Math 147 Notes

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Contents

1 Basics

1.1 The Language of Mathematics

Sets:

$x \in X$ (x is an element of X)

$y \subseteq X$ (y is a subset of X)

X=Y

Mathematical Statements are either true or false.

Combine statements to make new ones.

Statement p (true):

- not: not p (false)
- and: $a|b$ and $b|a$
- implication: If p then q. $\rightarrow$ means p implies q, if p is true then q must also be true.
- converse: If q then p. $\rightarrow$
- contrapositive: $p \Rightarrow q \rightarrow \neg q \Rightarrow \neg p$. (the two are equivalent)
- $\forall$: e.g. $\forall n \in \mathbb{N}$, $n^2 + n$ is even. (F)
- $\exists$: e.g. $\exists n \in \mathbb{N}$, s.t. $n \cdot 0 \neq 0$. (F)

Note that the order of the statements matters: $\forall n \in \mathbb{N}$, $\exists m \in \mathbb{N}$, s.t. $13|n^2 + m^2$. (T)

$\exists m \in \mathbb{N}$, $\forall n \in \mathbb{N}$, s.t. $13|n^2 + m^2$. (F)

1.2 Proof

1.2.1 Direct Proof

work through to the conclusion

Example: If $x \in \mathbb{R}$, with decimal expansion $x = x_0.\, x_1\, x_2\, x_3$, $(x_0 \in \mathbb{Z}, x_n \in \{0, 1, 2, 3 \cdots, 9\})$. Say this expansion is eventually periodic, then $\exists \mathbb{N}$, $a \in \mathbb{N}$, $x_i + d = x$.

Theorem 1.2.1. *If the decimal expansion of x is eventually periodic, then x is rational.*

Proof. Since x has a periodic expansion, we have $N, d \in \mathbb{N}$, such that $x_{i+d} = x_i$ for all $i \leq N$.

$$\begin{aligned} 10^{N+d} x &= c.\ x_{N+1+d}\ x_{N+2+d}\ x_{N+3+d} \\ 10^N x &= b.\ x_{N+1}\ x_{N+2}\ x_{N+3} \\ (10^{N+d} - 10^N) x &= c - b \\ x &= \frac{c-b}{10^{N+d} - 10^N} \in \mathbb{Q} \end{aligned}$$

□

Theorem 1.2.2 (**Pigenhole Principle**). *If $kn+1$ or more objects are divided into n groups then at least 1 group will contain $\geq k+1$ objects.*

Theorem 1.2.3. *If x is rational, then it has periodic decial expansion.*

Proof. Write x as $\frac{p}{q}$, $q \in \mathbb{N}$, $p \in \mathbb{Z}$. (Without loss of generality, $x \geq 0$).

Divide 10^k by q, the remainder $r_k \in \{0, 1, 2, \cdots, q-1\}$.

Then r_0, r_1, r_2, r_3, $\cdots$, $r_q \in \{0, 1, \cdots, q-1\}$, By the Pigeonhole Principle, there are $0 \leq i < j \leq q$, so that $r_j = r_q$.

Hence q divides $10^j - rq$ and $10^q - rq$. Therefore q divides $(10^i - ri) - (10^j - rj) = 10^j - 10^i = aq$. Therefore:

$$x = \frac{p}{q} = \frac{ap}{aq} = 10^{-i}$$
$$\frac{ap}{10^j - 10^i} = 10^{-i}(b + \frac{r}{10^d - 1})$$
$$d = j - i,\ 0 \leq r < 10^d - 1$$

The expansion of x is eventually periodic because the expansion of $\frac{r}{10^d - 1}$ is eventually periodic. let y be a number that has periodic decimal expansion,

such that $y = 0.r_{d-1}\ r_{d-2}\ r_1\ r_0\ r_{d-1}\ r_{d-2}\cdots$

$$y = 0.r_{d-1}\ r_{d-2}\ r_1\ r_0\ r_{d-1}\ r_{d-2}\cdots$$
$$10y = r_{d-1}\ r_{d-2}\ \cdots r_0.r_{d-1}r_{d-2}\ldots$$
$$y = \frac{r}{10^d - 1}$$

Therefore, $\frac{4}{10^d-1}$ has periodic decimal expansion and so does x. □

1.2.2 Proof by Contradiction

To prove (A) we can assume $\neg(A)$ and find that it is impossible.

Theorem 1.2.4. *If $d \geq 2$, $d \in \mathbb{N}$ is not a perfect square then $\sqrt{d}$ is irrational.*

Proof. Find $m \in \mathbb{N}_0$ such that $m^2 < d < (m+1)^2$.

Assume that $\sqrt{d}$ is rational ($\sqrt{d} = \frac{p}{q}$, $q\sqrt{d} = p$).

let $A = \{n \in \mathbb{N} : n\sqrt{d} \in \mathbb{N}\}$. A is not empty, by Well Ordering Principle, this set has a smallest number a.

$$m < \sqrt{3} < m+1 \qquad a\sqrt{d} \in \mathbb{N}$$
$$a < \sqrt{d} - m < 1$$

$$\text{Let } b = (\sqrt{d} - m)a \Rightarrow 0 < b < a$$
$$= a\sqrt{d} - am \Rightarrow b \in \mathbb{N}$$
$$b\sqrt{d} = ad = am\sqrt{d} \qquad \in \mathbb{N}$$

so $b \in A$ but $b < a$, contradiction, so $\sqrt{d}$ is irrational. □

1.2.3 Proof by Induction

Given $P(n)$ for $n \geq N_0$ (usually 0 or 1), if $P(n)$ is true and whenever $P(n)$ is true for $n_0 \leq n < m$, then $P(m)$ is true.

Theorem 1.2.5. *Every $\mathbb{Z} \geq 2$ can be factored as a product of 1 and primes.*

Proof. Let $P(n)$ be the statement that n is a product of prime numbers.

We knwo that $P(2)$ is true because 2 is a prime.

Suppose $P(k)$ is true for all $s \leq k \leq n-1$. Consider $P(n)$,

Case 1: n is a prime number, and $P(n)$ is true.

Case 2: p is not a prime then $n = a \times b$, $1 < a$, $b < n$. Since we assumed that $p(a)$ and $P(b)$ are true, so $a = p_1 \cdots p_\alpha$ and $b = q_1 \cdots q_\beta$, both product of primes. so $n = p_1 \cdots p_\alpha \cdots q_1 \cdots q_\beta$, which is a product of primes. Hence by Induction, $P(n)$ is true for all n.

$\square$

Definition 1.2.1. *The FIBONACCI numbers $F(0) = F(1) = 1$, and $F(n) = F(n-1) = F(n-2)$ for $n \geq 2$. (1, 1, 2, 3, 5, 8 ...).*

$\varphi = \frac{1+\sqrt{5}}{2}$ $\qquad \frac{1}{\varphi} = \frac{\sqrt{5}-1}{2}$

$\varphi^2 = \frac{3+\sqrt{5}}{2} = 1 + \varphi$

$\frac{1}{\varphi^2} = 1 - \frac{1}{\varphi}$

Theorem 1.2.6. *Let $F(n)$ be a Fibonacci Sequence, then $F(n) = \frac{\varphi^{n+1} - (\frac{-1}{\varphi})^{n+1}}{\sqrt{5}}$*

Proof. Let $P(n)$ be that $F(n) = \frac{\varphi^{n+1} - (\frac{-1}{\varphi})^{n+1}}{\sqrt{5}}$.

$P(0) = \frac{\varphi - (\frac{-1}{\sqrt{\varphi}})^1}{\sqrt{5}} = 1$, $P(n)$ is true.

$P(1) = \frac{\varphi^2 - (\frac{-1}{\varphi})^2}{\sqrt{5}} = \frac{1+\varphi-\frac{1}{\varphi^2}\sqrt{5}}{\sqrt{5}} = 1$, so $P(1)$ is true.

Assume $P(k)$ is true for $0 \leq k < n$, $n \geq 2$.

$$\begin{aligned}
F(n) &= F(n-1) + F(n-2) \\
&= \frac{\varphi^{n-1+1} - (\frac{-1}{\varphi})^{n-1+1}}{\sqrt{5}} + \frac{\varphi^{n-1} - (\frac{-1}{\varphi})^{n-1}}{\sqrt{5}} \\
&= \frac{1}{\sqrt{5}}(\varphi^{n-1}(\varphi+1) - (\frac{-1}{\varphi})^{n-1})(\frac{-1}{\varphi}+1)) \\
&= \frac{1}{\sqrt{5}} \cdot (\varphi^{n+1} \cdot \varphi^2 - (\frac{-1}{\varphi})^{n-1} \cdot (\frac{-1}{\varphi})^2) \\
&= \frac{\varphi^{n+1} - (\frac{-1}{\varphi})^{n+1}}{\sqrt{5}}
\end{aligned}$$

hence, by induction, $P(n)$ is true for all $n \geq 0$. $\square$

1.3 Real Numbers

A real number x has an infinite decimal expansion.

$x = x_0.x_1x_2x_3\cdots,\ x_0 \in \mathbb{Z},\ xi \in \{0,1,2,\cdots 9\},\ \ i \geq 1.$

Think "$\sum_{n=1}^{\infty} \frac{x^n}{10^n} = 1$" as a formal name for x.

Example:

$-1.25 = -1.2500000...$

Some real numbers have two names. Ex : $1.000\cdots = 0.999\cdots$

1.4 Order

The real number $\mathbb{R}$ is an ordered field.

If $x, y \in \mathbb{R}$, then $x < y \wedge x = y \wedge x > y$.

If $x \neq y$ then the decimal expansion diverges at some point.

If $x < y$, then there is $r \in \mathbb{Q}$ with a finite expansion such that $x < r < y$.

This says that the rationals are dense in $\mathbb{R}$.

Theorem 1.4.1. *Archimedean Property If $x \in \mathbb{R}$ and $0 \leq x \leq 10^{-n}$ for all $n \geq 1$. then x=0.*

Proof. Let $x = x_0.\ x_1x_2x_3\cdots$. If $0 < x$, then $x \geq 0$, $x = 0.000\cdots x_n$, then $x \geq 0 > 10^{-n-1}$, so the only real number satisfying the hypothesis is 0. □

2 Sequence

2.1 Limits

$\lim_{n\to\infty} x_n = L$

Example:

- $a_n = \frac{1}{n}$, $n \geq 1 \Rightarrow \lim_{n\to\infty} a_n = 0$
- $a_n = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$
- 0 , 1, 0, $\frac{1}{2}$, 0, $\frac{1}{3}$ $\cdots$ $\Rightarrow \lim_{n\to\infty} a_n = 0$
- $1, -1, 1, -1, 1, \cdots \Rightarrow$ limit D.N.E.

Definition 2.1.1. $\lim_{n\to\infty} x_n = L$ *mean for every* $\epsilon > 0$*, exists* N *so that* $|a_n - L| < \epsilon$ *for all* $n \geq N$*.*

$\forall\ \epsilon > 0$ *,* $\exists N$*,* $\forall n \geq N$*,* $|a_n - L| < \epsilon$*.*

Example 1: -1**,** 0**,** $\frac{1}{2}$**,** 0**,** $-\frac{1}{3}$**,** $\cdots$**,** $a_n = \begin{cases} \frac{(-1)^k}{k}, & n = 2k-1 \\ 0, & n = 2k \end{cases}$

If $\epsilon > 0$, there is $10^{-k} < \epsilon$.

Let $N = 2 \cdot 10^k$, if $n \geq N$, and n is even, then $a_n = 0$, $|a_n - 0| = 0 < \epsilon$,

if n is odd, let $N = \frac{2}{\epsilon}$, then $a_n = \pm\frac{1}{k}$, $\left|\pm\frac{1}{k} - 0\right| = \frac{1}{k} \leq \frac{1}{N} < \frac{\epsilon}{2}$.

Example 2: -1**,** 1**,** -1**,** 1**,** $\cdots$**,** $a_n = (-1)^n$

$\lim_{a_n} \neq L$ means for $\exists \epsilon_0 > 0$ and $|a_{n_k} - L| \geq \epsilon_0$.

Take $\epsilon = 1$,

if $L > 0$, since $a_{2n+1} = -1$, $|a_{2n+1} - L| = |-1 - L| > 1 = \epsilon$.

if $L \leq 0$, since $a_{2n} = 1$, $|a_{2n} - L| = |1 - L| \geq 1 = \epsilon$.

Hence no limit exists.

Example 3: Let $a_n = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$**, with** n **2s.**

$a_1 = \frac{1}{2}$, $a_2 = \frac{1}{2+\frac{1}{2}} = \frac{2}{5}$ $\cdots$, $a_{n+1} = \frac{1}{2+a_n}$, $n \geq 1$.

Suppose the limit exists, $\lim_{n\to\infty} a_n = L$, then $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \dfrac{1}{2+L}$.

Solve for L, $L^2 + 2L - 1 = 0 \Rightarrow L = -1 \pm \sqrt{2}$.

Since $a_n \geq 0$, hence $L \geq 0$ and $L = -1 + \sqrt{2}$.

Theorem 2.1.1. ***Squeeze Theorem***
If $a_n \le x_n \le b_n$, and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$, then $\lim_{n\to\infty} x_n = L$.

Proof. let $\epsilon > 0$, since $\lim_{n\to\infty} a_n = L$,

$\exists\ N_1$, $n \ge N_1 \Rightarrow |a_n - L| < \epsilon$, so $L - \epsilon < a_n < L + \epsilon$.

$\exists\ N_2$, $n \ge N_2 \Rightarrow |b_n - L| < \epsilon$, so $L - \epsilon < b_n < L + \epsilon$.

Let $N = \max\{N_1, N_2\}$, $n \ge N \Rightarrow L - \epsilon < a_n \le x_n \le b_n < L + \epsilon \Rightarrow |x_n - L| < \epsilon$ so $\lim_{n\to\infty} x_n = L$. □

Theorem 2.1.2. *A sequence can and can only have one limit.*

Proof. Assume a contradiction that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$, $M > L$.

Let $\epsilon = \frac{|L-M|}{2}$, since $\exists N$, $\forall n \ge N$, $|x_n - L| < \epsilon$. Hence $L - \epsilon < x_n < L + \epsilon = L + \frac{M-L}{2} = \frac{L+M}{2} < M$ Contradiction. □

Proposition 2.1.1. *Suppose $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences and that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then*

1. *of $k \in \mathbb{R}$, $\lim_{n\to\infty} k \cdot a_n = k \cdot L$*

   Proof. We know that there is an N such that for all $n \ge N$,

   $|a_n - L| < \frac{\epsilon}{|k^2+1|}$,

   therefore, $|ka_n - kL| = |k|\,|a_n - L| < |k|\left|\frac{\epsilon}{k^2+1}\right| < \epsilon$,

   hence, by defnition of limit, $\lim_{n\to\infty} ka_n = kL$. □

2. $\lim_{n\to\infty} a_n \pm b_n = L \pm M$

   Proof. Let $\epsilon > 0$, $|a_n + b_n - L - M| \le |a_n - L| + |b_n - M|$.

   Since $\lim_{n\to\infty} a_n = L$, $\exists N_1$, $n \ge N_1 \Rightarrow |a_n - L| < \frac{\epsilon}{2}$

   and $\lim_{n\to\infty} b_n = M$, so $\exists N_2$, $n \ge N_2 \Rightarrow |b_n - L| < \frac{\epsilon}{2}$

   Take $N = \max\{N_1, N_2\}$, so if $n \ge N$, then both are true.

   $\therefore |a_n + b_n - L - M| \le |a_n - L| + |b_n - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

   $\therefore \lim_{n\to\infty} a_n + b_n = L + M$

   □

3. $\lim_{n\to\infty} a_n b_n = LM$

Proof. Let $\epsilon > 0$, since $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 0$,

there is N_1 such that for all $n \geq N_1$, $|a_n - L| < \frac{\epsilon}{2|b_n^2+1|}$

and there is N_2 such that for all $n \geq N_2$, $|b_n - L| < \frac{\epsilon}{2|L^2+1|}$

therefore, let $N = \max N_1, N_2$, for all $n \geq N$, we have

$$\begin{aligned}
|a_n b_n - LM| &= |a_n b_n - b_n L + b_n L - LM| \\
&= |b_n(a_n - L) + L(b_n - M)| \\
&\leq |b_n|\,|a_n - L| + |L|\,|b_n - M| \\
&< |b_n| \cdot \frac{\epsilon}{2\,|b_n^2+1|} + |b_n| \cdot \frac{\epsilon}{2\,|L^2+1|} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad\qquad\qquad\qquad = \epsilon
\end{aligned}$$

Hence, by definition of limit, $\lim_{n\to\infty} a_n b_n = LM$.

$\square$

4. *if* $M \neq 0$, *then* $\exists\ N$, *s.t.* $\forall\ n \geq N$, $b_n \neq 0$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$

Proof. Take $\epsilon = \frac{|M|}{2} > 0$, find N_1, $n \geq N_1$

$$\begin{aligned}
|b_n - M| < \epsilon = \frac{|M|}{2}& \\
|b_n| &= |M - (M - b_n)| \geq |M| - |M - b_n| \\
&> |M| - \frac{|M|}{2} = \frac{|M|}{2} \neq 0
\end{aligned}$$

Hence $b_n \neq 0$ if $n \geq N$, and $\frac{a_n}{b_n}$ is defined.

Estimate:

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{L}{M}\right| &= \left|\frac{a_n \cdot M - b_n \cdot L}{b_n \cdot M}\right| \\
&= \left|\frac{a_n \cdot M + LM - LM - b_n \cdot L}{b_n \cdot M}\right| \\
&\leq \frac{|a_n - L|\,|M| + |M - b_n|\,|L|}{\frac{|M|}{2} + |M|} \\
&= \frac{|M|}{2} \cdot |a_n - L| + \frac{|2L|}{M^2} \cdot |M - b_n|
\end{aligned}$$

Take $\epsilon_1 = \frac{\epsilon \cdot |M|}{4}$, so $\frac{2}{|M|} \cdot \epsilon_1 = \frac{\epsilon}{2}$. Since $\lim_{n\to\infty} a_n = L$, $\exists N_2$, s.t. $\forall n \geq N_2 \Rightarrow |a_n - L| < \epsilon_1 = \frac{\epsilon \cdot |M|}{4}$.

Take $\epsilon_2 = \frac{\epsilon \cdot M^2}{4 \cdot |L|}$, so $\frac{2|L|}{M^2} \cdot \epsilon_2 = \frac{\epsilon}{2}$. Since $\lim_{n\to\infty} b_n = M$, $\exists N_3$, s.t. $\forall n \geq N_2 \Rightarrow |b_n - M| < \epsilon_2 = \frac{\epsilon \cdot M^2}{4|L|}$.

Let $N = \max\{N_1, N_2, N_3\}$, for $n \geq N$, means that

$$\begin{aligned}\left|\frac{a_n}{b_n} - \frac{L}{M}\right| &< \frac{2}{|M|}\,|a_n - L| + \frac{2\,|L|}{M^2}\cdot|b_n - M| \\ &< \frac{2}{|M|}\cdot\frac{|M|}{4}\cdot\epsilon + \frac{2\,|L|}{M^2}\cdot\frac{M^2}{4\,|L|}\cdot\epsilon \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

□

2.2 Upper and Lower Bounds

Definition 2.2.1. *If $\varnothing \neq S \subseteq \mathbb{R}$.*

If S is bounded above. ($\exists M$, $\forall s \in S$, $S \leq M$), then the least upper bound, or supremum is the smallest possible upper bound, written as $\sup S$*.*

Similarly, if S is bounded below, the greatest lower bound or infinium, written as $\inf S$*.*

Example:

- $S = \{2, e, \pi, 4, 17, \frac{37}{\sqrt{47}}\}$, $\inf S = 2$, $\sup S = 17$. * when finite set, $\inf S = \min$, $\sup S = \max$.
- $S = \{x \in \mathbb{Q} : x^2 < 2\}$, $\sup S = \sqrt{2}$, $\inf S = -\sqrt{2}$.
- $S = \{\sin\ n : n \in \mathbb{N}\}$, $\sup S = 1$, $\inf S = -1$

Theorem 2.2.1. ***Least Upper Bound Principle***
If $S \subseteq \mathbb{R}$ is bounded above, then S has a least upper bound $\sup S$*; if $S \subseteq \mathbb{R}$ is bounded below, then S has a greatest lower bound* $\inf S$*. It suffices to prove one statement.*

Proof. Pick a_0 which is a lower bound for S, $a_0 + 1$ is not a lower bound. Pick $S_0 \in S$, $s_0 < a_0 + 1$ (witness).

Pick $a_1 = \{0, 1, 2, \cdot, 9\}$, such that $a_0.\, a_1$ is a lower bound but $a_0.(a_1 + 1)$ is not. pick $s_1 < a_0.\ a_1 + 0.1$.

Proceed recursively, at stage n we have $a_0.a_1\ a_2\ a_3\ \cdots a_n$ as a lower bound and $a_0.a_1\ a_2\ a_3\ \cdots a_n + \frac{1}{10^n}$ is not.

Again split into 10 pieces,then choose $a_{n+1} \in \{0, 1, 2, \cdots, 9\}$, so $a_0.a_1a_2\cdots a_na_{n+1}$ is a lower bound and $a_0.a_1a_2\cdots a_na_{n+1} + \frac{1}{10^{n+1}}$ is not and pick witness $s^{n+1} < a_0.a_1a_2\cdots a_na_{n+1} + \frac{1}{10^{n+1}}$.

Let $s \in S$, then $s \geq a_0.a_1a_2a_3\cdots a_n$ for every n. Therefore $S \geq \lim\limits_{n\to\infty} a_0.a_1a_3a_4\cdots a_n = L$, so L is a lower bound of S.

Suppose $L_1 > L$, $L_1 - L > 0$. Since $\mathbb{R}$ is Archimedean, so there $\exists N$, $L_1 \geq L + \frac{1}{10^N} \geq a_0.a_1a_2\cdots a_n + \frac{1}{10^N}$.

But there is $s_N \in S$ $S_N < a_0.a_1\cdots a_N + \frac{1}{10^N} \leq L_1$, therefore, $L1$ is not a lower bound, therefore $L = \inf S$. □

Lemma 2.2.1. $\inf S = -(\sup(-S))$ *and* $\sup S = -(\inf(-S))$

Definition 2.2.2. *A sequence* $(a_n)_{n=1}^{\infty}$ *is monotone increasing if* $a_n \leq a_{n_1}$*, and strictly increasing if* $a_n < a_{n+1}$*.*

Theorem 2.2.2. ***Monotone Convergence Theorem***
If $(a_n)_{n=1}^{\infty}$ *is monotone increasing and bounded above, then* $\lim\limits_{n\to\infty} a_n$ *exists.*

Proof. Let $(a_n)_{n=1}^{\infty}$ be a monotone increasing sequence. Then by the LUBP, $\exists L = \sup\{a_n, n \geq 1\}$ $a_n \leq L$ for all $n \geq 1$.

Let $\epsilon > 0$, then $L - \epsilon$ is not an upper bound. $\therefore \exists N$, $L - \epsilon < a_N$.

Since for $n \geq N$, $L - \epsilon < a_N \leq a_n \leq L$. Therefore $0 < L - a_n < L - (L - \epsilon) = \epsilon$, so $|L - a_n| < \epsilon$. □

Example 1: Let $a_1 = 1$ and $a_{n+1} = \sqrt{2 + \sqrt{a_n}}$ for $n \geq 1$. Find limit for a_n.

Claim: a_n is monotone increasing. $a_n < a_{n+1}$

$a_2 = \sqrt{2 + \sqrt{1}} > a_1$, proceed by induction, if $a_{n-1} < a_n$, then $a_{n+1} = \sqrt{2 + \sqrt{a_n}} > \sqrt{2 + \sqrt{a_{n-1}}} = a_n$
Hence by induction, $a_n > a_{n-1}$ for all n.

Claim: $a_n \leq 2$ for all n.

First, $a_1 = 1 < 2$. Assume $a_n \leq 2$, then $a_{n+1} = \sqrt{2 + \sqrt{a_n}} < \sqrt{2 + 2} = 2$. Hence by indcution, $a_n < 2$ for all n.

Therefore, a_n is monotone increasing and bounded above, and by Monotone Convergence Theorem, $\lim\limits_{n\to\infty} a_n = L$ exists.

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + \sqrt{a_n}} = \sqrt{2 + \sqrt{L}}$$

$L^4 - 4L^2 - L + 4 = 0$, solve for L, $L = 1$ or $L = 1$ $L = 1.8312$.

Since $L \geq a_2 = \sqrt{3}$, $L \neq 1$, so $L = 1.8312$.

Proposition 2.2.1. *If* $\lim\limits_{n\to\infty} a_n = L$ *and* $(a_{n_i})_{i=1}^{\infty}$ *is a subsequence, then* $\lim\limits_{n\to\infty} (a_{n_i})_{i=1}^{\infty} = L$*.*

Proof. $\lim\limits_{n\to\infty} a_n = L$ means that there exists N s.t. $n \geq N \Rightarrow |a_n - L| < \epsilon$

If $i \geq N$, then $n_i \geq i \geq N \Rightarrow |a_{n_i} - L| < \epsilon$

□

Proposition 2.2.2. *If* $\lim\limits_{n\to\infty} a_n = L$*, then* a_n *is bounded.*

Proof. Let $\epsilon = 1$, Find N s.t. $n \geq N \Rightarrow |a_n - L| < 1$

$\therefore L - 1 < a_n < L + 1$,

$\therefore |a_n| \leq |L| + 1$

Let $R = \max\{|L| + 1, |a_1|, |a_2| \cdots |a_N - 1|\}$, $\therefore |a_n| \leq R$ for all $n \geq 1$. □

Theorem 2.2.3. ***Bolzano-Weierstrass Theorem***
If $(a_n)_{n=1}^{\infty}$ is a bounded sequence of real numbers, then it has a convergent subsequence.

Proof. Let a_n be a sequence bounded by B, thus the interval $I = [-B, B]$ contains the whole infinite sequence.

Then of the interval $[-B, 0]$ and $[0, B]$, one of these halfs must contain infinitely many elements of a_n, let it be I_2.

Similarly, divide I_2 into two closed intervals of length $\frac{B}{2}$, and choose the interval I_3 of the two which contains infinitely many elements of a_n.

Again divide I_3 into two closed intervals of length $B/4$ and choose the interval I_4 of the two which contains infinitely many elements of a_n.

Therefore we have intervals I_1 to I_k in which the length of I decreases as k increases. $I_{k-1} \subset I_k$

Let the left endpoint of I_k be l_k and the right endpoint of I_k be r_k. Observe that $l_k \leq l_{k+1} < r_{k+1} < r_k \leq r_{k+1}$.

Hence l_k is a increasing sequence that is bounded by r_1 and r_k is a decreasing sequence which is bounded by l_1.

Hence by monotone convergence theorem, there is $\lim\limits_{k\to\infty} l_k = L$ and $\lim\limits_{k\to\infty} r_k = M$.

Since the length of $I_k = \frac{B}{2^{k-2}}$, hence $M - L = \lim\limits_{k\to\infty} r_k - l_k = \lim\limits_{k\to\infty} \dfrac{B}{2^{k-2}} = 0$.

Therefore, $M = L$.

Choose a increasing sequence a_{n_k} that a_{n_k} belong to I_k, This is possible because each I_k contains infinitely many elements of a_n and n_{k-1} only fnitely many have index at most n_{k-1}.

Then $l_k \leq a_{n_k} \leq r_k$, by Squeeze Theorem, $\lim\limits_{k\to\infty} a_{n_k} = L$.

□

Definition 2.2.3. *A* ***Cauchy Sequence*** *is a sequence $(a_n)_{n=1}^{\infty}$ that if $\epsilon > 0$, $\exists\ N$ s.t. if $m, n \geq N$, then $|a_n - a_m| < \epsilon$.*

Proposition 2.2.3. *Every convergent sequence of real numbers are Cauchy.*

i.e. Let $(a_n)_{n=0}^{\infty}$ be a sequence converging to L. For every $\epsilon > 0$, there is an N such that for all $m, n > 0$ $|a_n - a_m| < \epsilon$.

Proof. Let $\epsilon > 0$, and use the value $\frac{\epsilon}{2}$.

Then by the definition of limit, there is an N_1, such that for all $n > N_1$, $|a_n - L| < \frac{\epsilon}{2}$,

and so $|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

□

Proposition 2.2.4. *Every Cauchy sequence is bounded.*

Proof. Let $(a_n)_{n=0}^{\infty}$ be a Cauchy sequence,

then there exists N such that for all $n \geq N$, $|a_n - a_N| < 1$.

It follows that the sequence is bounded by $\max\{|a_1|, |a_2|, \cdots, |a_N| + 1\}$ □

Definition 2.2.4. *A subset $S \subseteq \mathbb{R}$ is* ***complete*** *if a Cauchy Sequenece $(a_n)_{n=1}^{\infty}$ with $a_n \in S$ has a limit $\lim_{n\to\infty} a_n = L \in S$.*

Theorem 2.2.4. *Completeness Theorem*
$\mathbb{R}$ is complete. i.e. every Cauchy Sequence of $\mathbb{R}$ converges to $\mathbb{R}$.

Proof. Let the sequence be $(a_n)_{n=0}^{\infty}$.

By Proposition 2.2.4, Cauchy Sequences are bounded.

Therefore, by Bolzano-Weierstrass Theorem, there is a subsequence a_{n_k} which has $\lim_{k\to\infty} a_{n_k} = L$.

Hence, there exists an K such that for all $k > K$, $|a_{n_k} - L| < \frac{\epsilon}{2}$.

Since this is a Cauchy sequence N such that for all $n, m > N$, $|a_n - a_m| < \frac{\epsilon}{2}$.

Pick $k > K$ such that $n_k > N$, then for all $n > N$,

$|a_n - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Hence $\lim_{n\to\infty} a_n = L$. □

3 Function

Definition 3.0.1. *For A, B sets, $f : A \to B$ is a function if for each $A \in B$, a value for $f(a) \in B$ is assigned by some rule.*

Definition 3.0.2. *Given two nonempty sets A and B, a function f from A to B is a subset of $A \times B$, denoted $G(f)$, so that*

1. *for each $a \in A$, there is some $b \in B$ so that $(a, b) \in G(f)$,*
2. *for each $a \in A$, there is only one $b \in B$ so that $(a, b) \in G(f)$.*

That is for each $a \in A$, there is exactly one elements $b \in B$ with $(a, b) \in G(f)$. We then write $f(a) = b$. A concise way to specify the function f and the sets A and B all at once is to write $f : A \to B$. We call $G(f)$ the graph of the function f.

The property of a subset of $A \times B$ that makes it the graph of a function is that $\{b \in B : (a, b) \in G(f)\}$ has precisely one element for each $a \in A$.

We call A the domain of the function $f : A \to B$ and B is the condomain. The range of the function is $Ran(f) := \{b \in B : b = f(a) \text{ for some } a \in A\}$.

Example:

- $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ by $f(x) = \frac{1}{x}$
- $g : (0, 1) \to (0, \infty)$ by $g(x) = \frac{1}{x}$
- $f(x) = x^2$

Definition 3.0.3.

$f : A \to B$ is one to one (injective) if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$

$f : A \to B$ is onto (surjective) if for every $b \in B$, there is an $a \in A$ with $f(a) = b$.

$f : A \to B$ is one-to-one and onto then it is bijective.

3.1 Limits of Functions

Definition 3.1.1. $\lim_{x \to a} f(x) = L$, $\exists\, b < a < c$, $f : (b, a) \cup (a, c) \to \mathbb{R}$. *$f$ is defined near a, not necessarily at a.*

Example 1: $\lim_{x \to 3} \sqrt[3]{x} = \sqrt[3]{3}$

Proof. Need to estimate: $\left|\sqrt[3]{x} - \sqrt[3]{3}\right| = \dfrac{|x-3|}{x^{\frac{2}{3}} + \sqrt[3]{3x} + \sqrt[3]{9}}$.

Let's decide that $\delta \leq 1$, so $|x-3| < 1 \Rightarrow 2 < x < 4$. Then $\dfrac{|x-3|}{x^{\frac{2}{3}} + \sqrt[3]{3x} + \sqrt[3]{9}} > \sqrt[3]{4} + \sqrt[3]{6} + \sqrt[3]{9} > 4$.

Hence $\left|\sqrt[3]{x} - \sqrt[3]{3}\right| = \dfrac{|x-3|}{x^{\frac{2}{3}} + \sqrt[3]{3x} + \sqrt[3]{9}} < \dfrac{|x-3|}{4}$.

Let $\delta = \min\{1, 4\epsilon\}$, if $|x-3| < \delta$, then $\left|\sqrt[3]{x} - \sqrt[3]{\delta}\right| < \frac{|x-3|}{4} \leq \frac{4\epsilon}{4} = \epsilon$

Hence $\lim_{x \to 3} \sqrt[3]{x} = \sqrt[3]{3}$. □

Example 2: $g(x) = \begin{cases} x, & x \in \mathbb{Q}\backslash\{0\} \\ -x, & x \in \mathbb{R}\backslash Q \\ 3, & x = 0 \end{cases}$, $\lim_{x \to 0} g(x) = 0$

Proof. Let $\epsilon > 0$, take $\delta = \epsilon$, if $0 < |x - 0| < \delta = \epsilon$, then $g(x) = \pm x$, so $|g(x) - 0| = |\pm x| = |x| < \epsilon$.
$\therefore \lim_{x \to \infty} g(x) = 0$

□

Remark: The definition of $\lim_{x \to a} f(x) = L$ does not depend on $f(a)$.

Variant:

$\lim_{x \to a^+} f(x) = R$ limit from the left refers only to $x \in (a,\ a + \delta)$.

$\lim_{x \to a^-} f(x) = L$ limit from the right refers only to $x \in (a,\ a - \delta)$.

Definition 3.1.2. $\lim_{x \to x_0} f(x) = L$ *means* $\forall \epsilon > 0$, *there is an* δ *s.t. if* $|x - x_0| < \delta$, *then* $|f(x) - L| < \epsilon$.

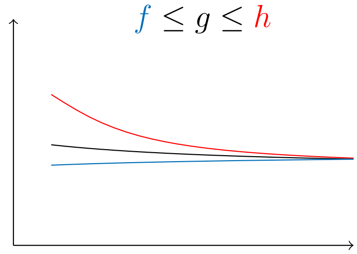
Example: $f(x) = \frac{1}{1+x^2}$, $\lim_{x \to \infty} \dfrac{1}{1+x^2} = 0$.

Given $\epsilon > 0$, let $N = \left|\frac{1}{x^2}\right| + 1$, $x > N \Rightarrow \frac{1}{1+x^2} < \frac{1}{1+N^2} < \frac{1}{1+\frac{1}{\epsilon^2}} < \epsilon$.

Theorem 3.1.1. ***Squeeze Theorem***
If $f(x) \leq g(x) \leq h(x)$, on (a, b) and $\lim_{x \to a^+} f(x) = L = \lim_{x \to a^+} h(x)$, then $\lim_{x \to a^+} f(x) = L$.

Proof. If $\epsilon > 0$,

Figure 1: The limit of $g(x)$ is squeezed by f and h.



$\lim\limits_{x \to a^+} f(x) = L$, means that $\exists\ \delta_1 > 0$, $a < x < a + \delta_1 \Rightarrow |f(x) - L| < \epsilon$.

$\lim\limits_{x \to a^+} g(x) = L$, means that $\exists\ \delta_2 > 0$, $a < x < a + \delta_2 \Rightarrow |h(x) - L| < \epsilon$.

so if $a < x < a + \min\{\delta_1, \delta_2\}$, then $L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon \Rightarrow |g(x) - L| < \epsilon$ $\square$

Example 1: $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

Example 2:

$$\begin{aligned}
\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta^2} &= \lim_{\theta \top 0} \frac{(1 - \cos\theta)(1 + \cos\theta)}{\theta^2(1 + \cos\theta)} \\
&= \lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\theta^2(1 + \cos\theta)} \\
&= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta^2} \frac{1}{1 + \cos\theta} \\
&= \frac{1}{2}
\end{aligned}$$

Proposition 3.1.1. *If* $\lim\limits_{x \to a} f(x) = L$ *and* $\lim\limits_{x \to a} f(x) = M$, *then*

1. $\lim\limits_{x \to a} c \cdot f(x) = c \cdot L$
2. $\lim\limits_{x \to a} f(x) + g(x) = L + M$

   Proof. Let $\epsilon > 0$, use value $\epsilon/2$

   There exists N_1 such that for all $n \geq N_1$, $|f(x) - L| < \frac{\epsilon}{2}$; there exists N_2 such that for all $n \geq N_2$, $|g(x) - M| < \frac{\epsilon}{2}$;

   Hence $|f(x) + g(x) - L - M| \leq |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ $\square$

3. $\lim\limits_{x \to a} f(x)g(x) = LM$
4. $\lim\limits_{x \to a}$ *if* $M \neq 0$, *then* $\exists\delta > 0$, *s.t.* $g(x) \neq 0$, $(a - \delta, b + \delta) \backslash \{a\}$ *and* $\frac{f(x)}{g(x)} = \frac{L}{M}$

Problem: graph $\sin x/x$, $x \sin \frac{1}{x}$

3.2 The Natural Logarithm and e

For $0 < a < b < \infty$, let $A(a,b)$ denote the area under $y = \frac{1}{x}$ from $x = a$ to $x = b$.

If $0 < b < a < \infty$, Let $A(a,b) = -A(a,b)$.

Define $L(x) = A(1,x)$, then

- $A(a,a) = 0$
- $A(a,b) + A(b,c) = A(a,c)$
- if $s > 0$, then $A(s_a, s_b) = A(a,b)$

Proposition 3.2.1. *If $a, b > 0$, $L(a) + L(b) = L(ab)$.*

Proof. $L(a) + L(b) = A(1,a) + A(1,b) = A(1,a) + A(a,ab) = A(1,ab) = L(ab)$

□

Corollary 3.2.1. *If $a > 0$, $L(a^n) = nL(a)$ for $n \in \mathbb{Z}$.*

Proof. $n = 0$, $L(a^0) = 0L(a) = A(1,1) = 0 = 0L(a)$.

If we have shown that $L(a^n) = nL(a)$, $n \geq 0$. then $L(a^{n+1}) = L(a^n \cdot a) = L(a^n) + L(a) = nL(a) + L(a) = (n+1)L(a)$.

By Induction, true for all $n \in \mathbb{N}$.

$0 = L(1) = L(a^n \cdot a^{-n}) = L(a^n) + L(a^{-n}) = nL(a) + L(a^{-n}) \Rightarrow L(a^n) = nL(a)$

□

Remark: $L(x)$ is strictly monotone increasing.

$\lim_{x \to +\infty} L(x) = \lim_{n \to +infty} L(2^n) = \lim_{n \to +\infty} nL(2) = \infty$

Definition 3.2.1. *There is a unique number e such that $L(e) = 1$, $e \approx 2.71828\cdots$*

$L(2.5) < 1\frac{1+1/2}{2} + \frac{1}{2}\frac{0.5+2/3}{2} = \frac{37}{40} < 1$

$L(3) < A(1,2) + (A2,3) = \frac{16}{15} > 1$

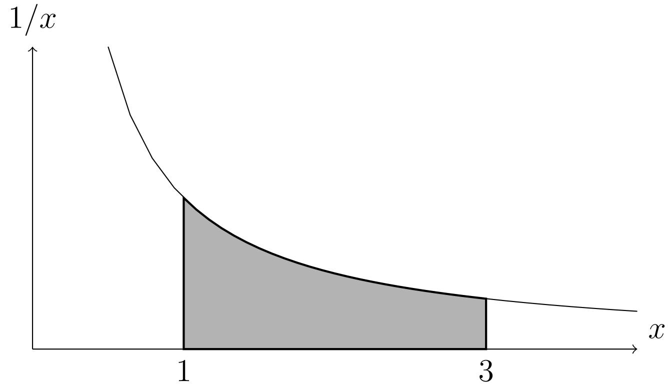
$\therefore 2.5 < e < 3$

Proposition 3.2.2. *L is differentiable and $\frac{d}{dx}\ L(x) = \frac{1}{x}$*

Definition 3.2.2. *Natural Logarithm* *The natrual logarithm $\ln x$ or $\log x$ is the function $L(x)$ defined.*

- $\ln x$ *is monotone increasing.*
- $\ln a^n = n \ln a$
- $\ln 1 = 0$ *and* $\ln e = 1$
- $\lim_{x\to\infty} \ln x = +\infty$
- $\lim_{x\to 0^+} \ln x = -\lim_{x\to 0^+} \ln \frac{1}{x} = -\lim_{y\to\infty} \ln y = -\infty$

Figure 2: The area under the function $\frac{1}{x}$ is $\log x$



Example 1: $\lim_{x\to\infty} \frac{\ln x}{x} = 0$

Proof. Since $2^n \to \infty$. Let $a_n = \frac{\ln x^n}{2^n} = \frac{n \ln 2}{2^n}$

If $n \geq 2$, then $\frac{a_{n+1}}{a_n} = \frac{(n+1)\ln 2}{2^{n+1}} \cdot \frac{2^n}{n \ln 2} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n} \leq \frac{3}{4}$.

$0 \leq a_{2+k} \leq \frac{3}{4} \cdot a_{2+k-1} \leq \cdots \leq (\frac{3}{4})^k \cdot a_2$, $\left|\frac{3}{4}\right| < 1$, so $(\frac{3}{4})^n \to 0$.

If $x^n \leq x \leq 2^{n+1}$, then $\ln 2^n \leq \ln x \leq \ln 2^{n+1}$,

$\therefore \quad \frac{n \ln 2}{2^{n+1}} \leq \frac{\ln x}{x} \leq (\frac{(n+1)\ln 2}{2^{n+1}})2.$

By Squeeze Theorem, $\lim_{x\to\infty} \frac{\ln x}{x} = 0$.

□

Example 2: If $a > 0$, $\lim_{x\to\infty} \frac{\ln x}{x^a} = \lim_{y\to\infty} \frac{\ln y^{\frac{1}{a}}}{y} = \frac{1}{a} \cdot \lim_{x\to\infty} \frac{\ln y}{y} = 0$

Example 3: $\lim_{x\to 0^+} x^a \ln x = \lim_{y\to\infty} y^{-a} \ln \frac{1}{y} = \lim_{y\to\infty} \frac{-\ln y}{y^a} = 0$

Definition 3.2.3. ***Inverse Function*** *If* $f : x \to y$ *is* $1:1$, *Ran*$(f) : Z \leq y$, *then there is an inverse function* $f^{-1} : Z \to y$, *s.t.* $f(x) = Z \iff f^{-1}(Z) = x$.

Definition 3.2.4. *The inverse of* $\ln x$*, in* $(0,\infty) \to (-\infty,+\infty)$ *is the exponential function* $f(x) = e^x$*.*

$f(x) = e^x : (-\infty,+\infty) \to (0,\infty)$.

Example 1: $\lim\limits_{x\to+\infty} \dfrac{e^x}{x^a}$

Let $x = \ln y$, then $\lim\limits_{x\to+\infty} \dfrac{e^x}{x^a} = \lim\limits_{y\to+\infty} \dfrac{e^{\ln y}}{(\ln y)^a} = \lim\limits_{y\to\infty} \dfrac{y}{(\ln y)^n} = +\infty$

Proposition 3.2.3. $\frac{d}{dx}e^x = e^x$

Proof.

$$\begin{aligned} \ln(e^x) &= x \\ \frac{1}{e^x} \cdot \frac{d}{dx}e^x &= 1 \\ \therefore \ \frac{d}{dx}e^x &= e^x \end{aligned}$$

$\square$

Remark:

$\lim\limits_{x\to+\infty} e^{\frac{1}{x}}x = \infty$

$\lim\limits_{x\to-\infty} e^{\frac{1}{x}}x = -\infty$

$\lim\limits_{x\to+\infty} e^{\frac{1}{x}}x - x = \lim\limits_{x\to+\infty} x(e^{\frac{1}{x}} - 1) = \lim\limits_{y\to0^+} \dfrac{e^y - 1}{y} = 1$

Proposition 3.2.4. $\lim\limits_{n\to\infty}(1+\dfrac{x}{n})^n = e^x$ $\qquad$ *e is monotone increasing*

Proof. Case 1: $x \geq 0$,

$$\begin{aligned} \frac{x}{n} \cdot \frac{1}{1+\frac{x}{n}} &< \ln(1+\frac{x}{n}) < 1 \cdot \frac{x}{n} \\ \frac{x}{1+\frac{x}{n}} &< n\ln(1+\frac{x}{n}) < x \\ \therefore \exp\frac{x}{1+\frac{x}{n}} &< \exp(n\ln 1+\frac{x}{n}) < e^x \\ e^x &< (1+\frac{x}{n})^n < e^x \end{aligned}$$

$$\therefore \ \lim_{n\to\infty}(1+\frac{x}{n})^n = e^n \text{ when } x \geq 0$$

Case 2: $x < 0$, let $h = \frac{-x}{n}$, $0 < h < 1$,

$$\begin{aligned}
1 \cdot h \leq A(1-h, 1) &\leq \frac{1}{1-h} + h \\
1 \leq \frac{\ln(1-h)}{-h} = \frac{-A(1-h,1)}{-h} &\leq \frac{1}{1-h} \\
1 \leq \frac{\ln(1+\frac{x}{n})}{\frac{x}{n}} &\leq \frac{1}{1+\frac{x}{n}} \\
x \geq n\ln(1+\frac{x}{n}) &\geq \frac{x}{1+\frac{x}{n}} \\
e^x \geq (1+\frac{x}{n})^n &\geq \exp\frac{x}{1+\frac{x}{n}}
\end{aligned}$$

$$\therefore \lim_{n\to\infty}(1+\frac{x}{n})^n = e^n \text{ when } x < 0.$$

Hence $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$. □

3.3 Continuity

Definition 3.3.1. *$f(a,b) \to \mathbb{R}$, $a < c < b$, then f is continuous at c if $\lim_{x\to c} f(x) = f(c)$.*

$(x - \delta version) \forall \epsilon > 0$, $\exists \delta > 0$ s.t. $|x - c| < \delta \to |f(x) - f(c)| < \epsilon$.

If $f : (a,b) \to \mathbb{R}$ say is continuous at a if $\lim_{x\to a^+} f(x) = f(a)$; and continuous at b if $\lim_{x\to b^-} f(x) = f(b)$.

Say f is continuous on a set A if f continuous at every $a \in A$.

Example 1: $f(x) = \frac{1}{x}$, for $x \neq 0$, $a \neq 0$, given $\epsilon > 0$, estimate $|f(x) - f(a)|$.

Want $\delta > 0$, s.t. $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$.

Set $\epsilon_1 = \frac{|a|}{2}$. If $|x - a| < \frac{|a|}{2}$, then $|x| = |x - a + a| \geq |a| - |x - a| > |a| - \frac{|a|}{2} = \frac{|a|}{2}$.

Let $\delta_2 = \frac{\epsilon|a|^2}{2}$, $\therefore \delta = \min\{\frac{|a|}{2}, \frac{\epsilon|a|^2}{2}\}$,

then $|f(x) - f(a)| = \left|\frac{1}{x} - \frac{1}{a}\right| = \left|\frac{a-x}{ax}\right| \leq \frac{|a-x|}{|a|\cdot\frac{|a|}{2}} = \frac{2|a-x|}{|a|^2} < \frac{2}{|a|^2} \cdot \frac{\epsilon|a|^2}{2} = \epsilon$

Example 2: $f(x) = \sin x$, let $x = a + h$, estimate $|f(x) - f(a)|$.

Since $|x - a| < \delta \iff |h| < \delta$.

Let $\epsilon > 0$, $\sin(a + h) = \sin a \cos h + \sin h \cos a$.

$|\sin(a + h) - \sin a| \leq |\sin a(\cos h - 1)| + |\cos a \cdot \sin h|$.

Showed:

$h \in [0, \frac{\pi}{2}]$, $1 - h^2 \leq \cos h \leq 1$ and $0 < \sin h < h$.

$h \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $-h^2 \leq \cos h - 1 \leq 0$ and $h < \sin h < 0$.

If $|h| < 1$, $|\sin(a + h) - \sin a| \leq |\sin a(\cos h - 1)| + |\cos a \cdot \sin h| \leq 1 \cdot h^2 + 1 \cdot |h| \leq 2\,|h|$.

Take $\delta = \min\{\frac{\epsilon}{2}, 1\}$.

Hence, $|\sin(a + h) - \sin a| \leq 2\,|h| < 2\delta \leq \epsilon$.

Example 3: $f(x) = \cos x = \sin(-x + \frac{\pi}{2})$ is continuous.

Example 4: $f(x) = \ln x$, $x > 0$, $|x - a| < \delta$.

Let $\delta = \min\{\frac{a\epsilon}{2}, \frac{a}{2}\}$, $|f(x) - f(a)| = |A(x,a)| \leq |x - a| \cdot \max\{\frac{1}{a}, \frac{1}{x}\} < \frac{2}{a}\,|x - a|$.

Example 5: $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$\lim_{x\to 0} \dfrac{\sin x}{x} = 1 = f(0) \qquad \therefore$ continuous.

Example 6: $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

$\lim\limits_{x \to 0^+} f(x) = 1 \neq \lim\limits_{x \to 0^-} f(x) = -1 \qquad \therefore$ not continuous.

Example 7: $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$

continuous at $x = 0$, discontinuous at every $x \neq 0$.

Example 8: $f(x) = \begin{cases} 0, & x \in \mathbb{R} \backslash \mathbb{Q} \\ \frac{1}{q}, & \textbf{if } x = \frac{p}{q},\ q > 0,\ \gcd(p, q) = 1,\ p, q \in \mathbb{Z} \end{cases}$

$x \in \mathbb{Q}$, $\forall \delta > 0$, $\exists y \notin \mathbb{Q}$, $|x - y| < \delta$. not continuous.

$x \in \mathbb{R}$, $\epsilon > 0$, $\exists \frac{1}{N} < \epsilon$, if $0 < |x| < \frac{1}{N}$, $x \notin \mathbb{Q}$, or $x \neq \frac{p}{q}$, $q \geq N$,

so $|x| \leq \frac{1}{N} < \epsilon$. $\lim\limits_{x \to 0} f(x) \neq 0$.

$f(x)$ is continuous on $\mathbb{R} \backslash \mathbb{Q}$, discontinuous on $\mathbb{Q}$.

Example 9: $f(x) = \sin \frac{1}{x}$**,** $x \neq 0$**, continuous.**

Proposition 3.3.1. *If f, g are functions on (a, b), which are continuous at c, $a < c < b$, then*

1. *$\alpha f(x)$ is continuous,*
2. *$f(x) + g(x)$ is continuous,*
3. *$f(x)g(x)$ is continuous,*
4. *$g(x) \neq 0$, $\frac{f(x)}{g(x)}$ is continuous at x if $x \neq 0$,*
5. *$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_n x^n$ is continuous.*

3.4 Composite Functions

Definition 3.4.1. ***Composition of Functions***
$(g \cdot f)(x) = g(f(x))$, $\operatorname{img} f \subseteq \operatorname{dom} g$

Proposition 3.4.1. *If* $\operatorname{img} f \subseteq \operatorname{dom} g$, $\lim\limits_{x\to a} f(x) = b$, *then* $\lim\limits_{y\to b} g(y) = c = g(b)$, *then* $\lim\limits_{x\to a} g(f(x)) = c$.

In particular, if f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

Proof. Let $\epsilon > 0$, since $\lim_{y\to b} g(y) = c$, $\exists\delta > 0$, s,t, $0 \le |y - b| < \delta \Rightarrow |g(y) - c| < \epsilon$.

Since $\lim_{x\to a} f(x) = b$, using $\delta > 0$, $\exists\delta > 0$, s.t.

$$\begin{aligned} 0 < |x - a| < \delta \\ |f(x) - b| < \delta \\ |g(f(x)) - c| < \epsilon \end{aligned}$$

In particular, if f continuous at a, then $b = f(a)$, $c = g(b) = g(f(a))$, and $\lim_{x\to a} g(f(x)) = c = g(f(a))$, so $g \circ f$ is continuous at a.

$\square$

Proposition 3.4.2. *If $f(x)$ and $g(x)$ are continuous,* $\operatorname{img} f \subseteq \operatorname{dom} g$, *then $g \circ f$ is continuous.*

Theorem 3.4.1. *If $f(x)$ is continuous then $|f(x)|$ is continuous.*

Proof. Since $f(x)$ is continuous, therefore, when $x_n \to x_0$,

$$||f(x_n)| - |f(x_0)|| \le |f(x_n) - f(x_0)| \to 0$$

Therefore, $||f(x_n)| - |f(x_0)|| \to 0$, hence $|f(x)|$ is continuous. $\square$

Theorem 3.4.2 (**Bolzano's Theorem**). *Let f be a continuous function on $[a, b]$ with $f(a)f(b) < 0$, then there is at least one $c \in (a, b)$ for which $f(c) = 0$.*

Proof. The proof can be done with IVT. $\square$

3.5 EVT and IVT

Theorem 3.5.1. ***Extreme Value Theorem*** *$f : [a, b] \to \mathbb{R}$ is continuous, then f attains its* min, max *values.*

i.e. $\exists x_1, x_1 \in [a, b], s.t. f(x_1) = \sup f(x)$ *and* $f(x_2) = \inf f(x)$.

Proof. Let $L = \sup f(x)$, $a \le x \le b$, (possibly $+\infty$).

Choose $L_1 < L_2 < L_3 < L_n < L_{n+1} \cdots$ s.t. $\lim_{n\to\infty} L_n = L$.

If $L < \infty$, let $L_n = L - \frac{1}{n}$. If $L = +\infty$, let $L_n = n$.

$L_n < \sup f(x)$, so we can pick $x_n \in [a, b]$ such that $f(x_n) > L_n$, and $(x_n)_{n=1}^{+\infty}$ is a bounded sequence.

By the Bolzano-Weierstrass Theorem, there is a subsequence x_{n_1}, x_{n_2},x_{n_3}, $\cdots$ s.t. $\lim_{i\to\infty} x_{n_i} = x_0$ exists.

$a \le x_{n_i} \le b$, so $a \le x_0 \le b$.

Since f is continuous, $\sup f(x) = L \ge f(x_0) = \lim\limits_{i\to\infty} f(x_{n_i}) \ge \lim\limits_{n\to\infty} L_n = L$.

$f(x_0) = L = \sup f(x) < \infty$ because $f(x_0) \in \mathbb{R}$.

For the minimum, either repeat proof using $M = \inf f(x)$ or find the max of $-f(x)$.

□

Example 1: $f(x) = 1 - |x|$ **on** $[-2, 2]$

- $f(0) = 1 = \sup f(x)$
- $f(-2) = -1 = f(2) = \inf f(x)$

Example 2: $f(x) = \frac{1}{1+x^2}$**,** $f : (-\infty, \infty)$

- $\sup f(x) = 1$
- $\inf f(x) = 0$ not attained

Example 3: $f : (0, 1) \to \mathbb{R}$**,** $f(x) = \cot \pi x$

- $\sup f(x) = +\infty$
- $\inf f(x) = +\infty$

Example 4: $f : [0, 1]$**,** $f(x) = \begin{cases} x, & 0 < x < 1 \\ \frac{1}{2}, & x = 0 \quad \textit{or} \quad x = 1 \end{cases}$

f is not continuous and the theorem does not apply.

Theorem 3.5.2. ***Intermediate Value Theorem*** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there is a point c, $a < c < b$ s.t. $f(c) = L$.*

Proof. Let $x = \{x \in [a, b] : f(x) < L\}$.

Assume $f(a) < L < f(b)$, then $a \in X$, $b \notin X$, By LUBP, $c = \sup X$ exists.

Claim $a < c < b$,

f is continuous at a, so let $\epsilon = L - f(a) > 0$, there is a $\delta_1 > 0$, $a \le x < a + \delta_1 \Rightarrow f(x) < f(a) + \epsilon = L$, so $[a, a + \delta_1) \subseteq X$.

f is continuous at b, so let $\epsilon = L - f(b) > 0$, there is a $\delta_2 > 0$, $b - \delta_2 < x \le b \Rightarrow f(x) < f(a) - \epsilon = L$, so $(b - \delta_2, b]$ is disjoint from X.

$a + \delta_1 \le c \le b - \delta_2$,

Find $x_n \in X$, $c - \frac{1}{n} < x_n \le c$, so $x_n \to c$.

f is continuous, $f(c) = \lim_{n\to\infty} f(x_n) \le L$, if $x > c$, $f(x) \ge L$, (not in X).

$\lim_{n\to\infty} y_n = c$, $f(y_n) \ge L$, $f(c) = \lim_{n\to\infty} f(y_n) \ge L$.

$\therefore f(c) = L$.

□

Example 1: Every polynomial $P(x)$ of degree $P \ge 1$ with odd degree has a real root.

$P(x) = a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1 x + a_0$, $a_{2n+1} \ne 0$.

$$\begin{aligned}
\lim_{x\to\infty} P(x) &= \lim_{x\to+\infty} a_{2n+1}x^{2n+1}(1 + \frac{a_{2n}}{x} + \cdots + \frac{a_0}{x^{2n+1}}) \\
&= \begin{cases} +\infty, \text{if } a_{2n+1} > 0 \\ -\infty, \text{if } a_{2n+1} < 0 \end{cases} \\
\lim_{x\to-\infty} P(x) &= \begin{cases} -\infty, \text{if } a_{2n+1} > 0 \\ +\infty, \text{if } a_{2n+1} < 0 \end{cases}
\end{aligned}$$

Say $a_{2n+1} > 0$,

$\exists x_1$ s.t. $P(x_1) > 157$,

$\exists x_2$ s.t. $P(x_2) < -341$,

By IVT, $\exists c \in (x_2, x_1)$ s.t. $P(c) = 0$.

Corollary 3.5.1. ***for IVT***
If $f : [a, b] \to \mathbb{R}$ is continuous, then $\operatorname{img} f = f([a, b])$ is a closed bounded interval.

Proof. By EVT, $\exists x_1, x_2 \in [a, b]$,

$f(x_1) = d = \sup f(x)$, $f(x_2) = c = \inf f(x)$

If $c \le L < d$, by $I.V.T$, $\exists x$ s.t. $f(x) = L$.

$\therefore f([a, b]) = [c, d]$. □

3.6 Monotone Functions

Definition 3.6.1 (Monotone). *A function f is called increasing on an interval (a,b) if $f(x) \leq f(y)$ where $a < x \leq y < b$.*

It is strictly increasing on (a,b) if $f(x) < f(y)$ whenever $a < x \leq y < b$.

Similarly, we define decreasing and strictly decreasing function. All of these functions are called monotone.

Proposition 3.6.1. *Let $f : (a,b) \to \mathbb{R}$ be monotone increasing (decreasing).*

If $a < b < c$, then $\lim\limits_{x \to c^-} f(x) = L$ exists, and $\lim\limits_{x \to c^+} f(x) = M$ exists.

$L \leq f(x) \leq M$ (or $L \geq f(c) \geq M$)

Definition 3.6.2. *If $\lim\limits_{x \to c^-} f(x) = L$, and $dlim_{x \to c^+} f(x) = M$, and $L \neq M$, this is called a **jump discontinuity.***

Corollary 3.6.1. *The only discontinuities that a monotone function can have are jump discontinuities.*

Corollary 3.6.2. *If $f : (a,b) : \mathbb{R}$ is monotone, then f is continuous $\Leftrightarrow$ $\operatorname{img} f$ is an interval.*

Proof. If f is continuous, then $\operatorname{img} f$ is an interval by IVT.

If f is discontinuous, then it has a jump discontinuity, say at $x = c$, $dlim_{x \to c^-} f(x) = L < M dlim_{x \to c^+} f(x)$.

The $\operatorname{img} f \subseteq (\lim\limits_{x \to a^+} f(x), L) \cup \{f(c)\} \cup (M, \lim\limits_{x \to a^-} f(x))$

$\square$

Corollary 3.6.3. *Let $f : (a,b) : \mathbb{R}$ bet strictly monotone increasing(decreasing) and continuous, with $\operatorname{img} f = (c,d)$. Then, the inverse function $f^{-1} : (c,d) \to (a,b)$ is also continuous.*

Proof. If $f(x) < f(y)$, so f is $1-1$. f is continuous, so $\operatorname{img} f$ is an interval.

So $f^{-1} : (c,d) \to (a,b)$ is defined by $f^{-1}(x)$ if $f(x) = y$. Since $\operatorname{img} f^{-1}$ is an interval, f^{-1} is continuous by Cor 3.6.2.

If $f(x_1) = y_1$, $f(x_2) < y_2 \Rightarrow x_1 < x_2$, so $y_1 < y_2 \Rightarrow f^{-1}(y_1) < x_1 < x_2 = f^{-1}(y_2)$.

Therefore, f^{-1} is strictly monotone incresing(decreasing).

$\square$

Example 1:

$\sec : [0, \frac{\pi}{2}) \to [1, -\infty), (\frac{\pi}{2}, \pi] \to (-\infty, 1]$.

$\sec^{-1} : [1, \infty) \to [0, \frac{\pi}{2}), (-\infty, -1] \to (\frac{\pi}{2}, \pi]$

Example 2: Cantor Ternary Function (continuous)

3.7 Cardinality and Countable Sets

Definition 3.7.1. *Say two sets A and B have the same cardinality.*

If $\exists f : A \to B$ which is $1-1$ and onto, write $|A| = |B|$.

Say $|A| \leq |B|$, if $\exists f : A \to B$ which is $1-1$, then A is countable if $|A| = \mathbb{N}$.

Example 1:

$2\mathbb{N} = \{2, 4, 6, 8, \cdots\}$ $|2\mathbb{N}| = |N|$

$f : 2\mathbb{N} \to \mathbb{N}$, $f(2n) = n$

Example 2:

$\mathbb{Z} = \{0, 1, -1, 2, -2, \cdots\} \qquad |\mathbb{Z}| = |\mathbb{N}|$

$$f(b) = \begin{cases} 2n+1, & n \leq 0 \\ 2\,|n|\,, & n < 0 \end{cases}$$

Example 3: $\mathbb{N} \times \mathbb{N} = \{(m, n),\ m, n \in \mathbb{N}\}$

Example 4: $\mathbb{Q} = \{\frac{p}{q}, q \in \mathbb{N}, p \in \mathbb{Z}\}, \gcd(p, q) = 1$.

$f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}, \qquad |\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{N}| \leq |\mathbb{Z}| = |\mathbb{N}|$

Proposition 3.7.1. *If A is an infinite set, and $|A| \leq |\mathbb{N}|$, then $|A| = |\mathbb{N}|$.*

Proof. Let $f : A \to \mathbb{N}$ be $1-1$, then $|A| = \mathbb{N}$, $\square$

4 Differentiation

4.1 Basics

Definition 4.1.1. *$f : (a, b) \to \mathbb{R}$, say f is differentiable at x if $\dim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists. The limit is $f'(x)$ or $\frac{d}{dx}f(x)$.*

Say f is differentiable on (a, b), if it is differentiable at x, for every $x \in (a, b)$.

Definition 4.1.2. *The tangent line to f at x is*

$$T(x + h) = f(x) + f'(x)h$$
$$T(y) = f(x) + f'(x)(y - x)$$

This is the best linear approximation near x. Line through $(x, f(x))$ with slope $f'(x)$.

Theorem 4.1.1. *If f is differentiable of $x = c$, then f is continuous at $x = c$.*

Proof.

$$\begin{aligned}
\lim_{x\to c} f(x) - f(c) &= \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \cdot (x - c) \\
&= \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \lim_{x\to c}(x - c) \\
&= \lim_{h\to 0} \frac{f(c + h) - f(c)}{h} \cdot \lim_{x\to c}(x - c) \\
&= f'(c) \cdot 0 = 0
\end{aligned}$$

$\therefore \lim_{x\to c} f(x) = f(c)$ so f is continuous at c.

□

Proposition 4.1.1. *Let $f : (a, b) \to \mathbb{R}$, $x_0 \in (a, b)$, TFAE:*

1. *f is differentiable at x_0, $f'(x_0) = m$*
2. *let $T(x) = f(x_0) + f'(x_0)(x - x_0)$, then $\lim_{h\to 0} \frac{f(x_0 + h) - T(x - x_0)}{h} = 0$*
3. *$f(x) = T(x)+e(x)$ (e stand for error), $\lim_{h\to 0} \frac{e(x_0 + h)}{h} = 0$. (the difference from the line is small relative to $h = x - x_0$)*

Proof. Proving equivalent means proving they imply each other.

From 1 to 2: $\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = m$.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x_0 + h) - T(x_0 + h)}{h} &= \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0) - mh}{h} \\
&= \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} - m = 0
\end{aligned}$$

From 2 to 3: Define $e(x) = f(x) - T(x)$.

$$\lim_{h\to 0} \frac{e(x_0+h)}{h} = \lim_{h\to 0} \frac{f(x_0+h) - T(x_0+h)}{h} = 0$$

From 3 to 1: $f(x) = T(x) + e(x) = f(x_0) + m(x - x_0) + e(x)$ and $\lim_{h\to 0} \dfrac{e(x_0+h)}{h} = 0$,

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0) + mh + e(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} m + \frac{e(x_0+h)}{h} = m$$

$\therefore f'(x_0) = m$ and f is differentiable at x_0.

$\square$

Example 1: $f(x) = x^n$

$$\begin{aligned}
\lim_{x\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{x\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{x\to 0} \frac{x^n + \binom{n}{1} \cdot x^{n-1}h + \binom{n}{2} \cdot x^{n-2}h^2 \cdots + \binom{n}{n} \cdot h^n - x^n}{h} \\
&= nx^{n-1}
\end{aligned}$$

Example 2: $f(x) = e^x$**,**

$$\lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h}$$

Let $u = e^h$, $h \to 0 \iff u \to 1$

$$\begin{aligned}
e^x \lim_{h\to 0} \frac{e^h - 1}{h} &= e^x \lim_{u\to 1} \frac{u-1}{\ln u} \\
&= \frac{e^x}{\lim\limits_{u\to 1} \dfrac{\ln u - \ln 1}{u - 1}} \\
&= \frac{e^x}{\lim\limits_{k\to 0} \dfrac{\ln(1+k) - \ln 1}{k}} = e^x
\end{aligned}$$

Example 3: $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$f'(x) = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \cdot -\frac{1}{x^2} = 2x \sin \frac{1}{x} + \cos \frac{1}{x}$

$x = 0$,

$$\lim_{h\to 0} \frac{f'(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h\to 0} h \sin \frac{1}{h}$$

f is differentiable everywhere but is not continuous at 0.

Remark: as $x \to \infty$, $x^2 \sin\frac{1}{x} \approx x^2 \frac{1}{x} = x$

$$\begin{aligned}
\lim_{x\to\infty} x - f(x) &= \lim_{x\to\infty} x - x^2 \sin\frac{1}{x} \\
&= \lim_{u\to 0^+} \frac{1}{u} - \frac{\sin u}{u^2} \\
&= \lim_{u\to 0^+} \frac{u - \sin u}{u^2} \\
&= \lim_{u\to 0^+} \frac{1 - \frac{\sin u}{u}}{u} \\
&= 0
\end{aligned}$$

Proposition 4.1.2. *If f, g are differentiable at x_0, then*

1. $(cf)'(x_0) = cf'(x_0)$
2. $(f+g)'(x_0) = f'(x_0) + g'(x_0)$
3. *product rule* $(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$
4. *quotient rule if* $g(x_0) \neq 0$, $(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$

Proof.

$$\begin{aligned}
(fg)'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0+h)f(x_0+h) - f(x_0)g(x_0+h) + f(x_0)g(x_0+h) - f(x_0)g(x_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \lim_{h\to 0} g(x_0+h) + f(x_0) \lim_{x\to 0} \frac{g(x_0+h) - g(x_0)}{h} \\
&= f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)
\end{aligned}$$

$$\begin{aligned}
(\frac{f}{g})'(x_0) &= \lim_{h\to 0} \frac{\frac{f'(x_0+h)}{g}(x_0+h) - \frac{f'(x_0)}{g'(x_0)}}{h} \\
&= \lim_{h\to 0} \frac{f(x_0+h)g(x_0) - f(x_0)g(x_0+h)}{hg(x_0+h)g(x_0)} \\
&= \lim_{h\to 0} \frac{f(x_0+h)g(x_0) - f(x0_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x_0+h)}{hg(x_0+h)g(x_0)} \\
&= \lim_{h\to 0} \frac{f(x_0) + h - f(x_0)}{h} \frac{g(x_0)}{g(x_0+h)g(x_0)} - \frac{f(x_0)}{g(x_0+h)g(x_0)} \frac{g(x_0+h) - g(x_0)}{h} \\
&= (\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}
\end{aligned}$$

□

Proposition 4.1.3. *TFAE*

1. *f is differentiable at x_0*

2. *$f(x) = f(x_0) + \varphi(x)(x - x_0)$ and φ is continuous at x_0*

3. *In this case, $\varphi(x_0) = f'(x_0)$*

Proof. 1 to 4: $\varphi(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} f'(x_0) & \text{if } x \neq x_0, \end{cases}$

$\lim_{x \to x_0} \varphi(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) = \varphi(x_0)$

4 to 1:

$$\begin{aligned} f'(x_0) &= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \\ &= \lim_{h \to 0} \frac{f(x_0) + \varphi(x_0 + h) - f(x_0)}{h} \\ &= \lim_{h \to 0} \varphi(x_0 + h) \\ &= \varphi(x_0) \end{aligned}$$

$\therefore f$ is differentiable at x_0 and $f'(x_0) = \varphi(x_0)$.

□

Theorem 4.1.2. ***Chain Rule*** $f : (a, b) \to (c, d),\ g : (c, d) \to \mathbb{R}$

consider $h(x) = g(f(x))$,

If f is differentiable at x_0 and g is differentiable at $y_0 = f(x_0)$,

then h is differentiable at x_0 and $h'(x_0) = g'(f(x_0))f'(x_0)$

Proof.
Pseudo Proof:

$$\begin{aligned} h'(x_0) &= \lim_{k \to 0} \frac{h(x_0 + h) - h(x_0)}{k} \\ &= \lim_{k \to 0} \frac{g(f(x_0 + k)) - g(f(x_0))}{f(x_0 + k) - f(x_0)} \cdot \frac{f(x_0 + h) - f(x_0)}{k} \\ &= \lim_{f(k) \to 0} \frac{g(f(x_0) + f(x)) - g(f(x_0))}{f(k)} \cdot \lim_{k \to 0} \frac{f(x_0 + k) f(x_0)}{k} \\ &= g'(f(x_0)) f'(x_0) \end{aligned}$$

Real Proof:

Write $f(x) = f(x_0) + \varphi(x)(x - x_0)$ $\lim_{x \to x_0} \varphi(x_0) = f'(x_0)$

$y_0 = f(x_0)$ Write $g(y) = g(y_0) + \psi(y)(y - y_0)$

$$\begin{aligned} h(x) = g(f(x)) &= g(y_0) + \psi(f(x))(f(x) - f(x_0)) \\ &= g(f(x_0)) + \psi(f(x))\varphi(x)(x - x_0) \\ &= g(f(x_0)) + \omega(x) \end{aligned}$$

$$\begin{aligned} \lim_{x \to x_0} \omega(x) &= \lim_{x \to x_0} \psi(f(x))\varphi(x) \\ &= \lim_{y = f(x) \to y_0} \psi(y) \lim_{x \to x_0} \varphi(x) \\ &= \psi(y_0)\psi(x_0) \\ &= g'(f(x_0))f'(x_0) \end{aligned}$$

$\square$

Example 1: $g(x) = \ln x = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$

$x > 0,\ g'(x) = \frac{1}{x}$

$x < 0,\ \frac{d}{dx}(\ln(-x))\frac{1}{-x}(-1) = \frac{1}{x}$

$\therefore g'(x) = \frac{1}{x}$.

Example 2:

1. $\tan x$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}(\frac{\sin x}{\cos x}) = \frac{\cos x(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

2. $\csc x$

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}(\frac{1}{\sin x}) = \frac{0 \sin x - 1 \cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x}\frac{1}{\sin x} = -\cot x \cdot \csc x$$

3. $\sec x$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

4. $\cot x$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

Example 3: $f(x) = x^a, a \in \mathbb{R} \backslash \{0\}$

$f(x) = e^{a \ln x}$

$f'(x) = e^{a \ln x} \cdot \frac{a}{x} = \frac{ax^a}{x} = ax^{a-1}$

Proposition 4.1.4. *f is monotone from (a,b) onto (c,d) and differentiable at x_0, AND $f'(x_0) \neq 0$, then f^{-1} is differentiable at $f(x_0)$, $f^{-1}f(x_0) = \frac{1}{f'(x_0)}$*

Rewrite $f^{-1}(y) = \frac{1}{f'(f^{-1}(y))}$

Proof. $f(x^{-1}(y)) = y$

$1 = \frac{d}{dx}y = f'(f^{-1}(y)) \cdot (f^{-1})'(y)$

Solve $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$. □

Example 1: $f(x) = \tan^{-1} x$

$f'(x) = \frac{1}{\sec^2(\tan^{-1} x)} = \cos^2(\tan^{-1} x) = \frac{1}{1+x^2}$

Proposition 4.1.5. *$f : (a,b) \to (c,d)$ monotone differentiable, if $f'(f^{-1}(x)) \neq 0$,*

$$f^{-1} : (c,d) \to (a,b), (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Example 1: $\sin x : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$. $f^{-1}x = \sin^{-1} x$ (also called $\arcsin x$)

$$(f^{-1})'(x)) = \frac{1}{\cos(\sin^{-1} x}) = \frac{1}{\sqrt{1-x^2}}$$

Remark: at $x \pm 1$, the derivative is undefined, because there is vertical tangent.

Example 2: $f(x) = \sec x$, $x \in [0,\pi] \backslash \{\frac{\pi}{2}\}$, $f^{-1}x = \sec^{-1} x$

$$\begin{aligned} (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ \sec^{-1} x &= \frac{1}{\sec(\sec^{-1} x)\tan(sec^{-1}x)} \\ &= \begin{cases} \dfrac{1}{x\sqrt{x^2-1}}, & x > 0 \\ \dfrac{1}{-x\sqrt{x^2-1}}, & x < 0 \end{cases} \end{aligned}$$

4.2 Maximum and Minimum

Theorem 4.2.1. *Fermat*
Let $f : [a, b] \to \mathbb{R}$ be a continuous function, if f attains its maximum or minimum value at x_0, then

1. *x_0 is an endpoint, $x \in \{a, b\}$, or*
2. *$f'(x_0)$ is undefined, or*
3. *$f'(x_0)$ is 0*

Proof. Either 1 x_0 is an endpoint, or 2 $f'(x_0)$ is undefined.

WLOG, $f(x_0) = \max f(x)$ for $a \le x \le b$.

$$\begin{aligned} L &= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h \quad > 0} \le 0 \\ &= \lim_{x\to 0} \frac{f(x_0+h) - f(x_0)}{h \quad < 0} \ge 0 \end{aligned}$$

□

Theorem 4.2.2. *Rolle's Theorem*
let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f(a) = f(b)$, then $\exists x_0 \in (a, b)$ s.t. $f'(x_0) = 0$.

Proof. Case 1: f is constant, then $f'(x) = 0$ for all $a < x < b$.

Case 2: f is not constant, then $\exists f(x_1) \ne f(a)$.

WLOG, $f(x_1) > f(x)$, by EVT, $\exists x_0 \in [a, b]$ s.t. $f(x_0) = \sup f(x) \ge f(x_1) > f(a)$.

$\therefore a < x_0 < b$. Apply Fermat's Theorem, $f'(x_0) = 0$

□

Theorem 4.2.3. *Mean Value Theorem*
If $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b), then $\exists x_0 \in (a, b)$ such that

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Let $g(x) = f(x) - \frac{f(b)-f(a)}{b-a} \cdot x$,

$$\begin{aligned} g(b) - g(a) &= f(b) - \frac{f(b) - f(a)}{b - a} \cdot b - f(a) + \frac{f(b) - f(a)}{b - a} \cdot a \\ &= (f(b) - f(a)) - \frac{f(b) - f(a)}{b - a} \cdot (b - a) = 0 \end{aligned}$$

g is continuous on $[a, b]$ and differentaible on (a, b), because $\frac{f(b)-f(a)}{b-a} \cdot x$ is differentiable everywhere.

Hence, by Rolle's Theorem, $\exists x_0 \in (a, b)$ such that $0 = g'(x_0) = f'(x_0) - \frac{f(b)-f(a)}{b-a} \cdot 1$

$\therefore f'(x_0) = \frac{f(b)-f(a)}{b-a}$.

□

Corollary 4.2.1. *If f is continuous on $[a,b]$, differentiable on (a,b), and $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing.*

Proof. If $a \le c < d \le b$, apply MVT on $[c,d]$,

so $\exists\ x_0,\ c < x_0 < d,\ \frac{f(d)-f(c)}{d-c} = f'(x_0) > 0 \Rightarrow f(d) > f(c)$.

<u>**Variants:**</u>

If $f'(x) \ge 0$ on (a,b), then $f(x)$ is increasing.

If $f'(x) > 0$ on (a,b), then $f(x)$ is strictly increasing.

If $f'(x) < 0$ on (a,b), then $f(x)$ is strictly decreasing.

If $f'(x) \le 0$ on (a,b), then $f(x)$ is decreasing.

$\square$

Corollary 4.2.2. *If f is C^1 ($f'(x)$ is continuous) and $f'(x_0) > 0$.*

then $\exists \delta > 0$, such that f is strictly increasing on $(x_0 - \delta, x_0 + \delta)$.

Proof. Let $\epsilon = f'(x_0) > 0$, by Continuity of $f'(x)$,

$\exists\ \delta > 0$ such that if $|x - x_0| < \delta \Rightarrow |f'(x) - f'(x_0)| < \epsilon = f'(x_0) \Rightarrow f'(x) > 0$

$\therefore f'(x) > 0$ on $(x_0 - \delta, x_0 + \delta) \Rightarrow f$ is strictly increasing on $(x_0 - \delta, x_0 + \delta)$.

$\square$

Example 1: $f(x) = \begin{cases} \frac{x}{2} + x^2 \sin\frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

$f'(0) = \frac{1}{2} + 0 = \frac{1}{2} > 0$

$x \ne 0,\ f'(x) = \frac{1}{2} + 2x\sin\frac{1}{x} + x^2(\cos\frac{1}{x})(\frac{-1}{x^2}) = \frac{1}{2} + 2x\sin\frac{1}{x} - s\cos\frac{1}{x}$. (discontinuity at $x = 0$)

$f''(\frac{1}{2n\pi} = \frac{1}{2} + \frac{1}{n\pi}) \cdot 0 - 1 = -\frac{1}{2}$.

Example 2: $\sin x$**,** $0 < x \le \frac{\pi}{2}$

Let $x_0 \in (0,x)$

$$\frac{\sin x}{x} = \frac{\sin x - \sin 0}{x} = cos(x_0)$$

$\therefore 0 < \sin x < x$

Let $g(x) = \frac{1}{2}x^2 + \cos x,\ g'(x) = x - \sin x > 0$ on $(0, \frac{\pi}{2})$, so $x_1 \in (0,x)$

$$\frac{g(x)-g(0)}{x} = g'(x) = x_1 - \sin x_1 > 0$$
$$\frac{\frac{1}{2}x^2 + \cos x - 1}{x} > 0$$
$$\therefore 1 > \cos x > 1 - \frac{x^2}{2}$$

Let $h(x) = \sin x - x + \frac{x^3}{6}$, $h'(x) = \cos x - 1 + \frac{x^2}{2} > 0$.

Apply MVT, $x_2 \in (0, x)$

$$\begin{aligned}\frac{h(x) - h(0)}{x} &= h'(x_2) > 0 \\ \sin x - x + \frac{x^3}{6} - 0 &> 0 \\ \therefore x - \frac{x^3}{6} &< \sin x < x.\end{aligned}$$

Let $k(x) = -\cos x - \frac{x}{2} + \frac{x^4}{2}4$, $k'(x) = \sin x - x + \frac{x^3}{6} > 0$.

By MVT, let $x_3 \in (0, x)$

$$\begin{aligned}\frac{k(x) - k(x_0)}{x} &= k'(x_3) > 0 \\ -\cos x - \frac{x^2}{2} + \frac{x^4}{24} + 1 &> 0 \\ \therefore 1 - \frac{x^2}{2} &< \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}\end{aligned}$$

Example 3:

Know $0 < \sin x < x < \tan x$ on $(0, \frac{\pi}{2})$, is $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ on $(0, \frac{\pi}{2})$?

Is x, $\sin x$, $\tan x > 0$, on $(0, \frac{\pi}{2})$ equivalent to $f(x) = \tan x \sin x - x^2 > 0$ on $(0, \frac{\pi}{2})$?

$$f(0) = 0$$

$$f'(x) = \sec^2 \sin x + \tan x \cos x - 2x$$

$$\begin{aligned}f''(x) &= 2\sec^2 x \tan x + \sec^2 x \cos x + \cos x - 2 \\ &= \frac{2\sin^2 x}{\cos^3 x} + (\sec x + \cos x - 2) \\ &= \frac{2\sin^2 x}{\cos^3 x} + (\cos x + \sec x)^2 \\ &> 0\end{aligned}$$

$f''(x)$ is strictly positive on $(0, \frac{\pi}{2})$, $f'(0) = 0 \implies f'(x)$ is strictly increasing, $\therefore f'(x) > 0$ if $x > 0$,

$\therefore f'(x)$ is strictly increasing, $f(0) = 0$

$\therefore f(x) > 0$ on the interval,

$\therefore \dfrac{\tan x}{x} > \dfrac{x}{\sin x}$

4.3 Second Derivative and Convexity

Definition 4.3.1. ***Convexity***

A function $f:(a,b)\to\mathbb{R}$ is convex, if $\forall\ a<c<d<b, \forall\ 0<t<1$,

$$f(tc+(1-t)d)\le tf(c)+(1-t)f(d),$$

i.e. the graph of f from c to d lies below the line between $c, f(c)$, and $(d, f(d))$.

Suppose $c\le x\le d$, define

$$t=\frac{d-x}{d-c}\in[0,1]$$

$$tc+(1-t)d=\frac{d-x}{d-x}c+\frac{x-c}{d-c}d=\frac{dc-xc+xd-cd}{d-c}=x$$

$$L(x)=f(c)+\frac{f(d)-f(1)}{d-c}(x-c)$$

$$L(tc+(1-t)d)=f(c)+\frac{f(d)-f(c)}{d-c}(1-t)(d-c)=tf(c)+(1-t)f(d).$$

Definition 4.3.2. *f is concave, then $-f$ is convex.*

Proposition 4.3.1.

- *If $f''(x)\ge 0$ on (a,b), then $f'(x)$ is increasing, if $f'(x)$ is increasing on (a,b), then f is convex.*
- *If $f''(x)\le 0$ on (a,b), then $f'(x)$ is decreasing, if $f'(x)$ is decreasing on (a,b), then f is concave.*

Proof. By MVT, $f''\ge 0 \ \Rightarrow\ f'$ is increasing on (a,b).

Suppose $f'(x)$ is increasing on (a,b).

Fix $a<c<d<b$, let $L(x)$ line through $(c,f(c))$ and $(d,f(d))$,

$g(c)=f(c)-f(c)=0$, and $g(d)=f(d)-f(d)=0$.

By Rolle's Theorem, $\exists x_0\in(c,d)$ such that $g'(x_0)=0$,

$g'(x)=f'(x)-L'(x)=f'(x)-\frac{f(d)-f(c)}{d-c}$ (the fraction part is constant).

So $g'(x)$ is increasing.

so on (c,x_0), $g'(x)\le g'(x_0)=0$, so g is decreasing;

on (x_0,d), $0\le g'(x)\le g'(x)$, so g is incresing.

$\therefore\ g(x)\ge 0$ on $(c,d)\Rightarrow f(x)\le L(x)\Rightarrow$, convex.

$\square$

Definition 4.3.3.

- *f is C_1 on (a,b) if $f'(x)$ exists and is continuous.*
- *f is C_2 on (a,b) if $f'(x)$, $f''(x)$ exist and are continuous.*

Corollary 4.3.1. *If f is C^2, $f'(x_0) = 0$, and $f''(x_0) < 0$, then, $(x_0, f(x_0))$ is a local maximum.*

Proof. By continuity, $\exists\, \delta > 0$, such that $f''(x) < 0$ in $(x_0 - \delta, x_0 + \delta) \Rightarrow f'$ decreasing $\Rightarrow f$ is concave.

$x \in (x_0 - \delta, x_0)$, $f'(x) \geq f'(x_0) = 0$, so f is increasing.

$x \in (x_0, x_0 + \delta)$, $f'(x) \leq f'(x_0) = 0$, so f is decresing.

□

Remark: $f''(x)$ measures 'curvature' of the graph.

$f''(x) > 0 \Rightarrow f'$ is increasing $\Rightarrow$ graph is curving upwards.

$f''(x) < 0 \Rightarrow f'$ is decreasing $\Rightarrow$ graph is curving downwards.

When $f''(x)$ changes sign, say from $+ \to -$, then $f(x)$ is an inflection point.

Graph $f(x) = \dfrac{x^{\frac{1}{3}}}{x-1}$.

Lemma 4.3.1. ***Secant Lemma***
Let $f : (a, b) \to \mathbb{R}$ be a convex function. Suppose that $a < x < y < z < b$, then

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}$$

Proof. We will just prove the first $\leq$, let $t = \frac{z-y}{z-x}$. Then, $y \neq t \cdot x + (a - t)z$.

By convexity, $f(tx + (1-t)z) \leq tf(x) + (1-t)f(z)$,

$$\therefore \frac{f(y) - f(x)}{y - x} \leq \frac{tf(x) + (1-t)f(z) - f(a)}{y - x} = (1-t)\frac{f(z) - f(x)}{(1-t)(z-x)} = \frac{f(z) - f(a)}{z - x}$$

□

Proposition 4.3.2. *If f is lipschitz, $|f(y) - f(x)| \leq K\,|y - x|$, for some constant K, then f is continuous.*

Proof. $\epsilon > 0$, let $\delta = \frac{\epsilon}{k}$, then if $|y - x| < \varepsilon$,

$$|f(y) - f(x)| \leq K\,|y - x| < K \cdot \frac{\varepsilon}{k} = \varepsilon$$

$\therefore f$ is continuous.

□

Theorem 4.3.1. *If $f : (a, b) \to \mathbb{R}$ is convex, then f is continuous.*

Proof. Let $a < c < d < c$, prove f is continuous on $[c, d]$.

Pick d and d' such that $a < c' < c$, $d < c' < b$, if $c \leq x < y \leq d$.

By the secant lemma,

$$\frac{f(c)-f(c')}{c-c'} \leq \frac{f(x)-f(c)}{x-c} \leq \frac{f(y-f(x)}{y-x} \leq \frac{f(d)-f(y)}{d-y} \leq \frac{f(d')-f(d)}{d'-d}$$

$$L \leq \frac{f(y)-f(x)}{y-x} \leq M \qquad \forall x, y \in [c,d]$$

$$\therefore \left|\frac{f(y)-f(x)}{y-x}\right| \leq \max\{|L|, |M|\} = K$$

$$\therefore |f(y)-f(x)| \leq K|y-x| \qquad \text{(lipschitz condition)}$$

so, f is continuous.

□

Corollary 4.3.2. *If f is C^1 on $[a,b]$, then f is Lipschitz.*

Proof. $f'(x)$ is continuous on [a,b].

By EVT, $\sup_{a\leq x\leq b} f'(x) = M$, $\inf_{a\leq x\leq b} f'(x) = L$.

By MVT, if $a \leq x < y \leq b$, $L \leq \frac{f(y)-f(x)}{y-x} = f'(c) \leq M$, for some $c \in (x,y) \subseteq [a,b]$ □

Example 1: $\begin{cases} x^2-1, & -1 < x < 1 \\ 1, & x = \pm 1 \end{cases}$ **Graph**

Definition 4.3.4. *If $f : (a,b) \to \mathbb{R}$,*

- *let* $D_-f(x) = \lim_{h\to o^-} \dfrac{f(x+h)-f(x)}{h}$ *if it exists.*
- *let* $D_+f(x) = \lim_{h\to 0^+} \dfrac{f(x+h)-f(x)}{h}$ *if it exists.*

Note f is differentiable at x if $D_-f(x) = D_+f(x)$ exists.

Theorem 4.3.2. *Let $f : (a,b) \to \mathbb{R}$ be a convex function,*

then f has left and right derivatives $D_\pm f(x)$ at every point,

and if $a < x < y < b$, $D_-f(x) \leq D_+f(x) \leq D_-f(y) \leq D_+f(y)$.

Proof. If $0 < k < g$, $a < x-h$, $y+h < b$, make $h < \frac{y-x}{2}$, then the Secant Lemma says that

$$\frac{f(x)-f(x-h)}{h} \leq \frac{f(x)-f(x-k)}{k} \leq \frac{f(x+k)-f(x)}{k} \leq \frac{f(x+h)-f(x)}{h} \leq \frac{f(y)-f(y-h)}{h}$$

$g(h) = \frac{f(x+h)-f(x)}{h}$ is increasing on $(-\delta, \delta)$.

$\sup_{h<0} g(h) = L \leq M = \inf_{h>0} g(h)$.

□

Lemma 4.3.2. *If $g : (\delta, 0) \to \mathbb{R}$ is increasing and bounded above, then $\lim_{h \to o^-} g(h) = L$ exists.*

Proof. $\{g(h) : -\delta < h < 0\} = S$ is bounded above.

By LUBP, $\sup S = L < \infty$,

let $\varepsilon > 0$, $L - \varepsilon$ is not an upper bound, $\therefore \exists h_0, g(h_0) > L - \varepsilon$.

Then if $h_0 \leq h < 0$, $L - \varepsilon \leq g(h) \leq L$,

$\therefore |L - g(h)| < \varepsilon$ if $h \in (h_0, 0) \Rightarrow \lim_{h \to 0^-} g(h) = L$ ◻

So get $D_- f(x) = \lim_{h \to 0^-} \dfrac{f(x+h) - f(x)}{h} = L$ exists.

If $k > 0$, $h < 0$,

$$\frac{f(x+h) - f(x)}{h} \leq \frac{f(x+k) - f(x)}{k} \Rightarrow L \leq \frac{f(x+k) - f(x)}{k}$$

is bounded below by L, and decreases as $k \to 0^+$.

$\therefore D_+ f(x) = \lim_{k \to 0^+} \dfrac{f(x+k) - f(x)}{k} = M \geq L$.

Theorem 4.3.3. *If $f : (a, b) \to \mathbb{R}$ is convex, then f is differentiable except on a countable set.*

Proof. $D_- f(x)$ is monotone increasing $\Rightarrow$ only has jump discontinuities.

Look at $[a + \frac{1}{n}, b - \frac{1}{n}]$, how many jump discontinuities with jump $\geq \frac{1}{n}$ can there be?

D_f runs from $D_f(a + \frac{1}{n})$ up to $D_f(b - \frac{1}{n})$ range has length:

$$\frac{D_- f(b - \frac{1}{n}) - D_- f(a + \frac{1}{n})}{\frac{1}{n}} = (D_- f(b - \frac{1}{n}) - D_- f(a + \frac{1}{n}))n < \infty$$

number of jumps of height $\geq \frac{1}{n}$: $\{$ discontinuity of $D_- f$ $\} = \underbrace{\cup_{n \geq 1} \{\underbrace{jump \geq \frac{1}{n} \in [a + \frac{1}{n}, b - \frac{1}{n}]}_{finite}\}}_{countable}$

◻

Theorem 4.3.4. *If $f : (a, b) \to \mathbb{R}$ is convex, then $D_- f(x) \leq D_+ f(x) \leq D_- f(x)$ exists for all $a < x < y < b$.*

Proof. If $g : [c, d] \to \mathbb{R}$ is monotone increasing, then g has at most a countable number of discontinuities.

All discontinuities are jump discontinuities.

Fix n, countable number of jumps of height $\geq \frac{1}{n}$.

$$n \leq \left| \frac{f(d) - f(c)}{1/n} \right| \leq h(f(d) - f(c)) < \infty$$

$\{x : g \text{ has a jump discontinuity at } x\ \} = \cup_{n\geq 1}\{x : g \text{ has a jump of height } \geq \frac{1}{n} \text{ at } x\}$.

$\therefore g$ is countable. $\square$

Corollary 4.3.3. *$D_-f(x)$ and $D_-f(x)$ are continuous except at at most a countable set.*

Proof. $\{$ jump discontinuities of $D_-f\} = \cup_{k\geq 1}\{$ jump discontinuities $m[a+\frac{1}{k}, b-\frac{1}{k}]\}$ $\square$

Theorem 4.3.5. *If f is convex, then f is differentiable, except on a countable set.*

Proof. If D_-f is continuous at x, then $D_-f(x) \leq D_+f(x) \leq D_-f(x+h)\ \forall h > 0$.

By Squeeze Theorem, $D_-f(x) \leq D_+f(x) \leq \lim\limits_{h\to 0^+} D_-f(x+h) = D_-f(x)$.

$\therefore D_+f(x) = D_-f(x) = f'(x)$ exists. $\square$

Example 1: Construct a function that is monotone increasing and discontinuous at every rational.

List rationals on $\mathbb{Q}$ as a list $r_1, r_2, r_3, \cdots$.

Pick a Cauchy sequence like $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Define $f(x) = \sum_{\{n,m<x\}} \frac{1}{2^n}$, if $x < y$, $\exists q \in \mathbb{Q}$, $x < q = r_{n_0} < y \Rightarrow f(y) > f(x) + \frac{1}{2^{n_0}} > f(x)$

$f(f_m) = \sum_{\{n:f_n<f_{n_i}\}} \frac{1}{2^n}$, if $x > r_m$, then $r_m < x \Rightarrow f(r_m) + \frac{1}{2^m} \leq f(x)$

$\therefore \lim\limits_{x\to r_m^-} f(x) = f(r_m) + \dfrac{1}{2^m}$

If $x \notin \mathbb{Q}$, f is continuous at x, let $\varepsilon > 0$, $\exists N \sum_{n=N+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^N} < \varepsilon$

Since $x \notin \{r_1, r_2, r_3, \cdots\}$, so $\min\limits_{1\leq i\leq N} |x - r_2| = \delta > 0$.

If $|x-y| < \delta$, say $y > x$, $f(y) - f(x) = \sum_{\{n:x<r_n<y\}} \frac{1}{2^n} \leq \sum_{n=N+1}^{\infty} \frac{1}{2^n} < \varepsilon$.

Remark:

Monotone functions are integrable. $g(x) = \int_0^x f(x)$.

Fundamental Theorem of Calculus: $g'(x) = f(x)$ where f is continuous.

Theorem 4.3.6. ***Jensen's Inequality***
If f is a convex function on (a,b),

$x_1, \cdots, x_n \in (a,b)$, $t_1, t_2, \cdots, t_n \in [0,1]$ such that $\sum_{i=1}^{n} t_i = 1$.

Then $f(t_1x_1 + t_2x_2 + \cdots + t_nx_n) \leq \sum_{i=1}^{n} t_i f(x_i)$.

Moreover, if f is strictly convex, and $0 < t_i < 1$, then equality holds only when $x_1 = x_2 = x_3 = \cdots$.

Proof. Let $P(n)$ be the statement for $n \in N$,

$n = 1$, $f(1, x_1) \le 1f(x_1)$ hence equality holds.

$n = 2$, $t_1 + t_2 = 1$, so $t_2 = 1 - t$, $f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2)$ (true by convexity).

If f is strictly convex, $0 < t < 1$, then inequality is strict, so equality holds (for $0 < t < 1$) only when $x_1 = x_2$.

Assume $P(n)$ is true for $1 \le k \le n-1$, $n \ge 3$, let $x - 1, \cdots x_n$. $t_1, \cdots, t_n$ be givem, WLOG, $t_n \ne 1$.

Let $y = \dfrac{t_1x_1 + \cdots + t_{n-1}x_{n-1}}{t_1 + t_2 + \cdots + t_{n-1}}$, let $t = t_1 + \cdots + t_{n-1}$, note $t_n = 1 - t$.

$$ty + (1-t)x_n = t_1x_1 + \cdots + t_{n-1}x_{n-1} + t_nx_n$$

$$\begin{aligned}\therefore f(\sum t_ix_i) &= f(ty + (1-t)x_n) \le tf(y) + (1-t)f(x_n)\\ f(\sum t_ix_i) &\le tf(\sum_{i=1}^{n-1} \frac{t_i}{t}x_i) + t_nf(x_n)\\ &\le t(\sum_{i=1}^{n-1}\frac{t_i}{t}f(x_i)) + t_nf(x_n) = \sum_{i=1}^{n} f(x_i)\end{aligned}$$

□

Example 1: let

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x > 0$

$\therefore f$ is strictly convex.

Let $a_1, a_2, \cdots, a_n \ge 0$, $t_1, t - 2, \cdots, t_n \in [0, 1]$, $t_1 + t_2 + \cdots + t_n = 1$.

By Jensen's inequality,

$$exp(\sum_{i=1}^{n} t_ix_i) \le \sum_{i=1}^{n} t_ie^{x_i} = \sum_{i=1}^{n} t_ia_i$$

$$a_1^{t_1}a_2^{t_2}\cdots a_n^{t_n} = \exp(t_1x_1 \cdot \exp(t_2x_2)\cdots \exp(t_nx_n) \le \sum_{i=1}^{n} t_ia_i$$

** Generalized AM-GM inequality. Let $t_i = \frac{1}{n}$, $\sqrt[n]{a_1a_2\cdots a_n} \le \frac{a_1 \cdot a_2 \cdots a_n}{n}$.

Example 2: Let $0 < s < t$**,** $a - 1, \cdots, a_n \ge 0$

Claim: $(\frac{1}{n}\sum_{i=1}^{n} a_i^s)^{\frac{1}{s}} \le (\frac{1}{n}\sum_{i=1}^{n} a_i^t)^{\frac{1}{t}}$

Let

$f(x) = x^{t/s}\ x \ge 0$

$f'(x) = \frac{t}{s} x^{t/s-1} \ x \geq 0$

$f''(x) = (\frac{t}{s})(\frac{t}{s} - 1) x^{t/s-2} \ x > 0$

Therefore, f is strictly convex.

Let $x_i = a_i^s$, $1 \leq i \leq n$, let $t_i = \frac{1}{n}$, by Jensen's Inequality,

$$
\begin{aligned}
f(\frac{1}{n}\sum x_i) &\leq \frac{1}{n}\sum f(x_i) \\
(\frac{1}{n}\sum a_i^s)^{\frac{t}{s}} &\leq (\frac{1}{n}\sum a_i^s)^{\frac{t}{s}} && \text{(take } t\text{th root.)} \\
(\frac{1}{n}\sum a_i^s)^{\frac{1}{s}} &\leq (\frac{1}{n}\sum a_i^t)^{\frac{1}{t}}
\end{aligned}
$$

Given positive real nuimbers $a_1, a_2, \cdots, a_n$, what is the n-gon of greatest area with sides $a_1, \cdots, a_n$?

need: $a_j < \sum_{i \neq j} a_i$ for $1 \leq j \leq n$.

$n = 4$,

4.4 Continuity, Differentiality, and Limits

Theorem 4.4.1. ***Darboux's Theorem: IVT for derivatives***
If f is differentiable on (a,b), and $x,y \in (a,b)$, $f'(x) < L < f'(y)$, then $\exists z \in (x,y)$, such that $f'(z) = L$.

Proof. Assignment 8.

□

Theorem 4.4.2. ***Cauchy's Mean Value Theorem*** *Suppose f, $g : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, differentiable on (a,b), then $\exists x_0 \in (a,b)$ s.t.*

$$\frac{f(a)-f(b)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

Proof. If $\exists x_1, x_2$, $g'(x_1) < 0 < g'(x_2)$, then by Darboux's Theorem, $\exists y \in (x_1, x_2)$ with $g'(y) = 0$

$\therefore$ sign $(g'(x))$ is constant.

positive $\implies$ monotone(strictly) increasing

negative $\implies$ monotone(strictly) decreasing

$\therefore g(b) \neq g(a)$.

Let $h(x) = (f(b) - f(a)) \cdot g(x) - (g(a) - g(b))f(x)$,

h is constant on $[a,b]$, differentiable on $[a,b]$

$h(a) = f(b)g(a) - f(a)g(a) - g(b)f(a) + g(a)f(a) = f(b)g(a) - g(b)f(a)$

$h(b) = g(a) + f(b) - f(a)g(b) = h(a)$,

By Rolle's Theorem, $\exists x_0 \in (a_1, b_1)$, $0 = h'(x_0) = (f(b) - f(a))g'(x_0) - (g(b) - g(a))f'(x_0)$

solve to get the equality we want.

□

Theorem 4.4.3. ***L'Hopital's Rule***
Let f, g be differentiable functions on an interval J with c at one endpoint (allow $\pm\infty$).

Suppose:

1. $g(x) \neq 0$ *and* $g'(x) \neq 0$
2. $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = 0$ *or* $\lim\limits_{x\to c} |g(x)| = \infty$
3. $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)} = L$ *exists (finite)*

Then, $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = L$.

Proof.

<u>Case 1:</u> $c \in \mathbb{R}$, $\lim\limits_{x\to c} f(x) = 0 = \lim\limits_{x\to c} g(x)$.

We define $f(c) = 0 = g(c)$, then f, g are continuous on $J \cup \{c\}$. Let $\varepsilon > 0$. Use 3 to find $\delta > 0$, $c < x < c + \delta$. (or $c - \delta < x < c$).

then $\left|\frac{f'(x)}{g'(x)} - L\right| < \epsilon$

Apply CMVT to $[c, x]$, then

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(x_0)}{g'(x_0)}$$

for some $c < x_0 < x$.

If $c < x < c + \delta$, then

$$\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f'(x_0)}{g(x_0)} - L\right| < \epsilon$$

i.e.

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L$$

<u>Case 2:</u> $c \in \mathbb{R}$, $\lim\limits_{x \to \infty} |g(x)| = \infty$.

Let $\varepsilon > 0$. Find $\delta > 0$. If $c < x < c + \delta \Rightarrow \left|\frac{f'(x)}{g'(x)} - L\right| < \epsilon$.

Let $c < x < y < c + \delta$, apply CMVT on $[x, y]$

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(x_0)}{g'(x_0)} \in (L - \varepsilon, L + \varepsilon)$$

divide numerator and denominator by $g(x) \neq 0$.

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{\frac{f(x)}{g(x)} - \frac{f(y)}{g(x)}}{1 - \frac{g(y)}{g(x)}} \approx \frac{f(x)}{g(x)}$$

if $|g(x)|$ is large enough, so if x is close enough to c, then

$$\frac{\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)}}{1 - \frac{g(y)}{g(x)}} - \frac{f(x)}{g(x)} = \frac{f(x)}{g(x)} \left[\frac{1 - (1 - \frac{g(y)}{g(x)})}{1 - \frac{g(y)}{g(x)}}\right] - \frac{f(y)}{g(x) - g(y)}$$

$\therefore \lim \dfrac{f(x)}{g(x)} = L$.

<u>Case 3, 4:</u> $c = \pm\infty$, let $F(n) = f(\frac{1}{u})$, $G(u) = g(\frac{1}{u})$.

$\lim\limits_{x \to o^{\pm}} F(u) = \lim\limits_{x \to \pm\infty} f(x) = 0$ or ∞.

$\lim\limits_{x \to o^{\pm}} G(u) = \lim\limits_{x \to \pm\infty} g(x)b = 0$ or ∞.

$G(u) = g(\frac{1}{u}) \neq 0$.

$G'(u) = \frac{-1}{u^2} g'(u) = \neq 0$.

$$\lim_{u \to 0^\pm} \frac{F'(u)}{G'(u)} = \lim_{u \to \pm} \frac{-\frac{1}{u^2} f'(\frac{1}{u})}{-\frac{1}{u^2} g'(\frac{1}{u})} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L$$

$$\therefore \lim_{u \to 0^\pm} \frac{F(u)}{G(u)} = L = \lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$$

□

Example 1: $a > 0$,

$$\lim_{x \to a^+} \frac{\sqrt{x} + \sqrt{x-a} - \sqrt{a}}{\sqrt{x^2 - a^2}} = \lim_{x \to a^+} \frac{\dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{x-a}}}{\dfrac{2x}{2\sqrt{x^2-a^2}}} = \lim_{x \to a^+} \frac{(\sqrt{x-a} + \sqrt{x})\sqrt{x^2 - a^2}}{x 2\sqrt{x}\sqrt{x-a}}$$

$$= \frac{(0 + \sqrt{a})\sqrt{2a}}{2a\sqrt{a}} = \frac{1}{\sqrt{2a}}$$

or

$$\begin{aligned}
\lim_{x \to a^+} \frac{\sqrt{x} + \sqrt{x-a} - \sqrt{a}}{\sqrt{x^2 - a^2}} &= \lim_{x \to a^+} \frac{\sqrt{x-a}}{\sqrt{x^2 - a^2}} + \frac{\sqrt{x} - \sqrt{a}}{\sqrt{x^2 - a^2}} \\
&= \lim_{x \to a^+} \frac{1}{\sqrt{x+1}} + \frac{x-a}{\sqrt{(x-a)(x+a)}} \cdot \frac{1}{\sqrt{x} + \sqrt{a}} \\
&= \lim_{x \to a^+} \frac{1}{\sqrt{x+a}} + \frac{\sqrt{x-a}}{\sqrt{x+a} \cdot (\sqrt{x} + \sqrt{a})}
\end{aligned}$$

Example 2: $\lim_{x \to 0} (\frac{\tan x}{x})^{1/x^2}$

Take log,

$$\ln((\frac{\tan x}{x})^{1/x^2}) = \frac{1}{x^2} \ln(\frac{\tan x}{x}) = \frac{\ln \tan x - \ln x}{x^2}$$

$\lim_{x \to 0^+} \frac{\tan x}{x} = 1$, so $\lim_{x \to 0^+} \ln \frac{\tan x}{x} = 0$.

$$\frac{\ln \tan x - \ln x}{x^2} = \lim_{x \to 0^+} \frac{\frac{1}{\tan x} \sec^2 x - \frac{1}{x}}{2x} = \lim_{x \to 0^+} \frac{\frac{1}{\sin x \cos x} - \frac{1}{x}}{2x} = \lim_{x \to 0^+} \frac{x - \sin x \cos x}{2x^2 - \sin x \cos x}$$

$$= \lim_{x \to 0^+} \frac{1 - \cos^2 x + \sin^2 x}{4x \sin x \cos x + 2x^2 \cos^2 x - 2x^2 \sin^2 x} = \lim_{x \to 0^+} \frac{2 \sin^2 x}{2x \sin 2x + 2x^2 \cos x}$$

$$= \lim_{x \to 0^+} (\frac{\sin x}{x})^2 \cdot (\frac{1}{\frac{\sin 2x}{x} + \cos 2x}) = 1^2 \cdot \frac{1}{2+1} = \frac{1}{3}$$

Example 3:

$$\lim_{x\to\infty}\frac{x-\sin x}{x+\sin x}=\lim_{x\to\infty}\frac{1-\frac{\sin x}{x}}{1+\frac{\sin x}{x}}=1$$

Very Special Case of L'Hopital's Rule

Suppose f, g have continuous derivatives on $[a,b]$ and $f(c)=g(c)=0$, and $g'(c)\neq 0$,

then by continuity $\exists\varepsilon>0$, $|x-c|<\varepsilon\Rightarrow|g'(x)-g'(c)|<\frac{|g'(c)|}{2}$

$\therefore\ sign(g'(x))=sign(g'(c))\in\{\pm 1\}$ is constant

$\therefore\ g$ is strictly monotone on $[c,c+\varepsilon]\Rightarrow g(x)\neq 0$ on $(c,c+\varepsilon]$ and $g'(x)\neq 0$ on $(c,c+\varepsilon]$

$\lim\limits_{x\to c^+}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to c^+}\dfrac{f'(x)}{g'(x)}=\dfrac{f'(c)}{g'(c)}$ (** $\frac{0}{0}$ situation $g,g'\neq 0$ on $(c,c+\varepsilon]$)

Explanation:

1^{st} order derivatives

$f(x)=T(x)+\varepsilon(x)(x-c)=(f'(c)+\varepsilon(x))(x-c)$, $\lim\limits_{x\to c^+}\varepsilon(x)=0$, $T(x)=f(c)+f'(c)(x-c)=f'(c)(x-c)$.

Similarly, $g(x)=(g'(c)+\delta(x))(x-c)$ $\lim\limits_{x\to c^+}\delta(x)=0$.

$\lim\limits_{x\to c^+}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to c^+}\dfrac{(f'(c)+\varepsilon(x))(x-c)}{g'(c)+\delta(x)(x-c)}=\dfrac{f'(c)}{g'(c)}$.

Often 1 differentiable on LHR is not enough.

Example 1:

$$\begin{aligned}\lim_{x\to 0}\frac{\sinh x-\sin x}{x^3}&=\lim_{x\to 0}\frac{\cosh x-\cos x}{x^3} && \text{(LHR)}\\ &=\lim_{x\to 0}\frac{\sinh x+\sin x}{6x} && \text{(LHR)}\\ &=\lim_{x\to 0}\frac{\cosh x+\cos x}{6} && \text{(LHR)}\end{aligned}$$

Example 2:

$$\lim_{x\to 0}\frac{\frac{1}{3}x^3-x^4}{x^3}=\lim_{x\to 0}\frac{x^2-4x^3}{3x^2}=\lim_{x\to 0}\frac{2x-12x^2}{6x}=\lim_{x\to 0}\frac{2-24x}{6}=\frac{1}{3}$$

Example 3: Putnam Type Problem

f is differentiable on $(0,\infty)$, $\lim\limits_{x\to\infty}f(x)+f'(x)=a$.

Find $\lim_{x\to\infty} f(x)$.

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{xf(x)}{x} \\ &= \lim_{x\to\infty} \frac{f(x)+xf'(x)}{1} && \text{(LHR fail)}\end{aligned}$$

Different Way:

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{e^x f(x)}{e^x} \\ &= \lim_{x\to\infty} \frac{e^x f(x)+e^x f'(x)}{e^x} && (\tfrac{\infty}{\infty},\ e^x \neq 0) \\ &= \lim_{x\to\infty} f(x)+f'(x) = a\end{aligned}$$

4.5 Big O and Little o

Big O and Little o Notation:

Say that $f(x)$ is $O(g(x))$ as $x \to a$ (or $x \to \infty$ or $x \to -\infty$)

if $\exists \delta > 0$, $C < \infty$, such that $\left|\frac{f(x)}{g(x)}\right| \leq C$ if $|x-a| < \delta$.

Say that $f(x)$ is $o(g(x))$ as $x \to a$ if $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = 0$.

If $\left|f^{(n+1)}(x)\right| \leq C < \infty$, then Taylor's Theorem states that $f(x) - P_{n,a}(x) = O((x-a)^{n+1})$

$$|f(x) - P_{n,a}(x)| = \left|\frac{f^{(n+1)}(x_0)(x-a)^{n+1}}{(n+1)!}\right| \leq \frac{C}{(n+1)!}\,|x-a|^{n+1}$$

Big O Arithmetic:

as $x \to 0$ $O(x^2) + O(x^3) = O(x^2)$.

Let $f(x) = O(x^2)$, $g(x) = O(x^3)$, $|f(x)| \leq c_1 O^2$, $|g(x)| \leq c_2 O(x^3)$,

$$|f(x) + g(x)| \leq c_1 x^2 + c_2 x^3 \leq (c_1 + c_2) x^2 \qquad when\,|x| < 1$$

4.6 Taylor Polynomial:

Definition 4.6.1. *If $f'(x)$ has n derivative f', $f^{(2)}$, $f^{(3)}$, $\cdots$, $f^{(n)}$, the Taylor Polynomial of degree n at $x = a$ is*

$$P_{n,a}(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f^{(3)}(x)}{3!}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=0}^{n}\frac{f^{(k)}(x)}{k!}(x-a)^k$$

<u>Convention:</u> $f^{(0)}(x) = f(x)$

Lemma 4.6.1. *$P_{n,a}^{(k)}(a) = f^{(k)}(a)$ for $0 \leq k \leq n$.*

Proof. Write $P_{n,a}(x) = \sum_{i=1}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$

$g = (x-a)^2$

$g' = i(x-a)^{i-1}$

$g^{(2)} = i(i-1)(x-a)^{-2}$

$\vdots$

$g^{(k)} = i(i-1)(i-2) + \cdots + (i+1-k)(x-a)^{i-k} = 0$ for $k > i$.

$$P_{n,a}^{(k)}(x) = \sum_{i=k}^{n}\frac{f^{(i)}(a)}{i!}i(i-1)\cdots(i+1-k)(x-a)^{i-k}$$

$$P_{n,a}^{(k)}(a) = \frac{f^k(a)}{k!}k(k-1)\cdots 1(x-a)^0 + \sum_{i=k}^{n}\frac{f^{(i)}(a)}{i!}i(i-1)\cdots(i+1-k)(a-a)^{i-k} = f^{(k)}(a)$$

□

$P_{n,a}(x)$ is like a higher order tangent line

$T(x) = f(a) + f'(a)(x-a) = P_{1,a}(x)$ $1st$ Taylor Polynomial.

Theorem 4.6.1. *Taylor's Theorem*
Suppose that $f : [a,b] \to \mathbb{R}$ has $n+1$ derivatives and let $P_{n,a}(x)$ be the Taylor polynomial of degree n at $x = a$.

If $x \in (a,b)$, there is $x_0 \in (a,x)$ such that

$$f(x) - P_{n,a}(x) = \frac{f^{(n+1)}(x_0)\cdot(x-a)^{n+1}}{(n+1)!}$$

Proof.
Let

$R(x) = f(x) - P_{n,a}(x)$

$R^{(k)}(a) = f^{(k)}(a) - P_{n,a}^{(k)}(a) = 0$ for $0 \leq k \leq n$

Idea: apply Cauchy MVT with $g(x) = (x-a)^{n+1}$

$$\begin{aligned}
\frac{R(x)}{(x-a)^{n+1}} &= \frac{R(x)-R(a)}{g(x)-g(a)} \\
&= \frac{R'(x_1)}{g'(x_1)} && (a < x_1 < x) \\
&= \frac{R'(x_1)}{(n+1)(x_1-a)^n} \\
&= \frac{R'(x_1)-R'(a)}{g'(x_1)-g'(a)} \\
&= \frac{R^{(2)}(x_2)}{g^{(2)}(x_2)} && (a < x_2 < x_1 < x) \\
&= \frac{R^{(2)}(x_2)}{(n+1)n(x_2-1)^{n-1}} \\
&= \frac{R^{(3)}(x_3)}{g^{(3)}(x)} && (a < x_3 < x_2 < x_1 < x)
\end{aligned}$$

$$\begin{aligned}
g &= (x-a)^{n+1} \\
g' &= (n+1)(x-a)^n \\
g'' &= (n+1)n(x-a)^{n-1} \\
g''' &= (n+2)(n+1)n(x-a)^{n-2}
\end{aligned}$$

repeat n times, at nth stage,

$$\begin{aligned}
\frac{R(x)}{(x-a)^{n+1}} &= \frac{R^{(n)}(x_n)-R^{(n)}(a)}{g^{(n)}(x_n)-g^{(n)}(a)} && (a < x_n < \cdots < x_1 < x) \\
&= \frac{R^{(n+1)}(x_{n+1})}{g^{(n+1)}(x_{n+1})} \\
&= \frac{f^{n+1}(x_{n+1})-0}{(n+1)!}
\end{aligned}$$

$$\therefore R(x) = \frac{f^{(n+1)}(x_{n+1})}{(n+1)!}(x-a)^{n+1} \qquad (a < x_{n+1} < x)$$

□

Example 1: $f(x) = e^x$, $a = 0$,

$$P_{n,0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}}{k!}(x-0)^k = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$$

Taylor's Thm says:

$$\left| e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right| = \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-0)^{n+1}$$

Try $x = 1$,

$$\left| e - \sum_{k=0}^{n} \frac{1}{k!} \right| = \frac{e^{x_0} \cdot 1^{n+1}}{(n+1)!} < \frac{e}{(n+1)!} < \frac{3}{(n+1)!}$$

Take $x = 13$,

$$\left| e - (1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{13!}) \right| < \frac{3}{14!} < 4 \cdot 10^{-11}$$

<u>BETTER:</u>

$$\left| e^{\frac{1}{16}} - \sum_{k=0}^{n} \frac{1}{k! 16^k} \right| < \frac{e^{\frac{1}{16}}}{11!} \cdot (\frac{1}{16})^{11} < 1.6 \cdot 10^{-21}$$

$$\left| e - (((a^2)^2)^2)^2 \right| < e^{\frac{15}{16}} < 7 \cdot 10^{-20}$$

$$e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \leq e^{x_0} \frac{|x|^{n+1}}{(n+1)!} \leq \max\{1, e^x\} \frac{|x|^{n+1}}{(n+1)!}$$

$$\lim_{x \to \infty} \frac{|x|^{n+1}}{(n+1)!} = \frac{|x|^{n_0}}{(n_0 + 1)}$$

Example 2: $f(x) = \sin x$

$f'(x) = \cos x = f^{(4n+1)}(x) \qquad f''(x) = -\sin x = f^{(4n+2)}(x)$

$f''(x) = -\cos x = f^{(4n+3)}(x) \qquad f'''(x) = \sin x = f^{(4n)}(x)$

$$P_{2n,0}(x) = \sum_{k=1}^{2n} \frac{f^{(k)}(0)}{k!} \cdot (x - 0)^k = \frac{1}{1!} x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots + \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!}$$

$$\left| \sin x - \sum_{k=0}^{n-1} \frac{(-1)^k \cdot x^{2k+1}}{(2k+1)!} \right| < \frac{|f^{2n+1}(x_n)|}{(2n+1)!} \left| (x - 0)^{2n+1} \right|$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{x}^{2k+1} (2k+1)! = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

$\sin(31°) = \sin(\frac{\pi}{6} + \frac{\pi}{1}80)$

$$\begin{aligned} P_{n,\frac{\pi}{6}}(x) &= \frac{\sin \frac{\pi}{6}}{0!} \cdot 1 + \frac{\cos \frac{\pi}{6}}{1!}(x - \frac{\pi}{6}) + \frac{-\sin \frac{\pi}{6}}{2!}(x - \frac{\pi}{6})^2 - \frac{\cos \frac{\pi}{6}}{3!}(x - \frac{\pi}{6})^3 + \cdots + \frac{?}{n!}(x - \frac{\pi}{6})^n \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) - \frac{1}{4}(x - \frac{\pi}{6})^2 - \frac{\sqrt{3}}{12}(x - \frac{\pi}{6})^3 + \cdots \end{aligned}$$

$$P_{n,\frac{\pi}{6}}(\frac{\pi}{6} + \frac{\pi}{180}) = \frac{1}{2} + \frac{\sqrt{3}\pi}{360} - \frac{\pi^2}{4(180)^2} - \frac{\sqrt{3}\pi^3}{12(180)^3} + \cdots$$

$$\left| \sin(\frac{\pi}{6} + \frac{\pi}{180}) - P_{n,\frac{\pi}{6}}(\frac{\pi}{6} + \frac{\pi}{180}) \right| < \left| \frac{f^{(n+1)(x_0)}}{(n+1)!} (\frac{\pi}{180})^{n+1} \right| < \frac{1}{(n+1)!} (\frac{1}{50})^{n+1}$$

When $n = 3$, $\frac{1}{24} \cdot \frac{1}{50^4} < 5 \cdot 10^{-4}$.

Exmaple 3: $\cos x$

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \\ &= \underbrace{1 - \frac{x^2}{2!}}_{P_{3,0}(x)} + \underbrace{O(x^4)}_{|R(x)| = \left|\frac{\cos x_0}{25} x^4\right|}\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0} \frac{\sec x - \cos x}{x^2} &= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2 \cos x} = \lim_{x\to 0} \frac{1-(1-x^2+O(x^4))^2}{x^2(1-\frac{x^2}{2}+O(x^4))} \\ &= \lim_{x\to 0} \frac{1-(1-x^2+O(x^4))}{x^2 - \frac{x^4}{4} + O(x^6)} \\ &= \lim_{x\to 0} \frac{x^2+O(x^4)}{x^2+O(x^4)} \\ &= 1\end{aligned}$$

Proposition 4.6.1. *If $f \in C^{(n)}[a-\delta, a+\delta]$ and $P(x)$ is a polynomial, deg $P \le n$, and $\lim_{x\to a} \frac{f(x)-f(a)}{(x-a)^n} = o$ ($f(x) = P(x) + o(x-a)^n$).*

then $P_{n,a}(x) = P(x)$.

Proof. proof 1: Suppose $f \in C^{n+1}[a-\delta, a+\delta]$, then by Taylor's Thm, $f(x) = P_{n,a}(x) + O((x-a)^{n+1})$. Then

$$\lim_{x\to a} \frac{f(x)-P_{n,a}(x)}{(x-a)^n} = \lim_{x\to a} \frac{O((x-a)^{n+1})}{(x-a)^n} = \lim_{x\to a} O(x-a) = 0$$

If we don't know about $f^{(n+1)}(x)$, instead use L'Hôpital's Rule

$$\lim_{x\to a} \frac{f(x)-P_{n,a}(x)}{(x-a)^n} = \lim_{x\to a} \frac{f'(x)-P'_{n,a}(x)}{n(x-a)^{n-1}} = \cdots = \lim_{x\to a} \frac{f^{(n)} - P^{(n)}_{n,a}(x)}{n!} = 0$$

If $\lim_{x\to a} \frac{f(x)-P(x)}{(x-a)^n} = 0 \Rightarrow \lim_{x\to a} \frac{P_{n,a}(x)-P(x)}{(x-a)^n} = 0$.

$P_{n,a}(x) - P(x) = \sum_{k=0}^{n} (\frac{f^{(k)}(a)}{k!} - bk)(x-a)^k$.

If $P_{n,a}(x) \ne P(x)$, $\exists$ smallest K_0, $\frac{f^{(k_0)}(a)}{k!} - b_{k_0} \ne 0$.

$$\lim_{x\to a} \frac{P_{n,a}(x)-P(x)}{(x-a)^n} = \lim_{x\to a} \frac{C_{k_0}\cdot(x-a)^{k_0} + h.o.t}{(x-a)^n} = \lim_{x\to a} \frac{C_{k_0} + h.o.t.}{(x-a)^{n-k_0} \to 0} = \begin{cases} \pm\infty, if k_0 < n \\ C_n, of k_0 \ge n \end{cases}$$

□

Example 1:

$$\begin{aligned} f(x) &= \tan x & f(0) &= 0 \\ f'(x) &= \sec^2 x & f'(0) &= 1 \\ f''(x) &= 2\sec^2 x \tan x & f''(0) &= 0 \\ f'''(x) &= 4\sec^2 x \tan^2 x + 2\sec^4 x & f'''(0) &= 2 \end{aligned}$$

since

$\sin x = x - \frac{x^3}{6} + \frac{x^5}{5!} + O(x^7)$

$\cos x = x - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)$

$$\therefore \tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{120} + O(x^7)}{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)} \text{Nov 18 14:01} = x + \frac{x^3}{3} + \frac{2}{15}x^5 + O(x^7)$$

Example 2:

$$\begin{aligned} \lim_{x\to 0} \cot^2 x - \frac{1}{x^2} &= \lim_{x\to 0} \frac{1}{\tan^2 x} - \frac{1}{x^2} \\ &= \lim_{x\to 0} \frac{1}{(x + \frac{x^3}{3} + O(x^5))^2} - \frac{1}{x^2} \\ &= \lim_{x\to 0} \frac{1}{x^2 + (1 + \frac{x^2}{3} + O(x^4))^2} - \frac{1}{x^2} \\ &= \lim_{x\to 0} \frac{1 - (1 + \frac{x^3}{3} + O(x^4))^2}{x^2(1 + \frac{x^2}{3} + O(x^4))^2} \\ &= \lim_{x\to 0} \frac{1 - (1 + \frac{2x^3}{3} + O(x^4))^2}{x^2(1 + \frac{x^2}{3} + O(x^4))^2} \\ &= \lim_{x\to 0} \frac{-\frac{2}{3}x^2 + O(x^4)}{x^2(1 + O(x^2))} \\ &= \lim_{x\to 0} \frac{-2}{3} \cdot \frac{1 + O(x^2)}{1 + O(x^2)} = \frac{-2}{3} \end{aligned}$$

Example 3: $f(x) = \tan^{-1} x$

$$f'(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \cdots = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

$$f'(x) - \underbrace{\sum_{k=0}^{n} (-1)^k x^{2k}}_{P_{2n+1,0}(x)} = \frac{(-1)^n x^{2n+2}}{1+x^2} = O(x^{2n+2})$$

<u>Claim:</u> Taylor Polynomial for $\arctan x$ is

$$\underbrace{0}_{\arctan 0} + x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} = Q_{2n+1,0}(x)$$

$$
\begin{aligned}
\arctan x - \underbrace{Q_{2n+1,0}(x)}_{TP of \deg 2n+2} &= (f(x) - Q_{2n+1,0}(x)) - (f(0) - Q_{2n+1,0}(0)) \\
&= (f'(x) - Q'_{2n+1,0}(x)) - (x - 0)) \\
&= (\frac{1}{1+x^2} - P_{2n+1,0}(x))x = O(x^{2n+2})x = O(x^{2n+3})
\end{aligned}
$$

pic: Nov 18, 14:14; Nov 18, 14:17

Example 4: $\lim\limits_{x\to 0} \dfrac{(1+x)^{1/x} - e}{x}$

$(1+x)^{1/x} = \exp(\frac{\ln(1+x)}{x})$

$e^x = 1 + x + \frac{x^2}{2!} + O(x^2)$

$f(x) = \ln(1+x) \qquad f(0) = 0$

$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2} \qquad f''(0) = -1$

$\ln(1+x) = o + x - \frac{x^2}{2!} + O(x^3)$

$$
\begin{aligned}
(1+x)^{\frac{1}{x}} &= e^{\frac{\ln(1+x)}{x}} \\
&= e^{\frac{x - \frac{x^2}{2} + O(x^3)}{2}} \\
&= e^{1 - \frac{x}{2} + O(x^2)} \\
&= e \cdot e^{-x/2 + O(x^2)} \\
&= e(1 + (-\frac{2}{x} + O(x^2)) + \frac{(-\frac{x}{2} + O(x^3))^2}{2!} + O((\quad)^3)) \\
&= e(1 - \frac{x}{2} + O(x^2))
\end{aligned}
$$

$$
\begin{aligned}
\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x} &= \lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x} \\
&= \lim_{x\to 0} \frac{e - \frac{e}{2} \cdot x + O(x^2) - 3}{x} \\
&= \lim_{x\to 0} -\frac{e}{2} + O(x) = \frac{-e}{2}
\end{aligned}
$$

5 Other

5.1 Alternating Sequence

Definition 5.1.1. *Say a series of the form* $\pm \sum_{k=0}^{\infty}(-1)^k a_k$ *is an alternating series, if* $a_0 \geq a_1 \geq a_2 \geq \cdots$

Theorem 5.1.1. *Alternating series converges* $\iff$ $a_n = 0$. *(this is special to this type of series)*

If $\sum_{k=0}^{\infty}(-1)^k a_k = L$, *then* $\left|L - \sum_{k=0}^{n}(-1)^k a_n\right| \leq a_{k+1}$.

Proof. $\sum_{k=0}^{\infty}(-1)^k a_k$ converges means $s_n = \sum_{k=0}^{n} a_k$ is a convergent sequence.

$$s_1 = a_0 - a_2 \leq 3_4 = s_2 - a_3 = s_1 + (a_2 - a_3)$$

$$\begin{aligned} s_2 = a_0 - (a_1 - a_2) \leq s_0 = a_0 = s_q 1 + a_2 \geq a_1 \\ = s_1 + a_2 \geq s_1 \end{aligned}$$

Claim: $s_1 \leq s_3 \leq s_{2n-1} \leq s_{2n-2} \leq \cdots \leq s_2 \leq s_0$.

For $n = 1$, $s_1 \leq s_0$

for $n = 2$, $s_1 \leq s_3 \leq s_2 \leq s_0$

Assume true for n,

$$\begin{aligned} s_{2n} &= s_{2n+1} + (-1)^{2n} a_{2n} \geq s_{2n-1} \\ &= s_{2n-2} - a_{2n-1} + a_{2n} = s_{2n-2} - (a_{2n-1} - a_n) \leq s_{2n-2} \end{aligned}$$

$$\begin{aligned} s_{2n+1} &= s_{2n} - a_{2n+1} \leq s_{2n} \\ &= s_{2n-1} + (a_{2n} - a_{2n+1}) \geq s_{2n-1} \end{aligned}$$

By induction, therefore true. $s_1 \leq s_3 \leq s_5 \leq \cdots \leq s_4 \leq s_2 \leq s_0$

By the MCT, $\lim\limits_{n\to\infty} s_{2n-1} = L$ exists, $s_2 \geq s_4 \geq s_{2n+2}$ decreasing sequence and bounded below by any x_{2n-1}.

$\therefore MCT \Rightarrow \lim\limits_{exists} s_{2n} = M$,

$M - L = \lim\limits_{n\to\infty} s_{2n} - s_{2n-1} = \lim\limits_{n\to\infty} a_{2n}$

if $\lim\limits_{n\to\infty} a_n = 0 \Rightarrow L = M \Rightarrow \lim\limits_{n\to\infty} s_n = L$

conversely, if limit exists, then $L = M$ so $\lim a_n = 0$ ◻

$s_{2n+1} \leq L \leq s_{2n}$

If k is even, $k = 2n$, $s_{2n} - a_{2n+1} \leq L \leq s_n \Rightarrow |L - s_{2n}| \leq a_{2n+1} = a_{k+1}$.

If k is odd, $k = 2n = 1$, $s_{2n+1} \leq L \leq s_{2n+2} = s_{2n+1} + a_{2n+2}$.

$\therefore |L - s_{2n+1}| \leq a_{2n+2} = a_{k+1}$.

Example 1:

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + - \qquad \text{Alternating}$$

$|x| \le \frac{\pi}{2}$ this is an alternating sequence.

need:

$$\begin{aligned} && \frac{|x|^{2n+1}}{(2n+1)!} \le \frac{|x|^{2n+1}}{(2n-1)!} \\ \iff && x^2 \le (2n)(2n+1) \\ \iff && x^2 \le 2 \cdot 3 = 6 \\ && |x| \le \sqrt{6} \end{aligned}$$

Example 2:

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$

alternating when $0 \le x \le 1$, not when $x < 0$.

$$e^x = 1 + x + \frac{x^2}{x} + \frac{x^3}{x} + \cdots$$

Example 3: *** Graph Nov 20 14:10 *** let $f(x) = \Big\{ e^{-1/x^2}, x \neq 00, x = 0$

Claim:

1. f is C^∞ (all derivatives are continuous).
2. $f^{(k)}(0) = 0$ for all $k \ge 0$.

Taylor Polynomial of degree n is $P_{n,a}(x) = \sum_{k=0}^{k} \frac{f^{(k)}(0)}{k!}(x-0)^k = 0$.

$$\begin{aligned} \lim_{x \to 0} \frac{f(x)}{x^{2n}} &= \lim_{x \to \infty} \frac{e^{-1/x^2}}{x^{2n}} \\ &= \lim_{x \to \infty} \frac{e^{-u}}{(\frac{1}{u})^n} \\ &= \lim_{x \to \infty} \frac{u^n}{e^u} = 0 \end{aligned}$$

$|f(x) - 0| = O(x)^{2n} \Rightarrow 0 = P_{2n-1,0}(x)$.

$f'(x) = \frac{2e^{-1/x62}}{x^3}$

Claim:

$$f^{(n)}(x) = \frac{q_n(x) e^{-1/x^2}}{x^{3n}} \qquad \deg q_n \le 2n$$

True for $n = 0$ and $n = 1$, assume true for n, then

$$\begin{aligned} f^{(n+1)}(x) &= e^{-1/x^2} \left(\frac{q_n'}{x^{3n}} + \frac{q_n \frac{2}{x^3}}{x^{3n}} - \frac{3nq_n}{x^{3n+1}} \right) \\ &= \frac{e^{-1/x^2}}{x^{3n+3}} (x^3 q_n' + 2q_n - 3x^2 q_n) \end{aligned}$$

$$\lim_{x\to 0} f^{(n)}(x) = \lim_{x\to 0} \frac{q_n(x)e^{-\frac{1}{x^2}}}{x^{3n+2}} \tag{1}$$

$$= \lim_{x\to 0} \sum_{k=0}^{2n} a_i \frac{e^{-\frac{1}{x^2}}}{x^{3n+3-i}} \tag{2}$$

$$= \sum_{k=0}^{2n} a_i(0) = 0 \tag{3}$$

$$f^{(n+1)}(0) = \lim_{h\to 0} \frac{f^{(h)} - 0}{h} = \lim_{x\to 0} \frac{q_n(h)\cdot e^{-\frac{1}{h^2}}}{h^{3n+4}} = 0$$

5.2 Newton's Method

__Need:__

1. f is c^2,
2. $\exists x_*,\ f(x_*) = 0$, and $f'(x_*) \neq 0$.
3. Need to start "close" enough

__Idea:__

Pick x_0 starting point, solve for the zero of the tangent line at x_0.

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

Solve $0 = f(x) + f'(x_0)(x - x_0)$

$$x - x_0 = \frac{f(x_0)}{f'(x_0)}$$
$$\therefore x_1 = x_0 - \frac{x_0}{f'(x_0)}$$
$$\text{Repeat} \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = T(x_n)$$

Define $T(x) = x - \frac{f(x)}{f'(x)}$

Note $T(x_*) = x_* - \frac{f(x_*)}{f'(x_*)} = x_*$ fixed point of T.

$$T'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$$

$$x_{n+1} - x_* = T_{x_n} - T_{x_*} \overset{MVT}{=} T'(c)(x_n - x_*) \qquad \text{(c between } x_n \text{ and } x_*)$$

$T'(x)$ is continuous near x_*, so $\exists \epsilon > 0$ such that $|T'(x)| \leq \frac{1}{2}$ on $[x_0 - \epsilon, x_0 + \epsilon]$.

If $|x_0 - x_*| \leq \epsilon$, then

$|x_1 - x_*| \leq \frac{x_0 - x_*}{2} \Rightarrow x_1 \in [x_* - \epsilon, x^* + \epsilon]$

$|x_2 - x_*| \leq \frac{x_1 - x_*}{2} \leq \frac{1}{4}|x_0 - x_*|$

$|x_n - x_*| \leq \frac{x_0 - x_*}{2^n}$

$\therefore$, $x_n \to x_*$.

Let $m = \min\limits_{a \leq x \leq b} |f'(x)|$ where $x_* \in [a, b]$ and $f'(x) \neq 0$ on $[a, b]$; and let $C = \max\limits_{a \leq x \leq b} |f''(x)|$

For $x \in [a.b]$, $f(x) = f(x) - f(x_*) = f'(c)(x - x_*)$ c between x and x_0 .

$$\therefore\ |x - x_*| = \left|\frac{f(x)}{f'(c)}\right| \leq \frac{|f(x)|}{m}$$

$$
\begin{aligned}
x_{n+1} - x_* &= x_{n+1} - x_n + x_n - x_* \\
&= -\frac{f(x_n)}{f'(x_n)} + \frac{f(x_n)}{f'(c)} \\
&= \frac{f(x_n)f'(x_n) - f(x_n)f'(c)}{f'(x_n)f'(c)} \\
&= \frac{f(x_n)(f'(x_n) - f'(c))}{f'(x_n)f'(c)} && d \text{ between } x_n \text{ and c} \\
& && d \text{ between } x_n \text{ and } x_* \\
&\overset{MVT}{=} \frac{f(x_n)}{f'(x_c)} \frac{f''(d)(x_n - c)}{f'(n)} \\
&= (x_n - x_*)(x_n - c)\frac{f''(d)}{f'(s_n)} \\
&\leq |x_n - x_*|\, |x_n - x_*| \, \frac{C}{m}
\end{aligned}
$$

$$
\therefore |x_{n+1} - x_*| \leq \frac{C}{m} |x_n - x_*|^2
$$

Quadratic Convergence

Once $|x_n - x_*|$ is sufficiently small, this goes to zero very fast, almost double a numbered accurate decimal each step.

Example 1: Sqaure root, $\sqrt{149}$, let $f(x) = x^2 - 149$, $f'(x) = 2x$

$$
\begin{aligned}
T(x) &= x - \frac{f(x)}{f'(x)} \\
&= x - \frac{x^2 - 149}{2x} \\
&= x - \frac{x}{2} + \frac{149}{2x}
\end{aligned}
$$

$$
T(x) = \frac{x + \frac{149}{2}}{2}
$$

Try $x_0 = 12$, $x_1 = 12.2083$. work on $[12, 13]$,

$C = \sup f''(x) = 2$, $m = \min f'(x) = 24$.

$$
|x_* - 12| = \left| \frac{f(12)}{} \right| \leq \frac{5}{m} = \frac{5}{24} \approx 0.21
$$

$$
|x_1 - x_*| \leq \frac{1}{12}(0.21)^2 < (3.7)10^{-3}
$$

$x_2 = 12.2065557$

$|x_2 - x_*| \leq \frac{1}{12}(3.7 \cdot 10^{-3})^2 < 1.2 \times 10^{-6}$.

$|x_3 - x_*| \leq \frac{1}{12}(1.2 \times 10^{-6})^2 < 4 \cdot 10^{-13}$

Example 2:

$f(x) = (x-r)^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}(x-r)^{\frac{-2}{3}}$ $f'(r)$ is undefined.

$$x_{n+1} = x_n - \frac{(x-r)^{\frac{1}{3}}}{\frac{1}{3}(x-r)^{\frac{2}{3}}} = x_n - 3(x_n - r) = 3r - 2x_n$$

$$|x_{n=1} - r| = |2r - 2x_n| = 2\,|x_n - r|$$

Example 3: $f(\pm 1) = \pm 1$ and $f'(\pm 1) = 2$.

$f(x) = x^5 - \frac{3}{2}x^3 + \frac{3}{2}x$

5.3 Uniform Continuity

Definition 5.3.1. *$f : [a,b] \to \mathbb{R}$ is continuous if $\forall x \in [a,b]$, $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.*

Definition 5.3.2. *$f : [a,b] \to \mathbb{R}$ is uniformly continuous if $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall x, y \in [a,b]$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.*

Example 1: Assume that f is C^1 on $[a,b]$, since $f'(x)$ is continuous, EVT, $M = \max_{a \leq x \leq b} |f'(x)| < \infty$.

By MVT

$$\left| \frac{f(y) - f(x)}{y - x} \right| = |f'(c)| \leq M$$

$$\underbrace{\therefore |f(y) - f(x)| \leq M |y - x|}_{\textit{Lipschitz Condition}}$$

$C \in (x, y)$.

Let $\varepsilon > 0$ be given, let $\delta = \frac{\varepsilon}{M}$,

$$|y - x| < \delta \Rightarrow |f(y) - f(x)| \leq M |y - x| < M\delta = \varepsilon$$

Example 2:

Let $f(x) = x^2$ on $(-\infty, \infty)$,

$f'(x) = 2x \to \pm\infty$ as $x \to \infty$

$y^2 - x^2 = (y - x)(y + x)$

If $1 \geq \varepsilon > 0$, let $\delta_n = \frac{1}{n}$, $n \in \mathbb{N}$.

Let $x = \frac{n}{\varepsilon}$, $y = \frac{n}{\epsilon} + \frac{1}{n+1}$. $|y - x| = \frac{1}{n+1} < \epsilon_n$.

$$y^2 - x^2 = (y - x)(y + x) = \frac{1}{n+1}(\frac{2n}{\varepsilon} + \frac{1}{n+1}) > \frac{1}{\epsilon} \geq 1$$

so $\delta = \frac{1}{n}$ does not work in definition of uniform continuity. for any n.

Therefore f is not uniformly conitnuous on $\mathbb{R}$.

Example 3: $f(x) = \sqrt{x}$ on $[0, \infty)$.

$0 \leq x \ < y$,

$$\sqrt{y} - \sqrt{x} = \frac{y - x}{\sqrt{y} + \sqrt{x}}$$

Let $\varepsilon > 0$ be given.

If $0 \leq x,\, y \leq \epsilon^2 \Rightarrow 0 \leq \sqrt{x}, \sqrt{y} \leq \varepsilon$.

$$\left| \sqrt{y} - \sqrt{x} \right| \leq \epsilon$$

If $\frac{\epsilon^2}{2} \leq x, y$, then

$$\left|\sqrt{y} - \sqrt{x}\right| = \frac{|y-x|}{\sqrt{y}-\sqrt{x}} < \frac{|y-x|}{2\sqrt{\frac{\epsilon^2}{2}}} = \frac{|y-x|}{\sqrt{2}\varepsilon}$$

Let $\delta = \frac{\varepsilon^2}{2}$, $|x-y| < \delta = \frac{\varepsilon^2}{2}$.

Case 1: Either $x \leq \frac{\varepsilon^2}{2}$ or $y \leq \frac{\varepsilon^2}{2} \Rightarrow 0 \leq x, y \leq \varepsilon^2$.

$\therefore \left|\sqrt{x} - \sqrt{y}\right| < \varepsilon$

Case 2: $x > \frac{ep^2}{2}$, $y > \frac{ep^2}{2}$.

$$\therefore \left|\sqrt{x} - \sqrt{y}\right| < \frac{|y-x|}{\sqrt{2}\varepsilon} < \frac{\varepsilon^2}{2\sqrt{2}\varepsilon} < \varepsilon$$

Example 4: $f(x) = \frac{1}{x}$ on $(0, \infty)$

$1 \geq \varepsilon > 0$, for any $\delta > 0$, let $x = \min\{\varepsilon, \delta, 1\}$, $y = \frac{x}{10}$,

$x - y = \frac{9}{10}x < \delta$.

$\frac{1}{y} - \frac{1}{x} = \frac{10}{x} - \frac{1}{x} = \frac{9}{x} \geq 9$

On $[1, \infty]$, $|f'(x)| = \left|\frac{-1}{x^2}\right| \leq 1$ Lipschitz.

$\therefore$ uniformly continuous. $|f(x) - f(y)| \leq M\,|y-x|$.

Theorem 5.3.1. *If $f(x)$ is continuous on a closed bounded interval $[a, b]$, then f is uniformly continuous.*

Proof. Let $\varepsilon > 0$.

For $\delta = \frac{1}{n}$, definition fails.

Then $\exists x_n, y_n \in [a, b]$ with $|x_n - y_n| < \frac{1}{n}$, but $|f(x_n) - f(y_n)| \geq \varepsilon$.

Apply the Bolzano-Weierstrass Theorem to sequence x_n.

Get a convergent subsequence $(x_{n_1}, x_{n_2}, \cdots,)$, $\lim\limits_{k \to 0} x_{n_k} = x_0$ exists.

$\lim\limits_{k \to \infty} y_{n_k} = \lim\limits_{k \to \infty} x_{n_k} + (y_{n_k} - x_{n_k}) = x_0$.

f is continuous at x_0.

$f(x_0) = \lim\limits_{k \to \infty} f(x_{n_k}) = \lim\limits_{k \to 0} f(y_{n_k})$

$0 = \lim\limits_{x \to \infty} |f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$. Contradiction.

$\therefore$ there is a $\delta > 0$ which works. $\square$

Example 5:

$f(x) = \left\{ x \sin \frac{1}{x}, x \neq 0 0, x = 0 \right.$.

$\lim_{x\to 0} f(x) = 0 = f(0)$ squeeze theorem

f is differentiable (continuous) at x if $x \neq 0$.

f is uniformly continuous on $[-1,1]$ by Thm.

$f'(x) = \sin\frac{1}{x} + x\frac{-1}{x^2}\cos\frac{1}{x} = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}$

On $[\frac{1}{2},\infty] \cup (-\infty,-\frac{1}{2})$, $|f'(x)| \leq 1 + 2\cdot 1 = 3$.

$\therefore$ f is uniformly continuous on $(-\infty,-\frac{1}{2}) \cup (\frac{1}{2},\infty)$.

Let $\varepsilon > 0$. If $|x| < \frac{\varepsilon}{2}$, $|y| < \frac{\varepsilon}{2} \Rightarrow |f(x) - f(y)| \leq |f(x)| + |(y)| \leq |x| + |y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

if $|x| \geq \frac{\varepsilon}{4}$, $|y| \geq \frac{\varepsilon}{4}$,

$$|f'(x)| = \left|\sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}\right| \leq 1 + \frac{1}{|x|} \leq 1 + \frac{4}{\varepsilon} < \frac{5}{\varepsilon}$$

$$|x-y| < \frac{\varepsilon}{\varepsilon/5} = \frac{\varepsilon^2}{5} \Rightarrow |f(x) - f(y)| \leq \frac{5}{\varepsilon}|x-y| < \varepsilon$$

$\delta = \min\{\frac{\varepsilon^2}{5}, \frac{\varepsilon}{4}\}$

<u>Case 1:</u> $|x| < \frac{ep}{4}$ or $|y| < \frac{\varepsilon}{4} \Rightarrow |y| \leq |x| + |y-x| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} \Rightarrow |f(x) - f(y)| < \varepsilon$

<u>Case 2:</u> $|x| \geq \frac{\varepsilon}{4}$, $|y| \geq \frac{\varepsilon}{4}$

5.4 Tutorial Exam Review

Example 1:

Let $f : [a,b] \to \mathbb{R}$ be cont. f diff on (a,b), let $c \in (a,b)$, so $f'(c)$ exists.

T/F Do there exist $a < x < c < y < b$ so that $\frac{f(x)-f(y)}{x-y} = f'(c)$?

Answer: false.

If $f'(c) = 0$. e.g. $f(x) = x^3$ then $f'(0) = 0$

Definition 5.4.1.