# Math 146 Notes

velo.x

Instructor: Ross Willard NC 5006
Section: 001
Tutorial: Monday, 4:30, RCH 305

# Contents


**1 Vector Space** **2**

1.1 Vector Space - Jan 6 . . . . . 2
1.2 Linear Combination - Jan 8 . . . . . 6
1.3 Subspace - Jan 10 . . . . . 7
1.4 Span - Jan 13 . . . . . 8
1.5 Basis Jan 17 . . . . . 11
1.6 Dimension - Jan 20 . . . . . 13
1.7 Direct Sum - Tutorial Jan 20 . . . . . 15
1.8 Jan 22 . . . . . 16
1.9 Quotient Space - Jan 24 . . . . . 18

**2 Linear Transformations** **20**

2.1 Tutorial - Jan 27 . . . . . 23
2.2 Null Spance and Range . . . . . 24
2.3 Jan 31 . . . . . 25
2.4 Feb 3 . . . . . 27
2.5 Tutorial - Feb 3 . . . . . 28
2.6 Feb 5 . . . . . 30

# 1 Vector Space

## 1.1 Vector Space - Jan 6

**Definition 1.1.1** (**Pseudo-**Field)**.** *A field is an algebraic system* $\mathbb{F}$ *having:*

- *two elements* $0$ *and* $1$
- *operations* $+, \times, -$, *and* $()^{-1}$*(defined on nonzero elements)*

*satisfying "the obvious" properties.*

*See appendix of the textbook.*

*Examples:* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Z}_{prime}$. $\mathbb{Q}(x) = \{\frac{f(x)}{g(x)} : f, g\ polynomials, g \neq 0\}$

*NonExamples:* $\{0\}$, $\mathbb{Z}_m$*(m not prime), Quaternions.*

**Definition 1.1.2** (**Vector Space**)**.** *A vector space over* $\mathbb{F}$ *is a set* $V$ *with two operations:*

- *Addition:* $V \times V \to V$ $x + y$
- *Scalar Multiplication:* $\mathbb{F} \times V \to V$ $ax$

*satisfying* $8$ *properties:* $\forall x, y, z \in V$, $\forall a, b \in \mathbb{F}$

- *V1:* $x + y = y + x$
- *V2:* $x + (y + z) = (x + y) + z$
- *V3:* $\exists$ *a "zero vector"* $0 \in V$ *s.t.* $x + 0 = x$
- *V4:* $\forall x \in V$, $\exists u \in V$, *s.t.* $x + u = 0$
- *V5:* $1x = x$
- *V6:* $(ab)x = a(bx)$ **let* $\cdot$ *denote scalar multiplication*
- *V7:* $a(x + y) = ax + ay$
- *V8:* $(a + b)x = ax + bx$

**Objective 1.1.1.**

- *Defining/Constructing*
- *Proving that a system is a vector space*

**Example 1:** $\mathbb{R}$ def: set of all $n - tuples$ of real numbers

$(x_1, \cdots, x_n) + (y_1, \cdots, y_n) =$

$a(x_1, \cdots, x_n)$ defined $(ax_1, \cdots, ax_n)$ Claim: $\mathbb{R}^n$ is a vector space over $\mathbb{R}$

*Proof.* Check V1:

$$\begin{aligned}(x_1, \cdots, x_n) + (y_1, \cdots, y_n) &= (x_1 + y_1, \cdots, x_n + y_n) \\ &= (y_1 + x_1, \cdots, y_n + x_n) \\ &= (y_1, \cdots, y_n) + (x_1, \cdots, x_n)\end{aligned}$$

$\square$

More generally, for any field $\mathbb{F}$, $\mathbb{F}^n$ is a field over $\mathbb{F}$.

**Example 2:** $\mathbb{R}^{[0,1]} = \{all\ functions\ f : [0, 1] \to \mathbb{R}\}$

- $(f + h)(x) \stackrel{def}{=} f(x) + g(x)$
- $(af)(x) = af(x)$

Claim: $\mathbb{R}^[0, 1]$ is a vector space $/\mathbb{R}$.

*Proof.* V3: Let $\overline{0}$ be the constant 0 function, i.e., $\overline{0}(x) = 0\ \forall x \in [0, 1]\ \overline{0} \in \mathbb{R}^{[0,1]}$

Check: $f + \overline{0} = f\ \forall f \in \mathbb{R}^{[0,1]}$

$$\begin{aligned}(f + \overline{0})(x) &= f(x) + \overline{0}(x) \\ &= f(x) + 0 = f(x)\end{aligned}$$

Since $x \in [0, 1]$ arbitrary, $f + \overline{o} = f$.

More generally, for any set D, and any field $\mathbb{F}$, $\mathbb{F}^D$ is a vector space over $\mathbb{F}$.

$\square$

**Example 3:** let $\mathbb{F} = \mathbb{Z}_2$.

Define $W = \{APPLE\}$,

- $APPLE + APPLE \stackrel{def}{=} APPLE$
- $0APPLE \stackrel{def}{=} APPLE$
- $1APPLE \stackrel{def}{=} APPLE$

Claim: W is a vector space over $\mathbb{Z}_2$.

**Examples 4:** 1. $\mathbb{R}^n : \mathbb{F}^n$, 2. $\mathbb{R}^{[0,1]}, : \mathbb{F}^D$, 3. $\{APPLE\}$.

4. Fix a field $\mathbb{F}$, for $n \geq 0$, $P_n(\mathbb{F})$ is the set of all polynomials, of degree $\leq n$, in variable $x$, with coefficients from $\mathbb{F}$,

$$= \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_i \in \mathbb{F}\}$$

Addition, scalar mult are "obvious", using op's of $\mathbb{F}$.

Claim: $P_n(\mathbb{F})$ is a vecor space $/\mathbb{F}$.

5. $\mathbb{F}[x]$ = the set of all polynomials in $x$ with coefficients from $\mathbb{F} = \Cup_{n=0}^{\infty} P_n(\mathbb{F})$

Claim: with the "obvious" op's $\mathbb{F}[x]$ is a V.S. $/\mathbb{F}$.

**Theorem 1.1.1 (Cancellation Law).** *Let $V$ be a V.S., $/\mathbb{F}$, if $x, y, z \in V$, and $x + z = y + z$, then $x = y$.*

*Proof.* Let $u \in V$ be such that $z + u = 0$ (from V4).

Then

$$\begin{aligned}
x &= x + 0 && \text{(V3)} \\
x &= x + (z + u) && \text{(Choice of u)} \\
x &= (x + z) + u && \text{(hypothesis)} \\
x &= (y + z) + u && \text{(V2)} \\
x &= y + (z + u) && \text{(V2)} \\
x &= y + 0 && \text{(choice of u)} \\
x &= y
\end{aligned}$$

□

**Corollary 1.1.1.** *Suppose $V$ is a V.S., there is exactly one "zero vector". i.e. a vector satisfy V3. in V.*

*Proof.* Assume $0_1, 0_2 \in V$, both satisfying V3, i,e, $x + 0_1 = x$ and $x + 0_2 = x$, $\forall x \in V$.

$$\begin{aligned}
0_1 &= 0_1 + 0_1 \\
0_1 &= 0_1 + 0_2
\end{aligned}$$

$$\begin{aligned}
0_1 + 0_1 &= 0_1 + 0_2 \\
&= 0_2 + 0_1 && \text{(V1)} \\
0_1 &= 0_2 && \text{(By Cancellation)}
\end{aligned}$$

□

**Corollary 1.1.2.** *Suppose $V$ is a V.S. and $x \in V$, then the vector u in V4 is unique.*

*Proof.* Assume $u_1, u_2 \in V$ both satisfy $x + u_1 = 0 = x + u_2$, then

$$\begin{aligned}
u_1 + x &= u_2 + x && \text{(V1)} \\
u_1 &= u_2 && \text{(By Cancellation)}
\end{aligned}$$

□

**Definition 1.1.3.** *Given a V.S. $V$ and $x \in V$,*

- *the unique vector $u \in V$ s.t. $x + u = 0$ is denoted $-x$.*
- *$x - y$ denotes $x + (-y)$*

**<u>Note:</u>** V2 justifies $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_kx_k$ not worry about parentheses.

## 1.2 Linear Combination - Jan 8

**Definition 1.2.1 (Linear Combination).** $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_kx_k$ *is called a linear combination of* $x_1, \cdots, x_k$.

**Basic Problem:** Given a V.S. $V/\mathbb{F}$, and $u_1, u_2, \cdots, u_n \in V$ and $x \in V$ to decide whether $x$ is a linear combination of $u_1, \cdots, u_n$.

**Example:** $V = \mathbb{Q}[x]$ over $\mathbb{Q}$. Let $p = 4x^4 + 7x^2 - 2x + 3$.

- $u_1 = x^4 - x^2 + 2x + 1$
- $u_2 = 2x^4 + 3x^2 + 2x$
- $u_3 = x^4 + 4x^2 + 1$
- $u_4 = 2x^3 + 3$
- $u_5 = x^4 + 1$

Is $p$ a linear combination of $u_1, \cdots, u_5$? Solution: search for $a_1, \cdots, a_5 \in \mathbb{Q}$ s.t.

$$p = a_1u_1 + a_2u_2 + \cdots + a_5u_5$$

$$\begin{aligned} 4x^4 + 7x^2 - 2x + 3 =& a_1(x^4 - x^2 + 2x - 1) + a_2(2x^4 + 3x^2 + 2x) + a_3(x^4 + 4x^2 + 1) \\ & + a_4(2x^3 + 3) + a_5(x^4 + 1) \\ =& (a + 1 + 2a_2 + a - 3 + a_5)x^4 + (2a^4)x^3 + (-a_1 + 3a_2 + 4a_3)x^2 \\ & + (2a_1 + 2a_2)x + (-a_1 + a_3 + 3a_4 + a_5) \end{aligned}$$

$$\begin{cases} a_1 + 2a_2 + a_3 + a_5 = 4 \\ 2a_4 = 0 \\ -a_1 + 3a_2 + 4a_3 = 7 \\ 2a_2 + 2a_2 = -2 \\ -a_1 + a_3 + 3a_4 + a_5 = 3 \end{cases}$$

No solution.

## 1.3 Subspace - Jan 10

**Notation 1.3.1.**

- $0$ *denote the unique vector in* $V$
- $x$ *denote the unique* $u \in V$ *satisfying* $V4$

**Theorem 1.3.1.** *Suppose* $V$ *is a VS /* $\mathbb{F}$*,* $X \in V$*,* $a \in \mathbb{F}$*.*

1. *0x=0, the first* $0$ *is scalar, the second* $0$ *is a vector*
2. *(-a)x=a(-x)=-(ax)*
3. *a0=0*

**Definition 1.3.1.** *Suppose* $V$ *is a* $V.S.$ *over* $\mathbb{F}$*,* $S \subseteq V$*,*

- ***Closed under Addition:*** *if* $x, y \in S$*,* $x + y \in S$*.*
- ***Closed under Scalar Multiplication:*** *if* $x \in S \Rightarrow ax \in S$*,* $\forall a \in \mathbb{F}$*.*

**Definition 1.3.2 (Subspace).** *Let* $V$ *be a VS /* $\mathbb{F}$*,* $S \subseteq V$*, say* $S$ *is a* ***Subspace*** *of* $V$ *if*

1. $S$ *is closed under addition and scalar multiplication*
2. $S \neq \varnothing$

**Theorem 1.3.2.** *Suppose* $V$ *is a vector space /* $\mathbb{F}$ *and* $S$ *is a subspace of* $V$*, then* $S$*, together the operations of* $V$ *restricted to* $S$*.*

- $+_S : S \times S \to S$
- $\cdot_S : \mathbb{F} \times S \to S$

*Proof.* Given $V$, $S$, must prove: $S$ with restricted operations of $V$, satisfying $V1$ to $V8$.

**V1**: must show: if $X, y \in S$, then $x + y = y + x$. Since $S \in V$, hence $x, y \in S \Rightarrow x, y \in V$, and $V \models V1$.

Same proof works for V2, 5, 6, 7, 8.

**V3:** know $S \neq \varnothing$, take any $x \in S$, consider $0x = 0 \in S$. (S is closed under scalar multiplication)

Hence there eixst a zero vector in $S$.

**V4:** fix $x \in S$, let $u = (-x)x \in S$, then $x + u = 1x + (-1)x = (1 + (-1))x = 0x = 0$. □

<u>**Note:**</u> in every $\mathbb{F}$, $\forall a \in \mathbb{F}$, $\exists c \in \mathbb{F} a + c = 0$, $c = -a$. Since $1 \in \mathbb{F}$, $-1 \in \mathbb{F}$.

**Theorem 1.3.3.** *If* $V$ *is a vector space over* $\mathbb{F}$ *and* $S \subseteq V$*, and* $S$ *with the operations of* $V$*, is itself a V.S. /* $\mathbb{F}$*, then* $V$ *is a subspace of* $V$*.*

## 1.4 Span - Jan 13

**Recall:** If $V$ is a V.S. / $\mathbb{F}$, and $u_1, \cdots, u_n, x \in V$, then $x$ is a linear combination (lin. combo.) of $u_1, \cdots, u_n$ if $\exists a_1, \cdots, a_n$ such that $x = a_1u_1 + a_2u_2 + \cdots + a_nu_n$.

**Definition 1.4.1.** *Suppose $V$ is a V.S. / $\mathbb{F}$, $x \in V$, and $\varnothing \neq S \subseteq V$.*

1. *Say $x$ is a lin. combo. of $S$ if $\exists$ finitely many $u_1, \cdots, u_n \in S$, s.t. $x$ is a lin. combo. of $u_1, \cdots, u_n$. $S = \{u_1, u_2, \cdots, u_n\}$, $x = \sum_{n=0}^{\infty} a_nu_n$, converge.*
2. *The **Span** of $S$ written $\text{span}(S)$, is the set of all linear combinations of $S$.*
3. $\text{span}(\varnothing) \overset{df}{=} \{0\}$

### Examples

- In $\mathbb{R}^2$, $S = \{(1,1)\}$, what is $\text{span}(S)$? the
- In $\mathbb{R}^3$, $S = \{(1,0,0),(1,1,0)\} = \{a(1,0,0) + b(1,1,0) : a, b \in \mathbb{R}\} = \{(a+b, b, 0) : a, b \in \mathbb{R}\} = (s,t,0) : s,t, \in \mathbb{R}$ =the plane given by $z = 0$
- In $\mathbb{R}[x]$, let $S = \{x, x^2, x^3, \cdots\}$, $span(S) = \{f \in \mathbb{R}[x] : f(0) = 0\}$.

**Proposition 1.4.1.** $(\varnothing \neq S \subseteq V)$. *Suppose $u_1, \cdots, u)n \in S$, $x \in V$. Suppose $x$ is a linear combination of $u_1, \cdots, u_n$. If $v_1, \cdots, v_n$ are more vectors from $S$, then $x$ is also a linear combination of $u_1, \cdots, u_n$, $v_1, \cdots, v_n$.*

**Proposition 1.4.2.** *If $S = \{u_1, \cdots, u_n\}$, then $\text{span}(S) = \{a_1u_1, \cdots, a_ku_k, a_1, \cdots, a_k \in \mathbb{F}\}$.*

**Proposition 1.4.3.** *If $S \subseteq T \subseteq V$, then $\text{span}(S) \subseteq \text{span}(T)$.*

**Proposition 1.4.4.** *If $S$ is infinite, if $x, y \in \text{span}(S)$, say $x$ is a linear combo of $u_1, \cdots, u_n \in S$, $y$ is a linear combo of $v_1, \cdots, v_n \in S$, then $x, y$ are linear combos of $u_1, \cdots, u_n, v_1, \cdots, v_n$.*

**Generalization 1.4.1.** *If $x_1, \cdots, x_k \in \text{span}(S)$, then $\exists u_1, \cdots, u_n \in S$, s.t. each $x_l$ is a linear combo of $u_1, \cdots, u_n$.*

**Theorem 1.4.1.** *Suppose $V$ is a V.S / $\mathbb{F}$, $S \subseteq V$, then $\text{span}(S)$ is the (unique) smallest subspace of $V \supseteq S$. i.e.*

1. $\text{span}(S)$ *is a subspace of $V$.*
2. $S \subseteq \text{span}(S)$
3. *If $W$ is any subspace of $V$ containing $S$, then $\text{span}(S) \subseteq W$.*

*Proof.* 1. Let $x \in S$, $x = 1x$, a linear combination of finitely many vectors in $S$.

2. i) Closure under scalar multiplication: let $x \in \text{span}(S)$, $c \in \mathbb{F}$, $\Rightarrow \exists u_1, \cdots, u_n \in S$, s.t. $x = a_1x_1 + \cdots + a_nx_n$, so

$$cx = c(a_1u_1 + \cdots + a_mu_m) = (ca_1)u_1 + \cdots + (ca_n)u_n$$

ii) Closure under vector addition: let $x, y \in \text{span}(S)$, want to prove that $x + y \in \text{span}(S)$.

By the technical remark, $\exists u_1, \cdots, u_n \in S$ s.t. $x = a_1u_1 + \cdots + a_nu_n$, $y = b_1u_1 + \cdots + a_nu_n$, $a_i, b_i \in \mathbb{F}$,

Then, $x + y = (a_1u_1 + \cdots + a_nu_n) + (b_1u_1 + \cdots + b_nu_n) = (a_1 + b_1)u_1 + \cdots + (a_n + b_n)u_n$.

So $x + y \in \text{span}(S)$.

Finally, if $S = \varnothing$, then $\text{span}(S) = \{0\}$, if $S \neq \varnothing$, then $S \subseteq \text{span}(S)$,

either case, $\text{span}(S) \neq \varnothing$, so $\text{span}(S)$ is a subspace of $V$.

3. Let $W$ be a subspace

$\square$

<u>Intuition:</u> Redundancies in span. Example: V / $\mathbb{F}$, suppose $S = \{u_1, \cdots, u_5\} \subseteq V$.

Assume $u_3$ is a linear combination of $u_2, u_4, u_5$.

$$u_3 = c_2u_2 + c_4u_4 + c_5u_5$$

<u>Claim:</u> $(S) = \text{span}(S - \{u_3\})$.

*Proof.* RTP $\subseteq$ and $\supseteq$.

$\text{span}(S)$ is

- a subspace of $V$
- which contains $S \backslash \{u_3\} = \{u_1, u_2, \cdots, u_3\}$

By the theorem, the samllest subspace of $V$ containing $S \backslash \{u_3\}$ is $\text{span}(S \backslash \{u_3\})$. hence $\text{span}(S) \supseteq \text{span}(S \backslash \{u_3\})$.

To prove that $\text{span}(S) \subseteq \text{span}(S \backslash \{u_3\})$,

let $x \in \text{span}(S)$, i.e.

$$\begin{aligned} x &= a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 + a_5u_5 \\ &= a_1u_1 + a_2u_2 + a_3(c_2u_2 + c_4u_4 + c_5u_5) + q_4u_4 + a_5u_5 \\ &= a_1u_1 + (a_2 + a_3c_2)u_2 + (a_4 + a_3c_4)u_4 + (a_5 + a_3c_5)u_5 \end{aligned}$$

$x \in Span(\{u_1, u_2, u_4, u_5\})$ $\square$

<u>Also Observe:</u>

$$0u_1 + c_2u_2 + (-1)u_3 + c_4u_4 + c_5u_5 = 0$$

A linear combination of $u_1, \cdots, u_5$ equally the $0$ vector with coefficients not all $0$.

So we code redundacies formally with definition:

**Definition 1.4.2.** *($V\mathbb{F}, S \subseteq V$), $S$ is linearly dependent if $\exists$ distinct vectors $u_1, \cdots, u_n \in S$, and $\exists a_1, \cdots, a_n \in \mathbb{F}$, not all $0$, such that*

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0(zero\ vector)$$

*$S$ is linearly independent if $S$ is not linearly dependent.*

*$S$ is linearly dependent* $\iff$ ($\exists$ *distinct* $u_1, \cdots, u_n \in S)(\exists a_1, \cdots, a_n \in \mathbb{F},\ \not\forall\, 0)(a_1u_1 + \cdots + a_nu_n) = 0$

$\equiv (\forall\ distinct\ u_1, \cdots, u_n \in S)()$

**Technical Remark:** when $S = \{u_1, \cdots, u_n\}$ without reports

- Can drop ($\forall\ distinct\ u_1. \cdots, u_n \in S$) in choice of linear independence.

-Can drop ($\exists\ distinct\ u_1 \cdots u_1, \cdots, u_n \in S$) in choice of linear dependence.

**Example 2:** Is $S = \{(0,1,1), (1,0,1), (1,2,3)\}$ linear dependent? (in $\mathbb{R}^3$)

Try to find: $a, b, c \in \mathbb{R}$ s.t.

$$a\begin{pmatrix}0\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\0\\1\end{pmatrix} + c\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}-2\\-1\\1\end{pmatrix}$$

Shows $S$ is linearly dependent.

**Question:** If $S = \varnothing$, $S$ is **linearly dependent**.

**Question 2:** If $S = \{0\}$, $S$ linearly dependent. Can write $1 \cdot 0 = 0$.

More Generally, if $0 \in S \subseteq V$, then $S$ is linearly dependent.

**Theorem 1.4.2 (Linear Dependence).** *$V\mathbb{F}$, $S \subseteq V$, then $S$ is linearly dependent, iff $S = \{0\}$ or $\exists x \in S$, s.t. $x$ is a linear combination of some vectors in $S \backslash \{x\}$.*

## 1.5 Basis Jan 17

**Recall** If $V$ is a V.S. / $\mathbb{F}$, $S \subseteq B$.

1. $\text{span}(S) =$ set of all linear combinations of $S$
2. $S$ is linearly dependent if $\exists u_1, u_2, \cdots, u_n \in S$ (distinct), $\exists a_1, \cdots, a_n \in \mathbb{F}$ not all $0$, s,t, $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$.

   - else, $S$ is linearly independent.

**Definition 1.5.1.** *$V$ is V.S. / $\mathbb{F}$,*

1. *A set $S \subseteq V$ is a spanning set of* $\text{span}(S) = V$*. Also say $S$ spans $V$.*
2. *$V$ is finitely spanned if $V$ has a finite spanning set.*
   *$V$ is countably spanned if $V$ has a countable spanning set.*

**Examples:**

$\mathbb{R}^3$ is finitely spanned, e.g. by $\{e_1, e_2, e_3\}$.

so is $\mathbb{R}^n$ e.g. by $\{e_1, e_2, \cdots, e_n\}$, $e_i = (0, 0, \cdots, 1, 0, \cdots, 0)$ with $1$ at $i_{th}$ spot.

$\mathbb{R}[x]$ is countably spanned e.g. by $\{1, x, x^2, x^3, \cdots\}$. not finitely spanned.

$\mathbb{R}^[0, 1]$ not countably spanned.

**Definition 1.5.2.** *$V$ is a V.S. / $\mathbb{F}$.*

*A <u>basis</u> for $V$ is any $S \subseteq V$, which*

- *spans $V$, and*
- *$S$ is linearly independent*

**Examples:** $\{e_1, \cdots, e_n\} \subseteq \mathbb{F}^n$ is a basis for $\mathbb{F}^n$.

$\{1, x, x^2, x^3, \cdots\} \subseteq \mathbb{R}[x]$ is a basis for $\mathbb{R}[x]$.

**Theorem 1.5.1.** *Every countably spanned V.S. has a basis.*

*Proof.* Suppose V.S. V is spanned by countable set $S$, so either $S = \{v_1, v_2, \cdots, v_n\}$, or $S = \{v_1, v_2, \cdots\}$, WLOG, we assume $0 \notin S$, define

$$T = \{v_j \in S, v_j \notin span(v_1, v_2, \cdots, v_{j-1})\},$$

Claim that $T$ is a basis for $V$.

<u>Proof of Claim:</u> $1^{st}$ show $T$ is linearly independent, by contradiction, assume $T$ is linearly dependent.

Then, $\exists k$, and scalars $a_1, a_2, \cdots, a_n$ (not all $0$), s,t,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

Choose least $k$ for which this is true.

Claim: $k \neq 1$, if $k = 1$, $a_1 v_1 = 0 \Rightarrow v_1 = 0$, but $0 \notin T$, contradiction.

so $k > 1$, Assume $a_k = 0$, then

$$a_1 v_1 + a_2 v_2 + a_{k-1} v_{k-1} = 0$$

Not all of $a_1, a_2, \cdots, a_{k-1} = 0$.

Next, show $span(S) = V$.

$$S = \{v_1, v_2, v_3, \cdots, v_n\}$$

$$T = \{v_{i_1}, v_{i_2}, v_{i_3}, \cdots\}$$

Know $\text{span}(S) = V$, intuitively $\text{span}(T) = \text{span}(S)$.

$$T = \{v_j \in S : v_j \notin \text{span}(\{v_1, v_2, \cdots, v_{j-1}\})\}$$

Therefore, $T$ is a basis of $V$.

□

**Remark:**

1. Every Vector Space has a basis. proof: some version of axiom of choice
2. bases is not unique, every V.S. except $\{0\}$, has multiple bases.
3. What is a basis for $V = \{0\}$? $\varnothing$

**Theorem 1.5.2 (Axiom of Choice).** *Suppose $A, B$ are sets, $f : A \to$.*

## 1.6 Dimension - Jan 20

**Remark:** Given a vector space $V$, the basis is not unique.

Relation between two basis of a vector space. (finitely spanned vector spaces)

**Theorem 1.6.1.** *Let $V$ be a finitely spanned vector space over a field $\mathbb{F}$, let $\{v_1, \cdots, v_m\}$ be a basis of $V$, let $\{w_1, \cdots, w_n\} \subset V$ and $n > m$. Then $\{w_1, \cdots, w_n\}$ is linearly dependent.*

*Sketch.* Idea: Replace successfully $v_1, v_2, \cdots, v_n$, by $w_1, w_2, \cdots, w_n$ so that

$$\operatorname{span}(\{w_1, w_2, \cdots, w_i, v_{i+1}, \cdots, v_m\}) = \operatorname{span}(\{v_1, v_2, \cdots, v_i, v_{i+1}\})$$

$1 \leq i \leq m - 1$. □

*Proof.* Assume $\{w_1, \cdots, w_n\}$ is linearly dependent. Prove the statement by induction.

Base Case: (i=1), since $\{v_1, \cdots, v_m\}$ is a basis for $V$ and $w_1 \in V$, there exist $a_1, \cdots, a_m \in \mathbb{F}$ s.t. $w_1 = a_1 v_1 + \cdots + a_m v_m$.

By the assumption, $w_1 \neq 0$, hence one of the $a_k'$s is nonzero.

By renumbering $v_1, \cdots, v_m$, WLOG, we can assume $a_1 \neq 0$. We can solve for $v_1$.

$$\begin{aligned} a_1 v_1 &= w_1 - a_2 v_2 - \cdots - a_m v_m \\ v_1 &= a_1^{-1} w_1 - a_1^{-1} a_2 v_2 - \cdots - a_1^{-1} a_m v_m \end{aligned}$$

so, $\operatorname{span}(\{v_1, v_2, \cdots, v_m\}) \subset \operatorname{span}(\{w_1, w_2, \cdots, w_m\}) = V$.

Induction Assumption: Assume that the statement is true for $r$. It means after renumbering, $v_1, v_2, \cdots, v_m$ we have

$$\operatorname{span}(\{w_1, w_2, \cdots, w_i, v_{i+1}, \cdots, v_m\}) = V.$$

*replace $w_{i+1}$.

Prove for $r+1$: Rewrite $w_{i+1}$ as a linear combination of $\{w_1, \cdots, w_r, v_{r+1}, \cdots, v_m\}$.

$$w_{i+1} = c_1 w_1 + \cdots + c_r w_r + d_{i+1} v_{i+1} + \cdots + d_m v_m$$

Observation: One of the $d_{r+1}, \cdots, d_m$ must be nonzero. Because if $d_{i+1} = \cdots = d_m = 0$, then

$$\begin{aligned} w_{r+1} &= c_1 w_1 + \cdots + c_r 2_r \\ 0 &= c_1 w_1 + \cdots + c_r w_r - w_{r+1} \end{aligned}$$

Contradiction since $\{w_1, \cdots, w_{r+1}\}$ is linearly independent.

WLOG, we can assume $d_{i+1} \neq 0$,

$$d_{r+1} v_{r_1} = w_{r+1} - c_1 w_1 - \cdots - a_r w_r - d_{r+2} v_{r+2} - \cdots - d_m v_m$$

Since $n > m$, $w_n = a_i w_i + \cdots + a_m w_m$, so $\{w_1, \cdots, w_n\}$ is linearly dependent.

It completes the proof. □

**Theorem 1.6.2.** *Let $V$ be a finitely spanned vector space, having one basis of $m$ elements having another basis of n elements. Then $m = n$.*

*Proof.* We could not have $m < n$, or $m > n$. If it happends, the other set must be linearly dependent. $\square$

**Definition 1.6.1.** *Let $V$ be a vector: space having a basis consisting of $n$ elements, we say $n$ is the dimensioning of $V$.*

$$\dim_{\mathbb{F}} V = n$$

$$\lim\{0 = 0\}$$

*A vector space that has a basis consisting of $n$ elements, zero elememts, zero vector space, is called finite dimensional. Otherwise, $V$ is called infinite dimensional(Hamel Basis)*

**Example:**

- $\dim \mathbb{F}^n = n$

  Since

  $$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

  is a basis for $\mathbb{F}^n$.

- $\dim P_n(\mathbb{F}) = n + 1$

  Since $\{1, x, \cdots, x^n\}$ is a basis for $P_n(\mathbb{F})$.

- $\dim \mathbb{F}[x] = \infty$

**Corollary 1.6.1.** *Let $V$ be an $n-$dimensional space, then*

- *If $\{v_1, \cdots, v_n\} \subset V$ is linearly independent, then $\{v_1, \cdots, v_n\}$ is a basis for $V$.*
- *If $\{v_1, \cdots, v_n\} \subset V$, $k < n$ is linearly we can add $v_{k+1}, \cdots, v_n$ so that $\{v_1, \cdots, v_n\}$ is a basis for $V$.*
- *If $W$ is a subspace of $V$, then $\dim W \leq \dim V$, if furthermore, $\dim W = \dim V$. Then $W = V$.*

## 1.7 Direct Sum - Tutorial Jan 20

**Corollary 1.7.1.** *If $V$ is finitely spanned, and $\beta\{v_1, \cdots, v_n\}$ is linearly independent, then $\beta$ can be extended to a basis for $V$, i.e. $\exists w_1, \cdots, w_n \in V$, s.t. $\{v_1, \cdots, v_n, w_1, \cdots, w_r\}$ is a basis for $V$*

*Proof.* Let $m = dim = V$. So $n \leq m$ by theorem.

Case 1: $\beta$ is alreayd a basis. (n=m)

Case 2: $\beta$ is not a basis. □

## 1.8 Jan 22

**Corollary 1.8.1.** *If $V$ is finitely spanned, and $\mathfrak{B} = \{v_1, \cdots, v_n\}$ is linearly independent, then $\mathfrak{B}$ can be extended to a basis for $V$.*

*i.e.* $\exists w_1, \cdots w_r \in V$, *s.t.* $\{v_1, \cdots, v_n, w_1, \cdots, w_n\}$ *is a basis for* $V$.

*Proof.* Let $m = \dim V$, so $n \leq m$. (By theorem).

**case 1:** $\mathfrak{B}$ is already a basis (n = m). done

**Case 2:** $\mathfrak{B}$ is not a basis, so $\text{span}\mathfrak{B} \neq V$, so $\exists w_1 \in V \backslash \mathfrak{B}$.

□

**Theorem 1.8.1.** *For any V.S. $V$, if $\mathfrak{B} \subseteq V$ is linearly independent, then $\mathfrak{B}$ can be extended to a basis for $V$. [use axiom of choice]*

**Example:** Let $\mathfrak{B} = \{\cos(nx), n \geq 0\} \cup \{\sin(nx) : n > 0\} \cup \{e^x\}$.

This $\mathfrak{B}$ can be extended to a basis $\mathfrak{B}'$ for $\mathbb{R}^{[0,1]}$.

$$|\mathfrak{B}'| = 2^{2^{\aleph_0}}$$

**Recall:** If $\{v_1, \cdots, v_n\} \subseteq V$ is linearly independent. Say $\{v_1, \cdots, v_n\}$ is a maximal linearly independent set, if $\forall w \in V \backslash \{v_1, \cdots, v_n\}$, $\{v_1, \cdots, v_n, w\}$ is linearly dependent.

**Corollary 1.8.2.** *If $V$ is a finitely spanned set, then every basis is a maximal linearly independent set, and vice versa.*

More generally,

**Definition 1.8.1.** *Let $V$ be a V.S., a subset $\mathfrak{B} \subseteq V$ is a* ***maximal linearly independent set*** *if*

- *$\mathfrak{B}$ is linearly independent*
- *$\forall w \in V \backslash \mathfrak{B}$, $\mathfrak{B} \cup \{w\}$ is linearly dependent.*

**Theorem 1.8.2.** *In any V.S. $V$, every basis is a maximal linearly independent set, and vice versa.*

**Definition 1.8.2.** *A* ***mininal spanning set*** *is a set $\mathfrak{B}$ such that*

- $\text{span}\mathfrak{B} = V$
- $\forall w \in \mathfrak{B}$, $\text{span}(\mathfrak{B} \backslash \{w\}) \neq V$

**Theorem 1.8.3.** *In every vector space V,*

1. *Every bassi is a minimal spanning set and vice versa*

2. *Every spanning set can be "shrunk" to a basis*
   *i.e. if* $\operatorname{span}\mathfrak{B} = V$*, then* $\exists \mathfrak{B}' \subseteq \mathfrak{B}$ *s.t.* $\mathfrak{B}'$ *is a basis for* $V$*.*

*Proof.* For (2), already proved when $\mathfrak{B}$ is countable. Can extend the proof to uncountable "well-ordering $\mathfrak{B}$".

To find a basis for $\mathbb{R}^{[0,1]}$

1. start with $\mathfrak{B} = \mathbb{R}^{[0,1]}$
2. well-order $\mathfrak{B}$ ("enumerates" $\mathfrak{B}$)
3. use the enumeration to shrink $\mathfrak{B}$ to a basis

$\square$

## 1.9 Quotient Space - Jan 24

**Review:** $\mathbb{Z}_n$ = the set of the congruence classes, $x \equiv y \pmod m \iff m|x-y$

**Revisit:** $[0] = \{qm : a \in \mathbb{Z}\} = m\mathbb{Z}$.

-$m\mathbb{Z}$ is collapsed to become zero

-$x \equiv y \pmod n \iff x = y \in m\mathbb{Z}$.

-advanced notation: $\mathbb{F}/m\mathbb{Z}$.

Version of this:

- $(\mathbb{Z}, +, \cdot) \to$ a vector space $V$.
- $(m\mathbb{Z}) \to$ a subspace of $V$.

**Definition 1.9.1.** *Fix a V.S. $V$ over $\mathbb{F}$, and a subspace $W$. For $x, y \in V$ say $x \equiv y \pmod W$, if $x - y \in W$.*

**Claim:** $\equiv \pmod W$ is an equivalence relation on $V$.

*Proof.* For transitivity:

Assume $x, y, z \in V$, $x \equiv y \pmod W$ and $y \equiv z \pmod W$, by definition, $x - y \in W$, $y - z \in W$.

Then $x - z = (x - y) + (y - z) \in W$ since $W$ is closed under addition.

Then by definition, $x \equiv z \pmod W$.

□

**Definition 1.9.2.** *Define $V, W$ as before:*

*For $x \in V$,*

$$x + W := \{x + w : w \in W\}$$

*($x$ is fixed, add $x$ to every vector on $W$). $x + W$ is called **translation of** $W$ **by** $x$, or **coset of** $W$ **through** $x$.*

**Lemma 1.9.1.** *$V, W$ as before, for any $x \in V$, the equivalence class (congruence class) of $\equiv \pmod W$ containing $x$ is $x + W$. If $y \equiv x \pmod W$, and $w \in W$, then $y \equiv x + w \pmod W$.*

*Proof.* For any $y \in V$, $y \in$ the equiv of $\equiv \pmod V$ containing $x$.

$$\begin{aligned}
&\iff \quad y \equiv x \pmod W \\
&\iff \quad y - x \in W \\
&\iff \quad y - x = w, for\ some\ w \in W \\
&\iff \quad y = x + w \\
&\iff \quad y \in x + W
\end{aligned}$$

□

**Corollary 1.9.1.** *With $V$ and $W$ as above, for any $x, y \in V$,*

$$x + W = y + W \iff x \equiv y \pmod{W} \qquad i.e. x - y \in W.$$

**Remark:** For $x \in V$, the span class of $\equiv \pmod{W}$ containing $x$ is

$$\{y \in V, y \equiv x \pmod{W}\}$$

**Definition 1.9.3.**

$$\begin{aligned} V/W &:= \textit{the set of all equiv classes of the } \equiv \pmod{W} \textit{ relation} \\ &:= \textit{the set of all translations of } W \\ &:= \{x + W : x \in V\} \neq V \end{aligned}$$

Next, we turn $V/W$ into a vector space over $\mathbb{F}$,

$$(x + W) \oplus (y + W) := (x + y) + W$$

$$c(x + w) := (cx) + W$$

**Issue:** Are the operations well-defined? Yes

E.g. check scalar multiplication:

assume $x + W = x_1 + W$, $x \equiv x_1 \pmod{W} \iff x - x_1 \in W$ .

need to know: $\forall c \in \mathbb{F}$,

$$\begin{aligned} & (cx + W) = (cx_1) + W \\ \Leftrightarrow \quad & cx \equiv cx_1 \pmod{W} \\ \Leftrightarrow \quad & (cx) - (cx_1) \in W \\ \Leftrightarrow \quad & c(x - x_1) \in W \end{aligned}$$

**Definition 1.9.4.** *$V/W$ with the natural opeartions is called the **quotient space** of $V$ modulo $W$.*

## 2 Linear Transformations

**Definition 2.0.1.** *Let $V$, $W$ be vector spaces over $\mathbb{F}$, a function $T : V \to W$ is a linear transformation (or is linear) if*

1. $T(x+y) = T(x) + T(y)$, $\forall x, y \in V$
2. $T(ax) = aT(x)$, $\forall x \in V$, $\forall a \in \mathbb{F}$

**Example**

$V = W = \mathbb{R}$ (as $V.S./\mathbb{R}$)

Fix $\lambda \in \mathbb{R}$,

$$T : \mathbb{R} \to \mathbb{R} \qquad T(x) = \lambda x$$

$T$ is a linear transformation.

Check: Let $x, y \in \mathbb{R}$, $a \in \mathbb{R}$

1. $T(x+y) = \lambda(x+y) = \lambda x + \lambda y = T(x) + T(y)$
2. $T(ax) = \lambda(ax) = a(\lambda x) = aT(x)$

fact: Every linear transformation from $\mathbb{R} \to \mathbb{R}$ has this form.

**Generalization 2.0.1.** *let $V = X = \mathbb{F}$, (field) considered as $V.S/\mathbb{F}$, every linear transformation $T : \mathbb{F} \to \mathbb{F}$ is of form $T(x) = \lambda x$ for some $\lambda \in \mathbb{F}$.*

**Example:** $V = W = \mathbb{R}^2$

define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T((x_1, x_2)) = (-x_2, x_1)$,

$T((1,0)) = (0,1)$

$T((0,1)) = (-1,0)$

Actually, $T$ is "rotation" by $90°$ c.c.w centered at $(0,0)$.

Claim: $T$ is a linear transformation.

*Proof.* $T((x_1, x_2) + (y_1, y_2)) = T((x_1 + y_1, x_2 + y_2)) = T(-(x_2 + y_2), x_1 + y_1) = (-x_2, z_1) + (-y_2, y_1) = T((x_1, x_2)) + T((y_1, y_2))$

Similarly, can check $T(a(x_1, x_2)) = aT((x_1, x_2))$ □

**Generalization 2.0.2.** *Fix $A \in M\mathbb{R}$, set of all $m \times n$ matrices with entries from $\mathbb{R}$,*

*so*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

*Define* $L_A : \mathbb{R}^n \to \mathbb{R}^n$, $L_A(x) = Ax$. $x$ *is a column vector* $nx_1$ *matrix*

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_n + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

**Claim:** $L_A$ is a linear transformation.

*Proof.* By example, $m = n = 2$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$L_A(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = (-x_2, x_1)$

□

**Generalization 2.0.3.** *Fix a field* $\mathbb{F}$*, fix* $A \in M_{m\times n}(\mathbb{F})$*,*

*define* $L_A; \mathbb{F}^n \to \mathbb{F}^m$ *by* $L_A(x) = Ax$*,*

***Claim:*** $L_A$ *is a linear transformation.*

**Recall:** $C([-1,1])$ = all continuous functions$f : [-1,1] \to \mathbb{R}$, define $T : C([-1,1]) \to \mathbb{R}$, by $T(f) = \int_{-1}^{1} f(x)dx$.

**Claim:** $T$ is a linear transformation.

*Proof.*

$$\begin{aligned} T(f+g) &= \int_{-1}^{1} (f+g)dx \\ &= \int_{-1}^{1} fdx + \int_{-1}^{1} gdx \\ &= T(f) + T(g) \end{aligned}$$

$T(af) = \int_{-1}^{1} afdx = a\int_{-1}^{1} fdx = aT(f)$ □

$D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ (set of all $f \in C(\mathbb{R})$),

$f^{(n)}$ exists, and is continuous $\forall n$.

Define $D(f) = f'$, $D$ is linear.

Some easy properties of all linear transformations, suppose $T : V \to W$ linear.

1. $T(0) = 0$

   *Proof.* (a) $T(x+0) = T(x) + T(0)$

   (b) $T(0 \cdot x) = 0T(x) = 0$

□

2. $T(x-y) = T(x) - T(y)$

   *Proof.* $T(x-y) = T(x+(-1)y) = T(x) + T((-1)y) = T(x) - T(y)$ □

3. $T(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = a_1T(x_1) + \cdots + a_nT(x_n)$

**Common Mistake:**

$$T(ax+by) = T(a)T(x) + T(b)T(y)$$

**More Examples:**

$M_{m\times n}\mathbb{F}$ is a vector space over $\mathbb{F}$, -add matrices componentwise -scalar multiply by multiplying all components

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

$$c\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix}$$

$T : M_{m\times n}(\mathbb{F}) \to M_{n\times m}(\mathbb{F})$ by $T(A) = A^t$. (transpose of $A$)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}^t = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

$(V = W)$ define $I_v : V \to V$ by $I_v(x) = x$ its linear.

## 2.1 Tutorial - Jan 27

**Goals:**

- Be able to describe the quotient space
- Be able to find a basis and the dimension of the quotient space

**Recall that:**

**Definition 2.1.1.** *$V$ is a V.S. $W \leq V/\mathbb{R}$, we call $V/W$ a quotient space if*

$$\begin{cases} (x+W)+(y+W)=(x+y)+W \\ c(x+W)=cx+W \end{cases}$$

*which $x, y \in V$, $c \in \mathbb{R}$.*

**Example:**

$V=\mathbb{R}^3$, $W=\operatorname{span}\{(0,0,1)\}$. $\mathbb{R}^3/W$ is a quotient space.

Question: What are the elements in $\mathbb{R}^3/W$?

A: $p+W$, $p \in \mathbb{R}^3$.

B: $[p+W]=\{x \in \mathbb{R}^3 | x-p \in w\}$

C: All lines that are parallel to $Z$-axis

## 2.2 Null Spance and Range

**Definition 2.2.1.** *Suppose $T : V \to W$ is a linear transformation.*

1. *The **null space** of $T$ denoted $N(T)$, is*

$$N(T) = \{x \in V : T(x) = 0\}$$

2. *The **range** of $T$ denoted as $R(T)$*

$$R(T) = \{T(x) : x \in V\} \subseteq W$$

**Example:** $D_n : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ $D_n(f) = f'$. It's linear.

What is $N(D_n)$?

$$N(D_n) = \{f \in P_n(\mathbb{R}) : f' = 0\} = \{c : c \in \mathbb{R}\}$$

$R(D_n) = P_n(\mathbb{R})$

**Theorem 2.2.1.** *Suppose $T : V \to W$ is linear*

1. *$N(T)$ is a subspace of $V$.*
2. *$R(T)$ is a subspace of $W$.*

*Proof.*

1. $T(0_v) = 0_w$ so $0_v \in N(T)$ so $N(T) \neq \varnothing$

   -closure under addition: let $x, y \in N(T)$,

$$T(x + y) = T(x) + T(y) = 0 + 0 = 0 \in N(T)$$

   -closure under scalar multiplication: let $x \in N(T)$, $c \in \mathbb{F}$

$$T(cx) = cT(x) = ca = 0 \in N(T)$$

2. $R(T) \neq \varnothing$ because $V \neq \varnothing$

   -closure under addition: let $u, v \in R(T) \subset W$, can write $u = T(x)$, $v = T(y)$, (for some $x, y \in V$), so $u + v = T(x) + T(y) = T(x + y) \in R(T)$.

   -Similar argument shows that $R(T)$ is closed under scalar multiplication.

□

**Algorithm 2.2.1 (Useful Trick).** *Suppose $T : V \to W$ is a linear transformation, suppose we know* $\text{span}\{v_1, \cdots, v_k\}$*, then*

$$\begin{aligned} R(T) =& \{T(x), x \in V\} \\ =& \{T(x) : x = a_1v_1 + \cdots a_kv_k, a_i \in \mathbb{F}\} \\ =& \{T(a_1v_1 + \cdots + a_kv_k) : a_1, \cdots, a_k \in \mathbb{F}\} \\ =& \{a_1T(v_1) + \cdots a_kT(v_k) : a_1, \cdots, a_k \in \mathbb{F}\} \\ =& \text{span}\{T(x_1), \cdots, T(x_k)\} \end{aligned}$$

**Example 1:** $D_n : P_n(\mathbb{R}) \to P_n(\mathbb{R})$

A spanning set for $P_n(\mathbb{R})$ is

$$\{1, x, x^2, x^3, \cdots x^n\}$$

so

$$\begin{aligned}\mathbb{R}(D_n) =& \text{span}\{D_n(1), D_n(x), D_n(x^2), \cdots, D_n(x^n)\}\\ =& \text{span}\{0, 1, 2x, \cdots, nx^{n-1}\}\\ =& \text{span}\{1, x, x^2, \cdots, x^{n-1}\} = P_{n-1}(\mathbb{R})\end{aligned}$$

**Example 2:** Fix $A \in M_{m\times n}(\mathbb{F})$. $L_A : \mathbb{R}^n \to \mathbb{F}^m$ by $L_A(x) = Ax$.

The "standard basis" for $\mathbb{F}^n$ is

$$\{(1, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)\}$$

$\mathbb{F}^n = \text{span}\{e_1, e_2, \cdots, e_n\}$

Say $A = \begin{pmatrix} a_{11} \cdots a_{1n} \\ \vdots \\ a_{m1} \cdots a_{mn} \end{pmatrix}$

$$L_A(e_1) =$$

**Two Basic Questions about Linear Transformation**

Question 1: Is it injective?

Question 2: Is it surjective?

**Theorem 2.2.2.** *Suppose $T : V \to W$ is linear, then $T$ is injective $\iff$ $N(T) = \{0\}$.*

*Proof.* ($\Rightarrow$) Assume $T$ is injective. i.e. $\forall x, y \in V, T(x) = T(y) \Rightarrow x = y$.

Obviously $0 \subseteq N(T)$. (Since $N(T)$ is a subspace)

For $N(T) \subseteq \{0\}$, let $x \in N(T)$ so $T(x) = 0 = T(0) \Rightarrow x = 0$.

($\Leftarrow$) Assume $N(T) = \{0\}$, prove injectively, assume $x, y \in V$ and $T(x) = T(y)$.

$\Rightarrow T(x) - T(y) = 0 \Rightarrow T(x - y) = 0 \Rightarrow x - y \in N(T) = \{0\} \Rightarrow x = y$. □

## 2.3 Jan 31

**Definition 2.3.1.** *A linear transformation $T : V \to W$ is an isomorphism if it is a bijection.*

*We also write $T : V \cong W$.*

*We say $V, W$ are **isomorphic**. (and write $V \cong W$) if $\exists T : V \cong W$.*

**Example 1:** $P_n(\mathbb{R}) \cong \mathbb{R}^{n+1}$

An example of an isomorphism $T : P_n(\mathbb{R}) \cong \mathbb{R}^{n+1}$ is

$$T(a_0 + a_1 + \cdots + a_n x^n) = (a_0, a_1, \cdots, a_n)$$

Easy facts:

1. For every V.S. V, $V \cong V$.
2. If $V \cong W$ then $W \cong V$.

**Definition 2.3.2.** *Given a linear tranformation $T : V \to W$ the*

***nullity** of $T$:* $\mathrm{nullity}(T) := \dim(N(T))$

***rank** of $T$:* $\mathrm{rank}(T) := \dim(R(T))$

**Theorem 2.3.1.** *Suppose $T : V \to W$ is linear and $dimV < \infty$, then $rank(T) + null(T) = dim(V)$.*

*Proof.* First step find basis for $N(T)$ and $R(T)$

Let $S$ be a basis for $N(T)$ let $n = dimV$, as $N(T) \subseteq V$, $S$ is linearly independent in $V$

$\Rightarrow |S| < n$. Write $S = \{v_n, \cdots, n_k\}$, $k < n$.

□

Special Case: when $T : V \cong W$, $dimV = n$

$T$ is injective $\Rightarrow N(T) = \{0\}$
$\Rightarrow null(T) = 0$
$\Rightarrow S = \varnothing$

$B = \{x_n, \cdots\}$

### 2.4 Feb 3

**Proposition 2.4.1.** *Suppose $\{v_1, \cdots, v_n\}$ is a basis for V.S. / $\mathbb{F}$.*

*Then $\forall x \in V$, $x$ can be uniquely written*

$$x = a_1 v_1 + \cdots + a_n v_n \qquad a_i \in \mathbb{F}$$

*Proof.* $\{v_1, \cdots, v_n\}$ span $V$ so every $x \in V$ can be written in this way.

For uniqueness, assume $x = a_1 v_1 + \cdots + a_n v_n = b_1 v_1 + \cdots b_n v_n$

Get $0 = (a_1 - b_1)v_1 + \cdots + (a_n b_n)v_n$. As $\{v_1, \cdots, v_n\}$ is linearly independent, get $a_1 = b_1, \cdots, a_n = b_n$. ☐

**Example:**

Let $V = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$. A plane in $\mathbb{R}^3$. $V$ is a subspace of $\mathbb{R}$.

Let $v_1 = (-1, 1, 0)$, $v_2 = (0, -1, 1)$.

$\{v_1, v_2\}$ is a basis for $V$

$x = (-3, 1, 2) \in V \Rightarrow x = 3v_1 + 2v_2$

The **coordinates** of $x$ relative to $\{v_1, v_2\}$ are $(3, 2)$.

**Definition 2.4.1.** *Let $V$ be a V.S. $\dim V = n$. An* ***Ordered Basis*** *for $V$ is an n-tuple $(v_1, \cdots, v_n)$ where $\{v_1, \cdots, v_n\}$ is a basis.*

**Notation 2.4.1.** *$\alpha, \beta, \gamma$ for ordered bases, $A, B, C$ for basis.*

**Definition 2.4.2.** *Suppose $V$ is a V.S., $\dim V = n$, $\beta$ is an ordered basis for $V$.*

*The coordinate vector of $x$ relative to $\beta$ is the unique n-tuple $(a_1, \cdots, a_n) \in \mathbb{F}^n$ s.t.*

$$x = a_1 v_1 + \cdots + a_n v_n$$

**Notation 2.4.2.** *The coordinate of $x$ relative to $\beta$ is denoted as: $[x]_\beta := (a_1, \cdots, a_n)$*

Fix $V, \mathbb{F}, \beta = (v_1, \cdots, v_n)$ as in definition.

Define

$$[\,]_\beta : V \to \mathbb{F}^n, \qquad\qquad x \mapsto [x]_p$$

**Theorem 2.4.1.** *$[\,]_\beta : V \to \mathbb{F}^n$ is an isomorphism.*

*Proof.* Let $x, y \in V$, (must show $[x + y]_\beta = [x]_\beta + [y]_\beta$)

Write

$[x]_\beta = (a_1, \cdots, a_n) \Rightarrow x = a_1 v_1 + \cdots + a_n v_n$

$[y]_\beta = (b_1, \cdots, b_n) \Rightarrow y = b_1 v_1 + \cdots + b_n v_n$

$[x + y]_\beta = (c_1, \cdots, c_n) \Rightarrow x + y = c_1 v_1 + \cdots + c_n v_n$

$$\Rightarrow (a_1 + b_1)v_1 + \cdots + (a_n + b_n)v_n = c_1v_1 + \cdots c_nv_n$$

By prop,

$$\begin{cases} a_1 + b_1 = c_1 \\ a_2 + b_2 = c_2 \\ \cdots \\ a_n + b_n = c_n \end{cases} \Rightarrow (a_1, \cdots, a_n) + (b_1, \cdots, b_n) = (c_1, \cdots, c_n) = [x]_\beta + [y]_\beta = [x+y]_\beta$$

Similarly, $[\quad]_\beta$ presents scalar multiplication, so it is linear.

**Bijection:**

**Injective:**

$$N([\quad]_\beta = \{x \in V : [x]_\beta = (0, \cdots, 0)\})$$

To show $[\quad]_\beta$ is surjective, first find a spanning set for $V = \{v_1, \cdots, v_n\}$

$$\begin{aligned} R([\quad]_\beta) =& \text{span}\{[v_1]_\beta, \cdots, [v_n]_\beta\} \\ =& \{x \in V : x = 0\} \\ =& \{0\} \end{aligned}$$

What is $[v_1]_\beta = (1, 0, \cdots, 0) = e_1$. $\square$

## 2.5 Tutorial - Feb 3

let $V$ be a V.S. / $\mathbb{F}$, a linear functional on $V$ is a linear map $f : V \to \mathbb{F}$.

The collection of all linear functionals is denoted $V^*$ and is called the dual space of $V$.

**Example 1:**

Let $V f = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$, $f(x) = f(x \cdot 1) = xf(1)$, $x \in \mathbb{R}$.

so the linear maps $f : \mathbb{R} \to \mathbb{R}$ are given by $f(x) = ax$ for some $a \in \mathbb{R}$.

**Exampel 2:**

$V = \mathbb{R}^3$, $\mathbb{F} = \mathbb{R}$, let $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$.

$$f_{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = x_1a + x_2b + x_3c = [abc] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Then $f_{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}$ is linear.

Let $f \in (T\mathbb{R}^3)^*$ recall that a linear map $f$ is determined by its values on a basis $B$.

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ so $x = x_1 e_1 + x_2 e_2 + x_3 e_3$, $e$ : the standard unit basis.

$f(x) = f(x_1 e_1) + f(x_2 e_3) + f(x_3 e_3) = x_1 f(e_1) + x_2 f(e_2) + x_3 f(e_3)$.

The values of $f$ on the basis vectors determine $f$.

Let $a_1 = f(e_1)$, then $f(x_1 e_1 + x_2 e_2 + x_3 e_3) = (a_1, a_2, a_3)^T (x_1, x_2, x_3)^T$.

so $f(x) = f_{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}$

## 2.6 Feb 5

**Proposition 2.6.1.** *Suppose $V, W$ are vector spaces over $\mathbb{F}$, $B$ is a basis for $V$. (note $V$ is not finite dimensional) and $T : V \to W$ is a linear. Then $T$ is determined by its values on vectors in $B$.*

*Direct Proof.* The claim is that if $T' : V \to W$ is another linear transformation and $T'(v) = T(v)\ \forall v \in B$.

i.e. $T'|_B = T|_B$, then $T' = T$.

Let $x \in V$. (show that $T'(x) = T(x)$)

$\Rightarrow x \in \operatorname{span}(B)$

$\Rightarrow \exists v_1, v_2, \cdots, v_n \in B, \exists a_1, \cdots, a_n \in \mathbb{F}$

s.t. $x = a_1 v_1 + \cdots + a_n v_n$.

Then

$$\begin{aligned} T'(x) =& T'(a_1 v_1 + \cdots + a_n v_n) \\ =& a' T(v_1) + \cdots + a_n T'(v_n) \\ =& a_1 T(v_1) + \cdots + a_n T(v_n) \\ =& \cdots \\ =& T(x) \end{aligned}$$

Since $x$ was arbitrary, $T' = T$. □

*Proof.* Define $D = T - T'$.

i.e. $D : V \to W$ given by $D(x) = T(x) - T'(x)$.

$D$ is linear transformation. I'll prove that $D$ is constant $0$ function by provign $N(D) = V$.

Observe $B \subseteq N(D)$, therefore, $\operatorname{span}(B) \subseteq N(D)$, i.e. $V \subseteq N(D) \Rightarrow N(D) = V$.

□

**Proposition 2.6.2.** *Suppose $V, W, \mathbb{F}, B$ as before, $B$ is a basis for $V$. **Every** function $\tau : B \to W$ extends to a unique linear transformation $T : V \to W$. (i.e. $T|_B = \tau$)*

*Proof.* Given $\tau : B \to W$, define $T : V \to W$ as follows:

given $x \in V$, write

$$x = a_1 v_1 + \cdots + a_n v_n \qquad (v_1, \cdots, v_n \in B, a_1, \cdots a_n \in \mathbb{F})$$

Let $T(x) := a_1 \tau(v_1) + \cdots + a_n \tau(v_n) \in W$.

Check $T|B = \tau$. Suppose $x \in B$, then $x = 1 \cdot x$, so $T(x) = 1\tau(x) = \tau(x)$.

<u>Check:</u> $T$ is linear.

Additivity: let $x, y \in V$, $\exists v_1, \cdots, v_n \in B$, such that

$$x = a_1 v_1 + \cdots + a_n v_n$$

$$y = b_1 v_1 + \cdots + b_n v_n$$

for sone $a_i, b_i \in \mathbb{F}$.

So $x + y = (a_1 + b_1)v_1 + \cdots + (a_n + b_n)v_n$.

$$\begin{aligned} T(x+y) =& (a_1+b_1)\tau(v_1) + \cdots + (a_n+b_n)\tau(v_n) && \text{(def of T)} \\ =& (a_1\tau(v_1) + \cdots + a_n\tau(v_n)) + (b_1\tau(v_1) + \cdots + b_n\tau(v_n)) \\ =& T(x) + T(y) && \text{(def of T)} \end{aligned}$$

Similar proof shows that $T$ preserves scalar multiplication.

So $T$ is linear. □

**Example:** $V = \mathbb{R}^3$, $W = \mathbb{R}^3$, $B = \{v_1, v_2, v_3\}$, where

$v_1 = (1, 0, 1)$

$v_2 = (1, 0, -1)$

$v_3 = (1, 1, 1)$

$B$ is a basis for $\mathbb{R}^3$ (exercise)

Define $\tau : \{v_1, v_2, v_3\} \to \mathbb{R}^2$ by

$\tau(v_1) = (1, 0)$

$\tau(v_2) = (1, 0)$

$\tau(v_3) = (\pi, e)$

Define $\tau : \mathbb{R}^3 \to \mathbb{R}^3$ extending $\tau$.

$T(a, b, c) = (a + b(\pi - 1), be)$

$T = L_{\begin{pmatrix} 1 & \pi - 1 & 0 \\ 0 & e & 0 \end{pmatrix}}$

$T(v_1) = T(1, 0, 1) = (1, 0)$

$T(v_2) = (1, 0)$

$T(1, i, 1) = (\pi, e)$

**Example 2:**

$V$ V.S. / $\mathbb{F}$, $\dim V = n$, let $\beta = (v_1, \cdots, v_n)$ be an ordered basis.

Define $\tau : \{v_1, \cdots, v_n\} \to \mathbb{F}^n$ by $\tau(v_i) = e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$.

$\tau$ extends uniquely to a linear transformation $T : V \to \mathbb{F}^n$.

$T : [\quad]_\beta$.

**Example 3:**

Same $V, \beta$.

Pick $\bar{a} = (a_1, \cdots, a_n) \in \mathbb{F}^n$.

Define $\tau_{\bar{a}} : \{v_1, \cdots, v_n\} \to \mathbb{F}$,

$\tau_{\bar{a}}(v_i) = a_i$.

$T(\bar{a})$ extends to a linear transformation. $f_{\bar{a}} : V \to \mathbb{F}$.

Exercise: What is $f_{e_i}$?