# Math 247 Notes

velo.x

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# Contents

# 1 Rectangles in n-dimen space (Lec 1-6)

## 1.1 volumn on half-open rectangles

**Definition 1.1.1.** *Let $n > 0, n \in \mathbb{Z}$.*

- ***Half-open rectangle*** *in $\mathbb{R}^n$ to refer to sets of the form:*

$$(a_1, b_1] \times \cdots \times (a_n, b_n] := \{(x_1, \dots, x_n) \mid a_1 < x_1 \le b_1, \ \dots, \ a_n < x_n \le b_n\} \subseteq \mathbb{R}^n \ ,$$

  *with $a_i < b_i$ in $\mathbb{R}$.*

  *Denote*

$$\mathcal{P}_n := \{(a_1, b_1] \times \cdots \times (a_n, b_n] \mid a_1 < b_1, \dots, a_n < b_n \text{ in } \mathbb{R}\} \cup \{\varnothing\}$$

- ***Volumn Function:*** *define the function $\mathrm{vol}_n : P_n \to [0, \infty)$ that $\mathrm{vol}_n(\varnothing) = 0$ and*

$$\mathrm{vol}_n\big((a_1, b_1] \times \cdots \times (a_n, b_n]\big) = (b_1 - a_1) \cdots (b_n - a_n) \ .$$

  *For $P \in \mathcal{P}_n$, the number $\mathrm{vol}_n(P)$ is called the **n-dimensional volume** of $P$.*

***Remark:*** In the above Equation, $\mathcal{P}_n$ is a collection of subsets of $\mathbb{R}^n$, it is a set of sets (rectangles). And $\mathrm{vol}_n$ is a function defined on this $\mathcal{P}_n$.

**Remark 1.1.1** (Set operations with half-open rectangles)**.**

1. *For Intersection: $P_n$ is a $\pi$**-system.** For $k \ge 2$, $A_1, \dots, A_k \in \mathcal{P}_n$, $A_1 \cap \cdots \cap A_k \in \mathcal{P}_n$. i.e., the intersection of rectangles is still a rectangle.*
2. *For Union: $A, B \in \mathcal{P}_n$, the set $A \cup B$ may or may not be in $\mathcal{P}_n$. i.e., the union of rectangle may not be a rectangle.*
3. ***For Set Difference:*** *for any $P, Q \in \mathcal{P}_n$ one can find some $k \in \mathbb{N}$ and $R_1, \dots, R_k \in \mathcal{P}_n$ such that $R_i \cap R_j = \varnothing$ for $i \ne j$ and such that $R_1 \cup \cdots \cup R_k = P \setminus Q$.*

*We will elaborate on item 3.*

**Fact 1.1.1.** *Let $P$ be a half-open rectangle in $\mathcal{P}_n$ and let $\varepsilon > 0$ be given, one can find a decomposition $P = P_1 \cup \cdots \cup P_k$ with $P_1, \dots, P_k \in \mathcal{P}_n$ pairwise disjoint such that*

$$\max\{\mathrm{diam}(P_1), \dots, \mathrm{diam}(P_k)\} < \varepsilon.$$

*idea.* Decompose every $(a_1, b_i]$ into as a disjoint union of half-open intervals of length smaller than $\varepsilon$, then take Cartesian products of such intervals of small length. □

**Remark 1.1.2** ($\mathrm{vol}_n$ is decomposition-additive)**.**
*Let $P$ be a half-open re ctanlge in $\mathcal{P}_n$ and consider a decomposition $P = P_1 \cup \cdots \cup P_n$, $P_1, \dots, P_n \in \mathcal{P}_n$ pairwise disjoint, then it follows that*

$$\sum_{i=1}^{k} \mathrm{vol}_n(P_i) = \mathrm{vol}_n(P).$$

## 1.2 More abstract definitions

**Definition 1.2.1.** *Let $X$ be a non-empty set and let $\mathcal{C}$ be a collection of subsets of $X$. If $\mathcal{C}$ has the property that*

$$\left(A, B \in \mathcal{C}\right) \Rightarrow A \cap B \in \mathcal{C}$$

*then $\mathcal{C}$ is $\pi$-**system**, then induction on $k$ shows that*

$$A_1 \cap \cdots \cap A_k \in \mathcal{C} \text{ for all } k \geq 2$$

**Definition 1.2.2.** *Let $X$ be a non-empty set and let $\mathcal{C}$ be a collection of subsets of $X$. We say that $\mathcal{C}$ is a semi-ring of subseteq of $X$ to mean that it fullfills the following conditions:*

- *$SemiRing1$: $\varnothing \in \mathcal{C}$.*
- *$SemiRing2$: If $A, B \in \mathcal{C}$ then $A \cup B \in \mathcal{C}$.*
- *$SemiRing3$: for all $A, B \in \mathbb{R}^n$, exists $k \in \mathbb{N}$ and $C_1, \ldots, C_k \in \mathcal{C}$ that $C_i \cup C_j = \varnothing$ for $i \neq j$ and $C_1 \cup \cdots \cup C_k = A \setminus B$.*

## 1.3 Divisions and their refinement

In this part, we fix a non-empty set $X$ and a collection $\mathcal{C}$ of subsets of $X$.

**Definition 1.3.1** (Divisions and Their Refinement)**.** *Let $A$ be a non-empty set in $\mathcal{C}$.*

1. *A division of $A$ is a set $\Delta = \{A_1, \ldots, A_p\}$ where $A_i$ are non-empty sets in $\mathcal{C}$ such that $A_1 \cup \cdots \cup A_p = A$ and such that $A_i \cup A_j = \varnothing$ for $i \neq j$.*
2. *Let $\Delta = \{A_1, \ldots, A_p\}$ and $\Gamma = \{B_1, \ldots, B_q\}$ be two divisions of $A$.*

   *Say that $\Gamma$ **refines** $\Delta$, denoted $\Gamma \prec \Delta$ to mean that for every $1 \leq j \leq q$ there exists $1 \leq i \leq p$ such that $B_j \subseteq A_i$.*

Exercise: Let $A$ be a set in $\mathcal{C}$ and let $\Delta = \{A_1, \ldots, A_p\}$ and $\Gamma = \{B_1, \ldots, B_q\}$ be divisions of $A$ such that $\Gamma \prec \Delta$. Prove that one can re-denote the sets of $\Gamma$ in the form

$$\Gamma = \{B_{1,1}, \ldots, B_{1,q_1}, \ldots, B_{p,1}, \ldots, B_{p,q_p}\}$$

such that $\{B_{1,1}, \ldots, B_{1,q_i}\}$ is a division of $A_i$.

*Proof.* Let $A'_i = \bigcup_{\{B_j | B_j \subseteq A_i\}} B_j$, then $A'_i \subseteq A_i$. Then, since $\Gamma \prec \Delta$,

$$\bigcup_{i=1}^{p} A_i = A = \bigcup_{i=1}^{q} B_i = \bigcup_{i=1}^{p} \bigcup_{B_j \subseteq A_i} B_j = \bigcup_{i=1}^{p} A'_i \ .$$

Since each $A'_i \subseteq A_i$ and each $A_i$ disjoint, then, each $A'_i$ disjoint.

For any $A_i$, if $x \in A_i$, then $x \notin A_m$, $i \neq m$, then $x \notin A'_m$, then, $x \in A'_i$. Hence, $A'_i = A_i$. Then, each $A_i$ is a union of some $\{B_{1,1}, \ldots, B_{1,q_i}\}$. $\square$

**Definition 1.3.2.** *Let $A$ be a non-empty set from $\mathcal{C}$ and let $\Delta' = \{A'_1, \ldots, A'_p\}$, $\Delta'' = \{A''_1, \ldots, A''_q\}$ be two divisions of $A$. Then the set of sets*

$$\Delta' \wedge \Delta'' = \{A'_i \cap A''_j \mid 1 \leq i \leq p, 1 \leq j \leq q, A'_i \cap A''_j \neq \varnothing\}$$

*is a division of $A$ as well, called the **meet** of $\Delta'$ and $\Delta''$.*

*Notice that*

- *the sets in $\Delta' \wedge \Delta''$ are pairwise disjoint*
- *the union of these sets is $A$*
- $\Delta' \wedge \Delta'' \prec \Delta'$ *and* $\Delta' \wedge \Delta'' \prec \Delta''$

**Corollary 1.3.1** (Any two divisions have common refinements)**.** *Let $A$ be a non-empty set from $\mathcal{C}$ and let $\Delta' = \{A'_1, \ldots, A'_p\}$, $\Delta'' = \{A''_1, \ldots, A''_q\}$ be two divisions of $A$. There exists a division $\Gamma$ of $A$ such that $\Gamma \prec \Delta'$ and $\Gamma \prec \Delta''$.*

## 1.4 Integrable Functions on Rectangles

**Basic Idea:** extended the integrability in MATH 148 to integrability on $\mathcal{P}_n$, and $\mathcal{C}$($\pi$-system).

Lecture 2 & 3 & 4

## 1.5 Inequalities and Properties of Integrals

**Proposition 1.5.1** (Non-negative Property)**.** *Let $f \in \mathrm{Int}_b(A, \mathbb{R})$ be such that $f(x) \geq 0$ for all $x \in A$, then $\int_A f \geq 0$.*

**Proposition 1.5.2.**

- *Let $f_1, f_2 \in \mathrm{Int}_b(A, \mathbb{R})$ be such that $f_1 \leq f_2$, then $\int f_1 \leq \int f_2$.*
- *For all $f, g \in \mathrm{Int}_b(A, \mathbb{R})$ we have*

$$\int_A f \vee g \geq \max\left(\int_A f, \int_A g\right), \qquad \int_A f \wedge g \leq \min\left(\int_A f, \int_A g\right)$$

- *For any $f \in \mathrm{Int}_b(A, \mathbb{R})$ we have*

$$\left|\int_A f\right| \leq \int_A |f|$$

## 1.6 Lecture 6: Continuity and Integrability

In MATH 148, $f : \mathbb{R} \to \mathbb{R}$. Here, $f : A \to \mathbb{R}$ which $A \subseteq \mathbb{R}^n$.

**Definition 1.6.1.** *Let $A \subseteq \mathbb{R}^n, A \neq \varnothing$, let $\vec{a} \in A$, and let $f : A \to \mathbb{R}$,*

1. *$f$ is **CONT-at-$\vec{a}$**, if for all $\varepsilon > 0$, exists $\delta > 0$ that for all $\vec{x} \in A$, $||\vec{x}-\vec{a}|| < \delta$, $||f(\vec{x})-f(\vec{a})|| < \varepsilon$.*
2. *$f$ is **SEQ-CONT-at-$\vec{a}$** if whenever $(\vec{x}_k)_{k=1}^{\infty}$ is a sequence in $A$ with $\lim_{k\to\infty} \vec{x}_k = \vec{a}$, then $\lim_{k\to\infty} f(\vec{x}_k) = f(\vec{a})$.*

*Two definitions are equivalent. We say $f$ is continuous on $A$ to mean that $f$ is continuous at every point $\vec{a} \in A$.*

**Definition 1.6.2.** *Let $A \subseteq \mathbb{R}^n, A \neq \varnothing$, and let $f : A \to \mathbb{R}$,*

1. *$f$ is **UNIF-CONT** on $A$ to mean that for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $\vec{x}, \vec{y} \in A$ have $||\vec{x} - \vec{y}|| < \delta$, it follows that $|f(\vec{x}) - f(\vec{y})| < \varepsilon$.*
2. *Let $c \in [0, \infty)$. $f$ is **c-LIPSCHITZ** on $A$ if $|f(\vec{x}_1) - f(\vec{x}_2)| \leq c||\vec{x}_1 - \vec{x}_2||$, $\forall \vec{x}_1, \vec{x}_2 \in A$.*

   *$f$ is Lipschitz on $A$ to mean that there exists $c \in [0, \infty)$ such that $f$ is c-Lipschitz on $A$.*

**Theorem 1.6.1.** *Let $P$ be a half-open rectangle from $\mathcal{P}_n$, and let $f : P \to \mathbb{R}$ be a bounded function, then if $f$ is UNIF-CONT on $P$, then it is integrable on $P$.*

*Proof.* Let $V = \mathrm{vol}_n(P)$. Then, there is some $\delta > 0$ such that for all $\vec{x}, \vec{y} \in P$,

$$||\vec{x} - \vec{y}|| < \delta \qquad \Rightarrow \qquad |f(\vec{x}) - f(\vec{y})| < \frac{\varepsilon}{2V} .$$

then,

$$U(f, \Delta) - L(f, \Delta) = \sum_{i=1}^{n}$$

□

**Definition 1.6.3.** *Let $P \in \mathcal{P}_n$, and let $f : P \to \mathbb{R}$ be a bounded function. We say that $f$ is **Unif-Cont-mod-SmallVol** when for all $\varepsilon > 0$, there is some $E_1, \ldots, E_k \in \mathcal{P}_n$ such that $E_1, \ldots, E_k \subseteq P$ and $E_i \cup E_j = \varnothing$ for all $i \neq j$, such taht $\sum_{i=1}^{k} \mathrm{vol}_n(E_i) < \varepsilon$, and such that $f$ is uniformly continuous on $P \setminus \cup_{i=1}^{k} E_i$.*

**Theorem 1.6.2.** *Let $P \in \mathcal{P}_n$, and let $f : P \to \mathbb{R}$ be a bounded function. If $f$ has the UnifCont-mod-SmallVol property, then $f$ is integrable on $P$.*

*Proof.* Left to fill in for review. □

# 2 Topology of $\mathbb{R}^n$

## 2.1 Lecture 7: Interior, Closure and Boundry

**Definition 2.1.1** (Interior and Closure for a subset of $\mathbb{R}^n$)**.** *Let $A$ be a subset of $\mathbb{R}^n$.*

1. *An **interior point** is a point $\vec{a} \in A$ which there exists $r > 0$ such that $B(\vec{a}; r) \subseteq A$.*

   *(definition of Ball in <ins>section 1.1</ins>)*

   *The set of all interior points of $A$ is called the **interior** of denoted by $\operatorname{int}(A)$.*

2. *An **adherent point** is a point $\vec{b} \in \mathbb{R}^n$ which $B(\vec{b}; r) \cap A \neq \varnothing$ for every $r > 0$.*

   *The set of all points that are adherent to $A$ is called the **closure** of $A$, denoted by $\operatorname{cl}(A)$.*

**Remark 1:** A $\vec{p}$ that is adherent to $A$ is not necessarily in $A$, that is $\vec{p} \in \operatorname{cl}(A) \not\Rightarrow \vec{p} \in A$.

**Remark 2:** For every subset $A \subseteq \mathbb{R}^n$ we have $\operatorname{int}(A) \subseteq A \subseteq \operatorname{cl}(A)$.

**Definition 2.1.2** (boundry)**.** *The **boundry** of $A$ is denoted $\operatorname{bd}(A) := \operatorname{cl}(A) \setminus \operatorname{int}(A)$.*

**Proposition 2.1.1** (operation of closure and interior with respect to inclusion of sets)**.**
*For two subsets of $\mathbb{R}^n$ $M, N$ and suppose $M \subseteq N$ then*

$$\operatorname{int}(M) \subseteq \operatorname{int}(N) \qquad \textit{and} \qquad \operatorname{cl}(M) \subseteq \operatorname{cl}(N)$$

*For any two subsets of $\mathbb{R}^n$ $A, B$,*

*(1)* $\operatorname{int}(A \cup B) \supseteq \operatorname{int}(A) \cup \operatorname{int}(B)$,

*(2)* $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$,

*(3)* $\operatorname{cl}(A \cup B) = \operatorname{cl}(A) \cup \operatorname{cl}(B)$,

*(4)* $\operatorname{cl}(A \cap B) \subseteq \operatorname{cl}(A) \cap \operatorname{cl}(B)$,

*HW Oct 05 Q1*

**Example 1:** For a half-open rectangle $A = (0, 2] \times (0, 1]$,

- $\operatorname{int}(A) = (0, 2) \times (0, 1)$
- $\operatorname{cl}(A) = [0, 2] \times [0, 1]$

**Proposition 2.1.2** (duality between interior and closure)**.**

*For every $A \subseteq \mathbb{R}^n$ we have*

- $\operatorname{int}(\mathbb{R}^n \setminus A) = \mathbb{R}^n \setminus \operatorname{cl}(A)$ *The interior of the complement is the complement of the closure*
- $\operatorname{cl}(\mathbb{R}^n \setminus A) = \mathbb{R}^n \setminus \operatorname{int}(A)$ *The closure of the complement is the complement of the interior*

**Remark:** DeMorgan's Law: $(\mathbb{R}^n \setminus A) \cup (\mathbb{R}^n \setminus B) = \mathbb{R}^n \setminus (A \cap B)$

**Proposition 2.1.3** (Description of $\mathrm{cl}(A)$ by using sequences)**.** *Let $A \subseteq \mathbb{R}^n$ and let $\vec{b}$ be a point in $\mathbb{R}^n$, we have that*

$$\left( \vec{b} \in \mathrm{cl}(A) \right) \iff \left( \exists \text{ a sequence } (\vec{x}_k)_{k=1}^{\infty} \text{ in } A \text{ that } \lim_{k\to\infty} \vec{x}_k = \vec{b} \right)$$

*Note this must be "iff".*

**Corollary 2.1.1** (Descriptions of $\mathrm{bd}(A)$)**.**

*Let $A$ be a subset of $\mathbb{R}^n$ then*

1. $\mathrm{bd}(A) = \mathrm{cl}(A) \cap \mathrm{cl}(\mathbb{R}^n \setminus A)$

   *that is, each $\vec{b} \in \mathrm{bd}(A)$ if and only if there is a sequence $(\vec{x}_k)_{k=1}^{\infty}$ in $A$ that $\lim_{k\to\infty} \vec{x}_k = \vec{b}$ and $\exists$ a sequence $(\vec{y}_k)_{k=1}^{\infty}$ in $\mathbb{R}^n \setminus A$ that $\lim_{k\to\infty} \vec{y}_k = \vec{b}$*

2. $\mathrm{bd}(A) = \mathrm{bd}(\mathbb{R}^n \setminus A)$

## 2.2 Lecture 8: Open and closed subsets of Rn̂

**Definition 2.2.1** (open and closed sets)**.**

1. *A set $A \subseteq \mathbb{R}^n$ is **open** if $A = \mathrm{int}(A)$.*

$$\left( A \text{ open} \right) \iff \left( \forall \vec{a} \in A, \exists r > 0, B(\vec{a}; r) \subseteq A \right)$$

2. *A set $A \subseteq \mathbb{R}^n$ is **closed** if $A = \mathrm{cl}(A)$.*

$$\left( A \text{ closed} \right) \iff \left( \nexists\, \vec{b} \in (\mathbb{R}^n \setminus A) \text{ such that } \vec{b} \text{ is adherent to } A \right)$$

**Proposition 2.2.1.** *Let $A$ be a subset of $\mathbb{R}^n$ then*

$$\left( A \text{ is closed} \right) \iff \left( \mathbb{R}^n \setminus A \text{ is open} \right)$$

*Proof.* $\Rightarrow$: $\mathrm{cl}A = A$, then $\mathrm{int}(\mathbb{R}^n \setminus A) = \mathbb{R}^n \setminus \mathrm{cl}A = \mathbb{R}^n \setminus A$.

$\Leftarrow$: $\mathrm{int}(\mathbb{R}^n \setminus A) = \mathbb{R}^n \setminus A$, then,

$$\begin{aligned} \mathrm{cl}(A) = \mathrm{cl}(\mathbb{R}^n \setminus (\mathbb{R}^n \setminus A)) &= \mathbb{R}^n \setminus \mathrm{int}(\mathbb{R}^n \setminus A) && \text{(Proposition 2.1.2)} \\ &= \mathbb{R}^n \setminus (\mathbb{R}^n \setminus A) = A \end{aligned}$$

$\square$

**Definition 2.2.2** (NO-ESC)**.** *A set $A \subseteq \mathbb{R}^n$ is NO-ESC (no escape) if for all sequence $(\vec{x}_k)_{k=1}^{\infty}$ in $A$ such that $\lim_{k\to\infty} \vec{x}_k = \vec{b} \in \mathbb{R}^n$, then $\vec{b} \in A$.*

**Proposition 2.2.2.** *Let $A \subseteq \mathbb{R}^n$,*

$$A \text{ is closed} \iff A \text{ has NO-ESC property}.$$

*Proof.* $\Rightarrow$: If $A$ is closed, then $A = \mathrm{cl}(A)$. Then for all sequence $(\vec{x}_k)_{k=1}^{\infty}$ in $A$ that $\lim_{k\to\infty} \vec{x}_k = \vec{b} \in \mathbb{R}^n$, by Proposition 3.1.3, $\vec{b} \in \mathrm{cl}(A) = A$. Hence $A$ is NO-ESC.

$\Leftarrow$: have $A \subseteq \mathrm{cl}(A)$, left to prove $\mathrm{cl}(A) \subseteq A$. Again use Proposition 3.1.3. □

**Proposition 2.2.3** (open and closed ball)**.**

1. *Any open ball is an open subset of $\mathbb{R}^n$.*
2. *Any closed ball is an closed subset of $\mathbb{R}^n$.*

**Proposition 2.2.4.** *For a set $A \subseteq \mathbb{R}^n$, $\mathrm{int}(A)$ is open and $\mathrm{cl}(A)$ is closed.*

**Definition 2.2.3** (topology)**.** ***Topology*** *of $\mathbb{R}^n$ is:*

$$\mathcal{T}_n := \{G \subseteq \mathbb{R}^n \mid G \text{ is open}\}$$

- *TOP 1: $\mathbb{R}^n \in \mathcal{T}_n$*
- *TOP 2: $\varnothing \in \mathcal{T}_n$*
- *TOP 3: If $(G_i)_{i\in I}$ is a family of sets from $\mathcal{T}_n$, then $\bigcup_{i\in I} G_i$ is in $\mathcal{T}_n$.*
- *TOP 4: If $G_1, \ldots, G_p$ in $\mathcal{T}_n$, then $\bigcap_{i=1}^{p} G_i$ is in $\mathcal{T}_n$. (proof see HW Oct05)*

**Lemma 2.2.1.** *Let $A \subseteq \mathbb{R}^n$, Suppose that $G \subseteq A$, and $G$ is open, then $G \subseteq \mathrm{int}(A)$.*

**Lemma 2.2.2.** *$\forall A \subseteq \mathbb{R}^n$, $\mathrm{int}(A)$ is open, $\mathrm{cl}(A)$ is closed.*

**Lemma 2.2.3.**

- *$\mathrm{cl}(A)$ is the smallest closed set in $\mathbb{R}^n$ which contains $A$.*
- *$\mathrm{int}(A)$ is the largest open set which is inside $A$.*

**Example 1:** A linear space is a closed set.

## 2.3 Lecture 9: Compact Sets

**Definition 2.3.1.** *A set $A \subseteq \mathbb{R}^n$ is* ***compact*** *when it is closed and bounded.*

**Definition 2.3.2.** *A set $A \subseteq \mathbb{R}^n$ is* ***sequentially compact (SEQ-CP)*** *when for all sequence $(\vec{x}_k)_{k=1}^{\infty}$ in $A$, there is a convergent subsequence $(\vec{x}_{k(p)})_{p=1}^{\infty}$, where $\lim_{p\to\infty}(\vec{x}_{(p)})$ still belongs to $A$.*

**Proposition 2.3.1.** *Definition 1 and 2 are the equivalent.*

**Definition 2.3.3** (OPEN-COVER)**.** *A subseteq $A \subseteq \mathbb{R}^n$ is an open-cover whenever $(G_i)_{i\in I}$ is a family of open sets such that $A \subseteq \cup_{i\in I} G_i$, it is possible to find finitely many indices $i_1, \dots, i_p \in I$ such that $A \subseteq G_{i_1} \cup \cdots \cup G_{i_p}$.*

***Remark:*** *OPEN-COVER is often taken to be the definition for* ***compact****.*

**Proposition 2.3.2** (Automatic upgrade to uniform continuity on a compact domain)**.** *Let a set $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a function. If $f$ is continuous on $A$, then $f$ is uniformly continuous on $A$.*

Moving to an image space of $\mathbb{R}^m$, $m \in \mathbb{N}$.

**Definition 2.3.4** (Continuous and Uniformly Continuous)**.** *Let $n, m \in \mathbb{N}$, let $A$ be a subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a function. Denote $f_1, \dots, f_m : A \to \mathbb{R}$ the $m$ component functions of $f$. That is $f_1, \dots, f_m$ are defined via the requirement that one has*

$$f(\vec{x}) = \big(f_1(\vec{x}), \dots, f_m(\vec{x})\big) \in \mathbb{R}^m , \quad \forall \vec{x} \in A .$$

*then, we define*

- *$f$ is* ***continuous*** *on $A$ if all of $f_1, \dots, f_m$ are continuous on $A$.*
- *$f$ is* ***uniformly continuous*** *on $A$ if all of $f_1, \dots, f_m$ are uniformly continuous on $A$.*

**Proposition 2.3.3** (updated from last proposition)**.** *[Automatic upgrade to uniform continuity on a compact domain] Let a set $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$ be a function. If $f$ is continuous on $A$, then $f$ is uniformly continuous on $A$.*

**Proposition 2.3.4.** *Let $m, n \in \mathbb{N}$, let $A$ be a compact subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$ be a continuous function. Consider the image-set*

$$M := f(A) = \{\vec{y} \in \mathbb{R}^m \mid \exists \vec{x} \in A \text{ such that } f(\vec{x}) = \vec{y}\} .$$

*Then $M$ is a compact subset of $\mathbb{R}^m$.*

*Proof.* Let $(\vec{y}_k)_{k=1}^{\infty}$ be an arbitrary sequence in $M$, then $y_k = f(\vec{x}_k)$ for some $\vec{x}_k \in A$, then we have a sequence $(x_k)_{k=1}^{\infty}$ which since $A$ is compact, there is a subsequence of it that converges, let this subsequence be $(x_{k(p)})_{p=1}^{\infty}$, so $\lim_{p\to\infty} x_{k(p)} = \vec{a} \in A$. Since $f$ is continuous, $\lim_{p\to\infty} f(x_{k(p)}) = f(\vec{a}) \in M$, hence, we have found a subsequence of $(\vec{y}_k)_{k=1}^{\infty}$ which converges. Therefore, $M$ is sequentially compact and also compact. □

## 2.4 Extremum and EVT

**Definition 2.4.1.** *Let $A$ be a subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a function,*

- *A point $\vec{a} \in A$ is a global minimum of $f$ on $A$ if $f(\vec{a}) \leq f(\vec{x})$ for all $\vec{x} \in A$.*
- *A point $\vec{b} \in A$ is a global maximum of $f$ on $A$ if $f(\vec{b}) \geq f(\vec{x})$ for all $\vec{x} \in A$.*

**Theorem 2.4.1** (EVT)**.** *Let $A$ be a subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a function. If $A$ is compact and if $f$ is continuous, then $f$ must have at least one point of global minimum and one point of global maximum on $A$.*

## 2.5 $P$

## 2.6 Integrability of a bounded continuous function

**Lemma 2.6.1.** *Let* $P = (a_1, b_1] \times \cdots \times (a_n, b_n]$ *be a half-open rectangle in* $\mathcal{P}_n$*, for every* $\varepsilon > 0$*,*

- *there is an* $S \in \mathcal{P}_n$ *such that* $\mathrm{cl}(S) \subseteq P$ *and* $\mathrm{vol}_n(S) > \mathrm{vol}_n(P) - \varepsilon$.
- *there is a* $Q \in \mathcal{P}_n$ *such that* $\mathrm{cl}(P) \subseteq Q$ *and* $\mathrm{vol}_n(Q) < \mathrm{vol}_n(P) + \varepsilon$.

Lec 11.

## 2.7 Null Set

## 2.8 Lecture 13: Jordan Measurable Sets

**Definition 2.8.1.** *Let $A$ be s subset of $\mathbb{R}^n$. We say that $A$ is Jordan measurable to mean that*

1. *$A$ is bounded*
2. $\operatorname{bd}(A)$ *is a null-set*

*we use the notation*

$$\mathcal{J}_n := \{A \subseteq \mathbb{R}^n \,|\, A \text{ is Jordan measurable}\}$$

**Example 1:** (relating to balls)
Any set $A$ such that $B(\vec{0};1) \subseteq A \subseteq \bar{B}(\vec{0};1)$ is sure to be Jordan measurable.

**Example 2:** (relating to rectangles)
Any set $A$ such that $(a_1, b_1] \times \cdots \times (a_n, b_n] \subseteq A \subseteq [a_1, b_1] \times \cdots \times [a_n, b_n]$ is sure to be Jordan measurable.

**Example 3:**
Every null-set is Jordan measurable.

**Non-Example 1:**
$S = \{(s,t) \in \mathbb{R}^2 \,|\, 0 < s, t < 1 \text{ and } s, t \in \mathbb{Q}\}$

**Lemma 2.8.1.**

1. *For every $A, B \subseteq \mathbb{R}^n$ we have* $\operatorname{bd}(A \cup B) \subseteq \operatorname{bd}(A) \cup \operatorname{bd}(B)$.
2. *For every $A, B \subseteq \mathbb{R}^n$,* $\operatorname{bd}(A \cap B) \subseteq \operatorname{bd}(A) \cup \operatorname{bd}(B)$.
3. *For every $A, B \subseteq \mathbb{R}^n$,* $\operatorname{bd}(A \setminus B) \subseteq \operatorname{bd}(A) \cup \operatorname{bd}(B)$.

**Theorem 2.8.1.** *Let $A$ be a jordan measurable subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a function. If $f$ is bound and has the **cont-mod-Null** property, then $f$ is integrable on $A$.*

**Corollary 2.8.1.** *Let $A$ be a Jordan measurable subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a bounded continuous function. Then $f$ is integrable on $A$.*

# 3 Derivatives

## 3.1 Directional Derivatives

**Definition 3.1.1.** *Let $A$ be a subset of $\mathbb{R}^n$, and let $\vec{a}$ be an interior point of $A$, and let $f : A \to \mathbb{R}$ be a function. Consider a "direction" vector $\vec{v} \in \mathbb{R}^n$,*

- *1-dimensional reduction: $\varphi : (-c, c) \to \mathbb{R}$,*

$$\varphi(t) := f(\vec{a} + t\vec{v}), \quad -c < t < c \,,$$

  *where $c > 0$ is small enough to ensure that $\vec{a} + t\vec{v} \in A$ for all $t \in (-c, c)$.*

- *If $\varphi$ is differentiable at $0$, then the number $\varphi'(0)$ is called the* ***directional derivative*** *of $f$ at $\vec{a}$, in direction $\vec{v}$, and is denoted by $\partial_{\vec{v}} f(\vec{a})$,*

$$\partial_{\vec{v}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{v}) - f(\vec{a})}{t} \qquad \text{(when the limit exists)} \,.$$

**Proposition 3.1.1.** *Let $A, f, \vec{a}$ be defined in the above definition, then*

1. *The directional derivative $\partial_{\vec{0}} f(\vec{a})$ exists and is equal to $0$.*
2. *Let $\vec{v}$ be a non-zero vector in $\mathbb{R}^n$, let $\alpha \in \mathbb{R}$ and put $\vec{w} = \alpha\vec{v}$ then if directional derivative $\partial_{\vec{v}} f(\vec{a})$ exists. then $\partial_{\vec{w}} f(\vec{a})$ exists and*

$$\partial_{\vec{0}} f(\vec{a}) = \alpha \cdot \partial_{\vec{v}} f(\vec{a}) \,.$$

*Proof.* Left for review. □

Hence, we can focus on direction vector $\vec{v}$ that $||\vec{v}|| = 1$.

A multivariate version of MVT in MATH 147.

**Notation 3.1.1.** *For $\vec{x}, \vec{y} \in \mathbb{R}^n$,*

$$Co(\vec{x}, \vec{y}) := \{(1 - t)\vec{x} + t\vec{y} \,|\, 0 \le t \le 1\} \subseteq \mathbb{R}^n \,.$$

**Theorem 3.1.1** (Mean value THeorem)**.**

## 3.2 partial derivatives

## 3.3 Iterated Partial Derivatives

**Goal:** to prove that

$$\partial_1\partial_2 f = \partial_2\partial_1 f, \quad \forall f \in C^2(A,\mathbb{R}) \ .$$

**Lemma 3.3.1.** *Let $u, v : A \to \mathbb{R}$ be two continuous functions, suppose we are given that $\int_S u(\vec{x})d\vec{x} = \int_S v(\vec{x})d\vec{x}$ for every half-open rectangle $S = (a_1, b_1] \times (a_2, b_2]$ such that $\mathrm{cl}(S) \subseteq A$. Then it follows that $u(\vec{a}) = v(\vec{a})$ for all $\vec{a} \in A$.*

*Proof.* Assume for some $\vec{a} \in A$, $u(\vec{a}) \neq v(\vec{a})$, then for this $\vec{a}$ we denote $u(\vec{a}) = \alpha$ and $v(\vec{a}) = \beta$. The continuity of $u$ and of $v$ at $\vec{a}$ give us some values $\delta_1, \delta_2 > 0$ such that $\square$

## 3.4 The Hessian Matrix and Quadratic Approximation for C2-functions

**Definition 3.4.1.** *Let $A$ be an open subset of $\mathbb{R}^n$, let $f : A \to \mathbb{R}$ be a $C^2$-function, and let $\vec{a}$ be a point of $A$. The* ***Hessian matrix*** *is the $n \times n$ matrix which, for every $1 \leq i, j < n$, its $(i, j)$ entry is equal to $\partial_i\partial_j f(\vec{a})$. We denote it as $[Hf](\vec{a})$.*

**Remark:** from the above definition, we know that $H : [Hf](\vec{a})$ is a symmetry matrix ($H = H^T$) as $\partial_i\partial_j f(\vec{a}) = \partial_j\partial_i f(\vec{a})$.

**Definition 3.4.2** (quadratic function associated with H)**.** *The* ***quadratic function*** *associated to $H$ is defined by*

## 3.5 Local Extremum and Second Derivative Test

**Definition 3.5.1** (Points of Local Extremum). *Let $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a function.*

- ***local minimum**: $\vec{p} \in A$, which there exists $r > 0$ such that $f(\vec{x}) \geq f(\vec{p})$, $\forall x \in B(\vec{p}; r) \cap A$.*
- ***local maximum**: $\vec{p} \in A$, which there exists $r > 0$ such that $f(\vec{x}) \leq f(\vec{p})$, $\forall x \in B(\vec{p}; r) \cap A$.*
- ***local extremum**: local minimum or local maximum*

**Definition 3.5.2.** *Let $A \subseteq \mathbb{R}^n$ be an open set, let $A \to \mathbb{R}$ be a $C^1$-function, and let $\vec{a}$ be a point in $A$. Consider the gradient vector of $f$ at $\vec{a}$, $\nabla f(\vec{a}) := (\partial_1 f(\vec{a}), \ldots, \partial_n f(\vec{a})) \in \mathbb{R}^n$.*

- *If $\nabla f(\vec{a}) = \vec{0}$, then it is a critical point for $f$;*
- *otherwise, $\vec{a}$ is a **regular point** for $f$.*

**Proposition 3.5.1.** *Let $A \subseteq \mathbb{R}^n$ be open and let $f : A \to \mathbb{R}$ be a $C^1$-function. Let $\vec{a} \in A$ be a point of local extremum for $f$. Then $\vec{a}$ is a critical point for $f$.*

## 3.6 Chain Rule

Key idea: the CR formula will remain the same, only that instead of multiplying numbers, we will now multiply matrices.

**Definition 3.6.1.** *Let $m, n \in \mathbb{N}$, let $A$ be an open subset of $\mathbb{R}^n$, and let $f : A \to \mathbb{R}^m$ be a function. For every $\vec{x} \in A$ write*

$$f(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \dots, f_m(\vec{x})) \in \mathbb{R}^m ,$$

*and if each $f_1, \dots, f_m : A \to \mathbb{R}$ are in $C_1(A, \mathbb{R})$, then $f \in C^1(A, \mathbb{R}^m)$.*

**Definition 3.6.2** (Jacobian Matrix)**.** *Let $A$ be an open subset of $\mathbb{R}^n$ and let $f$ be a function in $C^1(A, \mathbb{R}^m)$ with components $f_1, \dots, f_m \in C^1(A, \mathbb{R})$. For every $\vec{a} \in A$, the matrix*

$$[Jf](\vec{a}) := \begin{bmatrix} \partial_1 f_1(\vec{a}) & \cdots & \partial_n f_1(\vec{a}) \\ \vdots & & \vdots \\ \partial_1 f_m(\vec{a}) & \cdots & \partial_n f_m(\vec{a}) \end{bmatrix} \in \mathcal{M}_{m\times n}(\mathbb{R}) .$$

*is the* ***Jacobian Matrix*** *of $f$ at $\vec{a}$.*

*Note for $m = 1$,*

$$[Jf](\vec{a}) := [\partial_1 f(\vec{a}), \dots, \partial_m f(\vec{a})] ,$$

*and for $n = 1$,*

$$[Jf](a) := [f_1'(a), \dots, f_m'(a)] .$$

**Proposition 3.6.1** (Linear Approximation for $f \in C^1(A, \mathbb{R}^m)$)**.** *Let $A$ be an open subset of $\mathbb{R}^n$, and let $f$ be a function in $C^1(A, \mathbb{R}^m)$. Then for every $\vec{a} \in A$ we have*

$$\lim_{\vec{x}\to\vec{a}, \vec{x}\neq\vec{a}} \frac{||f(\vec{x}) - f(\vec{a}) - [Jf](\vec{a}) \cdot (\vec{x} - \vec{a})||}{||\vec{x} - \vec{a}||} = 0 .$$

**Proposition 3.6.2.** *Let $A$ be an open subset of $\mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$ be a function, with components $f_1, \dots, f_m : A \to \mathbb{R}$. Let $\vec{a}$ be a point in $A$ and suppose a matrix $M \in \mathcal{M}_{m\times n}(\mathbb{R})$ has that*

$$\lim_{\vec{x}\to\vec{a}, \vec{x}\neq\vec{a}} \frac{||f(\vec{x}) - f(\vec{a}) - M(\vec{a}) \cdot (\vec{x} - \vec{a})||}{||\vec{x} - \vec{a}||} = 0 .$$

*then, every component function $f_i$ has partial derivatives at $\vec{a}$ and we have the equality*

$$\partial_j f_i(\vec{a}) = [(i, j) - \text{ entry of } M], \forall 1 \leq i \leq m \text{ and } 1 \leq j \leq n .$$

## 3.7 o

**Definition 3.7.1.** *Let $A$ be a subset of $\mathbb{R}^n$, let $\varphi : A \to \mathbb{R}$ be a function, and let $c \in \mathbb{R}$ be one of the values attained by the function $\varphi$. The level of $\varphi$ corresponding to the value $c$ is defined as $L := \{\vec{x} \in A \,|\, \varphi(\vec{x}) = c\}$.*

**Definition 3.7.2.** *Let $A$ be an open subset of $\mathbb{R}^n$ and let $\varphi$ be a function in $C^1(A, \mathbb{R})$. Let $c \in \mathbb{R}$ be one of the values attained by $\varphi$ and consider the level set of $\varphi$ corresponding to the value $c$. We fix a point $\vec{a} \in L$ which is regular for $\varphi$ (so we have $\varphi(\vec{a}) = c$ and $\nabla\varphi(\vec{a}) \neq \vec{0}$). Then,*

1. *A vector $\vec{w} \in \mathbb{R}^n$ is said to be a normal vector to $L$ at the point $\vec{a}$ when it is a scalar multiple of $\nabla\varphi(\vec{a})$.*
2. *Let $\vec{v}$ be a vector $\mathbb{R}^n$. We say that $\vec{v}$ is a tangent vector to $L$ at the point $\vec{a}$ to mean that it is possible to find an open interval $I \subseteq \mathbb{R}$ and that a $C^1-$path $\gamma : I \to \mathbb{R}^n$ such that the following conditions hold*
   *(a) $\gamma(t) \in L$ for all $t \in I$;*
   *(b) $0 \in I$ and we have $\gamma(0) = \vec{a}$ nad $\gamma'(0) = \vec{v}$.*

## 3.8

**Objective:** extend the substitution formula to multi-variables.

**Remark:** Let $n \in \mathbb{N}$, we will be concerned with integrals

$$\int_A f(\vec{x})\, d\vec{x} \ .$$

where $A \subseteq \mathbb{R}^n$ is open and Jordan measurable, and $f : A \to \mathbb{R}$ is a bounded continuous function.