Math 146 Notes

velo.x

Instructor: Ross Willard NC 5006
Section: 001
Tutorial: Monday, 4:30, MC1085

Contents

1 VECTOR SPACE

1.1 Vector Space - Jan 6

Definition 1.1.1 (**Pseudo-**Field)**.** *A field is an algebraic system $\mathbb{F}$ having:*

- *two elements* 0 *and* 1
- *operations* $+, \times, -$, *and* $()^{-1}$*(defined on nonzero elements)*

satisfying "the obvious" properties.

See appendix of the textbook.

Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Z}_{prime}$. $\mathbb{Q}(x) = \{\frac{f(x)}{g(x)} : f, g \; polynomials, g \neq 0\}$

NonExamples: $\{0\}$, $\mathbb{Z}_m$*(m not prime), Quaternions.*

Definition 1.1.2 (**Vector Space**)**.** *A vector space over* $\mathbb{F}$ *is a set* V *with two operations:*

- *Addition:* $V \times V \to V \; x + y$
- *Scalar Multiplication:* $\mathbb{F} \times V \to V \; ax$

satisfying 8 *properties:* $\forall x, y, z \in V$, $\forall a, b \in \mathbb{F}$

- *V1:* $x + y = y + x$
- *V2:* $x + (y + z) = (x + y) + z$
- *V3:* $\exists$ *a "zero vector"* $0 \in V$ *s.t.* $x + 0 = x$
- *V4:* $\forall x \in V$, $\exists u \in V$, *s.t.* $x + u = 0$
- *V5:* $1x = x$
- *V6:* $(ab)x = a(bx)$ **let* $\cdot$ *denote scalar multiplication*
- *V7:* $a(x + y) = ax + ay$
- *V8:* $(a + b)x = ax + bx$

Objective 1.1.1.

- *Defining/Constructing*
- *Proving that a system is a vector space*

Example 1: $\mathbb{R}$ def: set of all $n - tuples$ of real numbers

$$(x_1, \cdots, x_n) + (y_1, \cdots, y_n) = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$$
$$a(x_1, \cdots, x_n) = (ax_1, \cdots, ax_n)$$

Claim: $\mathbb{R}^n$ is a vector space over $\mathbb{R}$

Proof. Check V1:

$$\begin{aligned}(x_1, \cdots, x_n) + (y_1, \cdots, y_n) &= (x_1 + y_1, \cdots, x_n + y_n) \\ &= (y_1 + x_1, \cdots, y_n + x_n) \\ &= (y_1, \cdots, y_n) + (x_1, \cdots, x_n)\end{aligned}$$

□

More generally, for any field $\mathbb{F}$, $\mathbb{F}^n$ is a field over $\mathbb{F}$.

Example 2: $\mathbb{R}^{[0,1]} = \{all\ functions\ f : [0, 1] \to \mathbb{R}\}$

- $(f + h)(x) \overset{def}{=} f(x) + g(x)$
- $(af)(x) = af(x)$

Claim: $\mathbb{R}^[0, 1]$ is a vector space $/\mathbb{R}$.

Proof. V3: Let $\overline{0}$ be the constant 0 function, i.e., $\overline{0}(x) = 0\ \forall x \in [0, 1]\ \overline{0} \in \mathbb{R}^{[0,1]}$

Check: $f + \overline{0} = f\ \forall f \in \mathbb{R}^{[0,1]}$

$$\begin{aligned}(f + \overline{0})(x) &= f(x) + \overline{0}(x) \\ &= f(x) + 0 = f(x)\end{aligned}$$

Since $x \in [0, 1]$ arbitrary, $f + \overline{o} = f$.

More generally, for any set D, and any field $\mathbb{F}$, $\mathbb{F}^D$ is a vector space over $\mathbb{F}$.

□

Example 3: let $\mathbb{F} = \mathbb{Z}_2$.

Define $W = \{APPLE\}$,

- $APPLE + APPLE \overset{def}{=} APPLE$
- $0APPLE \overset{def}{=} APPLE$
- $1APPLE \overset{def}{=} APPLE$

Claim: W is a vector space over $\mathbb{Z}_2$.

Examples 4: 1. $\mathbb{R}^n : \mathbb{F}^n$, 2. $\mathbb{R}^{[0,1]}, : \mathbb{F}^D$, 3. $\{APPLE\}$.

4. Fix a field $\mathbb{F}$, for $n \geq 0$, $P_n(\mathbb{F})$ is the set of all polynomials, of degree $\leq n$, in variable x, with coefficients from $\mathbb{F}$,

$$= \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_i \in \mathbb{F}\}$$

Addition, scalar mult are "obvious", using op's of $\mathbb{F}$.

Claim: $P_n(\mathbb{F})$ is a vecor space $/\mathbb{F}$.

5. $\mathbb{F}[x] =$ the set of all polynomials in x with coefficients from $\mathbb{F} = \Cup_{n=0}^{\infty} P_n(\mathbb{F})$

Claim: with the "obvious" op's $\mathbb{F}[x]$ is a V.S. $/\mathbb{F}$.

Theorem 1.1.1 (**Cancellation Law**)**.** *Let V be a V.S., $/\mathbb{F}$, if $x, y, z \in V$, and $x + z = y + z$, then $x = y$.*

Proof. Let $u \in V$ be such that $z + u = 0$ (from V4).

Then

$$\begin{aligned}
x &= x + 0 && \text{(V3)}\\
x &= x + (z + u) && \text{(Choice of u)}\\
x &= (x + z) + u && \text{(hypothesis)}\\
x &= (y + z) + u && \text{(V2)}\\
x &= y + (z + u) && \text{(V2)}\\
x &= y + 0 && \text{(choice of u)}\\
x &= y
\end{aligned}$$

□

Corollary 1.1.1. *Suppose V is a V.S., there is exactly one "zero vector". i.e. a vector satisfy V3. in V.*

Proof. Assume $0_1, 0_2 \in V$, both satisfying V3, i,e, $x + 0_1 = x$ and $x + 0_2 = x, \forall x \in V$.

$$\begin{aligned}
0_1 &= 0_1 + 0_1\\
0_1 &= 0_1 + 0_2
\end{aligned}$$

$$\begin{aligned}
0_1 + 0_1 &= 0_1 + 0_2\\
&= 0_2 + 0_1 && \text{(V1)}\\
0_1 &= 0_2 && \text{(By Cancellation)}
\end{aligned}$$

□

Corollary 1.1.2. *Suppose V is a V.S. and $x \in V$, then the vector u in V4 is unique.*

Proof. Assume $u_1, u_2 \in V$ both satisfy $x + u_1 = 0 = x + u_2$, then

$$\begin{aligned}
u_1 + x &= u_2 + x && \text{(V1)}\\
u_1 &= u_2 && \text{(By Cancellation)}
\end{aligned}$$

□

Definition 1.1.3. *Given a V.S.* V *and* $x \in V$*,*

- *the unique vector* $u \in V$ *s.t.* $x + u = 0$ *is denoted* $-x$*.*
- $x - y$ *denotes* $x + (-y)$

Note: V2 justifies $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_kx_k$ not worry about parentheses.

1.2 Linear Combination - Jan 8

Definition 1.2.1 (Linear Combination). $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_kx_k$ *is called a linear combination of* $x_1, \cdots, x_k$.

Basic Problem: Given a V.S. $V/\mathbb{F}$, and $u_1, u_2, \cdots, u_n \in V$ and $x \in V$ to decide whether x is a linear combination of $u_1, \cdots, u_n$.

Example: $V = \mathbb{Q}[x]$ over $\mathbb{Q}$. Let $p = 4x^4 + 7x^2 - 2x + 3$.

- $u_1 = x^4 - x^2 + 2x + 1$
- $u_2 = 2x^4 + 3x^2 + 2x$
- $u_3 = x^4 + 4x^2 + 1$
- $u_4 = 2x^3 + 3$
- $u_5 = x^4 + 1$

Is p a linear combination of $u_1, \cdots, u_5$? Solution: search for $a_1, \cdots, a_5 \in \mathbb{Q}$ s.t.

$$p = a_1u_1 + a_2u_2 + \cdots + a_5u_5$$

$$\begin{aligned}4x^4 + 7x^2 - 2x + 3 =& a_1(x^4 - x^2 + 2x - 1) + a_2(2x^4 + 3x^2 + 2x) + a_3(x^4 + 4x^2 + 1)\\ &+ a_4(2x^3 + 3) + a_5(x^4 + 1)\\ =&(a + 1 + 2a_2 + a - 3 + a_5)x^4 + (2a^4)x^3 + (-a_1 + 3a_2 + 4a_3)x^2\\ &+ (2a_1 + 2a_2)x + (-a_1 + a_3 + 3a_4 + a_5)\end{aligned}$$

$$\begin{cases} a_1 + 2a_2 + a_3 + a_5 = 4 \\ 2a_4 = 0 \\ -a_1 + 3a_2 + 4a_3 = 7 \\ 2a_2 + 2a_2 = -2 \\ -a_1 + a_3 + 3a_4 + a_5 = 3 \end{cases}$$

No solution.

1.3 Subspace - Jan 10

Notation 1.3.1.

- 0 *denote the unique vector in* V
- x *denote the unique* $u \in V$ *satisfying* $V4$

Theorem 1.3.1. *Suppose* V *is a VS /* $\mathbb{F}$*,* $X \in V$*,* $a \in \mathbb{F}$*.*

1. *0x=0, the first* 0 *is scalar, the second* 0 *is a vector*
2. *(-a)x=a(-x)=-(ax)*
3. *a0=0*

Definition 1.3.1. *Suppose* V *is a* $V.S.$ *over* $\mathbb{F}$*,* $S \subseteq V$*,*

- ***Closed under Addition:*** *if* $x, y \in S$*,* $x + y \in S$*.*
- ***Closed under Scalar Multiplication:*** *if* $x \in S \Rightarrow ax \in S$*,* $\forall a \in \mathbb{F}$*.*

Definition 1.3.2 (Subspace). *Let* V *be a VS /* $\mathbb{F}$*,* $S \subseteq V$*, say* S *is a* ***Subspace*** *of* V *if*

1. S *is closed under addition and scalar multiplication*
2. $S \neq \varnothing$

Theorem 1.3.2. *Suppose* V *is a vector space /* $\mathbb{F}$ *and* S *is a subspace of* V*, then* S*, together the operations of* V *restricted to* S*.*

- $+_S : S \times S \to S$
- $\cdot_S : \mathbb{F} \times S \to S$

is a vector space over $\mathbb{F}$*.*

Proof. Given V, S, must prove: S with restricted operations of V, satisfying $V1$ to $V8$.

V1: must show: if $X, y \in S$, then $x + y = y + x$. Since $S \in V$, hence $x, y \in S \Rightarrow x, y \in V$, and $V \models V1$.

Same proof works for V2, 5, 6, 7, 8.

V3: know $S \neq \varnothing$, take any $x \in S$, consider $0x = 0 \in S$. (S is closed under scalar multiplication)

Hence there eixst a zero vector in S.

V4: fix $x \in S$, let $u = (-x)x \in S$, then $x + u = 1x + (-1)x = (1 + (-1))x = 0x = 0$. □

<u>**Note:**</u> in every $\mathbb{F}$, $\forall a \in \mathbb{F}$, $\exists c \in \mathbb{F} a + c = 0$, $c = -a$. Since $1 \in \mathbb{F}$, $-1 \in \mathbb{F}$.

Theorem 1.3.3. *If* V *is a vector space over* $\mathbb{F}$ *and* $S \subseteq V$*, and* S *with the operations of* V*, is itself a V.S. /* $\mathbb{F}$*, then* V *is a subspace of* V*.*

1.4 Span - Jan 13

Recall: If V is a V.S. / $\mathbb{F}$, and $u_1, \cdots, u_n, x \in V$, then x is a linear combination (lin. combo.) of $u_1, \cdots, u_n$ if $\exists a_1, \cdots, a_n$ such that $x = a_1u_1 + a_2u_2 + \cdots + a_nu_n$.

Definition 1.4.1. *Suppose V is a V.S. / $\mathbb{F}$, $x \in V$, and $\varnothing \neq S \subseteq V$.*

1. *Say x is a linear combination of S if x is a linear combination of some finite list of vectors from S. (Note that S might be infinite)*
2. *The **span** of S written $\operatorname{span}(S)$, is the set of $x \in V$ which are linear combinations of S.*
3. $\operatorname{span}(\varnothing) \overset{def}{=} \{0\}$

Examples

- In $\mathbb{R}^2$, $S = \{(1,1)\}$, what is $\operatorname{span}(S)$?
- In $\mathbb{R}^3$,

$$\begin{aligned} S =& \{(1,0,0),(1,1,0)\} \\ =& \{a(1,0,0)+b(1,1,0) : a, b \in \mathbb{R}\} \\ =& \{(a+b, b, 0) : a, b \in \mathbb{R}\} \\ =& \{(s,t,0) : s, t, \in \mathbb{R}\} \\ =& \text{the plane given by } z = 0 \end{aligned}$$

- In $\mathbb{R}[x]$, let $S = \{x, x^2, x^3, \cdots\}$, $\operatorname{span}(S) = \{f \in \mathbb{R}[x] : f(0) = 0\}$.

Proposition 1.4.1. $(\varnothing \neq S \subseteq V)$.

- *Suppose $u_1, \cdots, u_n \in S$, $x \in V$. Suppose x is a linear combination of $u_1, \cdots, u_n$.*

$$x = a_1u_1 + a_2u_2 + \cdots + a_nu_n \quad ,$$

If $v_1, \cdots, v_n$ are more vectors from S, then x is also a linear combination of $u_1, \cdots, u_n$, $v_1, \cdots, v_n$.

$$x = a_1u_1 + a_2u_2 + \cdots + a_nu_n + 0v_1 + 0v_2 + \cdots + 0v_n$$

- *If S is finite, say $S = \{u_1, u_2, \cdots, u_n\}$, then $x \in \operatorname{span}(S)$ iff x is a linear combination of $u_1, \cdots, u_n$.*
- *If S is infinite, we can say the following. Suppose $x, y \in \operatorname{span}(S)$. Then x is a linear combination of a finite list $u_1, \cdots, u_m$ from S and y is a linear combination of a finite list $v_1, \cdots, v_n$ from S. By the earlier remark, we can view both x and y as linaer combinations of the same list*

$$\{u_1, \cdots, u_m, v_1, \cdots, v_m\}$$

- *If $S = \{u_1, \cdots, u_n\}$, then $\operatorname{span}(S) = \{a_1u_1, \cdots, a_ku_k, a_1, \cdots, a_k \in \mathbb{F}\}$.*
- *If $S \subseteq T \subseteq V$, then $\operatorname{span}(S) \subseteq \operatorname{span}(T)$.*

Generalization 1.4.1. *If $x_1, \cdots, x_k \in \operatorname{span}(S)$, then $\exists u_1, \cdots, u_n \in S$, s.t. each x_l is a linear combo of $u_1, \cdots, u_n$.*

Theorem 1.4.1. *Suppose V is a V.S / $\mathbb{F}$, $S \subseteq V$, then $\operatorname{span}(S)$ is the (unique) smallest subspace of $V \supseteq S$. i.e.*

1. $\operatorname{span}(S)$ *is a subspace of V.*
2. $S \subseteq \operatorname{span}(S)$
3. *If W is any subspace of V containing S, then $\operatorname{span}(S) \subseteq W$.*

Proof.

1. Let $x \in S$, $x = 1x$, a linear combination of finitely many vectors in S.
2. i) Closure under scalar multiplication: let $x \in \operatorname{span}(S)$, $c \in \mathbb{F}$, $\Rightarrow \exists u_1, \cdots, u_n \in S$, s.t. $x = a_1x_1 + \cdots + a_nx_n$, so

   $$cx = c(a_1u_1 + \cdots + a_mu_m) = (ca_1)u_1 + \cdots + (ca_n)u_n$$

   ii) Closure under vector addition: let $x, y \in \operatorname{span}(S)$, want to prove that $x + y \in \operatorname{span}(S)$.

   By the technical remark, $\exists u_1, \cdots, u_n \in S$ s.t. $x = a_1u_1 + \cdots + a_nu_n$, $y = b_1u_1 + \cdots + a_nu_n$, $a_i, b_i \in \mathbb{F}$,

   Then, $x + y = (a_1u_1 + \cdots + a_nu_n) + (b_1u_1 + \cdots + b_nu_n) = (a_1 + b_1)u_1 + \cdots + (a_n + b_n)u_n$.

   So $x + y \in \operatorname{span}(S)$.

   Finally, if $S = \varnothing$, then $\operatorname{span}(S) = \{0\}$, if $S \neq \varnothing$, then $S \subseteq \operatorname{span}(S)$,

   either case, $\operatorname{span}(S) \neq \varnothing$, so $\operatorname{span}(S)$ is a subspace of V.
3. Let W be a subspace of V, $W \supseteq S$. RTP: $\operatorname{span}(S) \subseteq W$.

   Let $x \in \operatorname{span}(S)$, pick $u_1, \cdots, u_n \in S$, so that x is linear combination of it. that means

   $$x = a_1u_1 + \cdots + a_nu_n$$

   hence, $u_i \in S \subseteq W \Rightarrow a_1u_1 + \cdots + a_nu_n \in W \Rightarrow x \in W$.

   Hence, $\operatorname{span}(S) \subseteq W$.

$\square$

1.5 Span(continued) - Jan 15

Theorem 1.5.1 (Redundancies in span.). *Example: $V/\mathbb{F}$, suppose $S = \{u_1, \cdots, u_5\} \subseteq V$.*

Assume u_3 is a linear combination of u_2, u_4, u_5.

$$u_3 = c_2u_2 + c_4u_4 + c_5u_5$$

Claim: $(S) = \text{span}(S - \{u_3\})$.

Proof. RTP $\subseteq$ and $\supseteq$.

$\text{span}(S)$ is

- a subspace of V
- which contains $S\backslash\{u_3\} = \{u_1, u_2, \cdots, u_3\}$

By the theorem, the samllest subspace of V containing $S\backslash\{u_3\}$ is $\text{span}(S\backslash\{u_3\})$. hence $\text{span}(S) \supseteq \text{span}(S\backslash\{u_3\})$.

To prove that $\text{span}(S) \subseteq \text{span}(S\backslash\{u_3\})$,

let $x \in \text{span}(S)$, i.e.

$$\begin{aligned} x &= a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 + a_5u_5 \\ &= a_1u_1 + a_2u_2 + a_3(c_2u_2 + c_4u_4 + c_5u_5) + q_4u_4 + a_5u_5 \\ &= a_1u_1 + (a_2 + a_3c_2)u_2 + (a_4 + a_3c_4)u_4 + (a_5 + a_3c_5)u_5 \end{aligned}$$

$x \in Span(\{u_1, u_2, u_4, u_5\})$.

$\square$

Also Observe:

$$0u_1 + c_2u_2 + (-1)u_3 + c_4u_4 + c_5u_5 = 0$$

A linear combination of $u_1, \cdots, u_5$ equally the 0 vector with coefficients not all 0.

So we code redundacies formally with definition:

Definition 1.5.1. *($V\mathbb{F}, S \subseteq V$), S is linearly dependent if $\exists$ distinct vectors $u_1, \cdots, u_n \in S$, and $\exists a_1, \cdots, a_n \in \mathbb{F}$, not all 0, such that*

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0(zero\ vector).$$

Thus a set S is linearly dependent:

$\Leftrightarrow \quad (\exists$ *distinct* $u_1, \cdots, u_n \in S)(\exists a_1, \cdots, a_n \in \mathbb{F})(a_1u_1 + \cdots + a_nu_n = 0)\ and\ \neg(a_1 = \cdots = a_n = 0)$

Thus a set S is linearly independent if S is not linearly dependent. i.e.

$\Leftrightarrow \quad \neg(\exists$ *distinct* $u_1, \cdots, u_n \in S)(\exists a_1, \cdots, a_n \in \mathbb{F})(a_1u_1 + \cdots + a_nu_n = 0)\ and\ \neg(a_1 = \cdots = a_n = 0)$

$\Leftrightarrow \quad (\forall$ *distinct* $u_1, \cdots, u_n \in S)(\forall a_1, \cdots, a_n \in \mathbb{F})(a_1u_1 + \cdots + a_nu_n \neq 0)or(a_1 = \cdots = a_n = 0)$

$\equiv (\forall\ distinct\ u_1, \cdots, u_n \in S)()$

Technical Remark: when $S = \{u_1, \cdots, u_n\}$ without reports

- Can drop ($\forall\ distinct\ u_1. \cdots, u_n \in S$) in choice of linear independence.

-Can drop ($\exists\ distinct\ u_1 \cdots u_1, \cdots, u_n \in S$) in choice of linear dependence.

Example 2: Is $S = \{(0,1,1),(1,0,1),(1,2,3)\}$ linear dependent? (in $\mathbb{R}^3$)

Try to find: $a, b, c \in \mathbb{R}$ s.t.

$$a\begin{pmatrix}0\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\0\\1\end{pmatrix} + c\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}-2\\-1\\1\end{pmatrix}$$

Shows S is linearly dependent.

Question: If $S = \varnothing$, S is **linearly independent**.

Question 2: If $S = \{0\}$, S linearly dependent. Can write $1 \cdot 0 = 0$.

More generally, if $0 \in S \subseteq V$, then S is linearly dependent.

Theorem 1.5.2 (Linear Dependence). *$V\mathbb{F}$, $S \subseteq V$, then S is linearly dependent, iff $S = \{0\}$ or $\exists x \in S$, s.t. x is a linear combination of some vectors in $S \backslash \{x\}$.*

1.6 Basis Jan 17

Recall If V is a V.S. / $\mathbb{F}$, $S \subseteq B$.

1. $\text{span}(S) =$ set of all linear combinations of S
2. S is linearly dependent if $\exists u_1, u_2, \cdots, u_n \in S$ (distinct), $\exists a_1, \cdots, a_n \in \mathbb{F}$ not all 0, s,t, $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$.

   - else, S is linearly independent.

Definition 1.6.1. *V is V.S. / $\mathbb{F}$,*

1. *A set $S \subseteq V$ is a spanning set if $\text{span}(S) = V$. Also say S spans V.*
2. *V is finitely spanned if V has a finite spanning set.*
   V is countably spanned if V has a countable spanning set.

Examples:

$\mathbb{R}^3$ is finitely spanned, e.g. by $\{e_1, e_2, e_3\}$.

so is $\mathbb{R}^n$ e.g. by $\{e_1, e_2, \cdots, e_n\}$, $e_i = (0, 0, \cdots, 1, 0, \cdots, 0)$ with 1 at i_{th} spot.

$\mathbb{R}[x]$ is countably spanned e.g. by $\{1, x, x^2, x^3, \cdots\}$. not finitely spanned.

$\mathbb{R}^[0, 1]$ not countably spanned.

Definition 1.6.2. *V is a V.S. / $\mathbb{F}$. A **basis** for V is any $S \subseteq V$,*

- *spans V, and*
- *S is linearly independent*

Examples: $\{e_1, \cdots, e_n\} \subseteq \mathbb{F}^n$ is a basis for $\mathbb{F}^n$.

$\{1, x, x^2, x^3, \cdots\} \subseteq \mathbb{R}[x]$ is a basis for $\mathbb{R}[x]$.

Theorem 1.6.1. *Every countably spanned V.S. has a basis.*

Proof. Suppose V.S. V is spanned by countable set S, so either $S = \{v_1, v_2, \cdots, v_n\}$, or $S = \{v_1, v_2, \cdots\}$, WLOG, we assume $0 \notin S$, define

$$T = \{v_j \in S, v_j \notin span(v_1, v_2, \cdots, v_{j-1})\},$$

Claim that T is a basis for V.

<u>Proof of Claim: </u> 1^{st} show T is linearly independent, by contradiction, assume T is linearly dependent.

Then, $\exists k$, and scalars $a_1, a_2, \cdots, a_n$ (not all 0), s,t,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

Choose least k for which this is true.

Claim: $k \neq 1$, if $k = 1$, $a_1 v_1 = 0 \Rightarrow v_1 = 0$, but $0 \notin T$, contradiction.

so $k > 1$, Assume $a_k = 0$, then

$$a_1 v_1 + a_2 v_2 + a_{k-1} v_{k-1} = 0$$

Not all of $a_1, a_2, \cdots, a_{k-1} = 0$.

Next, show $span(S) = V$.

$$S = \{v_1, v_2, v_3, \cdots, v_n\}$$

$$T = \{v_{i_1}, v_{i_2}, v_{i_3}, \cdots\}$$

Know $\text{span}(S) = V$, intuitively $\text{span}(T) = \text{span}(S)$.

$$T = \{v_j \in S : v_j \notin \text{span}(\{v_1, v_2, \cdots, v_{j-1}\})\}$$

Therefore, T is a basis of V.

$\square$

Remark:

1. Every Vector Space has a basis. proof: some version of axiom of choice
2. bases is not unique, every V.S. except $\{0\}$, has multiple bases.
3. What is a basis for $V = \{0\}$? $\varnothing$

1.7 Dimension - Jan 20

Remark: Given a vector space V, the basis is not unique.

Relation between two basis of a vector space. (finitely spanned vector spaces)

Theorem 1.7.1. *Let V be a finitely spanned vector space over a field $\mathbb{F}$, let $\{v_1, \cdots, v_m\}$ be a basis of V, let $\{w_1, \cdots, w_n\} \subset V$ and $n > m$. Then $\{w_1, \cdots, w_n\}$ is linearly dependent.*

Sketch. Idea: Replace successfully $v_1, v_2, \cdots, v_n$, by $w_1, w_2, \cdots, w_n$ so that

$$\text{span}(\{w_1, w_2, \cdots, w_i, v_{i+1}, \cdots, v_m\}) = \text{span}(\{v_1, v_2, \cdots, v_i, v_{i+1}\})$$

$1 \leq i \leq m-1$. $\square$

Proof. Assume $\{w_1, \cdots, w_n\}$ is linearly dependent. Prove the statement by induction.

Base Case: (i=1), since $\{v_1, \cdots, v_m\}$ is a basis for V and $w_1 \in V$, there exist $a_1, \cdots, a_m \in \mathbb{F}$ s.t. $w_1 = a_1v_1 + \cdots + a_mv_m$.

By the assumption, $w_1 \neq 0$, hence one of the a_k's is nonzero.

By renumbering $v_1, \cdots, v_m$, WLOG, we can assume $a_1 \neq 0$. We can solve for v_1.

$$\begin{aligned} a_1v_1 &= w_1 - a_2v_2 - \cdots - a_mv_m \\ v_1 &= a_1^{-1}w_1 - a_1^{-1}a_2v_2 - \cdots - a_1^{-1}a_mv_m \end{aligned}$$

so, $\text{span}(\{v_1, v_2, \cdots, v_m\}) \subset \text{span}(\{w_1, w_2, \cdots, w_m\}) = V$.

Induction Assumption: Assume that the statement is true for r. It means after renumbering, $v_1, v_2, \cdots, v_m$ we have

$$\text{span}(\{w_1, w_2, \cdots, w_i, v_{i+1}, \cdots, v_m\}) = V.$$

*replace w_{i+1}.

Prove for $r+1$: Rewrite w_{i+1} as a linear combination of $\{w_1, \cdots, w_r, v_{r+1}, \cdots, v_m\}$.

$$w_{i+1} = c_1w_1 + \cdots + c_rw_r + d_{i+1}v_{i+1} + \cdots + d_mv_m$$

Observation: One of the $d_{r+1}, \cdots, d_m$ must be nonzero. Because if $d_{i+1} = \cdots = d_m = 0$, then

$$\begin{aligned} w_{r+1} &= c_1w_1 + \cdots + c_r2_r \\ 0 &= c_1w_1 + \cdots + c_rw_r - w_{r+1} \end{aligned}$$

Contradiction since $\{w_1, \cdots, w_{r+1}\}$ is linearly independent.

WLOG, we can assume $d_{i+1} \neq 0$,

$$d_{r+1}v_{r_1} = w_{r+1} - c_1w_1 - \cdots - a_rw_r - d_{r+2}v_{r+2} - \cdots - d_mv_m$$

Since $n > m$, $w_n = a_iw_i + \cdots + a_mw_m$, so $\{w_1, \cdots, w_n\}$ is linearly dependent.

It completes the proof. $\square$

Theorem 1.7.2. *Let V be a finitely spanned vector space, having one basis of m elements having another basis of n elements. Then $m = n$.*

Proof. We could not have $m < n$, or $m > n$. If it happends, the other set must be linearly dependent. $\square$

Definition 1.7.1. *Let V be a vector: space having a basis consisting of n elements, we say n is the dimensioning of V.*

$$\dim_{\mathbb{F}} V = n$$

$$\dim\{0\} = 0$$

A vector space that has a basis consisting of n elements, zero elememts, zero vector space, is called finite dimensional. Otherwise, V is called infinite dimensional(Hamel Basis)

Example:

- $\dim \mathbb{F}^n = n$

  Since

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

  is a basis for $\mathbb{F}^n$.

- $\dim P_n(\mathbb{F}) = n + 1$

  Since $\{1, x, \cdots, x^n\}$ is a basis for $P_n(\mathbb{F})$.

- $\dim \mathbb{F}[x] = \infty$

Definition 1.7.2. *Let $\{v_1, \cdots, v_n\}$ be linearly independent elements of a vector space V. We say that $\{v_1, \cdots, v_n\}$ is a **maximal set of linearly independent elements** of V if given any $w \in V$, the set $\{w, v_1, \cdots, v_m\}$ is linearly dependent.*

Corollary 1.7.1. *Let V be an $n-$dimensional space, then*

- *If $\{v_1, \cdots, v_n\}$ is a maximal set of linearly independent elements of V, then $\{v_1, \cdots, v_n\}$ is a basis of V.*
- *If $\{v_1, \cdots, v_n\} \subset V$ is linearly independent, then $\{v_1, \cdots, v_n\}$ is a basis for V.*
- *If $\{v_1, \cdots, v_n\} \subset V$, $k < n$ is linearly we can add $v_{k+1}, \cdots, v_n$ so that $\{v_1, \cdots, v_n\}$ is a basis for V.*
- *If W is a subspace of V, then $\dim W \leq \dim V$, if furthermore, $\dim W = \dim V$. Then $W = V$.*

1.8 Direct Sum - Tutorial Jan 20

Corollary 1.8.1. *If V is finitely spanned, and $\beta = \{v_1, \cdots, v_n\}$ is linearly independent, then β can be extended to a basis for V, i.e. $\exists w_1, \cdots, w_n \in V$, s.t. $\{v_1, \cdots, v_n, w_1, \cdots, w_r\}$ is a basis for V*

Proof. Let $m = dim = V$. So $n \leq m$ by theorem.

Case 1: β is already a basis. (n=m)

Case 2: β is not a basis. $\square$

Theorem 1.8.1. *Let S, T be linearly independent sets, then $S \cup T$ is linearly independent if and only if* $\text{span}(S) \cap \text{span}(T) = \{0\}$.

1.9 Jan 22

Corollary 1.9.1. *If V is finitely spanned, and $\mathfrak{B} = \{v_1, \cdots, v_n\}$ is linearly independent, then $\mathfrak{B}$ can be extended to a basis for V.*

i.e. $\exists w_1, \cdots w_r \in V$, *s.t.* $\{v_1, \cdots, v_n, w_1, \cdots, w_n\}$ *is a basis for* V.

Proof. Let $m = \dim V$, so $n \leq m$. (By theorem).

case 1: $\mathfrak{B}$ is already a basis (n = m). done

Case 2: $\mathfrak{B}$ is not a basis, so span$\mathfrak{B} \neq V$, so $\exists w_1 \in V \backslash \mathfrak{B}$. □

Theorem 1.9.1. *For any V.S. V, if $\mathfrak{B} \subseteq V$ is linearly independent, then $\mathfrak{B}$ can be extended to a basis for V. [use axiom of choice]*

Example: Let $\mathfrak{B} = \{\cos(nx), n \geq 0\} \cup \{\sin(nx) : n > 0\} \cup \{e^x\}$.

This $\mathfrak{B}$ can be extended to a basis $\mathfrak{B}'$ for $\mathbb{R}^{[0,1]}$.

$$|\mathfrak{B}'| = 2^{2^{\aleph_0}}$$

Recall: If $\{v_1, \cdots, v_n\} \subseteq V$ is linearly independent. Say $\{v_1, \cdots, v_n\}$ is a maximal linearly independent set, if $\forall w \in V \backslash \{v_1, \cdots, v_n\}$, $\{v_1, \cdots, v_n, w\}$ is linearly dependent.

Corollary 1.9.2. *If V is a finitely spanned set, then every basis is a maximal linearly independent set, and vice versa.*

More generally,

Definition 1.9.1. *Let V be a V.S., a subset $\mathfrak{B} \subseteq V$ is a* ***maximal linearly independent set*** *if*

- *$\mathfrak{B}$ is linearly independent*
- *$\forall w \in V \backslash \mathfrak{B}$, $\mathfrak{B} \cup \{w\}$ is linearly dependent.*

Theorem 1.9.2. *In any V.S. V, every basis is a maximal linearly independent set, and vice versa.*

Definition 1.9.2. *A* ***mininal spanning set*** *is a set $\mathfrak{B}$ such that*

- span$\mathfrak{B} = V$
- $\forall w \in \mathfrak{B}$, span$(\mathfrak{B} \backslash \{w\}) \neq V$

Theorem 1.9.3. *In every vector space V,*

1. *Every basis is a minimal spanning set and vice versa*

2. *Every spanning set can be "shrunk" to a basis*
   i.e. if $\operatorname{span}\mathfrak{B} = V$*, then* $\exists \mathfrak{B}' \subseteq \mathfrak{B}$ *s.t.* $\mathfrak{B}'$ *is a basis for* V*.*

Proof. For (2), already proved when $\mathfrak{B}$ is countable. Can extend the proof to uncountable "well-ordering $\mathfrak{B}$".

To find a basis for $\mathbb{R}^{[0,1]}$

1. start with $\mathfrak{B} = \mathbb{R}^{[0,1]}$
2. well-order $\mathfrak{B}$ ("enumerates" $\mathfrak{B}$)
3. use the enumeration to shrink $\mathfrak{B}$ to a basis

$\square$

1.10 Quotient Space - Jan 24

Review: $\mathbb{Z}_n$ = the set of the congruence classes, $x \equiv y \pmod m \iff m|x-y$

Revisit: $[0] = \{qm : a \in \mathbb{Z}\} = m\mathbb{Z}$.

-$m\mathbb{Z}$ is collapsed to become zero

-$x \equiv y \pmod n \iff x = y \in m\mathbb{Z}$.

-advanced notation: $\mathbb{F}/m\mathbb{Z}$.

Version of this:

- $(\mathbb{Z}, +, \cdot) \to$ a vector space V.
- $(m\mathbb{Z}) \to$ a subspace of V.

Definition 1.10.1. *Fix a V.S. V over $\mathbb{F}$, and a subspace W. For $x, y \in V$ say $x \equiv y \pmod W$, if $x - y \in W$.*

Claim: $\equiv \pmod W$ is an equivalence relation on V.

Proof. For transitivity:

Assume $x, y, z \in V$, $x \equiv y \pmod W$ and $y \equiv z \pmod W$, by definition, $x - y \in W$, $y - z \in W$.

Then $x - z = (x - y) + (y - z) \in W$ since W is closed under addition.

Then by definition, $x \equiv z \pmod W$.

□

Definition 1.10.2. *Define V, W as before:*

For $x \in V$,

$$x + W := \{x + w : w \in W\}$$

(x is fixed, add x to every vector on W). $x + W$ is called ***translation of** W **by** x, or **coset of** W **through** x.*

Lemma 1.10.1. *V, W as before, for any $x \in V$, the equivalence class (congruence class) of $\equiv \pmod W$ containing x is $x + W$. If $y \equiv x \pmod W$, and $w \in W$, then $y \equiv x + w \pmod W$.*

Proof. For any $y \in V$, $y \in$ the equiv of $\equiv \pmod V$ containing x.

$$\begin{aligned}
&\iff && y \equiv x \pmod W \\
&\iff && y - x \in W \\
&\iff && y - x = w, for\ some\ w \in W \\
&\iff && y = x + w \\
&\iff && y \in x + W
\end{aligned}$$

□

Corollary 1.10.1. *With V and W as above, for any $x, y \in V$,*

$$x + W = y + W \iff x \equiv y \pmod W \qquad i.e. x - y \in W.$$

Remark: For $x \in V$, the span class of $\equiv \pmod W$ containing x is

$$\{y \in V, y \equiv x \pmod W\}$$

Definition 1.10.3.

$$\begin{aligned} V/W &:= \textit{the set of all equiv classes of the } \equiv \pmod W \textit{ relation} \\ &:= \textit{the set of all translations of } W \\ &:= \{x + W : x \in V\} \neq V \end{aligned}$$

Next, we turn V/W into a vector space over $\mathbb{F}$,

$$(x + W) + (y + W) := (x + y) + W$$

$$c(x + w) := (cx) + W$$

Claim: on the above situation, the operations well-defined, and the set V/W is a vector space over $\mathbb{F}$.

E.g. check scalar multiplication:

assume $x + W = x_1 + W$, $x \equiv x_1 \pmod W \iff x - x_1 \in W$.

need to know: $\forall c \in \mathbb{F}$,

$$\begin{aligned} & (cx + W) = (cx_1) + W \\ \Leftrightarrow \quad & cx \equiv cx_1 \pmod W \\ \Leftrightarrow \quad & (cx) - (cx_1) \in W \\ \Leftrightarrow \quad & c(x - x_1) \in W \end{aligned}$$

Definition 1.10.4. *V/W with the natural opeartions is called the **quotient space** of V modulo W.*

2 LINEAR TRANSFORMATION

2.1 Introduction to Linear Transformation - Jan 27

Definition 2.1.1. *Let V, W be vector spaces over $\mathbb{F}$, a function $T : V \to W$ is a linear transformation (or is linear) if*

1. *$T(x+y) = T(x) + T(y)$, $\forall x, y \in V$*
2. *$T(ax) = aT(x)$, $\forall x \in V$, $\forall a \in \mathbb{F}$*

Example

$V = W = \mathbb{R}$ (as $V.S./\mathbb{R}$)

Fix $\lambda \in \mathbb{R}$,

$$T : \mathbb{R} \to \mathbb{R} \qquad T(x) = \lambda x$$

T is a linear transformation.

Check: Let $x, y \in \mathbb{R}$, $a \in \mathbb{R}$

1. $T(x+y) = \lambda(x+y) = \lambda x + \lambda y = T(x) + T(y)$
2. $T(ax) = \lambda(ax) = a(\lambda x) = aT(x)$

fact: Every linear transformation from $\mathbb{R} \to \mathbb{R}$ has this form.

Generalization 2.1.1. *let $V = X = \mathbb{F}$, (field) considered as $V.S/\mathbb{F}$, every linear transformation $T : \mathbb{F} \to \mathbb{F}$ is of form $T(x) = \lambda x$ for some $\lambda \in \mathbb{F}$.*

Example: $V = W = \mathbb{R}^2$

define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T((x_1, x_2)) = (-x_2, x_1)$,

$T((1,0)) = (0,1)$

$T((0,1)) = (-1,0)$

Actually, T is "rotation" by $90°$ c.c.w centered at $(0,0)$.

Claim: T is a linear transformation.

Proof. $T((x_1,x_2)+(y_1,y_2)) = T((x_1+y_1, x_2+y_2)) = T(-(x_2+y_2), x_1+y_1) = (-x_2, z_1)+(-y_2, y_1) = T((x_1,x_2)) + T((y_1,y_2))$

Similarly, can check $T(a(x_1,x_2)) = aT((x_1,x_2))$ □

Generalization 2.1.2. *Fix $A \in M\mathbb{R}$, set of all $m \times n$ matrices with entries from $\mathbb{R}$,*

so

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Define $L_A : \mathbb{R}^n \to \mathbb{R}^n$, $L_A(x) = Ax$. x *is a column vector* nx_1 *matrix*

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

Claim: L_A is a linear transformation.

Proof. By example, $m = n = 2$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$L_A(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = (-x_2, x_1)$

□

Generalization 2.1.3. *Fix a field* $\mathbb{F}$*, fix* $A \in M_{m \times n}(\mathbb{F})$*,*

define $L_A ; \mathbb{F}^n \to \mathbb{F}^m$ *by* $L_A(x) = Ax$*,*

Claim: L_A *is a linear transformation.*

Recall: $C([-1,1])$ = all continuous functions$f : [-1,1] \to \mathbb{R}$, define $T : C([-1,1]) \to \mathbb{R}$, by $T(f) = \int_{-1}^{1} f(x)dx$.

Claim: T is a linear transformation.

Proof.

$$\begin{aligned} T(f+g) &= \int_{-1}^{1} (f+g)dx \\ &= \int_{-1}^{1} f dx + \int_{-1}^{1} g dx \\ &= T(f) + T(g) \end{aligned}$$

$T(af) = \int_{-1}^{1} af dx = a \int_{-1}^{1} f dx = aT(f)$ □

$D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ (set of all $f \in C(\mathbb{R})$),

$f^{(n)}$ exists, and is continuous $\forall n$.

Define $D(f) = f'$, D is linear.

Some easy properties of all linear transformations, suppose $T : V \to W$ linear.

1. $T(0) = 0$

   Proof. (a) $T(x + 0) = T(x) + T(0)$

   (b) $T(0 \cdot x) = 0T(x) = 0$

   □

2. $T(x-y) = T(x) - T(y)$

   Proof. $T(x-y) = T(x+(-1)y) = T(x) + T((-1)y) = T(x) - T(y)$ □

3. $T(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = a_1T(x_1) + \cdots + a_nT(x_n)$

Common Mistake:

$$T(ax+by) = T(a)T(x) + T(b)T(y)$$

More Examples:

Example 1: $M_{m\times n}\mathbb{F}$ is a vector space over $\mathbb{F}$, -add matrices componentwise -scalar multiply by multiplying all components

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

$$c\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix}$$

$T : M_{m\times n}(\mathbb{F}) \to M_{n\times m}(\mathbb{F})$ by $T(A) = A^t$. (transpose of A)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}^t = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

$(V = W)$ define $I_v : V \to V$ by $I_v(x) = x$ its linear.

Example 2: Given any V and W the function $T_0 : V \to W$ which maps every $x \in V$ to the 0 vector in W. (zero transformation)

Example 3: Given any V, the function $I_V : V \to V$ defined by $I_v(x) = x$ for all $x \in V$. (identity function)

2.2 Tutorial - Jan 27

Goals:

- Be able to describe the quotient space
- Be able to find a basis and the dimension of the quotient space

Recall that:

Definition 2.2.1. *V is a V.S. $W \leq V/\mathbb{R}$, we call V/W a quotient space if*

$$\begin{cases} (x+W)+(y+W)=(x+y)+W \\ c(x+W)=cx+W \end{cases}$$

which $x, y \in V$, $c \in \mathbb{R}$.

Example:

$V=\mathbb{R}^3$, $W=\text{span}\{(0,0,1)\}$. $\mathbb{R}^3/W$ is a quotient space.

Question: What are the elements in $\mathbb{R}^3/W$?

A: $p+W$, $p \in \mathbb{R}^3$.

B: $[p+W]=\{x \in \mathbb{R}^3 | x-p \in w\}$

C: All lines that are parallel to Z-axis

2.3 Null Spance and Range

Definition 2.3.1. *Suppose $T : V \to W$ is a linear transformation.*

1. *The* ***null space*** *of T denoted $N(T)$, is*
$$N(T) = \{x \in V : T(x) = 0\}$$

2. *The* ***range*** *of T denoted as $R(T)$*
$$R(T) = \{T(x) : x \in V\} \subseteq W$$

Example: $D_n : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ $D_n(f) = f'$. It's linear.

What is $N(D_n)$?
$$N(D_n) = \{f \in P_n(\mathbb{R}) : f' = 0\} = \{c : c \in \mathbb{R}\}$$
$R(D_n) = P_n(\mathbb{R})$

Theorem 2.3.1. *Suppose $T : V \to W$ is linear*

1. *$N(T)$ is a subspace of V.*
2. *$R(T)$ is a subspace of W.*

Proof.

1. $T(0_v) = 0_w$ so $0_v \in N(T)$ so $N(T) \neq \varnothing$

   -closure under addition: let $x, y \in N(T)$,
$$T(x + y) = T(x) + T(y) = 0 + 0 = 0 \in N(T)$$
   -closure under scalar multiplication: let $x \in N(T)$, $c \in \mathbb{F}$
$$T(cx) = cT(x) = ca = 0 \in N(T)$$

2. $R(T) \neq \varnothing$ because $V \neq \varnothing$

   -closure under addition: let $u, v \in R(T) \subset W$, can write $u = T(x)$, $v = T(y)$, (for some $x, y \in V$), so $u + v = T(x) + T(y) = T(x + y) \in R(T)$.

   -Similar argument shows that $R(T)$ is closed under scalar multiplication.

$\square$

Theorem 2.3.2 (Useful Trick). *Suppose $T : V \to W$ is a linear transformation, suppose we know $V = \operatorname{span}\{v_1, \cdots, v_k\}$, then*

$$\begin{aligned}
R(T) =& \{T(x), x \in V\} \\
=& \{T(x) : x = a_1 v_1 + \cdots a_k v_k, a_i \in \mathbb{F}\} \\
=& \{T(a_1 v_1 + \cdots + a_k v_k) : a_1, \cdots, a_k \in \mathbb{F}\} \\
=& \{a_1 T(v_1) + \cdots a_k T(v_k) : a_1, \cdots, a_k \in \mathbb{F}\} \\
=& \operatorname{span}\{T(x_1), \cdots, T(x_k)\}
\end{aligned}$$

Example 1: $D_n : P_n(\mathbb{R}) \to P_n(\mathbb{R})$

A spanning set for $P_n(\mathbb{R})$ is
$$\{1, x, x^2, x^3, \cdots x^n\}$$

so

$$\begin{aligned}\mathbb{R}(D_n) =& \text{span}\{D_n(1), D_n(x), D_n(x^2), \cdots, D_n(x^n)\}\\ =& \text{span}\{0, 1, 2x, \cdots, nx^{n-1}\}\\ =& \text{span}\{1, x, x^2, \cdots, x^{n-1}\} = P_{n-1}(\mathbb{R})\end{aligned}$$

Example 2: Fix $A \in M_{m\times n}(\mathbb{F})$. $L_A : \mathbb{R}^n \to \mathbb{F}^m$ by $L_A(x) = Ax$.

The "standard basis" for $\mathbb{F}^n$ is

$$\{(1, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)\}$$

$\mathbb{F}^n = \text{span}\{e_1, e_2, \cdots, e_n\}$

Then $R(L_A) = \text{span}(L_A(e_1), \cdots, L_A(e_n))$ by the Useful Trick Theorem, then,

$$L_A(e_i) = \begin{pmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mi} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} = \text{the ith column of } A$$

Hence, $R(L_A)$ is the subspace of $\mathbb{F}^m$ spanned by the columns of A.

Two Basic Questions about Linear Transformation

Question 1: Is it injective?

Question 2: Is it surjective?

Theorem 2.3.3. *Suppose $T : V \to W$ is linear, then T is injective $\iff$ $N(T) = \{0\}$.*

Proof. $(\Rightarrow)$ Assume T is injective. i.e. $\forall x, y \in V, T(x) = T(y) \Rightarrow x = y$.

Obviously $0 \subseteq N(T)$. (Since $N(T)$ is a subspace)

For $N(T) \subseteq \{0\}$, let $x \in N(T)$ so $T(x) = 0 = T(0) \Rightarrow x = 0$.

$(\Leftarrow)$ Assume $N(T) = \{0\}$, prove injectively, assume $x, y \in V$ and $T(x) = T(y)$.

$\Rightarrow T(x) - T(y) = 0 \Rightarrow T(x - y) = 0 \Rightarrow x - y \in N(T) = \{0\} \Rightarrow x = y$. $\square$

2.4 Jan 31

Definition 2.4.1. *A linear transformation $T : V \to W$ is an isomorphism if it is a bijection.*

We also write $T : V \cong W$.

We say V, W are ***isomorphic****. (and write $V \cong W$) if $\exists T : V \cong W$.*

Example 1: $P_n(\mathbb{R}) \cong \mathbb{R}^{n+1}$

An example of an isomorphism $T : P_n(\mathbb{R}) \cong \mathbb{R}^{n+1}$ is

$$T(a_0 + a_1 + \cdots + a_n x^n) = (a_0, a_1, \cdots, a_n)$$

Easy facts:

1. For every V.S. V, $V \cong V$.
2. If $V \cong W$ then $W \cong V$.

Definition 2.4.2. *Given a linear tranformation $T : V \to W$ the*

nullity *of T:* $\text{nullity}(T) := \dim(N(T))$

rank *of T:* $\text{rank}(T) := \dim(R(T))$

Theorem 2.4.1. *Suppose $T : V \to W$ is linear and $dim(V) < \infty$, then $rank(T) + null(T) = dim(V)$.*

Proof. First step find basis for $N(T)$ and $R(T)$

Let S be a basis for $N(T)$ let $n = dim(V)$, as $N(T) \subseteq V$, S is linearly independent in V

$\Rightarrow |S| \leq n$. Write $S = \{v_1, \cdots, v_k\}$, $k < n$.

Since S is linearly independent in V and V is countably spanned, by A2Q2, S can be extended to a basis B_i for V.

$$B = \{v_1, \cdots, v_k, x_1, \cdots, x_m\}, \qquad k + m = n$$

Let $C = \{T(x_1), \cdots, T(x_m)\}$,

Claim

1. $|C| = m$
2. C is a basis for $R(T)$

It will follow that $rank(T) = m = n - k = \dim(V) - null(T)$.

First prove that C is linearly independent. Assume

$$\begin{aligned}
& a_1 T(x_1) + \cdots + a_m T(x_m) = 0 \\
\Rightarrow \quad & T(a_1 x_1) + \cdots + T(a_m x_m) = 0 \\
\Rightarrow \quad & a_1 x_1 + \cdots + a_m x_m \in N(T) \\
\Rightarrow \quad & a_1 x_1 + \cdots + a_m x_m = b_1 v_1 + \cdots + b_k v_k \\
\Rightarrow \quad & a_1 x_1 + \cdots + a_m x_m - b_1 v_1 - \cdots - b_k v_k = 0
\end{aligned}$$

As $\{x_1, \cdots, x_m, v_1, \cdots, v_k\}$ is linearly independent, $a_1 = \cdots = a_m = b_1 = \cdots = b_k = 0$.

Therefore, C is linearly independent.

To prove that C spans $R(T)$, since $\{v_1, \cdots, v_k, x_1, \cdots, x_m\}$ spans V.

$$R(T) = \text{span}(T(v_1), \cdots, T(v_k), T(x_1), \cdots, T(x_m)) = \text{span}(C)$$

$\square$

2.5 Ordered Basis - Feb 3

Proposition 2.5.1. *Suppose* $\{v_1, \cdots, v_n\}$ *is a basis for V.S. /* $\mathbb{F}$.

Then $\forall x \in V$, x *can be uniquely written*

$$x = a_1 v_1 + \cdots + a_n v_n \qquad a_i \in \mathbb{F}$$

Proof. $\{v_1, \cdots, v_n\}$ span V so every $x \in V$ can be written in this way.

For uniqueness, assume $x = a_1 v_1 + \cdots + a_n v_n = b_1 v_1 + \cdots b_n v_n$

Get $0 = (a_1 - b_1)v_1 + \cdots + (a_n b_n)v_n$. As $\{v_1, \cdots, v_n\}$ is linearly independent, get $a_1 = b_1, \cdots, a_n = b_n$. ☐

Example:

Let $V = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$. A plane in $\mathbb{R}^3$. V is a subspace of $\mathbb{R}$.

Let $v_1 = (-1, 1, 0)$, $v_2 = (0, -1, 1)$.

$\{v_1, v_2\}$ is a basis for V

$x = (-3, 1, 2) \in V \Rightarrow x = 3v_1 + 2v_2$

The **coordinates** of x relative to $\{v_1, v_2\}$ are $(3, 2)$.

Definition 2.5.1. *Let* V *be a V.S.* $\dim V = n$. *An* ***Ordered Basis*** *for* V *is an basis* $(v_1, \cdots, v_n)$, *ordered as an n-tuple.*

Notation 2.5.1. α, β, γ *for ordered bases,* A, B, C *for basis.*

Definition 2.5.2. *Suppose* V *is a V.S.,* $\dim V = n$, β *is an ordered basis for* V.

The ***coordinate vector of*** x ***relative to*** β *is the unique n-tuple* $(a_1, \cdots, a_n) \in \mathbb{F}^n$ *s.t.*

$$x = a_1 v_1 + \cdots + a_n v_n$$

We denote $(a_1, \cdots, a_n)$ *by* $[x]_\beta$.

Example: In the previous example, let $\beta = (v_1, v_2)$ where $v_1 = (-1, 1, 0)$, and $v_2 = (0, -1, 1)$. If $x = (-3, 1, 2)$, then $[x]_\beta = (3, 2)$.

Definition 2.5.3. *Fix* $V, \mathbb{F}, \beta = (v_1, \cdots, v_n)$ *as in definition.*

Define

$$[\quad]_\beta : V \to \mathbb{F}^n, \qquad\qquad x \mapsto [x]_p$$

Therefore, we can view $[\quad]_\beta$ *as a function* $V \to \mathbb{F}^n$.

Theorem 2.5.1. *Let* V *be a finite dimensional vector space over* $\mathbb{F}$, $\dim(V) = n$, *and let* β *be an ordered basis, then the map* $[\quad]_\beta : V \to \mathbb{F}^n$ *is an isomorphism (i.e. a bijective linear transformation).*

Proof. Let $x, y \in V$, (must show $[x+y]_\beta = [x]_\beta + [y]_\beta$)

Write

$$\begin{aligned} [x]_\beta &= (a_1, \cdots, a_n) &\Rightarrow \quad x &= a_1 v_1 + \cdots + a_n v_n \\ [y]_\beta &= (b_1, \cdots, b_n) &\Rightarrow \quad y &= b_1 v_1 + \cdots + b_n v_n \\ [x+y]_\beta &= (c_1, \cdots, c_n) &\Rightarrow \quad x+y &= c_1 v_1 + \cdots + c_n v_n \end{aligned}$$

$$\Rightarrow (a_1+b_1)v_1 + \cdots + (a_n+b_n)v_n = c_1 v_1 + \cdots c_n v_n$$

By prop,

$$\begin{cases} a_1 + b_1 = c_1 \\ a_2 + b_2 = c_2 \\ \quad\vdots \\ a_n + b_n = c_n \end{cases} \Rightarrow (a_1, \cdots, a_n) + (b_1, \cdots, b_n) = (c_1, \cdots, c_n) = [x]_\beta + [y]_\beta = [x+y]_\beta$$

Similarly, $[\quad]_\beta$ presents scalar multiplication, so it is linear.

Bijection:

To show $[\quad]_\beta$ is injective,

$$\begin{aligned} N([\quad]_\beta =& \{x \in V : [x]_\beta = (0, \cdots, 0)\}) \\ =& \{x \in V : x = 0\} \\ =& \{0\} \end{aligned} \qquad ([\quad]_\beta \text{ is injective})$$

To show $[\quad]_\beta$ is surjective, first find a spanning set for $V = \{v_1, \cdots, v_n\}$

$$R([\quad]_\beta) = \text{span}\{[v_1]_\beta, \cdots, [v_n]_\beta\}$$

$[v_1]_\beta = (1, 0, \cdots, 0) = e_1$, as $v_1 = 1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n$.

In $\mathbb{C}^2$, $e_1 = (1, 0)$, $e_2 = (0, 1)$, $[v_2]_\beta = e_2, \cdots$.

So

$$\begin{aligned} R([\quad]_\beta) =& \text{span}\{[v_1]_\beta, \cdots, [v_n]_\beta\} \\ =& \text{span}\{e_1, \cdots, e_n\} \\ =& \mathbb{F}^n \end{aligned}$$

So $[\quad]_\beta$ is surjective.

So $[\quad]_\beta : V \cong \mathbb{F}^n$.

□

2.6 Tutorial - Feb 3

let V be a V.S. / $\mathbb{F}$, a linear functional on V is a linear map $f : V \to \mathbb{F}$.

The collection of all linear functionals is denoted V^* and is called the dual space of V.

Example 1:

Let $Vf = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$, $f(x) = f(x \cdot 1) = xf(1)$, $x \in \mathbb{R}$.

so the linear maps $f : \mathbb{R} \to \mathbb{R}$ are given by $f(x) = ax$ for some $a \in \mathbb{R}$.

Exampel 2:

$V = \mathbb{R}^3$, $\mathbb{F} = \mathbb{R}$, let $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$.

$$f_{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = x_1a + x_2b + x_3c = [abc] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Then $f_{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}$ is linear.

Let $f \in (T\mathbb{R}^3)^*$ recall that a linear map f is determined by its values on a basis B.

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ so $x = x_1e_1 + x_2e_2 + x_3e_3$, e : the standard unit basis.

$f(x) = f(x_1e_1) + f(x_2e_3) + f(x_3e_3) = x_1f(e_1) + x_2f(e_2) + x_3f(e_3)$.

The values of f on the basis vectors determine f.

Let $a_1 = f(e_1)$, then $f(x_1e_1 + x_2e_2 + x_3e_3) = (a_1, a_2, a_3)^T(x_1, x_2, x_3)^T$.

so $f(x) = f_{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}$

2.7 Linear Transformation and Basis and Extension- Feb 5

Proposition 2.7.1. *Suppose V, W are vector spaces over $\mathbb{F}$, B is a basis for V. (note V is not finite dimensional) and $T : V \to W$ is a linear. Then T is determined by its values on vectors in B.*

Proof #1. The claim is that if $T' : V \to W$ is another linear transformation and $T'(v) = T(v)\ \forall v \in B$.

i.e. $T'|_B = T|_B$, then $T' = T$.

Let $x \in V$. (show that $T'(x) = T(x)$)

$\Rightarrow x \in \mathrm{span}(B)$

$\Rightarrow \exists v_1, v_2, \cdots, v_n \in B, \exists a_1, \cdots, a_n \in \mathbb{F}$

s.t. $x = a_1v_1 + \cdots + a_nv_n$.

Then

$$\begin{aligned} T'(x) =& T'(a_1v_1 + \cdots + a_nv_n) \\ =& a'T(v_1) + \cdots + a_nT'(v_n) \\ =& a_1T(v_1) + \cdots + a_nT(v_n) \\ =& \cdots \\ =& T(x) \end{aligned}$$

Since x was arbitrary, $T' = T$. $\square$

proof #2. **Claim:** the set of all linear transformation from V to W is a subspace of W^V. This set is called $\mathrm{Hom}(V, W)$

Define $D = T - T'$. i.e. $D : V \to W$ given by $D(x) = T(x) - T'(x)$. T

D is linear transformation. I'll prove that D is constant 0 function by showing $N(D) = V$.

Observe $B \subseteq N(D)$, therefore, $\mathrm{span}(B) \subseteq N(D)$, i.e. $V \subseteq N(D) \Rightarrow N(D) = V$.

$\square$

Proposition 2.7.2. *Suppose $V, W, \mathbb{F}, B$ as before, B is a basis for V. **Every** function $\tau : B \to W$ extends uniquely to a linear transformation $T : V \to W$. (i.e. $T|_B = \tau$) We call this "freely extending " tau.*

Proof. Given $\tau : B \to W$, define $T : V \to W$ as follows:

given $x \in V$, write

$$x = a_1v_1 + \cdots + a_nv_n \qquad (v_1, \cdots, v_n \in B, a_1, \cdots a_n \in \mathbb{F})$$

Let $T(x) := a_1\tau(v_1) + \cdots + a_n\tau(v_n) \in W$.

Check $T|B = \tau$. Suppose $x \in B$, then $x = 1 \cdot x$, so $T(x) = 1\tau(x) = \tau(x)$.

<u>Check: </u> T is linear.

Additivity: let $x, y \in V$, $\exists v_1, \cdots, v_n \in B$, such that

$$x = a_1v_1 + \cdots + a_nv_n$$

$$y = b_1v_1 + \cdots + b_nv_n$$

for sone $a_i, b_i \in \mathbb{F}$.

So $x + y = (a_1 + b_1)v_1 + \cdots + (a_n + b_n)v_n$.

$$\begin{aligned} T(x+y) =& (a_1+b_1)\tau(v_1) + \cdots + (a_n+b_n)\tau(v_n) && \text{(def of T)} \\ =& (a_1\tau(v_1) + \cdots + a_n\tau(v_n)) + (b_1\tau(v_1) + \cdots + b_n\tau(v_n)) \\ =& T(x) + T(y) && \text{(def of T)} \end{aligned}$$

Similar proof shows that T preserves scalar multiplication.

So T is linear. $\square$

Example: $V = \mathbb{R}^3$, $W = \mathbb{R}^3$, $B = \{v_1, v_2, v_3\}$, where

$v_1 = (1, 0, 1)$

$v_2 = (1, 0, -1)$

$v_3 = (1, 1, 1)$

B is a basis for $\mathbb{R}^3$ (exercise)

Define $\tau : \{v_1, v_2, v_3\} \to \mathbb{R}^2$ by

$\tau(v_1) = (1, 0)$

$\tau(v_2) = (1, 0)$

$\tau(v_3) = (\pi, e)$

Define $\tau : \mathbb{R}^3 \to \mathbb{R}^3$ extending τ.

$T(a, b, c) = (a + b(\pi - 1), be)$

$T = L_{\begin{pmatrix} 1 & \pi - 1 & 0 \\ 0 & e & 0 \end{pmatrix}}$

$T(v_1) = T(1, 0, 1) = (1, 0)$

$T(v_2) = (1, 0)$

$T(1, i, 1) = (\pi, e)$

Example 2:

V V.S. / $\mathbb{F}$, $\dim V = n$, let $\beta = (v_1, \cdots, v_n)$ be an ordered basis.

Define $\tau : \{v_1, \cdots, v_n\} \to \mathbb{F}^n$ by $\tau(v_i) = e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$.

τ extends uniquely to a linear transformation $T : V \to \mathbb{F}^n$.

$T : [\quad]_\beta$.

Example 3:

Same V, β.

Pick $\bar{a} = (a_1, \cdots, a_n) \in \mathbb{F}^n$.

Define $\tau_{\bar{a}} : \{v_1, \cdots, v_n\} \to \mathbb{F}$,

$\tau_{\bar{a}}(v_i) = a_i$.

$T(\bar{a})$ extends to a linear transformation. $f_{\bar{a}} : V \to \mathbb{F}$.

Exercise: What is f_{e_i}?

3 MATRIX

3.1 Introduction - Feb 7

Proposition 3.1.1. *Suppose $T : V \to W$ linear over $\mathbb{F}$, let $\beta = (v_1, \cdots, v_n)$ be an ordered basis for V, and $\gamma = (w_1, \cdots, w_m)$ be an ordered basis for W.*

- *T is completely determined by $T(v-1), \cdots, T(v_n)$*
- *Each $T(v_j)$ is determined by its coordinate vector $[T(v_j)]_\gamma \in \mathbb{F}^m$*

Definition 3.1.1. *In this context, the **matrix representation for** T for β and γ is the matrix $A \in M_{m\times n}(\mathbb{F})$, whose j^{th} column is $[T(v_j)]_\gamma \in \mathbb{F}^m$, thought of as a column vector.*

We write $[T]_\beta^\gamma$ for A.

$$[T]_\beta^\gamma = \begin{bmatrix} | & | & & | \\ [T(v_1)]_\gamma & [T(v_2)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & | & & | \end{bmatrix}$$

Example 1:

$D_3 : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ linear, the ordered basis

$$\begin{aligned} \beta &= (1, x, x^2, x^3) && \text{for} \quad P_3(\mathbb{R}) \\ \alpha &= (1, x, x^2) && \text{for} \quad P_2(\mathbb{R}) \end{aligned}$$

Let's find $[D_3]_\beta^\gamma$.

Apply D_3 to vectors in β.

$D_3(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$D_3(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$D_3(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$

$D_3(x^3) = 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2$

then,

$[D_3(1)]_\gamma = [0]_\gamma = (0, 0, 0)$

$[D_3(x)]_\gamma = [1]_\gamma = (1, 0, 0)$

$[D_3(x^2)]_\gamma = [2x]_\gamma = (0, 2, 0)$

$[D_3(x^3)]_\gamma = [3x^2]_\gamma = (0, 0, 3)$

Hence,

$$[D_3]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Algorithm 3.1.1. *Fix $\mathbb{F}$, $m, n \geq 1$, pick $A \in M_{m\times n}(\mathbb{F})$.*

$T = L_A : \mathbb{F}^n \to \mathbb{F}^m$. $T(x) = Ax$.

Let σ_n =standard ordered basis for $\mathbb{F}^n$; σ_m = standard ordered basis for $\mathbb{F}^m$.

$$\sigma_n = (e_1, \cdots, e_n), \qquad e_j \in \mathbb{F}^n$$
$$\sigma_m = (e_1, \cdots, e_m), \qquad e_i \in \mathbb{F}^m$$

Recall: *if $A \in M_{m\times n}(\mathbb{F})$,*

$e_j \in \mathbb{F}^n$ $(e_j = (0, \cdots, 0, 1, 0, \cdots, 0))$, $A_{e_j} \in \mathbb{F}^m$ *is the j^{th} column of A*

If σ_n is the standard basis for $\mathbb{F}^n$, and $x \in \mathbb{F}^n$, then $[x]_{\sigma_n} = x$

Proof. $x = (a_1, \cdots, a_n) = a_1e_1 + a_2e_2 + \cdots + a_ne_n$, so $[x]_{\sigma_n} = (a_1, a_2, \cdots, a_n) = x$. □

Now I'll prove

$$[L_A]_{\sigma_n}^{\sigma_m} = A$$

Proof. I will show, for each $j = 1, \cdots, n$, that $[L_A]_{\sigma_n}^{\sigma_m}$ and A have same j^{th} columns.

By definition, j^{th} column of $[L_A]_{\sigma_n}^{\sigma_m}$ is $[L_A(e_j)]_{\sigma_m}$

$$L_A(e_j) = Ae_j = j^{th}\ column\ of\ A$$
$$[L_A(e_j)]_{\sigma_n} = L_A(e_j) = j^{th}\ column\ of\ A$$

□

Theorem 3.1.1. *Suppose V, M are finite dimensional vector spaces $/\mathbb{F}$, and $T : V \to W$ is a linear transformation.*

$\alpha(v_1, \cdots, v_n)$ *an ordered basis for V*

$\gamma(w_1, \cdots, w_n)$ *an ordered basis for W*

then $\forall x \in V$,

$$\underbrace{[T]_\beta^\gamma}_{m\times n} \cdot \underbrace{[x]_\beta}_{n\times 1} = [T(x)]_\gamma$$

Proof. Write

$$[T]_\beta^\gamma = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

i.e.

$$\text{for } j = 1, ..., n, \qquad \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} = [T(v_j)]_\gamma$$

i.e.

$$T(v_j) = a_{ij}w_1 + a_{2j}w_2 + \cdots + a_{nj}w_m$$

Also write

$$[x]_\beta = (c_1, \cdots, c_n) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

i.e.

$$x = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

On the one hand,

$$[T]_\beta^\gamma \cdot [x]_\beta = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n \\ a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n \\ a_{m1}c_1 + a_{n2}c_2 + \cdots + a_{mn}c_n \end{pmatrix}$$

On the other hand,

$$\begin{aligned} T(x) =& T(c_1v_1 + \cdots + c_nv_n) \\ =& c_1T(v_1) + \cdots + c_nT(v_n) \\ =& c_1(a_{11}w_1 + \cdots + a_{m1}w_m) + c_2(a_{12}w_1 + \cdots + a_{m2}w_m) + \cdots + c_n(a_{1n}w_1 + \cdots + a_{mn}w_m) \\ =& (c_1a_{11} + c_2a_{12} + \cdots + c_na_{1n})w_1 + \cdots + (c_1a_{m1} + \cdots + c_na_{mn})w_m \end{aligned}$$

Hence, we can see that $[T]_\beta^\gamma \cdot [x]_\beta$ is the coordinate vector of $T(x)$ relative to γ, proving the theorem. $\square$

3.2 Feb 10

Recall: if $T : V \to W$ is linear,

$$\beta = (v_1, \cdots, v_n) ordered basis for V$$

$$\gamma = (w_1, \cdots, w_n) ordered basis for W$$

then, $[T]_\beta^\gamma \in M_{m\times n}(\mathbb{F})$. then,

$$[T]_\beta^\gamma = \begin{bmatrix} | & | & & | \\ [T(v_1)]_\gamma & [T(v_2)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & | & & | \end{bmatrix}$$

For any matrix,

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \mathrm{Col}_1(A) & \mathrm{Col}_2(A) & \cdots & \mathrm{Col}_n(A) \\ | & | & & | \end{pmatrix} = \begin{bmatrix} -- & Row_1(A) & -- \\ -- & Row_2(A) & -- \\ \vdots & & \vdots \\ -- & Row_3(A) & -- \end{bmatrix}$$

$$Col_j([T]_\beta^\gamma) = [T(v_j)]_\gamma$$

Definition 3.2.1. *Let $\mathbb{F}$ be a field, $m, n, p \geq 1$, $A \in M_{m\times n}(\mathbb{F})$, $B \in M_{n\times p}(\mathbb{F})$.*

The matrix product AB is the $m \times p$ matrix such that the (row i, column j) entry of AB is the linear combination of the entries in $\mathrm{Col}_j(B)$ using entries of $Row_i(A)$ as scalars.

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2p} \\ \vdots & \vdots & \vdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & a_{1p} \\ c_{21} & \cdots & a_{2p} \\ \vdots & c_{ij} & \vdots \\ c_{m1} & \cdots & a_{mp} \end{pmatrix}$$

$$c_{ij} = a_{i1}b_{ij} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Example: in $\mathbb{Z}_5$,

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 4 & 2 \\ 3 & 2 \end{pmatrix}$$

Comment: When $p = 1$, (B is a column vector, x)

Our definition here agrees with earlier definition of $A_x(A : m \times n)x \in \mathbb{F}^n$.

In fact, AB is such that, for each $j = 1, \cdots, p$, $Col_j(AB) = A \cdot Col_j(B)$.

So

$$A \cdot \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_p \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ Ax_1 & Ax_2 & \cdots & Ax_p \\ | & | & & | \end{pmatrix}$$

Suppose we have V, W, Z all finite dimensional vector spaces over $\mathbb{F}$. $T : V \to W$, and $U : W \to Z$ both linear transformations.

Let

$$\alpha = (v_1, \cdots, v_n)\text{ordered basis for}V$$
$$\beta = (w_1, \cdots, w_n)\text{ordered basis for}W$$
$$\gamma = \text{ordered basis for Z}$$

$$[T]_\alpha^\beta \Rightarrow n \times p$$

$$[U]_\beta^\gamma \Rightarrow m \times n$$
$$[U \circ T]_\alpha^\gamma \Rightarrow m \times p$$

Theorem 3.2.1. *In this situation,*

$$[U]_\beta^\gamma \cdot [T]_\alpha^\beta = [U \circ T]_\alpha^\gamma$$

Proof. LHS and RHS are both $m \times p$ matrices.

Suffices to show $\mathrm{Col}_j(LHS) = \mathrm{Col}_j(RHS)$. $\forall j = 1, \cdots, p$.

Let $\alpha(v_1, \cdots, v_p)$.

$$\begin{aligned}
\mathrm{Col}_j(RHS) =& \mathrm{Col}_j([U \circ T]_\alpha^\gamma) \\
=& [(U \circ T)(v_j)]_\gamma \\
=& [U(T(v_j))]_\gamma
\end{aligned}$$

$$\begin{aligned}
\mathrm{Col}_j(LHS) =& \mathrm{Col}_j([U]_\beta^\gamma \cdot [T]_\alpha^\beta) \\
=& [U]_\beta^\gamma \cdot \mathrm{Col}_j([T]_\alpha^\beta) \\
=& [U]_\beta^\gamma \cdot [T(v_j)]_\beta \\
=& [U]_\beta^\gamma \cdot [x]_\beta && (x = T(v_j)) \\
=& [U(x)]_\gamma \\
=& [\mathrm{Col}_j(RHS)]
\end{aligned}$$

$\square$

3.3 Feb 12

Theorem 3.3.1. *Suppose V, W, Z are all finite dimensional, $V \xrightarrow{T} W \xrightarrow{U} Z$ is linear. α, β, γ are ordered bases for V, W, Z respectively, then*

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma \cdot [T]_\alpha^\beta$$

Proof. Suppose $\mathbb{F}$ a field, $A \in M_{m\times n}(\mathbb{F})$, $B \in M_{n\times p}(\mathbb{F})$,

$L_A : \mathbb{F}^n \to \mathbb{F}^m$ $L_A(x) = Ax$

$L_B : \mathbb{F}^p \to \mathbb{F}^n$ $L_B(x) = Bx$

$$\overset{\sigma_p}{\mathbb{F}^p} \xrightarrow{L_B} \overset{\sigma_n}{\mathbb{F}^n} \xrightarrow{L_A} \overset{\sigma_m}{\mathbb{F}^m}$$

theorem gives

$$[L_A L_B]_{\sigma_p}^{\sigma_m} = [L_A]_{\sigma_n}^{\sigma_m} [L_B]_{\sigma_p}^{\sigma_n} = AB$$

□

Corollary 3.3.1. *In this situation*

$$L_A \cdot L_B = L_{AB}$$

Proof. It suffices to show that

$$[L_A \circ L_B]^{\sigma_m} = [L_{AB}]_{\sigma_p}^{\sigma_m}$$

□

Corollary 3.3.2. *Matrix multiplication(when defined) is associative. If $A \in M_{m\times n}(\mathbb{F})$, $B \in M_{n\times p}(\mathbb{F})$, $C \in M_{p\times r}(\mathbb{F})$, then*

$$\underset{m\times p}{(AB)}\underset{p\times r}{C} = \underset{m\times n}{A}\underset{n\times r}{(BC)} =$$

Proof. Suffices to prove

$$L_{(AB)C} = L_{A(BC)}$$

** not only the matrix determines the linear transformation, but also the linear transformation determines the matrix.

Well, by the first corollary,

$$L_{(AB)C} = L_{AB} \circ L_C = (L_A \circ L_B) \circ L_C$$

Similarly,

$$L_{A(BC)} = L_A \circ (L_B \circ L_C)$$

Since composition of functions is associastive, therefore,

$$L_{(AB)C} = L_{A(BC)}$$

□

$$\begin{array}{rcl} Fin.DimV.S.V & \leftrightsquigarrow & \mathbb{F}^n \\ v & \leftrightarrow & [v]_\beta \\ LinearTransformation & \leftrightarrow & matrices \\ T & \longrightarrow & [T]_\beta^\gamma \end{array}$$

Definition 3.3.1 (Invertible Matrices). *A square matrix $A \in M_{m\times n}(\mathbb{F})$ is invertible if $\exists$ $B \in M_{n\times n}(\mathbb{F})$s.t. $AB = BA$.*

Call B an inverse of A.

Observe: If B exists then it is unique.

i.e. if $B_1, \cdots, B_2 \in M_{m\times n}(\mathbb{F})$ and

$$\begin{aligned} & AB_1 = B_1A = I_n \\ and \quad & AB_2 = B_2A = I_n \end{aligned}$$

then $B_1 = I_nB_1 = (B_2A)B = B_2(AB_1) = B_2I_n = B_2$,

when A is invertible, use A^{-1} for the unique inverse of A. So $AA^{-1} = A^{-1}A = I_n$

Theorem 3.3.2. *Suppose V, W are finite dimensional spaces $/\mathbb{F}$, α, β are ordered bases for V, W respectively,*

$$T : V \to W, \qquad A = [T]_\alpha^\beta$$

then T is an isomorphism $\iff$ A is invertible.

In which case

$$A^{-1} = [T_{-1}]_\beta^\alpha.$$

Proof. $\Rightarrow$ Assume T is an isomorphism, (bijection), Jan 31: so $\dim(W) = \dim(V)$(say $= n$).

So A is $n \times n$. Let

$$B = [T^{-1}]_\beta^\alpha \in M_{n\times n})(\mathbb{F})$$

$$\begin{aligned} AB =& [T]_\alpha^\beta \cdot [T^{-1}]_\beta^\alpha \\ =& [T \circ T^{-1}]_\beta^\beta && \text{(By Monday's Theorem)} \\ =& [I_v]_\beta^\beta = I_n \end{aligned}$$

A similar proof shows $BA = I_n$, so by definition, A is invertible with $A^{-1} = B$.

$\Leftarrow$ exercise. $\square$

Lemma 3.3.1. *If $A, B \in M_{n\times n}(\mathbb{F})$ are invertible, then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*

Proof. Let $C = B^{-1}A^{-1}$, it suffices to show that $(AB)C = C(AB) = I_n$, for then it will follow that AB is invertible and its inverse is C. Then,

$$(AB)C = (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$$

The proof of $C(AB) = I_n$ is similar. $\square$

3.4 Feb 14

Corollary 3.4.1. *Suppose $T : V \to W$ is linear, and $\dim(V) = \dim(W) = n$, then T is injective $\Leftrightarrow$ T is surjective $\Leftrightarrow$ T is an isomorphism.*

Proposition 3.4.1. *Suppose $f : X \to Y$, $g : Y \to Z$, so $gf := g \circ f : X \to Z$. If gf is bijection, then :*

- *f is injective, and*
- *g is surjective.*

Exercise: cannot expect f or g to be bijections.

Example: Let $f, g : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$.

$$g(x) = \begin{cases} \ln x, & if\ x > 0 \\ 0, & if\ x \leq 0 \end{cases}$$

$g \circ f\ g(f(x)) = g(e^x) = \ln(e^x) = x.$

$f \circ f = id_{\mathbb{R}}$, a bijection.

Theorem 3.4.1. *Suppose $A, B \in M_{n\times n}(\mathbb{F})$, if AB is invertible, then A and B are also invertible.*

Proof. Assume AB is invertible, $\Rightarrow L_{AB} : \mathbb{F}^n \to \mathbb{F}^n$ is an isomorphism (bijection).

By wednesday's theorem, $L_{AB} = L_A L_B$.

$$\mathbb{F}^n \overset{L_B}{\to} \mathbb{F}^n \overset{L_A}{\to} \mathbb{F}^n$$

By the fact, L_B is injective, L_A surjective, so by Jan 3, Cor, L_A, L_B are isomophisms $\Leftrightarrow$ A, B are invertible ☐

Corollary 3.4.2. *If $A, B \in M_{n\times n}(\mathbb{F})$ and $AB = I_n$, then $BA = I_n$.*

Proof. Assume $AB = I_n$, I_n is invertible.

$$I_n I_n = I_n I_n = I_n$$

i.e. AB is invertible. A and B are invertible by the theorem.

$$\begin{aligned} & AB = I_n \\ \Rightarrow \quad & A^{-1}(AB) = A^{-1} I_n \\ \Rightarrow \quad & B = A^{-1} \\ & BA = A^{-1}A = I_n \end{aligned}$$

☐

B is an inverse to A if $AB = BA = I_n$

B is a left inverse to A if $BA = I_n$

B is a right inverse to A if $AB = I_n$

Exercise:
Prove if B is a left inverse of A.
Prove if C is a right inverse of A then $B = C$.

Back to Coordinatization

Recall from Feb 3,

$$\begin{aligned} W =& \{(x, y, z) \in \mathbb{R}^3, x + y + z = 0\}, \qquad 2 - \dim & \text{(subspace of } \mathbb{R}^3\text{)} \\ v_1 =& (-1, 1, 0), \qquad v_2 = (0, -1, 1) & \\ \beta =& (v_1, v_2) & \text{(an ordered basis for } W\text{)} \\ w =& (-3, 1, 2) \in W. \qquad [w]_\beta = (3, 2) & \end{aligned}$$

In general, if $(a, b, c) \in W$, then $[(a, b, c)]_\beta = (-a, c)$.

We might prefer a different ordered basis.

Let $\gamma = (u_1, u_2)$ where

$$\begin{aligned} u_1 &= (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) & \in W \\ u_2 &= (\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}) & \in W \end{aligned}$$

What is $[w]_\gamma$? In general, what is $[(a, b, c)]_\gamma$ for $(a, b, c) \in W$?

Theorem 3.4.2. *Suppose V is a finite dimensional V.S. over $\mathbb{F}$, β, γ two ordered basis for V. In this situation, let $Q = [I_v]_\gamma^\beta$,*

$I_v : V \to V$, $I_v : V \to V$, $I_v(x) = x$, then

1. *Q is invertible*
2. *$Q[x]_\beta = [x]_\gamma$, $\forall x \in V$*
3. *$Q^{-1}[x]_\gamma = [x]_\beta$*

Definition 3.4.1. *Q is called the **change of coordinate matrix from β to γ**.*

Proof. $I_v : V \to V$ is an isomorphism, $\Rightarrow Q$ is invertible.

Let $x \in V$,

$$\begin{aligned} Q[x]_\beta =& [I_v]_\beta^\gamma \cdot [x]_\beta \\ =& [I_v(x)]_\gamma \end{aligned}$$

(Thm, Feb 7).

Multiply on left by Q^{-1},

$$Q^{-1}/Q[x]_\beta = Q^{-1}[x]_\gamma$$

□

Notation 3.4.1. *If $T : V \to V$, β an ordered basis for V, then*

$$[T]_\beta = [T]_\beta^\beta$$

Theorem 3.4.3. *Suppose V is a finite dimensional vector space over $\mathbb{F}$, β, γ two orderd bases, and $T : V \to V$ is linear, let $Q = [I_v]_\beta^\gamma$, then*

$$[T]_\beta = Q[T]_\gamma Q$$

Proof. Suffices to show that

$$\begin{aligned}
Q[T]_\beta =& [T]_\gamma Q \\
[I_v]_\beta^\gamma [T]_\beta^\beta =& [T]_\gamma^\gamma [I_v]_\beta^\gamma \\
[I_v \circ T]_\beta^\gamma =& [T \circ I_v]_\beta^\gamma \\
T =& T
\end{aligned}$$

□

4 Chapter 3?

Facts:

1. $\text{Col}_j(AB) = A\text{Col}_j(B)$
2. $A_{e_j} = \text{Col}_j(A)$
3. $Ax = \sum_{j=1}^{n} x_j \text{Col}_j(A)$ $x \in \mathbb{F}^n$
4. $\text{Row}_i(AB) = \text{Row}_i(A)B$
5. $(e_j)^t B = \text{Row}_i(N)$
6. $x^t B = \sum_{i=1}^{n} \text{Row}_i(B)$ $x \in \mathbb{F}^n$

Notes: when $x \in \mathbb{F}^b$and use it as a matrix, always consider x as $n \times 1$ matrix.

Definition 4.0.1. *Let $A \in M_{m\times n}(\mathbb{F})$. An elementary row operation (an A) is any one of*

1. *Switching two rows* $\quad R_i \rightleftharpoons R_j \quad c_i \rightleftharpoons C_j, \quad (i \neq j)$
2. *Multiplying a row by an **nonzero** scalar* $\quad R_i \leftarrow cR_i \quad c_i \leftarrow cC_i \quad (c \neq 0)$
3. *Adding a scalar multiple of one row to another* $\quad R_i \leftarrow R_i + cR_j \quad C_i \leftarrow C_i + aC_j \quad (i \neq j)$

Elementary Column Operations are defined similarly, with columns instead of rows.

Operations have types (1) (2) or (3).

Proposition 4.0.1 (Newton's Third Law of Operations). *To every elementary operation, there is an equal but opposite elementary operation.*

Example: *$R_i \leftarrow R_i + aR_j$ can be undone by $R_i \leftarrow R_i + (-a)R_j$*

Notation 4.0.1. *If O is an elementary operation and O applied A gives B, then write $A \xrightarrow{O} B$.*

Definition 4.0.2. *An elementary matrix is an $n \times n$ matrix (over $\mathbb{F}$), which can be obtained by applying **one** elementary operation to I_n.*

Example 1: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is an elementary matrix. $R_2 \rightleftharpoons R_3$ $C_2 \rightleftharpoons C_3$

Example 2: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$ $R_3 \leftarrow aR_3$ $a \in \mathbb{F}$, $a \neq 0$ $\qquad C_3 \leftarrow aC_3$

Example 3:

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xLeftarrow{R_2 \leftarrow R_2 + aR_1} I_3$$

Example 4:

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xLeftarrow{C_1 \leftarrow C_1 + aC_2} I_3$$

Example 5:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \; elementary?$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow aR_2 + R_1} \begin{pmatrix} 1 & 0 & 0 \\ 1 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

NOT AN ELEM ROW OP

Theorem 4.0.1. *Fix m, n, let O be an elementary operation (for $m \times n$ matrix).*

Let E be the elementary matrix corresponding to O. (i.e. $I_n \xrightarrow{O} E$), then $\forall A \in M_{m\times n}(\mathbb{F})$, $A \xrightarrow{O} AE$.

Proof. For $j = 1, \cdots, n$, let $A_j = \mathrm{Col}_j(A)$ so $A = \begin{bmatrix} | & | & & | \\ A_1 & A_2 & \cdots & A_n \\ | & | & & | \end{bmatrix}$, $T = \begin{bmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{bmatrix}$

Case 1: O is $C_i \rightleftharpoons C_j$ $(i < j)$,

$$E = \begin{bmatrix} | & | & & | & & | & & & | \\ e_1 & e_2 & \cdots & e_j & \cdots & e_i & \cdots & e_n & \\ | & | & & | & & | & & & | \end{bmatrix}$$

Say $A \xrightarrow{O} \begin{bmatrix} | & & | & & | & & | \\ A_1 & \cdots & A_j & \cdots & A_i & \cdots & A_n \\ | & & | & & | & & | \end{bmatrix}$ To show that $AE = B_t$,

$$\mathrm{Col}_t(AE) = \mathrm{Col}_t(B) \qquad \forall t = 1, \cdots, n$$

By fact 1

$$\mathrm{Col}_t(AE) = A \cdot \mathrm{Col}_t(E) = A \begin{cases} e_t & if\ t \neq i, j \\ e_j & if\ t = i \\ e_i & if\ t = j \end{cases} \overset{Facts}{=} \begin{cases} \mathrm{Col}_t(A) & if\ t \neq i, j \\ \mathrm{Col}_j(A) & if\ t = i \\ \mathrm{Col}_i(A) & if\ t = j \end{cases} = \mathrm{Col}_t(B)$$

So $AE = B$. ◻

Theorem 4.0.2. *Let O be an elementary row operation, for $m \times \_$ matrices. Let E be its elementary matrix. Then $\forall A \in M_{m\times n}(\mathbb{F})$, $A \xrightarrow{O} EA$.*

Proof. Can be proved similarly, if $A \xrightarrow{O} B$, show

$$\mathrm{Row}_i(B) = \mathrm{Row}_j(EA) \qquad \forall i = 1, \cdots, m$$

◻

Theorem 4.0.3. *Elementary matrices are invertible. Moreover, if E is the elementary matrix, corresponding to an elementary operation O, then E^{-1} is the elementary matrix corresponding to the elementary operation O^{-1}. "inverse to" O.*

Proof. Say E corresponding to O, elementary column operation, so

$$I_n \xrightarrow{O} E \xrightarrow{O^{-1}} I_n$$

By Theorem 1,

$$EE' = I_n \Rightarrow E' = E^{-1}$$

and E is invertible. $\square$

4.1 Feb 24 Tutorial

Goal: Permutation: definition, notation, and permutation matrix

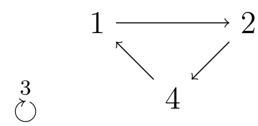
Definition 4.1.1 (General Definition). ***Permutation*** *is an order of the set* $\{1, 2, \cdots, n\}$, *e.g.* $1, 2, 3, \cdots, n$ *or* $2, 1, 3, 5, 4, \cdots, n$.

Definition 4.1.2 (Our Definition). ***Permutation*** *is a bijection between* $\{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$.

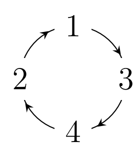
e.g.

$$\begin{cases} 1 \to 2 \\ 2 \to 4 \\ 3 \to 3 \\ 4 \to 1 \end{cases}$$

or



define τ :



Definition 4.1.3 (Composition of σ and τ). $\sigma \circ \tau(i) = \sigma(\tau(i))$

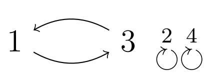
$\sigma \circ \tau(1) = \sigma(3) = 3$

$\sigma \circ \tau(2) = \sigma(1) = 2$

$\sigma \circ \tau(3) = \sigma(4) = 1$

$\sigma \circ \tau(4) = \sigma(2) = 4$

still a bijection! $\sigma \circ \tau$ *is a bijection.*

Expression $\sigma \circ \tau$: 

Notation 4.1.1 ("cycles").

$$\begin{aligned} \sigma &= (124)(3) = (241)(3) \\ \tau &= (1342) \\ \sigma \circ \tau &= (13)(2)(4) \end{aligned}$$

Definition 4.1.4. S_n *is the set of all permutations of* $\{1, \cdots, n\}$.

Definition 4.1.5. *Given* $r \in S_n$, $A, B \in M_n(\mathbb{F})$, *we write* $A \xrightarrow{R:\sigma} B$ *to mean that* B *is obtained from* A *by moving*

$$\mathrm{Row}_i(A) + \mathrm{Row}_\sigma(i)(B), \qquad \textit{for } 0 \le i \le n$$

Example: $\sigma = (124)$

$$A = \begin{pmatrix} -- & r_1 & -- \\ -- & r_2 & -- \\ == & r_3 & -- \\ -- & r_4 & -- \end{pmatrix} \xrightarrow{R:\sigma} \begin{pmatrix} -- & r_4 & -- \\ -- & r_1 & -- \\ == & r_3 & -- \\ -- & r_2 & -- \end{pmatrix} = B$$

$\sigma(a) = 1 \; a = \sigma^{-1}(1)$

B can also be written as

$$\begin{pmatrix} -- & r_{\sigma^{-1}}(1) & -- \\ -- & r_{\sigma^{-1}}(2) & -- \\ == & r_{\sigma^{-1}}(3) & -- \\ -- & r_{\sigma^{-1}}(4) & -- \end{pmatrix}$$

Definition 4.1.6. *Given $\sigma \in S_n$, the permutation matrix associated to σ is the matrix P_σ which is obtained from I_n by $\xrightarrow{R:\sigma}$.*

$$I_n \xrightarrow{R:\sigma} P_\sigma$$

$$I_n = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} \xrightarrow{R:\sigma} P_\sigma = \begin{pmatrix} -- & e_{\sigma^{-1}}(1) & -- \\ -- & e_{\sigma^{-1}}(2) & -- \\ & \vdots & \\ -- & e_{\sigma^{-1}}(n) & -- \end{pmatrix}$$

Recall: $\sigma = (124)(3)$

$$P_\sigma = \begin{pmatrix} -- & e_{\sigma^{-1}}(1) & -- \\ -- & e_{\sigma^{-1}}(2) & -- \\ == & e_{\sigma^{-1}}(3) & -- \\ -- & e_{\sigma^{-1}}(4) & -- \end{pmatrix} = \begin{pmatrix} -- & e_4 & -- \\ -- & e_1 & -- \\ == & e_3 & -- \\ -- & e_2 & -- \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} | & | & | & | \\ e_2 & e_4 & e_3 & e_1 \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | & | \\ e_{\sigma(1)} & e_{\sigma(2)} & e_{\sigma(3)} & e_{\sigma(4)} \\ | & | & | & | \end{pmatrix}$$

P_σ is a very special matrix

Each row and column has exactly one "1"

For this matrix $\mathrm{Row}_i(P_\sigma) = e_{\sigma^{-1}}(i)$ and $\mathrm{Col}_i(P_\sigma) = e_\sigma(i)$

$P_{\sigma^{-1}} = (P_\sigma)^t$

$P_\sigma e_j = e_\sigma(j)$

Theorem 4.1.1. $\sigma \in S_n$, $A \in M_{n\times n}(\mathbb{F})$

1. *$P_\sigma A$ is the result of applying σ to the rows of $A : A \xrightarrow{R:\sigma} P_\sigma A$*
2. *AP_σ is the result of applying σ to the column of $A : A \xrightarrow{C:\sigma} AP_\sigma$*

Proof. (2) Suppose $(\sigma^{-1})^{-1} = \tau$ (bijection), $(\sigma^{-1}) = (\tau^{-1}) \to \sigma = \tau$, $1 \xrightarrow{\sigma} i \xrightarrow{\sigma^{-1}} 1$.

$$A = \begin{pmatrix} | & | & & | \\ C_1 & C_2 & \cdots & C_n \\ | & | & & | \end{pmatrix} \xrightarrow{\sigma^{-1}} \begin{pmatrix} | & | & & | \\ C_{(\sigma^{-1})^{-1}}(1) & C_{(\sigma^{-1})^{-1}}(2) & \cdots & C_{(\sigma^{-1})^{-1}}(n) \\ | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & & | \\ C_\sigma(1) & C_\sigma(2) & \cdots & C_\sigma(n) \\ | & | & & | \end{pmatrix} = B$$

We want $B = AP_\sigma$, $\mathrm{Col}_j(AP_\sigma) = A \cdots \mathrm{Col}_j(P_\sigma) = A \cdot e_\sigma(j) = A_\sigma(j)$

$\mathrm{Col}_j(B) = \mathrm{Col}_j$. ◻

Corollary 4.1.1. $P_\sigma P_\tau = P_{\sigma\tau}$

Proof.

$$I_n \xrightarrow{R:\sigma\tau} P_{\sigma\tau} = \begin{pmatrix} -- & r_{\sigma\tau^{-1}}(1) & -- \\ -- & r_{\sigma\tau^{-1}}(2) & -- \\ & \vdots & \\ -- & r_{\sigma\tau^{-1}}(n) & -- \end{pmatrix}$$

$$I_n \xrightarrow{R:\tau} P_\tau = \begin{pmatrix} -- & r_{\tau^{-1}}(1) & -- \\ -- & r_{\tau^{-1}}(2) & -- \\ & \vdots & \\ -- & r_{\tau^{-1}}(n) & -- \end{pmatrix} \xrightarrow{R:\sigma} \begin{pmatrix} -- & r_{\sigma^{-1}\tau^{-1}}(1) & -- \\ -- & r_{\sigma^{-1}\tau^{-1}}(2) & -- \\ & \vdots & \\ -- & r_{\sigma^{-1}\tau^{-1}}(n) & -- \end{pmatrix}$$

$(\sigma\tau)^{-1} = \sigma^{-1}\tau^{-1}$ ◻

4.2 Feb 26

Definition 4.2.1. *If A, B matrices of some size, write $A \rightsquigarrow B$ to mean we can obtain B from A by some sequence of elements row and/or solumn operations.*

Example: $\mathbb{F} = \mathbb{R}$

$$A = \begin{pmatrix} 2 & 4 & 1 & 0 \\ -1 & -2 & 1 & 3 \\ 3 & 6 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A \xrightarrow{R_1 \rightleftarrows R_2} \begin{pmatrix} -1 & -2 & 1 & 3 \\ 2 & 4 & 1 & 0 \\ 3 & 6 & 0 & -3 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix} -1 & -2 & 1 & 3 \\ 0 & 0 & 3 & 6 \\ 3 & 6 & 0 & -3 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 + 3R} \begin{pmatrix} -1 & -2 & 1 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 3 & 6 \end{pmatrix} \xrightarrow{C_1 \leftarrow (-1)C_1} \begin{pmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 3 & 6 \end{pmatrix} \xrightarrow{C_2 \rightleftarrows C_3} \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 3 & 0 & 6 \\ 0 & 3 & 0 & 6 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 + (-1)R_2} \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{1}{3}R_2} \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{C_3 \leftarrow C_3 + 2C_1} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{C_4 \leftarrow C_4 + (-2)C_1} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 + (-1)R_2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{C_1 \leftarrow C_1 + (-1)C_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For $i = 1, \cdots, , 11$, let E_i be the elementary corresponding to step i,

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\underbrace{E_{10}E_7E_6E_3E_2E_1}_{P} A \underbrace{E_4E_5E_8E_9E_{11}}_{Q} = D$. P and Q are invertible.

Theorem 4.2.1. *Suppose $A, B \in M_{m \times n}(\mathbb{F})$, if $A \rightsquigarrow B$ then $\exists$ invertible $P \in M_{m \times m}(\mathbb{F})$ and invertible $Q \in M_{n \times n}(\mathbb{F})$ s.t. $PAQ = B$.*

Proof. Pick a sequence of elementary row/col operations taking A to B. Let Q_1, Q_k be the row operations in this sequence, $Q'_1, \cdots, Q'_2$ be the column operations.

Let E_i be the elementary matrix corresponding to Q_i. $I_m \xrightarrow{Q_i} E_i$.

Let E_j be the elementary matrix corresponding to Q'_j.

$P = E_k \cdots, E_2 E_1$
$Q = E_1' E_2' \cdots E_e'$

Then $PAQ = B$. $\square$

Theorem 4.2.2. $\forall A \in M_{m\times n}(\mathbb{F})$, $\exists D \in M_{m\times n}(\mathbb{F})$ *of the form*

$$D = \left(\begin{array}{c|c} I_r & O_{r\times(n-r)} \\ \hline O_{(m-r)\times r} & O_{(m-r)\times(n-r)} \end{array}\right) \qquad (\textit{for some } r \geq 0)$$

s.t. $A \rightsquigarrow D$.

Proof. If $A = O_{m\times n}$, done. Else, A has a nonzero entry somewhere, use Type (1) operations, can move this entry to $(1,1)$ position, making this entry = 1, with a type (2) operations, apply type 3 operations to get

$$\rightsquigarrow \left(\begin{array}{c|cccc} 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right)$$

$\square$

Corollary 4.2.1. $\forall A$, $\exists$ *invertible* P, Q *s.t.*

$$PAQ = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$$

Proof. Find P and Q,

1. find $E_1, \cdots, E_k, E_1', \cdots, E_e'$ multiply...

$\square$

4.3 Feb 28

Corollary 4.3.1. *If $A \rightsquigarrow B$, then*

1. $B \rightsquigarrow A$
2. $A^t \rightsquigarrow B^t$

Proof. 1. Row and Column operations are reversible

2. Change row operations to column operations and vice versa

$\square$

Definition 4.3.1. *Let A be an $m \times n$ matrix over $\mathbb{F}$.*

1. *The row space of A is the span in $\mathbb{F}^n$, of the rows of A.*

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

2. *The column space of A is the span in $\mathbb{F}^n$ of the columns of A.*

   Recall: $L_A : \mathbb{F}^n \to \mathbb{F}^n$ $R(L_A) =$ *the column space of* A

3. *The null space of A, denoted $N(A)$ is $N(L_A) = \{x \in \mathbb{F}^n : Ax = 0\}$, a subsapce $\mathbb{F}^n$*

   Recall: $\underbrace{\dim(\textit{column space of } A)}_{rank(L_A)} + \underbrace{\dim(N(A))}_{nullity(L_A)} = \dim \mathbb{F}^n = n.$

Definition 4.3.2. ***rank of*** *A is* $\operatorname{rank}(L_A)$

nullity of *A is* $\operatorname{nullity}(L_A)$

Theorem 4.3.1. *If $A \in M_{m\times n}(\mathbb{F})$ and $Q \in M_{m\times n}(\mathbb{F})$, with Q invertible, then $R(L_AQ) = R(L_A)$.*

Proof. AQ is $m \times n$, L_{AQ} :

$$\begin{tikzcd} \mathbb{F}^n \arrow[dr] & & \mathbb{F}^m \arrow[ll] \\ & \mathbb{F}^n \arrow[ur] & \end{tikzcd}$$

$L_{AQ} = L_A \circ L_Q$, L_Q is an isomorphism hence is surjective.

$$\begin{aligned} R(L_{AQ}) =& \{L_{AQ} : x \in \mathbb{F}^n\} \\ =& \{L_A(L_Q(x)) : x \in \mathbb{F}^n\} \\ =& \{L_A(y) : y \in \mathbb{F}\} && \text{(as } L_Q \text{ is surjective)} \\ =& R(L_A) \end{aligned}$$

$\square$

Corollary 4.3.2. *IF $A \rightsquigarrow B$, entirely by column operations, then A and B have the same column space.*

Proof. $A \rightsquigarrow B$ by column operations $\Rightarrow B = AQ$, for some invertible Q.

Then,

$$\begin{aligned} \text{Column Space } of\ B =& \text{Column Space } of\ AQ \\ =& R(L_{AQ}) \\ =& R(L_A) && \text{(Thm 1)} \\ =& \text{Column Space } of\ A \end{aligned}$$

□

Corollary 4.3.3. *IF $A \rightsquigarrow B$, entirely by row operations, then A and B have the same row space.*

Proof. $A \rightsquigarrow B$ by row operations

$\Rightarrow A^t \rightsquigarrow B^t$ by column operations

$\Rightarrow A^t, B^t$ have same column space

$\Rightarrow A, B$ have same row space □

Lemma 4.3.1. *Suppose V is finite dimensional, $T : V \cong V'$, and W is a subspace of V.*

Let $W' = \{T(w) : w \in W\}$, a subspace of V', then $\dim(W) = \dim(W')$.

Proof. Let B_w be a basis for $W = \{w_1, \ldots, w_k\}$.

Claim: $\{T(w_1), \ldots, T(w_k)\}$ is a basis for W.

$$\begin{aligned} & a_1T(w_1) + \ldots + a_kT(w_k) =& 0 \\ \Rightarrow\quad & T(a_1w_1 + \ldots + a_kw_k) =& 0 \\ \Rightarrow\quad & a_1w_1 + \ldots + a_kw_k \in N(T) =& \{0\} \\ \Rightarrow\quad & a_1w_1 + \ldots + a_kw_k =& 0 \\ \Rightarrow\quad & a_1 = \ldots = a_k =& 0 \end{aligned}$$

□

Theorem 4.3.2. *Suppose $A \in M_{m\times n}(\mathbb{F})$. $P \in M_{m\times m}(\mathbb{F})$, P invertible. Then* $\dim(\text{Column Space } of\ A) = \dim(\text{Column Space } of\ PA)$. *i.e.* $\text{rank}(A) = \text{rank}(PA)$.

Proof. $L_{PA} : \mathbb{F}^n \to \mathbb{F}^m$,

Let $W = R(L_A)$, let $W' = \{L_P(y) : y \in W\}$.

We know $\dim(W) = \dim(W')$. (lemma)

Note:

$$\begin{aligned} W' =& \{L_P(y) : y \in W\} \\ =& \{L_P(L_A(x)) : x \in \mathbb{F}^n\} \\ =& \{L_{PA}(x) : x \in \mathbb{F}^n\} \\ =& R(L_{PA}) \end{aligned}$$

So $\dim(\text{Column Space } of\ A) = \dim(R(L_A)) = \dim(R(L_{PA})) = \dim(\text{Column Space } of\ PA)$. □

Corollary 4.3.4. *IF* $A \rightsquigarrow B$ *entirely by row operations then* $\operatorname{rank}(A) = \operatorname{rank}(B)$.

Proof. $A \rightsquigarrow B$ by row operations, $\Rightarrow$ $B = PA$ for some invertible $P \Rightarrow \operatorname{rank}(A) = \operatorname{rank}(PA) = \operatorname{rank}(B)$. □

Corollary 4.3.5. *If* $A \rightsquigarrow B$ *then* $\operatorname{rank}(A) = \operatorname{rank}(B)$.

Proof. If $A \rightsquigarrow B$, then $B = PAQ$ then $\operatorname{rank}(A) = \operatorname{rank}(PA) = \operatorname{rank}(PAQ)$. □

Corollary 4.3.6. *If* $A \rightsquigarrow \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ *then* $\operatorname{rank}(A) = r$.

Corollary 4.3.7. *For any* A*,* $\operatorname{rank}(A) = \operatorname{rank}(A^t)$. *i.e.* Column Space *of* A *and row space of* A *have same dimension.*

Proof.

$$A \rightsquigarrow \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right) \qquad r = \operatorname{rank} A$$

$$A^t \rightsquigarrow \left(\begin{array}{c|c} I_r^t & 0 \\ \hline 0 & 0 \end{array}\right) \qquad r = \operatorname{rank} A^t$$

□

4.4 March 2

Suppose $A \in M(\mathbb{F})$, A can be transformed:

$$A^t \rightsquigarrow \left(\begin{array}{c|c} I_r^t & 0 \\ \hline 0 & 0 \end{array}\right) \qquad r = \operatorname{rank} A^t$$

$r = \operatorname{rank}(A)$.

Theorem 4.4.1 (Invertible Matrix Theorem). *For $A \in M_{n\times n}(\mathbb{F})$, TFAE*

1. *A is invertible*
2. $\operatorname{rank}(A) = n$
3. *A can be written as a product of elementary matrices.*
4. *$A \rightsquigarrow I_n$*
5. *$A \rightsquigarrow I_n$ by row operations*

Proof. From 5 to 1,

$I_n = E_k \cdots E_1 A$

$$I_n = EA \Rightarrow AE = I_n$$

So A is invertible, and $A^{-1} = E$. Hence, E is invertible.

$\Rightarrow E^{-1} I_n = E^{-1}(EA) \Rightarrow E^{-1} = A$.

$E^{-1} = (E_k \cdots E_1)^{-1}$.

$$E^{-1} = (E_k \cdots E_1)^{-1} = E^{-1} \cdots E^{-1}$$

and each $^{-1}$ is an elementary operation, hence proves 3.

If $A \rightsquigarrow I_n = \left(\begin{array}{c|c} I_n & \cdots \\ \hline \cdots & \cdots \end{array}\right)$.

$1 \Rightarrow L_A$ is an isomorphism.

$\Rightarrow L_A$ is surjective.

$\Rightarrow R(L_A) = \mathbb{F}^n$

$\Rightarrow \dim(R(L_A)) = n \Rightarrow \operatorname{rank}(A) = n$.

4 to 1, 3, Assume (4), then $I_n = PAQ$, $\Rightarrow P^{-1} I_n Q^{-1} = P^{-1}(PAQ)Q^{-1}$,

$\Rightarrow P^{-1}Q^{-1} = A$ so A is invertible.

4 to 5, $A \rightsquigarrow I_n \Rightarrow I_n = (PA)Q$, P, Q invertible.

$QI_nQ^{-1} = Q(PAQ)Q^{-1}$.

$I_n = QPA \Rightarrow A \rightsquigarrow I_n$ by row operations.

$\square$

Suppose $A \rightsquigarrow I_n$ by row operations, then $I_n = E_k \cdots E_2 E_1 A$

$\Rightarrow I_n A^{-1} = E_k \cdots E_2 E_1 A A^{-1}$

$\Rightarrow A^{-1} = E_k \cdots E_2 E_1 I_n$.

Show: exactly the same sequence of row operations, transforming $A \rightsquigarrow I_n$ also transforms $I_n \rightsquigarrow A^{-1}$.

Algorithm 4.4.1. *To find A^{-1} (when it exists)*

1. *Form $n \times 2n$ matrix AI_n*
2. *Apply row operations transform to $(I_n \blacksquare)$ $\blacksquare$: will be A^{-1}.*

Example:

$$A = \begin{pmatrix} 1 & -3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -3 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftarrow R_1 + 3R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 3 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 & 1 & -3 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrows R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftarrow R_1 - R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 6 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & -3 \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 2 & -1 & 6 \\ 0 & 0 & 1 \\ -1 & 1 & -3 \end{pmatrix}$$

4.5 March 4

Consier a system of m linear equations in n variables.

$$S \begin{cases} a_{11}x_1 + a_{12}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_1 + \cdots + a_{2n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_1 + \cdots + a_{nn}x_n = b_1 \end{cases}$$

$a_{ij}, b_i \in$ some field $\mathbb{F}$.

Then we want solutions $x = (x_1, \cdots, x_n) \in \mathbb{F}^n$.

Can write (S) as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

or compactly as $AX = b$.

A is the coefficient matrix of (S).

$A \in M_{m\times n}(\mathbb{F})$, $b \in \mathbb{F}^m$, the RHS vector.

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is a vector of formal variables.

Solutions: vectors $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$, such that $Ax = b$.

(S) is homogenous if $b = 0$ else, (S) is nonhomogeneous.

If $b \neq 0$, then the system $AX = 0$ is the homogeneous system associated to $AX = b$.

The solution set to $AX = b$ is $\{x \in \mathbb{F}^n : Ax = b\} =: Sol(AX = b)$.

Definition 4.5.1. *$AX = b$ is consistent if $Sol(AX = b) \neq \varnothing$, else $AX = b$ is inconsistent.*

Theorem 4.5.1. *Let $A \in M_{m\times n}(\mathbb{F})$, $b \in \mathbb{F}^m$, consider the system $AX = b$,*

1. *If $b = 0$, then $Sol(AX = 0) = N(A)$.*
2. *$AX = b$ is consistent $\Leftrightarrow$ $b \in$ column space of A $(= R(L_A))$.*
3. *If $AX = b$ is consistent, then $Sol(AX = b)$ is a translation of $N(A)$, i.e. $Sol(AX = b) = u + N(A)$, where u can be any solution to $AX = b$.*

Proof. (1) For $x \in \mathbb{F}^n$,

$$\begin{aligned} & x \in Sol(AX = 0) \\ \Leftrightarrow \quad & Ax = 0 \\ \Leftrightarrow \quad & L_A(x) = 0 \\ \Leftrightarrow \quad & x \in N(L_A) = N(A) \end{aligned}$$

(2) $AX = b$ is consistent

$$\begin{aligned} & AX = b \text{ is consistent} \\ \Leftrightarrow \quad & Sol(AX = b) \neq \varnothing \\ \Leftrightarrow \quad & \exists x \in \mathbb{F}^n . Ax = b \\ \Leftrightarrow \quad & \exists x \in \mathbb{F}^b, L_A(x) = b \\ \Leftrightarrow \quad & b \in R(L_A) = \text{Column Space } of \ A \end{aligned}$$

(3) Assume $AX = b$ is consistent, pick a solution, say $u \in \mathbb{F}^n$, (So $Au = b$).

I'll prove that $Sol(AX = b) \subseteq u + N(A)$. So $Ax = b = Au$, so $A(x - u) = Ax - Au = 0$.

$\Rightarrow x - u \in N(A)$

$\Rightarrow x = u + (X - u) \Rightarrow x \in u + N(A)$.

$u + N(A) \subseteq Sol(AX = b)$, suppose $x \in u + N(A)$,

$$\begin{aligned} \Rightarrow \quad x =& u + v \\ \Rightarrow \quad Ax =& A(u + v) \\ =& Au + Av \\ =& b + 0 = b \\ \Rightarrow \quad x \in& Sol(AX = b). \end{aligned}$$

□

Goal: Given $AX = b$,

1. Determine whether $AX = b$ is consistent
2. If it is, then find one solution u and find basis $\{x_1, \cdots, x_k\}$ for $N(A)$. Then $Sol(AX = b) = u + \text{span}(\{x_1, \cdots, x_k\}) = \{u + c_1 x_1 + \cdots + c_k x_k, c_1 \cdots c_k \in \mathbb{F}\}$.

Definition 4.5.2. *Suppose $A \in M_{m \times n}(\mathbb{F})$, $b \in \mathbb{F}^m$, the $m \times (n + 1)$ matrix $(A|b)$ is the* ***augmented matrix*** *of $AX = b$.*

Lemma 4.5.1. *Given $A \in M_{m \times n}(\mathbb{F})$, $b \in \mathbb{F}^m$, if $(A|b) \rightsquigarrow (A'|b')$ using only row operations, then*

$$Sol(AX = b) = Sol(A'X = b')$$

Proof. Suppose $(A|b) \rightsquigarrow (A'|b')$ via row operations,

so $\exists$ invertible $P \in M_{m\times m}(\mathbb{F})$ s.t. $P(A|b) = (A'|b')$. $\Rightarrow PA = A' \qquad and \qquad Pb = b'$ and $A = P^{-1}A'$ and $b = P^{-1}b'$.

Claim:

$$Sol(AX = b) \subseteq Sol(A'X = b')$$

Let $x \in Sol(AX = b)$, i.e. $Ax = b$,

$\Rightarrow (PA)x = Pb$

$\Rightarrow A'x = b'$.

i.e. $x \in Sol(A'X = b')$. $\square$

Definition 4.5.3. *A matrix is in* ***Reduced Row Echelon Form*** *(RREF) if all of the following hold:*

1. *If a row has a nonzero entry, the 1st such = 1. (called the* ***leading one*** *of the row)*
2. *If a column contains a leading one, all other entries in that column = 0.*
3. *Lower (nonzero) columns have leadings further to right.*
4. *All zero rows of any are at bottom*

Non-Examples of RREF:

$$\begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example:

$$\begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

4.6 RREF and Solving Linear Equations - March 6

Theorem 4.6.1. *Every $A \in M_{m\times n}(\mathbb{F})$ can be converted to a matrix in RREF, by a sequence of elementary row operations.*

Proof. If $A = O_{m\times n}$, done.

Else, pick first column, say $\mathrm{Col}_j(A)$, which is nonzero, using row operations, move nonzero entry in column j, to position $(1, j,)$, change it to 1, clear all other entries in $\mathrm{Col}j$,

get

$$\rightsquigarrow A' = \left(\begin{array}{cccc|ccc} 0 & \cdots & 0 & 1 & * & \cdots & * \\ \hline 0 & \cdots & 0 & 0 & & & \\ & & & \vdots & & B & \\ 0 & \cdots & 0 & 0 & & & \end{array}\right)$$

Next, if $B = O_{(m-1)\times(n-j)}$, we are done, else find $1st$ column of A, say $\mathrm{Col}_j(A)$, which meets a nonzero entry of B, pick a nonzero entry of B in that column, move it to position $(2, j_2)$, change it to 1, clear all other entries in that column.

$$\rightsquigarrow A'' = \left(\begin{array}{cccccccccc} 0 & \cdots & 1 & * & \cdots & * & 0 & * & \cdots & * \\ \hline 0 & \cdots & 0 & * & \cdots & 0 & 1 & * & \cdots & * \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & & & \\ & & & & & & & & & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & & & \end{array}\right)$$

Eventually stops, resulting matrix is in RREF. ∎

Proposition 4.6.1. *If R is in RREF, then* $\mathrm{rank}(R) = \#$ *of leading 1s.*

Example:

$$R = \begin{pmatrix} 1 & 0 & 2 & 0 & -3 \\ 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In general, if R is in RREF and has leading 1s in columns $j_1, \cdots, j_n$, then.

$$\mathrm{Col}_{j_1}(R) = e_1$$
$$\vdots$$
$$\mathrm{Col}_{j_r}(R) = e_r$$

and these column span Column Space of R, so $\mathrm{rank}(R) = r$.

4.6.1 Solving Linear Equations using matrix

Solving $AX = b$.

Form augmented matrix $(A|b) \overset{rowops}{\rightsquigarrow} \underbrace{(R|s)}_{in\ RREF}$.

$$\mathrm{Sol}(AX = b) = \mathrm{Sol}(RX = s)$$

Example: Say

$$(R|s) = \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -3 & s_1 \\ 0 & 1 & -1 & 0 & 4 & s_2 \\ 0 & 0 & 0 & 1 & -2 & s_3 \\ 0 & 0 & 0 & 0 & 0 & s_4 \end{array}\right)$$

Either $s_4 = 0$ or $s = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Column Space of R = $\mathrm{span}\{e_1, e_2, e_3\}$.

So $RX = s$ is consistent, $\Leftrightarrow$ $s_4 = 0$.

Assume $s_4 = 0$, write the equations

$$\begin{cases} x_1 + 2x_3 - 3x_5 = s_1 \\ x_2 - x_3 + 4x_5 = s_2 \\ x_4 - 2x_5 = s_3 \\ 0 = 0 \end{cases}$$

Variable $\sim$ Leading 1s: dependent variables, x_1, x_2, x_4

Other variables: free variables x_3, x_5,

Next, express every variable in terms of free variables

$$\begin{aligned} x_1 &= s_1 - 2x_3 + 3x_5 \\ x_2 &= s_2 + x_3 - 4x_5 \\ x_3 &= x_3 \\ x_4 &= s_3 + 2x_5 \\ x_5 &= x_5 \end{aligned}$$

Rewrite as vector equation:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ 0 \\ s_3 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ -4 \\ 0 \\ 2 \\ 1 \end{pmatrix} \qquad s, t \in \mathbb{F}$$

If $s = t$, then we use u is one solution to $RX = s$.

Consider the homogeneous case: $RX = 0$,

$$\mathrm{Sol}(RX = 0) = \mathrm{Sol}(AX = 0) = N(A)$$

We see $N(A) = \text{span}\{v_1, v_2\}$.

$$\begin{aligned}\dim(N(A)) =& \text{nullity}(A)\\ =& n - \text{rank}(A)\\ =& n - (\#\text{ of leading 1s in} R)\\ =& \#\quad\text{of free variables}\end{aligned}$$

In genreal

$$(A|b) \overset{row\ ops}{\rightsquigarrow} \underbrace{(R|S)}_{RREF}$$

If $(R|s)$ has a row $(0\cdots 0|1)$, then $AX = b$ has no solution, otherwise we write equations corresponding to $RX = s$, and express all variables in terms of free variables.

Write in vector form, $x = u + s_1 v - 1 + \cdots + s_k v_k$, $k = \#$ of free variables $= n - \text{rank}(A) = \text{nullity}(A)$.

Proposition 4.6.2. *Given A, there is only one unique RREF R s.t. $A \overset{row\ ops}{\rightsquigarrow} R$.*

Proof. Understand what info R encodes. A4Q5b, if R is RREF for A, then R has a leading one in column j, if and only if the $\text{Col}_j(A) \notin \text{span}\{\text{Col}_1(A), \cdots, \text{Col}_{j-1}(A)\}$. A determines where leading 1s in R go.

$$A = \begin{pmatrix} | & | & & | \\ A_1 & A_2 & \cdots & A_s \\ | & | & & | \end{pmatrix} \rightsquigarrow R = \begin{pmatrix} 1 & 0 & 2 & 0 & -3 \\ 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} R_1 & R_1 & R_3 & \cdots \end{pmatrix}$$

$A_1 \notin \text{span}(\varnothing)$
$A_2 \notin \text{span}(A_1)$
$A_3 \notin \text{span}(A_1, A_2)$, $A_3 = 2A_1 - A_2$

It's true, Hint A4Q5(a). $\square$

5 Determinants

5.1 March 9

In 2×2 case:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det(A) = ad - bc \in \mathbb{F}(\text{or}|A|)$$

A is invertible $\iff \det(A) \neq 0$.

When $det(A) \neq 0$,

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
$$\det(AB) = \det(A)\det(B)$$

IN $n \times n$, we assign $(-1)^{i+j}$ to (i, j) position (of any $n \times n$ matrix).

Definition 5.1.1. *Suppose $A \in M_{n\times n}(\mathbb{F})$, $1 \leq i, j \leq n$, $\tilde{A}_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i, column j,*

*$\tilde{A}_{ij}$ is called the (i, j) **submatrix** of A.*

When dets *are defined,*

- $\det(\tilde{A}_{ij})$ *is the (i, j) **minor** of A*
- $(-1)^{i+j}\det(\tilde{A}_{ij})$ *is the (i, j) **cofactor** of A.*

Definition 5.1.2 (Determinants). *recursive on n, we use **cofactor expansion on** 1^{st} column,*

1. *If A is 1×1 ($A = (a)$), then $\det(A) = a$.*
2. *If A is $n \times n$, $n > 1$,*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$\begin{aligned} \det(A) =& a_{11}\det(\tilde{A}_{11}) - a_{21}\det(\tilde{A}_{21}) + \cdots + (-1)^{1+n} a_{n1}\det(\tilde{A}_{n1}) \\ =& \sum_{i=1}^{n} a_{i1} \underbrace{(-1)^{i+1}\det(\tilde{A}_{i1})}_{(i,1) \quad \textit{cofactor of} A} \end{aligned}$$

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \tilde{A}_{11} = (a_{22}) \qquad \tilde{A}_{21} = (a_{12})$$
$$\det(A) = a_{11} \cdot \det(\tilde{A}_{11}) - a_{21}\det(\tilde{A}_{21}) a_{11}a_{22} - a_{21}a_{12}$$

Lemma 5.1.1. *If $A \in M_{n\times n}(\mathbb{F})$ is upper-triangle, say*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \qquad \textit{then} \qquad \det A = \prod_{i=1}^{n} a_{ii}$$

Proof. By induction on n,

Base Case: $n = 1, A = (a_{11}), \det(A) = a_{11} = \prod_{i=1}^{1} a_{ii}$ ✓.

Inductive Step: Assume $n > 1$, by definition,

$$\begin{aligned} \det A =& a_{11}\det(\tilde{A}_{11}) - 0\cdot\det(\tilde{A}_{21}) + 0\cdot\det(\tilde{A}_{31}) - \cdots \\ =& a_{11}\cdot\det(\tilde{A}_{11}) \\ =& a_{11}(\prod_{i=2}^{n} a_{ii}) && \text{(by IH)} \\ =& \prod_{i=1}^{n} a_{1i} \end{aligned}$$

□

Corollary 5.1.1. $\det(I_n) = 1$.

Theorem 5.1.1. *If $A \in M_{n\times n}(\mathbb{F})$ hsa a zero row, then* $\det(A) = 0$.

Proof. By induction on n,

$n = 1$, then A is the zero matrix, $\det(A) = 0$.

$n > 1$, assume its $\mathrm{Row}_{i_0}(A) = (0, 0, \ldots, 0)$, then

$$A = a_{11}\det(\tilde{A}_{11}) - a_{21}\det(\tilde{A}_{21}) + (-1)^{i_0+1}\det(\tilde{A}_{i_01}) + \cdots + (-1)^{n+1}a_{n1}\det(\tilde{A}_{n1})$$

Claim: $\forall i \neq i_0$, $\tilde{A}_{i1}$ also has a zero row, by induction, $det(\tilde{A}_{i1}) = 0$, $\forall i \neq i_0$. □

Theorem 5.1.2. *If $A \in M_{n\times n}(\mathbb{F})$ has a zero column, then* $\det(A) = 0$.

Proof. $n = 1$,

$n > 1$, *case 1:* $\mathrm{Col}_1(A) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$A = \begin{pmatrix} 0 & a_{12} & \cdots \\ 0 & a_{22} & \cdots \\ \vdots & \vdots & \\ 0 & a_{m2} & \cdots \end{pmatrix}$$

Then

$$\det(A) = 0 \cdot \det(\tilde{A}_{11}) - 0 \cdot \det(\tilde{A}_{21}) + 0 \cdots \det(\tilde{A}_{31}) - \cdots = 0$$

case 2:

$$\mathrm{Col}_j(A) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad j > 1$$

$$A = \begin{pmatrix} a_{11} & \dots & 0 & \dots & a_{1n} \\ a_{11} & \dots & 0 & \dots & a_{2n} \\ \vdots & \dots & 0 & \dots & \\ a_{11} & \dots & 0 & \dots & a_{mn} \end{pmatrix}$$

$$\det(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + \cdots$$

each $\tilde{A}_{i1}$ itself has a zero column. $\square$

5.2 March 11

Theorem 5.2.1. *If $A \in M_{n\times n}(\mathbb{F})$, and A has two equal* ***adjacent*** *rows, then,* $\det(A) = 0$.

Proof. Suppose rows i_0, $i_0 + 1$ are equal.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ r_1 & r_1 & \cdots & r_n \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$

$$\begin{aligned} \det(A) =& a_{11}\det(\tilde{A}_{11}) - a_{21}\det(\tilde{A}_{21}) + \cdots \\ =& (-1)^{i_0+1} r_1 \det(\tilde{A}_{i_0,1}) + (-1)^{i_0+1} r_1 \det(\tilde{A}_{i_0,1}) \\ =& 0 \end{aligned}$$

Observe: If $i \neq i_0, i_0 + 1$, then $\tilde{A}_{i1}$ has 2 equal adjacent rows so $\det(\tilde{i}1) = 0$ by IH. Also $\tilde{A}_{i_0,1} = \tilde{A}_{i_0+1,1}$. □

Theorem 5.2.2. *For fixed n, $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ is "linear in each row" i.e. for each $i_0 \in \{1, \cdots, n\}$, $\forall u_1, \cdots, u_n \in \mathbb{F}^n$, $\forall r, s \in \mathbb{F}^n$, $\forall c \in \mathbb{F}$,*

$$\det\begin{pmatrix} -- & u_1 & -- \\ & \vdots & \\ -- & r+s & -- \\ & \vdots & \\ -- & u_n & -- \end{pmatrix} = \det\begin{pmatrix} -- & u_1 & -- \\ & \vdots & \\ -- & r & -- \\ & \vdots & \\ -- & u_n & -- \end{pmatrix} + \det\begin{pmatrix} -- & u_1 & -- \\ & \vdots & \\ -- & s & -- \\ & \vdots & \\ -- & u_n & -- \end{pmatrix}$$

and

$$\det\begin{pmatrix} -- & u_1 & -- \\ & \vdots & \\ -- & cr & -- \\ & \vdots & \\ -- & u_n & -- \end{pmatrix} = c\det\begin{pmatrix} -- & u_1 & -- \\ & \vdots & \\ -- & r & -- \\ & \vdots & \\ -- & u_n & -- \end{pmatrix}$$

Proof. By example, $n = 4$, $i_0 = 3$,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ r_1 & r_2 & r_3 & r_4 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \qquad B = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ s_1 & s_2 & s_3 & s_4 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$$C = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ r_1 + s_1 & r_2 + s_2 & r_3 + s_3 & r_4 + s_4 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

Claim: $\det C = \det A + \det B$.

$$\tilde{C}_{11} = \begin{pmatrix} a_{22} & a_{23} & a_{24} \\ r_2 + s_2 & r_3 + s_3 & r_4 + s_4 \\ a_{42} & a_{43} & a_{44} \end{pmatrix}$$
$$\tilde{A}_{11} = \begin{pmatrix} & same & \\ r_2 & r_3 & r_4 \\ & same & \end{pmatrix}$$
$$\tilde{B}_{11} = \begin{pmatrix} & same & \\ s_2 & s_3 & s_4 \\ & same & \end{pmatrix}$$

By induction, $\det(\tilde{C}_{11}) = \det(\tilde{A}_{11}) + \det(\tilde{B}_{11})$, similarly,

$$\begin{aligned} \det(\tilde{C}_{21}) &= \det(\tilde{A}_{21}) + \det(\tilde{B}_{21}) \\ \det(\tilde{C}_{41}) &= \det(\tilde{A}_{41}) + \det(\tilde{B}_{41}) \end{aligned}$$

so,

$$\det(C) = a_{11}\det(\tilde{C}_1) - a_{21}\det(\tilde{C}_{21}) + (r_1 + s_1)\det(\tilde{C}_{31}) - a_{41}\det(\tilde{C}_{41}) = \det A + \det B$$

$\square$

Theorem 5.2.3. *Suppose* $A \in M_{n\times n}(\mathbb{F})$*, then* $A \overset{R_i \leftarrow R_i + cR_j}{\longrightarrow} B$*, where* $j = i \pm 1$*, then* $\det A = \det B$.

Proof. Assuem $j = i + 1$, let $r = \text{Row}_i(A)$, $s = \text{Row}(i+1)(A)$,

$$A = \begin{pmatrix} -- & u_1 & -- \\ & \vdots & \\ -- & r & -- \\ -- & s & -- \\ & \vdots & \\ -- & u_n & -- \end{pmatrix} \qquad B = \begin{pmatrix} -- & u_1 & -- \\ & \vdots & \\ -- & r + cs & -- \\ -- & s & -- \\ & \vdots & \\ -- & u_n & -- \end{pmatrix}$$

Use linearity in row i,

$$\det B = \det \begin{pmatrix} -- & u_1 & -- \\ & \vdots & \\ -- & r & -- \\ -- & s & -- \\ & \vdots & \\ -- & u_n & -- \end{pmatrix} + c\det \begin{pmatrix} -- & u_1 & -- \\ & \vdots & \\ -- & s & -- \\ -- & s & -- \\ & \vdots & \\ -- & u_n & -- \end{pmatrix} = \det A$$

$\square$

Theorem 5.2.4. *Suppose* $A \in M_{n\times n}(mF)$, $1 \le i \le n-1$, *and* $A \overset{R_1 \leftrightharpoons R_{i+1}}{\longrightarrow} B$, *then* $\det B = -\det A$.

Proof.

$$A = \begin{pmatrix} & \vdots & \\ -- & r & -- \\ -- & s & -- \\ & \vdots & \end{pmatrix}$$

so

$$B = \begin{pmatrix} & \vdots & \\ -- & s & -- \\ -- & r & -- \\ & \vdots & \end{pmatrix}$$

$$\det B = \det \begin{pmatrix} & \vdots & \\ -- & s & -- \\ -- & r & -- \\ & \vdots & \end{pmatrix} = \det \begin{pmatrix} & \vdots & \\ -- & s-r & -- \\ -- & r & -- \\ & \vdots & \end{pmatrix} = \det \begin{pmatrix} & \vdots & \\ -- & s-r & -- \\ -- & r+(s-r) & -- \\ & \vdots & \end{pmatrix}$$

$$= \det \begin{pmatrix} & \vdots & \\ -- & s-r & -- \\ -- & s & -- \\ & \vdots & \end{pmatrix} = \begin{pmatrix} & \vdots & \\ -- & s-r-s & -- \\ -- & s & -- \\ & \vdots & \end{pmatrix} = \det \begin{pmatrix} & \vdots & \\ -- & -r & -- \\ -- & s & -- \\ & \vdots & \end{pmatrix}$$

$$= (-1)\det \begin{pmatrix} & \vdots & \\ -- & r & -- \\ -- & s & -- \\ & \vdots & \end{pmatrix} = \det(A)$$

□

Theorem 5.2.5. *If* A *has* 2 *equal rows, then* $\det A = 0$.

Proof. Suppose $A = \begin{pmatrix} & \vdots & \\ -- & r & -- \\ & \vdots & \\ -- & r & -- \\ & \vdots & \end{pmatrix}$, By a sequence of adjacent row switches, $A \rightsquigarrow A' = \begin{pmatrix} & \vdots & \\ -- & r & -- \\ -- & r & -- \\ & \vdots & \end{pmatrix}$,
By theorem 4.8, $\det A' = \pm \det A = 0$, by theorem 4.5,

□