Math 148 Notes

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Section: 002

Contents

1 INTEGRATION

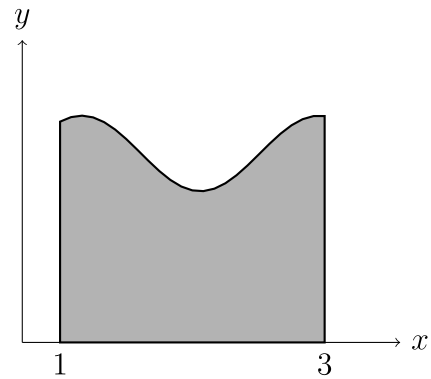
MOTIVATION: area, let $a < b$ in $\mathbb{R}$, and let $f : [a, b] \to [0, \infty]$, let

$$S_f = \{(x, y) : 0 \leq y \leq f(x), x \in [a, b]\}(\text{"}subgraph\text{"})$$

IDEA: area of rectangel = height * width

1.

Figure 1: The area under the function $\frac{1}{x}$ is $\log x$



2. approximate S_f by rectangle from below 13:43 JAN 6,

   $j = 1, \cdots, 4$, $m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$

$$\sum_{j=1}^{4} m_{j-1}(x_i - x_{j-1}) \leq area(s_f)$$

3. approximate S_f by rectangle from above, $M_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\}$

$$area \leq \sum_{j=1}^{4} M_j(x_j - x_{j-1})$$

4. if we can arrange lower sum $\approx$ upper sum, then we have some good approximation

1.1 Partition, Upper and Lower Sum

Let $a < b \in \mathbb{R}$, $f : [a, b] \in \mathbb{R}$,

Definition 1.1.1 (Partition). *A **partition** of $[a, b]$ is any finite set of points including the endpoints.*

$$P : \{x_0, x_1, \cdots, x_n\} s.t. a = x_0 < x_1 < \cdots < x_n = b$$

often for convenience, we write $P = \{a = x_0 < \cdots < x_n = b\}$.

*A **Refinement** of P is any partition Q of $[a, b]$ s,t, $P \subseteq Q$.*

Now, fix a partition P of $[a,b]$ and let $f:[a,b]\to\mathbb{R}$ be bounded on $[a,b]$, i.e. $\underset{x\in[a,b]}{sup}|f(x)|\leq M<\infty$. Write $P=\{a=x_0<\cdots<x_n=b\}$. For $j=l,\cdots,n$,

$$\begin{aligned}m_j=m_j(P)=\inf\{f(x):x\in[x_{j-1},x_j]\}\\ M_j=M_j(P)=\sup\{f(x):x\in[x_{j-1},x_j]\}\end{aligned}$$

Notice that $-M\leq m_k\leq M_j\leq M$ for each j, and these "inf", "sup" exist. (Using that $\mathbb{R}$ is complete.)

We then define after Riemann-Darboux for P and f as above.

Definition 1.1.2.

- ***Lower Sum:*** $L(f,P)=\sum_{j=1}^{n}m_j\underbrace{(x_j-x_{j-1})}_{width\ of\ [x_{j-1},x_j]}$
- ***Upper Sum:*** $U(f,P)=\sum_{j=1}^{n}M_j(x_j-x_{j-1})$

Remark:

1. if f is not bounded, at least one of $L:(f,P)$ or $U(f,P)$ cannot be defined.
2. we have $L(f,P)\leq U(f,P)$, Indeed, for each $j=l,\cdots,n$, $m_j\leq M_j$. (exactly from definition),

$$L(f,P)=\sum_{j=1}^{n}m_j(x_j-x_{j-1})\leq\sum_{j=1}^{n}M_j(x_j-x_{j-1})=U(f,P)$$

Lemma 1.1.1. *If P is a partition of $[a,b]$, $f:[a,b]\to\mathbb{R}$ is bounded, and Q is a refinement of P, then*

$$L(f,P)\leq L(f,Q)\qquad U(f,Q)\leq U(f,P)$$

Proof.

- Case 0: $Q=P$ obvious
- Case 1: $Q=P\cup\{q\}$ where $q\notin P$,

  write $P=\{a=x_0<\cdots,x_n=b\}$ so $Q=\{a=x_0<\cdots<x_{k-1}<q<x_k<\cdots<x_n=b\}$ Then,

$$\begin{aligned}m_k(P)&=\inf\{f(x):x\in[x_{k-1}],x_k\}\qquad[x_{k-1},x_k]=[x_{k-1},q]\cup[q,x_k]\\&=\min\{\inf\{f(x):x\in[x_{k-1},q]:x\in[x_{k-1},q]\}\inf f(x):x\in[q,x_k]\}\\&=\min\{m_k(Q),m_k'(Q)\}\leq m_k(Q),m_k'(Q)\end{aligned}$$

Thus,

$$\begin{aligned}
L(f,P) &= \sum_{j=1}^{m} m_j(P)(x_j - x_{j-1}) \\
&= \sum_{j=1}^{k-1} m_j(P)(x_j - x_{j-1}) + m_k(P)(x_k - x_{k-1}) + \sum_{j=k+1}^{n} m_j(P)(x_j - x_{j-1}) \\
&\leq \sum_{j=1}^{k-1} m_j(P)(x_j - x_{j-1}) + m_k
\end{aligned}$$

- Case 2: $Q = P \cup \{q_1, \cdots, q_m\}, q_1, \cdots, q_m$ distinct, $q_u \notin P$, by case 1, we have

  $$L(f,P) \leq L(f, P \cup \{q_1\}) \leq L(f, P \cup \{q_1, q_2\}) \leq \cdots \leq L(f, P \cup \{q_1, \cdots, q_m\}) = L(f,Q)$$

  The case $U(f,Q) \leq U(f,P)$ is similar.

□

Corollary 1.1.1. *let P, Q be any partition of $[a,b]$ and $f : [a,b] \to \mathbb{R}$ be bounded, then*

$$L(f,P) \leq U(f,Q)$$

Proof. We have $P, Q \subseteq P \cup Q$, i.e. $P \cup Q$ refines each of P and Q. Thus,

$$L(f,P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f,Q)$$

□

1.2 Integral and Upper and Lower Sum

Definition 1.2.1. *Given a bounded $f : [a,b] \to \mathbb{R}$, define*

- *lower integral :* $\underline{\int} a^b f = \sup\{L(f,P) : P \textit{ is a partition of } [a,b]\}$
- *Upper Integral:* $\overline{\int}_a^b f = \inf\{U(f,Q) : Q \textit{ is a partition of } [a,b]\}$

Note: $\underline{\int}_a^b f = \sup\{L(f,P) : P \textit{ is a partition of } [a,b]\} \leq \inf\{U(f,Q) : Q \textit{ is a partition of } [a,b]\} = \overline{\int}_a^b f$

*We say that f is **integrable** on $[a,b]$ provided that*

$$\underline{\int}_a^b f = \overline{\int}_a^b f$$

In this case, we write $\int_a^b f = \overline{\int}_a^b f = \underline{\int}_a^b f$

Notation: Write
$$\int_a^b f = \int_a^b f(x)d(x) = \int_a^b f(t)dt$$

Non-Example 1: not every bounded function is integrable.

Define: $\chi_{\mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$.

Let $P = \{0 = x_0 < \cdots < x_n = 1\}$ be any partition of $[0,1]$, We have that

- $\mathbb{Q}$ is dense in $\mathbb{R}$, so there is $q_j \in \mathbb{Q} \cap (x_{j-1}, x_j), j = l, \cdots, n$
- $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$, so there is $r_j \in (\mathbb{R}\backslash\mathbb{Q}) \cap (x_{j-1}, x_j), j = l, \cdots, n$

$$0 \leq L(\chi_{\mathbb{Q},P}) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}) \leq \sum_{j=1}^{n} \chi_{\mathbb{Q}}(r_j)(x_j - x_{j-1}) = 0 \Rightarrow \underline{\int_0^1} = 0$$

Likewise,

$$1 \geq U(\chi_{\mathbb{Q}}, P) \geq \sum_{j=1}^{n} \chi_{\mathbb{Q}}(q_j)(x_j - x_{j-1}) = \sum_{j=1}^{n}(x_j - x_{j-1}) = 1 - 0 = 1 \Rightarrow \overline{\int_0^1} = 1$$

hence,
$$\underline{\int_0^1} \chi_{\mathbb{Q}} = 0 < 1 = \overline{\int_0^1} \chi_{\mathbb{Q}}$$

so $\chi_{\mathbb{Q}}$ is not integrable on $[0,1]$.

Theorem 1.2.1 (**Cauchy Criterion For Integrability**). *Let $a < b \in \mathbb{R}$, $f : [a,b] \to \mathbb{R}$ be bounded, then TFAE,*

1. *f is integrable on $[a,b]$*
2. *given $\varepsilon > 0$, there exists a partition P_ε of $[a,b]$ s,t,*
$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$
and
3. *given $\varepsilon > 0$, there exists a partition P_ε of $[a,b]$ so for every refinement P of P_ε*
$$U(f, P) - L(f, P) < \varepsilon$$

Proof. 1 to 2: we assume that

$$\sup\{L(f,P) : P \, \text{partition} \ \ of \ [a,b]\} = \underline{\int_a^b} f = \overline{\int_a^b} \inf\{U(f,P) : P \, \text{partition} \ \ of \ [a,b]\}$$

Let $\varepsilon > 0$, by first equality above, there is a partition P_1 of $[a,b]$ s.t.

$$\underline{\int_a^b} f - \frac{\varepsilon}{2} < L(f, P_1)$$

and by the third equality, there is a partition P_2 s.t.

$$U(f,P_2) < \overline{\int_a^b} f + \frac{\varepsilon}{2}$$

Let $P_\varepsilon = P_1 \cup P_2$, a refinement of P_1 and P_2, then since $\underline{\int_a^b} f = \overline{\int_a^b} = \int_a^b f$ we find

$$\int_a^b f - \frac{\varepsilon}{2} < L(f,P_1) \le L(f,P_\varepsilon) \le U(f,P_\varepsilon) \le U_{f,P_2} < \int_a^b f + \frac{\varepsilon}{2}$$

$$\Rightarrow U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$$

2 to 3: we use the lemma.

If $P_\varepsilon \le P$, we have

$$L(f,P_\varepsilon) \le L(f,P) \le U(f,P) \le U(f,P_\varepsilon)$$

Hence,

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon \Rightarrow U(f,P) - L(f,P) < \varepsilon$$

3 to 2: $P_\varepsilon \subseteq P_\varepsilon$ i.e. P_ε self-defines itself

2 to 1: Given $\varepsilon > 0$, there is P_ε, a partition of $[a,b]$, so $U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon$. We have

$$L(f,P_\varepsilon) \le \underline{\int_a^b} \le \overline{\int_a^b} f \le U(f,P_\varepsilon) \Rightarrow$$

□

1.3 Continuity and Integrability

Definition 1.3.1 (Continuous). *$f : I \to \mathbb{R}$ is continuous if for every x in I, for every $\varepsilon > 0$, there exists $\delta > 0$ s.t. for all $|x - x'| < \delta$, $x' \in I$,*

$$|f(x) - f(x')| < \varepsilon$$

this definition is local. first we choose x, ε, then δ

Definition 1.3.2 (uniform Continuity). *$f : I \to \mathbb{R}$ is uniformly continuous if for every $\varepsilon > 0$, there is $\delta > 0$ so $|f(x) - f(x')| < \varepsilon$ whenever $|x - x'| < \delta$ for $x, x' \in I$.*

Proposition 1.3.1 (Sequential Test of Continuity). *Let $f : I \to \mathbb{R}$, then f is uniformly continuous $\Rightarrow$ for any sequences $(x_n)_{n=1}^\infty$, $(x'_n)_{n=1}^\infty \subset I$, with $\lim_{n\to\infty} |x_n - x'_n| = 0$, we have $\lim\limits_{n\to\infty} |f(x_n) - f(x'_n)| = 0$.*

[Fact $\Leftarrow$ also true]

Proof. Given $\varepsilon > 0$, let δ be as in def'n of uniform continuity. Since $\lim\limits_{n\to\infty} |x_n - x'_n| = 0$, there is $N \in \mathbb{N}$, so for $n \ge N$, we have $|x_n - x'_n| < \delta$.

But then, for $n \ge N$, we also have that $|f(x_n) - f(x'_n)| < \varepsilon$. i.e. $\lim\limits_{n\to\infty} |f(x_n) - f(x'_n)| = 0$. □

Example 1 $f:(0,1]\to\mathbb{R}$, $f(x)=\frac{1}{x}$. Notice that f is continuous.

Let $x_n=\frac{1}{n}$, $x'_n=\frac{1}{2n}$, $|x_n-x'_n|=\frac{1}{2n}n\overrightarrow{\to}\infty 0$.

$$|f(x_n)-f(x'_n)|=|n-2n|=n$$

Hence, not uniformly continuous.

Example 2: $g:(0,1]\to\mathbb{R}$, $g(x)=\sin\frac{1}{x}$, then g is continuous.

$x_n=\frac{1}{\pi n}$, $x'_n=\frac{2}{(2n+1)\pi}$, $|x_n-x'_n|=\frac{1}{\pi n(2n+1)}n\overrightarrow{\to}\infty 0$,

$$|g(x_n)-g(x'_n)|=\left|\sin(n\pi)-\sin(\frac{2n+1}{2}\pi)\right|=1$$

For $\varepsilon=1$, uniform continuity fails.

Theorem 1.3.1. *Let $f:[a,b]\to\mathbb{R}$ be continuous, then f is uniformly continuous.*

Proof. Let us suppose that f is continuous, but not uniformly continuous, hence there exist $\varepsilon>0$, such that for any $\delta>0$, there are $x,x'\in[a,b]$ so

$$|f(x)-f(x')|\geq\varepsilon \textit{ while } |x-x'|<\delta$$

Let us consider $\delta=\frac{1}{n}$, so there are x_n,x'_n in $[a,b]$ such that

$$|f(x_n)-f(x'_n)|\geq\varepsilon \textit{ while } |x-x'|<\frac{1}{n}$$

By Bolzano Weierstrass Theorem, there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$, such that $x=\lim\limits_{k\to\infty}x_{n_k}$ exists in $[a,b]$.

Then, notice that

$$\left|x-x'_{n_k}\right|\leq|x_n-x_{n_k}|+\left|x_{n_k}-x'_{n_k}\right|<|x-x_{n_k}|+\frac{1}{n_k}$$

hence, by Squeeze Theorem, $\lim\limits_{k\to\infty}x'_{n_k}=x$. Since f is continuous, we have that

$$\lim_{k\to\infty}f(x_{n_k})=f(x)=\lim_{k\to\infty}f(x'_{n_k})$$

$\Rightarrow$

$$\lim_{k\to\infty}\left|f(x_{n_k})-f(x'_{n_k})\right|=0$$

This contradicts that each $\left|f(x_{n_k})-f(x'_{n_k})\right|\geq\varepsilon$. Thus by contradiction argument, f' must be uniformly continuous. $\square$

Theorem 1.3.2 (**Continuous on a Closed Bounded Interval and Integrability**). *let $f:[a,b]\to\mathbb{R}$ be continuous, then f is integrable.*

Proof. Let $\varepsilon>0$, then by uniform continuity of f, there exists a δ such that whenever $|x-x'|<\delta$, for $x,x'\in[a,b]$,

$$|f(x)-f(x')|>\varepsilon$$

Thus, we let $P = \{a = x_0 < \cdots < x_n = b\}$ be any partition with length $l(P) = \max_{j=1,\cdots,n}(x_j - x_{j-1}) < \delta$.

Example: $P_n = \{a_i < a + \frac{b-a}{n} < a + 2\frac{b-a}{n} < \cdots < a + (n-1)\frac{b-1}{n} << b\}$, then $\lim_{n\to\infty} l(P_n) = 0$.

Now, by EVT, we have

$$\begin{aligned} x_j^* \in [x_{j-1}, x_j] \ s.t. \ f(x_j^*) = \inf\{f(x) : x \in [x_{j-1}, x_j]\} = m_j \\ x_j^{**} \in [x_{j-1}, x_j] \ s.t. \ f(x_j^{**}) = \sup\{f(x) : x \in [x_{j-1}, x_j]\} = m_j \end{aligned}$$

Then

$$\begin{aligned} L(f, P) &= \sum_{j=1}^{n} m_j(x_j - x_{j-1}) = \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}) \\ U(f, P) &= \sum_{j=1}^{n} f(x_j^{**})(x_j - x_{j-1}) \end{aligned}$$

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{j=1}^{n}(f(x_j^{**}) - f(x_j^*))(x_j - x_{j-1}) \\ &= \sum_{j=1}^{n} \left|f(x_j^{**}) - f(x_j^*)\right| (x_j - x_{j-1}) < \sum_{j=1}^{n} \frac{\varepsilon}{b-a}(x_j - x_{j-1}) \\ &= \frac{\varepsilon}{b-a} = \varepsilon \end{aligned}$$

Hence, we have satisfied the Cauchy Criterion for integrability. $\square$

Corollary 1.3.1. *if $f : [a, b] \to \mathbb{R}$ is continuous, then*

$$\int_a^b f = \lim_{n\to\infty} \sum_{j=1}^{n} f(a + j\frac{b-a}{n})\frac{b-a}{n}$$

Proof. We have $a + j\frac{b-a}{n} \in [a + \frac{b-a}{n}(j-1), a + \frac{b-a}{n}(j-1)]$, $j = 1, \cdots, n$.

So,

$$m_j \le f(a + j\frac{b-a}{n}) \le M_j$$

and thus

$$L(f, P_n) \le \sum_{j=1}^{n} f(a + j\frac{b-a}{n})\frac{b-a}{n} \le U(f, P_n)$$

$\lim_{n\to\infty}(U(f, P_n) - L(f, P_n)) = 0$ as $\lim_{n\to\infty} l(P_n) = 0$.

where $P_n = \{a < a + \frac{b-a}{n} < a + 2\frac{b-a}{n} < \cdots < b\}$, then proof of theorem shows that $\lim_{n\to\infty} L(f, P_a) = \int_a^b f = \lim_{n\to\infty} U(f, P_n)$ as $\lim_{n\to\infty} l(P_n) = \lim_{n\to\infty} \frac{b-a}{n} = 0$.

and hence Cauchy Criterion is satisfied, hence $\int_a^b f$ exists and is $\lim_{n\to\infty} L(f, P_n)$, apply Squeeze Theorem. $\square$

1.4 Basic Properties of Integrals

Example 1: We will let $a > 0$ and compute $\int_0^a x^p dx$ for $p = 0, 1, 2$.

1. $p = 0$, $x^p = 1$, $P = \{0 = x_0 < x_1 = a\}$, $L(1, P) = a = U(1, P)$

   $[P'$ refines P, then $L(1, P) \leq L(l, P') \leq U(1, P') \leq U(1, P) = a]$

   It follows that $\int_0^a 1dx = a$.

2. From last corollary

$$\int_0^a xdx = \lim_{n\to\infty} \sum_{j=1}^n (j\frac{a}{n})\frac{a}{n} = \lim_{n\to\infty} \frac{a^2}{n^2} \sum_{j=1}^n j = \lim_{n\to\infty} \frac{a^2}{n^2} \frac{n(n+1)}{2} = \frac{a^2}{2}$$

3. We need a fomula for $\sum_{j=1}^n j^2$.

   Trick:

$$\begin{aligned}
(n+1)^3 - 1 &= \sum_{j=1}^n [(j-1)^3 - j^3] && \text{(telescope)} \\
&= \sum_{j=1}^n [\sum_{k=0}^3 \binom{3}{k} j^k - j^3] && \text{(binomial theorem)} \\
&= \sum_{j=1}^n \sum_{k=0}^2 \binom{3}{k} j^k \\
&= \sum_{k=0}^3
\end{aligned}$$

$$\begin{aligned}
\int_0^a x^2 dx &= \lim_{n\to\infty} \sum_{j=1}^n (j\frac{a}{n})^2 \frac{a}{n} \\
&= \lim_{n\to\infty} \frac{a^3}{n^3} \sum_{j=1}^n j^2 \\
&= \lim_{n\to\infty} \frac{a^3}{3n^3} a[(n+1)^3 - 1 - n - \frac{n(n+1)}{2}] \\
&= \frac{a^3}{3}
\end{aligned}$$

Algorithm 1.4.1 (Basic Properties Of Integrals).

Proposition 1.4.1 (Additivity over intervals). *Let $a < b < c \in \mathbb{R}$, and $f : [a, c] \to \mathbb{R}$ satisfies that f is integrable on each of $[a, b]$, $[b, c]$, then*

- *f is integrable on $[a, c]$ and $\int_a^c f = \int_a^b f + \int_b^c f$.*

Proof. Given $\varepsilon > 0$, the Cauchy Criterion provides that

- a partition P_1 of $[a,b]$ s.t. $U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}$
- a partition P_2 of $[b,c]$ s.t. $U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{2}$

Let P be any refinement of $P_1 \cup P_2$. Then

$$L(f,P) \geq L(f,P_1 \cup P_2) = L(f,P_1) + L(f,P_2)$$

$$U(f,P) \leq U(f,P_1 \cup P_2) = U(f,P_1) + U(f,P_2)$$

Then

$$U(f,P) = L(f,P) \leq U(f,P_1) - L(f,P_1) + U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

hence, f is integrable on $[a,b]$.

Let P as above, be written $P = \{a = x_0 < \cdots < x_n = c\}$.

Let $Q_1 = \{a = x_0 < \cdots < x_m = b\}$, $Q_2 = \{b = x_m < \cdots < x_n = c\}$.

We have

$$L(f,Q_1) \leq \int_a^b f \leq U(f,Q_1) \qquad L(f,Q_2) \leq \int_b^c f \leq U(f,Q_2)$$

from the proof above, we have

$$L(f,P) = L(f,Q_1) + L(f,Q_2) \leq \int_a^b f + \int_b^c f \leq U(f,Q_1) + U(f,Q_2) = U(f,P)$$

Since f is integrable on $[a,c]$, we have

$$\int_a^c = \sup\{L(f,P) : P\; partition\; of [a,c]\} \leq \int_a^b f + \int_a^c f \leq \inf\{U(f,P) : P\; partition\; of [a,c]\} = \int_a^c f$$

$\Rightarrow$

$$\int_a^c f = \int_a^b f + \int_b^c f$$

$\square$

1.5 Riemann Sum - Jan 13 Mon, Jan 15 Wed

Definition 1.5.1 (Riemann Sums). *Let $f : [a, b] \to \mathbb{R}$, $P = \{a = x_0 < \cdots = x_n = b\}$.*

*A **Riemann Sum** is any sum of the following form:*

$$S(f, P) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) \qquad t_j \in [x_{j-1}, x_j], j = 1, \cdots, n$$

Left Sum:

$$S_l(f, P) = \sum_{j=1}^{n} f(x_{j-1})(x_j - x_{j-1})$$

Right Sum:

$$S_r(f, P) = \sum_{j=1}^{n} f(x_j)(x_j - x_{j-1})$$

Mid-point Sum:

$$S_m(f, P) = \sum_{j=1}^{n} f(\frac{x_{j-1} + x_j}{2})(x_j - x_{j-1})$$

Trapezoid Sum

$$\begin{aligned} T(f, P) = \frac{1}{2}[S_l(f) + S_r(f)] &= \sum_{j=1}^{n} \frac{f(x_j) + f(x_j)}{2}(x_j - x_{j-1}) \\ &= \frac{1}{2} f(a)(x_1 - a) + \sum_{j=1}^{n-1} f(x_j)(x_j - x_{j-1}) \\ &\quad + \frac{1}{2} f(b)(b - x_{n-1}) \end{aligned}$$

Theorem 1.5.1. *If $f : [a, b] \to \mathbb{R}$, then TFAE,*

1. *f is integrable and*
2. *there is a number I_f satisfying the following: given any $\varepsilon > 0$, there exists a partition P_ε of $[a, b]$ such that*

   for any refinement of P of P_ε, any Riemann Sum of $S(f, P)$ we have

   $$|S(f, P) - I(f)| < \varepsilon$$

   Furthermore, $I_f = \int_a^b f$.

Proof.

- (i)$\Rightarrow$ (ii) Given $\varepsilon > 0$, the Cauchy Criterion provides that P_ε so for any refinement P of P_ε,

  $$U(f, P) - L(f, P) < \varepsilon$$

  Write $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, and let for $j = 1, \cdots, n$, $t_j = [x_{j-1}, x_j]$.

We observe that

$$m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\} \leq \sup\{f(x) : x \in [x_{j-1}, x_j]\} = M_j$$

and hence

$$\sum_{j=1}^{n} m_j(x_j - x_{j-1}) \leq \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) \leq \sum_{j=1}^{n} M_j(x_j - x_{j-1})$$

i.e.

$$L(f, P) \leq S(f, P) \leq U(f, P) \qquad (2)$$

Also,

$$L(f, P) \leq \int_a^b f \leq U(f, P) \qquad (3)$$

(1), (2) & (3) $\Rightarrow$

$$\left| S(f, P) - \int_a^b f \right| < \varepsilon$$

In particular, take $I_f = \int_a^b f$.

- (ii) $\Rightarrow$ (i), we let for $\varepsilon > 0$, given $P_{\varepsilon/4}$ be a partition s.t.

$$|S(f, P) - I_f| < \frac{\varepsilon}{4}$$

For P a refinement of $P_{\varepsilon/4}$, $S(f, P)$ a Riemann Sum. We fix such $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$.

FOr $j = 1, \cdots, n$, let m_j, M_j be as below, we then find for each j,

$$x_j^*, x_j^{**} \in [x_{j-1}, x_j] s.t. f(x_j^*) < m_j + \frac{\varepsilon}{4(b-a)} \ and \ M_j - \frac{\varepsilon}{4(b-a)} < f(x_j^{**})$$

We then consider Riemann Sums

$$S^*(f, P) = \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}), \qquad S^{**}(f, P) = \sum_{j=1}^{n} f(x_j^{**})(x_j - x_{j-1})$$

Notice that

$$\begin{aligned} S^*(f, P) - L(f, P) &= \sum_{j=1}^{n} [f(x_j^*) - m_j](x_j - x_{j-1}) \\ &< \sum_{j=1}^{n} \frac{\varepsilon}{4(b-a)}(x_j - x_{j-1}) \\ &= \frac{\varepsilon}{4(b-a)}(b-a) = \frac{\varepsilon}{4} \end{aligned}$$

and likewise,

$$U(f, P) - S^{**}(f, P) < \frac{\varepsilon}{4}$$

thus

$$\begin{aligned} U(f, P) - L(f, P) &= U(f, P) - S^{**}(f, P) + S^{**}(f, P) - I_f + I_f - S^*(f, P) + S^*(f, P) - L(f, P) \\ &< \frac{\varepsilon}{4} + |S^{**}(f, P) - I_f| + |I_f - S^*(f, P)| + \frac{\varepsilon}{4} < \varepsilon \end{aligned}$$

hence, by Cauchy's Criterion, f is integrable. □

Given $t_j \in [x_{j-1}, x_j]$ and $f, g : [a, b] \to \mathbb{R}$, we have for $\alpha, \beta \in \mathbb{R}$,

$$S(\alpha f + \beta g, P) = \alpha S(f, P) + \beta S(f, P)$$

Remark: If $f : [a, b] \to \mathbb{R}$ is continuous, then P a partition of $[a, b]$ then each of $L(f, P)$ and $U(f, P)$ are Riemann Sums, proof: See proof of integrability of continuous.

Proposition 1.5.1 (linearity of integration). *Let $f, g : [a, b] \to \mathbb{R}$ each be integrable and $\alpha, \beta \in \mathbb{R}$, then*

- $\alpha f + \beta g : [a, b] \to \mathbb{R}(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x)$
- $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$

Proof. Let $\varepsilon > 0$, then find partitions of $[a, b]$.

- P_1 s.t. for any refinement P of P_1, and any Riemann Sum $S(f, P)$

$$\left| S(f, P) - \int_a^b f \right| < \frac{\varepsilon}{2|\alpha| + 1}$$

- P_2 s.t. for any refinement of $\mathbb{Q}$ of P_2, and any Riemann Sum $S(g, P)$,

$$\left| S(g, Q) - \int_a^b g \right| < \frac{\varepsilon}{2|\beta| + 1}$$

Now we let $P = \{P_1 \cup P_2\}$, a refinement of each of P_1 and P_2, write $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, and choose $t_j \in [x_{j-1}, x_j]$ for each j. Then

$$\begin{aligned} S(\alpha f + \beta g, P) &= \sum_{j=1}^{n} (\alpha f(t_j) + \beta g(t_j))(x_j - x_{j-1}) \\ &= \alpha \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) + \beta \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) + \beta \sum_{j=1}^{n} g(t_j)(x_j - x_{j-1}) \end{aligned}$$

Then we have,

$$\begin{aligned} \left| S(\alpha f + \beta g, P) - [\alpha \int_a^b f + \beta \int_a^b g] \right| &\leq |\alpha| \left| S(f, P) - \int_a^b f \right| + |\beta| \\ \left| S(g, P) - \int_a^b g \right| &< |\alpha| \frac{\varepsilon}{2|\alpha| = 1} + |\beta| + \frac{\varepsilon}{2|\beta| + 1} \end{aligned}$$

□

Proposition 1.5.2 (Order Properties of Integrals). *Let $f, g : [a, b] \to \mathbb{R}$ each be integrable, then*

1. $f \geq 0 \Rightarrow f \geq 0$
2. $f \geq g \Rightarrow \int_a^b f \geq 0$
3. $f \geq g$ *on* $[a, b] \Rightarrow \int_a^b f \geq \int_a^b g$

4. $|f| : [a,b] \to \mathbb{R}(|f|(x) = |f(x)|)$ *is integrable, with* $\left|\int_a^b f\right| \leq \int_a^b |f|$

5. $g \vee g,\ f \wedge g : [a,b] \to \mathbb{R}$ *(*$f \vee g(x) = \max\{f(x), g(x)\}, f \vee g(x) = \min\{f(x), g(x)\}$*) are each integrable*

Proof.

1. for any partition P, $L(f,P) > 0$.
2. $f-g$ is integrable with $f-g \geq 0$, so $\int_a^b f - \int_a^b g = \int_a^b (f-g) \geq 0$, by 1.
3. let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, and for each $j = 1, \cdots, n$

$\square$

1.6 Fundamental Theorem Of Calculus - Jan 17 Friday

Proposition 1.6.1. *Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, define*

$$F : [a, b] \to \mathbb{R}, \qquad F(x) = \int_a^x f = \int_a^x f(t)dt$$

no $\int_a^x f(x)dx$.

We may call this "integral accumulation function".

1. *F is continuous on $(a, b]$*
2. $\lim_{x \to a^+} F(x) = 0$

hence, we define $F(a) = 0 = \int_a^a f$. Thus $F : [a, b] \to \mathbb{R}$, and is continuous on $[a, b]$.

Proof.

1. A1. Q5(c) assume that f is integrable on each $[a, x]$, $x \in [a, b]$, so $F(x) = \int_a^x f$ makes sense. Now, let $a < x < x' \leq b$, and we compute

$$\begin{aligned} F(x') - F(x) &= \int_a^{x'} f - \int_a^x f \\ &= \int_a^x f + \int_x^{x'} f - \int_a^x f && \text{(additivity)} \\ &= \int_x^{x'} f \end{aligned}$$

   Since f is integrable, it is bounded i.e. $\overset{\sup}{x \in [a, b]} |f(x)| = M < \infty$. Thus, $|f(x)| \leq M$ on $[a, b]$. Hence, by order properties,

$$|F(x') - F(x)| = \left| \int_x^{x'} f \right| \leq \int_x^{x'} |f| \leq \int_x^{x'} M = M(x' - x) = M |x' - x|$$

   Thus, given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{M+1}$, we have

$$|x' - x| < \delta \Rightarrow |F(x') - F(x)| \leq M\delta = M \frac{\varepsilon}{M + 1} < \varepsilon$$

   hence, F is uniformly continuous on $[a, b]$.

2. We use M as above

$$\left| \int_a^x f - 0 \right| = \left| \int_a^x f \right| \leq \int_a^b |f| \leq \int_a^x M = M(x - a)$$

   Porceed as above.

$\square$

Theorem 1.6.1 (**Mean Value For Integrals** or **Average Value for Integrals**). *Let $f : [a, b] \to \mathbb{R}$ be continuous (integrability follows), then there exists $c \in [a, b]$, s.t.*

$$\int_a^b f = f(c)(b-a)$$

Proof. We use two important facts about continuous functions.

By **EVT**, there exists $x^*, x^{**} \in [a, b]$ s.t.

$$f(x^*) = m = min\{f(x) : x \in [a, b]\} \qquad and \qquad f(x^* *) = M \max\{f(x) : x \in [a, b]\}$$

Then $m \leq f \leq M$, on $[a, b]$ so order properties provide

$$m(b-a) = \int_a^b m \leq \int_a^b f \leq \int_a^b M = M(b-a)$$

so

$$f(x^*) = m \leq \frac{1}{b-a}\int_a^b f \leq M = f(x^{**})$$

By **IVT**, Since $f(x^*) \leq \frac{1}{b-a}\int_a^b f \leq f(x^{**})$, there is c between x^* and x^{**}, and hence $c \in [a, b]$ s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f$$

□

f is integrable $\Rightarrow F(x) = \int_a^b f$ is a cts function. f cts $\Rightarrow F$ differentiable. (BELOW)

Theorem 1.6.2 (**Fundamental Theorem of Calculus (I)**). *Let $f : [a, b] \to \mathbb{R}$ be continuous, then*

$$F : [a, b] \to \mathbb{R}, \qquad F(x) = \int_a^x f$$

satisfies that

- *F is differentiable on $[a, b]$, with $F' = f$ on $[a, b]$*
-

Proof. Let $x \in [a, b]$, we want to examine the quotient

$$\frac{F(x+h) - F(x)}{h} \qquad when \qquad x + h \in [a, b]$$

$h > 0$,

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f = \frac{1}{h}f(c_h)(x+h-x) = f(c_h)$$

by M.V.T for I, where $c_h \in [x, x+h]$,

$h < 0$,

$$\frac{F(x+h) - F(x)}{h} = \frac{F(x) - F(x+h)}{-h} = \frac{1}{-h}\int_{x+h}^x f = \frac{1}{-h}f(c_h)(x - x(x_h)) = f(c_h)$$

hence,

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \underbrace{\lim_{h\to 0} f(c_h)}_{continuity} = \underbrace{f(\lim_{h\to 0} c_h)}_{squeeze} = f(x)$$

Thus, $F'(x)$ exists, and equals $f(x)$, for $x \in [a,b]$

Remark: Notice that we really found

- left derivative at $x = b$
- right derivative at $x = a$

$\square$