CHAPTER 3
Experiencing Meanings in Geometry

David W. Henderson and Daina Taimina

What geometrician or arithmetician could fail to take pleasure in the symmetries, correspondences, and principles of order observed in visible things? Consider, even, the case of pictures: those seeing by the bodily sense the products of the art of painting do not see the one thing in the one only way; they are deeply stirred by recognizing in the objects depicted to the eyes the presentation of what lies in the idea, and so are called to recollection of the truth – the very experience out of which Love rises. (Plotinus, *The Enneads,* II.9.16; 1991, p. 129)

In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward *abstraction* seeks to crystallize the *logical* relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward *intuitive understanding* fosters a more immediate grasp of the objects one studies, a live *rapport* with them, so to speak, which stresses the concrete meaning of their relations.

As to geometry, in particular, the abstract tendency has here led to the magnificent systematic theories of Algebraic Geometry, of Riemannian Geometry, and of Topology; these theories make extensive use of abstract reasoning and symbolic calculation in the sense of algebra. Notwithstanding this, it is still as true today as it ever was that intuitive understanding plays a major role in geometry. And such concrete intuition is of great value not only for the research worker, but also for anyone who wishes to study and appreciate the results of research in geometry. (David Hilbert, in Hilbert and Cohn-Vossen, 1932/1983, p. iii; *italics in original*)

It's a thing that non-mathematicians don't realize. Mathematics is actually an aesthetic subject almost entirely. (John Conway, in Spencer, 2001, p. 165)

The artist and scientist both live within and play active roles in constructing human mental and physical landscapes. That they should share structural intuitions is less surprising than inevitable. What is surprising and wonderful is how these intuitions have manifested themselves in the works of innovative artists and scientists in culturally apposite ways. (Kemp, 2000, p. 7)

The authors quoted above all stress the importance of the deep experience of meanings. It is these experiences in geometry (and indeed in all of mathematics, as well as in art and engineering) that we believe deserve to be called *aesthetic experiences.* Mathematics is a natural and deep part of human experience and experiences of meaning in mathematics should be accessible to everyone. Much of mathematics is not accessible through formal approaches except to those with specialized learning. However, through the use of non-formal experience and geometric imagery, many levels of meaning in mathematics can be opened up in a way that most people can experience and find intellectually challenging and stimulating.

A formal proof, as we normally conceive of it, is not the goal of mathematics—it is a tool, a means to an end. The goal is to understand meanings. Without understanding, we will never be satisfied—with understanding, we want to expand the meanings and to communicate them to others (see also Thurston, 1994). Many formal aspects of mathematics have now been mechanized and this mechanization is widely available on personal computers or even on hand-held calculators, but the experience of meaning in mathematics is still a human enterprise. Experiencing meanings is vital for anyone who wishes to understand mathematics or anyone wanting to understand something in their experience by means of the vehicle of mathematics. We observe in ourselves and in our students that such experiencing of meaning is, at its core, an aesthetic experience.

In this chapter, we recount some stories of our experience of meanings in geometry and art. David's story starts with art and ends with geometry, while Daina's story starts with geometry and ends with art. However, the bulk of what follows we both share.

David's Story: from Art to Mathematics

I have always loved geometry and have been thinking about geometric kinds of things ever since I was very young, as evidenced by a drawing I made when I was six years old (see Figure 1).

The drawing is of a cat drawing a picture of a cat (who is presumably drawing a picture of a cat …). Notice the perspective from the point of view of the cat—for example, the drawing shows the underside of the table. I was already experiencing geometric meanings.

But I did not realize then that the geometry that I experienced was mathematics or even that it was called 'geometry'. I did not call it 'geometry' —I called it 'drawing' or 'design' or perhaps failed to call it anything at all and just did it. I did not like mathematics in school, because it seemed very dead to me—just memorizing techniques for computing things and I was not very good at memorizing. I especially did not like my high-school geometry course, with its formal, two-column proofs.

However, I kept on doing geometry in various forms: in art classes, in carpentry, by woodcarving, when out exploring nature or by becoming

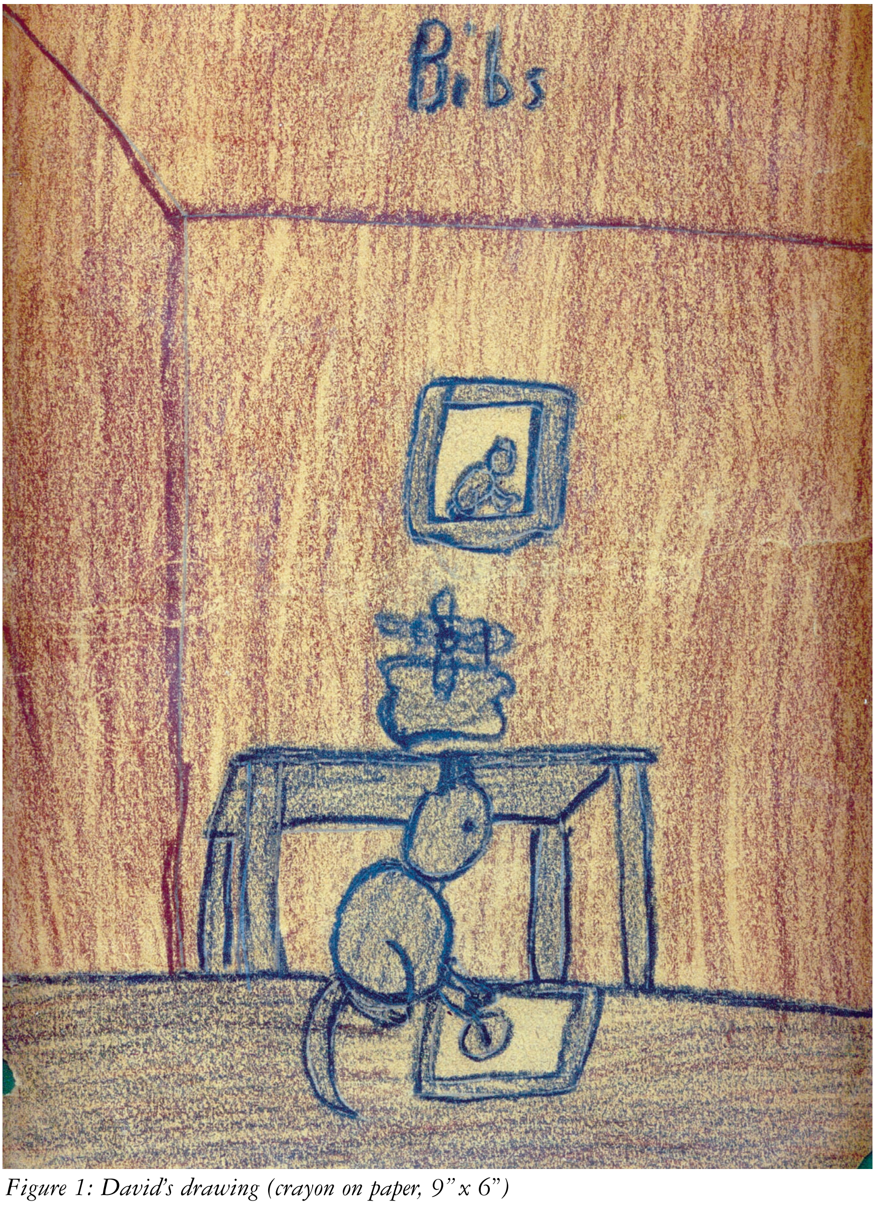

Figure 1: David's drawing (crayon on paper, 9" x 6")

involved in photography. This continued on into university where I became a joint physics and philosophy major, taking only those mathematics courses that were required for physics majors. I became absorbed by the geometry-based aspects of physics: mechanics, optics, electricity and magnetism, and relativity. On the other hand, my first mathematics research paper (on the geometry of Venn diagrams with more than four classes) evolved from a university course on the philosophy of logic. There were no geometry courses except for analytic geometry and linear algebra, which only lightly touched on anything geometric. So, it was not until my fourth and final year at the university that I switched into mathematics and I only did so then because I was finally convinced that the geometry that I loved really was a part of mathematics.

Since high school, I have never taken a course in geometry, because there were no geometry courses offered at the two universities I attended. Now I am a professional geometer and I started teaching an undergraduate Euclidean geometry course in the mid-1970s. My concern that both my students and I should experience meaning in the geometry quickly led me into conflict with traditional, formal approaches.

Daina's Story: from Mathematics to Art

I took a lot of geometry, both in grade school and at the university. But I only had a very few art lessons in school. From them, I developed the impression that I could not draw and that I had little artistic talent. But I liked geometry precisely for its aesthetic values. My mathematics teachers always paid a lot of attention to how we drew geometric diagrams; they encouraged Euclidean constructions with compass and straight-edge, but also supported the free-hand drawing of geometric figures, while insisting on accurate shapes and proportions. At university, besides other traditional geometry courses, I also took a course in descriptive geometry, as well as a short course on how to draw three-dimensional geometric diagrams—both of these latter courses contained a lot about perspective. I always enjoyed and excelled at the drawing aspects of geometry, but I did not think it had anything to do with art or aesthetic sensibilities.

When teaching the history of mathematics, I was particularly interested in the history of geometry and, because of my interest in art appreciation and art history, was happy to find so many connections between geometry and art. I was fascinated with the golden ratio, with the story of projective geometry arising from painters' perspective *prior* to it becoming a pure mathematical subject and with the considerable impact of mathematics on art in the twentieth century (for example, in cubism and, later, in the work of M. C. Escher). I was also teaching a university course on 'the psychology of mathematical thinking', which led me to wonder about all creative thinking.

I have had many students in my mathematics classes tell me that they were taking my class just to fulfill a distribution requirement. But they would also assert that they were no good at mathematics, because they are artists (poets, musicians, actors, painters) and their thinking is different. This made me wonder: is creative thinking really different in its very essence? So I decided as an experiment to take a watercolor class, knowing that I had never been any good at art. I wanted to get a glimpse of the emotions one goes through as a student in a subject for which one has no talent. I started the watercolor class not really understanding what techniques I should use for my brush, how to mix colors and other such technical details. But then I realized it was *only* the techniques I did not know.

I found that my aesthetic experiences with drawing in geometry gave me a feel for how to use my skill at geometric drawing in painting. Ideas of composition and perspective in painting are all so geometrical. I enjoyed reading books about composition and perspective, as well as finding out how much I already knew from my earlier geometry studies. Proportions (the golden ratio, particularly) and shapes are directly related to composition, but I had to learn about the use of colors. For perspective drawings, I already knew from three-dimensional geometric drawing how to draw in linear perspective, but I had to learn how to create an atmospheric perspective. It was crucial for me to find out that I had had similar experiences already—albeit ones obtained in different ways and for different purposes.

Below in Figure 2 is the painting I did after attending only eight watercolor classes. I started it in class and later the same day finished it at home because I could not stop. When it was dry, I looked at it and could not believe I had painted it.

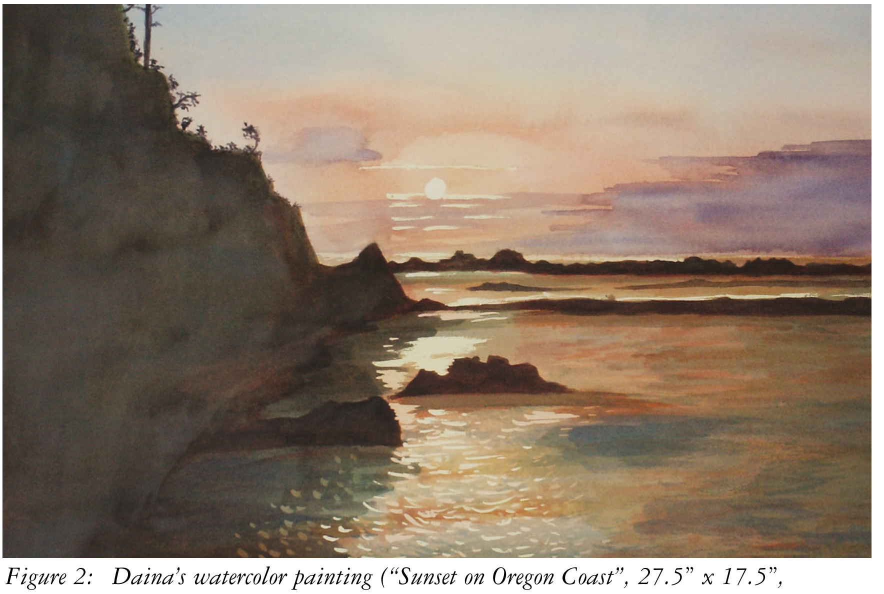

Figure 2: Daina's watercolor painting ("Sunset on Oregon Coast", 27.5" x 17.5", photograph by Daina Taimina)

Experiencing 'Undefined' Terms

In geometry, 'point' and 'straight line' are usually referred to as "undefined terms". In a formal sense, something has to be undefined, because it is impossible to define everything without being circular. However, if we want to pay attention to meanings in geometry, then we must still ask what is the meaning of 'point' and what is the meaning of 'straight'? The standard formal approach of saying these are undefined terms pushes these questions away under the carpet.

What is the meaning of 'point'?

Euclid has one answer—according to Heath's (1926/1956) translation of *The Elements,* "A point is that which has no parts" (p. 153). This is one meaning of 'point'. 'Point' has another meaning in geometry and mathematics that can be experienced by imagining zooming in on the point. A Tibetan monk/artist/geometer explained this to one of us by saying:

> Imagine a poppy seed. Now imagine in this poppy seed a temple and in the middle of the temple a Buddha and in the navel of the Buddha another poppy seed. Now in that poppy seed imagine a temple and in the temple a Buddha and in the navel of the Buddha another poppy seed. Now in that poppy seed imagine … (and keep going). Where is the point?

As we write this, we notice some similarity between this zooming and ideas in David's picture of a cat drawing a picture of a cat … .

These meanings of 'point' are not the same and, thus, bring about the following question: why and how are these meanings related? This is a *why*-question that often confronts calculus students when looking at the meanings of 'tangent', 'limit' and the 'definite integral'.

What is the meaning of 'straight'?

This is the question that starts both of the geometry books that we have written (see Henderson with Taimina, 1998, 2001). Of course, whether a text or teacher allows this discussion or not, students (in fact, it appears, most human beings) have an experience of meanings of 'straight'. The meanings of 'straight' are part of the core foundation for meaning in geometry.

One common meaning for 'straight' is "shortest distance". This meaning can be used in practice to produce a straight line by stretching a string (or rubber band). There is another meaning in the realization that a straight line is very symmetric—for instance, "it does not turn or wiggle" or "in the plane, both sides are the same". Straight lines have at every point the following symmetries: reflection through the line, reflection perpendicular to the line, a half-turn about any point on the line, translation along the line, and so forth (see Figure 3).

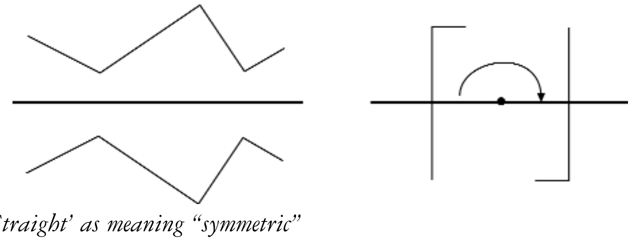

Figure 3: 'Straight' as meaning "symmetric"

This symmetry meaning is in line with Heath's (1926/1956) translation of Euclid's definition of straight line as "a line that lies evenly with the points on itself" (p. 153), which Heath then attempts to clarify in a footnote:

> we can safely say that the sort of idea which Euclid wished to express was that of a line [...] without any irregular or unsymmetrical feature distinguishing one part or side of it from another. (p. 167)

Using these experientially-based meanings of straightness, we can ask what are straight lines on the surface of a sphere. If we look at this question from a point of view outside of the sphere, then clearly the answer is that there are no straight lines on a sphere. This is the *extrinsic* point of view.

On the other hand, there is an *intrinsic* point of view. Imagine yourself to be a bug crawling on a sphere. The bug's universe is just the spherical surface. What paths on the sphere would the bug experience as straight? After some exploration, we can convince ourselves that the great circles on the sphere are the curves that have the same symmetries (with respect to the sphere) that ordinary straight lines have with respect to the plane. We thus say that the great circles are intrinsically straight. A much more usual approach in texts is simply to *define* straight lines on the sphere to be the great circles—but, again, this blocks contact with the meaning (and, thus, the potential for aesthetic experience).

So, again, why and in what way are these two meanings ("shortest" and "symmetric") related? On the sphere, we can see that (Figure 4), for two nearby points of the equator (a particular great circle), the shortest distance is along the equator. However, there is another straight path (in the sense of "symmetric") between the same two points that traverses the equator in the opposite direction (going the long way round). Thus, the "symmetric" meaning is not always the "shortest" meaning. In addition, there are surfaces with corners (see Figure 5) for which the shortest path is *not* symmetric.

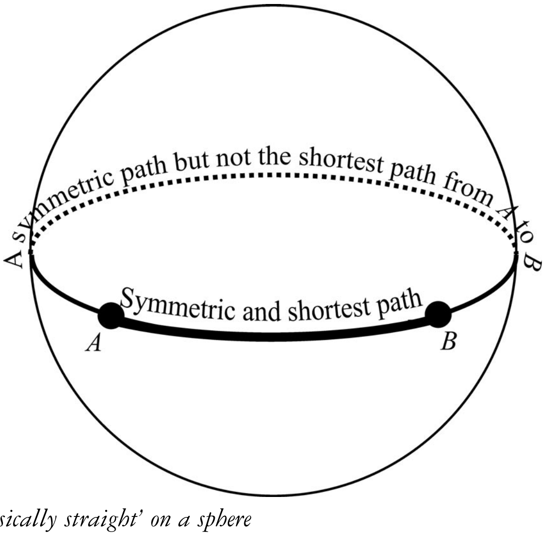

Figure 4: 'Intrinsically straight' on a sphere

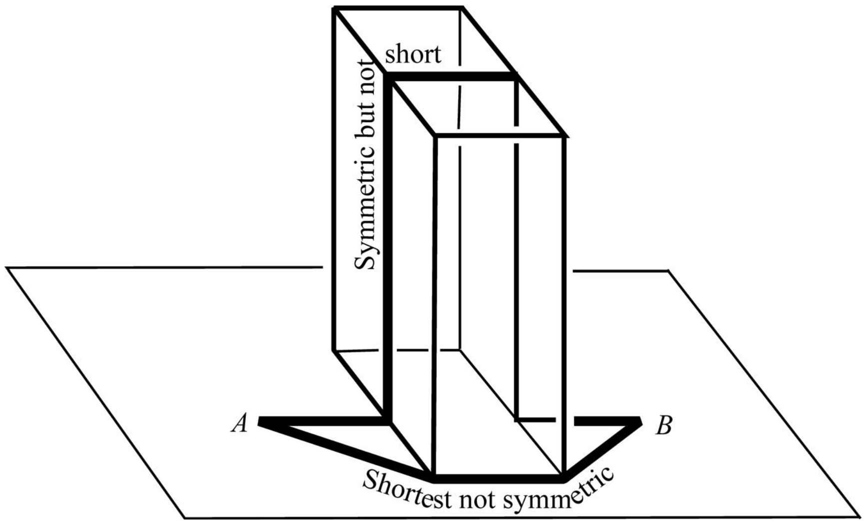

Figure 5: "Shortest" is not the same meaning as "symmetric"

A simple question that may seem intuitively straightforward at first glance, namely "what is the meaning of 'straight'?", reveals some deeper intuitions about symmetry and shortest distance, which may only become meaningful when explored in different geometrical contexts.

Proofs as Convincing Communications that Answer the Question *Why?*

Much of our own view of the nature of mathematics is intertwined with our notion of what a proof is. This is particularly true with geometry, which has traditionally been taught in high school in the context of 'two-column' proofs (see Herbst, 2002). Instead, we propose a different view of proof as "a convincing communication that answers a *why*-question".

The book entitled *Proofs Without Words* (Nelsen, 1993) contains numerous examples of visual proofs that provide an experience of *why* something is true—a experience that is, in most cases, difficult to obtain from the usual formal proofs. For example, Nelsen writes about the following result, which is usually attributed to Galileo (1615) – see Drake (1970).

$$\frac{1 + 3 + \ldots + (2n - 1)}{(2n + 1) + (2n + 3) + \ldots + (4n - 1)} = \frac{1}{3}$$

We can easily check that this is true by simply adding the numbers.

$$\frac{1 + 3}{5 + 7} = \frac{1}{3} \qquad \frac{1 + 3 + 5}{7 + 9 + 11} = \frac{1}{3}$$

These are the cases $n = 2$ and $n = 3$ of the more general equality.

So the question is whether the general equation holds and, if so, why it holds? One way to answer the first question is to apply an argument by mathematical induction, though such an argument is unlikely to satisfy the *why*-question. Instead, look at Figure 6.

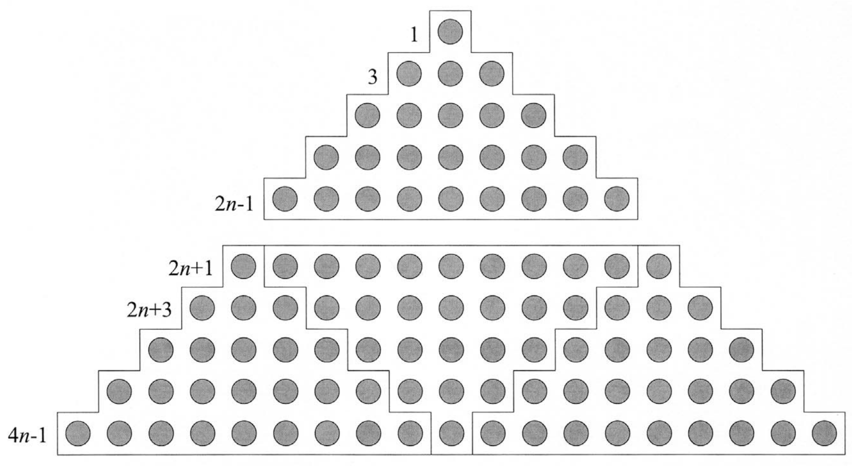

Figure 6: A proof without words (based on Nelsen, 1993, p. 115)

Through this picture, one can directly experience the meaning of Galileo's result and see both *that* is true and *why* it is true. The proof by induction would answer the question: how does Galileo's result follow from Peano's axioms? Most people (other than logicians) have little interest in that question.

> **Conclusion** In order for a proof to be an aesthetic experience for us, the proof must answer our *why*-question and relate our meanings of the concepts involved.

As further evidence toward this conclusion, many report the experience of reading a proof and following each step logically, but still not being satisfied because the proof did not lead them to experience the answers to their *why*-questions. In fact, most proofs in the literature are not written out in such a way that it is possible to follow each step in a logical, formal way. Even if they were so written, most proofs would be too long and too complicated for a person to check each step.

Furthermore, even among mathematics researchers, a formal logical proof that they can follow step-by-step is often not satisfying. For example, David's (1973) research paper ('A simplicial complex whose product with any ANR is a simplicial complex') has a very concise, simple (half-page) proof. This proof has provoked more questions from other mathematicians than any of his other research papers and most of the questions were of the sort: "Why is it true?", "Where did it come from?", "How did you see it?" They accepted the proof logically, yet were not satisfied.

Sometimes we have legitimate *why*-questions even with respect to statements traditionally accepted as axioms. One is Side-Angle-Side (or SAS):

> If two triangles have two sides and the included angle of one of them that are congruent to two sides and the included angle of the other, then the triangles themselves are congruent.

SAS is listed in some geometry textbooks as an axiom to be assumed; in others, it is listed as a theorem to be proved and in others still as a definition of the congruence of two triangles. But clearly one can ask: why is SAS true on the plane? This is especially true because SAS is false for (geodesic) triangles on the sphere. So naturally one can then ask: why is SAS true on the plane, but not on the sphere?

Here is another example – the vertical-angle theorem:

> If l and l′ are straight lines, then angle α is congruent to angle β.

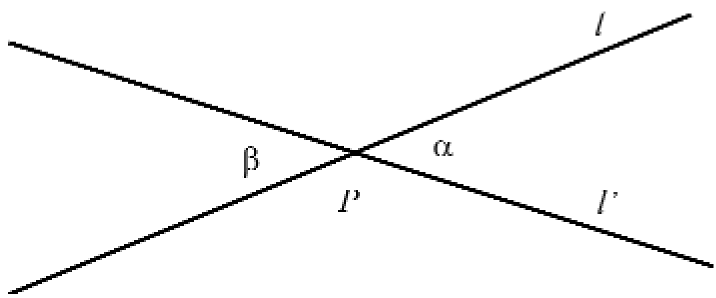

Figure 7: The vertical-angle theorem

The traditional proof of this in high-school geometry is to label the upper angle between α and β as γ, and then assert $\alpha + \gamma = 180^\circ$ and $\gamma + \beta = 180^\circ$. The usual proof then concludes that α is congruent to β because they are both equal to $180^\circ - \gamma$. This proof seems fine until one worries about whether the rules of arithmetic apply in this way to angles and their measures. The traditional solution in high school is to use several 'ruler and protractor' axioms to assert the properties needed. We do not know of anyone for whom this proof with the attendant axioms has aesthetic qualities (though it may be convincing). We do not usually perceive a proof as aesthetically pleasing when it is mostly repeating a list of axioms in a way that the meaning does not come through clearly. This proof seems to be an unnecessarily complicated answer to the question: why are vertical angles congruent to one another?

For about ten years of teaching this theorem in his geometry course, David was satisfied with the idea of this proof, though he managed to simplify and make more geometric the necessary assumptions contained in the 'ruler and protractor' axioms. But then one student suggested that the vertical angles were congruent because both lines had half-turn symmetry about their point of intersection, *P*. David's first reaction was that her argument could not possibly be a proof—it was too simple and did not involve everything in the standard proof. But she persisted patiently for several days and David's meanings deepened. Now her proof is much more convincing to him than the standard one, because it directly clarifies *why* the theorem is true.

Even more importantly, the meaning of the student's 'half-turn' proof is closer to the meaning in the statement of the theorem. To see this, look at the situation depicted in Figure 8.

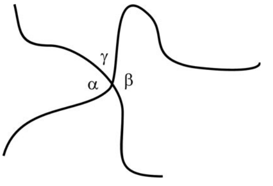

Figure 8: Are the opposite angles α *and* β *the same?*

Here, there is no symmetry: yet, the standard proof seems to apply and gives a misleading result. By means of either zooming in on the point of intersection until the curves are indistinguishable from straight-line segments (or by means of defining this angle to be the angle between the lines tangent to the curves at the intersection), symmetry arguments can be shown to apply and, hence, it is possible to argue that the angles α and β are congruent. However, the standard proof does not provide a way to discuss this, except by means of a discussion of when the 'ruler and protractor' axioms are valid.

One could ask:

> But, at least in plane geometry, isn't an angle an angle? Don't we all agree on what an angle is?

To which a reply could be:

> Well, yes and no.

Consider the acute angle depicted in Figure 9.

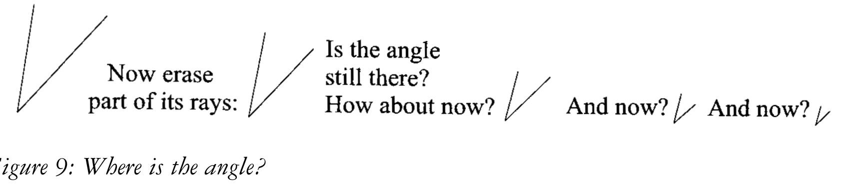

Figure 9: Where is the angle?

The angle is somehow *at the corner*, yet it is difficult to express this formally (note that the zooming meaning of 'point' seems to be involved here). As evidence of this difficulty, we have looked in all the plane geometry books in Cornell University's mathematics library for their definitions for 'angle'. We found nine different definitions. Each expressed a different meaning or aspect of 'angle' and, thus, each could potentially lead to a different proof for any theorem that crucially involves the meaning of 'angle'.

Experiencing the Hyperbolic Plane

Starting soon after Euclid's *Elements* were compiled (and continuing for the next 2000 years), mathematicians attempted either to prove Euclid's fifth postulate as a theorem (based on the other postulates) or to modify it in various ways. These attempts culminated around 1825 with Nicolai Lobachevsky and János Bolyai independently discovering a geometry that satisfies all of Euclid's postulates and common notions except that the fifth (parallel) postulate does not hold. It is this geometry that is called 'hyperbolic'. The first description of hyperbolic geometry was given in the context of Euclid's postulates and it was proved that all hyperbolic geometries are the same except for scale (in the same sense that all spheres are the same except for scale).

In the nineteenth century, mathematicians developed three so-called 'models' of hyperbolic geometry. During 1869-1871, Eugenio Beltrami and Felix Klein developed the first complete model of hyperbolic geometry (and were the first to call the specific geometry 'hyperbolic'). In the Beltrami–Klein model, the hyperbolic plane is represented by the interior of a circle, straight lines are (straight) chords of that circle and the circle's 'reflection' about a chord is a projective transformation that takes the circle to itself while still leaving the chord point-wise fixed.

Around 1880, Henri Poincaré developed two related models. In the Poincaré disc model, the hyperbolic plane is represented by the interior of a circle, with straight lines being circular arcs perpendicular to this circle. In the Poincaré upper-half-plane model, the hyperbolic plane is represented by half a plane on one side of a line, with straight lines being semi-circles that are perpendicular to this line. All three hyperbolic geometry models distort distances (in ways that are analytically describable), but the Beltrami–Klein model represents hyperbolic straight lines as Euclidean straight-line segments, while both of Poincaré's models represent angles accurately. For more details on these hyperbolic models, see Chapter 17 of Henderson and Taimina (2005).

These models of hyperbolic geometry have a definite aesthetic appeal, especially through the great variety of repeating patterns that are possible in the hyperbolic plane. The Dutch artist M. C. Escher used patterns based on these hyperbolic models in several well-known prints (see, for example, the one in Figure 10). Repeating patterns on the sphere have an aesthetic appeal through their simplicity and finiteness. However, in these various hyperbolic models, the patterns have an aesthetic appeal for us because of their connections with infinity—there are infinitely many such patterns and each also draws us to the infinity at the edge of the disc, leaving sufficient space for our imagination.

For more than a hundred and twenty-five years, these models have been very useful for studying hyperbolic geometry mathematically. However, many students and mathematicians (including the two of us) have desired a more direct experience of hyperbolic geometry—wishing for an

Figure 10: M. C. Escher's Circle Limit III *(based on the Poincaré disc model)*

experience similar to that of experiencing spherical geometry by means of handling a physical sphere. In other words, the experience of hyperbolic geometry available through the models did not directly include an experience of the *intrinsic* nature of hyperbolic geometry.

Mathematicians looked for surfaces that would posess the complete hyperbolic geometry, in the same sense that a sphere has the complete spherical geometry. A little earlier, in 1868, Beltrami had described a surface (called the 'pseudosphere', see Figure 11), which has hyperbolic geometry *locally*.

The pseudosphere also has a certain aesthetic appeal for us in the way (as with the Poincaré models) it points the imagination towards infinity. However, the pseudosphere allows only a very limited experience of hyperbolic geometry, because any patch on the surface that wraps around the surface or extends to the circular boundary does not have the geometry of any piece of the hyperbolic plane.

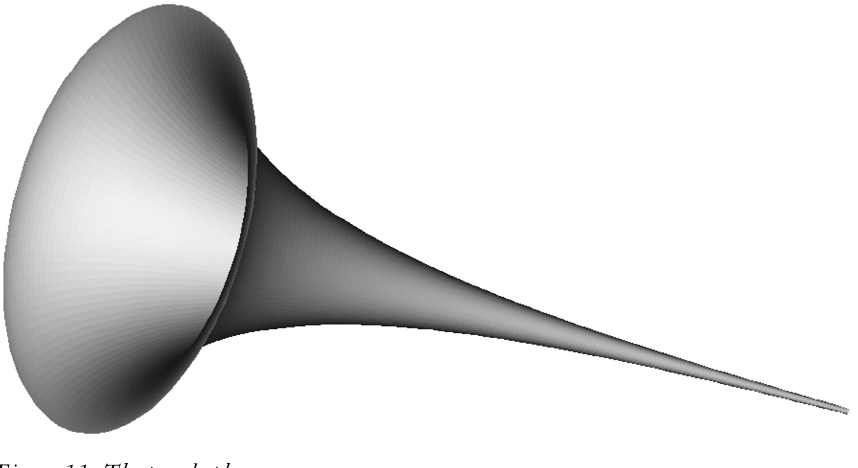

Figure 11: The pseudosphere

At the very beginning of the last century, David Hilbert (1901) proved that it is impossible to use real analytic equations to define a complete surface whose intrinsic geometry is the hyperbolic plane. In those days, 'surface' normally meant something defined by real analytic equations and so the search for a complete hyperbolic surface was abandoned. And N. V. Efimov (1964) extended Hilbert's result, by proving that there is no isometric embedding of the full hyperbolic plane into three-space, defined by functions whose first and second derivatives are continuous. Still, even today, many texts state incorrectly that a complete hyperbolic surface is impossible.

However, Nicolas Kuiper (1955) proved the existence of complete hyperbolic surfaces defined by continuously differentiable functions, although without giving an explicit construction. Then, in the 1970s, William Thurston described the construction of a surface (one that can be made out of identical paper annuli) that closely approximates a complete hyperbolic surface. (See Figure 12 and Thurston, 1997, pp. 49-50.) The actual hyperbolic plane is obtained by letting the width of the annular strips go to zero. In 1997, Daina worked out how to crochet the hyperbolic plane, following Thurston's annular construction idea. (See Figure 13.) Directions for constructing

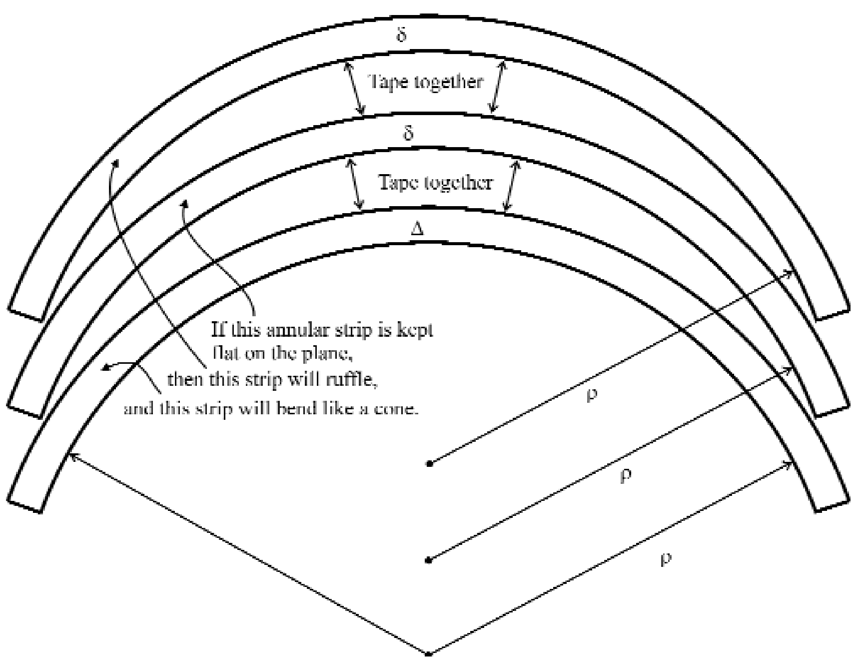

Figure 12: Construction of the annular hyperbolic plane

Thurston's surface out of paper or by crocheting can be found in Henderson and Taimina (2005) or in Henderson and Taimina (2001). In these refer-

Figure 13: A crocheted hyperbolic plane (made by Daina Taimina, photograph by David W. Henderson)

ences, there is also a description of an easily constructible polyhedral hyperbolic surface, called the 'hyperbolic soccer ball', comprising regular heptagons each surrounded by seven hexagons (the usual spherical soccer ball consists of regular pentagons each surrounded by five hexagons). This polyhedral surface was discovered by Keith Henderson (David's son) and provides a very accurate polyhedral approximation to the hyperbolic plane (see Figure 14).

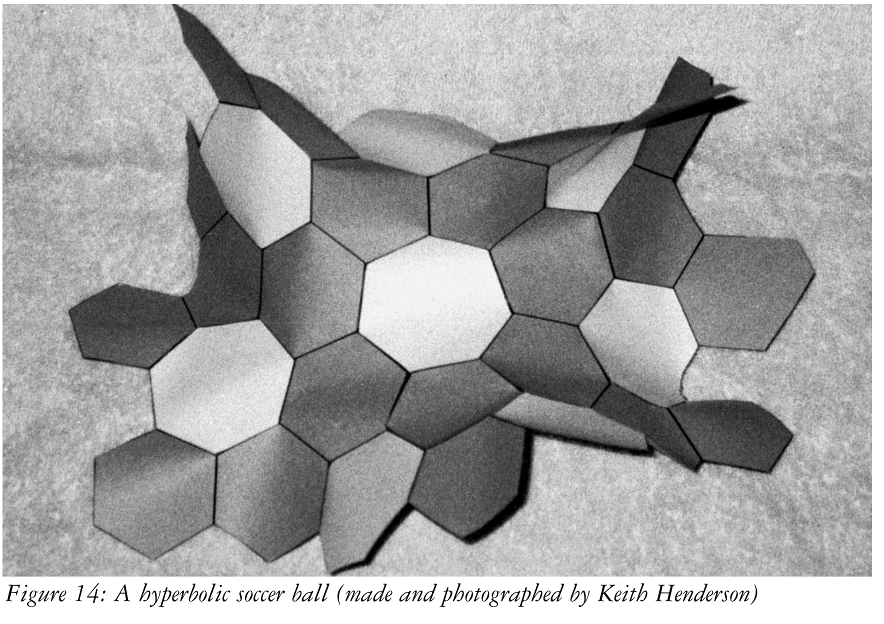

Figure 14: A hyperbolic soccer ball (made and photographed by Keith Henderson)

The geodesics ('intrinsic straight lines') on a hyperbolic surface can be found using the "symmetry" meaning of straightness discussed above: for example, the geodesics can be found by folding the surface (in the same way that folding a sheet of paper will produce a straight line on the paper). This folding also determines a reflection about that geodesic.

Now, by interacting with these surfaces, we can have a more direct experience of meanings in hyperbolic geometry. And, very importantly, we can experience the connections between these meanings and the three nineteenth-century models discussed above. These models can now be interpreted as projections (or maps) of the hyperbolic surface onto a region in the plane that distort the surface in a similar manner to the way projections (maps) of a sphere (such as the Earth) onto a region of the plane distort distances, areas and/or angles. This is important, because these models are used to study hyperbolic geometry in detail, while the surface itself allows us direct experience with the intrinsic geometry.

Before we had experience of these physical surfaces, our only experiences of hyperbolic geometry were through formal study with axiom systems and analytic study of the nineteenth-century models. The models provided aesthetic experiences that led our imagination to infinity, but this was not directly connected with geometric meanings. For example, the question that we (as well as most students) had was: why are geodesics in both Poincaré models represented by semi-circles or circular arcs?

To us, the nineteenth-century models were more like artistic abstractions. But, after constructing the surfaces, we could see how and why the geodesics are represented in the way they are. (See Henderson and Taimina, 2005, or Henderson and Taimina, 2001, for more details of these connections, including proofs that the intrinsic geometry of each of the surfaces is the same geometry as that represented by all of the models.)

Radius and curvature of the hyperbolic plane

Since all hyperbolic planes are the same up to scale, most treatments of the hyperbolic plane consider the curvature to be -1. It is very difficult to give meaning to the effects of the change of curvature without looking at actual physical hyperbolic surfaces with different curvatures. Each sphere has a radius r (which is extrinsic to the sphere) and its (Gaussian) curvature (as defined in differential geometry) is $1/r^2$. In a similar way, each hyperbolic plane has a radius r, which turns out to be the (*extrinsic*) radius of the annuli that go into Thurston's construction and the (Gaussian) curvature of the hyperbolic plane is $-1/r^2$. We were not aware of any meaning for the radius of a hyperbolic plane before experiencing these surfaces.

From a theoretical perspective, changing the radius or curvature is merely a change of scale and spheres, for example, of radii 4cm, 8cm and 16 cm look very much alike. However, we were shocked when we looked at the hyperbolic planes with these same radii (see Figures 15a, 15b and 15c, drawn with radii of 4 cm, 8 cm and 16 cm respectively).

Figure 15a-c: Hyperbolic planes with different radii (crocheted by Daina Taimina, photographed by David W. Henderson)

There is a felt difference that is not present in the spheres of the same radii (the main reason for this difference seems to come from the fact of exponential growth in the hyperbolic plane). This experience of the meaning of the radius of a hyperbolic plane was a profoundly aesthetic experience for us, because we were forced to look deeper mathematically into the meanings of both radius and curvature, as well as explore the local and global natures of the hyperbolic plane.

Ideal triangles

By exploring the possible shapes of large triangles on the hyperbolic surface (see Figure 16), we can see that they seem to become more and more the same shape as they become large. This leads on to the theorem (proved by using the models) that all *ideal* triangles (namely those with vertices at infinity) are congruent and have area equal to πr^2. (This is the same as the extrinsic area of the identical circles determined by the annuli in the construction.)

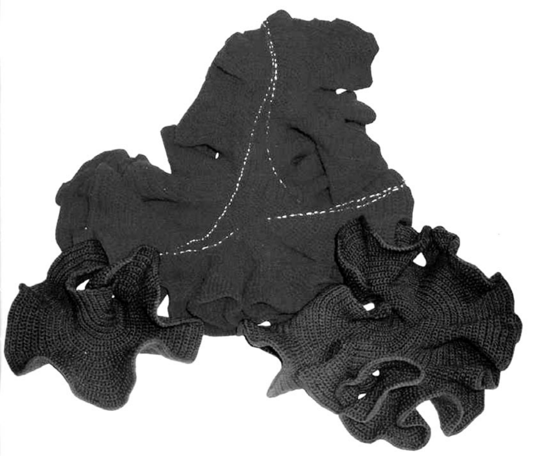

Figure 16: An ideal triangle on the hyperbolic plane (crocheted by Daina Taimina, photographed by David W. Henderson)

Horocycles or horocircles

By experiencing the annular construction (see Figure 12 once more), it is easy to see that curves perpendicular to the annuli (that is, curves that run in the radial direction) possess reflection symmetry and, thus, are geodesics. In addition, they are asymptotic to each other at infinity. Most treatments of hyperbolic geometry define *horocycles* as those curves that are orthogonal to a collection of asymptotic geodesics. Thus, the annuli (in the limit, as their width goes to zero) are horocycles. Both of us had studied hyperbolic geometry and its models; but exploring the hyperbolic surface was the first time we had experienced horocycles in a way that made clear their close connection with curvature and how, as many books simply assert, they can be described as circles with infinite (intrinsic) radii.

In the next section, we turn to look at the design of machines in the nineteenth century—at first sight, perhaps, a surprising leap. But in a curious way, these machines embody striking geometric principles and experiences in their design and the same questions we have been addressing (such as what is 'straight'?) reappear in exciting ways and, perhaps unexpectedly, horocycles reoccur once more.

Experiencing Geometry in Machines

Recently, we have been working on an NSF-funded project to examine the mathematics inherent in a collection of nineteenth-century mechanisms, as well as to see to the inclusion of these mechanisms (along with commentaries and learning modules) as part of the new National Science Digital Library (NSDL—see www.nsdl.org). Our experiences with these all of various mechanisms are offering us different perspectives on geometry, perspectives that arise from motion. For example, this work has brought us back to the question: what is 'straight'?

When using a compass to draw a circle, we are not starting with a figure we accept as circular: instead, we are using a fundamental property of circles, namely that the points on a circle are at a fixed distance from the center, as the basis for the tool. In other words, we are drawing on a mathematical definition of a circle. Is there a comparable tool (serving the equivalent role to a compass) that will draw a straight line? If, in this case, we want to use Euclid's definition ("a straight line is a line that lies evenly with the points on itself"), this will not be of much help.

One could say:

> We use a straight-edge for constructing a straight line.

To which a response might be:

> Well, how do you know that your straight-edge is straight? How do you know that anything is straight? How can you check that something is straight?

This question was important for James Watt. When he was thinking about improving steam engines, he needed a mechanism in order to convert circular motion into straight-line motion and *vice versa*. In 1784, Watt found a practical solution (which he called "parallel motion") that consisted of a linkage with six links. He described his parallel motion mechanism as being free of "untowardly frictions and other pieces of clumsiness", claiming it to be "one of the most ingenious simple pieces of mechanisms that I have contrived" (in Ferguson, 1962, p. 195). These expressions of smoothness and efficiency seem to be very close to what we are calling 'aesthetic'. However, Watt's mechanism produced only approximate straight-line motion: in fact, it actually produces a stretched-out figure of eight. Mathematicians were not satisfied with this approximate solution and worked for almost a hundred years to find exact solutions to the problem. A linkage that draws an exact straight line (see Figure 17a) was first reported by Peaucellier, in 1864. (See Henderson and Taimina, 2004 and 2005, for a discussion of relevant history.)

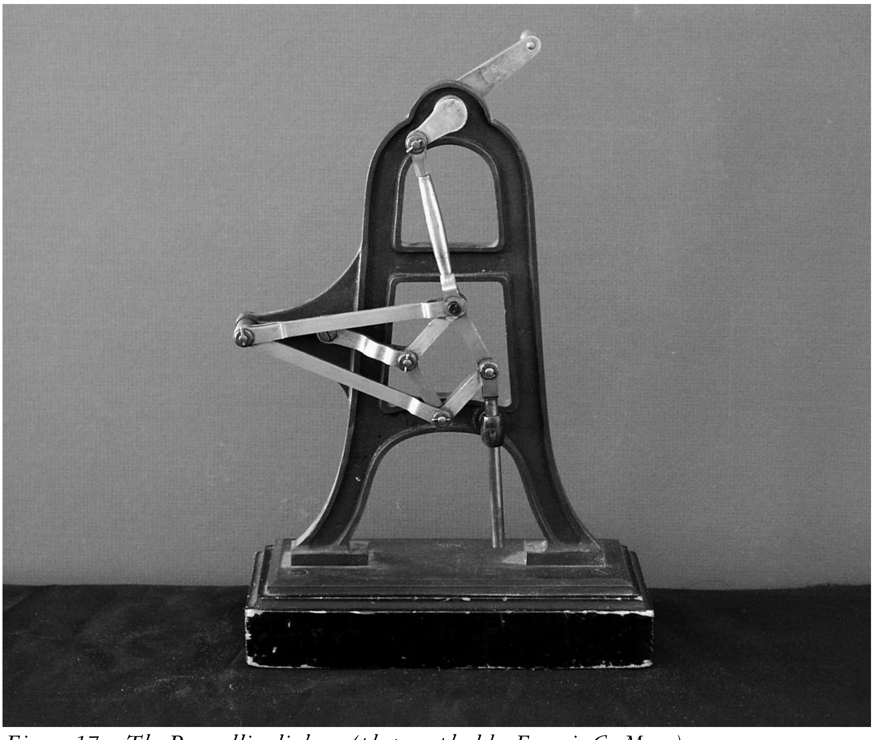

Figure 17a: The Peaucellier linkage (photographed by Francis C. Moon)

Why does the Peaucellier linkage draw a straight line? We suggest the reader connect to a web site where this linkage is depicted in motion (for example, see: KMODDL.library.cornell.edu). As an exercise in analytic geometry, one can verify *that* the point Q will always lie along a straight line—but this still does not answer the *why*-question. Especially difficult is being able to see any relationship with either the "shortest" or the "symmetric" meaning of straightness: is there perhaps a different meaning of straightness that is operative here?

In the 'inversor' (that is, the links joining *C, R, Q, S,* and *P* in Figure 17b), the points *P* and *Q* are inverse pairs with respect to a circle with center *C* and radius $r = \sqrt{s^2 - d^2}$. Analytically, this means that:

$$\text{distance } (C \text{ to } P) \times \text{distance } (C \text{ to } Q) = r^2.$$

Here, the crucial property of circle inversion is that it takes circles to circles. (For details on circle inversion, see Chapter 16 of Henderson and Taimina, 2005.) After experiencing the motion of the linkage, we now see that *P* is constrained (by means of its link to the stationary point *B*) to travel in a circle around *B*. Thus, *Q* must be traveling along the arc of a circle. The radius

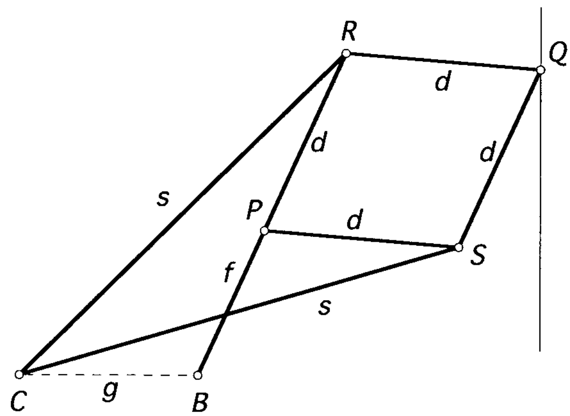

Figure 17b: The Peaucellier linkage diagram

and center of this circle can be varied by changing the position of the fixed point *B* and the length of the link *BP*.

Thus, the Peaucellier linkage draws (at *Q*) the arc of a circle without reference to the center of that circle. If the lengths of *CB* and *BP* are equal, then the circle on which *P* moves goes through the center *C*. Since points near *C* are inverted to points near infinity, the circle that *Q* lies on must go through infinity. How can a circle go through infinity? Answer: only if the circle has infinite radius. *A circle with infinite radius (and thus zero curvature) is a straight line*. We now have a third meaning for straight line—and the Peaucellier linkage is a tool for drawing a straight line that draws on this meaning.

In the previous section on hyperbolic geometry, we pointed out that the *horocycles* in the hyperbolic plane can be seen as circles of infinite radius. Thus, circles with infinite radius are not straight in the hyperbolic plane, even though they are straight in the Euclidean plane. This proves that seeing 'straight' as "circle of infinite radius" is a different meaning from either 'straight' as meaning "symmetric" or 'straight' as meaning "shortest".

Behind this discussion lies the theory of circle inversions, one of the most aesthetic geometric transformations that have also been used in modern art. The special aesthetic appeal here is that inversions (as seen in Figure 18) can draw out the imagination to infinity and can also bring out important geometric meanings. For example, the experience of the linkage as a mechanism that draws a circle without using its center allows one to understand how the linkage can draw a circle of infinite radius and, thus, a straight line.

Peaucellier's linkage is one of thirty-nine straight-line mechanisms in Cornell University's collection, which also has more then two hundred and twenty kinematic models designed by Franz Reuleaux. These models are a rediscovery of a lost, nineteenth-century machine design knowledge. Franz Reuleaux is often referred to as a 'father of modern machine design' (see, for example, Moon, 2003, p. 261). Reuleaux's two most important books

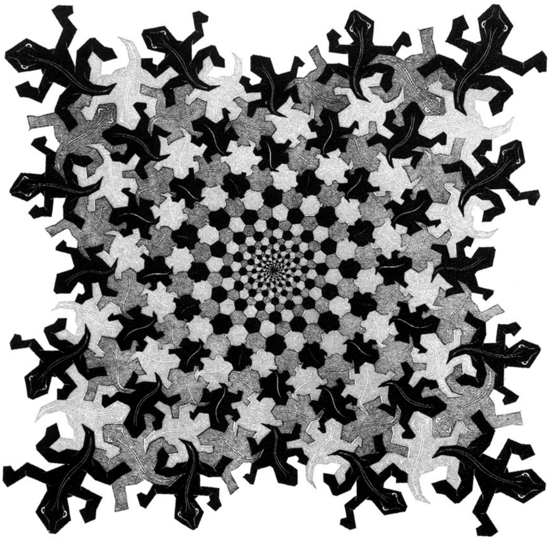

Figure 18: An example of inversion-based art (M. C. Escher's Development II*, woodcut, 1939)*

contain hundreds of drawings of machines and mechanisms. To complement his books, Reuleaux designed and built over eight hundred kinematic models to illustrate his theory of machines. The models in the Cornell collection clearly show the aesthetic style of Reuleaux. (To read more about Reuleaux, his mechanisms and his theory of machines, see Moon, 2003, which also contains many further references.)

As we have been exploring the mathematics behind the Reuleaux models for the NSDL project, we are repeatedly surprised how much aesthetic appeal we find there—not only in machine design itself, but also in the mathematics. These experiences caused us to ask about the relationships among mathematics, engineering and art. Leonardo da Vinci is a well-known embodiment of this interrelationship, but we have found that there seems to be a broader connection. For example, Reuleaux, in his book *The Kinematics of Machinery* (1876/1963), refers specifically to the artist and to experiences of deeper meanings in a manner similar to our discussion at the beginning of this chapter.

> He who best understands the machine, who is best acquainted with its essential nature, will be able to accomplish the most by its means. (p. 2)

> In each new region of intellectual creation the inventor works as does the artist. His genius steps lightly over the airy masonry of reasoning which it has thrown across to the new standpoint. It is useless to demand from either artist or inventor an account of his steps. (p. 6)

> The real cause of the insufficiency of [previous classification systems] is not, however, the classification itself; it must be looked for deeper. It lies [...] in the circumstance that the investigations have never been carried back far enough, – back to the rise of the ideas; that classification has been attempted without any real comprehension being obtained of the subjects to be classified. (p. 18)

In addition, in his article on the history of engineering, Eugene Ferguson (1992) wrote:

> Both the engineer and the artist start with a blank page. Each will transfer to it the vision in his mind's eye. The choice made by artists as they construct their pictures may appear to be quite arbitrary, but those choices are guided by the goal of transmitting their visions, complete with insights and meaning, to other minds. [...] The engineers' goal of producing a drawing of a device—a machine or structure or system—may seem to rule out most if not all arbitrary choices. Yet engineering design is surprisingly open-ended. A goal may be reached by many, many different paths, some of which are better than others but none of which is in all respects the one best way. (p. 23)

Ferguson also notes that Robert Fulton (of steamboat fame) and Samuel Morse (the inventor of the electrical telegraph) were both professional artists before they turned to careers in technology.

We have already mentioned the Peaucellier linkage. Another example is Reuleaux triangles, which are the most well-known of curves with constant width. If a closed convex curve is placed between two parallel lines and the lines are moved together until they touch the curve, the distance between the parallel lines is the curve's 'width' in one direction. Because a circle has the same width in all directions, it can be rotated between two parallel lines without altering the distance between the lines.

The simplest, non-circular, constant-width curve is known as the *Reuleaux triangle*. Mathematicians knew it earlier, but Reuleaux (1876/1963, pp. 131-146) was the first to study various motions determined by constant-width figures. A Reuleaux triangle can be constructed starting with an equilateral triangle of side *s* and then replacing each side by a circular arc using the other two sides as radii, as shown in Figure 19. The resulting figure bounded by these three arcs is the Reuleaux triangle. Its constant width is equal to *s,* the side length of the original equilateral triangle.

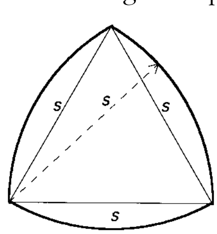

Figure 19: A Reuleaux triangle

In Reuleaux's collection, we find several applications of this triangle and other constant-width curves: see, for example, Figure 20.

Figure 20: A Reuleaux mechanism using a constant-width triangle (photographed by Francis C. Moon)

The Reuleaux triangle fits inside a square of side *s* and can be rotated a full 360° within the square—this is the idea behind drill bits that can drill (almost) a square hole: conversely, the square can rotate around the stationary Reuleaux triangle. Reuleaux did not give analytical descriptions of these motions. Instead, he produced many drawings that, in an aesthetically visual way, show the different paths of points during the motions.

Reuleaux was the first to describe properties of these motions accurately and, in his model collection, we find several applications, such as those illustrated above. For instance, he proved the following theorem geometrically: any relative motion between two shapes, S and R, in the plane can be realized as the motion of two other shapes, cS and cR, rolling on each other, with cS fixed to S and cR attached to R. He called the rolling shapes 'centroids' (the locus of instantaneous centers), but, in order to avoid confusion with the centroid of a triangle, the word 'centrode' was subsequently used.

Figures 21 and 22 (overleaf) show the centrodes (namely $O_1O_2O_3O_4$ and $m_1m_2m_3$) for the relative motions of the square and the Reuleaux triangle respectively. Since the *relative* motions are the same in the two figures, the centrodes are necessarily the same. But the real meaning of this rolling motion can be experienced only by actually looking at the models in motion.

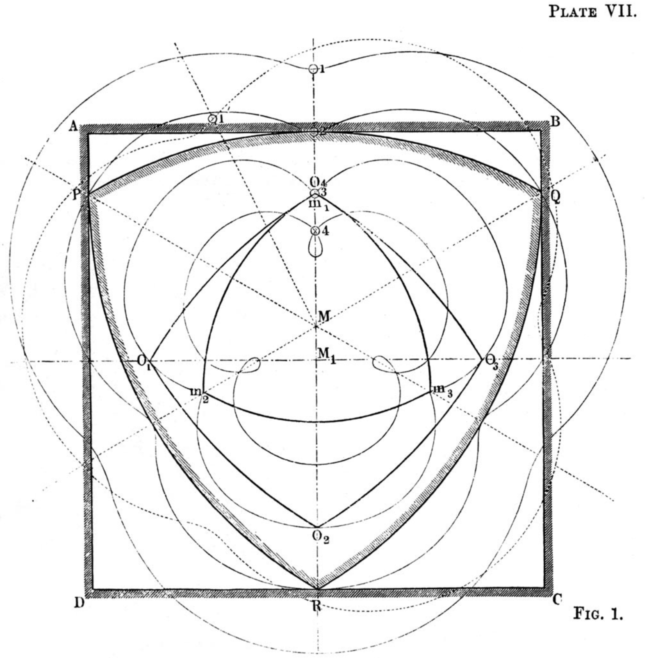

Figure 21: A Reuleaux triangle moving in a square (from Reuleaux, 1876/1963, p. 136)

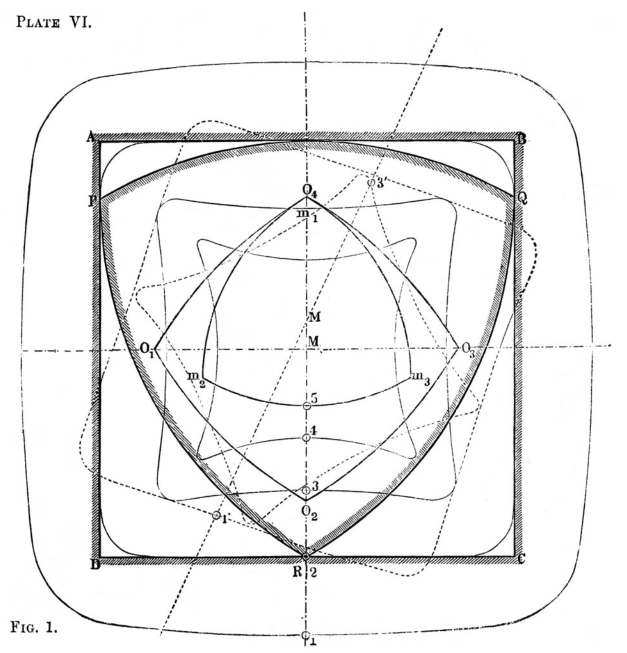

Figure 22: A square moving around a Reuleaux triangle (from Reuleaux, 1876/1963, p. 137)

Conclusion

Aesthetics has always been a driving force in our experiences of mathematics. We do not—as some mathematicians have claimed to do—carry with us a list of criteria by which we judge the aesthetic value of a proof. In fact, rarely do we find proofs, in and of themselves, to be aesthetic objects. Instead, we locate the aesthetic value of mathematics in the coming-to-understanding, in the *integration* of experience and meaning. We believe that the understanding of meanings in mathematics (often through aesthetic experiences) comes *before* an understanding of the analytic formalisms. We hope that the reader has gained, through our stories and our examples, a sense of the aesthetic component of our perception of mathematical meanings.

Acknowledgement

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