Math 146 Notes

velo.x

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Section: 001
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Contents

1 Vector Space

1.1 Vector Space - Jan 6

Definition 1.1.1 (**Pseudo-**Field)**.** *A field is an algebraic system* $\mathbb{F}$ *having:*

- *two elements* 0 *and* 1
- *operations* $+, \times, -$, *and* $()^{-1}$*(defined on nonzero elements)*

satisfying "the obvious" properties.

See appendix of the textbook.

Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Z}_{prime}$. $\mathbb{Q}(x) = \{\frac{f(x)}{g(x)} : f, g\ polynomials, g \neq 0\}$

NonExamples: $\{0\}$, $\mathbb{Z}_m$*(m not prime), Quaternions.*

Definition 1.1.2 (**Vector Space**)**.** *A vector space over* $\mathbb{F}$ *is a set* V *with two operations:*

- *Addition:* $V \times V \to V$ $x + y$
- *Scalar Multiplication:* $\mathbb{F} \times V \to V$ ax

satisfying 8 *properties:* $\forall x, y, z \in V$, $\forall a, b \in \mathbb{F}$

- *V1:* $x + y = y + x$
- *V2:* $x + (y + z) = (x + y) + z$
- *V3:* $\exists$ *a "zero vector"* $0 \in V$ *s.t.* $x + 0 = x$
- *V4:* $\forall x \in V$, $\exists u \in V$, *s.t.* $x + u = 0$
- *V5:* $1x = x$
- *V6:* $(ab)x = a(bx)$ **let* $\cdot$ *denote scalar multiplication*
- *V7:* $a(x + y) = ax + ay$
- *V8:* $(a + b)x = ax + bx$

Objective 1.1.1.

- *Defining/Constructing*
- *Proving that a system is a vector space*

Example 1: $\mathbb{R}$ def: set of all $n - tuples$ of real numbers

$(x_1, \cdots, x_n) + (y_1, \cdots, y_n) =$

$a(x_1, \cdots, x_n)$ defined $(ax_1, \cdots, ax_n)$ Claim: $\mathbb{R}^n$ is a vector space over $\mathbb{R}$

Proof. Check V1:

$$\begin{aligned}(x_1, \cdots, x_n) + (y_1, \cdots, y_n) &= (x_1 + y_1, \cdots, x_n + y_n) \\ &= (y_1 + x_1, \cdots, y_n + x_n) \\ &= (y_1, \cdots, y_n) + (x_1, \cdots, x_n)\end{aligned}$$

□

More generally, for any field $\mathbb{F}$, $\mathbb{F}^n$ is a field over $\mathbb{F}$.

Example 2: $\mathbb{R}^{[0,1]} = \{all\ functions\ f : [0,1] \to \mathbb{R}\}$

- $(f + h)(x) \overset{def}{=} f(x) + g(x)$
- $(af)(x) = af(x)$

Claim: $\mathbb{R}^[0, 1]$ is a vector space $/\mathbb{R}$.

Proof. V3: Let $\overline{0}$ be the constant 0 function, i.e., $\overline{0}(x) = 0\ \forall x \in [0, 1]\ \overline{0} \in \mathbb{R}^{[0,1]}$

Check: $f + \overline{0} = f\ \forall f \in \mathbb{R}^{[0,1]}$

$$\begin{aligned}(f + \overline{0})(x) &= f(x) + \overline{0}(x) \\ &= f(x) + 0 = f(x)\end{aligned}$$

Since $x \in [0, 1]$ arbitrary, $f + \overline{o} = f$.

More generally, for any set D, and any field $\mathbb{F}$, $\mathbb{F}^D$ is a vector space over $\mathbb{F}$.

□

Example 3: let $\mathbb{F} = \mathbb{Z}_2$.

Define $W = \{APPLE\}$,

- $APPLE + APPLE \overset{def}{=} APPLE$
- $0APPLE \overset{def}{=} APPLE$
- $1APPLE \overset{def}{=} APPLE$

Claim: W is a vector space over $\mathbb{Z}_2$.

1.2 Introduction to Linear Combination - Jan 8

Examples: 1. $\mathbb{R}^n : \mathbb{F}^n$, 2. $\mathbb{R}^{[0,1]}$, : $\mathbb{F}^D$, 3. $\{APPLE\}$.

4. Fix a field $\mathbb{F}$, for $n \geq 0$, $P_n(\mathbb{F})$ is the set of all polynomials, of degree $\leq n$, in variable x, with coefficients from $\mathbb{F}$,

$$= \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n : a_i \in \mathbb{F}\}$$

Addition, scalar mult are "obvious", using op's of $\mathbb{F}$.

Claim: $P_n(\mathbb{F})$ is a vecor space $/\mathbb{F}$.

5. $\mathbb{F}[x] =$ the set of all polynomials in x with coefficients from $\mathbb{F} = \Cup_{n=0}^{\infty} P_n(\mathbb{F})$

Claim: with the "obvious" op's $\mathbb{F}[x]$ is a V.S. $/\mathbb{F}$.

Theorem 1.2.1 (Cancellation Law). *Let V be a V.S., $/\mathbb{F}$, if $x, y, z \in V$, and $x + z = y + z$, then $x = y$.*

Proof. Let $u \in V$ be such that $z + u = 0$ (from V4).

Then

$$\begin{aligned}
x &= x + 0 && \text{(V3)}\\
x &= x + (z + u) && \text{(Choice of u)}\\
x &= (x + z) + u && \text{(hypothesis)}\\
x &= (y + z) + u && \text{(V2)}\\
x &= y + (z + u) && \text{(V2)}\\
x &= y + 0 && \text{(choice of u)}\\
x &= y
\end{aligned}$$

□

Corollary 1.2.1. *Suppose V is a V.S., there is exactly one "zero vector". i.e. a vector satisfy V3. in V.*

Proof. Assume $0_1, 0_2 \in V$, both satisfying V3, i,e, $x + 0_1 = x$ and $x + 0_2 = x, \forall x \in V$.

$$\begin{aligned}
0_1 &= 0_1 + 0_1\\
0_1 &= 0_1 + 0_2
\end{aligned}$$

$$\begin{aligned}
0_1 + 0_1 &= 0_1 + 0_2\\
&= 0_2 + 0_1 && \text{(V1)}\\
0_1 &= 0_2 && \text{(By Cancellation)}
\end{aligned}$$

□

Corollary 1.2.2. *Suppose V is a V.S. and $x \in V$, then the vector u in V4 is unique.*

Proof. Assume $u_1, u_2 \in V$ both satisfy $x + u_1 = 0 = x + u_2$, then

$$\begin{aligned} u_1 + x &= u_2 + x && \text{(V1)} \\ u_1 &= u_2 && \text{(By Cancellation)} \end{aligned}$$

$\square$

Definition 1.2.1. *Given a V.S. V and $x \in V$,*

- *the unique vector $u \in V$ s.t. $x + u = 0$ is denoted $-x$.*
- *$x - y$ denotes $x + (-y)$*

<u>**Note:**</u> V2 justifies $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_kx_k$ not worry about parentheses.

Definition 1.2.2 (Linear Combination). *$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_kx_k$ is called a linear combination of $x_1, \cdots, x_k$.*

Basic Problem: Given a V.S. $V/\mathbb{F}$, and $u_1, u_2, \cdots, u_n \in V$ and $x \in V$ to decide whether x is a linear combination of $u_1, \cdots, u_n$.

Example: $V = \mathbb{Q}[x]$ over $\mathbb{Q}$. Let $p = 4x^4 + 7x^2 - 2x + 3$.

- $u_1 = x^4 - x^2 + 2x + 1$
- $u_2 = 2x^4 + 3x^2 + 2x$
- $u_3 = x^4 + 4x^2 + 1$
- $u_4 = 2x^3 + 3$
- $u_5 = x^4 + 1$

Is p a linear combination of $u_1, \cdots, u_5$? <u>Solution:</u> search for $a_1, \cdots, a_5 \in \mathbb{Q}$ s.t.

$$p = a_1u_1 + a_2u_2 + \cdots + a_5u_5$$

$$\begin{aligned} 4x^4 + 7x^2 - 2x + 3 =& a_1(x^4 - x^2 + 2x - 1) + a_2(2x^4 + 3x^2 + 2x) + a_3(x^4 + 4x^2 + 1) \\ &+ a_4(2x^3 + 3) + a_5(x^4 + 1) \\ =&(a + 1 + 2a_2 + a - 3 + a_5)x^4 + (2a^4)x^3 + (-a_1 + 3a_2 + 4a_3)x^2 \\ &+ (2a_1 + 2a_2)x + (-a_1 + a_3 + 3a_4 + a_5) \end{aligned}$$

$$\begin{cases} a_1 + 2a_2 + a_3 + a_5 = 4 \\ 2a_4 = 0 \\ -a_1 + 3a_2 + 4a_3 = 7 \\ 2a_2 + 2a_2 = -2 \\ -a_1 + a_3 + 3a_4 + a_5 = 3 \end{cases}$$

No solution.

1.3 Subspace - Jan 10

Notation 1.3.1.

- 0 *denote the unique vector in* V
- x *denote the unique* $u \in V$ *satisfying* $V4$

Theorem 1.3.1. *Suppose* V *is a VS /* $\mathbb{F}$*,* $X \in V$*,* $a \in \mathbb{F}$*.*

1. *0x=0, the first* 0 *is scalar, the second* 0 *is a vector*
2. *(-a)x=a(-x)=-(ax)*
3. *a0=0*

Definition 1.3.1. *Suppose* V *is a* $V.S.$ *over* $\mathbb{F}$*,* $S \subseteq V$*,*

- ***Closed under Addition:*** *if* $x, y \in S$*,* $x + y \in S$*.*
- ***Closed under Scalar Multiplication:*** *if* $x \in S \Rightarrow ax \in S$*,* $\forall a \in \mathbb{F}$*.*

Definition 1.3.2 (Subspace). *Let* V *be a VS /* $\mathbb{F}$*,* $S \subseteq V$*, say* S *is a* ***Subspace*** *of* V *if*

1. S *is closed under addition and scalar multiplication*
2. $S \neq \varnothing$

Theorem 1.3.2. *Suppose* V *is a vector space /* $\mathbb{F}$ *and* S *is a subspace of* V*, then* S*, together the operations of* V *restricted to* S*.*

- $+_S : S \times S \to S$
- $\cdot_S : \mathbb{F} \times S \to S$

Proof. Given V, S, must prove: S with restricted operations of V, satisfying $V1$ to $V8$.

V1: must show: if $X, y \in S$, then $x + y = y + x$. Since $S \in V$, hence $x, y \in S \Rightarrow x, y \in V$, and $V \models V1$.

Same proof works for V2, 5, 6, 7, 8.

V3: know $S \neq \varnothing$, take any $x \in S$, consider $0x = 0 \in S$. (S is closed under scalar multiplication)

Hence there eixst a zero vector in S.

V4: fix $x \in S$, let $u = (-x)x \in S$, then $x + u = 1x + (-1)x = (1 + (-1))x = 0x = 0$. □

<u>**Note:**</u> in every $\mathbb{F}$, $\forall a \in \mathbb{F}$, $\exists c \in \mathbb{F} a + c = 0$, $c = -a$. Since $1 \in \mathbb{F}$, $-1 \in \mathbb{F}$.

Theorem 1.3.3. *If* V *is a vector space over* $\mathbb{F}$ *and* $S \subseteq V$*, and* S *with the operations of* V*, is itself a V.S. /* $\mathbb{F}$*, then* V *is a subspace of* V*.*

1.4 Span - Jan 13

Recall: If V is a V.S. / $\mathbb{F}$, and $u_1, \cdots, u_n, x \in V$, then x is a linear combination (lin. combo.) of $u_1, \cdots, u_n$ if $\exists a_1, \cdots, a_n$ such that $x = a_1u_1 + a_2u_2 + \cdots + a_nu_n$.

Definition 1.4.1. *Suppose V is a V.S. / $\mathbb{F}$, $x \in V$, and $\varnothing \neq S \subseteq V$.*

1. *Say x is a lin. combo. of S if $\exists$ finitely many $u_1, \cdots, u_n \in S$, s.t. x is a lin. combo. of $u_1, \cdots, u_n$. $S = \{u_1, u_2, \cdots, u_n\}$, $x = \sum_{n=0}^{\infty} a_nu_n$, converge.*
2. *The **Span** of S written $\operatorname{span}(S)$, is the set of all linear combinations of S.*
3. $\operatorname{span}(\varnothing) \overset{df}{=} \{0\}$

Examples

- In $\mathbb{R}^2$, $S = \{(1,1)\}$, what is $\operatorname{span}(S)$? the
- In $\mathbb{R}^3$, $S = \{(1,0,0),(1,1,0)\} = \{a(1,0,0) + b(1,1,0) : a, b \in \mathbb{R}\} = \{(a+b, b, 0) : a, b \in \mathbb{R}\} = (s,t,0) : s,t, \in \mathbb{R} =$the plane given by $z = 0$
- In $\mathbb{R}[x]$, let $S = \{x, x^2, x^3, \cdots\}$, $span(S) = \{f \in \mathbb{R}[x] : f(0) = 0\}$.

Proposition 1.4.1. $(\varnothing \neq S \subseteq V)$. *Suppose $u_1, \cdots, u)n \in S$, $x \in V$. Suppose x is a linear combination of $u_1, \cdots, u_n$. If $v_1, \cdots, v_n$ are more vectors from S, then x is also a linear combination of $u_1, \cdots, u_n$, $v_1, \cdots, v_n$.*

Proposition 1.4.2. *If $S = \{u_1, \cdots, u_n\}$, then $\operatorname{span}(S) = \{a_1u_1, \cdots, a_ku_k, a_1, \cdots, a_k \in \mathbb{F}\}$.*

Proposition 1.4.3. *If $S \subseteq T \subseteq V$, then $\operatorname{span}(S) \subseteq \operatorname{span}(T)$.*

Proposition 1.4.4. *If S is infinite, if $x, y \in \operatorname{span}(S)$, say x is a linear combo of $u_1, \cdots, u_n \in S$, y is a linear combo of $v_1, \cdots, v_n \in S$, then x, y are linear combos of $u_1, \cdots, u_n, v_1, \cdots, v_n$.*

Generalization 1.4.1. *If $x_1, \cdots, x_k \in \operatorname{span}(S)$, then $\exists u_1, \cdots, u_n \in S$, s.t. each x_l is a linear combo of $u_1, \cdots, u_n$.*

Theorem 1.4.1. *Suppose V is a V.S / $\mathbb{F}$, $S \subseteq V$, then $\operatorname{span}(S)$ is the (unique) smallest subspace of $V \supseteq S$. i.e.*

1. $\operatorname{span}(S)$ *is a subspace of V.*
2. $S \subseteq \operatorname{span}(S)$
3. *If W is any subspace of V containing S, then $\operatorname{span}(S) \subseteq W$.*

Proof. 1. Let $x \in S$, $x = 1x$, a linear combination of finitely many vectors in S.

2. i) Closure under scalar multiplication: let $x \in \text{span}(S)$, $c \in \mathbb{F}$, $\Rightarrow \exists u_1, \cdots, u_n \in S$, s.t. $x = a_1x_1 + \cdots + a_nx_n$, so

$$cx = c(a_1u_1 + \cdots + a_mu_m) = (ca_1)u_1 + \cdots + (ca_n)u_n$$

ii) Closure under vector addition: let $x, y \in \text{span}(S)$, want to prove that $x + y \in \text{span}(S)$.

By the technical remark, $\exists u_1, \cdots, u_n \in S$ s.t. $x = a_1u_1 + \cdots + a_nu_n$, $y = b_1u_1 + \cdots + a_nu_n$, $a_i, b_i \in \mathbb{F}$,

Then, $x + y = (a_1u_1 + \cdots + a_nu_n) + (b_1u_1 + \cdots + b_nu_n) = (a_1 + b_1)u_1 + \cdots + (a_n + b_n)u_n$.

So $x + y \in \text{span}(S)$.

Finally, if $S = \varnothing$, then $\text{span}(S) = \{0\}$, if $S \neq \varnothing$, then $S \subseteq \text{span}(S)$,

either case, $\text{span}(S) \neq \varnothing$, so $\text{span}(S)$ is a subspace of V.

3. Let W be a subspace

$\square$

<u>Intuition:</u> Redundancies in span. Example: V / $\mathbb{F}$, suppose $S = \{u_1, \cdots, u_5\} \subseteq V$.

Assume u_3 is a linear combination of u_2, u_4, u_5.

$$u_3 = c_2u_2 + c_4u_4 + c_5u_5$$

<u>Claim:</u> $(S) = \text{span}(S - \{u_3\})$.

Proof. RTP $\subseteq$ and $\supseteq$.

$\text{span}(S)$ is

- a subspace of V
- which contains $S\backslash\{u_3\} = \{u_1, u_2, \cdots, u_3\}$

By the theorem, the samllest subspace of V containing $S\backslash\{u_3\}$ is $\text{span}(S\backslash\{u_3\})$. hence $\text{span}(S) \supseteq \text{span}(S\backslash\{u_3\})$.

To prove that $\text{span}(S) \subseteq \text{span}(S\backslash\{u_3\})$,

let $x \in \text{span}(S)$, i.e.

$$\begin{aligned} x &= a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 + a_5u_5 \\ &= a_1u_1 + a_2u_2 + a_3(c_2u_2 + c_4u_4 + c_5u_5) + q_4u_4 + a_5u_5 \\ &= a_1u_1 + (a_2 + a_3c_2)u_2 + (a_4 + a_3c_4)u_4 + (a_5 + a_3c_5)u_5 \end{aligned}$$

$x \in Span(\{u_1, u_2, u_4, u_5\})$ $\square$

<u>Also Observe:</u>

$$0u_1 + c_2u_2 + (-1)u_3 + c_4u_4 + c_5u_5 = 0$$

A linear combination of $u_1, \cdots, u_5$ equally the 0 vector with coefficients not all 0.

So we code redundancies formally with definition:

Definition 1.4.2. *($V\mathbb{F}, S \subseteq V$), S is linearly dependent if $\exists$ distinct vectors $u_1, \cdots, u_n \in S$, and $\exists a_1, \cdots, a_n \in \mathbb{F}$, not all 0, such that*

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0(zero\ vector)$$

S is linearly independent if S is not linearly dependent.

S is linearly dependent $\iff$ ($\exists$ *distinct* $u_1, \cdots, u_n \in S)(\exists a_1, \cdots, a_n \in \mathbb{F},\ \not\forall\, 0)(a_1u_1 + \cdots + a_nu_n) = 0$

$\equiv (\forall\ distinct\ u_1, \cdots, u_n \in S)()$

Technical Remark: when $S = \{u_1, \cdots, u_n\}$ without reports

- Can drop ($\forall\ distinct\ u_1. \cdots, u_n \in S$) in choice of linear independence.

-Can drop ($\exists\ distinct\ u_1 \cdots u_1, \cdots, u_n \in S$) in choice of linear dependence.

Example 2: Is $S = \{(0,1,1), (1,0,1), (1,2,3)\}$ linear dependent? (in $\mathbb{R}^3$)

Try to find: $a, b, c \in \mathbb{R}$ s.t.

$$a\begin{pmatrix}0\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\0\\1\end{pmatrix} + c\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}-2\\-1\\1\end{pmatrix}$$

Shows S is linearly dependent.

Question: If $S = \varnothing$, S is **linearly dependent**.

Question 2: If $S = \{0\}$, S linearly dependent. Can write $1 \cdot 0 = 0$.

More Generally, if $0 \in S \subseteq V$, then S is linearly dependent.

Theorem 1.4.2 (Linear Dependence). *$V\mathbb{F}$, $S \subseteq V$, then S is linearly dependent, iff $S = \{0\}$ or $\exists x \in S$, s.t. x is a linear combination of some vectors in $S \backslash \{x\}$.*

1.5 Basis Jan 17

Recall If V is a V.S. / $\mathbb{F}$, $S \subseteq B$.

1. $\text{span}(S) =$ set of all linear combinations of S
2. S is linearly dependent if $\exists u_1, u_2, \cdots, u_n \in S$ (distinct), $\exists a_1, \cdots, a_n \in \mathbb{F}$ not all 0, s,t, $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$.

   - else, S is linearly independent.

Definition 1.5.1. *V is V.S. / $\mathbb{F}$,*

1. *A set $S \subseteq V$ is a spanning set of $\text{span}(S) = V$. Also say S spans V.*
2. *V is finitely spanned if V has a finite spanning set.*
   V is countably spanned if V has a countable spanning set.

Examples:

$\mathbb{R}^3$ is finitely spanned, e.g. by $\{e_1, e_2, e_3\}$.

so is $\mathbb{R}^n$ e.g. by $\{e_1, e_2, \cdots, e_n\}$, $e_i = (0, 0, \cdots, 1, 0, \cdots, 0)$ with 1 at i_{th} spot.

$\mathbb{R}[x]$ is countably spanned e.g. by $\{1, x, x^2, x^3, \cdots\}$. not finitely spanned.

$\mathbb{R}^[0, 1]$ not countably spanned.

Definition 1.5.2. *V is a V.S. / $\mathbb{F}$.*

A basis for V is any $S \subseteq V$, which

- *spans V, and*
- *S is linearly independent*

Examples: $\{e_1, \cdots, e_n\} \subseteq \mathbb{F}^n$ is a basis for $\mathbb{F}^n$.

$\{1, x, x^2, x^3, \cdots\} \subseteq \mathbb{R}[x]$ is a basis for $\mathbb{R}[x]$.

Theorem 1.5.1. *Every countably spanned V.S. has a basis.*

Proof. Suppose V.S. V is spanned by countable set S, so either $S = \{v_1, v_2, \cdots, v_n\}$, or $S = \{v_1, v_2, \cdots\}$, WLOG, we assume $0 \notin S$, define

$$T = \{v_j \in S, v_j \notin span(v_1, v_2, \cdots, v_{j-1})\},$$

Claim that T is a basis for V.

Proof of Claim: 1^{st} show T is linearly independent, by contradiction, assume T is linearly dependent.

Then, $\exists k$, and scalars $a_1, a_2, \cdots, a_n$ (not all 0), s,t,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

Choose least k for which this is true.

Claim: $k \neq 1$, if $k = 1$, $a_1 v_1 = 0 \Rightarrow v_1 = 0$, but $0 \notin T$, contradiction.

so $k > 1$, Assume $a_k = 0$, then

$$a_1 v_1 + a_2 v_2 + a_{k-1} v_{k-1} = 0$$

Not all of $a_1, a_2, \cdots, a_{k-1} = 0$.

Next, show $span(S) = V$.

$$S = \{v_1, v_2, v_3, \cdots, v_n\}$$

$$T = \{v_{i_1}, v_{i_2}, v_{i_3}, \cdots\}$$

Know $\text{span}(S) = V$, intuitively $\text{span}(T) = \text{span}(S)$.

$$T = \{v_j \in S : v_j \notin \text{span}(\{v_1, v_2, \cdots, v_{j-1}\})\}$$

Therefore, T is a basis of V.

$\square$

Remark:

1. Every Vector Space has a basis. proof: some version of axiom of choice
2. bases is not unique, every V.S. except $\{0\}$, has multiple bases.
3. What is a basis for $V = \{0\}$? $\varnothing$

Theorem 1.5.2 (Axiom of Choice). *Suppose* A, B *are sets,* $f : A \rightarrow$.

1.6 Dimension - Jan 20

Remark: Given a vector space V, the basis is not unique.

Relation between two basis of a vector space. (finitely spanned vector spaces)

Theorem 1.6.1. *Let V be a finitely spanned vector space over a field $\mathbb{F}$, let $\{v_1, \cdots, v_m\}$ be a basis of V, let $\{w_1, \cdots, w_n\} \subset V$ and $n > m$. Then $\{w_1, \cdots, w_n\}$ is linearly dependent.*

Sketch. Idea: Replace successfully $v_1, v_2, \cdots, v_n$, by $w_1, w_2, \cdots, w_n$ so that

$$\operatorname{span}(\{w_1, w_2, \cdots, w_i, v_{i+1}, \cdots, v_m\}) = \operatorname{span}(\{v_1, v_2, \cdots, v_i, v_{i+1}\})$$

$1 \leq i \leq m-1$. □

Proof. Assume $\{w_1, \cdots, w_n\}$ is linearly dependent. Prove the statement by induction.

Base Case: (i=1), since $\{v_1, \cdots, v_m\}$ is a basis for V and $w_1 \in V$, there exist $a_1, \cdots, a_m \in \mathbb{F}$ s.t. $w_1 = a_1 v_1 + \cdots + a_m v_m$.

By the assumption, $w_1 \neq 0$, hence one of the a_k's is nonzero.

By renumbering $v_1, \cdots, v_m$, WLOG, we can assume $a_1 \neq 0$. We can solve for v_1.

$$\begin{aligned} a_1 v_1 &= w_1 - a_2 v_2 - \cdots - a_m v_m \\ v_1 &= a_1^{-1} w_1 - a_1^{-1} a_2 v_2 - \cdots - a_1^{-1} a_m v_m \end{aligned}$$

so, $\operatorname{span}(\{v_1, v_2, \cdots, v_m\}) \subset \operatorname{span}(\{w_1, w_2, \cdots, w_m\}) = V$.

Induction Assumption: Assume that the statement is true for r. It means after renumbering, $v_1, v_2, \cdots, v_m$ we have

$$\operatorname{span}(\{w_1, w_2, \cdots, w_i, v_{i+1}, \cdots, v_m\}) = V.$$

*replace w_{i+1}.

Prove for $r+1$: Rewrite w_{i+1} as a linear combination of $\{w_1, \cdots, w_r, v_{r+1}, \cdots, v_m\}$.

$$w_{i+1} = c_1 w_1 + \cdots + c_r w_r + d_{i+1} v_{i+1} + \cdots + d_m v_m$$

Observation: One of the $d_{r+1}, \cdots, d_m$ must be nonzero. Because if $d_{i+1} = \cdots = d_m = 0$, then

$$\begin{aligned} w_{r+1} &= c_1 w_1 + \cdots + c_r 2_r \\ 0 &= c_1 w_1 + \cdots + c_r w_r - w_{r+1} \end{aligned}$$

Contradiction since $\{w_1, \cdots, w_{r+1}\}$ is linearly independent.

WLOG, we can assume $d_{i+1} \neq 0$,

$$d_{r+1} v_{r_1} = w_{r+1} - c_1 w_1 - \cdots - a_r w_r - d_{r+2} v_{r+2} - \cdots - d_m v_m$$

Since $n > m$, $w_n = a_i w_i + \cdots + a_m w_m$, so $\{w_1, \cdots, w_n\}$ is linearly dependent.

It completes the proof. □

Theorem 1.6.2. *Let V be a finitely spanned vector space, having one basis of m elements having another basis of n elements. Then $m = n$.*

Proof. We could not have $m < n$, or $m > n$. If it happends, the other set must be linearly dependent. $\square$

Definition 1.6.1. *Let V be a vector: space having a basis consisting of n elements, we say n is the dimensioning of V.*

$$\dim_{\mathbb{F}} V = n$$

$$\lim\{0 = 0\}$$

A vector space that has a basis consisting of n elements, zero elememts, zero vector space, is called finite dimensional. Otherwise, V is called infinite dimensional(Hamel Basis)

Example:

- $\dim \mathbb{F}^n = n$

  Since

  $$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

  is a basis for $\mathbb{F}^n$.

- $\dim P_n(\mathbb{F}) = n + 1$

  Since $\{1, x, \cdots, x^n\}$ is a basis for $P_n(\mathbb{F})$.

- $\dim \mathbb{F}[x] = \infty$

Corollary 1.6.1. *Let V be an $n-$dimensional space, then*

- *If $\{v_1, \cdots, v_n\} \subset V$ is linearly independent, then $\{v_1, \cdots, v_n\}$ is a basis for V.*
- *If $\{v_1, \cdots, v_n\} \subset V$, $k < n$ is linearly we can add $v_{k+1}, \cdots, v_n$ so that $\{v_1, \cdots, v_n\}$ is a basis for V.*
- *If W is a subspace of V, then $\dim W \leq \dim V$, if furthermore, $\dim W = \dim V$. Then $W = V$.*

1.7 Direct Sum - Tutorial Jan 20

Corollary 1.7.1. *If V is finitely spanned, and $\beta\{v_1, \cdots, v_n\}$ is linearly independent, then β can be extended to a basis for V, i.e. $\exists w_1, \cdots, w_n \in V$, s.t. $\{v_1, \cdots, v_n, w_1, \cdots, w_r\}$ is a basis for V*

Proof. Let $m = dim = V$. So $n \leq m$ by theorem.

Case 1: β is alreayd a basis. (n=m)

Case 2: β is not a basis. □

1.8 Jan 22

Corollary 1.8.1. *If V is finitely spanned, and $\mathfrak{B} = \{v_1, \cdots, v_n\}$ is linearly independent, then $\mathfrak{B}$ can be extended to a basis for V.*

i.e. $\exists w_1, \cdots w_r \in V$, s.t. $\{v_1, \cdots, v_n, w_1, \cdots, w_n\}$ is a basis for V.

Proof. Let $m = \dim V$, so $n \leq m$. (By theorem).

case 1: $\mathfrak{B}$ is already a basis (n = m). done

Case 2: $\mathfrak{B}$ is not a basis, so span$\mathfrak{B} \neq V$, so $\exists w_1 \in V \backslash \mathfrak{B}$. □

Theorem 1.8.1. *For any V.S. V, if $\mathfrak{B} \subseteq V$ is linearly independent, then $\mathfrak{B}$ can be extended to a basis for V. [use axiom of choice]*

Example: Let $\mathfrak{B} = \{\cos(nx), n \geq 0\} \cup \{\sin(nx) : n > 0\} \cup \{e^x\}$.

This $\mathfrak{B}$ can be extended to a basis $\mathfrak{B}'$ for $\mathbb{R}^{[0,1]}$.

$$|\mathfrak{B}'| = 2^{2^{\aleph_0}}$$

Recall: If $\{v_1, \cdots, v_n\} \subseteq V$ is linearly independent. Say $\{v_1, \cdots, v_n\}$ is a maximal linearly independent set, if $\forall w \in V \backslash \{v_1, \cdots, v_n\}$, $\{v_1, \cdots, v_n, w\}$ is linearly dependent.

Corollary 1.8.2. *If V is a finitely spanned set, then every basis is a maximal linearly independent set, and vice versa.*

More generally,

Definition 1.8.1. *Let V be a V.S., a subset $\mathfrak{B} \subseteq V$ is a* ***maximal linearly independent set*** *if*

- *$\mathfrak{B}$ is linearly independent*
- *$\forall w \in V \backslash \mathfrak{B}$, $\mathfrak{B} \cup \{w\}$ is linearly dependent.*

Theorem 1.8.2. *In any V.S. V, every basis is a maximal linearly independent set, and vice versa.*

Definition 1.8.2. *A* ***mininal spanning set*** *is a set $\mathfrak{B}$ such that*

- span$\mathfrak{B} = V$
- $\forall w \in \mathfrak{B}$, span$(\mathfrak{B} \backslash \{w\}) \neq V$

Theorem 1.8.3. *In every vector space V,*

1. *Every bassi is a minimal spanning set and vice versa*
2. *Every spanning set can be "shrunk" to a basis*
   i.e. if $\operatorname{span}\mathfrak{B} = V$*, then* $\exists \mathfrak{B}' \subseteq \mathfrak{B}$ *s.t.* $\mathfrak{B}'$ *is a basis for* V*.*

Proof. For (2), already proved when $\mathfrak{B}$ is countable. Can extend the proof to uncountable "well-ordering $\mathfrak{B}$".

To find a basis for $\mathbb{R}^{[0,1]}$

1. start with $\mathfrak{B} = \mathbb{R}^{[0,1]}$
2. well-order $\mathfrak{B}$ ("enumerates" $\mathfrak{B}$)
3. use the enumeration to shrink $\mathfrak{B}$ to a basis

□

1.9 Jan 24

Review: $\mathbb{Z}_n$ = the set of the congruence classes, $x \equiv y \pmod m \iff m|x-y$

Revisit: $[0] = \{qm : a \in \mathbb{Z}\} = m\mathbb{Z}$.

-$m\mathbb{Z}$ is collapsed to become zero

-$x \equiv y \pmod n \iff x = y \in m\mathbb{Z}$.

-advanced notation: $\mathbb{F}/m\mathbb{Z}$.

Version of this:

- $(\mathbb{Z}, +, \cdot) \to$ a vector space V.
- $(m\mathbb{Z}) \to$ a subspace of V.

Definition 1.9.1. *Fix a V.S.* V *over* $\mathbb{F}$*, and a subspace* W*.*

For $x, y \in V$ *say* $x \equiv y \pmod W$*, if* $x - y \in W$*.*

Claim: $\equiv \pmod W$ is an equiv relation on V.

Proof. For transitivity:

Assume $x, y, z \in V$, $x \equiv y \pmod W$ and $y \equiv z \pmod W$, by definition, $x - y \in W$, $y - z \in W$.

Then $x - z = (x - y) + (y - z) \in W$ since W is closed under addition.

Then by definition, $x \equiv z \pmod W$. □

Notation 1.9.1. *Define* V, W *as before:*

For $x \in V$*,*

$$x + W := \{x + w : w \in W\}$$

(x is fixed, add x to every vector on W). $x + W$ *is called* ***translation of*** W ***by*** x*, or* ***coset of*** W ***through*** x*.*

Claim: V, W as before, for any $x \in V$, the equivalence class (congruence class) of $\equiv \pmod{W}$ containing x is $x + W$.

if $y \equiv x \pmod{W}$, and $w \in W$, then $y \equiv x + w \pmod{W}$.

Proof. For any $y \in V$, $y \in$ the equiv of $\equiv \pmod{V}$ containing x

$\iff y \equiv x \pmod{W}$

$\iff y - x \in W$

$\iff y - x = w$, for some $w \in W$

$\iff y = x + w$

$\iff y \in x + W$ □

Remark: For $x ßn V$, the span class of $\equiv \pmod{W}$ containing x is

$$\{y \in V, y \equiv x \pmod{W}\}$$

Now define

$$\begin{aligned} V/W &:= \text{the set of all equiv classes of the} \equiv \pmod{W}\text{relation} \\ &:= \text{the set of all translations of} W \\ &:= \{x + W : x \in V\} \neq V \end{aligned}$$

Next, we turn V/W into a vector space over $\mathbb{F}$,

$$(x + W) \oplus (y + W) := (x + y) + W$$

$$c(x + w) := (cx) + W$$

Issue: Are the operations well-defined? Yes

E.g. check scalar multiplication:

assume $x + W = x_1 + W$, $x \equiv x_1 \pmod{W} \iff x - x_1 \in W$.

need to know: $\forall c \in \mathbb{F}$,

$$\begin{aligned} &(cx + W) = (cx_1) + W \\ \Updownarrow \quad & cx \equiv cx_1 \pmod{W} \\ \Updownarrow & (cx) - (cx_1) \in W \\ & c(x - x_1) \in W \end{aligned}$$

2 Linear Transformations

Definition 2.0.1. *Let V, W be vector spaces over $\mathbb{F}$, a function $T : V \to W$ is a linear transformation (or is linear) if*

1. $T(x+y) = T(x) + T(y)$, $\forall x, y \in V$
2. $T(ax) = aT(x)$, $\forall x \in V$, $\forall a \in \mathbb{F}$

Example

$V = W = \mathbb{R}$ (as $V.S./\mathbb{R}$)

Fix $\lambda \in \mathbb{R}$,

$$T : \mathbb{R} \to \mathbb{R} \qquad T(x) = \lambda x$$

T is a linear transformation.

Check: Let $x, y \in \mathbb{R}$, $a \in \mathbb{R}$

1. $T(x+y) = \lambda(x+y) = \lambda x + \lambda y = T(x) + T(y)$
2. $T(ax) = \lambda(ax) = a(\lambda x) = aT(x)$

fact: Every linear transformation from $\mathbb{R} \to \mathbb{R}$ has this form.

Generalization 2.0.1. *let $V = X = \mathbb{F}$, (field) considered as $V.S/\mathbb{F}$, every linear transformation $T : \mathbb{F} \to \mathbb{F}$ is of form $T(x) = \lambda x$ for some $\lambda \in \mathbb{F}$.*

Example: $V = W = \mathbb{R}^2$

define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T((x_1, x_2)) = (-x_2, x_1)$,

$T((1,0)) = (0,1)$

$T((0,1)) = (-1,0)$

Actually, T is "rotation" by $90°$ c.c.w centered at $(0,0)$.

Claim: T is a linear transformation.

Proof. $T((x_1, x_2) + (y_1, y_2)) = T((x_1 + y_1, x_2 + y_2)) = T(-(x_2 + y_2), x_1 + y_1) = (-x_2, z_1) + (-y_2, y_1) = T((x_1, x_2)) + T((y_1, y_2))$

Similarly, can check $T(a(x_1, x_2)) = aT((x_1, x_2))$ □

Generalization 2.0.2. *Fix $A \in M\mathbb{R}$, set of all $m \times n$ matrices with entries from $\mathbb{R}$,*

so

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Define $L_A : \mathbb{R}^n \to \mathbb{R}^n$, $L_A(x) = Ax$. x is a column vector nx_1 matrix

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_n + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

Claim: L_A is a linear transformation.

Proof. By example, $m = n = 2$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$L_A(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = (-x_2, x_1)$ □

Generalization 2.0.3. *Fix a field $\mathbb{F}$, fix $A \in M_{m\times n}(\mathbb{F})$,*

define $L_A; \mathbb{F}^n \to \mathbb{F}^m$ by $L_A(x) = Ax$,

Claim: *L_A is a linear transformation.*

Recall: $C([-1,1])$ = all continuous functions$f : [-1,1] \to \mathbb{R}$, define $T : C([-1,1]) \to \mathbb{R}$, by $T(f) = \int_{-1}^{1} f(x)dx$.

Claim: T is a linear transformation.

Proof.

$$\begin{aligned} T(f+g) &= \int_{-1}^{1} (f+g)dx \\ &= \int_{-1}^{1} fdx + \int_{-1}^{1} gdx \\ &= T(f) + T(g) \end{aligned}$$

$T(af) = \int_{-1}^{1} afdx = a\int_{-1}^{1} fdx = aT(f)$ □

$D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ (set of all $f \in C(\mathbb{R})$),

$f^{(n)}$ exists, and is continuous $\forall n$.

Define $D(f) = f'$, D is linear.

Some easy properties of all linear transformations, suppose $T : V \to W$ linear.

1. $T(0) = 0$

   Proof. (a) $T(x+0) = T(x) + T(0)$

   (b) $T(0 \cdot x) = 0T(x) = 0$

   □

2. $T(x-y) = T(x) - T(y)$

Proof. $T(x-y) = T(x+(-1)y) = T(x) + T((-1)y) = T(x) - T(y)$ □

3. $T(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = a_1T(x_1) + \cdots + a_nT(x_n)$

Common Mistake:

$$T(ax+by) = T(a)T(x) + T(b)T(y)$$

More Examples:

$M_{m\times n}\mathbb{F}$ is a vector space over $\mathbb{F}$, -add matrices componentwise -scalar multiply by multiplying all components

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

$$c\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix}$$

$T : M_{m\times n}(\mathbb{F}) \to M_{n\times m}(\mathbb{F})$ by $T(A) = A^t$. (transpose of A)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}^t = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

$(V = W)$ define $I_v : V \to V$ by $I_v(x) = x$ its linear.

2.1 Tutorial - Jan 27

Goals:

- Be able to describe the quotient space
- Be able to find a basis and the dimension of the quotient space

Recall that:

Definition 2.1.1. *V is a V.S. $W \leq V/\mathbb{R}$, we call V/W a quotient space if*

$$\begin{cases} (x+W)+(y+W)=(x+y)+W \\ c(x+W)=cx+W \end{cases}$$

which $x, y \in V$, $c \in \mathbb{R}$.

Example:

$V=\mathbb{R}^3$, $W=\operatorname{span}\{(0,0,1)\}$. $\mathbb{R}^3/W$ is a quotient space.

Question: What are the elements in $\mathbb{R}^3/W$?

A: $p+W$, $p \in \mathbb{R}^3$.

B: $[p+W]=\{x \in \mathbb{R}^3 | x-p \in w\}$

C: All lines that are parallel to Z-axis

2.2 Null Spance and Range

Definition 2.2.1. *Suppose $T : V \to W$ is a linear transformation.*

1. *The **null space** of T denoted $N(T)$, is*

$$N(T) = \{x \in V : T(x) = 0\}$$

2. *The **range** of T denoted as $R(T)$*

$$R(T) = \{T(x) : x \in V\} \subseteq W$$

Example: $D_n : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ $D_n(f) = f'$. It's linear.

What is $N(D_n)$?

$$N(D_n) = \{f \in P_n(\mathbb{R}) : f' = 0\} = \{c : c \in \mathbb{R}\}$$

$R(D_n) = P_n(\mathbb{R})$

Theorem 2.2.1. *Suppose $T : V \to W$ is linear*

1. *$N(T)$ is a subspace of V.*
2. *$R(T)$ is a subspace of W.*

Proof.

1. $T(0_v) = 0_w$ so $0_v \in N(T)$ so $N(T) \neq \varnothing$

   -closure under addition: let $x, y \in N(T)$,

$$T(x + y) = T(x) + T(y) = 0 + 0 = 0 \in N(T)$$

   -closure under scalar multiplication: let $x \in N(T)$, $c \in \mathbb{F}$

$$T(cx) = cT(x) = ca = 0 \in N(T)$$

2. $R(T) \neq \varnothing$ because $V \neq \varnothing$

   -closure under addition: let $u, v \in R(T) \subset W$, can write $u = T(x)$, $v = T(y)$, (for some $x, y \in V$), so $u + v = T(x) + T(y) = T(x + y) \in R(T)$.

   -Similar argument shows that $R(T)$ is closed under scalar multiplication.

□

Algorithm 2.2.1 (Useful Trick). *Suppose $T : V \to W$ is a linear transformation, suppose we know* $\text{span}\{v_1, \cdots, v_k\}$*, then*

$$\begin{aligned}
R(T) =& \{T(x), x \in V\} \\
=& \{T(x) : x = a_1 v_1 + \cdots a_k v_k, a_i \in \mathbb{F}\} \\
=& \{T(a_1 v_1 + \cdots + a_k v_k) : a_1, \cdots, a_k \in \mathbb{F}\} \\
=& \{a_1 T(v_1) + \cdots a_k T(v_k) : a_1, \cdots, a_k \in \mathbb{F}\} \\
=& \text{span}\{T(x_1), \cdots, T(x_k)\}
\end{aligned}$$

Example 1: $D_n : P_n(\mathbb{R}) \to P_n(\mathbb{R})$

A spanning set for $P_n(\mathbb{R})$ is

$$\{1, x, x^2, x^3, \cdots x^n\}$$

so

$$\begin{aligned}\mathbb{R}(D_n) =& \text{span}\{D_n(1), D_n(x), D_n(x^2), \cdots, D_n(x^n)\}\\ =& \text{span}\{0, 1, 2x, \cdots, nx^{n-1}\}\\ =& \text{span}\{1, x, x^2, \cdots, x^{n-1}\} = P_{n-1}(\mathbb{R})\end{aligned}$$

Example 2: Fix $A \in M_{m\times n}(\mathbb{F})$. $L_A : \mathbb{R}^n \to \mathbb{F}^m$ by $L_A(x) = Ax$.

The "standard basis" for $\mathbb{F}^n$ is

$$\{(1, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)\}$$

$\mathbb{F}^n = \text{span}\{e_1, e_2, \cdots, e_n\}$

Say $A = \begin{pmatrix} a_{11} \cdots a_{1n} \\ \vdots \\ a_{m1} \cdots a_{mn} \end{pmatrix}$

$$L_A(e_1) =$$

Two Basic Questions about Linear Transformation

Question 1: Is it injective?

Question 2: Is it surjective?

Theorem 2.2.2. *Suppose $T : V \to W$ is linear, then T is injective $\iff N(T) = \{0\}$.*

Proof. ($\Rightarrow$) Assume T is injective. i.e. $\forall x, y \in V, T(x) = T(y) \Rightarrow x = y$.

Obviously $0 \subseteq N(T)$. (Since $N(T)$ is a subspace)

For $N(T) \subseteq \{0\}$, let $x \in N(T)$ so $T(x) = 0 = T(0) \Rightarrow x = 0$.

($\Leftarrow$) Assume $N(T) = \{0\}$, prove injectively, assume $x, y \in V$ and $T(x) = T(y)$.

$\Rightarrow T(x) - T(y) = 0 \Rightarrow T(x - y) = 0 \Rightarrow x - y \in N(T) = \{0\} \Rightarrow x = y$. □

2.3 Jan 31

Definition 2.3.1. *A linear transformation $T : V \to W$ is an isomorphism if it is a bijection.*

We also write $T : V \cong W$.

*We say V, W are **isomorphic**. (and write $V \cong W$) if $\exists T : V \cong W$.*

Example 1: $P_n(\mathbb{R}) \cong \mathbb{R}^{n+1}$

An example of an isomorphism $T : P_n(\mathbb{R}) \cong \mathbb{R}^{n+1}$ is

$$T(a_0 + a_1 + \cdots + a_n x^n) = (a_0, a_1, \cdots, a_n)$$

Easy facts:

1. For every V.S. V, $V \cong V$.
2. If $V \cong W$ then $W \cong V$.

Definition 2.3.2. *Given a linear tranformation $T : V \to W$ the*

nullity *of T is* $\dim(N(T))$

rank *of T is* $\dim(N(T))$

Theorem 2.3.1. *Suppose $T : V \to W$ is linear and $dimV < \infty$, then $rank(T) + null(T) = dim(V)$.*

Proof. First step find basis for $N(T)$ and $R(T)$

Let S be a basis for $N(T)$ let $n = dimV$, as $N(T) \subseteq V$, S is linearly independent in V

$\Rightarrow |S| < n$. Write $S = \{v_n, \cdots, n_k\}$, $k < n$.

□

Special Case: when $T : V \cong W$, $dimV = n$

T is injective $\Rightarrow N(T) = \{0\}$
$\Rightarrow null(T) = 0$
$\Rightarrow S = \varnothing$

$B = \{x_n, \cdots\}$

2.4 Feb 3

Proposition 2.4.1. *Suppose $\{v_1, \cdots, v_n\}$ is a basis for V.S. / $\mathbb{F}$.*

Then $\forall x \in V$, x can be uniquely written

$$x = a_1 v_1 + \cdots + a_n v_n \qquad a_i \in \mathbb{F}$$

Proof. $\{v_1, \cdots, v_n\}$ span V so every $x \in V$ can be written in this way.

For uniqueness, assume $x = a_1 v_1 + \cdots + a_n v_n = b_1 v_1 + \cdots b_n v_n$

Get $0 = (a_1 - b_1)v_1 + \cdots + (a_n b_n)v_n$. As $\{v_1, \cdots, v_n\}$ is linearly independent, get $a_1 = b_1, \cdots, a_n = b_n$. $\square$

Example:

Let $V = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$. A plane in $\mathbb{R}^3$. V is a subspace of $\mathbb{R}$.

Let $v_1 = (-1, 1, 0)$, $v_2 = (0, -1, 1)$.

$\{v_1, v_2\}$ is a basis for V

$x = (-3, 1, 2) \in V \Rightarrow x = 3v_1 + 2v_2$

The **coordinates** of x relative to $\{v_1, v_2\}$ are $(3, 2)$.

Definition 2.4.1. *Let V be a V.S. $\dim V = n$. An **Ordered Basis** for V is an n-tuple $(v_1, \cdots, v_n)$ where $\{v_1, \cdots, v_n\}$ is a basis.*

Notation 2.4.1. *α, β, γ for ordered bases, A, B, C for basis.*

Definition 2.4.2. *Suppose V is a V.S., $\dim V = n$, β is an ordered basis for V.*

The coordinate vector of x relative to β is the unique n-tuple $(a_1, \cdots, a_n) \in \mathbb{F}^n$ s.t.

$$x = a_1 v_1 + \cdots + a_n v_n$$

Notation 2.4.2. *The coordinate of x relative to β is denoted as: $[x]_\beta := (a_1, \cdots, a_n)$*

Fix $V, \mathbb{F}, \beta = (v_1, \cdots, v_n)$ as in definition.

Define

$$[\,]_\beta : V \to \mathbb{F}^n, \qquad\qquad x \mapsto [x]_p$$

Theorem 2.4.1. *$[\,]_\beta : V \to \mathbb{F}^n$ is an isomorphism.*

Proof. Let $x, y \in V$, (must show $[x + y]_\beta = [x]_\beta + [y]_\beta$)

Write

$[x]_\beta = (a_1, \cdots, a_n) \Rightarrow x = a_1 v_1 + \cdots + a_n v_n$

$[y]_\beta = (b_1, \cdots, b_n) \Rightarrow y = b_1 v_1 + \cdots + b_n v_n$

$[x + y]_\beta = (c_1, \cdots, c_n) \Rightarrow x + y = c_1 v_1 + \cdots + c_n v_n$

$$\Rightarrow (a_1+b_1)v_1+\cdots+(a_n+b_n)v_n = c_1v_1+\cdots c_nv_n$$

By prop,

$$\begin{cases} a_1+b_1=c_1 \\ a_2+b_2=c_2 \\ \cdots \\ a_n+b_n=c_n \end{cases} \Rightarrow (a_1,\cdots,a_n)+(b_1,\cdots,b_n)=(c_1,\cdots,c_n)=[x]_\beta+[y]_\beta=[x+y]_\beta$$

Similarly, $[\quad]_\beta$ presents scalar multiplication, so it is linear.

Bijection:

Injective: align* N([$\ ]_\beta = \{x\in V : [x]_\beta = (0,\cdots,0)\}$)

To show $[\quad]_\beta$ is surjective, first find a spanning set for $V = \{v_1,\cdots,v_n\}$

$$\begin{aligned} R([\quad]_\beta) =& \text{span}\{[v_1]_\beta,\cdots,[v_n]_\beta\} \\ =& \{x\in V : x=0\} \\ =& \{0\} \end{aligned}$$

What is $[v_1]_\beta = (1,0,\cdots,0) = e_1$.

$\square$