

# Math 148 Notes

velo.x

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# 1 INTEGRATION, SUMMATION

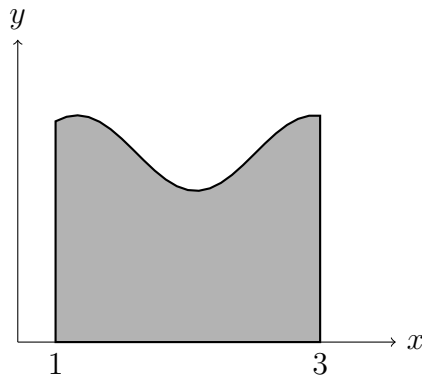
MOTIVATION: area, let  $a < b$  in  $\mathbb{R}$ , and let  $f : [a, b] \rightarrow [0, \infty]$ , let

$$S_f = \{(x, y) : 0 \leq y \leq f(x), x \in [a, b]\} ("subgraph")$$

IDEA: area of rectangel = height \* width

1.

Figure 1: The area under the function  $\frac{1}{x}$  is  $\log x$



2. approximate  $S_f$  by rectangle from below 13:43 JAN 6,

$$j = 1, \dots, 4, m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$$

$$\sum_{j=1}^4 m_{j-1}(x_j - x_{j-1}) \leq \text{area}(S_f)$$

3. approximate  $S_f$  by rectangle from above,  $M_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\}$

$$\text{area} \leq \sum_{j=1}^4 M_j(x_j - x_{j-1})$$

4. if we can arrange lower sum  $\approx$  upper sum, then we have some good approximation

## 1.1 Partition, Upper and Lower Sum

Let  $a < b \in \mathbb{R}$ ,  $f : [a, b] \in \mathbb{R}$ ,

**Definition 1.1.1 (Riemann-Darboux).**

A **partition** of  $[a, b]$  is any finite set of points including the endpoints.

$$P : \{x_0, x_1, \dots, x_n\} \text{ s.t. } a = x_0 < x_1 < \dots < x_n = b$$

often for convenience, we write  $P = \{a = x_0 < \dots < x_n = b\}$ .

A **Refinement** of  $P$  is any partition  $Q$  of  $[a, b]$  s.t.  $P \subseteq Q$ .

Now, fix a partition  $P$  of  $[a, b]$  and let  $f : [a, b] \rightarrow \mathbb{R}$  be bounded on  $[a, b]$ , i.e.  $\sup_{x \in [a, b]} |f(x)| \leq M < \infty$ .

Write  $P = \{a = x_0 < \cdots < x_n = b\}$ . For  $j = 1, \dots, n$ ,

$$\begin{aligned} m_j &= m_j(P) = \inf\{f(x) : x \in [x_{j-1}, x_j]\} \\ M_j &= M_j(P) = \sup\{f(x) : x \in [x_{j-1}, x_j]\} \end{aligned}$$

Notice that  $-M \leq m_k \leq M_j \leq M$  for each  $j$ , and these "inf", "sup" exist. (Using that  $\mathbb{R}$  is complete.)

**Definition 1.1.2.**

- **Lower Sum:**  $L(f, P) = \sum_{j=1}^n m_j \underbrace{(x_j - x_{j-1})}_{\text{width of } [x_{j-1}, x_j]}$
- **Upper Sum:**  $U(f, P) = \sum_{j=1}^n M_j(x_j - x_{j-1})$

**Remark:**

1. if  $f$  is not bounded, then at least one of  $L : (f, P)$  or  $U(f, P)$  cannot be defined.
2. we have  $L(f, P) \leq U(f, P)$ , Indeed, for each  $j = 1, \dots, n$ ,  $m_j \leq M_j$ . (exactly from definition),

$$L(f, P) = \sum_{j=1}^n m_j(x_j - x_{j-1}) \leq \sum_{j=1}^n M_j(x_j - x_{j-1}) = U(f, P)$$

**Lemma 1.1.1.** If  $P$  is a partition of  $[a, b]$ ,  $f : [a, b] \rightarrow \mathbb{R}$  is bounded, and  $Q$  is a refinement of  $P$ , then

$$L(f, P) \leq L(f, Q) \quad U(f, Q) \leq U(f, P)$$

*Proof.*

- Case 0:  $Q = P$  obvious
- Case 1:  $Q = P \cup \{q\}$  where  $q \notin P$ ,

write  $P = \{a = x_0 < \cdots, x_n = b\}$  so  $Q = \{a = x_0 < \cdots < x_{k-1} < q < x_k < \cdots < x_n = b\}$   
Then,

$$\begin{aligned} m_k(P) &= \inf\{f(x) : x \in [x_{k-1}, x_k]\} \quad [x_{k-1}, x_k] = [x_{k-1}, q] \cup [q, x_k] \\ &= \min\{\inf\{f(x) : x \in [x_{k-1}, q]\} \inf\{f(x) : x \in [q, x_k]\}\} \\ &= \min\{m_k(Q), m'_k(Q)\} \leq m_k(Q), m'_k(Q) \end{aligned}$$

Thus,

$$\begin{aligned}
L(f, P) &= \sum_{j=1}^m m_j(P)(x_j - x_{j-1}) \\
&= \sum_{j=1}^{k-1} m_j(P)(x_j - x_{j-1}) + m_k(P)(x_k - x_{k-1}) + \sum_{j=k+1}^n m_j(P)(x_j - x_{j-1}) \\
&\leq \sum_{j=1}^{k-1} m_j(P)(x_j - x_{j-1}) + m_k
\end{aligned}$$

- Case 2:  $Q = P \cup \{q_1, \dots, q_m\}$ ,  $q_1, \dots, q_m$  distinct,  $q_u \notin P$ , by case 1, we have

$$L(f, P) \leq L(f, P \cup \{q_1\}) \leq L(f, P \cup \{q_1, q_2\}) \leq \dots \leq L(f, P \cup \{q_1, \dots, q_m\}) = L(f, Q)$$

The case  $U(f, Q) \leq U(f, P)$  is similar.

□

**Corollary 1.1.1.** *let  $P, Q$  be any partition of  $[a, b]$  and  $f : [a, b] \rightarrow \mathbb{R}$  be bounded, then*

$$L(f, P) \leq U(f, Q)$$

*Proof.* We have  $P, Q \subseteq P \cup Q$ , i.e.  $P \cup Q$  refines each of  $P$  and  $Q$ . Thus,

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q)$$

□

## 1.2 Upper and Lower Sum

**Definition 1.2.1.** Given a bounded  $f : [a, b] \rightarrow \mathbb{R}$ , define

- **Lower Integral** :  $\int_a^b f = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}$
- **Upper Integral**:  $\int_a^b f = \inf\{U(f, Q) : Q \text{ is a partition of } [a, b]\}$

$$\int_a^b f = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\} \leq \inf\{U(f, Q) : Q \text{ is a partition of } [a, b]\} = \int_a^b f$$

We say that  $f$  is **integrable** on  $[a, b]$  provided that

$$\int_a^b f = \int_a^b f$$

In this case, we write  $\int_a^b f = \int_a^b f = \int_a^b f$

**Notation:** Write

$$\int_a^b f = \int_a^b f(x)dx = \int_a^b f(t)dt$$

**Non-Example 1:** not every bounded function is integrable.

$$\text{Define: } \chi_{\mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

Let  $P = \{0 = x_0 < \dots < x_n = 1\}$  be any partition of  $[0, 1]$ , We have that

- $\mathbb{Q}$  is dense in  $\mathbb{R}$ , so there is  $q_j \in \mathbb{Q} \cap (x_{j-1}, x_j), j = 1, \dots, n$
- $\mathbb{R} \setminus \mathbb{Q}$  is dense in  $\mathbb{R}$ , so there is  $r_j \in (\mathbb{R} \setminus \mathbb{Q}) \cap (x_{j-1}, x_j), j = 1, \dots, n$

$$0 \leq L(\chi_{\mathbb{Q}}, P) = \sum_{j=1}^n m_j(x_j - x_{j-1}) \leq \sum_{j=1}^n \chi_{\mathbb{Q}}(r_j)(x_j - x_{j-1}) = 0 \Rightarrow \int_0^1 = 0$$

Likewise,

$$1 \geq U(\chi_{\mathbb{Q}}, P) \geq \sum_{j=1}^n \chi_{\mathbb{Q}}(q_j)(x_j - x_{j-1}) = \sum_{j=1}^n (x_j - x_{j-1}) = 1 - 0 = 1 \Rightarrow \int_0^1 = 1$$

hence,

$$\int_0^1 \chi_{\mathbb{Q}} = 0 < 1 = \int_0^1 \chi_{\mathbb{Q}}$$

so  $\chi_{\mathbb{Q}}$  is not integrable on  $[0, 1]$ .

**Theorem 1.2.1 (Cauchy Criterion For Integrability).** *Let  $a < b \in \mathbb{R}$ ,  $f : [a, b] \rightarrow \mathbb{R}$  be bounded, then TFAE,*

1.  $f$  is integrable on  $[a, b]$
2. given  $\varepsilon > 0$ , there exists a partition  $P_\varepsilon$  of  $[a, b]$  s.t.

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

3. given  $\varepsilon > 0$ , there exists a partition  $P_\varepsilon$  of  $[a, b]$  so for every refinement  $P$  of  $P_\varepsilon$

$$U(f, P) - L(f, P) < \varepsilon$$

*Proof.* 1 to 2: we assume that

$$\sup\{L(f, P) : P \text{ partition of } [a, b]\} = \int_a^b f = \int_a^b \inf\{U(f, P) : P \text{ partition of } [a, b]\}$$

Let  $\varepsilon > 0$ , by first equality above, there is a partition  $P_1$  of  $[a, b]$  s.t.

$$\int_a^b f - \frac{\varepsilon}{2} < L(f, P_1)$$

and by the third equality, there is a partition  $P_2$  s.t.

$$\int_a^b f < U(f, P_2) - \frac{\varepsilon}{2}$$

Let  $P_\varepsilon = P_1 \cup P_2$ , a refinement of  $P_1$  and  $P_2$ , then since  $\int_a^b f = \int_a^b f = \int_a^b f$  we find

$$\begin{aligned} \int_a^b f - \frac{\varepsilon}{2} < L(f, P_1) &\leq L(f, P_\varepsilon) \leq U(f, P_\varepsilon) \leq U(f, P_2) < \int_a^b f + \frac{\varepsilon}{2} \\ \Rightarrow U(f, P_\varepsilon) - L(f, P_\varepsilon) &< \varepsilon \end{aligned}$$

2 to 3: we use the lemma.

If  $P_\varepsilon \leq P$ , we have

$$L(f, P_\varepsilon) \leq L(f, P) \leq U(f, P) \leq U(f, P_\varepsilon)$$

Hence,

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon \Rightarrow U(f, P) - L(f, P) < \varepsilon$$

3 to 2:  $P_\varepsilon \subseteq P_\varepsilon$  i.e.  $P_\varepsilon$  self-defines itself

2 to 1: Given  $\varepsilon > 0$ , there is  $P_\varepsilon$ , a partition of  $[a, b]$ , so  $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$ . We have

$$\begin{aligned} L(f, P_\varepsilon) &\leq \int_a^b f \leq \int_a^b f \leq U(f, P_\varepsilon) \quad \Rightarrow \quad \int_a^b f - \int_a^b f < \varepsilon \\ \int_a^b f &= \int_a^b f \quad \Rightarrow \quad f \text{ is integrable} \end{aligned}$$

□



### 1.3 Continuity and Inegrability

**Definition 1.3.1 (Continuous).**  $f : I \rightarrow \mathbb{R}$  is continuous if for every  $x$  in  $I$ , for every  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t. for all  $|x - x'| < \delta$ ,  $x' \in I$ ,

$$|f(x) - f(x')| < \varepsilon$$

this definition is local. first we choose  $x, \varepsilon$ , then  $\delta$ .

**Definition 1.3.2 (uniform Continuity).**  $f : I \rightarrow \mathbb{R}$  is uniformly continuous if for every  $\varepsilon > 0$ , there is  $\delta > 0$  so  $|f(x) - f(x')| < \varepsilon$  whenever  $|x - x'| < \delta$  for  $x, x' \in I$ .

**Proposition 1.3.1 (Sequential Test of Continuity).** Let  $f : I \rightarrow \mathbb{R}$ , then  $f$  is uniformly continuous  $\Rightarrow$  for any sequences  $(x_n)_{n=1}^\infty, (x'_n)_{n=1}^\infty \subset I$ , with  $\lim_{n \rightarrow \infty} |x_n - x'_n| = 0$ , we have  $\lim_{n \rightarrow \infty} |f(x_n) - f(x'_n)| = 0$ .

[Fact  $\Leftarrow$  also true]

*Proof.* Given  $\varepsilon > 0$ , let  $\delta$  be as in def'n of uniform continuity. Since  $\lim_{n \rightarrow \infty} |x_n - x'_n| = 0$ , there is  $N \in \mathbb{N}$ , so for  $n \geq N$ , we have  $|x_n - x'_n| < \delta$ .

But then, for  $n \geq N$ , we also have that  $|f(x_n) - f(x'_n)| < \varepsilon$ . i.e.  $\lim_{n \rightarrow \infty} |f(x_n) - f(x'_n)| = 0$ . □

**Example 1**  $f : (0, 1] \rightarrow \mathbb{R}, f(x) = \frac{1}{x}$ . Notice that  $f$  is continuous.

Let  $x_n = \frac{1}{n}, x'_n = \frac{1}{2n}, |x_n - x'_n| = \frac{1}{2n} \xrightarrow{\rightarrow} 0$ .

$$|f(x_n) - f(x'_n)| = |n - 2n| = n$$

Hence, not uniformly continuous.

**Example 2:**  $g : (0, 1] \rightarrow \mathbb{R}, g(x) = \sin \frac{1}{x}$ , then  $g$  is continuous.

$x_n = \frac{1}{\pi n}, x'_n = \frac{2}{(2n+1)\pi}, |x_n - x'_n| = \frac{1}{\pi n(2n+1)} \xrightarrow{\rightarrow} 0$ ,

$$|g(x_n) - g(x'_n)| = \left| \sin(n\pi) - \sin\left(\frac{2n+1}{2}\pi\right) \right| = 1$$

For  $\varepsilon = 1$ , uniform continuity fails.

**Theorem 1.3.1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous, then  $f$  is uniformly continuous.

*Proof.* Let us suppose that  $f$  is continuous, but not uniformly continuous, hence there exist  $\varepsilon > 0$ , such that for any  $\delta > 0$ , there are  $x, x' \in [a, b]$  so

$$|f(x) - f(x')| \geq \varepsilon \text{ while } |x - x'| < \delta$$

Let us consider  $\delta = \frac{1}{n}$ , so there are  $x_n, x'_n$  in  $[a, b]$  such that

$$|f(x_n) - f(x'_n)| \geq \varepsilon \text{ while } |x - x'| < \frac{1}{n}$$

By Bolzano Weierstrass Theorem, there is a subsequence  $(x_{n_k})_{k=1}^\infty$  of  $(x_n)_{n=1}^\infty$ , such that  $x = \lim_{k \rightarrow \infty} x_{n_k}$  exists in  $[a, b]$ .

Then, notice that

$$|x - x'_{n_k}| \leq |x_n - x_{n_k}| + |x_{n_k} - x'_{n_k}| < |x - x_{n_k}| + \frac{1}{n_k}$$

hence, by Squeeze Theorem,  $\lim_{k \rightarrow \infty} x'_{n_k} = x$ . Since  $f$  is continuous, we have that

$$\lim_{k \rightarrow \infty} f(x_{n_k}) = f(x) = \lim_{k \rightarrow \infty} f(x'_{n_k})$$

$\Rightarrow$

$$\lim_{k \rightarrow \infty} |f(x_{n_k}) - f(x'_{n_k})| = 0$$

This contradicts that each  $|f(x_{n_k}) - f(x'_{n_k})| \geq \varepsilon$ . Thus by contradiction argument,  $f'$  must be uniformly continuous. □

**Theorem 1.3.2 (Continuous on a Closed Bounded Interval and Integrability).** *let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous, then  $f$  is integrable.*

*Proof.* Let  $\varepsilon > 0$ , then by uniform continuity of  $f$ , there exists a  $\delta$  such that whenever  $|x - x'| < \delta$ , for  $x, x' \in [a, b]$ ,

$$|f(x) - f(x')| < \frac{\varepsilon}{b - a}$$

Thus, we let  $P = \{a = x_0 < \dots < x_n = b\}$  be any partition with length  $l(P) = \max_{j=1, \dots, n} (x_j - x_{j-1}) < \delta$ .

Example:  $P_n = \{a_i < a + \frac{b-a}{n} < a + 2\frac{b-a}{n} < \dots < a + (n-1)\frac{b-a}{n} < b\}$ , then  $\lim_{n \rightarrow \infty} l(P_n) = 0$ .

Now, by EVT, we have

$$\begin{aligned} x_j^* &\in [x_{j-1}, x_j] \text{ s.t. } f(x_j^*) = \inf\{f(x) : x \in [x_{j-1}, x_j]\} = m_j \\ x_j^{**} &\in [x_{j-1}, x_j] \text{ s.t. } f(x_j^{**}) = \sup\{f(x) : x \in [x_{j-1}, x_j]\} = M_j \end{aligned}$$

Then

$$\begin{aligned} L(f, P) &= \sum_{j=1}^n f(x_j^*)(x_j - x_{j-1}) & U(f, P) &= \sum_{j=1}^n f(x_j^{**})(x_j - x_{j-1}) \\ U(f, P) - L(f, P) &= \sum_{j=1}^n (f(x_j^{**}) - f(x_j^*))(x_j - x_{j-1}) \\ &= \sum_{j=1}^n |f(x_j^{**}) - f(x_j^*)| (x_j - x_{j-1}) < \sum_{j=1}^n \frac{\varepsilon}{b - a} (x_j - x_{j-1}) \\ &= \frac{\varepsilon}{b - a} \cdot (b - a) = \varepsilon \end{aligned}$$

Hence, we have satisfied the Cauchy Criterion for integrability. □

**Corollary 1.3.1.** *if  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then*

$$\int_a^b f = \lim_{n \rightarrow \infty} \sum_{j=1}^n f(a + j \frac{b-a}{n}) \frac{b-a}{n}$$

*Proof.* We have  $a + j \frac{b-a}{n} \in [a + \frac{b-a}{n}(j-1), a + \frac{b-a}{n}(j-1)]$ ,  $j = 1, \dots, n$ .

So,

$$m_j \leq f(a + j \frac{b-a}{n}) \leq M_j$$

and thus

$$L(f, P_n) \leq \sum_{j=1}^n f(a + j \frac{b-a}{n}) \frac{b-a}{n} \leq U(f, P_n)$$

$$\lim_{n \rightarrow \infty} (U(f, P_n) - L(f, P_n)) = 0 \text{ as } \lim_{n \rightarrow \infty} l(P_n) = 0.$$

where  $P_n = \{a < a + \frac{b-a}{n} < a + 2\frac{b-a}{n} < \dots < b\}$ , then proof of theorem shows that  $\lim_{n \rightarrow \infty} L(f, P_n) = \int_a^b f = \lim_{n \rightarrow \infty} U(f, P_n)$  as  $\lim_{n \rightarrow \infty} l(P_n) = \lim_{n \rightarrow \infty} \frac{b-a}{n} = 0$ .

and hence Cauchy Criterion is satisfied, hence  $\int_a^b f$  exists and is  $\lim_{n \rightarrow \infty} L(f, P_n)$ , apply Squeeze Theorem.  $\square$

## 1.4 Basic Properties of Integrals

**Example 1:** We will let  $a > 0$  and compute  $\int_0^a x^p dx$  for  $p = 0, 1, 2$ .

1.  $p = 0$ ,  $x^p = 1$ ,  $P = \{0 = x_0 < x_1 = a\}$ ,  $L(1, P) = a = U(1, P)$

$[P'$  refines  $P$ , then  $L(1, P) \leq L(1, P') \leq U(1, P') \leq U(1, P) = a]$

It follows that  $\int_0^a 1 dx = a$ .

2. From last corollary

$$\int_0^a x dx = \lim_{n \rightarrow \infty} \sum_{j=1}^n \left(j \frac{a}{n}\right) \frac{a}{n} = \lim_{n \rightarrow \infty} \frac{a^2}{n^2} \sum_{j=1}^n j = \lim_{n \rightarrow \infty} \frac{a^2}{n^2} \frac{n(n+1)}{2} = \frac{a^2}{2}$$

3. We need a formula for  $\sum_{j=1}^n j^2$ .

Trick:

$$\begin{aligned} (n+1)^3 - 1 &= \sum_{j=1}^n [(j+1)^3 - j^3] && \text{(telescope)} \\ &= \sum_{j=1}^n \left[ \sum_{k=0}^3 \binom{3}{k} j^k - j^3 \right] && \text{(binomial theorem)} \\ &= \sum_{j=1}^n \sum_{k=0}^2 \binom{3}{k} j^k \\ &= \sum_{k=0}^2 \sum_{j=1}^n \binom{3}{k} j^k \end{aligned}$$

$$\begin{aligned} \int_0^a x^2 dx &= \lim_{n \rightarrow \infty} \sum_{j=1}^n \left(j \frac{a}{n}\right)^2 \frac{a}{n} \\ &= \lim_{n \rightarrow \infty} \frac{a^3}{n^3} \sum_{j=1}^n j^2 \\ &= \lim_{n \rightarrow \infty} \frac{a^3}{3n^3} a \left[ (n+1)^3 - 1 - n - \frac{n(n+1)}{2} \right] \\ &= \frac{a^3}{3} \end{aligned}$$

**Algorithm 1.4.1 (Basic Properties Of Integrals).**

**Proposition 1.4.1 (Additivity over intervals).** Let  $a < b < c \in \mathbb{R}$ , and  $f : [a, c] \rightarrow \mathbb{R}$  satisfies that  $f$  is integrable on each of  $[a, b]$ ,  $[b, c]$ , then

- $f$  is integrable on  $[a, c]$  and  $\int_a^c f = \int_a^b f + \int_b^c f$ .

*Proof.* Given  $\varepsilon > 0$ , the Cauchy Criterion provides that

- a partition  $P_1$  of  $[a, b]$  s.t.  $U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2}$
- a partition  $P_2$  of  $[b, c]$  s.t.  $U(f, P_2) - L(f, P_2) < \frac{\varepsilon}{2}$

Let  $P$  be any refinement of  $P_1 \cup P_2$ . Then

$$L(f, P) \geq L(f, P_1 \cup P_2) = L(f, P_1) + L(f, P_2)$$

$$U(f, P) \leq U(f, P_1 \cup P_2) = U(f, P_1) + U(f, P_2)$$

Then

$$U(f, P) - L(f, P) \leq U(f, P_1) - L(f, P_1) + U(f, P_2) - L(f, P_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

hence,  $f$  is integrable on  $[a, c]$ .

Let  $P$  as above, be written  $P = \{a = x_0 < \cdots < x_n = c\}$ .

Let  $Q_1 = \{a = x_0 < \cdots < x_m = b\}$ ,  $Q_2 = \{b = x_m < \cdots < x_n = c\}$ .

We have

$$L(f, Q_1) \leq \int_a^b f \leq U(f, Q_1) \quad L(f, Q_2) \leq \int_b^c f \leq U(f, Q_2)$$

from the proof above, we have

$$L(f, P) = L(f, Q_1) + L(f, Q_2) \leq \int_a^b f + \int_b^c f \leq U(f, Q_1) + U(f, Q_2) = U(f, P)$$

Since  $f$  is integrable on  $[a, c]$ , we have

$$\int_a^c f = \sup\{L(f, P) : P \text{ partition of } [a, c]\} \leq \int_a^b f + \int_b^c f \leq \inf\{U(f, P) : P \text{ partition of } [a, c]\} = \int_a^c f$$

$\Rightarrow$

$$\int_a^c f = \int_a^b f + \int_b^c f$$

□

## 1.5 Riemann Sum - Jan 13 Mon, Jan 15 Wed

**Definition 1.5.1 (Riemann Sums).** Let  $f : [a, b] \rightarrow \mathbb{R}$ ,  $P = \{a = x_0 < \cdots < x_n = b\}$ .

A **Riemann Sum** is any sum of the following form:

$$S(f, P) = \sum_{j=1}^n f(t_j)(x_j - x_{j-1}) \quad t_j \in [x_{j-1}, x_j], j = 1, \dots, n$$

*Left Sum:*

$$S_l(f, P) = \sum_{j=1}^n f(x_{j-1})(x_j - x_{j-1})$$

*Right Sum:*

$$S_r(f, P) = \sum_{j=1}^n f(x_j)(x_j - x_{j-1})$$

*Mid-point Sum:*

$$S_m(f, P) = \sum_{j=1}^n f\left(\frac{x_{j-1} + x_j}{2}\right)(x_j - x_{j-1})$$

*Trapezoid Sum*

$$\begin{aligned} T(f, P) &= \frac{1}{2}[S_l(f) + S_r(f)] \\ &= \sum_{j=1}^n \frac{f(x_{j-1}) + f(x_j)}{2}(x_j - x_{j-1}) \\ &= \frac{1}{2}f(a)(x_1 - a) + \sum_{j=1}^{n-1} f(x_j)(x_j - x_{j-1}) + \frac{1}{2}f(b)(b - x_{n-1}) \end{aligned}$$

**Theorem 1.5.1.** If  $f : [a, b] \rightarrow \mathbb{R}$ , then TFAE,

1.  $f$  is integrable and
2. there is a number  $I_f$  satisfying the following: given any  $\varepsilon > 0$ , there exists a partition  $P_\varepsilon$  of  $[a, b]$  such that  
for any refinement of  $P$  of  $P_\varepsilon$ , any Riemann Sum of  $S(f, P)$  we have

$$|S(f, P) - I(f)| < \varepsilon$$

Furthermore,  $I_f = \int_a^b f$ .

*Proof.*

(i) $\Rightarrow$ (ii) Given  $\varepsilon > 0$ , the Cauchy Criterion provides that  $P_\varepsilon$  so for any refinement  $P$  of  $P_\varepsilon$ ,

$$U(f, P) - L(f, P) < \varepsilon \tag{1}$$

Write  $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ , and let for  $j = 1, \dots, n$ ,  $t_j = [x_{j-1}, x_j]$ .

We observe that

$$m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\} \leq \sup\{f(x) : x \in [x_{j-1}, x_j]\} = M_j$$

and hence

$$\sum_{j=1}^n m_j(x_j - x_{j-1}) \leq \sum_{j=1}^n f(t_j)(x_j - x_{j-1}) \leq \sum_{j=1}^n M_j(x_j - x_{j-1})$$

i.e.

$$L(f, P) \leq S(f, P) \leq U(f, P) \quad (2)$$

Also,

$$L(f, P) \leq \int_a^b f \leq U(f, P) \quad (3)$$

(1), (2) & (3)  $\Rightarrow$

$$\left| S(f, P) - \int_a^b f \right| < \varepsilon$$

In particular, take  $I_f = \int_a^b f$ .

(ii) $\Rightarrow$ (i) we let for  $\varepsilon > 0$ , given  $P_{\varepsilon/4}$  be a partition s.t.

$$|S(f, P) - I_f| < \frac{\varepsilon}{4}$$

For  $P$  a refinement of  $P_{\varepsilon/4}$ ,  $S(f, P)$  a Riemann Sum. We fix such  $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ .

For  $j = 1, \dots, n$ , let  $m_j, M_j$  be as below, we then find for each  $j$ ,

$$x_j^*, x_j^{**} \in [x_{j-1}, x_j] \quad \text{s.t.} \quad f(x_j^*) < m_j + \frac{\varepsilon}{4(b-a)} \quad \& \quad M_j - \frac{\varepsilon}{4(b-a)} < f(x_j^{**})$$

We then consider Riemann Sums

$$S^*(f, P) = \sum_{j=1}^n f(x_j^*)(x_j - x_{j-1}), \quad S^{**}(f, P) = \sum_{j=1}^n f(x_j^{**})(x_j - x_{j-1})$$

Notice that

$$\begin{aligned} S^*(f, P) - L(f, P) &= \sum_{j=1}^n [f(x_j^*) - m_j](x_j - x_{j-1}) \\ &< \sum_{j=1}^n \frac{\varepsilon}{4(b-a)}(x_j - x_{j-1}) \\ &= \frac{\varepsilon}{4(b-a)}(b-a) = \frac{\varepsilon}{4} \end{aligned}$$

and likewise,

$$U(f, P) - S^{**}(f, P) < \frac{\varepsilon}{4}$$

thus

$$\begin{aligned}
& U(f, P) - L(f, P) \\
&= U(f, P) - S^{**}(f, P) + S^{**}(f, P) - I_f + I_f - S^*(f, P) + S^*(f, P) - L(f, P) \\
&< \frac{\varepsilon}{4} + |S^{**}(f, P) - I_f| + |I_f - S^*(f, P)| + \frac{\varepsilon}{4} < \varepsilon
\end{aligned}$$

hence, by Cauchy's Criterion,  $f$  is integrable. □

**Remark:** If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then  $P$  a partition of  $[a, b]$  then each of  $L(f, P)$  and  $U(f, P)$  are Riemann Sums, proof: See proof of integrability of continuous.

**Proposition 1.5.1 (linearity of integration).** *Let  $f, g : [a, b] \rightarrow \mathbb{R}$  each be integrable and  $\alpha, \beta \in \mathbb{R}$ , then*

- $\alpha f + \beta g : [a, b] \rightarrow \mathbb{R} \quad (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x)$
- $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$

*Proof.* Let  $\varepsilon > 0$ , then find partitions of  $[a, b]$ .

- $P_1$  s.t. for any refinement  $P_p$  of  $P_1$ , and any Riemann Sum  $S(f, P_p)$

$$\left| S(f, P_p) - \int_a^b f \right| < \frac{\varepsilon}{2|\alpha| + 1}$$

- $P_2$  s.t. for any refinement of  $Q$  of  $P_2$ , and any Riemann Sum  $S(g, Q)$ ,

$$\left| S(g, Q) - \int_a^b g \right| < \frac{\varepsilon}{2|\beta| + 1}$$

Now we let  $P = \{P_1 \cup P_2\}$ , a refinement of each of  $P_1$  and  $P_2$ , write  $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ , and choose  $t_j \in [x_{j-1}, x_j]$  for each  $j$ . Then

$$\begin{aligned}
S(\alpha f + \beta g, P) &= \sum_{j=1}^n (\alpha f(t_j) + \beta g(t_j))(x_j - x_{j-1}) \\
&= \alpha \sum_{j=1}^n f(t_j)(x_j - x_{j-1}) + \beta \sum_{j=1}^n g(t_j)(x_j - x_{j-1}) \\
&= \alpha S(f, P) + \beta S(g, P)
\end{aligned}$$

Then we have,

$$\begin{aligned}
\left| S(\alpha f + \beta g, P) - \left[ \alpha \int_a^b f + \beta \int_a^b g \right] \right| &\leq |\alpha| \left| S(f, P) - \int_a^b f \right| + |\beta| \left| S(g, P) - \int_a^b g \right| \\
&< |\alpha| \frac{\varepsilon}{2|\alpha| + 1} + |\beta| \cdot \frac{\varepsilon}{2|\beta| + 1} < \varepsilon
\end{aligned}$$

□



**Proposition 1.5.2 (Order Properties of Integrals).** *Let  $f, g : [a, b] \rightarrow \mathbb{R}$  each be integrable, then*

1.  $f \geq 0 \Rightarrow \int_a^b f \geq 0$
2.  $f \geq g \Rightarrow \int_a^b f \geq \int_a^b g$
3.  $f \geq g$  on  $[a, b] \Rightarrow \int_a^b f \geq \int_a^b g$
4.  $|f| : [a, b] \rightarrow \mathbb{R} (|f|(x) = |f(x)|)$  is integrable, with  $\left| \int_a^b f \right| \leq \int_a^b |f|$
5.  $f \vee g, f \wedge g : [a, b] \rightarrow \mathbb{R} (f \vee g(x) = \max\{f(x), g(x)\}, f \wedge g(x) = \min\{f(x), g(x)\})$  are each integrable

*Proof.*

1. for any partition  $P$ ,  $L(f, P) \geq 0$ .
2.  $f - g$  is integrable with  $f - g \geq 0$ , so  $\int_a^b f - \int_a^b g = \int_a^b (f - g) \geq 0$ , by 1.
3. let  $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ , and for each  $j = 1, \cdots, n$

□

## 2 ANTIDERIVATIVE

### 2.1 Fundamental Theorem Of Calculus I - Jan 17 Friday

**Proposition 2.1.1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be integrable on  $[a, b]$ , define

$$F : [a, b] \rightarrow \mathbb{R}, \quad F(x) = \int_a^x f = \int_a^x f(t)dt$$

Note: no  $\int_a^x f(x)dx$ .

We may call this "integral accumulation function".

1.  $F$  is continuous on  $(a, b]$

2.  $\lim_{x \rightarrow a^+} F(x) = 0$

hence, we define  $F(a) = 0 = \int_a^a f$ . Thus  $F : [a, b] \rightarrow \mathbb{R}$ , and is continuous on  $[a, b]$ .

*Proof.*

1. A1. Q5(c) assume that  $f$  is integrable on each  $[a, x]$ ,  $x \in [a, b]$ , so  $F(x) = \int_a^x f$  makes sense. Now, let  $a < x < x' \leq b$ , and we compute

$$\begin{aligned} F(x') - F(x) &= \int_a^{x'} f - \int_a^x f \\ &= \int_a^x f + \int_x^{x'} f - \int_a^x f && \text{(additivity)} \\ &= \int_x^{x'} f \end{aligned}$$

Since  $f$  is integrable, it is bounded i.e.  $\sup_{x \in [a, b]} |f(x)| = M < \infty$ . Thus,  $|f(x)| \leq M$  on  $[a, b]$ .

Hence, by order properties,

$$|F(x') - F(x)| = \left| \int_x^{x'} f \right| \leq \int_x^{x'} |f| \leq \int_x^{x'} M = M(x' - x) = M|x' - x|$$

Thus, given  $\varepsilon > 0$ , let  $\delta = \frac{\varepsilon}{M+1}$ , we have

$$|x' - x| < \delta \Rightarrow |F(x') - F(x)| \leq M\delta = M \frac{\varepsilon}{M+1} < \varepsilon$$

hence,  $F$  is uniformly continuous on  $[a, b]$ .

2. We use  $M$  as above

$$\left| \int_a^x f - 0 \right| = \left| \int_a^x f \right| \leq \int_a^x |f| \leq \int_a^x M = M(x - a)$$

Porceed as above.

□

**Theorem 2.1.1 (Mean Value For Integrals or Average Value for Integrals).** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous (integrability follows), then there exists  $c \in [a, b]$ , s.t.*

$$\int_a^b f = f(c)(b - a)$$

*Proof.* We use two important facts about continuous functions.

By **EVT**, there exists  $x^*, x^{**} \in [a, b]$  s.t.

$$f(x^*) = m = \min\{f(x) : x \in [a, b]\} \quad \text{and} \quad f(x^{**}) = M = \max\{f(x) : x \in [a, b]\}$$

Then  $m \leq f \leq M$ , on  $[a, b]$  so order properties provide

$$m(b - a) = \int_a^b m \leq \int_a^b f \leq \int_a^b M = M(b - a)$$

so

$$f(x^*) = m \leq \frac{1}{b - a} \int_a^b f \leq M = f(x^{**})$$

By **IVT**, Since  $f(x^*) \leq \frac{1}{b-a} \int_a^b f \leq f(x^{**})$ , there is  $c$  between  $x^*$  and  $x^{**}$ , and hence  $c \in [a, b]$  s.t.

$$f(c) = \frac{1}{b - a} \int_a^b f$$

□

**Remark:**  $f$  is integrable  $\Rightarrow F(x) = \int_a^x f$  is a cts function.  $f$  cts  $\Rightarrow F$  differentiable. (BELOW)

**Theorem 2.1.2 (Fundamental Theorem of Calculus (I)).** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous, then*

$$F : [a, b] \rightarrow \mathbb{R}, \quad F(x) = \int_a^x f$$

*satisfies that  $F$  is differentiable on  $[a, b]$ , with  $F' = f$  on  $[a, b]$ .*

*Proof.* Let  $x \in [a, b]$ , we want to examine the quotient

$$\frac{F(x + h) - F(x)}{h} \quad \text{when} \quad x + h \in [a, b]$$

$h > 0$ ,

$$\frac{F(x + h) - F(x)}{h} = \frac{1}{h} \cdot \int_x^{x+h} f = \frac{1}{h} \cdot f(c_h^*)(x + h - x) = f(c_h^*)$$

by M.V.T for I, where  $c_h^* \in [x, x + h]$ ,

$h < 0$ ,

$$\frac{F(x + h) - F(x)}{h} = \frac{F(x) - F(x + h)}{-h} = \frac{1}{-h} \cdot \int_{x+h}^x f = \frac{1}{-h} \cdot f(c_h^{**})(x - (x + h)) = f(c_h^{**})$$

by M.V.T for I, where  $c_h^{**} \in [x + h, x]$ .

hence,

$$\lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = \underbrace{\lim_{h \rightarrow 0} f(c_h^*)}_{\text{continuity}} = \underbrace{\lim_{h \rightarrow 0} f(c_h^{**})}_{\text{squeeze}} = f(x)$$

Thus,  $F'(x)$  exists, and equals  $f(x)$ , for  $x \in [a, b]$ .

**Remark:** Notice that we really found

- left derivative at  $x = b$
- right derivative at  $x = a$

□

**Notation 2.1.1.** Let  $-\infty \leq a < b \leq \infty \in \mathbb{R}$ ,  $f : [a, b] \rightarrow \mathbb{R}$  be continuous, fix  $c \in (a, b)$ , define

$$F : (a, b) \rightarrow \mathbb{R}, F(x) = \begin{cases} \int_c^x f, & x \geq c \\ -\int_x^c f, & x < c \end{cases}$$

We know from FToCI, that  $F'(x) = f(x)$  for  $x > c$ .

**Proposition 2.1.2.** Let us compute  $F'(x)$  for  $x < c$ , let  $c' \in (a, c)$  and for  $x \in (c', c)$  we have

$$\begin{aligned} \int_{c'}^c f &= \int_{c'}^x f + \int_x^c f \\ \Rightarrow -\int_x^c f &= \int_{c'}^x f - \int_{c'}^c f \\ \Rightarrow F'(x) &= \frac{d}{dx} \int_{c'}^x f - \int_{c'}^c f = f(x) \end{aligned}$$

It will be convenient, hereafter, to let  $\int_c^x f = -\int_x^c f$  if  $x < c$ , and we have FToCI

$$\frac{d}{dx} \int_c^x f = f(x), \quad x \in (a, b).$$

## 2.2 Logrithm and Exponential Functions

**Definition 2.2.1.** For  $x \in (0, \infty)$ ,

$$L(x) = \int_1^x \frac{1}{t} dt$$

we shall use only integral & differentiation rates to gain theory of  $L$ .

**Proposition 2.2.1.** If  $a, b > 0$ , gthen  $L(ab) = L(a) + L(b)$ .

*Proof.* Let  $F(x) = L(ax)$ , then chain rule provides

$$F'(x) = \frac{1}{ax} \frac{d}{dx}(ax) = \frac{1}{x} = L'(x)$$

hence,  $F' - L' = 0 \Rightarrow F - L = C$  (constant), by MVT,  $F = L + C(*)$ . Then,

$$L(a) = F(1) = L(1) + C = C.$$

Also,  $L(ab) = F(b) = L(b) + L(a)$ . □

**Proposition 2.2.2.** For  $a > 0$ ,  $q \in \mathbb{Q}$ ,  $L(a^q) = qL(a)$ , (convention:  $a^0 = 1$ ).

*Proof.* First:  $n \in \mathbb{N}$ ,

$$L(a^n) = L(a) + L(a^{n-1}) = \cdots = \underbrace{L(a) + L(a) + \cdots + L(a)}_n = nL(a) \quad (1)$$

Second:

$$L(a) = L((a^{\frac{1}{n}})^n) = nL(a^{\frac{1}{n}}) \Rightarrow L(a^{\frac{1}{n}}) = \frac{1}{n}L(a) \quad (2)$$

Third:

$$0 = L(1) = L(aa^{-1}) = L(a) + L(a^{-1}) \Rightarrow L(a^{-1}) = -L(a) \quad (3)$$

Then, (1) & (2)  $\Rightarrow L(a^m) = mL(a)$ , for  $m \in \mathbb{Z}$ , for  $q = \frac{m}{n}$ ,  $m \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ .

We combine (1), (2), & (3) to get  $L(a^q) = mL(a^{\frac{1}{n}}) = \frac{m}{n}L(a)$ . □

**Proposition 2.2.3.**

1.  $L$  is inreasing:  $0 < x < x'$  then  $L(x) < L(x')$
2.  $\lim_{x \rightarrow 0^+} L(x) = -\infty$ ,  $\lim_{x \rightarrow \infty} L(x) = \infty$

*Proof.*

1.

$$L(x') - L(x) = \int_x^{x'} \frac{1}{t} dt \geq \int_x^{x'} \frac{1}{x'} dt = \frac{1}{x'}(x' - x) > 0$$

Alternatively:  $L'(x) = \frac{1}{x} > 0$ , MVT  $\Rightarrow L$  is strictly increasing.

2. To see that  $\lim_{x \rightarrow \infty} L(x) = \infty$ , it suffices to find  $(a_n)_{n=0}^{\infty} \subset (0, \infty)$  s.t.  $\lim_{n \rightarrow \infty} a_n = \infty$  and  $\lim_{n \rightarrow \infty} L(a_n) = \infty$ . Consider  $(2^n)_{n=0}^{\infty}$  and we have  $\lim_{n \rightarrow \infty} L(2^n) = \lim_{n \rightarrow \infty} nL(2) = \infty$ . Likewise,  $\lim_{n \rightarrow \infty} 2^{-n} = 0$ , and  $\lim_{n \rightarrow \infty} (2^{-n}) = \lim_{n \rightarrow \infty} (-n)L(2) = -\infty$ .

□

**Corollary 2.2.1.**  $L : (0, \infty) \rightarrow \mathbb{R}$  is one-to-one and onto.

*Proof.* Increasing  $\Rightarrow$  one-to-one, since  $\lim_{x \rightarrow 0^+} = -\infty$ ,  $\lim_{x \rightarrow \infty} L(x) = \infty$ , and IVT provides that  $L$  is onto.

□

**Definition 2.2.2.**  $E : \mathbb{R} \rightarrow (0, \infty)$  to be  $L^{-1}$ : inverse function. Hence,

$$E(L(x)) = x, x \in (0, \infty) \quad \text{and} \quad L(E(y)) = y \quad \text{if } y \in \mathbb{R}$$

**Proposition 2.2.4.** If  $y \in \mathbb{R}$ ,  $L(E(y)) = y$ , chain rule  $\frac{1}{E(y)} E'(y) = 1$   
 $\Rightarrow E'(y) = E(y)$

**Algorithm 2.2.1 (About  $E$ ).** Let  $c, d \in \mathbb{R}$ ,

1.  $E(c + d) = E(c)E(d)$
2.  $E(-c) = \frac{1}{E(c)}$
3.  $E(0) = 1$
4.  $E(qc) = E(c)^q, q \in \mathbb{Q}$

*Proof.* 1. Let  $c = L(a)$ ,  $d = L(b)$  ( $L$  is onto)  $E(c + d) = E(L(a) + L(b)) = E(L(ab)) = ab = E(a)E(b)$

2.  $L(1) = 0$  so  $E(0) = 1$

3. use (1) and (2)

4.  $E(qc) = E(qL(a)) = E(L(a^q)) = a^q = E(c)^q$ .

□

**What is  $E(1)$ ?** We note that

$$\lim_{h \rightarrow 0} \frac{L(1+h)}{h} = L'(1) = \frac{1}{1} = 1$$

Hence,

$$1 = \lim_{n \rightarrow \infty} \frac{L(1 + \frac{1}{n})}{\frac{1}{n}} = \lim_{n \rightarrow \infty} nL(1 + \frac{1}{n}) = \lim_{n \rightarrow \infty} L((1 + \frac{1}{n})^n)$$

Since  $E$  is continuous,

$$E(1) = E(\lim_{n \rightarrow \infty} L((1 + \frac{1}{n})^n)) = \lim_{n \rightarrow \infty} E(L((1 + \frac{1}{n})^n)) = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$$

From rule (iv),  $E(q) = e^q$  for  $q \in \mathbb{Q}$ , if  $x \in \mathbb{R}$ , write  $x = \lim_{n \rightarrow \infty} q_n$ , each  $q_n \in \mathbb{Q}$ , and we define

$$e^x = E(x) = \lim_{n \rightarrow \infty} E(q_n) = \lim_{n \rightarrow \infty} e^{q_n}$$

**Definition 2.2.3.** For  $a > 0$ , we have  $a = E(L(a)) = e^{L(a)}$ , and we let

$$a^x = E(L(a)x) = e^{L(a)x}$$

**Exercise With Chain Rule:**

1.  $\frac{d}{dx}(a^x) = L(a)a^x$ ,
2.  $L(a^x) = L(a)x = xL(a)$ ,
3.  $p \in \mathbb{R}$ ,  $x > 0$ ,  $x^p = e^{p(L(x))}$ ,  $\frac{d}{dx}(x^p) = px^{p-1}$

## 2.3 Fundamental Theorem of Calculus II - Jan 22

**Theorem 2.3.1 (Fundamental Theorem of Calculus II).** *Let  $f, F : [a, b] \rightarrow \mathbb{R}$  satisfy that*

- *$f$  is integrable*
- *$F$  is continuous on  $[a, b]$*
- *$F$  is differentiable on  $(a, b)$ , with  $F' = f$  on  $(a, b)$*

*Then,*

$$F(b) - F(a) = \int_a^b f$$

*Proof.* Let  $\varepsilon > 0$ , find a partition  $P_\varepsilon$  on  $[a, b]$ , so

- for every refinement  $P$  of  $P_\varepsilon$
- for every Riemann Sum  $S(f, P)$ , we have

$$\left| S(f, P) - \int_a^b f \right| < \varepsilon$$

Take  $P$  as above, write  $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ .

Now let us consider  $F$  on each  $[x_{j-1}, x_j]$

- $F$  is continuous on  $[x_{j-1}, x_j]$
- $F$  is differentiable on  $(x_{j-1}, x_j)$  [can be used in closed interval, except for  $j = 0, n$ ]

Thus MVT tells us there exists  $c_j \in (x_{j-1}, x_j) \subset [x_{j-1}, x_j]$  such that

$$F(x_j) - F(x_{j-1}) = F'(c_j)(x_j - x_{j-1}) = f(c_j)(x_j - x_{j-1}) \quad (*)$$

Now we consider

$$\begin{aligned} F(b) - F(a) &= \sum_{j=1}^n [F(x_j) - F(x_{j-1})] && \text{(telescope)} \\ &= \sum_{j=1}^n f(c_j)(x_j - x_{j-1}) && \text{(by *)} \\ &= S(f, P) && \text{(a Riemann Sum)} \end{aligned}$$

Hence,

$$\left| F(b) - F(a) - \int_a^b f \right| = \left| S(f, P) - \int_a^b f \right| < \varepsilon$$

Since  $\varepsilon > 0$  is arbitrary, we get desired result. □

**Remark:**

- Suppose  $F, G : [a, b] \rightarrow \mathbb{R}$ , both satisfy  $F' = f = G'$ , for integrable  $f$ , then

$$(F - G)' = F' - G' = f - f = 0 \xRightarrow{M.V.T} F - G = C(\text{constant})$$

hence,  $F(x) = G(x) + C$  for any  $x$  in  $[a, b]$ .



- If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then  $f$  is integrable (theorem from earlier) &  $F(x) = \int_a^x f$  defines an antiderivative.

Moral:  $f$  continuous  $\rightarrow$  an antiderivative exists.

**Notation 2.3.1.** If  $f$  is continuous, (on same intervals), and  $F$  is an antiderivative of  $f$ , i.e.  $F' = f$  (on interval of said intervals), write  $\int f(x)dx = F(x) + C$ .

### Antiderivatives of Basic Functions:

$$\begin{array}{ll} p \neq -1, & \int x^p dx = \frac{x^{p+1}}{p+1} + C \\ & \int \cos x dx = \sin x + C \\ & \int \sec^2 x dx = \tan x + C \end{array} \quad \begin{array}{l} \int e^x dx = e^x + C \\ \int \sin x dx = -\cos x + C \\ \int \sec^2 x dx = \tan x + C \end{array}$$

$$\begin{array}{ll} \int \frac{1}{x^2+1} dx = \arctan x + C [Tan = \tan|_{(\frac{\pi}{2}, \frac{\pi}{2})} : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}] & \text{one-to-one and onto} \\ \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C [Sin = \sin|_{(\frac{\pi}{2}, \frac{\pi}{2})} : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow [-1, 1]] & \text{one-to-one and onto} \end{array}$$

**Theorem 2.3.2 (Change of Variables/Substitution/Reverse Chain Rule).** Suppose

- $g : [a, b] \rightarrow \mathbb{R}$ , differentiable with  $g'$  continuous
- $f$  is defined on  $g([a, b])$  with  $f \circ g : [a, b] \rightarrow \mathbb{R}$  continuous

Then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Anti Derivative Form:

$$\int f(g(x))g'(x)dx = \int f(u)du|_{u=g(x)}$$

*Proof.* Let  $F$  be any antiderivative of  $f$  on  $g([a, b]) = [c, d]$ , let  $F(x) = \int_x^c f$ .

Let  $H : [a, b] \rightarrow \mathbb{R}$  be given by  $H(x) = F(g(x))$ . Then Chain Rule provides

$$H'(x) = F'(g(x))g'(x) = f(g(x))g'(x)$$

and F.T. of C II provides that

$$H(b) - H(a) = \int_a^b f(g(x))g'(x)dx$$

but F.T. of C provides that

$$\int_{g(a)}^{g(b)} f(u)du = F(g(b)) - F(g(a)) = H(b) - H(a)$$

□

**Example:**

1.

$$\begin{aligned}\int x e^{-x^2} dx &= -\frac{1}{2} \int e^{-x^2} (-2x) dx \\ &= -\frac{1}{2} \int e^u du \\ &= -\frac{1}{2} e^u + C \\ &= -\frac{1}{2} e^{-x^2} + C\end{aligned}$$

2.

$$\begin{aligned}\int_1^3 x(x^2 + 4)^{91} dx &= \frac{1}{2} \int_5^{13} u^{91} du \\ &= \frac{1}{2} \frac{u^{92}}{92} \Big|_5^{13} \\ &= \frac{1}{184} [(13)^{92} - 5^{92}]\end{aligned}$$

3.

$$\begin{aligned}\int \cos^m x \sin^n x dx &= \int \cos^m x \sin^{2k} x \sin x dx && (\text{n odd}) \\ &= \int \cos^m x (1 - \cos^2 x)^k \sin x dx && (u = \cos x, \ du = -\sin x dx) \\ &= - \int u^m (1 - u^2)^k du \Big|_{u=\cos x}\end{aligned}$$

## 2.4 Integration and Trigonometry - Jan 22 Wed, TUT

**Definition 2.4.1.**  $\pi = 2 \int_{-1}^a \sqrt{a - x^2} dx$

**Definition 2.4.2.** Let for  $-1 \leq x \leq 1$ ,

$$\arccos x = x\sqrt{1-x^2} + 2 \int_x^1 \sqrt{1-u^2} du$$

Then  $\frac{1}{2} \arccos x$  is the area of —graph—

**Note:**  $\frac{1}{2} \arccos x$  is proportional to the angle  $\theta$ , hence it is reasonable to measure.

$$\theta = \arccos x \quad \text{"radians"}$$

- $\arccos -1 = \pi$
- $\arccos 0 = 2 \int_0^1 \sqrt{1-u^2} du \stackrel{\text{symmetry}}{=} \int_{-1}^1 \sqrt{1-u^2} du = \frac{\pi}{2}$
- $\arccos 1 = 0$

**Derivatives:**

$$\begin{aligned} \arccos' x &= \sqrt{1-x^2} + x \frac{1}{2} (1-x^2)^{-\frac{1}{2}} (-2x) - 2\sqrt{1-x^2} \\ &= -\frac{x^2}{\sqrt{1-x^2}} - \sqrt{1-x^2} \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} = -\frac{1}{\sqrt{1-x^2}} \end{aligned}$$

hence,

- $\arccos' x < 0$  and by MVY, decreasing
- $\lim_{x \rightarrow -1^+} \arccos' x = -\infty = \lim_{x \rightarrow 1^-} \arccos' x$
- $\arccos' 0 = -1$
- $\arccos''(x) = -\frac{x}{(1-x^2)^{\frac{3}{2}}}$  hence,
  - $\arccos''(x) > 0$  if  $x < 0 \Rightarrow$  concave up
  - $\arccos''(x) < 0$  if  $x > 0 \Rightarrow$  concave down

**Definition 2.4.3.**

- $\text{Cos } x = \arccos^{-1} : [0, \pi] \rightarrow [-1, 1]$
- $\sin \theta = \sqrt{1 - \cos^2 \theta}$

Hence,  $\sin : [0, \pi] \rightarrow [0, 1]$ , with

- $\text{Sin}(0) = \sqrt{1 - 1^2} = 0$
- $\text{Sin}(\frac{\pi}{2}) = \sqrt{1 - 0^2} = 1$
- $\text{Sin}(\pi) = \sqrt{1 - (-1)^2} = 0$

## Derivatives of $\cos, \sin$

$$\arccos(\cos \theta) = \theta$$

$$\xRightarrow{\text{Chain Rule}} \frac{-1}{\sqrt{1 - \cos^2 \theta}} \cos' \theta = 1 \Rightarrow \cos' \theta = -\sin \theta$$

$$\sin' \theta = \frac{d}{d\theta} \sqrt{1 - \cos^2 \theta} = \frac{1}{x} (1 - \cos^2 \theta)^{-\frac{1}{2}} (-2 \cos \theta \cos' \theta) = \cos \theta$$

Hence,  $\sin'(0) = 1$ ,  $\sin' \frac{\pi}{2} = 0$ ,  $\sin'(\pi) = -1$ , and  $\sin''(\theta) = -\sin \theta < 0$  if  $0 < \theta < \pi \Rightarrow$  concave down

### Extension to $\mathbb{R}$

(a) we define  $\cos, \sin : [-\pi, \pi] \rightarrow [-1, 1]$

- $\cos$  is even:  $\cos(-\theta) = \cos \theta$ ,  $\theta \geq 0$
- $\sin$  is odd:  $\sin(-\theta) = -\sin \theta$ ,  $\sin \theta = \sin x$ , if  $\theta \geq 0$

(b) we define  $\cos, \sin : \mathbb{R} \rightarrow [-1, 1]$

$$\cos(\theta + 2\pi n) = \cos(\theta) \quad \sin(\theta + 2\pi n) = \sin(\theta) \quad \theta \in [-\pi, \pi], n \in \mathbb{Z}$$

**Lemma 2.4.1.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  is twice differentiable, then*

- $f(0) = f'(0) = 0$  and
- $f'' + f = 0$

then  $f = 0$ .

*Proof.* Let  $g = (f')^2 + f^2$  then

$$g(0) = 0 \quad \text{and} \quad g' = 2ff' + 2ff' = 2f[f'' + f] = 0$$

$\Rightarrow$  by MVT,  $g$  constant, hence,  $g = 0$ , then  $0 \leq f^2 \leq g$ . □

**Lemma 2.4.2.** *Double Angle Formula for Cos*

*Proof.* Let  $a, b \in \mathbb{R}$  be fixed, defined  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$f(t) = \cos(s + t) - a \sin t + b \cos t$$

Then

$$\begin{aligned} f'(t) &= -\sin(s + t) + a \sin t + b \cos t \\ f''(t) &= -\cos(s + t) + a \cos t - b \sin t \\ \Rightarrow f'' + f &= 0 \end{aligned}$$

Now we wish to choose  $a, b$  to satisfy

$$f(0) = 0, \text{ hence } 0 = f(0) = \cos s - a \Rightarrow a = \cos s$$

$$f'(0) = 0, \text{ hence } 0 = f'(0) = -\sin s + b \Rightarrow b = \sin s$$

With these choices of  $a, b$ , the lemma tells us that  $f(t) = 0$ , hence

$$0 = \cos(s + t) - [\cos s \cos t - \sin s \sin t]$$

□

**Double Angle Formula for cos:** Since  $\cos^2 t + \sin^2 t = 1$ , the angle sum formula gives

$$\cos 2t = \cos^2 t - \sin^2 t = \begin{cases} 1 - 2\sin^2 t \Rightarrow \sin^2 t = \frac{1}{2}[1 - \cos^2 t] \\ 2\cos^2 t - 1 \Rightarrow \cos^2 t = \frac{1}{2}[1 + \cos^2 t] \end{cases}$$

**Lemma 2.4.3.** *Double Angle Formula for sin:*  $\sin(s + t) = \cos s \sin t + \sin s \cos t$

*Proof.* Fix  $s \in \mathbb{R}$ , for  $t$  consider

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

and take  $\frac{d}{dt}$  to both sides. □

**Double Angle Formula for sin:**

$$\sin 2t = 2 \cos t \sin t$$

**Example 1:**

1.

$$\begin{aligned} \int \sin^2 x dx &= \frac{1}{2} \int (1 - \cos 2x) dx \\ &= \frac{1}{2} \left[ x - \frac{1}{2} \sin 2x \right] + C \\ &= \frac{1}{2} x - \frac{1}{4} \sin 2x + C \\ &= \frac{1}{2} x - \frac{1}{2} \sin x \cos x + C \end{aligned}$$

2.

$$\begin{aligned} \int \cos^4 x dx &= \int \left[ \frac{1}{2} (1 + \cos 2x) \right]^2 dx \\ &= \frac{1}{4} \int (1 + 2 \cos 2x + \cos^2 2x) dx \\ &= \frac{1}{4} \int (1 + 2 \cos 2x + \frac{1}{2} [1 + \cos 4x]) dx \end{aligned}$$

3.

$$\begin{aligned} &\int \sin x \cos^4 x dx && (u = \cos x, du = -\sin x dx) \\ &= - \int u^4 du \Big|_{u=\cos x} \\ &= - \frac{\cos^5 x}{5} + C \end{aligned}$$

4.

$$\begin{aligned}\int \sin^2 x \cos^4 x dx &= \int \sin^2 x \cos^2 x \cos^2 x dx \\ &= \int \left(\frac{1}{2} \sin 2x\right)^2 \frac{1}{2} [1 + \cos 2x] dx \\ &= \frac{1}{8} \int [\sin^2 2x + \sin^2 2x \cos 2x] dx\end{aligned}$$

### Change of Variables(Antiderivatives form)

$$\int f(g(x))g'(x)dx = \int f(u)du|_{u=g(x)}$$

$f, g'$  continuous.

**Inverse Form:** Suppose we try  $x = g(u)$ ,

$$\int f(x)dx = \int f(g(u))g'(u)du|_{x=g(u)}$$

### Algorithm 2.4.1 (Trig Substitution).

<i>Forms</i>	<i>Substitution</i>	<i>Main Identity</i>	<i><math>dx</math></i>
$a^2 - x^2$	$x = a \sin \theta$	$a^2 - x^2 = a^2 \cos^2 \theta$	$dx = a \cos \theta d\theta$
$x^2 + a^2$	$x = a \tan \theta$	$x^2 + a^2 = a^2 \sec^2 \theta$	$dx = a \sec^2 \theta d\theta$

### Examples

1.

$$\begin{aligned}\int \frac{dx}{(9 - x^2)^{3/2}} &= \int \frac{3 \cos \theta}{(9 \cos^2 \theta)^{3/2}} dx \\ &= \frac{1}{9} \int \sec^2 \theta d\theta = \frac{1}{9} \tan \theta + C \\ &= \frac{1}{9} \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} + C \\ &= \frac{1}{9} \frac{\frac{1}{3}x}{\sqrt{1 - (\frac{1}{3}x)^2}} + C = \frac{1}{9} \frac{x}{\sqrt{9 - x^2}} + C\end{aligned}$$

2.

$$\begin{aligned}\int \frac{dx}{x^2 + 2x + 5} &= \int \frac{dx}{(x + 1)^2 + 4} && (x + 1 = 2 \tan \theta, dx = 2 \sec^2 \theta d\theta) \\ &= \int \frac{2 \sec^2 \theta}{2^2 \sec^2 \theta} d\theta \\ &= \frac{1}{2} \int d\theta = \frac{1}{2} \theta + C \\ &= \frac{1}{2} \arctan \frac{x + 1}{2} + C\end{aligned}$$

3.

$$\begin{aligned}
 \int \sqrt{1-x^2} dx &= \int \cos^2 \theta d\theta \\
 &= \frac{1}{2} \int [1 + \cos 2\theta] d\theta \\
 &= \frac{1}{2} \left[ \theta + \frac{1}{2} \sin 2\theta \right] + C \\
 &= \frac{1}{2} [\arcsin x + \sin \theta \cos \theta] + C \\
 &= \frac{1}{2} [\arcsin x] + x\sqrt{1-x^2} + C \\
 \Rightarrow \arcsin(x) &= 2 \int \sqrt{1-x^2} dx - x\sqrt{1-x^2} + C' \\
 [\arcsin x = \frac{\pi}{2} - \arccos x] \checkmark
 \end{aligned}$$

4.

$$\begin{aligned}
 \int \frac{dx}{\sqrt{x^2+1}} &= \int \frac{\sec^2 \theta d\theta}{\sec \theta} && (x = \tan \theta, dx = \sec^2 \theta d\theta) \\
 &= \int \sec \theta d\theta \\
 &= \int \sec \theta \frac{\sec \theta \tan \theta}{\sec \theta + \tan \theta} d\theta \\
 &= \int \frac{\sec^2 \theta + \tan \theta \sec \theta}{\sec \theta + \tan \theta} d\theta \\
 &= \log |\sec \theta + \tan \theta| + C \\
 &= \log(\sqrt{x^2+1} + x) + C
 \end{aligned}$$

5.

$$\int \frac{dx}{\sqrt{x^2+1}} = \int \frac{\cosh t}{\cosh t} dt \quad (x = \tan \theta,)$$

## 2.5 Integration by Partial Fraction - Jan 27

Warm Up:

$$\begin{aligned}
 \int \sec \theta d\theta &= \int \frac{1}{\cos \theta} d\theta \\
 &= \int \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta} d\theta \\
 &= \int \frac{\sin^2 \theta}{\cos \theta} d\theta + \int \cos \theta d\theta \\
 &= \int \frac{\sin^2 \theta}{1 - \sin^2 \theta} \cos \theta d\theta + \int \cos \theta d\theta
 \end{aligned}$$

**Theorem 2.5.1.**

1. Let  $q \neq 0$  be a polynomial with  $\mathbb{R}$ -coefficients, then we may write

$$q(x) = a(x - r_1)^{m_1} \cdots (x - r_m)^{m_m} \cdot (x^2 + b_1x + c_a)^{n_1} \cdots (x^2 + b_Nx + C_N)^{n_N}$$

where  $a \neq 0$ ,  $r_1, \dots, r_M$  are the distinct  $\mathbb{R}$ -roots of  $q$ , and  $b_1, \dots, b_N, \dots, c_N \in \mathbb{R}$ .

$b_j^2 - 4c_j < 0$  for  $j = 1, \dots, N$ . Also,  $m_1, \dots, m_N, n_1, \dots, n_N \in \mathbb{N}$ .

2. Let  $p$  be  $\mathbb{R}$ -polynomial with

$$\deg p < \deg q$$

Then there are unique  $\mathbb{R}$ -numbers  $A_1, \dots, B_N, C_N$ . so

$$\frac{p(x)}{q(x)} = \sum_{j=1}^M \sum_{k=1}^M \frac{A_{j,k}}{(x - r_j)^k} + \sum_{j=1}^N \sum_{k=1}^{n_j} \frac{B_{j,k}x + C_{j,k}}{x^2 + b_jx + c_j}$$



## 2.6 Integration by parts - Jan 29

**Theorem 2.6.1 (Integration by Parts/"Reverse Product Rule").** Let  $f, gF : [a, b] \rightarrow \mathbb{R}$  satisfy

- $f$  is integrable on  $[a, b]$
- $F' = f$  on  $[a, b]$
- $g'$  is integrable on  $[a, b]$

Then

$$\int_a^b f(x)g(x)dx = F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)dx$$

**Antiderivative Form:**

$$\int f(x)g(x)dx = F(x)g(x) - \int F(x)g'(x)dx, \quad F(x) = \int f(x)dx \quad \text{Can choose } c = 0$$

$$\int f'g = fg - \int fg'$$

*Proof.* Product Rule:

$$\frac{d}{dx}[F(x)g(x)] = F'(x)g(x) + F(x)g'(x) = f'(x)g(x) + F(x)g'(x)$$

FToCII:

$$F(b)g(b) - F(a)g(a) = \int_a^b [f(x)g(x) + F(x)g'(x)]dx$$

$$\Rightarrow F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)dx = \int_a^b f(x)g(x)dx$$

□

**Example 1**

$$\begin{aligned} \int \arctan(x)dx &= \int 1 \cdot \arctan(x)dx \\ &= x \arctan(x) - \int x \frac{1}{1+x^2} dx \\ &= x \arctan(x) - \frac{1}{2} \log(1+x^2) + C \end{aligned}$$

**Example 2**

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - \int 2x \cdot e^x dx \\ &= x^2 e^x - 2[xe^x - \int e^x dx] \\ &= x^2 e^x - 2xe^x + 2e^x + C \end{aligned}$$

**Example 3**

$$\begin{aligned}
\int \cos^{2n}(x)dx \quad n \geq 1 &= \int \cos x \cos^{2n-1} x dx & (I_n) \\
&= \sin x \cos^{2n-1} x - \int \sin x (2n-1) \cos^{2n-2} (-\sin x) dx \\
&= \sin x \cos^{2n-1} x + (2n-1) \int (1 - \cos^2 x) \cos^{2n-2} x dx \\
&= \sin x \cos^{2n-1} x + (2n-1) \left[ \int \cos^{2n-2} x dx - \int \cos^{2n} x dx \right] \\
&= \sin x \cos^{2n-1} x + (2n-1) [I_{n-1}(x) - I_n(x)] \\
\Rightarrow 2nI_n(x) &= \sin x \cos^{2n-1} x + (2n-1)I_{n-1}(x) \\
I_n(x) &= \frac{1}{2n} \sin x \cos^{2n-1} x + \frac{2n-1}{2n} I_{n-1}(x) & (\text{"Reduction Formula"})
\end{aligned}$$

Specific Example:  $n = 0$ ,  $I_0(x) = \int \cos^0 x dx = \int 1 dx = x + C$  Hence

$$\begin{aligned}
\int \cos^2 x dx = I_1(x) &= \frac{1}{2} \sin x \cos x + \frac{1}{2} [x + C] \\
&= \frac{1}{2} \sin x \cos x + \frac{1}{2} x + C'
\end{aligned}$$

$$\begin{aligned}
\int \cos^2 x dx &= \frac{1}{2} \int [1 + \cos 2x] dx \\
&= \frac{1}{2} x + \frac{1}{4} \sin 2x + C
\end{aligned}$$

$$\begin{aligned}
\int \cos^4 x dx = I_2(x) &= \frac{1}{4} \sin x \cos^3 x + \frac{3}{4} \left[ \frac{1}{2} \sin x \cos x + \frac{1}{2} x \right] + C \\
&= \frac{1}{4} \sin x \cos^3 x + \frac{3}{8} \sin x \cos x + \frac{3}{8} x + C
\end{aligned}$$

**Exmample 3'**

$$\begin{aligned}
\int \frac{dt}{(t^2 + 1)^3} &= \int \frac{\sec^2 \theta}{(\sec^2 \theta)^3} d\theta \\
&= \int \cos^4 \theta d\theta \\
&= \frac{1}{4} \sin \theta \cos^3 \theta + \frac{3}{8} \sin \theta \cos \theta + \frac{3}{8} \theta + C \\
&= \frac{1}{4} \frac{t}{(1+t^2)^2} + \frac{3}{8} \frac{t}{1+t^2} + \frac{3}{8} \arctan(t) + C
\end{aligned}$$

## 2.7 Improper Integral - Jan 29

**Recall:** Integration involves upper and lower sums and hence requires

- bounded functions and
- bounded intervals

**Definition 2.7.1.** let  $a < b$  and  $f : (a, b] \rightarrow \mathbb{R}$

- $f$  is integrable on  $[x, b]$  for each  $x \in (a, b]$ .

Then we define the **improper integral** by

$$\int_a^b f = \lim_{x \rightarrow a^+} \int_x^b f, \quad \text{provided that limit exists}$$

**Example 1:**

$f(t) = \frac{1}{\sqrt{t}}$  on  $(0, 2]$ , notice that  $f$  is continuous, hence integrable on  $[x, 2]$ ,  $0 < x < 2$ .

Compute

$$\int_x^2 \frac{dt}{\sqrt{t}} = \int_x^2 t^{-1/2} dt = 2t^{1/2} \Big|_x^2 = 2\sqrt{2} - 2\sqrt{x}$$

Then

$$\int_0^2 \frac{dt}{\sqrt{t}} = \lim_{x \rightarrow 0^+} \int_x^2 \frac{dt}{\sqrt{t}} = 2\sqrt{2} - 2\sqrt{0} = 2\sqrt{2}$$

**Example 2:**

$g(t) = \frac{1}{t^2}$  on  $[0, 2]$ .  $g$  is cts, so integrable on each  $[x, 2]$ ,  $0 < x < 2$ .

$$\int_x^2 \frac{dt}{t^2} = -\frac{1}{t} \Big|_x^2 = \frac{1}{x} - \frac{1}{2}$$

$$\lim_{x \rightarrow 0^+} \int_x^2 \frac{dt}{t^2} = \lim_{x \rightarrow 0^+} \left[ \frac{1}{x} - \frac{1}{2} \right] = \infty$$

We write  $\int_0^2 \frac{dt}{t^2} = \infty$  or  $\int_0^2 \frac{dt}{t^2}$  D.N.E..

**Example 3:**

$h(t) = \frac{|\sin \frac{1}{t}|}{\sqrt{t}}$ ,  $t \in (0, 2]$ ,  $h$  is continuous on each  $[x, 2]$ ,  $0 < x < 2$ .

How can we show if this is improperly integrable?

**Comparison method**

$$\begin{aligned} 0 &\leq \left| \sin \frac{1}{t} \right| \leq 1 \\ \Rightarrow 0 &\leq \frac{|\sin \frac{1}{t}|}{\sqrt{t}} \leq \frac{1}{\sqrt{t}} \\ \Rightarrow 0 &\leq \int_x^2 \frac{|\sin \frac{1}{t}|}{\sqrt{t}} dt \leq \int_x^2 \frac{dt}{\sqrt{t}} = 2\sqrt{2} - 2\sqrt{x} \leq 2\sqrt{2} \end{aligned}$$

$H(x) = \int_x^2 \frac{|\sin \frac{1}{t}|}{\sqrt{t}} dt$  is nonincreasing.

If  $0 < x' < x < 2$ ,  $H(x') - H(x) = \int_{x'}^2 h - \int_x^2 h = \int_{x'}^x h + \int_x^2 h - \int_x^2 h = \int_{x'}^x h \geq 0$ .

$$H(x') \geq H(x)$$

$H'(x) - h(x) \leq 0$  by F.T. of C.I., *M.V.T.*  $\Rightarrow H$  is non-increasing,  $H(x)$  is bounded on  $(0, 2]$  and monotone.

$$\therefore \lim_{x \rightarrow 0^+} H(x) = \int_0^\infty \frac{|\sin(\frac{1}{t})|}{\sqrt{t}} dt \quad \text{exists}$$

## 2.8 Jan 31

Facts from MATH 147:

1.  $\lim_{x \rightarrow a} F(x) = L \Leftrightarrow$  for every sequence  $(a_n)_{n=1}^{\infty}$  s.t.  $\lim_{n \rightarrow \infty} a_n = a$ , provides that  $\lim_{n \rightarrow \infty} F(a_n) = L$ .
2. Let  $(a_n)_{n=1}^{\infty}$  be a sequence in  $\mathbb{R}$ , then  $\lim_{n \rightarrow \infty} a_n$  exists  $\Leftrightarrow$  for any  $\varepsilon > 0$ , there exists  $n_\varepsilon \in \mathbb{N}$  s.t.  $|a_m - a_n| < \varepsilon$  whenever  $m, n \geq n_\varepsilon$ .

Cauchy criterion[Deep Fact: Bolzano Weierstrass Theorem]

**Theorem 2.8.1 (Cauchy Criterion for limit of function).** *Let  $F : (a, b] \rightarrow \mathbb{R}$ , then  $\lim_{x \rightarrow a^+} F(x) \Leftrightarrow$  exists for any  $\varepsilon > 0$ , there is  $\delta > 0$  s.t.  $|F(u) - F(v)| < \varepsilon$  whenever  $|u - a| < \delta$  and  $|v - a| < \delta$  for  $u, v \in (a, b](*)$ .*

*Proof.*  $\Rightarrow$  Let  $L = \lim_{x \rightarrow \infty^+} F(x)$ , then, given  $\varepsilon > 0$ , there is  $\delta > 0$  s.t.

$$|F(u) - L| < \frac{\varepsilon}{2}$$

where  $|u - a| < \delta$  and  $u \in (a, b]$ .

Hence, if  $u, v \in (a, b]$ ,  $|u - a| < \delta$ ,  $|v - a| < \delta$ , then

$$|F(u) - F(v)| \leq |F(u) - L| + |L - F(v)| < \varepsilon$$

$\Leftarrow$  We will verify Fact 1. Hence, let  $(a_n)_{n=1}^{\infty} \subset (a, b]$  be any sequence s.t.  $\lim_{n \rightarrow \infty} a_n = a$ , we wish to see that  $(F(a_n))_{n=1}^{\infty}$  is Cauchy, hence, by fact 2, is convergent, let  $\varepsilon > 0$  be given, find  $\delta > 0$  as in  $(*)$

$\lim_{n \rightarrow \infty} a_n = a \Rightarrow$  there exists  $n_\delta \in \mathbb{N}$  s.t.  $|a - a_n| < \delta$  whenever  $n \geq n_\delta$ .

Hence, if  $m, n \geq n_\delta$ , we have

$$\left. \begin{array}{l} |a - a_m| < \delta \\ |a - a_n| < \delta \end{array} \right\} \Rightarrow |a_m - a_n| < \delta$$

note both  $a_n, a_m$  are to the right of  $a$ .

Thus  $(*)$  provide that  $|F(a_n) - F(a_m)| < \varepsilon$ . Summary, we have  $n_\varepsilon = n_\delta$  s.t.  $|F(a_n) - F(a_m)| < \varepsilon$  whenever  $n, m \geq n_\varepsilon$ .  $\square$

**Last Time:**

$$\int_0^2 \frac{|\sin(\frac{1}{t})|}{\sqrt{t}} dt = \lim_{x \rightarrow 0^+} \underbrace{\int_x^2 \frac{|\sin(\frac{1}{t})|}{\sqrt{t}} dt}_{H(x)}$$

$H$  is monotone and bounded  $\Rightarrow \lim_{x \rightarrow a^+} H(x)$  exists.

**Example 1:**

Consider

$$\begin{aligned} \int_0^1 \frac{\sin(\frac{1}{t})}{\sqrt{t}} dt &= \lim_{x \rightarrow 0^+} \int_x^1 \frac{\sin(\frac{1}{t})}{\sqrt{t}} dt \\ -1 &\leq \sin(\frac{1}{t}) \leq 1 \\ \Rightarrow -\frac{1}{\sqrt{y}} &\leq \frac{\sin(\frac{1}{t})}{\sqrt{t}} \leq \frac{1}{\sqrt{t}} \xrightarrow{\text{order properties}} -\int_x^1 \frac{dt}{\sqrt{t}} \leq \int_x^1 \frac{\sin(\frac{1}{t})}{\sqrt{t}} dt \leq \int_x^1 \frac{dt}{\sqrt{t}} \end{aligned}$$

Now we consider  $0 < u < v < 1$ , again order properties give:

$$\begin{aligned} -\int_u^v \frac{dt}{\sqrt{t}} &\leq \int_u^v \frac{\sin(\frac{1}{t})}{\sqrt{t}} dt \leq \int_u^v \frac{dt}{\sqrt{t}} \\ -2(\sqrt{v} - \sqrt{u}) &\leq F(v) - F(u) \leq 2(\sqrt{v} - \sqrt{u}) \\ |F(v) - F(u)| &\leq 2(\sqrt{v} - \sqrt{u}) \leq 2\sqrt{v} \end{aligned}$$

If  $\delta = \frac{\varepsilon^2}{4}$  and if  $0 < u < v < \delta$

$$|F(v) - F(u)| < 2\sqrt{\delta} = \varepsilon$$

hence,  $\lim_{x \rightarrow 0^+} F(x) = \int_0^1 \frac{\sin(\frac{1}{t})}{\sqrt{t}} dt$  exists.

**Example 2:**  $\int_0^\infty x^2 e^{-x} dx$  use integration by parts two times.

### Other Types of Integrals:

$\int_a^b f$ ,  $f$  is integrable on each  $[a, b]$ ,  $a < x < b$ , but unbounded.

**Example:**

$$\int_{-1}^1 \frac{1}{\sqrt{|t|}} dt = \int_{-1}^0 \frac{dt}{\sqrt{-t}} + \int_0^1 \frac{dt}{\sqrt{t}}$$

**Definition 2.8.1.** Let  $a \in \mathbb{R}$ ,  $f : [a, \infty) \rightarrow \mathbb{R}$  satisfy that  $f$  is integrable on each  $[a, x]$ ,  $a < x$ , let the improper integral be given by

$$\int_a^\infty f = \lim_{x \rightarrow \infty} \int_a^x f, \quad \text{if the limit exists}$$

## 2.9 Convergence and Comparison Test- Feb 3

**Notes on Comparison Test:** Consider the improper integral  $\int_a^b f$ , either  $f$  is unbounded at  $a$  or at  $b$ , just one of  $a, b$  is  $-\infty$ , or  $\infty$ .

**Case 1:**  $f \geq 0$  on  $(a, b)$ ,

- If we can find  $0 \leq f \leq g$  on  $(a, b)$  and  $\int_a^b g$  converges.  $\Rightarrow \int_a^b f$  converges.

[We use monotone convergence theorem, and boundedness]

- If we can find  $0 \leq g \leq f$  on  $(a, b)$  and  $\int_a^b g$  diverges, then  $\Rightarrow \int_a^b f$  diverges.

$$\int_a^x f \geq \int_a^x g \xrightarrow{x \rightarrow \infty} \infty$$

**Case 2:**  $f$  is not (necessarily) non-negative on  $(a, b)$ ,

- if we can find  $g, h \geq 0$  with  $-g \leq f \leq h$ , and let both  $\int_a^b g, \int_a^b h$  converge, then  $\int_a^b f$  converges.

[Need is Cauchy criterion]

**Theorem 2.9.1 (Cauchy Criterion for Limits at  $\infty$ ).** If  $F : [0, \infty] \rightarrow \mathbb{R}$ , then  $\lim_{n \rightarrow \infty} F(x)$  exists  $\iff$  given  $\varepsilon > 0$ , there is  $N > 0$  s.t.  $|F(u) - F(v)| < \varepsilon$  whenever  $u, v > N$ .

*Proof.* ( $\Leftarrow$ ) Let  $(a_n)_{n=1}^\infty \subset [a, \infty)$  with  $\lim_{n \rightarrow \infty} a_n = \infty$ , then there is  $n_0 \in \mathbb{N}$  so  $m, n \geq n_0 \Rightarrow |F(a_n) - F(a_m)| < \varepsilon$ .

Hence,  $F(a_n)_{n=1}^\infty$  is Cauchy, hence admits limit  $L$ , check that for any  $(b_n)_{n=1}^\infty \subset [a, \infty)$ ,  $\lim_{n \rightarrow \infty} a_n \Rightarrow \lim_{n \rightarrow \infty} F(b_n) = L$ .

Check that this implies that  $\lim_{x \rightarrow \infty} F(x) = L$ . □

## 2.10 Integration and Area

let  $f, g[a, b] \rightarrow \mathbb{R}$  be integrable with  $f \leq g$ .

Let  $S = \{(x, y) : y \text{ lies between } f(x) \text{ and } g(x), a \leq x \leq b\}$ .

Partition  $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ .  $s_j, t_j \in [x_{j-1}, x_j]$ .

$$\begin{aligned} \text{area}(S) &\approx \sum_{j=1}^n \underbrace{[g(t_j) - f(s_j)]}_{\text{height}} \underbrace{(x_j - x_{j-1})}_{\text{width}} \\ &= \sum_{j=1}^n g(t_j)(x_j - x_{j-1}) - \sum_{j=1}^n f(s_j)(x_j - x_{j-1}) \\ &\approx \int_a^b g - \int_a^b f = \int_a^b [g - f] \end{aligned} \quad (\text{Say } l(P) < \delta, \text{ by A2Q1})$$

hence we define

$$\text{area}(S) := \int_a^b [g(x) - f(x)] dx$$

if  $S$  is a nice region,

$$\text{area}(S) = \int_a^b h_S(x) dx = \int_c^d W_S(y) dy$$

**Warning Example:**

$$S = \{(x, y) : 0 \leq y \leq 1, \text{ if } x \in \mathbb{Q}, -1 \leq y \leq 0; \text{ if } x \in \mathbb{I}, 0 \leq y \leq 1\}$$

Notice "height function"  $h_S(x) = 1$ . But we should not imagine that

$$\text{area}(S) = \int_0^1 h_S(x) dx = 1$$

**Example 1:**  $S = \{(x, y) : y \text{ between } x^3, y = x^2 - 2x, -1 \leq x \leq 1\}$ .

$$\text{area}(S) = \int_{-1}^1 |x^3 - (x^2 - 2x)| dx = \int_{-1}^0 [x^2 - 2x - x^3] dx + \int_0^1 [x^3 - x^2 + 2x] dx$$

**Example 2:** Circle of radius  $a > 0$ :  $x^2 + y^2 = a^2$ .

$$\text{area}(C) = \int_{-a}^a [\sqrt{a^2 - x^2} - (-\sqrt{a^2 - x^2})] dx = 2 \int_{-a}^a \sqrt{a^2 - x^2} dx$$

Method 1: Substitute  $x = au$ ,  $dx = a \cdot du$ ,

$$= 2 \int_{-1}^1 \sqrt{a^2 - (au)^2} du = a^2 \dots 2 \int_{-1}^1 \sqrt{1 - u^2} du = a^2 \pi$$



Method 2:  $x = a \sin \theta$ ,  $dx = a \cos \theta d\theta$ ,  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ .

$$\begin{aligned}
 \text{area}(C) &= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2 - (a \sin \theta)^2} a \cos \theta d\theta \\
 &= 2a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta d\theta \\
 &= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1 + \cos 2\theta] d\theta \\
 &= a^2 \left[ \theta + \frac{1}{2} \sin 2\theta \right] \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = a^2 \pi + 0 = a^2 \pi
 \end{aligned}$$

**Example 3:** let  $W$  be a circular wedge:

$$\begin{aligned}
 \text{area}(W) &= \int_0^{a \cos \beta} (\tan \beta - \tan \alpha) x dx \\
 &= \int_0^{a \cos \beta} (\tan \beta) x dx \\
 &= \frac{a^2}{2} \sin \beta \cos \beta - \frac{a^2}{2} \sin \alpha \cos \alpha - a^2 \int_{\beta}^{\alpha} \sin^2 \theta d\theta \\
 &= \frac{a^2}{2} [\sin \beta \cos \beta - \sin \alpha \cos \alpha] + \frac{a^2}{2} \left[ (\beta - \alpha) - \frac{1}{2} [\sin(2\beta) - \sin(2\alpha)] \right] \\
 &= a^2 \frac{\beta - \alpha}{2} = a^2 \pi \frac{\beta - \alpha}{2\pi}
 \end{aligned}$$

Therefore, the area is

$$\text{area}(W) = \frac{a^2}{2} (\beta - \alpha)$$

### 2.10.1 Average Value:

$$\begin{aligned}
 a &= \{a_1, a_2, \dots, a_n\} \subset \mathbb{R} \\
 a_{ave} &= \frac{a_1 + \dots + a_n}{n}
 \end{aligned}$$

Let  $f : [a, b] \rightarrow \mathbb{R}$  be an integrable function, we wish to figure out the "average value"  $f_{ave}$ . Uniform partition

$$\left\{ a < a + \frac{b-a}{n} < a + 2\frac{b-a}{n} < \dots < b \right\} = P_n$$

Sample values

$$t_j \in \left[ a + (j-1)\frac{b-a}{n}, a + j\frac{b-a}{n} \right], \quad j = 1, \dots, n$$

**Expect:**

$$f_{ave} \approx \frac{\sum_{j=1}^n f(t_j)}{n} \frac{1}{b-a} \sum_{j=1}^n f(t_j) \frac{b-a}{n} = \frac{1}{b-a} S(f, P_n)$$

$$\text{A2Q1} \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{b-a} S(f, P_n) = \frac{1}{b-a} \int_a^b f.$$

**Definition 2.10.1.** *The average height of  $f$  is:*

$$f_{ave} = \frac{1}{b-a} \int_a^b f$$

### 2.10.2 Weighted Average

$a = \{a_1, \dots, a_n\} \subset \mathbb{R}$  weights  $w_1, \dots, w_n > 0$ .

$$a_{w,ave} = \frac{a_1 w_1 + \dots + a_n w_n}{w_1 + \dots + w_n}$$

We have integrable  $f : [a, b] \rightarrow \mathbb{R}$ ,  $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ , sample points  $t_j \in [x_{j-1}, x_j]$ .

$$\begin{aligned} f_{ave} &\approx \frac{f(t_1)(x_1 - x_0) + f(t_2)(x_2 - x_1) + \dots + f(t_n)(x_n - x_{n-1})}{(x_1 - x_0) + (x_2 - x_1) + \dots + (x_n - x_{n-1})} \\ &= \frac{\sum_{j=1}^n f(t_j)(x_j - x_{j-1})}{b - a} \\ &= \frac{1}{b - a} S(f, P) \end{aligned}$$

### 2.10.3 Centroid

$S$  is a "nice region",  $P = \{a = x_0 < \dots < x_n = b\}$ , tags:  $t_j \in [x_{j-1}, x_j], j = 1, \dots, n$ .

$x$ -center:  $\bar{x}_s$ ,

$$\begin{aligned} S_j &= \{(x, y) \in S : x_{j-1} \leq x \leq x_j\} \\ \bar{x}_S &\approx \frac{\sum_{j=1}^n t_j \text{area}(S_j)}{\sum_{j=1}^n \text{area}(S_j)} \end{aligned}$$

Then

$$\begin{aligned} \bar{x}_S &= \frac{1}{\text{area}(S)} \cdot \int_a^b x h_S(x) dx \\ \bar{y}_S &= \frac{1}{\text{area}(S)} \cdot \int_c^d y w_S(y) dy \end{aligned}$$

### 3 S M H

#### 3.1 Polar Coordinates

Euclidean Coordinates:  $(x, y) \in \mathbb{R}^2$ .

$$r = \sqrt{x^2 + y^2} \quad \text{distance from origin}$$

$$x = r \cos \theta \quad y = r \sin \theta$$

Find  $\theta$ :

- $x > 0, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\frac{y}{x} = \frac{r \sin \theta}{r \cos \theta} = \tan \theta \quad \Rightarrow \quad \arctan\left(\frac{y}{x}\right) = \theta$$

- $x < 0$ : check that  $\theta = \arctan\left(\frac{y}{x}\right) = \theta$

- $x = 0$ :

$$y > 0, \theta = \frac{\pi}{2}$$

$$y < 0, \theta = -\frac{\pi}{2}, \frac{3\pi}{2}$$

Given  $r > 0, 0 \leq \theta < 2\pi$ ,

$(x, y) = (r \cos \theta, r \sin \theta)$  is a unique point in  $\mathbb{R}^2 \setminus \{(0, 0)\}$

**Example 1:** Circle  $x^2 + y^2 = a^2, (a > 0)$ .

$$\begin{aligned} x &= r \cos \theta, y = r \sin \theta \\ a^2 &= (r \cos \theta)^2 + (r \sin \theta)^2 \\ &= r^2 (\cos^2 \theta + \sin^2 \theta) = r^2 \\ \Rightarrow a &= \pm r \end{aligned}$$

But  $a = r$  survives, as  $r \geq 0$ .

Circle, in polar coordinates,  $r = a$ .

**Example 2:** Vertical Lines:

- $x = a, a > 0$ ,

$$r \cos \theta = a \quad \Rightarrow \quad r = \frac{a}{\cos \theta} = a \sec \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

- $x = a, a < 0$ ,

$$r = \frac{a}{\cos \theta}, \quad \frac{\pi}{2} < \theta < \frac{3\pi}{2}$$

- $x = 0$ ,

$$\theta = \frac{\pi}{2} \quad \vee \quad \theta = \frac{3\pi}{2}$$

### 3.2 Arclength - Feb 7

**Definition 3.2.1.**  $f : [a, b] \rightarrow \mathbb{R}$  continuous,

$$\Gamma = \{(x, y) : y = f(x), a \leq x \leq b\}$$

$P = \{a = x_0 < \dots < x_n = b\}$ , then

$$\begin{aligned} \text{length}(L_i) &= \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2} \\ \text{length}(\Gamma) &\approx \sum_{j=1}^n \text{length}(L_j) = \sum_{j=1}^n \sqrt{(x_j - x_{j-1})^2 + [f(x_j) - f(x_{j-1})]^2} \end{aligned}$$

Add assumptions:

- $f'$  exists on  $[a, b]$  and is continuous on  $[a, b]$

$$M.V.T. \Rightarrow f(x_j - x_{j-1}) = f'(c_j)(x_j - x_{j-1}), \quad c_j \in (x_{j-1}, x_j)$$

$$\begin{aligned} &\Rightarrow \sqrt{(x_j - x_{j-1})^2 + [f(x_j) - f(x_{j-1})]^2} \\ &= \sqrt{(x_j - x_{j-1})^2 + [f'(c_j)(x_j - x_{j-1})]^2} \\ &= \sqrt{1 + f'(c_j)^2}(x_j - x_{j-1}) \end{aligned}$$

Then,

$$\text{length}(\Gamma) \approx \sum_{j=1}^n \sqrt{1 + [f'(c_j)]^2}(x_j - x_{j-1}) = S(\sqrt{1 + (f')^2}, P)$$

**Definition 3.2.2.** If  $f'$  exists and is continuous on  $[a, b]$ ,  $\Gamma = \{(x, y) : y = f(x), a \leq x \leq b\}$

$$\text{length}(\Gamma) = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

**Example 1:**

$$0 \leq \alpha < \beta \leq \pi, a > 0$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}}$$

$$\begin{aligned} \text{length}(\Gamma) &= \int_{a \cos \beta}^{a \cos \alpha} \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2} dx \\ &= \int_{a \cos \beta}^{a \cos \alpha} \frac{\alpha}{\sqrt{a^2 - x^2}} dx \\ &= \int_{\beta}^{\alpha} \frac{\alpha}{\sqrt{a^2 - a^2 \cos^2 \theta}} (-a \sin \theta) d\theta \\ &= \int_{\alpha}^{\beta} \alpha d\theta = \alpha(\beta - \alpha) \end{aligned}$$

**Example 2:**  $\Gamma = \{(x, y) : y = x^2, 0 \leq x \leq 2\}, \frac{dy}{dx} = 2x$

$$\text{length}(\Gamma) = \int_0^2 \sqrt{1 + (2x)^2} dx$$

$$2x = \sinh t = \frac{e^t - e^{-t}}{2}, dx = \frac{1}{2} \cosh t dt, \cosh t = \frac{e^t + e^{-t}}{2}.$$

$$\begin{aligned} \text{length}(\Gamma) &= \int_0^{\log(4+\sqrt{17})} \sqrt{1 + \sinh^2 t} \frac{1}{2} \cosh t dt & (t = \log(2x + \sqrt{(2x)^2 + 1})) \\ &= \frac{1}{2} \int_0^{\log(4+\sqrt{17})} \cosh^2 t dt & (\cosh^2 t = \frac{1}{2}[\cosh(2t) + 1]) \\ &= \frac{1}{2} \int_0^{\log(4+\sqrt{17})} [\cosh(2t) + 1] dt & (\sinh 2t = 2 \sinh t \cosh t) \\ ***check*** &= \frac{1}{4} \sinh(2t) + t \Big|_0^{\log(4+\sqrt{17})} \end{aligned}$$

**Method 2**  $2x = \tan t, dx = \frac{1}{2} \sec^2 t dt,$

$$\text{length}(\Gamma) = \int_0^{\arctan(4)} \sqrt{1 + \tan^2 t} \frac{1}{2} \sec^2 t dt = \frac{1}{2} \int_0^{\arctan(4)} \sec^2 t dt$$

$$\begin{aligned} \int \sec^3 t dt &= \int \sec^2 t \sec t dt \\ &= \tan t \sec t - \int \tan t \tan t \sec t dt \\ &= \tan t \sec t - \int (\sec^3 t - \sec t) dt \end{aligned}$$

$$\begin{aligned} \Rightarrow 2 \int \sec^3 t dt &= \tan t \sec t + \int \sec t dt \\ &= \tan t \sec t + \log |\sec t + \tan t| + C \\ \sec(\arctan 4) &= \sqrt{1 + \tan^2(\arctan 4)} = \sqrt{1 + 16} = \sqrt{17} \end{aligned}$$

### 3.3 Parameterization

We regard  $x, y \in [a, b] \rightarrow \mathbb{R}$  (coordinates are each functions)

$$\Gamma = \{y(t) : t \in [a, b]\}$$

Examples :

- Polar Curves:  $x(\theta) = r(\theta) \cos \theta$ ,  $y(\theta) = r(\theta) \sin \theta$ .
- Hyperbolic Coordinates:  $a, b > 0$ ,  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$   
 $x(t) = a \cosh t$   
 $y(t) = b \sinh t$

We wish to compute/define  $\text{length}(\Gamma)$ ,

assumption,

- $x'(t)$ ,  $y'(t)$  always exist on  $[a, b]$ ,  $x', y' : [a, b] \rightarrow \mathbb{R}$  are each continuous,

$$P = \{a = t_0 < t_1 < \dots < t_n = b\}$$

$$\begin{aligned} \text{length} &\approx \sum_{j=1}^n \text{length}(L_j) \\ &= \sum_{j=1}^n \sqrt{[x(t_j) - x(t_{j-1})]^2 + [y(t_j) - y(t_{j-1})]^2} \end{aligned}$$

$$\begin{aligned} M.V.T. \Rightarrow x(t_j) - x(t_{j-1}) &= x'(c_j)(t_j - t_{j-1}), c_j \in (t_{j-1}, t_j) \\ y(t_j) - y(t_{j-1}) &= y'(c_j^*)(t_j - t_{j-1}), c_j^* \in (t_{j-1}, t_j) \end{aligned}$$

$$\begin{aligned} \text{length}(\Gamma) &\approx \sum_{j=1}^n \sqrt{[x'(c_j)]^2 + [y'(c_j^*)]^2} (t_j - t_{j-1}) \\ &\approx \sum_{j=1}^n \sqrt{[x'(c_j)]^2 + [y'(c_j)]^2} (t_j - t_{j-1}) \\ &= S(\sqrt{(x')^2 + (y')^2}, P) \end{aligned}$$

$$\text{length}(\Gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

### 3.4 Volume and Integration

**Volume:**  $S \subset \mathbb{R}^3$  "nice region", typically bounded by definable surfaces with definable cross-sections.

$$Partition = \{a = x_0 < x_1 < \cdots < x_n = b\} = Q$$

$$\text{vol}(P) = \sum_{j=1}^n \text{vol}(P_j) \approx \sum_{j=1}^n A(x_j)(x_j - x_{j-1})$$

We define  $\text{vol}(P) = \int_a^b A(x)dx$ .

$$A(x) = \int_{c(x)}^{d(x)} [z_{top,x}(y) - z_{bot,x}(y)]dy$$

Hard Part: Figure out  $z_{top,x}, z_{bot,x}, c(x), d(x)$ .

**Remark:** we may interchange roles of  $x, y, z$ .

Circular Symmetry: circular symmetry about  $x$ -axis, cross sections are circles.

**Method of Disks**

$$A(x) = \pi[r(x)]^2$$

$$\text{vol}(S) = \pi \int_a^b [r(x)]^2 dx$$

**Method of Cylindrical Shells**

Suppose that  $R \subset \mathbb{R}^3$  is circularly symmetric about  $z$ -axis.

$$P = \{0 = x_0 < x_1 < \cdots < x_n = b\}$$

$$\begin{aligned} \text{vol}(R) &\approx \sum_{j=1}^n \text{vol}(S_j) = \sum_{j=1}^n 2\pi t_j h(t_j)(x_j - x_{j-1}) \\ &= \sum_{j=1}^n S(H, P)(x_j - x_{j-1}) \end{aligned}$$

$$\begin{aligned} \text{vol}(S_i) &= \text{vol}(\text{cylinder, height } h(t_j), \text{radius } x_j) - \text{vol}(\text{cylinder, height } h(t_j), \text{radius } x_{j-1}) \\ &= \pi x_j^2 h(t_j) - \pi x_{j-1}^2 h(t_j) \\ &= \pi(x_j^2 - x_{j-1}^2)h(t_j) \\ &= 2\pi \frac{x_j + x_{j-1}}{2}(x_j - x_{j-1})h(t_j) \\ &= 2\pi t_j h(t_j)(x_j - x_{j-1}) \end{aligned}$$

$$\begin{aligned}\text{vol}(R) &= 2\pi \int_0^b xh(x)dx \\ \text{vol}(R) &= 2\pi \int_0^b x[z_{top,0}(x) - z_{bot,0}(x)]dx\end{aligned}$$

**Example:**  $S$  a sphere, radius  $a > 0$ ,  $x^2 + y^2 + z^2 = a^2$ .

Compute volume  $(S)$ ,

**Disks:** Fix  $x$ , for the moment,  $-a \leq x \leq a$ . Set  $y = 0$ ,

$$x^2 + z^2 = a^2 \Rightarrow z^2 = a^2 - x^2 \Rightarrow r(x) = \sqrt{a^2 - x^2}$$

$$\text{vol}(S) = \pi \int_0^a (\sqrt{a^2 - x^2})^2 dx = \frac{4}{3}\pi a^3$$

**Cylindrical Shells:**

$$h(x) = \sqrt{a^2 - x^2} - (-\sqrt{a^2 - x^2}) = 2\sqrt{a^2 - x^2}$$

$$\begin{aligned}\text{vol}(S) &= 2\pi \int_0^a x 2\sqrt{a^2 - x^2} dx \\ &= 4\pi \int_0^a x \sqrt{a^2 - x^2} dx \\ &= \frac{4}{3}\pi a^3\end{aligned}$$



### 3.5 Application of Antiderivatives:

## 4 DIFFERENTIAL EQUATIONS

### 4.1 Differential Equations

1st order D.E., standard form:

$$y' = f(x, y) \quad , \quad \underbrace{y(x_0) = y_0}_{\text{initialvalue}}$$

**Facts:**

**Theorem 4.1.1 (Caratheodory Existence Theorem:).**  *$f(x, y)$  is continuous near  $(x_0, y_0) \Rightarrow$  solution to I.V.P. exists.*

**Theorem 4.1.2 (Picard-Lindelof Theorem).**

Nice assumption of 2nd variable of  $f$  near  $(x_0, y_0)$ . Caratheodory Existence Theroem:

**Example:**  $y' = x \cdot y^{\frac{1}{3}}, y(0) = x_0$

*Solution #1:*  $y(x) = 0$

*Solution #2:* assume  $y(0) \neq 0$ , hence  $y(x) \neq 0$  in neighborhood of  $x$ .

## 4.2 Feb 14

An object e.g. person with open parachute falls from a standstill to the earth from height  $H$ . (H large  $H < R$ . )

As the object falls, it experiences wind resistance proportional to velocity.

## 4.3 DE - Feb 24

### 4.3.1 First Order Linear Equation

**Definition 4.3.1 (First Order Linear D.E.).**

$$y' = p(x)y + q(x) \quad p, q \text{ cts functions on some domain}$$

**Facts:** Any I.V.P. with such a D.E. (i.e.  $y(x_0) = y_0$ ) always admits a unique solution, assuming that  $p, q$  are continuous in the neighborhood of  $x_0$ .

**Algorithm 4.3.1.**

1. **Homogeneous Case:**  $y' = p(x)y$ , i.e.  $q(x) = 0$ ,

$$\begin{aligned} \frac{y'}{y} &= p(x) \\ \Rightarrow \log |y| &= P(x) + C, P(x) = \int p(x) dx \\ \Rightarrow y &= k e^{P(x)}, k = e^C > 0 \end{aligned}$$

2. **Non Homogeneous Case:** Let  $P(x) = \int p(x) dx$ , as above,  $y' = p(x)y + \underbrace{q(x)}_{\text{forcing term}}$

**"Trick":**

$$\begin{aligned} (d^{-P(x)}y)' &= e^{-P(x)}y' + e^{-P(x)} \cdot (-p(x)) \\ &= e^{-P(x)}(y' - p(x)y) = e^{-P(x)}q(x) \\ \Rightarrow e^{-P(x)}y &= \int e^{-P(x)}q(x) dx \\ \Rightarrow y &= e^{P(x)} \int e^{-P(x)}q(x) dx \end{aligned}$$

Dont forget the integration constant.

$$e^{-P(x)} = e^{-\int p(x) dx} \text{ "integrating factor"}$$

**Example:** Solve  $xy' - ey = x^6$ ,  $y' = \frac{3}{x}y + x^5$

$$\begin{aligned} p(x) &= \frac{3}{x} && \text{(not defined at } x = 0) \\ P(x) &= \int \frac{3}{x} dx = 3 \log |x| = \log(|x|^3) && \text{(did not worry about C)} \\ e^{-P(x)} &= \frac{1}{|x|^3} \\ e^{P(x)} &= |x|^3 \end{aligned}$$

$$y = |x|^3 \int \frac{x^5}{|x|^3} dx = \begin{cases} x^3 [\frac{1}{3}x^3 + C], & x > 0 \\ -x^3 [\int \frac{x^5}{-x^3} dx], & x < 0 \end{cases} = \begin{cases} \frac{1}{3}x^6 + Cx^3, & x > 0 \\ \frac{1}{3}x^6 - Cx^3, & x < 0 \end{cases}$$

Note: equation does not allow  $x = 0$  in domain, we have for either  $x > 0$  or  $x < 0$ .

### 4.3.2 Second Order Linear Equation

**Definition 4.3.2** (Second Order Linear D.E.).

$$y'' + p(x)y' + q(x)y = r(x)$$

**Facts:**

1. if  $p, q, r$  are continuous on an open interval, then a "general solution" exist:  $\varphi_1 y_1 + \varphi_2 y_2$ ,  $y_1, y_2$  linearly independent,  $\varphi_1, \varphi_2$  differentiable functions, or constants
2. I.V.P  $y(x_0) = y_0, y'(x_0) = y_0' \Rightarrow$  solution unique.

**Algorithm 4.3.2** (Methods to Solve).

1. **Homogeneous Case:**

$$y'' + p(x)y' + q(x)y = 0$$

- can be very difficult to compute solution unless  $p, q$ , are constant (A4)
- general solution always exists: of form

$$c_1 y_1 + c_2 y_2$$

$y_1, y_2$  linearly independent solutions  $c_1, c_2$  constants.

In I.V.P. situation, use initial data to learn  $c_1, c_2$ .

2. **Variation of Parameters - L. Euler:**

$$y'' + p(x)y' + q(x)y = r(x) \quad \heartsuit$$

Idea: replace  $c_1, c_2$  from homogeneous case, with functions.

We assume: we have  $\varphi_1, \varphi_2$  differentiable with continuous  $\varphi_1', \varphi_2'$ , and we consider

$$y = \varphi_1' y_1 + \varphi_2' y_2 \quad (f)$$

$y_1, y_2$  are linearly independent solutions to homogeneous case, and

$$\varphi_1' y_1 + \varphi_2' y_2 = 0 \quad (*)$$

Let's consider for  $y$  in (f).

$$\begin{aligned} y' &= (\varphi_1 y_1 + \varphi_2 y_2)' \\ &= \varphi_1' y_1 + \varphi_2' y_2 + \varphi_1 y_1' + \varphi_2 y_2' \\ &= \varphi_1 y_1' + \varphi_2 y_2' \end{aligned} \quad (\text{by } (*))$$

then

$$\begin{aligned} y'' + p y' + q y &= \varphi_1' y_1' + \varphi_2' y_2' + \varphi_1 y_1'' + \varphi_2 y_2'' + p(\varphi_1 y_1' + \varphi_2 y_2') + q(\varphi_1 y_1 + \varphi_2 y_2) \\ &= \varphi_1' y_1' + \varphi_2' y_2' + \varphi_1 (y_1'' + p y_1' + q y_1) + \varphi_2 (y_2'' + p y_2' + q y_2) \\ &= \varphi_1' y_1' + \varphi_2' y_2' \end{aligned} \quad (**)$$

If we wish to solve  $\heartsuit$ , then we have

$$\left\{ \begin{array}{l} \varphi_1' y_1' + \varphi_2' y_2' = r, \\ \varphi_1' y_1 + \varphi_2' y_2 = 0 \end{array} \right. \quad \begin{array}{l} \text{by } \heartsuit \text{ and } ** \\ \text{by assumption*} \end{array}$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} \varphi_1' \\ \varphi_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix} \Rightarrow \begin{bmatrix} \varphi_1' \\ \varphi_2' \end{bmatrix} = \frac{1}{y_1 y_2' - y_1' y_2} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ r \end{bmatrix}$$

$$W = y_1 y_2' - y_1' y_2 \quad \text{Wronskian} \quad \Rightarrow \varphi_1' = -\frac{y_2 r}{W} \quad \varphi_2' = \frac{y_1 r}{W}$$

$$\varphi_1(x) = - \int \frac{y_2(x)r(x)}{W(x)} dx$$

$$\varphi_2(x) = \int \frac{y_1(x)r(x)}{W(x)} dx$$

dont forget integraiton constant General Solution:

$$y(x) = \varphi_1(x)y_1(x) + \varphi_2(x)y_2(x)$$

## 4.4 Feb 26

**Theorem 4.4.1 (Taylor's Theorem).** Let  $I \subseteq \mathbb{R}$  be an open interval,  $f : I \rightarrow \mathbb{R}$ , be  $(n+1)$ -times differentiable, then for  $a \in \mathbb{R}$ , we have

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{P_n(x)} + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for all  $x \in I$ ,  $c = c_x$  is between  $a$  and  $x$ .

$$R_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-a)^{n+1}$$

*Lagrange Remainder Theorem.*

*Proof.* Let  $C \in \mathbb{R}$  satisfy that  $f(x) - P_n(x) = C(x-a)^{n+1}$ , fix  $x$ , then for  $t$  between  $a$  and  $x$

$$\varphi(t) = f(x) - [f(t) + f'(t)(x-t) + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n + C(x-t)^{n+1}]$$

$\varphi(x) = 0 = \varphi(a)$ . Rolle's Theorem  $\Rightarrow \varphi'(c) = 0$  for some  $c$  between  $a$  and  $x$ .

Calculate:

$$\varphi'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)C(x-t)^n$$

Solve to get  $C = \frac{f^{(n+1)}(c_x)}{(n+1)!}$ .

□

**Theorem 4.4.2 (Taylor's Theorem version 2).** Let  $I \subseteq \mathbb{R}$  be an open interval,  $f : I \rightarrow \mathbb{R}$ , be  $(n+1)$ -times differentiable with  $f^{(n+1)}$  continuous, then for  $a \in I$ , we have

$$f(x) = \underbrace{\sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x-a)^k}_{P_n(x)} + \underbrace{\int_a^x \frac{f^{(n+1)}(t)}{n!}(t-a)^n dt}_{R_n(x), \text{Cauchy Form of Remainder for } x \in I}$$

*Proof.* We have

$$\begin{aligned} f(x) &= f(a) + \int_a^x f'(t) dt && \text{(F.T. of C)} \\ &= f(a) + \int_a^x f'(t)(x-t)^0 dt \\ &= f(a) - f'(t)(x-t) \Big|_{a=t}^{x=t} + \int_a^x f''(t)(x-t) dt && \text{(Integration by Parts)} \\ &= f(a) + f'(a)(x-a) + \int_a^x f''(t)(x-t) dt && (*) \end{aligned}$$

Inductive Step:

$$\begin{aligned}
\int_a^x f^{(m)}(t)(x-t)^{m-1} dt &= -\frac{1}{m} f^{(m)}(t)(x-t)^m \Big|_{t=a}^{t=x} + \frac{1}{m} \int_a^x f^{(m+1)}(t)(x-t)^m dt \\
&= \frac{1}{m} f^{(m)}(a)(x-a)^m + \frac{1}{m} \int_a^x f^{(m+1)}(t)(x-t)^m dt \\
* &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{1}{2} \int_a^x f^{(3)}(t)(x-t)^2 dt \\
&= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{1}{2} \left[ \frac{1}{3} f^{(3)}(a)(x-a)^3 + \frac{1}{3} \int_a^x f^{(4)}(t)(x-t)^3 dt \right] \\
&\vdots \\
&= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n dt
\end{aligned}$$

□

**Remark:** we assumed  $f^{(n+1)}$  is continuous, the  $M/AVT$  for integrals provides  $c = c_x$  between  $a$  and  $x$  s.t.

$$\frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n dt = \frac{f^{(n+1)}(c)}{n!} (x-c)^n (x-a)$$

above is the second version of Cauchy form of  $R_n(x)$ .

**Compare:** lagrange form

$$\frac{f^{(n+1)}(c_x)}{(n+1)!} (x-a)^{n+1} = R_n(x) = \frac{f^{(n+1)}(c_x^*)}{n!} (x-c_x^*)^n (x-a)$$

$c_x \neq c_x^*$  in general.

**Proposition 4.4.1.** Given  $f : I \rightarrow \mathbb{R}$ ,  $a$  as above,  $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$ , we have that

- $P_n$  is the unique polynomial with  $\deg P_n \leq n$  s.t.

$$\lim_{x \rightarrow a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0$$

*Proof.* Suppose  $Q$  is polynomial,  $\deg Q \leq n$  with

$$\lim_{x \rightarrow a} \frac{f(x) - Q(x)}{(x-a)^n} = 0$$

Then,

$$Q(x) + [f(x) - Q(x)] = f(x) = P_n(x) + R_n(x) \Rightarrow Q(x) - P_n(x) = R_n(x) - [f(x) - Q(x)]$$

$$\begin{aligned}
x \neq a \quad \frac{Q(x) - P_n(x)}{(x-a)^n} &= \frac{R_n(x)}{(x-a)^n} - \frac{f(x) - Q(x)}{(x-a)^n} \\
&= \frac{\frac{f^{(n+1)}(c_x)}{n!} (x - c_x)^n (x - a)}{(x-a)^n} - \frac{f(x) - Q(x)}{(x-a)^n} \\
&= \left[ \frac{f^{(n+1)}(c_x)}{n!} \cdot \frac{(x - c_x)^n}{(x-a)^n} (x-a) \right] \\
&= 0
\end{aligned}$$

and  $\deg(Q - P_n) \leq n$ , little effort  $\Rightarrow Q = P_n$ . □

**Example:**

$$e^x = \sum_{k=0}^n \frac{1}{k!} x^k + \frac{e^c}{(n+1)!} x^{n+1}$$

$f(x) = e^x$ ,  $f'(x) = e^x$  centered at  $a = 0$ .

Wish to examine  $e^{-x^2}$ .

$$\begin{aligned}
e^{-x^2} &= \sum_{k=0}^n \frac{1}{k!} (-x^2)^k + \frac{e^c}{(n+1)!} (-x^2)^{n+1} \\
&= \underbrace{\sum_{k=0}^n \frac{(-1)^k}{k!} x^{2k}}_{\text{degree } 2n} + \frac{e^c \cdot (-1)^{n+1}}{(n+1)!} x^{2n+2}
\end{aligned}$$

**Conclusion:**

$$\begin{aligned}
\frac{e^{-x^2} - \sum_{k=0}^N \frac{(-1)^k}{k!} x^{2k}}{x^{2n+1}} &= \frac{\frac{(-1)^{n+1} e^{c^k}}{(n+1)!} x^{2n+2}}{x^{2n+1}} \\
\lim_{x \rightarrow 0} \frac{\frac{(-1)^{n+1} e^{c^k}}{(n+1)!} x^{2n+2}}{x^{2n+1}} &= 0
\end{aligned}$$

We know that  $P_{2n}(x) = \sum_{k=0}^n \frac{(-1)^k}{k!} x^{2k}$ , for  $f(t) = e^{-t^2}$  around  $a = 0$ .

We can learn  $f^{(k)}(0)$  just from polynomial, for  $k = 0, \dots, n$ .



## 4.5 Error Estimation - Feb 28

**Example:** Integral Functions:

$$E(x) = \int_0^x e^{-t^2} dt$$

Wish to estimate  $E(1)$  with a polynomial in 1.

Wish to estimate  $E(x)$  with a polynomial in  $x$ ,  $x \in [0, 1]$ .

$a = 0$ ,

$$e^x = \sum_{k=0}^n \frac{x^k}{k!} + \frac{e^c}{(n+1)!} x^{n+1}$$

$$e^{-t^2} = \sum_{k=0}^n \frac{(-1)^k t^{2k}}{k!} + \frac{(-1)^{n+1} e^c}{(n+1)!} t^{2n+2}$$

$$E(x) = \int_0^x e^{-t^2} dt = \sum_{k=0}^n \frac{(-1)^k}{k!} \int_0^x t^{2k} dt + \frac{(-1)^{n+1}}{(n+1)!} \int_0^x e^c t^{2n+2} dt$$

$$\begin{aligned} \Rightarrow & \left| E(x) - \sum_{k=0}^n \frac{(-1)^k \cdot x^{2k+1}}{k! \cdot (2k+1)} \right| \\ &= \left| \frac{(-1)^{n+1}}{(n+1)!} \int_0^x e^c t^{2n+2} dt \right| \\ &\leq \frac{1}{(n+1)!} \int_0^x |e^c t^{2n+2}| dt \\ &\leq \frac{1}{(n+1)!} \int_0^x t^{2n+2} dt, \quad 0 \leq e^c \leq 1, \quad c \in [-1, 0] \\ &= \frac{x^{2n+3}}{(2n+3)(n+1)!} \leq \frac{1}{(2n+3)(n+1)!}, \text{ as } x \in [0, 1] \end{aligned}$$

**”Uniform Estimate”:** Estimate holds for any  $x \in [0, 1]$ .

**Rate of Decay of Estimate:**

Ratio of Estimates:

$$\frac{\frac{1}{(2(n+1)+3)((n+1)+1)!}}{\frac{1}{(2n+3)(n+1)!}} = \frac{(2n+3)}{(2n+5)(n+2)}$$

Better than exponential decay.

$e_n$  error in  $n$   $e_n \sim r^n$  ( $0 < r < 1$ ),  $\frac{e_{n+1}}{e_n} = r$  (fixed)

## 5 SERIES

### 5.1 Introduction to Series - Feb 28

**Definition 5.1.1.** Let  $(a_k)_{k=1}^{\infty} \subset \mathbb{R}$  be a sequence. We define the series

$$\sum_{k=1}^{\infty} a_k := \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k$$

provided the limit exists.

**Series = Improper Sum.**

**Terminology:** We say  $\sum_{k=1}^{\infty} a_k$  converges provide  $(\sum_{k=1}^n a_k)_{n=1}^{\infty}$  converges.

#### Essential Example: Geometric series

Let  $a \in \mathbb{R}$ , when does  $\sum_{k=0}^{\infty} a^k$  converges?

Let  $S_n = \sum_{k=0}^n a^k = 1 + a + a^2 + \dots + a^n$ ,

$$S_n(1 - a) = 1 + \dots + a^n - [a + a^2 + \dots + a^n + a^{n+1}] = 1 - a^{n+1},$$

$$S_n = \begin{cases} \frac{1-a^{n+1}}{1-a}, & \text{if } a \neq 1 \\ n+1, & \text{if } a = 1 \end{cases}$$

**Fact:**

$$\lim_{n \rightarrow \infty} a^{n+1} = \begin{cases} 0, & |a| < 1 \\ D.N.E., & |a| \geq 1, a \neq 1 \\ 1, & a = 1 \end{cases}$$

Hence,  $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$  if  $|a| < 1$ .

**Example:** (Sometimes we get lucky)

Consider  $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ ,

$$S_n = \sum_{k=1}^n \frac{1}{k(k+1)} = \sum_{k=1}^n \left[ \frac{1}{k} - \frac{1}{k+1} \right] = 1 - \frac{1}{2} + \frac{1}{2} - \dots + \frac{1}{n} - \frac{1}{n+1}$$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k(k+1)} = \lim_{n \rightarrow \infty} \left[ 1 - \frac{1}{n+1} \right] = 1$$

Series converges to 1.

## 5.2 Series Convergence Test I - March 2

Recall:

$$\sum_{k=1}^{\infty} a_k := \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k \quad \text{if limit exists.}$$

**Fundamental Question of Series:** Given  $\sum_{k=1}^{\infty} a_k$ , does it converge?

**Tests For Convergence:**

**Proposition 5.2.1 (Test #1: nth term test - weakest necessity result).**

$$\sum_{k=1}^{\infty} a_k \quad \text{converges} \Rightarrow \lim_{k \rightarrow \infty} a_k = 0$$

*Proof.* Let  $S_n = \sum_{k=1}^n a_k$ . Then  $a_n = S_n - S_{n-1}$ . We assume  $\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} S_n$  exists.

Hence  $\lim_{n \rightarrow \infty} S_{n-1} = \lim_{n \rightarrow \infty} S_n$  exists. Hence by taking differences of limits of sequences, we get

$$0 = \lim_{n \rightarrow \infty} S_n - \lim_{n \rightarrow \infty} S_{n-1} = \lim_{n \rightarrow \infty} [S_n - S_{n-1}] = \lim_{n \rightarrow \infty} a_n$$

□

**Example:**  $|a| \geq 1 \Rightarrow \sum_{k=0}^{\infty} a^k$  D.N.E. Indeed,  $\lim_{k \rightarrow \infty} a^k \neq 0$ . (or does not exist).

**Theorem 5.2.1 (Cauchy Criterion for Series Convergence).**  $\sum_{k=1}^{\infty} a_k$  converges  $\Leftrightarrow$  given  $\varepsilon > 0$ , there is a  $n_{\varepsilon}$  in  $\mathbb{N}$  s.t.  $|\sum_{k=m}^n a_k| < \varepsilon$ . whenever  $n > m \geq n_{\varepsilon}$ . *da*

*Proof.* let  $S_n = \sum_{k=1}^n a_k$ . then  $\sum_{k=1}^{\infty} a_k$  converges  $\Leftrightarrow \lim_{n \rightarrow \infty} S_n$  exists,  $\Leftrightarrow$  given  $\varepsilon > 0$ , there is  $n_{\varepsilon}$  in  $\mathbb{N}$  so  $|S_n - S_{m-1}| < \varepsilon$  whenever  $n > m \geq n_{\varepsilon}$ .

Note that  $S_n - S_{m-1} = \sum_{k=1}^n a_k - \sum_{k=1}^{m-1} a_k = \sum_{k=m}^n a_k$ .

□

**Example:** For  $\varepsilon = \frac{1}{2}$ , then Cauchy Criterion fails for  $\sum_{k=1}^{\infty} \frac{1}{k}$ .

**Proposition 5.2.2 (Linearity of Converging series).** Suppose  $\sum_{k=1}^{\infty} a_k$  and  $\sum_{k=1}^{\infty} b_k$  converges, then for  $\alpha, \beta \in \mathbb{R}$ ,

$$\sum_{k=1}^{\infty} (\alpha a_k + \beta b_k) \quad \text{converges}$$

with

$$\sum_{k=1}^{\infty} (\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{\infty} a_k + \beta \sum_{k=1}^{\infty} b_k$$

*Proof.* We use linearity of sums and of limits(when they exist).

$$\begin{aligned}
\sum_{k=1}^{\infty} (\alpha a_k + \beta b_k) &= \lim_{n \rightarrow \infty} \sum_{k=1}^n (\alpha a_k + \beta b_k) \\
&= \lim_{n \rightarrow \infty} \left( \alpha \sum_{k=1}^n a_k + \beta \sum_{k=1}^n b_k \right) \\
&= \alpha \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k + \beta \lim_{n \rightarrow \infty} \sum_{k=1}^n b_k && \text{(some limit exist)} \\
&= \alpha \sum_{k=1}^{\infty} a_k + \beta \sum_{k=1}^{\infty} b_k
\end{aligned}$$

□

**Theorem 5.2.2 (Comparison Test).** Suppose  $0 \leq a_k \leq b_k$ ,  $k \geq \mathbb{N}$ , for some  $N \in \mathbb{N}$ , then

1. If  $\sum_{k=1}^{\infty} b_k$  converges  $\Rightarrow \sum_{k=1}^{\infty} a_k$  converges.
2. If  $\sum_{k=1}^{\infty} a_k$  diverges  $\Rightarrow \sum_{k=1}^{\infty} b_k$  diverges.

*Proof.* 1. Assume  $\sum_{k=1}^{\infty} b_k$  converges, then for  $n \geq N$ ,

$$\begin{aligned}
\sum_{k=1}^n a_k &= \sum_{k=1}^{N-1} a_k + \sum_{k=N}^n a_k \\
&\leq \sum_{k=1}^n a_k + \sum_{k=N}^{\infty} b_k \\
&\leq \sum_{k=1}^n a_k + \underbrace{\lim_{n \rightarrow \infty} \sum_{k=1}^n b_k}_{\text{nondecreasing in } n} \\
&\leq \underbrace{\sum_{k=1}^{N+1} a_k}_{\text{finite}} + \underbrace{\sum_{k=1}^{\infty} b_k}_{< \infty} && \text{(added in } \sum_{k=1}^{\infty} b_k \geq 0)
\end{aligned}$$

Also  $S_{n+1} - S_n = \sum_{k=1}^{n+1} a_k - \sum_{k=1}^n a_k = a_{n+1} \geq 0 \Rightarrow (S_n)_{n=1}^{\infty}$  is non-decreasing.

By monotone convergence theorem  $\Rightarrow \sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} S_n$  exists.

2. Assume  $\sum_{k=1}^{\infty} a_k$  diverges, since  $S_n = \sum_{k=1}^n a_k$  is non-decreasing, we must have that  $\sum_{k=1}^{\infty} a_k = \infty$ .

Now for  $n \geq N$ , we have

$$\begin{aligned}
 \sum_{k=1}^n b_k &= \sum_{k=1}^{N-1} b_k + \sum_{k=N}^n b_k \\
 &\geq \sum_{k=1}^{N-1} b_k + \sum_{k=N}^n a_k \\
 &= \underbrace{\sum_{k=1}^{N-1} b_k}_{\text{independent of } n} - \underbrace{\sum_{k=1}^{N-1} a_k}_{S_n} + \sum_{k=1}^n a_k
 \end{aligned}$$

$$\lim_{n \rightarrow \infty} \rightarrow \infty \Rightarrow \sum_{k=1}^{\infty} b_k = \infty.$$

□

**Example:**

$$\sum_{k=2}^{\infty} \frac{1}{(\log k)^k}$$

$$\log k \geq 2 \Leftrightarrow k \geq e^2, \text{ i.e. } k = \lfloor e^2 \rfloor + 1$$

$$\frac{1}{\log k} \leq \frac{1}{2^k}$$

By geometric series of  $\frac{1}{2}$ , the series converges.

### 5.3 Series Convergence Test II - March 4

**Remark:** Let  $a_k \geq 0$ , and  $S_n = \sum_{k=1}^n a_k$

$$S_{n+1} - S_n = a_n \geq 0 \Rightarrow (S_n)_{n=1}^{\infty} \text{ is monotone increasing}$$

$\sum_{k=1}^{\infty} S_n$  converges  $\Leftrightarrow S_n$  is bounded.

**Corollary 5.3.1 (Limit Comparison Test).** If  $a_k \geq 0$  and  $b_k > 0$ , and  $0 \leq \lim_{k \rightarrow \infty} \frac{a_k}{b_k} = L$  exists, then,

1. If  $L > 0$ ,  $\sum_{k=1}^{\infty} b_k$  converges  $\Leftrightarrow \sum_{k=1}^{\infty} a_k$  converges.
2. If  $L = 0$ ,  $\sum_{k=1}^{\infty} b_k$  converges  $\Rightarrow \sum_{k=1}^{\infty} a_k$  converges.
3. If  $L = 0$ ,  $\sum_{k=1}^{\infty} a_k$  diverges  $\Rightarrow \sum_{k=1}^{\infty} b_k$  diverges. (Contrapositive of ii).

*Proof.* 1) We suppose  $L > 0$ , thus there is  $N \in \mathbb{N}$  such that

$$\begin{aligned} & \left| \frac{a_k}{b_k} - L \right| < \frac{L}{2} \quad \text{if } k \geq N \\ \Leftrightarrow & -\frac{L}{2} < \frac{a_k}{b_k} - L < \frac{L}{2} \quad \text{if } k \geq N \\ \Leftrightarrow & \frac{L}{2} < \frac{a_k}{b_k} < \frac{3L}{2} \quad \text{if } k \geq N \\ \Leftrightarrow & \frac{L}{2} b_k < a_k < \frac{3L}{2} b_k \quad \text{if } k \geq N \end{aligned}$$

We have  $\sum_{k=1}^{\infty} b_k$  converges  $\Leftrightarrow \sum_{k=1}^{\infty} \frac{L}{2} b_k$  converges, and  $\sum_{k=1}^{\infty} \frac{3L}{2} b_k$  converges.

We apply comparison test, twice. □

**Example:** Let us consider  $\sum_{k=1}^{\infty} \frac{1}{k^p}$ ,  $p \geq 2$ , recall that  $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$  converges (\*).

$$\frac{\frac{1}{k^p}}{\frac{1}{k(k+1)}} = \frac{k^2 + k}{k^p} = \frac{1 + \frac{1}{k}}{k^{p-2}} \xrightarrow{k \rightarrow \infty} \begin{cases} 1, & p = 2 \\ 0, & p > 2 \end{cases}$$

**Remark:** The limit comparison test is typically easier to compute than comparison test, and hence useful (you should remember this)

**Corollary 5.3.2 (Ratio Comparison Test).** If  $a_k > 0$  and  $b_k > 0$ , and

- $\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k}$  for  $k \geq N$ ,  $N \in \mathbb{N}$ .

Then  $\sum_{k=1}^{\infty} b_k$  converges  $\Rightarrow \sum_{k=1}^{\infty} a_k$  converges.

**Remark:** This is more difficult in practice than either comparison test or limit comparison test, we will see that it has strong theoretical value.

*Proof.* For  $k \geq N$ ,

$$\begin{aligned}
 & \frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k} \\
 \Rightarrow & \frac{a_{k+1}}{b_{k+1}} \leq \frac{a_k}{b_k} \\
 \Rightarrow & \frac{a_k}{b_k} \leq \frac{a_N}{b_N} = M \quad \text{for } k \geq N \\
 \Rightarrow & a_k \leq M b_k, \quad \text{for } k \geq N
 \end{aligned}$$

Then  $\sum_{k=1}^{\infty} b_k$  converges  $\Rightarrow \sum_{k=1}^{\infty} M b_k$  converges  $\Rightarrow$  comparison test  $\sum_{k=1}^{\infty} a_k$  converges. □

### Main Application of Ratio Comparison Test:

**Theorem 5.3.1 (Ratio test).** Suppose  $a_k > 0$  and that

$$\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = r \quad \text{exists}$$

Then  $r \geq 0$ , and

1. If  $r < 1 \Rightarrow \sum_{k=1}^{\infty} a_k$  converges,
2. If  $r > 1 \Rightarrow \sum_{k=1}^{\infty} a_k$  diverges.

**Remark:**

- test is easy to use, as no reference series are required
- case  $r = 1$  is ambiguous e.g.  $\sum_{k=1}^{\infty} \frac{1}{k}$  diverges and  $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$  converges.

*Proof.*

1. Say  $r < 1$ , pick any  $s$  so  $r < s < 1$ , then there is  $N$  in  $\mathbb{N}$ , so for  $k \geq N$ ,

$$\frac{a_{k+1}}{a_k} < r - (r - s) = s = \frac{s^{k+1}}{s^k}$$

We have that  $\sum_{k=1}^{\infty} s^k$  converges ( $0 < s < 1$ ), and hence by R.L.T.  $\sum_{k=1}^{\infty} a_k$  converges too.

2. Say  $r > 1$ , pick any  $s$  so  $1 < s < r$ , then there is  $N$  in  $\mathbb{N}$ , so for all  $k \geq N$ ,

$$\frac{a_{k+1}}{a_k} > r - (r - s) = s = \frac{s^{k+1}}{s^k}$$

However  $\sum_{k=1}^{\infty} s^k$  diverges, If we have that  $\sum_{k=1}^{\infty} a_k$  converges, then R.L.T. would imply  $\sum_{k=1}^{\infty} s^k$  converges, contradiction. □

**Example:** Consider  $\sum_{k=0}^{\infty} \frac{(1000)^k}{\sqrt{k!}}$

**Ratio Test:**

## 5.4 Series Convergence Test III - March 6

**Theorem 5.4.1 (Integral Test).** Let  $a_k > 0$ ,  $k \in \mathbb{N}$ , suppose there is a function  $f : [1, \infty) \rightarrow \mathbb{R}$  s.t.

- $f(k) = a_k$  for  $k \in \mathbb{N}$ , and
- $f$  is non-increasing

then

$$\sum_{k=1}^{\infty} a_k \text{ converges} \Leftrightarrow \int_1^{\infty} f(t) dt \text{ converges}$$

**Remark:**  $f$  nonincreasing  $\Rightarrow f$  is integrable on each  $[1, x]$ ,  $x \geq 1$ ,  $A_1$

*Proof.*  $f$  non-increasing, if  $t \in [1, \infty]$ , find  $k \in \mathbb{N}$ , so  $t \leq k$ , then  $f(t) \geq f(k) = a_k > 0$ , hence,  $f(t) > 0$  for  $t \in [1, \infty)$ .

If  $t \in [k, k+1]$ , then

$$a_k = f(k) \geq f(t) \geq f(k+1) = a_{k+1}$$

and hence,

$$a_k \geq \int_k^{k+1} f(t) dt \geq a_{k+1} \quad \text{since } k+1 - k = 1$$

$$\sum_{k=1}^{n+1} a_k \geq \int_1^{n+1} f(t) dt = \sum_{k=1}^n \int_k^{k+1} f(t) dt \geq \sum_{k=1}^n a_{k+1} = \sum_{k=2}^{n+1} a_k \quad (*)$$

If  $\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} \sum_{k=1}^{n+1} a_k$  converges, then, for  $x > 1$ ,

$$0 \leq \int_1^x f(t) dt \leq \int_0^{\lceil x \rceil} f(t) dt \leq \sum_{k=1}^{\lceil x \rceil} a_k \xrightarrow{x \rightarrow \infty} \sum_{k=1}^{\infty} a_k < \infty$$

hence  $F(x) = \int_1^x f(t) dt$  is increasing, as  $F'(x) = f(x) > 0$ , and  $F$  is bounded.

Thus  $\int_1^{\infty} f(t) dt = \lim_{x \rightarrow \infty} F(x)$  converges.

Conversely, if  $\int_1^{\infty} f(t) dt$  converges, Then for  $n \in \mathbb{N}$ ,

$$0 \leq \sum_{k=1}^{n+1} a_k = a_1 + \sum_{k=2}^{n+1} a_k \leq a_1 + \int_1^{n+1} f(t) dt \xrightarrow{x \rightarrow \infty} a_1 + \int_1^{\infty} f(t) dt$$

and thus  $S_{n+1} = \sum_{k=1}^{n+1} a_k$  is a bounded and non-decreasing sequence, hence,  $\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} S_{n+1}$  converges. □

**Remark:** Variant: we may mildly weaken assumptions on  $f$ , above, If there is  $M > 1$ , so  $f : [M, \infty) \rightarrow \mathbb{R}$  is nondecreasing,

- $f(k) = a_k$ , for  $k \in \mathbb{N}$ ,  $k \geq M$ ,

then  $\sum_{k=1}^{\infty} a_k$  converges  $\Leftrightarrow \int_1^{\infty} f(t) dt$  converges. [exercise]



**Corollary 5.4.1.** If  $p > 0$ ,  $\sum_{k=1}^{\infty} \frac{1}{k^p}$  converges  $\Leftrightarrow p > 1$ .

*Proof.*  $f(t) = \frac{1}{t^p}$  which is decreasing on  $[1, \infty)$ .

$f(t) = \frac{1}{k^p}$ , Integral Test:  $\sum_{k=1}^{\infty} \frac{1}{k^p}$  converges  $\Leftrightarrow \int_1^{\infty} \frac{dt}{t^p}$  converges.

$$\int_1^x \frac{dt}{t^p} = \int_1^x t^{-p} dt = \begin{cases} \frac{1}{1-p}(x^{1-p} - 1), & p \neq 1 \\ \log x, & p = 1 \end{cases} \xrightarrow{x \rightarrow \infty} \begin{cases} \infty, & p \leq 1 \\ \frac{1}{p-1}, & p > 1 \end{cases}$$

□

**Remark:** Indecisive part of ratio test:

$$\frac{\frac{1}{(k+1)^p}}{\frac{1}{k^p}} = \frac{k^p}{(1+k)^p} = \frac{1}{(\frac{1}{k} + 1)^p} \xrightarrow{k \rightarrow \infty} 1$$

**Example 1:**  $\sum_{k=1}^{\infty} \frac{k^3+1}{k^5+3k^3+1}$  converges? Use limit comparison test with  $\sum_{k=1}^{\infty} \frac{1}{k^2}$ . \* ratio test fails.

**Example 2:** Does  $\sum_{k=1}^{\infty} k e^{-k^2}$  converge?

1. integral test

$$\int_1^{\infty} t e^{-t^2} dt = \frac{1}{2e} \Rightarrow \sum_{k=1}^{\infty} k e^{-k^2} \text{ converges}$$

2. ratio test

$$\frac{(k+1)e^{-(k+1)^2}}{k e^{-k^2}} = \frac{k+1}{k} e^{-2k-1} \xrightarrow{k \rightarrow \infty} 0 \Rightarrow \text{series converges}$$

3. limit comparison test

$$\sum_{k=1}^{\infty} e^{-k} \text{ converges by geometric series}$$

Know that

## 5.5 Series Convergence Test IV - March 9

**Example:** Euler's Constant

$\gamma = \lim_{n \rightarrow \infty} [\sum_{k=1}^n \frac{1}{k} - \log n]$  exists.

*Recall:*

$$\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$$

$$\lfloor (\cdot) t \rfloor \leq t \leq \lfloor t \rfloor + 1; t \geq 1$$

$$\frac{1}{\lfloor t \rfloor} \geq \frac{1}{t} \geq \frac{1}{\lfloor t \rfloor + 1}$$

$$\Rightarrow \frac{1}{\lfloor t \rfloor} - \frac{1}{t} = 0$$

Consider

$$\begin{aligned} A_n &= \int_1^n \left( \frac{1}{\lfloor t \rfloor} - \frac{1}{t} \right) dt \\ &= \int_1^n \frac{1}{\lfloor t \rfloor} dt \\ &= \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{\lfloor t \rfloor} dt - \log n \\ &= \sum_{k=1}^{n-1} k = 1^{n-1} \frac{1}{k} - \log n \end{aligned}$$

$$(A_n = \sum_{k=1}^{n-1} \frac{1}{k} - \log n = [\sum_{k=1}^n \frac{1}{k} - \log n] - \frac{1}{n} \xrightarrow{n \rightarrow \infty} \lim_{n \rightarrow \infty} [\sum_{n=1}^{\infty} \frac{1}{k} - \log n])$$

When  $a_k > 0$ ,  $\lim_{n \rightarrow \infty} \frac{a_{k+1}}{a_k} = 1$ . (Indeterminate Case of Ratio Test)

**Proposition 5.5.1 (Raabe's Test).** Suppose  $\lim_{n \rightarrow \infty} k(1 - \frac{a_{k+1}}{a_k}) = p \in \mathbb{R}$ , then

1. If  $p > 1 \Rightarrow \sum_{k=1}^{\infty} a_k$  converges
2. If  $p < 1 \Rightarrow \sum_{k=1}^{\infty} a_k$  diverges
3. If  $p = 1$  and  $\left| k(1 - \frac{a_{k+1}}{a_k}) - 1 \right| \leq \frac{m}{k}$  for some  $M > 0$ , then  $\sum_{k=1}^{\infty} a_k$  converges

*Remark:* the case  $p = \infty$  also gives convergence, the proof is similar to  $p > 1$  case.

*Proof.*

1. Let  $q > 0 \in \mathbb{R}$ ,

$$\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} = 1 - \frac{1}{k} + \frac{B_k}{k^2}$$

where  $0 \leq B_k \leq (q+1)1$  (i.e. is bounded).

Indeed,

$$\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} = \frac{(k+1)^q}{k^q} = \frac{1}{(1 + \frac{1}{k})^q} = (1 + \frac{1}{k})^{-q}$$

Let  $f = (1+x)^{-q}$ ,  $f(x) = -q(1+x)^{-q-1}$ ,  $f''(x) = q(q+1)(1+x)^{-q-2}$ .

Taylor's Theorem about  $a = 0$ :  $f(x) = 1 - qx \frac{q(q+1)}{(1+cx)^{q+2}} x^2$ ,  $cx$  between 0 and  $x$ .

$$\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} = \left(1 + \frac{1}{k}\right)^{-q} = 1 - \frac{q}{k} + \underbrace{\frac{(q+1)q}{(q+c_k)^{q+2}}}_{B_k, 0 \leq B_k \leq q(q+1)} \frac{1}{k^2}$$

2. We write

$$\begin{aligned} \frac{a_{k+1}}{a_k} &= 1 - \frac{p}{k} + \frac{p}{k} - 1 + \frac{a_{k+1}}{a_k} \\ &= 1 - \frac{p}{k} \underbrace{\frac{1}{k} p - k \left(1 - \frac{a_{k+1}}{a_k}\right)}_{:= A_k} \\ &= 1 - \frac{p}{k} + \frac{A_k}{k} \end{aligned}$$

$$\lim_{k \rightarrow \infty} A_k = \lim_{k \rightarrow \infty} \left[ p - k \left(1 - \frac{a_{k+1}}{a_k}\right) \right] = p - \lim_{k \rightarrow \infty} \left(1 - \frac{a_{k+1}}{a_k}\right) = 0 \quad (\text{By Assumption})$$

3. Let we assume  $p > 1$ , find  $q$  with  $p > q > 1$ , then  $\sum_{k=1}^{\infty} \frac{1}{k^q}$  converges, and (1, 2) shows that

$$\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} - \frac{a_{k+1}}{a_k} = \left(1 - \frac{q}{k} + \frac{B_k}{k^2}\right) - \left(1 - \frac{p}{k} + \frac{A_k}{k}\right) = \frac{p - q + \frac{B_k}{k} - A_k}{k}$$

where  $\lim_{k \rightarrow \infty} \left(\frac{B_k}{k} - A_k\right) = 0$ .

Hence,  $\exists N \in \mathbb{N}$  s.t.  $-\frac{p-q}{2} < \frac{B_k}{k} - A_k < \frac{p-q}{2}$ , so for  $k \geq N$ .

$$\frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} - \frac{a_{k+1}}{a_k} > \frac{p-q}{2k} \Rightarrow \frac{\frac{1}{(k+1)^q}}{\frac{1}{k^q}} > \frac{a_{k+1}}{a_k}$$

Thus by ratio comparison test,  $\frac{a_{k+1}}{a_k}$  converges.

4. If  $p < 1$ , and find  $g$  so  $p < g < 1$ , thus  $\sum_{k=1}^{\infty} \frac{1}{k^g}$  diverges. As in II.

$$\frac{a_{k+1}}{a_k} - \frac{\frac{1}{(k+1)^g}}{\frac{1}{k^g}} = \frac{q - p + A_k - \frac{B_k}{k}}{k}$$

and as  $\lim_{k \rightarrow \infty} (A_k - \frac{B_k}{k}) = 0$ ,  $\exists N \in \mathbb{N}$ , so for  $k \geq N$ ,  $\frac{q-p}{2} < A_k - \frac{B_k}{k} < \frac{q-p}{2}$ , so

$$\frac{a_{k+1}}{a_k} - \frac{k^g}{(k+1)^g} > 0 \Rightarrow \frac{a_{k+1}}{a_k} > \frac{\frac{1}{(k+1)^g}}{\frac{1}{k^g}} \Rightarrow \sum_{k=1}^{\infty} a_k \text{ diverges}$$

5. proof of 3, We suppose  $p = 1$ , then

$$\left| f\left(1 - \frac{a_{k+1}}{a_k}\right) - 1 \right| \leq \frac{M}{k}, M > 0$$

$$\frac{a_{k+1}}{a_k} = 1 - \frac{1}{k} + \frac{1}{k} \left(1 - k \left(1 - \frac{a_{k+1}}{a_k}\right)\right) \geq 1 - \frac{1}{k} - \frac{M}{k^2}$$

Now  $\sum_{k=\lfloor M \rfloor + 2}^{\infty} \frac{1}{k-M+1}$  diverges.

□

**Example for Raabe's Test** Find  $a, b \geq 0$ , s.t.  $\sum_{k=1}^{\infty} \frac{(a+1)(a+2)\cdots(a+k)}{(b+1)(b+2)\cdots(b+k)}$  converges.

$$\frac{a_{k+1}}{a_k} = \sum_{k=1}^{\infty} \frac{\prod_{i=1}^{k+1} \frac{a+i}{b+i}}{\prod_{i=1}^k \frac{a+i}{b+i}} = \frac{a+k+1}{a+k} \xrightarrow{k \rightarrow \infty} 1$$

By Raabe's Test:

$$k(1 - \frac{a_{k+1}}{a_k}) = k(1 - \frac{a+k+1}{b+k+1}) = k(\frac{b-a}{b+k+1}) \xrightarrow{k \rightarrow \infty} b-a$$

$b-a > 1 \Rightarrow$  converges, and  $b-a < 1 \Rightarrow$  diverges.

If  $b-a = 1$ ,

$$k(1 - \frac{a_{k+1}}{a_k}) - 1 = \frac{k(b-a)}{b+k+1} = \frac{(b-a)k - (b+k+1)}{b+k+1} = -\frac{b+1}{k+b+1}$$

$$\left| k(1 - \frac{a_{k+1}}{a_k}) - 1 \right| = \frac{b+1}{k+b+1} = \frac{1}{k} [\frac{b+1}{1 + \frac{b+1}{k}}] < \frac{b+1}{k}$$

$b-a = 1 \Rightarrow$  converges.

## 5.6 Series Convergence Test V - March 11

### 5.6.1 Leibnitz Alternating Series Test

**Theorem 5.6.1** (leibnitz Alternating Series Test). *Suppose*

- $a_1 \geq a_2 \geq \cdots \geq 0$
- $\lim_{k \rightarrow \infty} a_k = 0$

*then,  $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$  converges.*

*Furthermore,  $|\sum_{k=1}^{\infty} (-1)^{k+1} a_k| \leq a_1$ .*

*Proof.* We let  $S_n = \sum_{k=1}^n (-1)^{k+1} a_k$ , then

$$\begin{aligned} S_{2n} &\leq S_{2n} + a_{2n+1} - a_{2n+2} \\ &= S_{2n+2} \\ &= a_1 - a_2 + a_3 - a_4 + \cdots - a_{2n} + a_{2n+1} - a_{2n+2} \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n} - a_{2n+1}) - a_{2n+2} \\ &\leq a_1 \end{aligned}$$

Hence,  $0 \leq S_2 \leq S_{2n+2} \leq a_1$ , i.e.  $(S_{2n})_{n=1}^{\infty}$  is non-negative, non-decreasing, and bounded.

Monotone Convergence  $\Rightarrow \alpha = \lim_{n \rightarrow \infty} S_{2n} \leq a_1$  exists.

$$|\alpha - S_{2k}| < \frac{\varepsilon}{2}, \quad a_{2k+1} < \frac{\varepsilon}{2}, \quad \text{whenever } k \geq N.$$

If  $n \geq 2N + 1$ , and with  $k = \lfloor \frac{n}{2} \rfloor \geq N$ , we have

$$S_n = \begin{cases} S_{2k}, & n \text{ even} \\ S_{2k+1}, & n \text{ odd} \end{cases}$$

and thus

$$|L - S_n| = \begin{cases} |L - S_{2k}|, & n \text{ even} \\ |L - S_{2k} - a_{2k+1}|, & n \text{ odd} \end{cases} < \begin{cases} \frac{\varepsilon}{2} \\ \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{cases} \leq \varepsilon$$

So we conclude that  $\lim_{n \rightarrow \infty} (-1)^{k+1} a_k = L$ . □

**Corollary 5.6.1.** *Let  $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$  satisfy that*

- $a_k$  is eventually non-increasing, non-negative, there is  $N \in \mathbb{N}$ ,  $a_k \geq a_{k+1} \geq 0$  if  $k \geq N$ .
- $\lim_{k \rightarrow \infty} a_n = 0$ ,

*then,*

1.  $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$  converges
2. Error estimate: if  $m \geq N$ ,  $|\sum_{k=m}^{\infty} (-1)^k a_k| \leq a_m$

*Proof.* Let  $n > m \geq N$ ,

$$\begin{aligned}\sum_{k=1}^n (-1)^{k+1} a_k &= \sum_{k=1}^n (-1)^{k+1} a_k + \sum_{k=m+1}^n (-1)^{k+1} a_k \\ &= \sum_{k=1}^n (-1)^{k+1} a_k + (-1)^m \sum_{k=m+1}^n (-1)^{k-m+1} a_k\end{aligned}$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \sum_{k=m+1}^n (-1)^{k-m+1} a_k &= \lim_{n \rightarrow \infty} \sum_{l=1}^{n-m} (-1)^{l+1} a_{l+m} \\ &= \sum_{l=1}^{\infty} (-1)^{l+1} a_{l+m} \in [0, a_{m+1}]\end{aligned}$$

□

**Example:** Let  $F(x) = \int_0^x \sin(\frac{1}{t}) dt$ .

FTofCI,  $x \neq 0$ ,  $F'(x) = \sin(\frac{1}{x})$ , as  $x \mapsto \sin(\frac{1}{x})$  is continuous away from 0.

Notice that hte integrand is not continuous at  $x = 0$ , can we evaluate  $F'(0)$ ?

*Answer:*  $F'(0) = 0$ .

Notice that

$$F(-x) = \int_0^{-x} \sin(\frac{1}{t}) dt = \int_0^x \sin(-\frac{1}{u}) du = \int_0^x \sin(\frac{1}{u}) du = F(x)$$

so  $F$  is even, also,  $F(0) = 0$ , want

$$\lim_{x \rightarrow 0} \frac{F(x) - F(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{F(x)}{x}$$

We will assess  $\lim_{x \rightarrow 0^+} \frac{F(x)}{x}$ .

Set  $x > 0$ , since  $t \mapsto \sin(\frac{1}{t})$  is bounded and continuous on  $(0, x]$ ,

$$F(x) = \int_0^x \sin(\frac{1}{t}) dt = \lim_{u \rightarrow 0^+} \int_u^x \sin(\frac{1}{t}) dt$$

Since

$$F(x) = \lim_{n \rightarrow \infty} \int_{\frac{1}{(n+1)\pi}}^x \sin(\frac{1}{t}) dt = \lim_{n \rightarrow \infty} \left[ \sum_{k=k_x}^n \int_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}} \sin(\frac{1}{t}) dt + \int_{\frac{1}{k_x\pi}}^x \sin(\frac{1}{t}) dt \right]$$

where  $k_x$  in  $\mathbb{N}$  satisfy that

$$\begin{aligned}\frac{1}{k_x\pi} &\leq x < \frac{1}{(k_x - 1)\pi} \\ \Rightarrow \frac{1}{k\pi} &\leq k_x \\ \Rightarrow k_x - 1 &< \frac{1}{k\pi} \\ \Rightarrow k_x &= \lceil \frac{1}{k\pi} \rceil \geq \frac{1}{x\pi}\end{aligned}$$

let

$$a_k = \int_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}} \left| \sin\left(\frac{1}{t}\right) \right| dt > 0$$

$$\sum_{k=k_x} \int_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}} \sin\left(\frac{1}{t}\right) dt = \sum_{k=k_x}^n (-1)^k a_k$$

Now

$$a_{k+1} = \int_{\frac{1}{(k+2)\pi}}^{\frac{1}{(k+1)\pi}} \left| \sin\left(\frac{1}{t}\right) \right| dt$$

$$= \frac{k}{k+2} \int_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}} \left| \sin\left(\frac{1}{u}\right) \right| du < a_k$$

$$u = \frac{\frac{1}{k\pi} - \frac{1}{(k+1)\pi}}{\frac{1}{(k+1)\pi} - \frac{1}{(k+2)\pi}} - \left(t - \frac{1}{(k+2)\pi}\right) + \frac{1}{(k+1)\pi}$$

$$= \frac{k+2}{k} \left(t - \frac{1}{(k+2)\pi}\right) + \frac{1}{(k+1)\pi} \Rightarrow \frac{k+2}{k} dt$$

Hence,r

## 5.7 March 12

**Proposition 5.7.1 (Arbitrary Rearrangement).** *If  $\sum_{k=1}^{\infty} a_k$  converges absolutely, and let  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$  be one-to-one and onto, (arbitrary rearrangement of  $\mathbb{N}$ ), then  $\sum_{k=1}^{\infty} a_{\sigma(k)}$  converges, to the value of  $\sum_{k=1}^{\infty} a_k$ .*

*Proof.* We start with Cauchy Criterion, given  $\varepsilon > 0$ , there is  $N$  in  $\mathbb{N}$ , s.t.

$$\sum_{k=m}^n |a_k| < \frac{\varepsilon}{2} \quad \text{whenever } n > m \geq N$$

We next choose  $N_{\sigma}$  s.t.

$$\{1, \dots, N\} \subseteq \{\sigma(1), \sigma(2), \dots, \sigma(N_{\sigma})\}$$

[ontoneess of  $\sigma$  is required]. Thus we let  $n > N_{\sigma} \geq N$ , then.

$$\begin{aligned} \left| \sum_{k=1}^n a_k - \sum_{k=1}^n a_{\sigma(k)} \right| &= \left| \sum_{k=N+1}^n a_k - \sum_{k=1}^n a_{\sigma(k)} \right| \\ &\leq \sum_{k=N+1}^n |a_k| + \sum_{k=N+1}^{\max\{\sigma(j):j=1,\dots,N\}} |a_k| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

it follows that  $\sum_{k=1}^{\infty} a_k = \lim_{a \rightarrow \infty} \sum_{k=1}^n a_k = \lim_{k \rightarrow \infty} \sum_{k=1}^n a_{\sigma(k)} = \sum_{k=1}^{\infty} a_{\sigma(k)}$ .

□

### Remark:

We say that  $\sum_{k=1}^{\infty} a_k$  is **conditionally convergent** if it is convergent, but not absolutely convergent.

**Example:**  $\sum_{k=1}^n \frac{(-1)^k}{k}$  (L.A.S.T  $\Rightarrow$  convergent), if fact,  $= -\log 2$  (A4)

if  $\sum_{k=1}^{\infty} a_k$  is conditionally convergent, and  $\alpha \in \mathbb{R}$  (any  $\alpha$  will do). then there exists a rearrangement  $\sigma_{\alpha} : \mathbb{N} \rightarrow \mathbb{N}$  s.t.

$$\sum_{k=1}^{\infty} a_{\sigma(k)} = \alpha$$

**Proposition 5.7.2 (Cauchy Product Formula).** *Let  $\sum_{k=0}^{\infty} a_k, \sum_{l=0}^{\infty} b_k$  each be absolutely convergent, then,*

$$\sum_{k=0}^{\infty} a_k \dots \sum_{l=0}^{\infty} b_l = \sum_{k=0}^{\infty} \left( \sum_{j=0}^k a_j b_{k-j} \right)$$

*Proof.* Let  $N \in \mathbb{N}$  be so for  $n > m \geq N$  we have

$$\sum_{k=m}^n |a_k| < \sqrt{\varepsilon}, \quad \sum_{l=m}^n |b_l| < \sqrt{\varepsilon}$$

$n \in \mathbb{N}$ ,

$$\sum_{k=0}^n a_k \cdot \sum_{l=0}^n b_l = \sum_{k=0}^n \sum_{l=0}^n a_k b_l$$



Now let  $n \geq 2N + 2$ ,  $\frac{n}{2} \geq N + 1 \Rightarrow \lfloor \frac{n}{2} \rfloor \geq N$ .

$$\begin{aligned}
& \left| \sum_{k=0}^n a_k \cdot \sum_{l=0}^n b_l - \sum_{k=0}^n \left( \sum_{j=0}^k a_j b_{k-j} \right) \right| \\
&= \left| \sum_{k,l=1, k+l > n}^n a_k b_l \right| \leq \sum_{k,l=1, k+l > n}^n |a_k| |b_l| \\
&\leq \sum_{k=\lfloor \frac{n}{2} \rfloor}^n |a_k| \cdot \sum_{l=\lfloor \frac{n}{2} \rfloor}^n < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{\infty} a_k \cdot \sum_{l=0}^{\infty} b_l &= \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_k \cdot \sum_{l=0}^n b_l \\
&= \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \left( \sum_{j=0}^k a_j b_{k-j} \right) \\
&= \sum_{k=0}^{\infty} \left( \sum_{j=0}^k a_j b_{k-j} \right)
\end{aligned}$$

□