Math 148 Notes

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Section: 002

Contents

1 INTEGRATION, SUMMATION

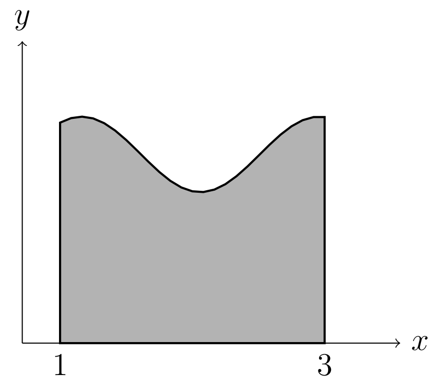
MOTIVATION: area, let $a < b$ in $\mathbb{R}$, and let $f : [a, b] \to [0, \infty]$, let

$$S_f = \{(x, y) : 0 \leq y \leq f(x), x \in [a, b]\}(\text{"}subgraph\text{"})$$

IDEA: area of rectangel = height * width

1.

Figure 1: The area under the function $\frac{1}{x}$ is $\log x$



2. approximate S_f by rectangle from below 13:43 JAN 6,

   $j = 1, \cdots, 4$, $m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$

$$\sum_{j=1}^{4} m_{j-1}(x_i - x_{j-1}) \leq area(s_f)$$

3. approximate S_f by rectangle from above, $M_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\}$

$$area \leq \sum_{j=1}^{4} M_j(x_j - x_{j-1})$$

4. if we can arrange lower sum $\approx$ upper sum, then we have some good approximation

1.1 Partition, Upper and Lower Sum

Let $a < b \in \mathbb{R}$, $f : [a, b] \in \mathbb{R}$,

Definition 1.1.1 (Riemann-Darboux).

A ***partition*** *of* $[a, b]$ *is any finite set of points including the endpoints.*

$$P : \{x_0, x_1, \cdots, x_n\} s.t. a = x_0 < x_1 < \cdots < x_n = b$$

often for convenience, we write $P = \{a = x_0 < \cdots < x_n = b\}$.

*A **Refinement** of P is any partition Q of $[a,b]$ s,t, $P \subseteq Q$.*

Now, fix a partition P of $[a,b]$ and let $f:[a,b]\to\mathbb{R}$ be bounded on $[a,b]$, i.e. $\sup\limits_{x\in[a,b]} |f(x)| \leq M < \infty$. Write $P = \{a = x_0 < \cdots < x_n = b\}$. For $j = l, \cdots, n$,

$$m_j = m_j(P) = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$$
$$M_j = M_j(P) = \sup\{f(x) : x \in [x_{j-1}, x_j]\}$$

Notice that $-M \leq m_k \leq M_j \leq M$ for each j, and these "inf", "sup" exist. (Using that $\mathbb{R}$ is complete.)

Definition 1.1.2.

- ***Lower Sum:*** $L(f,P) = \sum_{j=1}^{n} m_j \underbrace{(x_j - x_{j-1})}_{width\ of\ [x_{j-1},x_j]}$
- ***Upper Sum:*** $U(f,P) = \sum_{j=1}^{n} M_j(x_j - x_{j-1})$

Remark:

1. if f is not bounded, then at least one of $L:(f,P)$ or $U(f,P)$ cannot be defined.
2. we have $L(f,P) \leq U(f,P)$, Indeed, for each $j = l, \cdots, n$, $m_j \leq M_j$. (exactly from definition),

$$L(f,P) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}) \leq \sum_{j=1}^{n} M_j(x_j - x_{j-1}) = U(f,P)$$

Lemma 1.1.1. *If P is a partition of $[a,b]$, $f:[a,b]\to\mathbb{R}$ is bounded, and Q is a refinement of P, then*

$$L(f,P) \leq L(f,Q) \qquad U(f,Q) \leq U(f,P)$$

Proof.

- Case 0: $Q = P$ obvious
- Case 1: $Q = P \cup \{q\}$ where $q \notin P$,

  write $P = \{a = x_0 < \cdots, x_n = b\}$ so $Q = \{a = x_0 < \cdots < x_{k-1} < q < x_k < \cdots < x_n = b\}$ Then,

$$\begin{aligned} m_k(P) &= \inf\{f(x) : x \in [x_{k-1}], x_k\} \qquad [x_{k-1}, x_k] = [x_{k-1}, q] \cup [q, x_k] \\ &= \min\{\inf\{f(x) : x \in [x_{k-1}, q] : x \in [x_{k-1}, q]\} \inf f(x) : x \in [q, x_k]\} \\ &= \min\{m_k(Q), m'_k(Q)\} \leq m_k(Q), m'_k(Q) \end{aligned}$$

Thus,

$$\begin{aligned}
L(f,P) &= \sum_{j=1}^{m} m_j(P)(x_j - x_{j-1}) \\
&= \sum_{j=1}^{k-1} m_j(P)(x_j - x_{j-1}) + m_k(P)(x_k - x_{k-1}) + \sum_{j=k+1}^{n} m_j(P)(x_j - x_{j-1}) \\
&\leq \sum_{j=1}^{k-1} m_j(P)(x_j - x_{j-1}) + m_k
\end{aligned}$$

- Case 2: $Q = P \cup \{q_1, \cdots, q_m\}, q_1, \cdots, q_m$ distinct, $q_u \notin P$, by case 1, we have

  $$L(f,P) \leq L(f, P \cup \{q_1\}) \leq L(f, P \cup \{q_1, q_2\}) \leq \cdots \leq L(f, P \cup \{q_1, \cdots, q_m\}) = L(f,Q)$$

  The case $U(f,Q) \leq U(f,P)$ is similar.

□

Corollary 1.1.1. *let P, Q be any partition of $[a,b]$ and $f : [a,b] \to \mathbb{R}$ be bounded, then*

$$L(f,P) \leq U(f,Q)$$

Proof. We have $P, Q \subseteq P \cup Q$, i.e. $P \cup Q$ refines each of P and Q. Thus,

$$L(f,P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f,Q)$$

□

1.2 Upper and Lower Sum

Definition 1.2.1. *Given a bounded $f : [a,b] \to \mathbb{R}$, define*

- *lower integral* : $\underline{\int} a^b f = \sup\{L(f,P) : P \textit{ is a partition of } [a,b]\}$
- *Upper Integral:* $\overline{\int}_a^b f = \inf\{U(f,Q) : Q \textit{ is a partition of } [a,b]\}$

Note: $\underline{\int}_a^b f = \sup\{L(f,P) : P \textit{ is a partition of } [a,b]\} \le \inf\{U(f,Q) : Q \textit{ is a partition of } [a,b]\} = \overline{\int}_a^b f$

*We say that f is **integrable** on $[a,b]$ provided that*

$$\underline{\int}_a^b f = \overline{\int}_a^b f$$

In this case, we write $\int_a^b f = \overline{\int}_a^b f = \underline{\int}_a^b f$

Notation: Write

$$\int_a^b f = \int_a^b f(x)d(x) = \int_a^b f(t)dt$$

Non-Example 1: not every bounded function is integrable.

Define: $\chi_{\mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$.

Let $P = \{0 = x_0 < \cdots < x_n = 1\}$ be any partition of $[0,1]$, We have that

- $\mathbb{Q}$ is dense in $\mathbb{R}$, so there is $q_j \in \mathbb{Q} \cap (x_{j-1}, x_j), j = l, \cdots, n$
- $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$, so there is $r_j \in (\mathbb{R}\backslash\mathbb{Q}) \cap (x_{j-1}, x_j), j = l, \cdots, n$

$$0 \le L(\chi_{\mathbb{Q},P}) = \sum_{j=1}^n m_j(x_j - x_{j-1}) \le \sum_{j=1}^n \chi_{\mathbb{Q}}(r_j)(x_j - x_{j-1}) = 0 \Rightarrow \underline{\int}_0^1 = 0$$

Likewise,

$$1 \ge U(\chi_{\mathbb{Q}}, P) \ge \sum_{j=1}^n \chi_{\mathbb{Q}}(q_j)(x_j - x_{j-1}) = \sum_{j=1}^n (x_j - x_{j-1}) = 1 - 0 = 1 \Rightarrow \overline{\int}_0^1 = 1$$

hence,

$$\underline{\int}_0^1 \chi_{\mathbb{Q}} = 0 < 1 = \overline{\int}_0^1 \chi_{\mathbb{Q}}$$

so $\chi_{\mathbb{Q}}$ is not integrable on $[0,1]$.

Theorem 1.2.1 (**Cauchy Criterion For Integrability**). *Let $a < b \in \mathbb{R}$, $f : [a,b] \to \mathbb{R}$ be bounded, then TFAE,*

1. *f is integrable on $[a,b]$*
2. *given $\varepsilon > 0$, there exists a partition P_ε of $[a,b]$ s,t,*

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

   and

3. *given $\varepsilon > 0$, there exists a partition P_ε of $[a,b]$ so for every refinement P of P_ε*

$$U(f, P) - L(f, P) < \varepsilon$$

Proof. 1 to 2: we assume that

$$\sup\{L(f,P) : P \,\text{partition }\; of\; [a,b]\} = \underline{\int_a^b} f = \overline{\int_a^b} \inf\{U(f,P) : P \,\text{partition }\; of\; [a,b]\}$$

Let $\varepsilon > 0$, by first equality above, there is a partition P_1 of $[a,b]$ s.t.

$$\underline{\int_a^b} f - \frac{\varepsilon}{2} < L(f, P_1)$$

and by the third equality, there is a partition P_2 s.t.

$$U(f, P_2) < \overline{\int_a^b} f + \frac{\varepsilon}{2}$$

Let $P_\varepsilon = P_1 \cup P_2$, a refinement of P_1 and P_2, then since $\underline{\int_a^b} f = \overline{\int_a^b} = \int_a^b f$ we find

$$\int_a^b f - \frac{\varepsilon}{2} < L(f, P_1) \le L(f, P_\varepsilon) \le U(f, P_\varepsilon) \le U_{f, P_2} < \int_a^b f + \frac{\varepsilon}{2}$$

$$\Rightarrow U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

2 to 3: we use the lemma.

If $P_\varepsilon \le P$, we have

$$L(f, P_\varepsilon) \le L(f, P) \le U(f, P) \le U(f, P_\varepsilon)$$

Hence,

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon \Rightarrow U(f, P) - L(f, P) < \varepsilon$$

3 to 2: $P_\varepsilon \subseteq P_\varepsilon$ i.e. P_ε self-defines itself

2 to 1: Given $\varepsilon > 0$, there is P_ε, a partition of $[a,b]$, so $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$. We have

$$L(f, P_\varepsilon) \le \underline{\int_a^b} \le \overline{\int_a^b} f \le U(f, P_\varepsilon) \Rightarrow$$

$\square$

1.3 Continuity and Inegrability

Definition 1.3.1 (Continuous). *$f : I \to \mathbb{R}$ is continuous if for every x in I, for every $\varepsilon > 0$, there exists $\delta > 0$ s.t. for all $|x - x'| < \delta$, $x' \in I$,*

$$|f(x) - f(x')| < \varepsilon$$

this definition is local. first we choose x, ε, then δ

Definition 1.3.2 (uniform Continuity). *$f : I \to \mathbb{R}$ is uniformly continuous if for every $\varepsilon > 0$, there is $\delta > 0$ so $|f(x) - f(x')| < \varepsilon$ whenever $|x - x'| < \delta$ for $x, x' \in I$.*

Proposition 1.3.1 (Sequential Test of Continuity). *Let $f : I \to \mathbb{R}$, then f is uniformly continuous $\Rightarrow$ for any sequences $(x_n)_{n=1}^{\infty}$, $(x'_n)_{n=1}^{\infty} \subset I$, with $\lim_{n\to\infty} |x_n - x'_n| = 0$, we have $\lim\limits_{n\to\infty} |f(x_n) - f(x'_n)| = 0$.*

[Fact $\Leftarrow$ also true]

Proof. Given $\varepsilon > 0$, let δ be as in def'n of uniform continuity. Since $\lim\limits_{n\to\infty} |x_n - x'_n| = 0$, there is $N \in \mathbb{N}$, so for $n \geq N$, we have $|x_n - x'_n| < \delta$.

But then, for $n \geq N$, we also have that $|f(x_n) - f(x'_n)| < \varepsilon$. i.e. $\lim\limits_{n\to\infty} |f(x_n) - f(x'_n)| = 0$. $\square$

Example 1 $f : (0, 1] \to \mathbb{R}$, $f(x) = \frac{1}{x}$. Notice that f is continuous.

Let $x_n = \frac{1}{n}$, $x'_n = \frac{1}{2n}$, $|x_n - x'_n| = \frac{1}{2n} n \overset{\to}{\to} \infty 0$.

$$|f(x_n) - f(x'_n)| = |n - 2n| = n$$

Hence, not uniformly continuous.

Example 2: $g : (0, 1] \to \mathbb{R}$, $g(x) = \sin \frac{1}{x}$, then g is continuous.

$x_n = \frac{1}{\pi n}$, $x'_n = \frac{2}{(2n+1)\pi}$, $|x_n - x'_n| = \frac{1}{\pi n(2n+1)} n \overset{\to}{\to} \infty 0$,

$$|g(x_n) - g(x'_n)| = \left| \sin(n\pi) - \sin(\frac{2n+1}{2}\pi) \right| = 1$$

For $\varepsilon = 1$, uniform continuity fails.

Theorem 1.3.1. *Let $f : [a, b] \to \mathbb{R}$ be continuous, then f is uniformly continuous.*

Proof. Let us suppose that f is continuous, but not uniformly continuous, hence there exist $\varepsilon > 0$, such that for any $\delta > 0$, there are $x, x' \in [a, b]$ so

$$|f(x) - f(x')| \geq \varepsilon \text{ } while \text{ } |x - x'| < \delta$$

Let us consider $\delta = \frac{1}{n}$, so there are x_n, x'_n in $[a, b]$ such that

$$|f(x_n) - f(x'_n)| \geq \varepsilon \text{ } while \text{ } |x - x'| < \frac{1}{n}$$

By Bolzano Weierstrass Theorem, there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$, such that $x = \lim\limits_{k\to\infty} x_{n_k}$ exists in $[a, b]$.

Then, notice that

$$\left|x - x'_{n_k}\right| \leq |x_n - x_{n_k}| + \left|x_{n_k} - x'_{n_k}\right| < |x - x_{n_k}| + \frac{1}{n_k}$$

hence, by Squeeze Theorem, $\lim_{k\to\infty} x'_{n_k} = x$. Since f is continuous, we have that

$$\lim_{k\to\infty} f(x_{n_k}) = f(x) = \lim_{k\to\infty} f(x'_{n_k})$$

$\Rightarrow$

$$\lim_{k\to\infty} \left|f(x_{n_k}) - f(x'_{n_k})\right| = 0$$

This contradicts that each $\left|f(x_{n_k}) - f(x'_{n_k})\right| \geq \varepsilon$. Thus by contradiction argument, f' must be uniformly continuous. □

Theorem 1.3.2 (Continuous on a Closed Bounded Interval and Integrability). *let $f : [a, b] \to \mathbb{R}$ be continuous, then f is integrable.*

Proof. Let $\varepsilon > 0$, then by uniform continuity of f, there exists a δ such that whenever $|x - x'| < \delta$, for $x, x' \in [a, b]$,

$$|f(x) - f(x')| > \varepsilon$$

Thus, we let $P = \{a = x_0 < \cdots < x_n = b\}$ be any partition with length $l(P) = \max_{j=1,\cdots,n}(x_j - x_{j-1}) < \delta$.

Example: $P_n = \{a_i < a + \frac{b-a}{n} < a + 2\frac{b-a}{n} < \cdots < a + (n-1)\frac{b-1}{n} << b\}$, then $\lim_{n\to\infty} l(P_n) = 0$.

Now, by EVT, we have

$$x_j^* \in [x_{j-1}, x_j] \ s.t.\ f(x_j^*) = \inf\{f(x) : x \in [x_{j-1}, x_j]\} = m_j$$
$$x_j^{**} \in [x_{j-1}, x_j] \ s.t.\ f(x_j^{**}) = \sup\{f(x) : x \in [x_{j-1}, x_j]\} = m_j$$

Then

$$L(f, P) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}) = \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1})$$
$$U(f, P) = \sum_{j=1}^{n} f(x_j^{**})(x_j - x_{j-1})$$

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{j=1}^{n}(f(x_j^{**}) - f(x_j^*))(x_j - x_{j-1}) \\ &= \sum_{j=1}^{n} \left|f(x_j^{**}) - f(x_j^*)\right|(x_j - x_{j-1}) < \sum_{j=1}^{n} \frac{\varepsilon}{b-a}(x_j - x_{j-1}) \\ &= \frac{\varepsilon}{b-a} = \varepsilon \end{aligned}$$

Hence, we have satisfied the Cauchy Criterion for integrability. □

Corollary 1.3.1. *if $f : [a, b] \to \mathbb{R}$ is continuous, then*

$$\int_a^b f = \lim_{n\to\infty} \sum_{j=1}^n f(a + j\frac{b-a}{n})\frac{b-a}{n}$$

Proof. We have $a + j\frac{b-a}{n} \in [a + \frac{b-a}{n}(j-1), a + \frac{b-a}{n}(j-1)]$, $j = 1, \cdots, n$.

So,

$$m_j \le f(a + j\frac{b-a}{n}) \le M_j$$

and thus

$$L(f, P_n) \le \sum_{j=1}^n f(a + j\frac{b-a}{n})\frac{b-a}{n} \le U(f, P_n)$$

$\lim\limits_{n\to\infty}(U(f, P_n) - L(f, P_n)) = 0$ as $\lim\limits_{n\to\infty} l(P_n) = 0$.

where $P_n = \{a < a + \frac{b-a}{n} < a + 2\frac{b-a}{n} < \cdots < b\}$, then proof of theorem shows that $\lim_{n\to\infty} L(f, P_a) = \int_a^b f = \lim_{n\to\infty} U(f, P_n)$ as $\lim_{n\to\infty} l(P_n) = \lim_{n\to\infty} \frac{b-a}{n} = 0$.

and hence Cauchy Criterion is satisfied, hence $\int_a^b f$ exists and is $\lim_{n\to\infty} L(f, P_n)$, apply Squeeze Theorem. □

1.4 Basic Properties of Integrals

Example 1: We will let $a > 0$ and compute $\int_0^a x^p dx$ for $p = 0, 1, 2$.

1. $p = 0$, $x^p = 1$, $P = \{0 = x_0 < x_1 = a\}$, $L(1, P) = a = U(1, P)$

   $[P'$ refines P, then $L(1, P) \le L(l, P') \le U(1, P') \le U(1, P) = a]$

   It follows that $\int_0^a 1dx = a$.

2. From last corollary

$$\int_0^a xdx = \lim_{n\to\infty} \sum_{j=1}^n (j\frac{a}{n})\frac{a}{n} = \lim_{n\to\infty} \frac{a^2}{n^2} \sum_{j=1}^n j = \lim_{n\to\infty} \frac{a^2}{n^2}\frac{n(n+1)}{2} = \frac{a^2}{2}$$

3. We need a fomula for $\sum_{j=1}^n j^2$.

   Trick:

$$\begin{aligned}
(n+1)^3 - 1 &= \sum_{j=1}^n [(j-1)^3 - j^3] && \text{(telescope)} \\
&= \sum_{j=1}^n [\sum_{k=0}^3 \binom{3}{k} j^k - j^3] && \text{(binomial theorem)} \\
&= \sum_{j=1}^n \sum_{k=0}^2 \binom{3}{k} j^k \\
&= \sum_{k=0}^3
\end{aligned}$$

$$\begin{aligned}
\int_0^a x^2 dx &= \lim_{n\to\infty} \sum_{j=1}^{n} (j\frac{a}{n})^2 \frac{a}{n} \\
&= \lim_{n\to\infty} \frac{a^3}{n^3} \sum_{j=1}^{n} j^2 \\
&= \lim_{n\to\infty} \frac{a^3}{3n^3} a[(n+1)^3 - 1 - n - \frac{n(n+1)}{2}] \\
&= \frac{a^3}{3}
\end{aligned}$$

Algorithm 1.4.1 (Basic Properties Of Integrals).

Proposition 1.4.1 (Additivity over intervals). *Let $a < b < c \in \mathbb{R}$, and $f : [a, c] \to \mathbb{R}$ satisfies that f is integrable on each of $[a, b]$, $[b, c]$, then*

- *f is integrable on $[a, c]$ and $\int_a^c f = \int_a^b f + \int_b^c f$.*

Proof. Given $\varepsilon > 0$, the Cauchy Criterion provides that

- a partition P_1 of $[a, b]$ s.t. $U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2}$
- a partition P_2 of $[b, c]$ s.t. $U(f, P_2) - L(f, P_2) < \frac{\varepsilon}{2}$

Let P be any refinement of $P_1 \cup P_2$. Then

$$L(f, P) \geq L(f, P_1 \cup P_2) = L(f, P_1) + L(f, P_2)$$

$$U(f, P) \leq U(f, P_1 \cup P_2) = U(f, P_1) + U(f, P_2)$$

Then

$$U(f, P) = L(f, P) \leq U(f, P_1) - L(f, P_1) + U(f, P_2) - L(f, P_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

hence, f is integrable on $[a, b]$.

Let P as above, be written $P = \{a = x_0 < \cdots < x_n = c\}$.

Let $Q_1 = \{a = x_0 < \cdots < x_m = b\}$, $Q_2 = \{b = x_m < \cdots < x_n = c\}$.

We have

$$L(f, Q_1) \leq \int_a^b f \leq U(f, Q_1) \qquad L(f, Q_2) \leq \int_b^c f \leq U(f, Q_2)$$

from the proof above, we have

$$L(f, P) = L(f, Q_1) + L(f, Q_2) \leq \int_a^b f + \int_b^c f \leq U(f, Q_1) + U(f, Q_2) = U(f, P)$$

Since f is integrable on $[a, c]$, we have

$$\int_a^c = \sup\{L(f, P) : P\, partition\ of [a, c]\} \leq \int_a^b f + \int_a^c f \leq \inf\{U(f, P) : P\, partition\ of [a, c]\} = \int_a^c f$$

$\Rightarrow$

$$\int_a^c f = \int_a^b f + \int_b^c f$$

$\square$

1.5 Riemann Sum - Jan 13 Mon, Jan 15 Wed

Definition 1.5.1 (Riemann Sums). *Let $f : [a, b] \to \mathbb{R}$, $P = \{a = x_0 < \cdots = x_n = b\}$.*

*A **Riemann Sum** is any sum of the following form:*

$$S(f, P) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) \qquad t_j \in [x_{j-1}, x_j], j = 1, \cdots, n$$

Left Sum:

$$S_l(f, P) = \sum_{j=1}^{n} f(x_{j-1})(x_j - x_{j-1})$$

Right Sum:

$$S_r(f, P) = \sum_{j=1}^{n} f(x_j)(x_j - x_{j-1})$$

Mid-point Sum:

$$S_m(f, P) = \sum_{j=1}^{n} f(\frac{x_{j-1} + x_j}{2})(x_j - x_{j-1})$$

Trapezoid Sum

$$\begin{aligned} T(f, P) =& \frac{1}{2}[S_l(f) + S_r(f)] \\ =& \sum_{j=1}^{n} \frac{f(x_j) + f(x_j)}{2}(x_j - x_{j-1}) \\ =& \frac{1}{2}f(a)(x_1 - a) + \sum_{j=1}^{n-1} f(x_j)(x_j - x_{j-1}) + \frac{1}{2}f(b)(b - x_{n-1}) \end{aligned}$$

Theorem 1.5.1. *If $f : [a, b] \to \mathbb{R}$, then TFAE,*

1. *f is integrable and*
2. *there is a number I_f satisfying the following: given any $\varepsilon > 0$, there exists a partition P_ε of $[a, b]$ such that*

   for any refinement of P of P_ε, any Riemann Sum of $S(f, P)$ we have

   $$|S(f, P) - I(f)| < \varepsilon$$

   Furthermore, $I_f = \int_a^b f$.

Proof.

(i)⇒(ii) Given $\varepsilon > 0$, the Cauchy Criterion provides that P_ε so for any refinement P of P_ε,

$$U(f, P) - L(f, P) < \varepsilon \tag{1}$$

Write $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, and let for $j = 1, \cdots, n$, $t_j = [x_{j-1}, x_j]$.

We observe that

$$m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\} \leq \sup\{f(x) : x \in [x_{j-1}, x_j]\} = M_j$$

and hence

$$\sum_{j=1}^{n} m_j(x_j - x_{j-1}) \leq \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) \leq \sum_{j=1}^{n} M_j(x_j - x_{j-1})$$

i.e.

$$L(f, P) \leq S(f, P) \leq U(f, P \tag{2}$$

Also,

$$L(f, P) \leq \int_a^b f \leq U(f, P) \tag{3}$$

(1), (2) & (3) $\Rightarrow$

$$\left| S(f, P) - \int_a^b f \right| < \varepsilon$$

In particular, take $I_f = \int_a^b f$.

(ii)⇒(i) we let for $\varepsilon > 0$, given $P_{\varepsilon/4}$ be a partition s.t.

$$|S(f, P) - I_f| < \frac{\varepsilon}{4}$$

For P a refinement of $P_{\varepsilon/4}$, $S(f, P)$ a Riemann Sum. We fix such $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. For $j = 1, \cdots, n$, let m_j, M_j be as below, we then find for each j,

$$x_j^*, x_j^{**} \in [x_{j-1}, x_j] \qquad s.t. \qquad f(x_j^*) < m_j + \frac{\varepsilon}{4(b-a)} \qquad \& \qquad M_j - \frac{\varepsilon}{4(b-a)} < f(x_j^{**})$$

We then consider Riemann Sums

$$S^*(f, P) = \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1}), \qquad S^{**}(f, P) = \sum_{j=1}^{n} f(x_j^{**})(x_j - x_{j-1})$$

Notice that

$$\begin{aligned} S^*(f, P) - L(f, P) &= \sum_{j=1}^{n} [f(x_j^*) - m_j](x_j - x_{j-1}) \\ &< \sum_{j=1}^{n} \frac{\varepsilon}{4(b-a)}(x_j - x_{j-1}) \\ &= \frac{\varepsilon}{4(b-a)}(b-a) = \frac{\varepsilon}{4} \end{aligned}$$

and likewise,

$$U(f, P) - S^{**}(f, P) < \frac{\varepsilon}{4}$$

thus

$$\begin{aligned}
&U(f,P)-L(f,P)\\
=&U(f,P)-S^{**}(f,P)+S^{**}(f,P)-I_f+I_f-S^*(f,P)+S^*(f,P)-L(f,P)\\
<&\frac{\varepsilon}{4}+|S^{**}(f,P)-I_f|+|I_f-S^*(f,P)|+\frac{\varepsilon}{4}<\varepsilon
\end{aligned}$$

hence, by Cauchy's Criterion, f is integrable.

□

Remark: If $f:[a,b]\to\mathbb{R}$ is continuous, then P a partition of $[a,b]$ then each of $L(f,P)$ and $U(f,P)$ are Riemann Sums, proof: See proof of integrability of continuous.

Proposition 1.5.1 (linearity of integration). *Let $f,g:[a,b]\to\mathbb{R}$ each be integrable and $\alpha,\beta\in\mathbb{R}$, then*

- $\alpha f+\beta g:[a,b]\to\mathbb{R} \qquad (\alpha f+\beta g)(x)=\alpha f(x)+\beta g(x)$
- $\int_a^b(\alpha f+\beta g)=\alpha\int_a^b f+\beta\int_a^b g$

Proof. Let $\varepsilon>0$, then find partitions of $[a,b]$.

- P_1 s.t. for any refinement P of P_1, and any Riemann Sum $S(f,P)$
$$\left|S(f,P)-\int_a^b f\right|<\frac{\varepsilon}{2|\alpha|+1}$$
- P_2 s.t. for any refinement of $\mathbb{Q}$ of P_2, and any Riemann Sum $S(g,P)$,
$$\left|S(g,Q)-\int_a^b g\right|<\frac{\varepsilon}{2|\beta|+1}$$

Now we let $P=\{P_1\cup P_2\}$, a refinement of each of P_1 and P_2, write $P=\{a=x_0<x_1<\cdots<x_n=b\}$, and choose $t_j\in[x_{j-1},x_j]$ for each j. Then

$$\begin{aligned}
S(\alpha f+\beta g,P)&=\sum_{j=1}^n(\alpha f(t_j)+\beta g(t_j))(x_j-x_{j-1})\\
&=\alpha\sum_{j=1}^n f(t_j)(x_j-x_{j-1})+\beta\sum_{j=1}^n f(t_j)(x_j-x_{j-1})+\beta\sum_{j=1}^n g(t_j)(x_j-x_{j-1})
\end{aligned}$$

Then we have,

$$\begin{aligned}
\left|S(\alpha f+\beta g,P)-[\alpha\int_a^b f+\beta\int_a^b g]\right|\leq|\alpha|\left|S(f,P)-\int_a^b f\right|+|\beta|\\
\left|S(g,P)-\int_a^b g\right|<|\alpha|\frac{\varepsilon}{2|\alpha|=1}+|\beta|+\frac{\varepsilon}{2|\beta|+1}
\end{aligned}$$

□

Proposition 1.5.2 (Order Properties of Integrals). *Let $f, g : [a, b] \to \mathbb{R}$ each be integrable, then*

1. $f \geq 0 \Rightarrow f \geq 0$
2. $f \geq g \Rightarrow \int_a^b f \geq 0$
3. $f \geq g$ *on* $[a, b] \Rightarrow \int_a^b f \geq \int_a^b g$
4. $|f| : [a, b] \to \mathbb{R}(|f|(x) = |f(x)|)$ *is integrable, with* $\left|\int_a^b f\right| \leq \int_a^b |f|$
5. $g \vee g,\ f \wedge g : [a, b] \to \mathbb{R}$ *(*$f \vee g(x) = \max\{f(x), g(x)\}, f \vee g(x) = \min\{f(x), g(x)\}$*) are each integrable*

Proof.

1. for any partition P, $L(f, P) > 0$.
2. $f - g$ is integrable with $f - g \geq 0$, so $\int_a^b f - \int_a^b g = \int_a^b (f - g) \geq 0$, by 1.
3. let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, and for each $j = 1, \cdots, n$

$\square$

2 ANTIDERIVATIVE

2.1 Fundamental Theorem Of Calculus I - Jan 17 Friday

Proposition 2.1.1. *Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, define*

$$F : [a, b] \to \mathbb{R}, \qquad F(x) = \int_a^x f = \int_a^x f(t)dt$$

Note: *no $\int_a^x f(x)dx$.*

We may call this "integral accumulation function".

1. *F is continuous on $(a, b]$*
2. $\lim_{x \to a^+} F(x) = 0$

hence, we define $F(a) = 0 = \int_a^a f$. Thus $F : [a, b] \to \mathbb{R}$, and is continuous on $[a, b]$.

Proof.

1. A1. Q5(c) assume that f is integrable on each $[a, x]$, $x \in [a, b]$, so $F(x) = \int_a^x f$ makes sense. Now, let $a < x < x' \le b$, and we compute

$$\begin{aligned} F(x') - F(x) &= \int_a^{x'} f - \int_a^x f \\ &= \int_a^x f + \int_x^{x'} f - \int_a^x f && \text{(additivity)} \\ &= \int_x^{x'} f \end{aligned}$$

   Since f is integrable, it is bounded i.e. $\sup\limits_{x \in [a,b]} |f(x)| = M < \infty$. Thus, $|f(x)| \le M$ on $[a, b]$. Hence, by order properties,

$$|F(x') - F(x)| = \left| \int_x^{x'} f \right| \le \int_x^{x'} |f| \le \int_x^{x'} M = M(x' - x) = M\,|x' - x|$$

   Thus, given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{M+1}$, we have

$$|x' - x| < \delta \Rightarrow |F(x') - F(x)| \le M\delta = M\frac{\varepsilon}{M+1} < \varepsilon$$

   hence, F is uniformly continuous on $[a, b]$.

2. We use M as above

$$\left| \int_a^x f - 0 \right| = \left| \int_a^x f \right| \le \int_a^b |f| \le \int_a^x M = M(x - a)$$

   Porceed as above.

□

Theorem 2.1.1 (**Mean Value For Integrals** or **Average Value for Integrals**). *Let $f : [a, b] \to \mathbb{R}$ be continuous (integrability follows), then there exists $c \in [a, b]$, s.t.*

$$\int_a^b f = f(c)(b-a)$$

Proof. We use two important facts about continuous functions.

By **EVT**, there exists $x^*, x^{**} \in [a, b]$ s.t.

$$f(x^*) = m = min\{f(x) : x \in [a, b]\} \qquad and \qquad f(x^**) = M \max\{f(x) : x \in [a, b]\}$$

Then $m \leq f \leq M$, on $[a, b]$ so order properties provide

$$m(b-a) = \int_a^b m \leq \int_a^b f \leq \int_a^b M = M(b-a)$$

so

$$f(x^*) = m \leq \frac{1}{b-a}\int_a^b f \leq M = f(x^{**})$$

By **IVT**, Since $f(x^*) \leq \frac{1}{b-a}\int_a^b f \leq f(x^{**})$, there is c between x^* and x^{**}, and hence $c \in [a, b]$ s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f$$

□

Remark: f is integrable $\Rightarrow F(x) = \int_a^b f$ is a cts function. f cts $\Rightarrow F$ differentiable. (BELOW)

Theorem 2.1.2 (**Fundamental Theorem of Calculus (I)**). *Let $f : [a, b] \to \mathbb{R}$ be continuous, then*

$$F : [a, b] \to \mathbb{R}, \qquad F(x) = \int_a^x f$$

satisfies that F is differentiable on $[a, b]$, with $F' = f$ on $[a, b]$.

Proof. Let $x \in [a, b]$, we want to examine the quotient

$$\frac{F(x+h) - F(x)}{h} \qquad when \qquad x + h \in [a, b]$$

$h > 0$,

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \cdot \int_x^{x+h} f = \frac{1}{h} \cdot f(c_h)(x + h - x) = f(c_h)$$

by M.V.T for I, where $c_h \in [x, x + h]$,

$h < 0$,

$$\frac{F(x+h) - F(x)}{h} = \frac{F(x) - F(x+h)}{-h} = \frac{1}{-h} \cdot \int_{x+h}^{x} f = \frac{1}{-h} \cdot f(c_h)(x - x(x_h)) = f(c_h)$$

hence,

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\underbrace{\lim_{h\to 0}f(c_h)}_{continuity}=\underbrace{f(\lim_{h\to 0}c_h)}_{squeeze}=f(x)$$

Thus, $F'(x)$ exists, and equals $f(x)$, for $x\in[a,b]$.

Remark: Notice that we really found

- left derivative at $x=b$
- right derivative at $x=a$

□

Notation 2.1.1. *Let $-\infty\le a<b\le\infty\in\mathbb{R}$, $f:[a,b]\to\mathbb{R}$ be continuous, fix $c\in(a,b)$, dfeine*

$$F:(a,b)\to\mathbb{R}, F(x)=\begin{cases}\displaystyle\int_c^x f, & x\ge c\\ \displaystyle -\int_x^c f, & x<c\end{cases}$$

We know from FToCI, that $F'(x)=f(x)$ for $x>c$.

Proposition 2.1.2. *Let us compute $F'(x)$ for $x<c$, let $c'\in(a,c)$ and for $x\in(c',c)$ we have*

$$\begin{aligned} && \int_{c'}^c f &= \int_{c'}^x f+\int_x^c f\\ \Rightarrow && -\int_x^c f &= \int_{c'}^x f-\int_{c'}^c f\\ \Rightarrow && F'(x) &= \frac{d}{dx}\int_c^x f-\int_{c'}^c f=f(x)\end{aligned}$$

It will be convecient, hereafter, to let $\int_c^x f=-\int_x^c f$ if $x<c$, and we have FToCI

$$\frac{d}{dx}\int_c^x f=f(x),\qquad x\in(a,b).$$

2.2 Logrithm and Exponential Functions

Definition 2.2.1. *For $x \in (0, \infty)$,*

$$L(x) = \int_1^x \frac{1}{t} dt$$

we shall use only integral & differentiation rates to gain theory of L.

Proposition 2.2.1. *If $a, b > 0$, gthen $L(ab) = L(a) + L(b)$.*

Proof. Let $F(x) = L(ax)$, then chain rule provides

$$F'(x) = \frac{1}{ax} \frac{d}{dx}(ax) = \frac{1}{x} = L'(x)$$

hence, $F' - L' = 0 \Rightarrow F - L = C$ (constant), by MVT, $F = L + C(*)$. Then,

$$L(a) = F(1) = L(1) + C = C.$$

Also, $L(ab) = F(b) = L(b) + L(a)$.

$\square$

Proposition 2.2.2. *For $a > 0$, $q \in \mathbb{Q}$, $L(a^q) = qL(a)$, (convention: $a^0 = 1$).*

Proof. First: $n \in \mathbb{N}$,

$$L(a^n) = L(a) + L(a^{n-1}) = \cdots = \underbrace{L(a) + L(a) + \cdots + L(a)}_{n} = nL(a) \tag{1}$$

Second:

$$L(a) = L((a^{\frac{1}{n}})^n) = nL(a^{\frac{1}{n}}) \Rightarrow L(a^{\frac{1}{n}}) = \frac{1}{n} L(a) \tag{2}$$

Third:

$$0 = L(1) = L(aa^{-1}) = L(a) + L(a^{-1}) \Rightarrow L(a^{-1}) - L(a) \tag{3}$$

Then, (1) & (2) $\Rightarrow L(a^m) = mL(a)$, for $m \in \mathbb{Z}$, for $q = \frac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$.

We combine (1), (2), &, (3) to get $L(a^q) = mL(a^{\frac{1}{n}}) = \frac{m}{n} L(a)$.

$\square$

Proposition 2.2.3.

1. *L is inreasing: $0 < x < x'$ then $L(x) < L(x')$*
2. *$\lim_{x \to 0^+} L(x) = -\infty$, $\lim_{x \to \infty} L(x) = \infty$*

Proof.

1. $$L(x') - L(x) = \int_x^{x'} \frac{1}{t} dt \geq \int_x^{x'} \frac{1}{x'} dt = \frac{1}{x'}(x' - x) > 0$$

   Alternatively: $L'(x) = \frac{1}{x} > 0$, MVT $\Rightarrow L$ is strictly increasing.

2. To see that $\lim_{x\to\infty} L(x) = \infty$, it suffices to find $(a_n)_{n=0}^{\infty} \subset (0,\infty)$ s.t. $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} L(x_n) = \infty$. Consider $(2^n)_{n=0}^{\infty}$ and we have $\lim_{n\to\infty} L(2^n) = \lim_{n\to\infty} nL(2) = \infty$. Likewise, $\lim_{n\to\infty} 2^{-n} = 0$, and $\lim_{n\to\infty}(2^{-n}) = \lim_{n\to\infty}(-n)L(2) = -\infty$.

□

Corollary 2.2.1. *$L : (0,\infty) \to \mathbb{R}$ is one-to-one and onto.*

Proof. Increasing $\Rightarrow$ one-to-one, since $\lim_{x\to 0^+} = -\infty$, $\lim_{x\to\infty} L(x) = \infty$, and IVT provides that L is onto.

□

Definition 2.2.2. *$E : \mathbb{R} \to (0,\infty)$ to be L^{-1}: inverse function. Hence,*

$$E(L(x)) = x, x \in (0,\infty) \qquad \text{and} \qquad L(E(y)) = y \qquad if\ y \in \mathbb{R}$$

Proposition 2.2.4. *If $y \in \mathbb{R}$, $L(E(y)) = y$, chain $\underset{\Rightarrow}{}$ rule$\frac{1}{E(y)}E'(y) = 1$*
$\Rightarrow E'(y) = E(y)$

Algorithm 2.2.1 (About E). *Let $c, d \in \mathbb{R}$,*

1. $E(c+d) = E(c)E(d)$
2. $E(-c) = \frac{1}{E(c)}$
3. $E(0) = 1$
4. $E(qc) = E(c)^q$, $q \in \mathbb{Q}$

Proof.
1. Let $c = L(a)$, $d = L(b)$ (L is onto) $E(c+d) = E(L(a)+L(b)) = E(L(ab)) = ab = E(a)E(b)$
2. $L(1) = 0$ so $E(0) = 1$
3. use (1) and (2)
4. $E(qc) = E(qL(a)) = E(L(a^q)) = a^q = E(c)^q$.

□

What is $E(1)$? We note that

$$\lim_{h\to 0}\frac{L(1+h)}{h} = L'(1) = \frac{1}{1} = 1$$

Hence,

$$1 = \lim_{n\to\infty}\frac{L(1+\frac{1}{n})}{\frac{1}{n}} = \lim_{n\to\infty} nL(1+\frac{1}{n}) = \lim_{n\to\infty} L((1+\frac{1}{n})^n)$$

Since E is continuous,

$$E(1) = E(\lim_{n\to\infty} L((1+\frac{1}{n})^n)) = \lim_{n\to\infty} E(L((1+\frac{1}{n})^n)) = \lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

From rule (iv), $E(q) = e^q$ for $q \in \mathbb{Q}$, if $x \in \mathbb{R}$, write $x = \lim_{n\to\infty} q_n$, each $q_n \in \mathbb{Q}$, and we define

$$e^x = E(x) = \lim_{n\to\infty} E(q_n) = \lim_{n\to\infty} e^{q_n}$$

Definition 2.2.3. *For $a > 0$, we have $a = E(L(a)) = e^{L(a)}$, and we let*

$$a^x = E(L(a)x) = e^{L(a)x}$$

Exercise With Chain Rule:

1. $\frac{d}{dx}(a^x) = L(a)a^x$,
2. $L(a^x) = L(a)x = xL(a)$,
3. $p \in \mathbb{R}$, $x > 0$, $x^p = e^{p(L(x))}$, $\frac{d}{dx}(x^p) = px^{p-1}$

2.3 Fundamental Theorem of Calculus II - Jan 22

Theorem 2.3.1 (Fundamental Theorem of Calculus II). *Let $f, F : [a, b] \to \mathbb{R}$ satisfy that*

- *f is integrable*
- *F is continuous on $[a, b]$*
- *F is differentiable on (a, b), with $F' = f$ on (a, b)*

Then,

$$F(b) - F(a) = \int_a^b f$$

Proof. Let $\varepsilon > 0$, find a partition P_ε on $[a, b]$, so

- for every refinement P of P_ε
- for every Riemann Sum $S(f, P)$, we have

$$\left| S(f, P) - \int_a^b f \right| < \varepsilon$$

Take P as above, write $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$.

Now let us consider F on each $[x_{j-1}, x_j]$

- F is continuous on $[x_{j-1}, x_j]$
- F is differentiable on $[x_{j-1}, x_j]$ [can be used in closed interval, except for $j = 0, n$]

Thus MVT tells us there exists $c_j \in (x_{j-1}, x_j) \subset [x_{j-1}, x_j]$ such that

$$F(x_j) - F(x_{j-1}) = F'(c_j)(x_j - x_{j-1}) = f(c_j)(x_j - x_{j-1}) \qquad (*)$$

Now we consider

$$\begin{aligned} F(b) - F(a) &= \sum_{j=1}^{n} [F(x_j) - F(x_{j-1})] && \text{(telescope)} \\ &= \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) && \text{(by *)} \\ &= S(f, P) && \text{(a Riemann Sum)} \end{aligned}$$

Hence,

$$\left| F(b) - F(a) - \int_a^b f \right| = \left| S(f, P) - \int_a^b f \right| < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we get desired result. ☐

Remark:

- Suppose $F, G : [a, b] \to \mathbb{R}$, both satisfy $F' = f = G'$, for integrable f, then

$$(F - G)' = F' - G' = f - f = 0 \underset{\Rightarrow}{M.V.T} F - G = C(constant)$$

  hence, $F(x) = G(x) + C$ for any x in $[a, b]$.

- If $f : [a, b] \to \mathbb{R}$ is continuous, then f is integrable (theorem from earlier) & $F(x) = \int_a^b f$ defines on antiderivative.

  Moral: f continuous $\to$ an antiderivative exists.

Notation 2.3.1. *If f is continuous, (on same intervals), and F is an antiderivative of f, i.e. $F' = f$ (on interval of said intervals), write $\int f(x)dx = F(x) + C$.*

Antiderivatives of Basic Functions:

$$p \neq -1, \qquad \int x^p dx = \frac{x^{p+1}}{p+1} + C \qquad\qquad \int e^x dx = e^x + C$$

$$\int \cos x dx = \sin x + C \qquad\qquad \int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C \qquad\qquad \int \sec^2 x dx = \tan x + C$$

$$\int \frac{1}{x^2+1} dx = \arctan x + C[Tan = \tan|_{(\frac{\pi}{2}, \frac{-\pi}{2})]:(-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}} \qquad \text{one-to-one and onto}$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C[\text{Sin} = \sin|_{(\frac{\pi}{2}, \frac{-\pi}{2})]:(-\frac{\pi}{2}, \frac{\pi}{2}) \to [-1,1]} \qquad \text{one-to-one and onto}$$

Theorem 2.3.2 (**Change of Variables/Substitution/Reverse Chain Rule**). *Suppose*

- *$g : [a, b] \to \mathbb{R}$, differentiable with g' continuous*
- *f is defined on $g([a, b])$ with $f \circ g : [a, b] \to \mathbb{R}$ continuous*

Then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)d(u)$$

Anti Derivative Form:

$$\int f(g(x))g'(x)dx = \int f(u)du|_{u=g(x)}$$

Proof. Let F be any antiderivative of $f[g[(a, b)] = [c, d]$, let $F(x) = \int_x^c f]$.

Let $H : [a, b] \to \mathbb{R}$ be given by $H(x) = F(g(x))$. Then Chain Rule provides

$$H'(x) = F/(g(x))g'(x) = f(a(x))g'(x)$$

and F.T. of C II provides that

$$H(b) - H(a) = \int_a^b f(g(x))g'(x)dx$$

but F.T.of C provides that

$$\int_{g(a)}^{g(b)} f(u)d(u) = F(g(b)) - F(g(a)) = H(b) - H(a)$$

□

Example:

1.

$$\begin{aligned}\int xe^{-x^2}dx &= -\frac{1}{2}\int e^{-x^2}(-2x)dx \\ &= -\frac{1}{2}\int e^u du \\ &= -\frac{1}{2}e^u + C \\ &= -\frac{1}{2}e^{-x^2} + C\end{aligned}$$

2.

$$\begin{aligned}\int_1^3 x(x^2+4)^{91}dx &= \frac{1}{2}\int_5^{13} u^{91}dx \\ &= \frac{1}{2}\frac{u^{92}}{92}|_5^{13} \\ &= \frac{1}{184}[(13)^{92} - 5^{92}]\end{aligned}$$

3.

$$\begin{aligned}\int \cos^m x \sin^n x dx &= \int \cos^m x \sin^{2k} x \sin x dx && (\text{n odd}) \\ &= \int \cos^m x(1-\cos^2 x)^k \sin x dx && (u = \cos x,\ du = -\sin x dx) \\ &= -\int u^m(1-u^2)^k du|_{u=\cos x}\end{aligned}$$

2.4 Integration and Trignometry - Jan 22 Wed, TUT

Definition 2.4.1. $\pi = 2\int_{-1}^{a} \sqrt{a - x^2}dx$

Definition 2.4.2. *Let for* $-1 \leq x \leq 1$,

$$\arccos x = x\sqrt{1-x^2} + 2\int_x^1 \sqrt{1-u^2}du$$

Then $\frac{1}{2}\arccos x$ *is the area of —-graph—-*

Note: $\frac{1}{2}\arccos x$ is proportional to the angle θ, hence it is reasonable to measure.

$$\theta = \arccos x \qquad \text{"}radians\text{"}$$

- $\arccos -1 = \pi$
- $\arccos 0 = 2\int_0^1 \sqrt{1-u^2}du \overset{symmetry}{=} \int_{-1}^1 \sqrt{1-u^2}du = \frac{\pi}{2}$
- $\arccos 1 = 0$

Derivatives:

$$\begin{aligned}\arccos' x &= \sqrt{1-x^2} + x\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) - 2\sqrt{1-x^2} \\ &= -\frac{x^2}{\sqrt{1-x^2}} - \sqrt{1-x^2}\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} = -\frac{1}{\sqrt{1-x^2}}\end{aligned}$$

hence,

- $\arccos' x < 0$ and by MVY, decreasing
- $\lim_{x\to -1^+} \arccos' x = -\infty = \lim_{x\to 1^-} \arccos' x$
- $\arccos' 0 = -1$
- $\arccos''(x) = -\frac{x}{(1-x^2)^{\frac{3}{2}}}$ hence,
  - $\arccos''(x) > 0$ if $x < 0 \Rightarrow$ concave up
  - $\arccos''(x) < 0$ if $x > 0 \Rightarrow$ concave down

Definition 2.4.3.

- $\mathrm{Cos}\, x = \arccos^{-1} : [0, \pi] \to [-1, 1]$
- $\sin\theta = \sqrt{1 - \cos^2\theta}$

Hence, $\sin : [0, \pi] \to [0, 1]$, *with*

- $\mathrm{Sin}(0) = \sqrt{1-1^2} = 0$
- $\mathrm{Sin}(\frac{\pi}{2}) = \sqrt{1-0^2} = 0$
- $\mathrm{Sin}(\pi) = \sqrt{1-(-1)^2} = 0$

Derivatives of $\cos$, $\sin$

$\arccos(\cos\theta) = \theta$

$$\underset{\text{Chain Rule}}{\Rightarrow} \frac{-1}{\sqrt{1-\cos^2\theta}}\cos'\theta = 1 \Rightarrow \cos'\theta = -\sin\theta$$

$$\sin'\theta = \frac{d}{d\theta}\sqrt{1-\cos^2\theta} = \frac{1}{x}(1-\cos^2\theta)^{-\frac{1}{2}}(-2\cos\theta\cos'\theta) = \cos\theta$$

Hence, $\sin'(0) = 1$, $\sin'\frac{\pi}{2} = 0$, $\sin'(\pi) = -1$, and $\sin''(\theta) = -\sin\theta < 0$ if $0 < \theta < \pi \Rightarrow$ concave down

<u>Extension to $\mathbb{R}$</u>

(a) we define $\cos, \sin : [-\pi, \pi] \to [-1, 1]$

- $\cos$ is even: $\cos(-\theta) = \cos\theta$, $\theta \geq 0$
- $\sin$ is odd: $\sin(-\theta) = -\sin\theta$, $\sin\theta = \operatorname{Sin} x$, if $\theta \geq 0$

(b) we define $\cos, \sin : \mathbb{R} \to [-1, 1]$

$$\cos(\theta + 2\pi n) = \cos(\theta) \qquad \sin(\theta + 2\pi n) = \sin(\theta) \qquad \theta \in [-\pi, \pi], n \in \mathbb{Z}$$

Lemma 2.4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable, then*

- $f(0) = f'(0) = 0$ *and*
- $f'' + f = 0$

then $f = 0$.

Proof. Let $g = (f')^2 + f^2$ then

$$g(0) = 0 \qquad \textit{and} \qquad g' = 2ff' + 2ff' = 2f[f'' + f] = 0$$

$\Rightarrow$ by MVT, g constant, hence, $g = 0$, then $0 \leq f^2 \leq g$. □

Lemma 2.4.2. *Double Angle Fomula for Cos*

Proof. Let $a, b \in \mathbb{R}$ be fixed, defined $f : \mathbb{R} \to \mathbb{R}$,

$$f(t) = \cos(s+t) - a\sin t + b\cos t$$

Then

$$\begin{aligned} f'(t) &= -\sin(s+t) + a\sin t + b\cos t \\ f''(t) &= -\cos(s+t) + a\cos t - b\sin t \\ \Rightarrow f'' + f &= 0 \end{aligned}$$

Now we wish to choose a, b to satisfy

$f(0) = 0$, hence $0 = f(0) = \cos s - a \Rightarrow a = \cos s$
$f(0) = 0$, hence $0 = f'(0) = -\sin s + b \Rightarrow b = \sin s$

With these choices of a, b, the lamma tells us that $f(t) = 0$, hence

$$0 = \cos(s+t) - [\cos s\cos t - \sin s\sin t)$$

□

Double Angle Fomula for $\cos$**:** Since $\cos^2 t + \sin^2 t = 1$, the angle sum fomula gives

$$\cos 2t = \cos^2 t - \sin^2 t = \begin{cases} 1 - 2\sin^2 t \Rightarrow \sin^2 t = \frac{1}{2}[1 - \cos^2 t] \\ 2\cos^2 t - 1 \Rightarrow \cos^2 t = \frac{1}{2}[1 - \cos^2 t] \end{cases}$$

Lemma 2.4.3. *Double Angle Fomula for* $\sin$*:* $\sin(s + t) = \cos s \sin t + \sin x \cos t$

Proof. Fix $s \in \mathbb{R}$, for t consider

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

and take $\frac{d}{dt}$ to both sides. □

Double Angle Fomula for $\sin$**:**

$$\sin 2t = 2\cos t \sin t$$

Example 1:

1.

$$\begin{aligned} \int \sin^2 x dx =& \frac{1}{2}\int (1 - \cos 2x) dx \\ =& \frac{1}{2}[x - \frac{1}{2}\sin 2x] + C \\ =& \frac{1}{2}x - \frac{1}{4}\sin 2x + C \\ =& \frac{1}{2}x - \frac{1}{2}\sin x \cos x + C \end{aligned}$$

2.

$$\begin{aligned} \int \cos^4 x dx =& \int [\frac{1}{2}(1 + \cos 2x)]^2 dx \\ =& \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x) dx \\ =& \frac{1}{4}\int (1 + 2\cos 2x + \frac{1}{2}[1 + \cos 4x]) dx \end{aligned}$$

3.

$$\begin{aligned} & \int \sin x \cos^4 x dx && (u = \cos x, du = -\sin x dx) \\ =& -\int u^4 du|_{u=\cos x} \\ =& -\frac{\cos^5 x}{5} + C \end{aligned}$$

4.

$$\begin{aligned}\int \sin^2 x \cos^4 x dx =& \int \sin^2 x \cos^2 x \cos^2 x dx\\ =& \int (\frac{1}{2}\sin 2x)^2 \frac{1}{2}[1+\cos 2x]dx\\ =& \frac{1}{8}\int [\sin^2 2x + \sin^2 2x \cos 2x]dx\end{aligned}$$

Change of Variables(Antiderivatives form)

$$\int f(g(x))g'(x)dx = \int f(u)du|_{u=g(x)}$$

f, g' continuous.

Inverse Form: Suppose we try $x = g(u)$,

$$\int f(x)dx = \int f(g(u))g'(u)du|_{x=g(u)}$$

Algorithm 2.4.1 (**Trig Substitution**).

Forms	*Substitution*	*Main Identity*	dx
$a^2 - x^2$	$x = a\sin\theta$	$a^2 - x^2 = a^2\cos^2\theta$	$dx = a\cos\theta d\theta$
$x^2 + a^2$	$x = a\tan\theta$	$x^2 + a^2 = a^2\sec\theta$	$dx = a\sec^2\theta d\theta$

Examples

1.

$$\begin{aligned}\int \frac{dx}{(9-x^2)^{3/2}} =& \int \frac{3\cos\theta}{(9\cos^2\theta)^{3/2}}dx\\ =& \frac{1}{9}\int \sec^2\theta d\theta = \frac{1}{9}\tan\theta + C\\ =& \frac{1}{9}\frac{\sin\theta}{\sqrt{1-\sin^2\theta}} + C\\ =& \frac{1}{9}\frac{\frac{1}{3}x}{\sqrt{1-(\frac{1}{3}x)^2}} + C = \frac{1}{9}\frac{x}{\sqrt{9-x^2}} + C\end{aligned}$$

2.

$$\begin{aligned}\int \frac{dx}{x^2+2x+5} =& \int \frac{dx}{(x+1)^2+4} \qquad (x+1 = 2\tan\theta, dx = 2\sec^2\theta d\theta)\\ =& \int \frac{2\sec^2\theta}{2^2\sec^2\theta}d\theta\\ =& \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C\\ =& \frac{1}{2}\arctan\frac{x+1}{2} + C\end{aligned}$$

3.

$$\begin{aligned}
\int \sqrt{1-x^2}dx = &\int \cos^2\theta d\theta \\
=&\frac{1}{2}\int [1+\cos 2\theta]d\theta \\
=&\frac{1}{2}[\theta+\frac{1}{2}\sin 2\theta]+C \\
=&\frac{1}{2}[\arcsin x+\sin\theta\cos\theta]+C \\
=&\frac{1}{2}[\arcsin x]+x\sqrt{1-x^2}+C \\
&\Rightarrow \arcsin(x)=2\int\sqrt{1-x^2}dx-x\sqrt{1-x^2}+C'
\end{aligned}$$

$$[\arcsin x=\frac{\pi}{2}-\arccos x]\checkmark$$

4.

$$\begin{aligned}
\int\frac{dx}{\sqrt{x^2+1}} &= \int\frac{\sec^2\theta d\theta}{\sec\theta} \qquad (x=\tan\theta, dx=\sec^2\theta d\theta) \\
&=\int \sec\theta d\theta \\
&=\int \sec\theta\frac{\sec\theta\tan\theta}{\sec\theta+\tan\theta}d\theta \\
&=\int\frac{\sec^2\theta+\tan\theta\sec\theta}{\sec\theta+\tan\theta}d\theta \\
&=\log|\sec\theta+\tan\theta|+C \\
&=\log(\sqrt{x^2+1}+x)+C
\end{aligned}$$

5.

$$\int\frac{dx}{\sqrt{x^2+1}}=\int\frac{\cosh t}{\cosh t}dt \qquad (x=\tan\theta,)$$

2.5 Integration by Partial Fraction - Jan 27

Warm Up:

$$\begin{aligned}
\int \sec\theta d\theta &= \int \frac{1}{\cos\theta} d\theta \\
&= \int \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} d\theta \\
&= \int \frac{\sin^2\theta}{\cos\theta} d\theta + \int \cos\theta d\theta \\
&= \int \frac{\sin^2\theta}{1-\sin^2\theta} \cos\theta d\theta + \int \cos\theta d\theta
\end{aligned}$$

Theorem 2.5.1.

1. *Let $q \neq 0$ be a polynomial with $\mathbb{R}$-coefficients, then we may write*

$$q(x) = a(x-r_1)^{m_1} \cdots (x-r_m)^{m_m} \cdot (x^2+b_1x+c_a)^{n_1} \cdots (x^2+b_Nx+C_N)^{n_N}$$

   where $a \neq 0$, $r_1, \cdots, r_M$ are the distinct $\mathbb{R}$-roots of q, and $b_1, \cdots, b_N, \cdots, c_N \in \mathbb{R}$..

   $b_j^2 - 4c_j < 0$ for $j = 1, \cdots, N$. Also, $m_1, \cdots, m_N, n_1, \cdots, n_N \in \mathbb{N}$.

2. *Let p be $\mathbb{R}-$polynomial with*

$$\deg p < \deg q$$

   Then there are unique $\mathbb{R}-$numbers $A_1, \cdots, B_N, C_N$. so

$$\frac{p(x)}{q(x)} = \sum_{j=1}^{M} \sum_{k=1}^{M} \frac{A_j, k}{(x-r_j)^k} = + \sum_{j=1}^{N} \sum_{k=1}^{n_j} \frac{B_{j,k}x + C_{j,k}}{x^2+b_jx+c_j}^k$$

Example

$$\frac{x^2+4x+3}{(x-1)^2(x^2+3x+4)}$$

2.6 Integration by parts - Jan 29

Theorem 2.6.1 (Integration by Parts/"Reverse Product Rule"). *Let $f, gF : [a,b] \to \mathbb{R}$ satisfy*

- *f is integrable on $[a,b]$*
- *$F' = f$ on $[a,b]$*
- *g' is integrable on $[a,b]$*

Then

$$\int_a^b f(x)g(x)dx = F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)dx$$

Antiderivative Form:

$$\int f(x)g(x)dx = F(x)g(x) - \int F(x)g'(x)dx, \qquad F(x) = \int f(x)dx \qquad \textit{Can choose } c = 0$$

$$\int f'g = fg - \int fg'$$

Proof. Product Rule:

$$\frac{d}{dx}[F(x)g(x)] = F'(x)g(x) + F(x)g'(x) = f'(x)g(x) + F(x)g'(x)$$

FToCII:

$$F(b)g(b) - F(a)g(a) = \int_a^b [f(x)g(x) + F(x)g'(x)]dx$$

$$\Rightarrow F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)dx = \int_a^b f(x)g(x)dx$$

□

Example 1

$$\begin{aligned}\int \arctan(x)dx = &\int 1 \cdot \arctan(x)dx \\ =& x\arctan(x) - \int x\frac{1}{1+x^2}dx \\ =& x\arctan(x) - \frac{1}{2}\log(1+x^2) + C\end{aligned}$$

Example 2

$$\begin{aligned}\int x^2e^x dx =& x^2e^x - \int 2x \cdot e^x dx \\ =& x^2e^x - 2[xe^x - \int e^x dx] \\ =& x^2e^x - 2xe^x + 2e^x + C\end{aligned}$$

Example 3

$$\int \cos^{2n}(x)dx \qquad n \geq 1 = \int \cos x \cos^{2n-1} x dx \tag{I_n}$$

$$= \sin x \cos^{2n-1} dx - \int \sin x (2n-1) \cos^{2n-2}(-\sin x) dx$$

$$= \sin x \cos^{2n-1} x + (2n-1) \int (1 - \cos^2 x) \cos^{2n-2} x dx$$

$$= \sin x \cos^{2n-1} x + (2n-1)[\int \cos^{2n-2} x dx - \int \cos^{2n} x dx]$$

$$= \sin x \cos^{2n-1} x + (2n-1)[I_{n-1}(x) - I_n(x)]$$

$$\Rightarrow 2nI_n(x) = \sin x \cos^{2n-1} x + (2n-1) I_{n-1}(x)$$

$$I_n(x) = \frac{1}{2n} \sin x \cos^{2n-1} x + \frac{2n-1}{2n} I_{n-1}(x) \qquad \text{("Reduction Fomula")}$$

Specific Example: $n = 0$, $I_0(x) = \int \cos^0 x dx = \int 1 dx = x + C$ Hence

$$\int \cos^2 x dx = I_1(x) = \frac{1}{2} \sin x \cos x + \frac{1}{2}[x + C]$$

$$= \frac{1}{2} \sin x \cos x + \frac{1}{2} x + C'$$

$$\int \cos^2 x dx = \frac{1}{2} \int [1 + \cos 2x] dx$$

$$= \frac{1}{2} x + \frac{1}{4} \sin 2x + C$$

$$\int \cos^4 x dx = I_2(x) = \frac{1}{4} \sin x \cos^3 x + \frac{3}{4}[\frac{1}{2} \sin x \cos x + \frac{1}{2} x] + C$$

$$= \frac{1}{4} \sin x \cos^3 x + \frac{3}{8} \sin x \cos x + \frac{3}{8} x + C$$

Exmaple 3'

$$\int \frac{dt}{(t^2+1)^3} = \int \frac{\sec^2 \theta}{(\sec^2 \theta)^3} d\theta$$

$$= \int \cos^4 \theta d\theta$$

$$= \frac{1}{4} \sin \theta \cos^3 \theta + \frac{3}{8} \sin \theta \cos \theta + \frac{3}{8} \theta + C$$

$$= \frac{1}{4} \frac{t}{(1+t^2)^2} + \frac{3}{8} \frac{t}{1+t^2} + \frac{3}{8} \arctan(t) + C$$

2.7 Improper Integral - Jan 29

Recall: Integration involves upper and lower sums and hence requires

- bounded functions and
- bounded intervals

Definition 2.7.1. *let $a < b$ and $f : (a, b] \to \mathbb{R}$*

- *f is integrable on $[x, b]$ fro each $x \in (a, b]$.*

Then we define the ***improper integral*** *by*

$$\int_a^b f = \lim_{x \to a^+} f, \qquad \textit{provided that limit exists}$$

Example 1:

$f(t) = \frac{1}{\sqrt{t}}$ on $(0, 2]$, notice that f is continuous, hence integrable on $[x, 2]$, $0 < x < 2$.

Compute

$$\int_x^2 \frac{dt}{\sqrt{t}} = \int_x^2 t^{-1/2} dt = 2t^{1/2}\Big|_x^2 = 2\sqrt{2} - 2\sqrt{x}$$

Then

$$\int_0^2 \frac{dt}{\sqrt{t}} = \lim_{x \to 0^+} \int_x^2 \frac{dt}{\sqrt{t}} = 2\sqrt{2} - 2\sqrt{0} = 2\sqrt{2}$$

Example 2:

$g(t) = \frac{1}{t^2}$ on $[0, 2]$. g is cts, so integrable on each $[x, 2]$, $0 < x < 2$.

$$\int_x^2 \frac{dt}{t^2} = -\frac{1}{t}\Big|_x 62 = \frac{1}{x} - \frac{1}{2}$$

$$\lim_{x \to 0^+} \int_x 62 \frac{dt}{t^2} = \lim_{x \to 0^+} [\frac{1}{x} - \frac{1}{2}] = \infty$$

We write $\int_0^2 \frac{dt}{t^2} = \infty$ or $\int_0^2 \frac{dt}{t^2}$ D.N.E..

Example 3:

$h(t) - \frac{\left|\sin \frac{1}{t}\right|}{\sqrt{t}}$, $t \in (0, 2]$, h is continuous on each $[x, 2]$, $0 < x < 2$.

How can we show if this is improperly integrable?

Comparison method

$$\begin{aligned} & 0 \leq \left|\sin \frac{1}{t}\right| \leq 1 \\ \Rightarrow \qquad & 0 \leq \frac{\left|\sin \frac{1}{t}\right|}{\sqrt{t}} \leq \frac{1}{\sqrt{t}} \\ \Rightarrow \qquad & 0 \leq \int_x^2 \frac{\left|\sin \frac{1}{t}\right|}{\sqrt{t}} dt \leq \int_x^2 \frac{dt}{\sqrt{t}} = 2\sqrt{2} - 2\sqrt{x} \leq 2\sqrt{2} \end{aligned}$$

$H(x) = \int_x^2 \frac{\left|\sin\frac{1}{t}\right|}{\sqrt{t}} dt$ is nonincreasing.

If $0 < x' < x < 2$, $H(x') - H(x) = \int_{x'}^2 h - \int_x^2 h = \int_{x'}^x h + \int_x^2 h - \int_x^2 h = \int_{x'}^x h \geq 0$

2.8 Jan 31

1. $\lim_{x\to a} F(x) = L \Leftrightarrow$ for every sequence $(a_n)_{n=1}^{\infty}$ s.t. $\lim_{n\to\infty} a_n = a$, provides that $\lim_{n\to\infty} F(a_n) = L$.
2. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$, then $\lim_{n\to\infty} a_n$ exists $\Leftrightarrow$ for any $\varepsilon > 0$, there exists $n_\varepsilon \in N$ s.t. $|a_m - a_n| < \varepsilon$ whenever $m, n \geq n_\varepsilon$.

Cauchy criterion[Deep Fact: Bolzano Weierstrass Theorem]

Theorem 2.8.1 (Cauchy Criterion for limit of function). *Let $F : (a, b] \to \mathbb{R}$, then $\lim_{x\to a^+} F(x) \Leftrightarrow$ exists for any $\varepsilon > 0$, there is $\delta > 0$ s.t. $|F(u) - F(v)| < \varepsilon$ whenever $|u - a| < \delta$ and $|v - a| < \delta$ for $u, v \in (a, b]$.*

Last Time:

$$\int_0^2 \frac{\left|\sin(\frac{1}{t})\right|}{\sqrt{t}} dt = \lim_{x\to 0^+} \underbrace{\int_x^2 \frac{\left|\sin(\frac{1}{t})\right|}{\sqrt{t}} dt}_{H(x)}$$

H is monotone and bounded $\Rightarrow \lim_{x\to a^+} H(x)$ exists.

Example:

Consider

$$\int_0^1 \frac{\sin(\frac{1}{t})}{\sqrt{t}} dt = \lim_{x\to 0^+} \int_x^1 \frac{\sin(\frac{1}{t})}{\sqrt{t}} dt$$

$$-1 \leq \sin(\frac{1}{t}) \leq 1$$

$$\Rightarrow -\frac{1}{\sqrt{y}} \leq \frac{\sin(\frac{1}{t})}{\sqrt{t}} \leq \frac{1}{\sqrt{t}} \overset{\text{order properties}}{\Rightarrow} -\int_x^1 \frac{dt}{\sqrt{t}} \leq \int_x^1 \frac{\sin(\frac{1}{t})}{\sqrt{t}} dt \leq \int_x^1 \frac{dt}{\sqrt{t}}$$

Now we consider $0 < u < v < 1$, again order properties give:

$$-\int_u^v \frac{dt}{\sqrt{t}} \leq \int_u^v \frac{\sin(\frac{1}{t})}{\sqrt{t}} dt \leq \int_u^v \frac{dt}{\sqrt{t}}$$

$$-2(\sqrt{v} - \sqrt{u}) \leq F(v) - F(u) \leq 2(\sqrt{v} - \sqrt{u})$$

$$|F(v) - F(u)| \leq 2(\sqrt{v} - \sqrt{u}) \leq 2\sqrt{v}$$

If $\delta = \frac{\varepsilon^2}{4}$ and if $0 < u < v < \delta$

$$|F(v) - F(u)| < 2\sqrt{\delta} = \varepsilon$$

hence, $\lim_{x\to 0^+} F(x) = \int_0^1 \frac{\sin(\frac{1}{t})}{\sqrt{t}} dt$ exists.