise orientations.

Theorem 2.6.2 (Cauchy Integral Formula)**.**
Let f be a function holomorphic on a domain $D \subseteq \mathbb{C}$, and Γ a Jordan curve contained in D and whose interior is contained in D. Let $z_0 \in$ interior of Γ. Then

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} dz \ .$$

Proof. We can replace Γ with $C(r) = \{|z - z_0| = r\}$ for small enough r. Then

$$\begin{aligned}
\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-z_0} dz =& \frac{1}{2\pi i} \int_{C(r)} \frac{f(z)}{z-z_0} dz \\
=& \frac{1}{2\pi i} \left(\int_{C(r)} \frac{f(z_0)}{z-z_0} dz + \int_{C(r)} \frac{f(z)-f(z_0)}{z-z_0} dz \right) \\
=& f(z_0) + \int_{C(r)} \frac{f(z)-f(z_0)}{z-z_0} dz
\end{aligned}$$

Then, we claim that

$$\int_{C(r)} \frac{f(z)-f(z_0)}{z-z_0} dz = 0 \ .$$

Since $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists,

$$\left| \frac{f(z)-f(z_0)}{z-z_0} \right|$$

is bounded on $C(r)$ and its interior. So

$$\int_{C(r)} \frac{f(z) - f(z_0)}{z - z_0} dz \to 0 \text{ as } z \to 0 \ ,$$

so since this integral is independent of r, it must equal 0. $\square$

Example 2.6.1. Let $\Gamma = \{|z| = 1\}$. Compute the following:

- Consider $\int_\Gamma \frac{\cos z}{z} dz$,

$$\int_\Gamma \frac{\cos z}{z} dz = \int_\Gamma \frac{\cos z}{z - 0} dz = 2\pi i \cos(0) = 2\pi i \ .$$

- Consider $\int_\Gamma \frac{e^z}{z-2} dz$, because $2 \notin \{|z| \leq 1\}$, by Cauchy integral theorem, this equals 0.

$$\int_\Gamma \frac{e^z}{z - 2} dz = 2\pi i e^(2)$$

- consider $\int_\Gamma \frac{\cos(2\pi z)}{2z-1} dz$

Example 2.6.2. Say g is holomorphic on $0 < |z| < R$, which of the following implies that

$$\int_{C(r)} g(z) dz = 0?$$

1. g is holomorphic at 0
2. g identically 0 on $0 < |z| < R$
3. $|g|$ is bounded on $0 < |z| < R$
4. $g(z) = 2\pi i$ identically
5. g is defined and continuous at 0
6. $\lim_{z \to 0} g(z) = \infty$

2.7 June 6

Proposition 2.7.1 (Cauchy Integral Formula for Derivatives)**.** Note that $\left(\frac{1}{1-z}\right)'$ then by CIF

$$\begin{aligned} f(w) =& \frac{1}{2\pi i}\int_\Gamma \int_\Gamma \frac{f(z)}{z-w}dz \\ \Rightarrow \qquad \frac{d}{dw}f(w) =& \frac{1}{2\pi i}\frac{d}{dw}\int_\Gamma \frac{f(z)}{z-w}dz \\ =& \frac{1}{2\pi i}\int_\Gamma \frac{d}{dw}\frac{f(z)}{z-w}dz \\ =& \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-w)^2}dz \\ =& \frac{2}{\pi i}\int_\Gamma \frac{f(z)}{z-2}dz\ . \end{aligned}$$

Then, taking derivative n times we have

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(z)}{(z-w)^{n+1}}dz\ .$$

So f is infinitely differentiable.

2.8 June 8

Theorem 2.8.1 (Maximum Modulus Principle)**.**
Let f be holomorphic on a connected open set Ω. If f achieves its maximum on Ω, then f is constant. That is if there is some $z_0 \in \Omega$ s.t.

$$|f(z_0)| \geq f(z) \text{ for all } z \in \Omega$$

then f is constant.

Proof. Let z_0 be a local maximum of $|f|$ on Ω. Let

$$D = \{|z - z_0| \leq r\} \subset \Omega$$

be a disc around z_0. Then the Cauchy Integral Formula says let $C(r) = \partial D = \{|z - z_0| = r\}$, then

$$\begin{aligned}
f(z) =& \frac{1}{2\pi i}\int_{C(r)} \frac{f(z)}{z - z_0} dz \\
=& \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} d(z_0 + re^{i\theta}) \\
=& \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} re^{i\theta} d\theta \\
=& \frac{1}{2\pi i}\int_0^{2\pi} f(z) d\theta \\
\Rightarrow \qquad |f(z_0)| \leq& \frac{1}{2\pi}\int_0^{2\pi} |f(z)| d\theta \\
\leq& \max_{z \in C(r)} \{|f(z)|\} .
\end{aligned}$$

with equality if and only if $|f|$ is constant on $C(r)$, with $|f(z_0)| = \max_{z\in C(r)} |f(z)|$. Then because r was arbitrary, then around z_0, $|f|$ is constant on D.

Now to show that f is constant on D, write $f = u + iv$, then $u^2 + v^2$ is constant on D. Then

$$\begin{aligned}
2uu_x + 2vv_x &= 0 \\
2uu_y + 2vv_y &= 0 \\
-2uv_x + 2vu_x &= 0 \\
\Rightarrow \qquad \begin{pmatrix} u & v \\ v & -u \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

So either $u^2 + v^2 = 0$ or $u_x = v_x = 0$, $u^2 - v^2 = 0 \Rightarrow f = 0$ constant, or $u_x = v_x = -u_y = v_y = 0$ so f is constant on D.

WLOG, assume $\bar{\Omega}$ is compact. We can cover Ω with a set of open dics. Then,

therefore f is constant on Ω. □

Theorem 2.8.2 (Morera)**.**
Say f is continuous on a domain Ω, with

$$\int_\Gamma f(z) dz = 0$$

for all simple closed curves $\Gamma \subset \Omega$ whose interiors are contained in Ω. Then f is holomorphic on Ω.

Proof. We will find a holomorphic F with $F' = f$. This will prove that f is holomorphic. Since holomorphicity is local, we can assume that $\Omega = D$ is a dics. Choose $z \in D$, define □

2.9 June 10

THEOREM 2.9.1 (Cauchy Integral Formula, general form)**.**

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}} dw \; .$$

THEOREM 2.9.2.
Let f be holomorphic on and inside a circle C. Then for any z inside C, all derivatives $f^{(n)}(z)$ exist.

Note: this also means all partial derivatives exists and continuous.

Remark 2.9.3.

$$|f^{(n)}(z_0)| \leq \frac{n! \max_{z \in C_R(z_0)} |f(z)|}{R^n}$$
$$|f(z_0)| \leq \max_{z \in C_R(z_0)} |f(z)| \; .$$

THEOREM 2.9.4 (Liouville's Theorem)**.**
If f is entire and bounded, f is constant.

THEOREM 2.9.5 (Fundamental Theorem of Algebra)**.**
Every nonconstant polynomial over $\mathbb{C}$ has a zero in $\mathbb{C}$.

Corollary 2.9.1. A degree-n polynomial over $\mathbb{C}$ has exactly n zeros in $\mathbb{C}$, counted with multiplicity.

THEOREM 2.9.6 (Morera's Theorem)**.**
Let f be continuous on a simply connected cdomain D. If $\oint_\Gamma f = 0$ for every closed contour Γ in D, then f is holomorphic on D.

THEOREM 2.9.7.
Let f be continuous on a simply connected domain. Then TFAE

- f has a primitive on D
- $\oint_\Gamma f = 0$ for all closed Γ in D
- $\int_{\Gamma_1} f = \int_{\Gamma_2} f$ for any Γ_1, Γ_2 in D sharing same initial and terminal point
- f is holomorphic on D

2.10 June 13

Lemma 2.10.1 (Symmetry Principle)**.** Let D be a domain symmetric across $\mathbb{R}$. Let D^+, D^-, I be as indicated. Let f^+ be holomorphic on D^+, f^- be holomorphic on D, both extended continuously to I and $f^+(z) = f^-(z)$ for $z \in I$.

Then

$$f(z) = \begin{cases} f^+(z), z \in D^+ \\ f^+(z) = f^-(z), z \in I \\ f^-(z), z \in D^- \end{cases}$$

is holomorphic on D.

Proof. Note that f is continuous on D.

$$\left| \int_T f(z)dz - \int_{T_\varepsilon} f(z)dz \right| \leq \varepsilon\big(\max_{z\in T} |f'(z)|\big)(\text{length}(T)) \to 0 \text{ as } \varepsilon \to 0 \; .$$

□

Lemma 2.10.2 (Schwartz's Lemma)**.** Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Let f be holomorphic on D, $f(0) = 0$, and $|f(z)| < 1 \; \forall z \in D$. Then $|f(z)| \leq |z|$, $\forall z \in D$ and $|f'(0)| \leq 1$. Furthermore, if $|f(z)| = |z|$ for any $z \in D$, then f is a rotation $f(z) = \lambda z$, $|\lambda| \leq 1$ constant.

Proof. Let

$$g(z) = \begin{cases} \frac{f(z)}{z}, z \neq 0 \\ f'(0), z = 0 \end{cases}$$

Note that g is holomorphic on D, since $\lim_{z\to 0} \frac{f(z)}{z} = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = f'(0)$.

Consider g on $|z| < r < 1$. Then

$$g(z) \leq \max_{|w|=r} |g(w)| \leq \max_{|w|=r} \frac{f(w)}{|w|} \leq \frac{1}{r} \to 1 \text{ as } r \to 1$$

So $|g(z)| \leq 1$ on D.

For $z \neq 9$,

□

2.11 Midterm Review - June 15

COMPLEX NUMBER:

- $\bar{z} = x - iy$,
- $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2} = r$,
- $Re(z) = x$, $\text{Im}(z) = y$.
- $\arg z = \theta = \arctan \frac{y}{x}$, $\text{Arg}z = \theta \in (-\pi, \pi]$.

STANDRAD FUNCTIONS

- for $\theta \in \mathbb{R}$, $e^{\theta} = \cos\theta + i\sin\theta$
- $e^z = e^x(\cos y + i \sin y)$
- $\log z = \ln r + i\theta$
- $$\sin z = \frac{e^{iz} - e^{-iz}}{zi} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{z} \ .$$
- $$z^w = e^{w \log z}$$
- $$z^{1/n} = r^{1/n} e^{i\frac{\theta + 2\pi k}{n}}, k = 0, 1, 2, \ldots, n$$

DERIVATIVE:

$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \ .$$

f is holomorphic on a domain if it is differentiable at every point in the domain.

CR eqautsion: f is holomorphic on D if and only if $f = u + iv$ satisfies the CR equation:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x}$$

and these partials are continuous.

Also

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \ .$$

Also,

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial y} = e^{i\theta}(\frac{\partial u}{\partial r} - i\frac{\partial v}{\partial v}) = etc...$$

Harmonic: A function u is harmonic on a domain D if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } D \iff \forall z \in D, \exists f \text{ holomorphic at } z, \text{Re}(f(x + iy)) = u(x, y) \ .$$

A contour is a piecewise smooth directed curve.

- **simple:** no self-intersections
- **closed:** end point = initial point
- **smooth:** if a parametrization γ is continuously differentiable
- **positively oriented:** interior is on left side as you traverse

INTEGRATION:

For $\Gamma, \gamma : [a, b] \to \mathbb{C}$,

$$\int_\Gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

For f continuous on a simply connected domain D, TFAE:

- f is holomophic on D
- f has a primitive on D
- $\oint_\Gamma f = 0$ for all closed $\Gamma \subseteq D$
- $\int_{\Gamma_1} f = \int_{\Gamma_2} f$ for all $\Gamma_1, \Gamma_2 \subseteq D$ sharing start and end points.

Cauchy's Theorem: If f is holomorphic on a simply connected domain D, and let Γ be a closed contour in D. Then $\int_\Gamma f = 0$.

Cauchy's Integral Formula: If f is holomophic on and inside a simple, closed, positively oriented contour, then $\forall z_0$ inside the contour

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)(z - z_0)^{n+1}}{d} z \ ,$$

in particular for $n = 0$ case

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)} dz \ .$$

f holomorphic if and only if f is infinitely dfferentiable.

Liouville's Theorem: every bounded entire(holomorphic on all of $\mathbb{C}$) function is constant.

Fundamental Theorem of Algebra: every nonconstant polynomial has a zero in $\mathbb{C}$.

Maximum Modulus Principle: A nonconstant holomorphic f on a domain D cannot attain a maximum modulus in D.

Schwarz Reflection Principle: $f(z)$

3 Applications of The Cauchy Theory

3.1 Taylor Series - June 20

Remark 3.1.1. Consider a function f harmonic on a punctured disk $0 < |z - z_0| < r$. Let Γ be a simple closed, positively oriented contour wrapping about z_0 once. Consider

$$\int_\Gamma f(z)dz.$$

This will be a value independent of the choice of Γ. If f is holomorphic on a domain D, f is infinitely differentiable on D. In $\mathbb{R}$ an infinitely differentiable function has a Taylor series representation.

Definition 3.1.2 (Convergent Series)**.**
A series $\sum_{n=1}^{\infty} z_n \in \mathbb{C}$ is **convergent** if

$$\lim_{k\to\infty} \sum_{n=1}^{k} z_n$$

converges.

Definition 3.1.3 (Cauchy Series)**.**
A series is **Cauchy** if

$$\lim_{n\to\infty} \sum_{n=k}^{\infty} z_n = 0 \ .$$

Definition 3.1.4.
A sequence $\{f_n\}$ is **uniformly convergent** on a set S if for all $\varepsilon > 0$, exists some $N > 0 \in \mathbb{Z}$, such that $\forall z \in S$, $\exists L$, $\forall n > N$,

$$|f_n(z) - L| < \varepsilon \ .$$

Lemma 3.1.1. If $f_n \to f$ unfiromly on S, then

$$\int_S f_n \to \int_S f \ .$$

Proof. $\forall \varepsilon > 0$, $\exists N$, $\forall n > N$,

$$|f_n(z) - f(z)| < \varepsilon \text{ on } S \ .$$

thus

$$\left|\int_S f - \int_S f_n\right| \leq \left|\int_S f - f + n\right| \leq \int_S |f - f_n| \leq \varepsilon \text{length}(s) \to 0 \text{ as } \varepsilon \to 0 \ .$$

□

Definition 3.1.5.
A series is uniformly convergent if its sequence of partial sums is.

Definition 3.1.6.
A series $\sum_{n=0}^{\infty} z_n$ is **absolutely convergent** if the series $\sum_{n=0}^{\infty} |z_n|$ converges.

Definition 3.1.7.
Let $D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$ be the open disk of radius r centered at z_0. Let $\overline{D_r(z_0)} = D_r(z_0) \cup C_r(z_0)$ be its closure.

Definition 3.1.8.
Let $\{x_n\} \subseteq \mathbb{R}$,

$$\limsup x_n = \lim_{n\to\infty} \sup_{k>n} x_k \ .$$

Lemma 3.1.2 (Ratio Test)**.** If $\limsup \left|\frac{z_{n+1}}{z_n}\right| < 1$, then $\sum_{n=0}^{\infty} z_n$ converges absolutely. Otherwise if $\limsup \left|\frac{z_{n+1}}{z_n}\right| > 1$, then $\sum_{n=0}^{\infty} z_n$ diverges.

Lemma 3.1.3 (Root Test)**.** If $\limsup |z_n|^{1/n} < 1$, $\sum_{n=0}^{\infty} z_n$ converges absolutely. If $\limsup |z_n|^{1/n} > 1$, $\sum_{n=1}^{\infty} z_n$ diverges.

Lemma 3.1.4 (Comparison Test)**.** If $\sum_{n=0}^{\infty} x_n \in \mathbb{R}$ converges and $|z_n| \leq x_n$ for all n. Then $\sum_{n=0}^{\infty} z_n$ converges absolutely.

Lemma 3.1.5. Any Cauchy series is convergent, any uniformly cauchy series is uniformly convergent.

Lemma 3.1.6 (Weierstrass M-test)**.** Let $\{f_n\}_{n=1}^{\infty}$ satisfy $|f_n(z)| \leq M_n$ for all $z \in S$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on S.

Proof. Let $g_n(z) = \sum_{k=1}^{n} f_k(z)$, $\{g_n\}$ is uniformly Cauchy on S. $\square$

Definition 3.1.9 (Power Series)**.**
A **power series** about a point z_0 is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, a_n \in \mathbb{C} \ .$$

Theorem 3.1.10.
If a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges at a point z with $|z - z_0| = R$, then it converges absolutely on $D_R(z_0)$ and converges uniformly on any closed subdisk of $D_R(z_0)$.

Proof. Let $w \in D_R(z_0)$ and let $|w - z_0| < r < R$. Then

$$|a_n(w-z_0)^n| = \underbrace{|a_n(z-z_0)^n|}_{\to 0\text{, so is bounded } \leq M} \underbrace{\left|\frac{(w-z_0)^n}{(z-z_0)^n}\right|}_{\leq \frac{r}{R}} \leq M\big(\frac{r}{R}\big)^n ,$$

where $\frac{r}{R} < 1$. Since $M\left(\frac{r}{R}\right)^n$ is a convergent geometric series. So

$$\sum_{n=0}^{\infty} a_n(w-z_0)^n$$

converges absolutely by comparison test.

Apply the Weirerstrass M-test to the above to get uniform convergence on $\overline{D_r(z_0)}$. $\square$

3.2 June 22

THEOREM 3.2.1 (Taylor's Thereom)**.**
Let f be holomorphic on $D_R(z_0)$. Then $\forall z \in D_R(z_0)$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n .$$

Proof. Choose $z \in D_R(z_0)$ a nd let $|z - z_0|$ < r < R. By Cauchy's Integral Formula,

$$f(z) = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{w-z} dw .$$

$\forall w \in C_r(z_0)$,

$$\begin{aligned}\frac{f(w)}{w-z} &= \frac{f(w)}{(w-z_0)-(z-z_0)} = \frac{f(w)}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}} \\ &= \frac{f(w)}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n = \sum_{n=0}^{\infty} f(w)\frac{(z-z_0)^n}{(w-w_0)^{n+1}} .\end{aligned}$$

Now

$$\left|f(w)\frac{(z-z_0)^n}{(w-z_0)^{n+1}}\right| \leq \max_{w\in C_r(z_0)} |f(w)| \cdot \frac{|z-z_0|^n}{r^{n+1}} .$$

so by the Weierstrass M-test, this series converges uniformly on C Thus integrate term-by-term,

$$\begin{aligned} f(z) =& \frac{1}{2\pi i}\int_{Cr(z_0)}\sum_{n=0}^{\infty} f(w)\frac{(z-z_0)^n}{(w-z_0)^{n+1}}dw \\ =& \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{Cr(z_0)}\frac{f(w)}{(w-z_0)^{n+1}}dw\cdot(z-z_0)^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n .\end{aligned}$$

$\square$

Example 3.2.1.

- $R = \infty$
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots$$
- $R = \infty$,
$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$
- $R = \infty$,
$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$
- $R = 1$,
$$\frac{1}{1-z} = \sum_{n=0}^{\infty} = 1 + z + z^2 + \cdots$$

Example 3.2.2. Expand $\frac{1}{\frac{1}{2}z^2+1}$ about 0.

Let $w = -\frac{1}{2}z^2$. Then

$$\begin{aligned}
\frac{1}{\frac{1}{2}z^2 + 1} =& \frac{1}{1-w} = \sum_{n=0}^{\infty} w^n, |w| < 1 \\
=& \sum_{n=0}^{\infty} (-\frac{1}{2}z^2)^n, |-\frac{1}{2}z^2| < 1 \iff |z^2| < 2 \iff |z| < \sqrt{2}\,. \\
=& \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} z^n, |z| < \sqrt{2}\,.
\end{aligned}$$

Example 3.2.3. Taylor series for $\cos z + i \sin z$ about 0.

$$\begin{aligned}
\cos z + i \sin z =& 1 - \frac{z^2}{2} + \frac{z^4}{4!} + \cdots \\
& + i(z - \frac{z^3}{3!} + \frac{z^4}{5!} - \cdots) \\
=& 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \\
=& \sum_{n=0}^{\infty} \frac{i^n z^n}{n!} = e^{iz}\,.
\end{aligned}$$

3.3 Singularities - June 24

Remark 3.3.1. Recall a function f holomorphic on a disk $D_R(z_0)$ has a Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n ,$$

that converges on $D_R(z_0)$ and converges uniformly on any closed disk of $D_R(z_0)$.

THEOREM 3.3.2.
Let f be analytic on an open set D containing the annulus $\{z : r_1 \leq |z - z_0| \leq r_2\}$, $0 < r_1 < r_2 < \infty$, and let γ_1, γ_2 denote the positively oriented inner and outer boundaries of the annulus. Then for $r_1 < |z - z_0| < r_2$, we have

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w - z}dw - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w - z}dw .$$

DEFINITION 3.3.3.
Define

$$\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} a_n(z - z_0)^{-n} .$$

Note this means that e.g.

$$\cdots - \frac{1}{3} - \frac{1}{2} - 1 + 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

does not converge

THEOREM 3.3.4 (Laurent Series)**.**
Let f be holomorphic on an annulus $\{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$, then on that annulus, f has a unique **Laurent series** (generalization of Cauchy series),

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n ,$$

which converges on the annulus and converges uniformly on closed subannuli, and the coefficients a_n are given by

$$a_n = \frac{1}{2\pi i}\int_{C(z_0,r)} \frac{f(w)}{(w - z_0)^{n+1}}dw, n = 0, \pm 1, pm2, ...$$

Proof. HW □

DEFINITION 3.3.5 (Isolated singularity)**.**
f has an **isolated singularity** at z_0 if f is not analytic at z_0 but is analytic on the punctured disk $D(z_0, r) \setminus \{z_0\}$ for some $r > 0$.

Lemma 3.3.1. If $f(z)$ is analytic at z_0 and $f(z_0) = 0$, and f is not identically zero in any $D_r(z_0)$, then $\frac{1}{f(z)}$ has a singularity at z_0.

Definition 3.3.6.
An analytic function f has a **zero of order** m **at** z_0 if $\dfrac{f(z)}{(z-z_0)^m}$ is analytic at z_0 but $\dfrac{f(z)}{(z-z_0)^{m+1}}$ is not. Equivalently if $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, the order is the smallest n such that $a_n \neq 0$.

Example 3.3.1. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ about z_0. Then

$$f(z) = 2(z-z_0)^4 - (z-z_0)^5 + \cdots$$

has a zero of order 4.

Definition 3.3.7.
A **singularity** of f is a point where f is not analytic but is a limit point of points where f is analytic.

Definition 3.3.8.
Let z_0 be an isolated singularity of f. Let $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ be the Laurent series of f at z_0.

- If $a_{-m} \neq 0$ but $a_n = 0$ for all $n > m$, we call z_0 a **pole of order** m $\iff$ $(z-z_0)^m f(z)$ is analytic at z_0 and $(z-z_0)^{m-1} f(z)$ is not.
- If $a_{-n} = 0$ for all $n > 0$ we call this a **removable singularity.** In this case,

  $$f(z) = \begin{cases} f(z), z \neq z_0 \\ \lim_{z \to z_0} f(z), z = z_0 \end{cases}$$

  is analytic at z_0.
- If $a_{-n} \neq 0$ for infinitely many $n > 0$, we call this an **essential singularity**.

Example 3.3.2 (Removable Singularity)**.**

$$f(z) = \frac{\sin z}{z} = \frac{1}{z} \cdot \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots .$$

Lemma 3.3.2. Let $f(z) = \sum_{n=-m}^{\infty} a_n(z-z_0)^n$, $a_m \neq 0$ on a puntured disk $0 < |z-z_0| < r$. Let Γ be a simple, closed, positively oriented contour in the anaulus with z_0 inside the loop, then

$$\int_\Gamma f(z)dz = \int_\Gamma \sum_{n=-m}^{\infty} a_n(z-z_0)^n dz = \sum_{n=-m}^{\infty} a_n \int_\Gamma (z-z_0)^n dz = 2\pi i a_{-1}.$$

Proof.

$$\begin{aligned}
a_{-1} &= \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z_0)^{-1+1}}dw \\
\Rightarrow \qquad 2\pi i a_{-1} &= \int_\Gamma f(w)dw = \int_\Gamma \sum_{n=-m}^{\infty} a_n(z-z_0)^n dz
\end{aligned}$$

□

DEFINITION 3.3.9 (Residue)**.**
Given f, z_0, Γ as before, we define the **residue** of f at z_0 to be

$$\frac{1}{2\pi i}\int_\Gamma f(z)dz = a_{-1} = \mathrm{res}_{z_0}(f) \ .$$

DEFINITION 3.3.10 (Meromorphic)**.**
A function f is called **meromorphic** on a domain D if it is holomorphic on all of D except for a set of isolated poles.

THEOREM 3.3.11 (The Residue Theorem)**.**
Let f be meromorphic on a simply connected domain D and let Γ be a simple, closed, positively oriented contour lying in D. Let $z_1, \ldots z_k$ be the poles of f inside Γ. Then

$$\int_\Gamma f(z)dz = 2\pi i \sum_{j=1}^{k} \mathrm{res}_{z_j}(f) \ .$$

3.4 Singularities and Residue Theory - June 27

Recall let f be meromorphic with a pole at z_0 and let

$$\sum_{n=-m}^{\infty} a_n(z-z_0)^n, a_{-m} \neq 0$$

be the Laurent series for f valid in some punctured disk $0 < |z-z_0| < R$, $R > 0$. Then the order of the pole at z_0 is m and the residue is a_{-1}.

Example 3.4.1.

$$f(z) = \frac{3}{z^4} - \frac{5}{z^2} + \frac{7}{z} + 2 + z + \ldots$$

f has an order 4 pole at 0 with residue 7.

Example 3.4.2. Consider $f(z) = \frac{1}{z^2+z} = \frac{1}{z(z+1)}$,

- $0 < |z| < 1$. Then

$$\begin{aligned}\frac{1}{z(z+1)} =& \frac{1}{z}\left(\frac{1}{1-(-z)}\right)\\ =& \frac{1}{z}\sum_{n=0}^{\infty}(-z)^n, \quad |-z|<1\\ =& \frac{1}{z} - 1 + z - z^2 + z^3 - \cdots\end{aligned}$$

  order 1, residue 1

- $0 < |z+1| < 1$, then

$$\begin{aligned}\frac{z(z+1)}{=}\frac{1}{z+1}&\left(-\frac{1}{1-(z+1)}\right)\\ =& -\frac{1}{z+1}\sum_{n=0}^{\infty}(z+1)^n\\ =& -\frac{1}{z+1} - 1 - (z+1) - (z+1)^2 \,.\end{aligned}$$

  Simple pole, residue -1.

- $|z| > 1$,

$$\begin{aligned}\frac{1}{z(z+1)} =& \frac{1}{z}\frac{1}{z+1}\frac{1/z}{1/z}\\ =& \frac{1}{z^2}\frac{1}{1+\frac{1}{z}}\\ =& \frac{1}{z^2}\sum_{n=0}^{\infty}(\frac{1}{z})^n, \quad |\frac{1}{z}|<1\\ =& \cdots + \frac{1}{z^5} + \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{z^2}\end{aligned}$$

Definition 3.4.1 (Simple Pole)**.**
A pole of order 1 is called a **simple pole.**

Remark 3.4.2. Recall

$$\begin{aligned}& f \text{ has a pole of order } m \text{ at } z_0\\ \iff & (z-z_0)^m f(z) \text{ has a removable singularity at } z_0,\ (z-z_0)^{m-1}f(z) \text{ has a pole at } z_0.\end{aligned}$$

Therefore,

$$\begin{aligned}& f \text{ has a zero of order } m \text{ at } z_0\\ \iff & \frac{f(z)}{(z-z_0)^m} \text{ has a removable singularity at } z_0,\ \frac{f(z)}{(z-z_0)^{m+1}} \text{ has a pole at } z_0.\end{aligned}$$

$$\frac{1}{(z-z_0)^m}\left(a_m(z-z_0)^m + a_{n+1}(z-z_0)^{m+1} + \cdots\right) = a_m + a_{m+1}(z-z_0) + \cdots$$

Proposition 3.4.1. Let f and g be analytic at z_0, let f have a zero of order m at z_0 and let g have a zero of order n at z_0. Then

$$\frac{f(z)}{g(z)} = \frac{\frac{f(z)}{(z-z_0)^m}(z-z_0)^m}{\frac{g(z)}{(z-z_0)^n}(z-z_0)^n} = (z-z_0)^{m-n}h(z)$$

where $h(z_0) \neq 0$, h analytic at z_0.

$$\frac{f(z)}{g(z)} \text{ has } \begin{cases} \text{a zero of order } m-n \text{ at } z_0, & \text{if } m > n \\ \text{a pole of order } n-m \text{ at } z_0, & \text{if } m < n \\ \text{a removable singularity,} & \text{otherwise} \end{cases}$$

Example 3.4.3. $\frac{1}{z(z+1)}$ has simple poles at $z = 0, z = -1$. Then

$$\frac{1}{z^3(z+1)^2(z-2)} \text{ has } \begin{cases} \text{order 3 pole at } 0 \\ \text{order 2 pole at } -1 \\ \text{simple pole at } 2 \end{cases}$$

Example 3.4.4.

$$\frac{\cos z - 1}{z^2} = \frac{1}{z^2}(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots - 1) = -\frac{1}{2!} + \frac{z^2}{4!} - \cdots .$$

So $\frac{\cos z - 1}{z^2}$ has a removable singularity at $z = 0$.

Remark 3.4.3. Let f have a simple pole at z_0 and a Laurent series

$$f(z) = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

in some punctured disk about z_0. Then

$$\begin{aligned} (z-z_0)f(z) &= a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + \cdots \\ \Rightarrow \qquad \lim_{z - z_0}((z-z_0)f(z)) &= a_{-1} = \text{res}_{z_0} f \ . \end{aligned}$$

If f has a simple pole at z_0,

$$\text{res}_{z_0} f = \lim_{z \to z_0}(z-z_0)f(z) \ .$$

Example 3.4.5.

$$f(z) = \frac{1}{z(z+1)}. \quad res_0 f = \lim_{z\to 0 z f(z)} = \lim_{z\to 0} z\frac{1}{z(z+1)} = 1 \ .$$

$$res_{-1} f = \lim_{z\to -1}(z+1)\frac{1}{z(z+1)} = -1 \ .$$

Remark 3.4.4. Now let f have a pole of order m at z_0,

$$\begin{aligned}
f(z) =& \frac{a_{-m}}{(z-z_0)^{-m}} + \frac{a_{-m+1}}{(z-z_0)^{-m+1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + \cdots \\
\Rightarrow \qquad (z-z_0)^m f(z) =& a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{m-1} + \cdots \\
\Rightarrow \qquad \frac{d}{dz}\big((z-z_0)^m f(z)\big) =& a_{-m+1} + \cdots + (m-1)a_{-1}(z-z_0)^{m-2} + \cdots \\
\Rightarrow \qquad \frac{d^{m-1}}{dz^{m-1}}\big((z-z_0)^m f(z)\big) =& a_{-1}(m-1)! + \cdots \\
\lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\big((z-z_0)^m f(z)\big) =& a_{-1}(m-1)! \\
a_{-1} = res_{z_0} f =& \frac{1}{(m-1)!}\lim_{z-z_0}\frac{d^{m-1}}{dz^{m-1}}\big((z-z_0)^m f(z)\big)
\end{aligned}$$

Example 3.4.6. $f(z) = \frac{e^z+1}{z^3}$. $e^0+1 = 2 \neq 0$, so f has a order 3 pole at 0.

$$\text{res}_0 f = \frac{1}{(3-1)!}\lim_{z\to 0}\frac{d^2}{dz^2}(z^3\frac{e^z+1}{z^3}) = \frac{1}{2!}\lim_{z\to 0} e^z = \frac{1}{2} \ .$$

3.5 June 29

Proposition 3.5.1. Let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ be the Laurent series for f in some annulus.

$$a_{-1} = \frac{1}{2\pi i}\oint_\Gamma f(z)dz,$$

where Γ is a simple, closed, positively oriented contour looping around the inner circle of the annulus.

Now

$$(z-z_0)^{-m-1} f(z) = \cdots + \frac{a_m}{z-z_0} + \cdots \qquad \Rightarrow \qquad a_m = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-z_0)^{m+1}}dz \ .$$

Note for a Taylor series, this is equivalent to $\frac{f^{(n)(z)}}{n!}$ by Cauchy's Integral Theorem.

Proposition 3.5.2. Let $f(z) = \frac{g(z)}{h(z)}$, where g, h are analytic at z_0. Let $g(z_0) \neq 0, h(z) = 0$, $h'(z_0) \neq 0$, i.e., f has a simple pole at z_0. Then,

$$\begin{aligned}
\text{res}_{z_0}(f) &= \lim_{z \to z_0} (z - z_0) f(z) \\
&= \lim_{z \to z_0} \frac{g(z)}{h(z)} \\
&= g(z_0) \lim_{z \to z_0} \frac{z - z_0}{h(z)} \\
&= g(z_0) \lim_{z \to z_0} \frac{z - z_0}{h(z) - h(z_0)} \\
&= \frac{g(z_0)}{h'(z_0)} .
\end{aligned}$$

Example 3.5.1. Find residues of all poles of $f(z) = \frac{1}{z^3 - 1}$. Then

$$z^3 - 1 = 0 \iff z^3 = 1 \iff z \in \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$$

Thus f has 3 simple poles. Residue at a simple pole z is $\frac{1}{3z^2}$

$$\begin{aligned}
\text{res}_1 f &= \frac{1}{3(1^2)} = \frac{1}{3} \\
\text{res}_{e^{2\pi i/3}} f &= \frac{1}{3(e^{2\pi i}/3)^2} = \frac{1}{3} e^{2\pi i/3} \\
\text{res}_{e^{4\pi i/3}} f &= \frac{1}{3(e^{4\pi i}/3)^2} = \frac{1}{3} e^{4\pi i/3} .
\end{aligned}$$

Example 3.5.2. $\int_0^\infty \frac{1}{x^4+1} dz$

Let $I = \int_0^\infty \frac{1}{x^4+1} dx$, note that $2I = \int_{-\infty}^\infty \frac{1}{x^4+1} dz$.

Let Γ_R be the line segment running from $-R$ to $R \in \mathbb{R}$. Then $2I = \lim_{R \to \infty} \int_{\Gamma_R} \frac{1}{x^4+1} dz$. Let C_R be the upper semi circle running from R to $-R$. Not $\Gamma_R + C_R$ is a simple, closed, positively oriented contour, so we use the Residue Theorem. Consider

$$\left| \int_{C_R} \frac{1}{z^4+1} dz \right| \leq \left| \int_{C_R} \frac{1}{R^4} dz \right| \leq |\pi i R| \frac{1}{R^4} \leq \frac{\pi}{R^3} \to 0 \text{ as } R \to \infty .$$

Next we locate the poles of $\frac{1}{z^4+1}$ and find their residues. $z^4 + 1 = 0 \iff z^4 = -1 \iff z \in \{e^{i\pi/4}, e...\}$

3.6 Extended Complex Plane - July 4

Definition 3.6.1.
Extended complex plane $\mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}}$.

Definition 3.6.2.
Define the behavior of $f(z)$ at ∞ to behavior of $f(\frac{1}{z})$ at 0.

Example 3.6.1. Let $f(z) = z^2 + 1$, so that $f(\frac{1}{z}) = \frac{1}{z^2} + 1$, so that it has order 2 and residue 0 and

$$\lim_{R\to\infty} \int_{C_R(0)} z^2 + 1 dz = 0$$

also

$$\int_{-C_\infty(0)} f(z)dz = -2\pi i \text{res}_\infty(f) \ .$$

Example 3.6.2. $f(z) = \frac{z+1}{z-i}$

$$f(\frac{1}{z}) = \frac{\frac{1}{z}+1}{\frac{1}{z}-i} = \frac{1+z}{1-iz}$$

At $z = 0$, $f(\frac{1}{z}) = 1$, so f is analytic at ∞.

Example 3.6.3. $f(z) = \sin z$, $f(\frac{1}{z}) = \sin\frac{1}{z}$ does not converge as $z \to 0$. So $\sin z$ has an essential singularity at ∞.

$$\lim_{R\to\infty} \int_{C_R(0)} f(z)dz = 2\pi i \sum_{z_j \in \mathbb{C}, z \text{ pole of } f} \text{res}_{z_j} f = -2\pi i \cdot \text{res}_\infty f$$

Definition 3.6.3.
At an isolated singularity z_0,

- If $\lim_{z\to z_0} = c \in \mathbb{C}$, then f is analytic at z_0 (removable singularity)
- If $\lim_{z\to z_0} |f(z)| = \infty$, then f has a pole at z_0.
- If $\lim_{z\to z_0} f(z)$ does not exists in $\hat{\mathbb{C}}$, then f has essential singularity at z_0.

Example 3.6.4.

$$\int_0^\infty \frac{1}{x^3+1}dx$$

Let $f(z) = \frac{1}{z^3+1}$, f has poles at $z = -1, e^{i\pi/3}, e^{5i\pi/3}$.

$$\int_{\Gamma_2} \frac{1}{z^3+1} dz = \int_0^R \frac{1}{(te^{2\pi i/3})^3+1} e^{2\pi i/3} dt = \int_0^R \frac{e^{2\pi i/3}}{t^3+1} dt = e^{2\pi i/3} \int_0^R \frac{1}{t^3+1} dt = e^{2\pi i/3} \int_{\Gamma_1} f \ ,$$

parametrize Γ_1 by $\Gamma_1 : \gamma_1(t) = t, t \in [0, R]$. Then

$$\int_{\Gamma_1} f(z) dz = \int_0^R \frac{1}{t^3+1} dt \ .$$

3.7 Cauchy Principal Value - July 6

Definition 3.7.1.

$$\int_{-\infty}^{\infty} f(z)dz = \lim_{a\to-\infty}\int_a^0 f(x)dz + \lim_{b\to\infty}\int_0^b f(x)dz$$

if both limits converge.

Example 3.7.1. Note $\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$ and $\lim_0^\infty f(z)dz$ are not necessarily the same thing. e.g. $\int_{-\infty}^{\infty} xdx$ diverges but $\lim_\infty \int_R^R xdx = 0$.

Definition 3.7.2 (Cauchy Principal Value)**.**
Given a continuous function $f:\mathbb{R}\to\mathbb{R}$, define the **Cauchy principal value** of $\int_{-\infty}^{\infty} f(x)dx$ as

$$p.v.\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dz\ .$$

Remark 3.7.3. If $\int_{-\infty}^{\infty} f(x)dx$ exists, then

$$\int_{-\infty}^{\infty} f(x)dx = p.v.\int_{-\infty}^{\infty} f(x)dx$$

Example 3.7.2. $p.v.\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}dx$.

Let $f(z)=\frac{\cos z}{1+z^2}$,

$$\left|\int\frac{\cos z}{1+z^2}dz\right| = \left|\int\frac{\frac{1}{2}(e^{iz}+e^{-iz})}{1+z^2}dz\right|$$

- Consider

  $$I_1 = p.v.\int_{-\infty}^{\infty}\frac{e^{iz}}{1+z^2}dz$$

  We have

  $$\left|\int_{C_r}\frac{e^{iz}}{1+z^2}dz\right| \sim \frac{1}{R^2}R\sim\frac{1}{R}\to 0 \text{ as } R\to\infty\ ,$$

  then

  $$\int_{C_R+\Gamma} f(z)dz = 2\pi i\ \text{res}_i(f) = 2\pi i\left.\frac{e^{iz}}{2z}\right|_{z=i} = 2\pi i\frac{e^{-1}}{2i} = \frac{\pi}{e}\ .$$

  Hence, $I=\frac{\pi}{e}-0=\frac{\pi}{e}$.

- Consider

  $$I_2 = p.v.\int_{-\infty}^{\infty}\frac{e^{iz}}{1+z^2}dz\ .$$

  We have

  $$\left|\int_{C_R}\frac{e^{-iz}}{1+z^2}\right|dz \sim \frac{1}{R^2}\cdot R\sim\frac{1}{R}\to 0 \text{ as } R\to\infty\ .$$

$$\int_{C_R+\Gamma} f = -2\pi i \text{res}_{-i} f = -2\pi i \frac{e^{-iz}}{2z}\Big|_{z=-i} = 2\pi i \frac{e^{-1}}{-2i} = \frac{\pi}{e} \ .$$

Hence $I_2 = \frac{\pi}{e}$.

Therefore,

$$p.v. \int_{-\infty}^{\infty} \frac{\cos z}{1+z^2} dz = \frac{1}{2}\Big(p.v. \int_{-\infty}^{\infty} \frac{e^{iz}}{1+z^2} dz + p.v. \int_{-\infty}^{\infty} \frac{e^{-iz}}{1+z^2} dz \Big) = \frac{1}{2}(\frac{\pi}{e} + \frac{\pi}{e}) = \frac{\pi}{e} \ .$$

Example 3.7.3.

$$\int_0^{2\pi} \sin^2 \theta d\theta$$

Let $z = e^{i\theta} = \cos\theta + i \sin\theta$.

$$\sin\theta = \frac{e^{i\theta} - e^{i\theta}}{2i} = \frac{1}{2i}(z + \frac{1}{z}) \ .$$

thus

$$\int_0^{2\pi} \sin^2 \theta d\theta = \int_{C_1(0)} (\frac{1}{2i}(z+\frac{1}{z}))^2 \frac{d\theta}{dz} dz = \int_{C_1(0)} (\frac{z+\frac{1}{z}}{2i})^2 \frac{1}{iz} dz \ .$$

$$\begin{aligned} \int_{C_1(0)} \frac{1}{-4i}(z^2 + 2 + \frac{1}{z^2})(\frac{1}{z}) dz &= -\frac{1}{4i} \int_{C_1(0)} z - \frac{2}{z} + \frac{1}{z^3} dz \\ &= -\frac{1}{4i} 2\pi i \ \text{res}_0 (z - \frac{2}{z} + \frac{1}{z^3}) \\ &= -\frac{1}{4i}(2\pi i)(-2i) = \pi \ . \end{aligned}$$

Definition 3.7.4.
Let f be continuous on $[a,b]$ except at c, $a < c < b$, Then

$$p.v. \int_a^b f(z) dz = \lim_{\varepsilon \to 0} \Big(\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x) dx \ . \Big)$$

Combine those definitions.

Proposition 3.7.1. Let $p(z), q(z)$ be polynomial with $\deg(p) \le \deg(q) - 2$, then for any arc C_R of $C_R(0)$.

$$\lim_{R\to\infty} \Big| \int_{C_R} \frac{p(z)}{q(z)} dz \Big| = 0 \ ,$$

this is because

$$\Big| \int_{C_R} \frac{p(z)}{q(z)} dz \Big| \sim R \cdot \frac{R^{\deg(p)}}{R^{\deg(q)}} = R \cdot R^{-2} = \frac{1}{R} \to 0 \text{ as } R \to \infty$$

Lemma 3.7.1 (Jordan's Lemma)**.** Let $a > 0$ and $\deg(q) \geq 1 + \deg(p)$, let C_R be the upper half of $C_R(0)$, then

$$\lim_{R\to\infty} \int_R e^{iaz} \frac{p(z)}{q(z)} dz = 0 \ .$$

Proof. Parametrize C_R by Re^{it}, $t \in [0, \pi]$, then

$$\int_R e^{iaz} \frac{p(z)}{q(z)} dz = \int_R e^{iaRe^{it}} \frac{p(Re^{it})}{q(Re^{it})} iRe^{it} dt$$

Then

- $$\left| e^{ia} R^{e^{it}} \right| = |\exp(iaR(\cos t + i \sin t))| = \exp(-aR \sin t)$$

- For sufficiently large R, $\exists K \in \mathbb{R}$ such that

$$\left| \frac{p(Re^{it})}{q(Re^{it})} \right| \leq \frac{K}{R} \ .$$

Therefore,

$$\begin{aligned}
\left| \int_R e^{iaRe^{it}} \frac{p(Re^{it})}{q(Re^{it})} iRe^{it} dt \right| \leq & \int_0^\pi e^{-aR\sin t} \frac{K}{R} \cdot R dt \\
=& K \int_0^\pi e^{-aR\sin t} dt \\
=& 2K \int_0^{\pi/2} e^{-aR\sin t} dt \\
\leq & 2K \int_0^{pi/2} e^{-aR\frac{2t}{\pi}} dt. & (\sin t \geq \tfrac{2t}{\pi} \text{ on } [0, \tfrac{\pi}{2}]) \\
=& 2K(-\frac{\pi}{2aR})(e^{-aR} - 1) \to 0 \text{ as } R \to \infty \ .
\end{aligned}$$

$\square$

Lemma 3.7.2. Let f be meromorphic with a simple pole at z_0 and let Γ_r be parametrized by $\gamma(t) = z_0 + re^{i\theta}$, $\theta_1 < \theta < \theta_2$. Then,

$$\lim_{r\to 0^+} \int_{\Gamma_r} f(z)dz = i(\theta_2 - \theta_1)dz = (\theta_2 - \theta_1)\text{res}_{z_0} f \ .$$

Proof.

$$f(z) = \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n \ .$$

unfinished

$\square$

3.8 July 11

Lemma 3.8.1. Let $a > 0$, deg$(q) \geq 1 +$ deg(p), let C_R be the upper half of $C_R(0)$, then

$$\lim_{R\to\infty} e^{-az}\frac{p(z)}{q(z)}dz = 0 \ .$$

Proof. Parameterize C_R by Re^{it} with $t \in [0, \pi]$, now □

Remark 3.8.1.

$$\int_{-\infty}^{\infty} \frac{p(z)}{q(z)}dz \qquad \text{Need deg}(q) \geq 2 + \text{deg}(p)$$

$$\int_{-\infty}^{\infty} \cos(z)\frac{p(z)}{q(z)}dz \qquad \text{Need deg}(q) \geq 1 + \text{deg}(p)$$

Lemma 3.8.2. Let f be meromorphic with a simple pole at z_0, and Γ_r be parameterized by $r(t) = z_0 + re^{i\theta}$ with $\theta_1 < \theta < \theta_2$ then

$$\int_{r\to 0^+}\int_{\Gamma_r} f(z)dz = i(\theta_2 - \theta_1)\text{res}_{z_0} f \ .$$

Proof.

$$f(z) = \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n(z - z_0)^n = \frac{a_{-1}}{z - z_0} + g(z) \ ,$$

where g is analytic so g is continuous then $\exists R$ such that or $0 < r \leq R$, $\exists M > 0$ s.t. $|g(z)| \leq M$, so that

$$\left|\int_{\Gamma_r} g(z)dz\right| \leq M \cdot \text{length}(\Gamma_r) = M \cdot (\theta_2 - \theta_1)r \to 0 \text{ as } r \to 0^+ \ ,$$

then

$$\int_{\Gamma_r} f(z)dz = \int_{\Gamma_r} \frac{a_{-1}}{z - z_0}dz + 0 = a_{-1}\int_{\theta_1}^{\theta_2} \frac{1}{re^{i\theta}}rie^{i\theta}d\theta = a_{-1}\int_{\theta_1}^{-\theta_2} id\theta i \cdot \text{res}_2(f) \ .$$

□

3.9 July 11

Recall

Lemma 3.9.1. Let f be meromorphic with a simple pole at z_0. Then if C_r is parametrized by $\gamma(t) = z_0 + re^{it}$, $t \in [\theta_1, \theta_2]$,

$$\lim_{r\to 0^+}\int_{C_r} f(z)dz = i(\theta_2 - \theta_1)\text{res}_{z_0}(f) \ .$$

Example 3.9.1.

$$p.v. \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \lim_{R\to\infty, r\to 0^+} \left(\int_{-R}^{-r} \frac{e^{ix}}{x} dx + \int_{r}^{R} \frac{e^{ix}}{x} dx \, , \right)$$

$$\Rightarrow \qquad \int_{C_R} f(z) dz \to 0 \text{ by Jordan's Lemma .}$$

$$\begin{aligned} \int_{C_r} \frac{e^{iz}}{z} dz =& i(0-\pi)\text{res}_0 f = -\pi i(1) = i\pi i \, , \\ \oint_{C_R+C_r+\Gamma} f(z)dz =& 0 \end{aligned}$$

$$\Rightarrow \qquad \lim_{R\to\infty, r\to 0^+} \int_{\Gamma} f(z)dz = \oint_{C_R+C_r+\Gamma} f - \int_{C_R} f - \int_{C_r} f = 0 - 0 - (-\pi i) = \pi i \, .$$

Example 3.9.2. $\int_0^\infty \frac{x^{1/3}}{1+x^2} dx$, let $f(z) = \frac{z^{1/3}}{1+z^2}$ with branch cut along the positive real axis.

$$\left| \int_{C_R} \frac{z^{1/3}}{z^2+1} dz \right| \sim \frac{R^{1/3}}{R^2} R \sim R^{-2/3} \to 0 \text{ as } R \to \infty$$

$$\left| \int_{C_r} \frac{z^{1/3}}{z^2+1} dz \right| \sim r \cdot \frac{r^{1/3}}{1} R \sim r^{4/3} \to 0 \text{ as } r \to 0^+ \, .$$

$$\lim_{R\to\infty, r\to 0^+} \int_{\Gamma_1} f \to \int_0^\infty f(z)dz = I \, .$$

$$\begin{aligned} \int_{\Gamma_2} f(z)dz &= \int_{\Gamma_2} \frac{z^{1/3}}{1+z^2} dz \\ &= \int_{\Gamma_1} \frac{(ze^{2\pi i})^{1/3}}{1+z^2} dz \\ &= \int_{\Gamma_1} \frac{z}{1+z^2} e^{2\pi i/3} dz = I e^{\frac{2\pi i}{3}} \, . \end{aligned}$$

$f(z)$ has simple poles at $z = \pm i$ with residues.

$$\begin{aligned} \text{res}_i f =& \frac{i^{1/3}}{2i} = \frac{(e^{i\pi/2})^{1/3}}{2i} = \frac{e^{i\pi/6}}{2i} = \frac{\frac{\sqrt{3}}{2} + i\frac{1}{2}}{2i} = \frac{1}{4} - i\frac{\sqrt{3}}{4} \\ \text{res}_{-i} f =& \frac{(-i)^{1/3}}{-2i} = \frac{(e^{i\pi/2})^{1/3}}{-2i} = \frac{e^{i\pi/6}}{-2i} = -\frac{1}{2} \, . \end{aligned}$$

Then

$$\begin{aligned} 2\pi i(-\frac{1}{4} - i\frac{\sqrt{3}}{4}) =& \pi i(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) = \pi i e^{-2\pi i/3} = I(1 - e^{2\pi i/3}) \\ I =& \frac{\pi i e^{-2\pi i/3}}{1 - e^{2\pi i/3}} = \frac{\pi}{\sqrt{3}} \, . \end{aligned}$$

Example 3.9.3. $\int_0^\infty \frac{1}{1+x^3}dx$

Let $f(z) = \frac{\log z}{1+z^3}$, branch cut along the positive real axis.

$$\int_{\Gamma_1} \frac{\log z}{1+z^3}dz = \int_r^R \frac{\ln x}{1+x^3}dx \ ,$$

$$\int_{\Gamma_2} \frac{\log z}{1+z^3}dz = \int_r^R \frac{\log(xe^{2\pi i})}{1+x^3}dx = \int_r^R \frac{\ln x + 2\pi i}{1+x^3}dx \ .$$

Therefore,

$$\int_{\Gamma_1} f dz - \int_{\Gamma_2} f dz = \int_r^R \frac{\ln x}{1+x^3}dx - \int_r^R \frac{\ln x + 2\pi i}{1+x^3}dx = -2\pi i \int_r^R \frac{1}{1+x^3}dx = 2\pi i I \ .$$

3.10 July 13

Remark 3.10.1. Let $f \neq 0$ be meromorphic on D and let Γ be a simple, positively oriented closed contour with Γ and its interior he is in D, consider $\frac{f'(z)}{f(z)}$ is meromorphic and its poles can only lie at poles and zeros of f.

Let z_0 be an order$-m$ zero of f. Then

$$f(z) = (z-z_0)^m g(z), g(z_0) \neq 0,\ g \text{ analytic} \ .$$

Now

$$f'(z) = m(z-z_0)^{m-1}g(z) + (z-z_0)^m g'(z),$$

so

$$\frac{f'(z)}{f(z)} = \frac{m(z-z_0)^{m-1}g(z) + (z-z_0)^m g'(z)}{(z-z_0)^m g(z)} = \frac{m}{z-z_0} + \frac{g'(z)}{g(z)} \ .$$

Let z_0 be an order$-m$ pole of f, then

$$f(z) = \frac{h(z)}{(z-z_0)^n}, h(z_0) \neq 0, h \text{ analytic} \ ,$$

$$f' = \frac{(z-z_0)^m h'(z) - m(z-z_0)^{m-1}h(z)}{(z-z_0)^{2m}} \ .$$

so

$$\frac{f'}{f} = \frac{-m(z-z_0)^{m-1}h(z) + (z-z_0)^m h'(z)}{(z-z_0)^m h(z)} = -\frac{m}{(z-z_0)} + \frac{h'(z)}{h(z)} \ .$$

3.11 July 15

THEOREM 3.11.1 (The Argument Principle)**.**
Let f be meromorphic on and inside a simple, closed, positively oriented contour Γ. Let $N_0(f)$ and $N_p(f)$ be the member of zeros and number of poles of f inside Γ. (both counted wih multiplicity) Then

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz = N_0(f) - N_p(f)\ .$$

Note if f is analytic on and inside Γ this becomes

$$\frac{1}{2\pi i}\frac{f'}{f} = N_0(f)\ .$$

DEFINITION 3.11.2.
Let Γ be a closed contour and let $z_0 \neq \Gamma$. The **curling number** of Γ about z_0, denoted $n(\Gamma, z_0)$ is the unique integer n such that Γ is homeomorphic to $\underbrace{C_1(z_0) + C_1(z_0) + \cdots + C_1(z_0)}_{n \text{ copies}}$ in $\mathbb{C} \setminus \{z_0\}$.

Lemma 3.11.1. For $z_0 \in \Gamma$,

$$\oint_\Gamma \frac{1}{z - z_0}dz = 2\pi i n(\Gamma, z_0)\ ,$$

Proposition 3.11.1. Let $f(\Gamma)$ be which $\Gamma : \gamma(t) : [a, b] \to \Gamma$,

$$\begin{aligned}
&\int_\Gamma \frac{1}{z - z_0}dz = \int_a^b \frac{1}{\gamma(t) - z_0}\gamma'(t)dt \\
\Rightarrow \quad &\int_\Gamma \frac{1}{f(z) - z_0}dz = \int_a^b \frac{1}{f(\gamma(t)) - z_0}f'(\gamma(t))\gamma'(t)dt = \int_{f(\Gamma)} \frac{f'(z)}{f(z) - z_0}dz\ . \\
\Rightarrow \quad &\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z) - z_0}dz = n(f(\Gamma), z_0)\ .
\end{aligned}$$

Note for $z_0 = 0$, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = n(f(\Gamma), 0) = N_0(f) - N_p(f)\ .$$

Example 3.11.1.

$$\frac{d}{dz}\log(f(z)) = \frac{f'(z)}{f(z)}\ ,$$

then

$$\oint_{\Gamma_{i\theta}} \frac{f'(z)}{f(z)}dz = \log(f(z))\Big|_{z_0}^{z_1} = 2\pi i n(f(\Gamma), 0)\ .$$

Let $f = re^{i\theta}$,

$$\log(f(z)) = \ln r + i\theta\ .$$

Lemma 3.11.2 (The Dog-walking Theorem)**.** Let Γ_1, Γ_2 be parametrized by $\gamma_1, \gamma_2 : [a, b] \to \mathbb{C}$, and $\forall t \in [a, b]$, let $|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)|$. Then $n(\Gamma_1, 0) = n(\Gamma_2, 0)$.

Proof. Note that $\gamma_1(t), \gamma_2(t) \neq 0$, let $\Gamma : \gamma(t) = \frac{\gamma_2(t)}{\gamma_1(t)}$. Then

$$\begin{aligned} |1 - \gamma(t)| &= \left|1 - \frac{\gamma_2(t)}{\gamma_1(t)}\right| \\ &= \left|\frac{\gamma_1(t) - \gamma_2(t)}{\gamma_1(t)}\right| < 1 \ . \end{aligned}$$

Thus Γ lies in $D_1(1)$ so $n(\Gamma, 0) = 0$. Let $\gamma_1 = r_1 e^{i\theta_1}, \gamma_2 = r_2 e^{i\theta_2}$, where $r_1, r_2, \theta_1, \theta_2$ are functions of t. Then

$$\gamma = \frac{\gamma_2}{\gamma_1} = \frac{r_2}{r_1 e^{i(\theta_2 - \theta_1)}} \ .$$

$$n(\Gamma_1, 0) = \theta_1(b) - \theta_1(a) \qquad n(\Gamma_2, 0) = \theta_2(b) - \theta_2(a)$$

so

$$0 = n(\Gamma, 0) = \theta_2(b) - \theta_2(a) - (\theta_1(b) - \theta_1(a)) = n(\Gamma_2, 0) - n(\Gamma_1, 0) \ .$$

So $n(\Gamma_1, 0) = n(\Gamma_2, 0)$. □

Lemma 3.11.3 (Generalized Dog-walking Theorem)**.** Let Γ_1, Γ_2 be parametrized by γ_1, γ_2: $[a, b] \to \mathbb{C}$ and $\forall t \in [a, b]$ let

$$|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)| + |\gamma_2(t)|.$$

Then $n(\Gamma_1, 0) = n(\Gamma_2, 0)$.

Proof. Let $\gamma(t) = \frac{\gamma_1(t)}{\gamma_2(t)}$. Assume for contradiction that $\exists c > 0$, $\exists t \in [a, b]$, $\gamma(t) = -c$. Then $\gamma_1(t) = -c\gamma_2(t)$.

$$|\gamma_1(t) - \gamma_2(t)| = |(-c - 1)\gamma_2(t)| = (c + i)|\gamma_2(t)| \ .$$

But $|\gamma_2(t)| + |\gamma_1(t)| = |\gamma_2(t)| + |-c\gamma_2(t)| = (1 + c)|\gamma_2(t)|$. This contradicts the condition of the lemma, so no such c exists. So $\Gamma : \gamma(t)$ lies in the $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. So $n(\Gamma, 0) = 0$, so $n(\Gamma_1, 0) = n(\Gamma_2, 0)$. □

3.12 July 18

Theorem 3.12.1 (Rouche's Theorem)**.**
Let f, g be analytic on and inside a simple, positively oriented closed contour Γ. Let $|f(z)| > |g(z)|$ for all $z \in \Gamma$. Then f and $f+g$ have the same number of zeros inside Γ. (counted with multiplicity).

Proof. Let $h = f + g$. Then

$$|h(z) + (-f(z))| = |g(z)| < |-f(z)| \text{ on } \Gamma .$$

Then by argument principle and dog-walking theorem,

$$n(h(\Gamma), 0) = n(f(\Gamma), 0) \Rightarrow N_0(h) = N_0(f) .$$

□

Example 3.12.1. All 5 zeros of $h(z) = z^5 + 3z + 1$ lie inside $|z| < 2$.

Example 3.12.2. How many zeros does $z + 3 + 2e^z$ have in the left half-plane $\mathrm{Re}(z) < 0$?

Let Γ_R be the closed left semi circle .

Let $f(z) = z + 3$, $g(z) = 2e^z$. Then

- $$|g(z)| = |2e^z| = 2|e^z| = 2e^{\mathrm{Re}(z)} .$$
  So $|g(z)| \leq 2$ on Γ_R for all R.
- $|f(z)| = |z + 3|$, so
  $$|f(z)| = |z+3| \geq \begin{cases} |3+iy|, & z = iy \\ R - e, & |z| = R \end{cases} \geq \begin{cases} 3, & z = iy \\ R-3, |z| = R \end{cases}$$

So $\forall R > 5$, $|f(z)| > |g(z)|$ on Γ_R. Thus f has the same number of zeros inside Γ_R as $z + 3 + 2e^z$, $f(z) = z + 3$ has one zero inside Γ_R, namely 3. So $z + 3 + 2e^z$ has exactly one zero in the left half-plane.

Definition 3.12.2.
A point z is a limit point of a set S if there exists a sequence $\{z_n\}_{n=1}^{\infty} \subseteq S$, $z_n \neq z$, $\lim_{n\to\infty} z_n = z$.

Theorem 3.12.3.
Let f be holomorphic on a domain D. Let $Z \subseteq D$ be the set of zeros of f in D. if Z has a limit point in D, f is identically zero on D.

Proof. Let z_0 be the limit of $\{w_n\}_{n=1}^{\infty} \subseteq Z$, $z_0 \neq w_n$ for all n. Consider $D_\varepsilon(z_0)$ for some sufficiently small $\varepsilon > 0$, $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ on $D_\varepsilon(z_0)$.

If f is not identically zero on $D_\varepsilon(z_0)$. then there exists a minimal $m \geq 0$ such that $a_m \neq$. Write

$$f(z) = a_m(z-z_0)^m\big(1+g(z-z_0)\big) \ ,$$

where $g(z-z_0) \to 0$ as $z \to z_0$.

Let k be sufficiently large that $w_K \in D_\varepsilon(z_0)$ for all $K \geq k$. Now $f(w_k) = 0$, but

$$0 = f(w_k) = a_m(w_k - z_0)^n(1 + g(w - K - z_0))$$

and $a_m \neq 0$, $(w_k - z_0)^n \neq 0$, and $g(w_k - z_0) \to 0$ as $w_k \to z_0$, $k \to \infty$. So for sufficiently large k,

$$|g(w_k - z_0)| < 1,$$

so $1 + g(w_k - z_0) \neq 0$. This is a contradiction, so $f = 0$ on $D_\varepsilon(z_0)$.

Let U be the interior of Z. We just showed that U is nonempty. U is open by definition. Let $\{z_n\} \subseteq U$ converging $z_n \to z$. f is continuous, so $f(z) = 0$. By the earlier argument, $z \in U$. □

Corollary 3.12.1. Let f, g be analytic on D and $f(z) = g(z)$ on $S \subseteq D$ where S has limit point in D, then $f(z) = g(z)$ on D.

3.13 July 20

Theorem 3.13.1 (open mapping theorem)**.**
If f is holomorphic on a domain D, then f is an open map on D. (i.e. it maps open set to open sets).

Proof. It suffices to show that $f(D)$ is open. Let $z_0 \in D$, $f(z_0) = w_0$. Let $w \in \mathbb{C}$ and

$$g(z) = f(z) - w = f(z) - w_0 + w_0 - w \ .$$

Choose $\delta > 0$ such that $D_\delta \subseteq D$ and such that $f(z) \neq w_0$ on the circle $|z - z_0| = \delta$, which exists by the previous corollary.

Choose $\varepsilon > 0$ such that $|f(z) - w_0| \geq \varepsilon$ on $|z - z_0| = \delta$. Now for all $|w - w_0| < \varepsilon$, we have

$$|f(z) - w_0| \geq \varepsilon > |w - w_0|$$

on the circle $|z - z_0| = \delta$.

So by Rouche's Theorem, g and $f(z) - w_0$ have the same number of zeros in $D_\delta(z_0)$, namely by one. Thus $\exists z \in D_\delta(z_0) \subseteq D$, $g(z) = 0 = f(z) - w \Rightarrow f(z) = w \Rightarrow w \in f(D)$. Thus $D_\varepsilon(w) \subseteq f(D)$. □

Example 3.13.1. Let f be analytic on a domain D and Re$f(z)$ is constant. Then f is constant. Re$(f(z)) = K$ contains no open sets. So f must be constant by the contrapositive of the open mapping theorem.

Definition 3.13.2.
The **gamma function** is defined for $s > 0$ in $\mathbb{R}$ by

$$\Gamma(s) = \int_0^\infty e^{-f} t^{s-1} dt \ .$$

Lemma 3.13.1. Γ extends to an analytic function on Re$(s) > 0$ and $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$ still holds here.

Proof. It suffices to show the lemma on

$$S = \{z \in \mathbb{C} : \delta < \text{Re}(s) < M\} \text{ for any } 0 < \delta < M < \infty \ .$$

Let Re$(s) = \sigma$. Now

$$\int_0^\infty e^{-t} t^{-s-1} dt = \lim_{\varepsilon \to 0} \int_\varepsilon^{1/\varepsilon} e^{-t} t^{s-1} dt \ .$$

Let $F_\varepsilon(s) = \int_\varepsilon^{1/\varepsilon} e^{-t} t^{s-1} dt$. Note that $F_\varepsilon(s)$ is analytic. Recall that the limit of a uniformly convergent sequence of analytic functions is analytic, consider

$$\begin{aligned}
|\Gamma(s) - F_\varepsilon(w)| &= \left| \int_0^\infty e^{-t} t^{s-1} dt - \int_\varepsilon^{1/\varepsilon} e^{-t} t^{s-1} dt \right| \\
&= \left| \int_0^\varepsilon e^{-t} t^{s-1} dt + \int_{1/\varepsilon}^\infty e^{-t} t^{s-1} dt \right| \\
&\leq \int_0^\varepsilon e^{-t} t^{\sigma-1} dt + \int_{1/\varepsilon}^\infty e^{-t} t^{\sigma-1} dt \ .
\end{aligned}$$

□

3.14 July 22

Recall

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \ ,$$

is an analytic function on the half-plane $\text{Re}(s) > 0$.

Lemma 3.14.1. If $\text{Re}(s) > 0$, $\Gamma(s+1) = s\Gamma(s)$.

Proof. Consider

$$\int_\varepsilon^{1/\varepsilon} e^{-t} t^{s-1} dt = \int_\varepsilon^{1/ep} e^{-t} \frac{1}{s} \frac{d}{dt}(t^s) dt = \frac{1}{s}(e^{-t} t^s \Big|_\varepsilon^{1/\varepsilon} - \int_\varepsilon^{1/\varepsilon} -e^{-t} t^s dtt) \ .$$

Taking $\varepsilon \to 0$, this is

$$\Gamma(s) = \frac{1}{s} \int_0^\infty e^{-t} t^{(s+1)-1} = \frac{1}{s}\Gamma(s+1) \ .$$

□

Note 3.14.1.
Start with $\text{Re}(s) > -1$, let $F_1(s) = \frac{\Gamma(s+1)}{s}$. $\Gamma(s+1)$ is analytic on $\text{Re}(s) > -1$, so $F_1(s)$ is meromorphic on $\text{Re}(s) > -1$ with possible pole at only 0.

Since $\Gamma(0+1) = \Gamma(1) = 1$, $F_1(s)$ has a simple pole of residue 1 at $s = 0$. For $\text{Re}(s) > 0$, $F_1(s) = \frac{\Gamma(s+1)}{s} = \Gamma(s)$.

So $F_1(s)$ is the unique analytic continuation of $\Gamma(s)$ onto $\{\text{Re}(s) > -1\} \setminus \{0\}$.

Repeat, define $F_m(s)$ on $\text{Re}(s) > -m$. $m > 0$, $m \in \mathbb{Z}$ as

$$F_m(s) = \frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\cdots(s)} \ .$$

F_m is meromorphic on $\text{Re}(s) > m$ with simple poles at $s = 0, -1, -2, \ldots, -m+1$, $F_m(s) = \Gamma(s)$ on $\text{Re}(s0 > 0$.

So we have an analytic continuation of γ onto $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$. For $m > n$, Γ has residues

$$\begin{aligned}\text{res}_{-n} F_m(s) = & \lim_{n \to -n} (s+m) \frac{\Gamma(-n+m)}{(-n+m-1)(-n+m-2)\cdots(-n+1)(-n)} \\ = & \frac{\Gamma(-n+m)}{(-n+m)}\end{aligned}$$

Theorem 3.14.2.

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \ .$$

Proof. If suffices to prove this on $0 < \text{Re}(s) < 1$.

$$\Gamma(1-s) = \int_0^\infty e^{-u}$$

$\square$

Lemma 3.14.2. For $0 < \text{Re}(a) < 1$,

$$\int_0^\infty \frac{v^{a-1}}{1+v} dv = \frac{\pi}{\sin(\pi a)} \ .$$

Proof. Let $v = e^x$.

$$\int_0^\infty \frac{v^{a-1}}{1+v} dv = \int_0^\infty \frac{e^{a-1}x}{1+e^x} e^x dx = \int_{-\infty}^\infty \frac{e^{ax}}{1+e^x} dx \ .$$

Let $f(z) = \frac{e^{az}}{1+e^z}$ and integrate over

$$\begin{aligned}\left|\int_{\Gamma_2} f(z)dz\right| =& \left|\int_0^{2\pi} \frac{e^a(R+it)}{1+e^{R+it}} dt\right| \\ \leq& C\frac{e^{aR}}{e^R} \sim Ce^{(a-1)R} \to 0 \text{ as } R \to \infty \ .\end{aligned}$$

$$\left|\int_{\Gamma_4} f(z)dz\right| \leq Ce^{-aR} \to 0 \text{ as } R \to \infty \ .$$

$$\int_{\Gamma_1} f(z)dz = \int_{-R}^R \frac{e^{ax}}{1+e^x} dx \ .$$

$$\int_{\Gamma_3} f(z)dz = \int_{-R}^R \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}} dx = -e^{2\pi i a} \int_{-R}^R \frac{e^{ax}}{1+e^x} dx \ .$$

f has a pole at $z = \pi i$.

$$\begin{aligned}\lim_{z\to\pi i}(z-\pi i)f(z) = \lim_{z\to\pi i}(z-\pi i)\frac{e^{az}}{1+e^z} =& \qquad\qquad \lim_{z\to\pi i} e^{az}\left(\frac{z-\pi i}{e^z - e^{\pi i}}\right) \\ =& e^{a\pi i}\left(\frac{e^z - e^{\pi i}}{z - \pi i}\right)^{-1} \\ =& e^{a\pi i}\left(\frac{\frac{d}{dz}(e^z)\big|_{z=\pi i}}{}\right)^{-1} \\ =& e^{a\pi i}(e^{\pi i})^{-1} = -e^{a\pi i} = \text{res}_{\pi i} f\end{aligned}$$

Thus

$$\int f(z)dz = 2\pi i(-e^{a\pi i}) =$$

$\square$

3.15 July 25

DEFINITION 3.15.1.
Define the Riemann **zeta function** for real $s > 1$ as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} .$$

ζ immediately has an analytic continuation to $\mathrm{Re}(s) > 1$ and the formula $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is still valid.

If $s = \sigma + it$, $\sigma, t \in \mathbb{R}$. If $\sigma > 1 + \delta > 1$, then

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^s} \right| \leq \sum_{n=1}^{\infty} \left| \frac{1}{n^5} \right| = \sum_{n=1}^{\infty} \left| \frac{1}{e^{s \log n}} \right| = \sum_{n=1}^{\infty} \frac{1}{e^{\sigma \log n}} = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} \text{ uniformly} .$$

Thus $\zeta(s)$ is analytic on $\mathrm{Re}(s) > 1$.

Consider the Euler product

$$\prod_{p \text{ prime}} \frac{1}{p^{-s}} .$$

For $\mathrm{Re}(s) > 1$.

$$\frac{1}{1 - p^{-s}} = \sum_{n=1}^{\infty} \frac{1}{p^{ns}} .$$

Thus unique prime factorization

THEOREM 3.15.2.
$\zeta(s) - \frac{1}{s-1}$ has an analytic continuation to $\mathrm{Re}(s) > 0$. Thus $\zeta(s)$ is meromorphic on $\mathrm{Re}(s) > 0$ with a simple pole at $s = 1$ with residue 1.

Proof. Consider

$$\sum_{1 \leq n < N} \frac{1}{n^s} - \int_1^N \frac{1}{x^s} dx .$$

Let $\zeta_n(s) = \int_n^{n+1} \frac{1}{n^s} - \frac{1}{x^s} dx$. By the mean value theorem

$$\left| \frac{1}{n^s} - \frac{1}{x^5} \right| \leq \frac{|s|}{n^{\sigma - 1}} \text{ , on } n \leq x \leq n+1 .$$

Thus we have uniform convergence on $\delta_n(s)$ on $1 + \sigma > 0 \iff \mathrm{Re}(s) > 0$. Thus $\sum_{n=1}^{\infty} \delta_n(s)$ is analytic on $\mathrm{Re}(s) > 0$. Now

$$\sum_{1 \leq n < N} \frac{1}{n^s} = \sum_{n=1}^{N} \delta_n(s) + \int_1^N \frac{1}{x^s} dx .$$

Now
$$\lim_{n\to\infty}\int_1^N \frac{1}{x^s}dx = \int_1^\infty \frac{1}{x^s}dx = \frac{1}{1-s}x^{1-s}\Big|_0^\infty .$$
On Re$(s) > 1$, this = $\frac{1}{s-1}$. Thus $\sum_{n=1}^N \delta_n(s) + \int_1^N \frac{1}{x^s}dx$ converges uniformly and matches $\zeta(s)$ on Re$(s) > 1$, so $\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^\infty \delta_n(s)$ is analytic on Re$(s) > 0$.

Thus $\zeta(s)$ is meromorphic on Re$(s) > 0$ with a simple pole of residue 1 at $s = 1$. This argument can be extended to get ζ meromorphic on $\mathbb{C}$. □

Theorem 3.15.3.
$\zeta(s)$ has no zeros on the line Re$(s) = 1$.

Proof. Let $x, y \in \mathbb{R}$, $y \neq 0$ and define $h(x) = \zeta(x)^3$, $\zeta(s+iy)^4\zeta(x+2yi)$. Now
$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$$

So
$$\ln|\zeta(s)| = \ln\prod_p \left|\frac{1}{1-p^{-s}}\right| = -\sum_p \ln|1-p^{-s}| = -\text{Re}\sum_p \text{Log}(1-p^{-s}) .$$

Now
$$-\text{Log}(1-w) = \sum_{n=1}^\infty \frac{w^n}{n} \text{ for } |w| < 1 .$$
so $\ln|\zeta(s)| = \text{Re}\sum_p\sum_n \frac{1}{n}p^{-sn}$. Thus
$$\begin{aligned}\ln|h(x)| =& 3\ln|\zeta(x)| + 4\ln|\zeta(x+iy)| + \ln|\zeta(x+2iy)| \\ =& 3\text{Re}\sum_p\sum_n \frac{1}{n}p^{-sn} + 4\text{Re}\sum_p\sum_n \frac{1}{n}p^{-s}\end{aligned}$$

□

3.16 Final Review

Let $i^2 = -1$. Define
$$f'(z) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} .$$
If f is holomorphic on a domain D, f is infinitely differentiable on D.

Theorem 3.16.1.
Let f be continuous on a simply connected domain D. Then TFAE:

- f is holomorphic on D

- $\oint_\Gamma f = 0$ for any closed $\Gamma \subseteq D$.
- f has a primitive on D
- $\int_{\Gamma_1} f = \int_{\Gamma_2} f$ for $\Gamma_1, \Gamma_2 \subseteq D$ sharing initial and terminal points.

Cauchy's Integral Formula:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint \frac{f(w)}{(z-w)^{n+1}} dw \ .$$

Maximum Modulus Principle: a nonconstant analytic function can only at the boundary of a region.

Liouville's Theorem: A bounded entire function is constant.

Fundamental Theorem of Algebra: every polynomial in $\mathbb{C}$ has a zero in $\mathbb{C}$.

A function holomorphic in an open disk has a Taylor series representation in that disk.

A function analytic in an annulus has a laurent series representaion in the annulus.

Classification of Singuliarities:

- Removable singularities: $\lim_{z\to z_0} |f(z)| < \infty$.
- Pole: $\lim_{z\to z_0} |f(z)| = \infty$.
- Essential singularities: $\lim_{z\to z_0} |f(z)|$ erratic.

Order-m pole: $(z-z_0)^m f(z)$ is analytic at z_0

Residue:

$$\text{res}_{z_0} f = \frac{1}{2\pi i} \oint_\Gamma f(z) dz \ .$$

Computing Residues: $\text{res}_{z_0} = a_{-1}$, where $\sum_{n=-m}^{\infty} a_n (z-z_0)^n$ is the Laurent series in a punctured disk.

$$\text{res}_{z_0} f = (m - 10! \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} ((z-z_0)^m f(z)) \ .$$

If $m = 1$,

$$\lim_{z\to z_0} (z -_0) f(z) = \text{res}_{z_0} f \ ,$$

if $m = 1$, if $f(z) = \frac{g(z)}{h(z)}$, g, h holomorphic, then $\text{res}_{z_0} f = \frac{g(z)}{h'(z_0)}$.

Lemma 3.16.1. If z_0 is a simple pole,

$$\int_\Gamma f = i(\theta_2 - \theta_1)\mathrm{res}_{z_0} f \ .$$

$$\lim_{R\to\infty} \int_{C_R} \frac{p(z)}{q(z)} dz = 0 \text{ if } \deg(q) \geq 2 + \deg(p) \ .$$

$$\lim_{R\to\infty} \int_{C_R} e^{iaz} \frac{p(z)}{q(z)} = 0 \text{ if } \deg(q) \geq 1 + \deg(p) \ .$$

Theorem 3.16.2 (The Argument Principle)**.**

$$\frac{1}{2\pi i} \int_f f' = N_0(f) - N_p(f) \ .$$