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Extended Hyperbolic Surfaces in $\mathbb{R}^3$

D. W. Henderson[1]

Received August 2003

Abstract—In this paper, I will describe the construction of several surfaces whose intrinsic geometry is hyperbolic geometry, in the same sense that spherical geometry is the geometry of the standard sphere in Euclidean 3-space. I will prove that the intrinsic geometry of these surfaces is, in fact, (a close approximation of) hyperbolic geometry. I will share how I (and others) have used these surfaces to increase our own (and our students') experiential understanding of hyperbolic geometry. (How to find hyperbolic geodesics? What are horocycles? Does a hyperbolic plane have a radius? Where does the area formula πr^2 fit in hyperbolic geometry?)

Starting soon after the Euclid's *Elements* were written and continuing for the next 2000 years, mathematicians attempted either to prove Euclid's Fifth (parallel) Postulate as a theorem (based on the other postulates) or to modify it in various ways. These attempts culminated around 1825, when Nikolai Lobachevsky (1792–1856) and János Bolyai (1802–1860) independently discovered a geometry that satisfies all of Euclid's Postulates and Common Notions except that the Fifth Postulate does not hold. It is this geometry that is called *hyperbolic geometry.*

One of the first open questions about hyperbolic geometry was whether it is the intrinsic geometry of any surface in Euclidean space. In the mid-19th century, beginnings of differential geometry, it was shown that hyperbolic surfaces would be precisely surfaces with constant negative curvature. If there is such a surface, then hyperbolic geometry studies the geometric properties of the surface that are intrinsic—properties of the surface that a bug crawling on the surface could detect. Intrinsic properties include geodesics (intrinsically straight lines), length of paths, shortest paths (which are almost always segments of geodesics), angles, surface area, and so forth.

Students (and mathematicians) desire to touch and feel a hyperbolic surface in order to experience its intrinsic properties. Many people have trouble with standard "models" and pictures of hyperbolic geometry in textbooks, because the intrinsic meanings of geodesics, lengths, angles, and areas cannot be directly seen and experienced. However, it is a common misconception that you cannot have a surface in 3-space whose intrinsic geometry is hyperbolic geometry.

Mathematicians looked for surfaces whose intrinsic geometry is complete hyperbolic geometry in the same sense that the intrinsic geometry of a sphere has the complete spherical geometry. In 1868, Eugenio Beltrami (1835–1900) described a surface, called the pseudosphere, whose local intrinsic geometry is hyperbolic geometry but is not complete in the sense that some geodesics (intrinsically straight lines) cannot be continued indefinitely, and it is not simply connected (there are loops which cannot be shrunk on the surface to a point). (See [3, p. 218] for photos of the surfaces that Beltrami constructed and further discussion of the pseudosphere.) In 1901, David Hilbert (1862–1943) proved [7] that it is impossible to define by (real analytic) equations a complete hyperbolic surface. In those days, "surface" normally meant one defined by real analytic equations, and so the search for a complete hyperbolic surface was abandoned. Still today many works state that a

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complete hyperbolic surface is impossible. For popularly written examples, see Robert Osserman's *Poetry of the Universe* [11, p. 158] and David Hilbert and S. Cohn-Vossen's *Geometry and the Imagination* [8, p. 243]. All of the references implicitly assume surfaces embedded with some conditions of differentiability and refer (implicitly or explicitly) to Hilbert's 1901 theorem. For a detailed discussion and proof of Hilbert's theorem, see Spivak's *A Comprehensive Introduction to Differential Geometry* [12, Vol. III, pp. 373 and 381].

Hilbert's arguments also work to show that there is no isometric embedding whose derivatives up to order 4 are continuous. In 1964, N.V. Efimov [2] (Russian; discussed in English in Tilla Milnor's [10]) extended Hilbert's result by proving that there is no isometric embedding defined by functions whose first and second derivatives are continuous. However, in 1955, N. Kuiper proved [9] that there is an isometric embedding with continuous derivatives of the hyperbolic plane onto a closed subset of 3-space. Then, in some summer workshops that I attended in the 1970's, William Thurston described the construction of extendible hyperbolic surfaces (that can be made out of paper) (see [13, pp. 49–50]). For a more detailed discussion of these ideas, see Thurston [13, pp. 51–52].

In the next section, I will give directions for making Thurston's construction. I will also describe the method (invented by Daina Taimiņa) for crocheting Thurston's surfaces. These finite surfaces can apparently be extended indefinitely.

ANNULAR HYPERBOLIC PLANE

This is the paper and tape surface described by William Thurston. It may be constructed as follows: Cut out many identical annular strips as in Fig. 1. (An *annulus* is the region between two concentric circles, and we call an *annular strip* a portion of an annulus cut off by an angle from the center of the circles.) Attach the strips together by attaching the inner circle of one to the outer circle of the other or the straight ends together. (When the straight ends of annular strips are attached together, you get annular strips with increasing angles and eventually the angle will

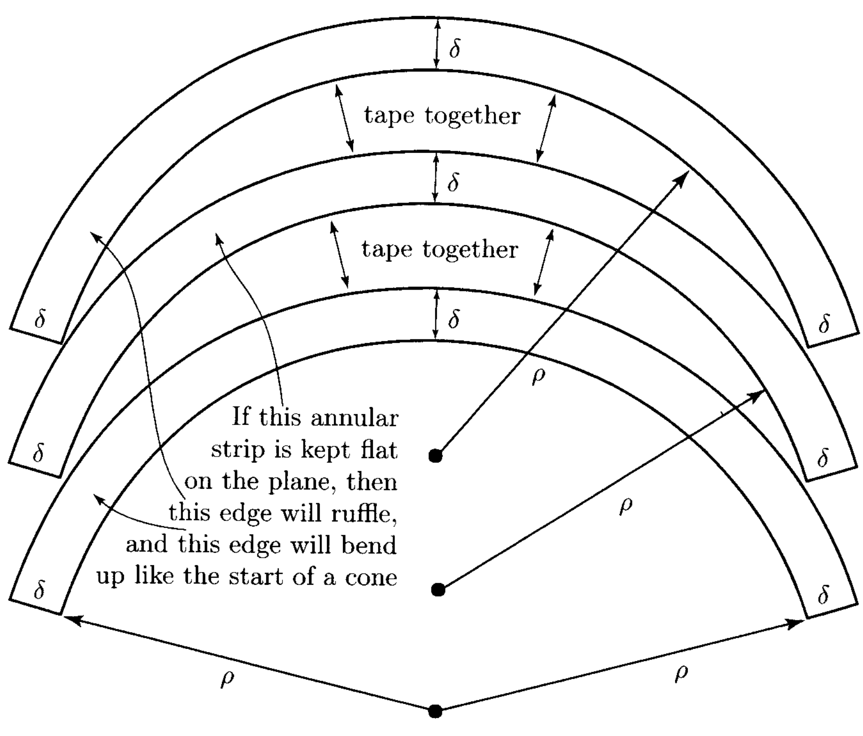


Fig. 1. Annular strips for making an annular hyperbolic plane

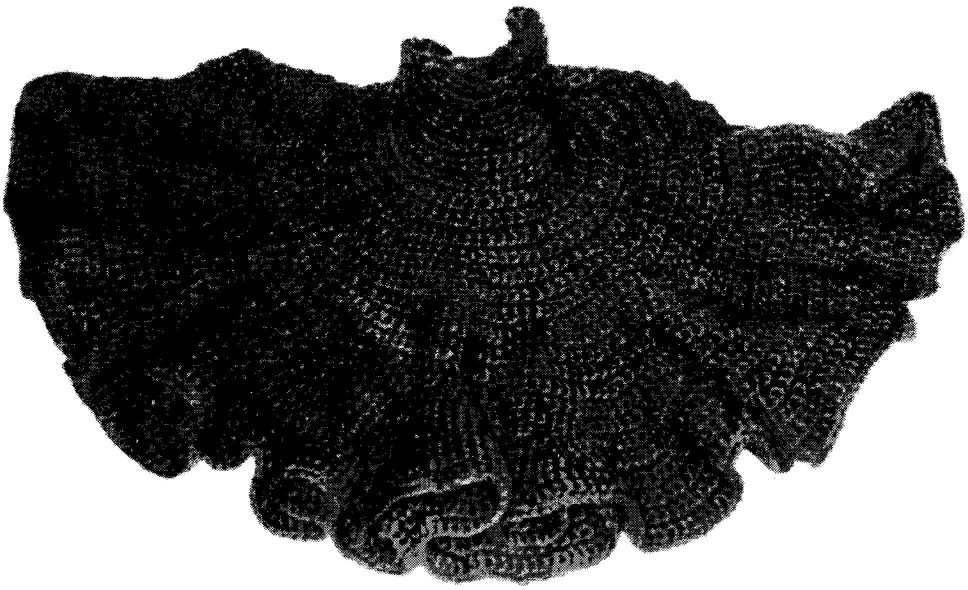


Fig. 2. A crocheted annular hyperbolic plane

be more than 2π.) The resulting surface is, of course, only an approximation of the desired surface. The actual annular hyperbolic plane is obtained by letting $\delta \to 0$ while holding ρ fixed. (We show below, in several ways, that this limit exists.) Note that since the surface is constructed the same everywhere (as $\delta \to 0$), it is homogeneous (that is, intrinsically and geometrically, every point has a neighborhood that is isometric to a neighborhood of any other point). We will call the result of this construction the *annular hyperbolic plane*. We urge the reader to try this by cutting out a few identical annular strips and taping them together as in Fig. 1.

If you tried to make your annular hyperbolic plane from paper annuli, you certainly realized that it takes a lot of time. Also, later you will have to play with it carefully because it is fragile and tears and creases easily—you may want just to have it sitting on your desk. But there is another way to get a sturdy model of the hyperbolic plane which you can work and play with as much as you wish. This is the crocheted hyperbolic plane invented by Daina Taimiņa. See [5] or [6] for directions for the crocheting. Crocheting will take some time, but later you can work with this model without worrying about destroying it. The completed product is pictured in Fig. 2.

HYPERBOLIC PLANES OF DIFFERENT RADII (CURVATURE)

Note that the construction of an annular hyperbolic plane is dependent on ρ (the radius of the annuli), which can be called the *radius of the hyperbolic plane*. As in the case of spheres, we get different hyperbolic planes depending on the value of ρ. In Figs. 3–5, there are crocheted hyperbolic planes with radii approximately 4, 8, and 16 cm. These photos were all taken from approximately the same perspective, and in each picture, there is a centimeter rule in order to indicate the scale.

Note that, as ρ increases, the hyperbolic plane becomes flatter and flatter (has less and less curvature). For both the sphere and the hyperbolic plane, as ρ goes to infinity, they become indistinguishable from the ordinary flat (Euclidean) plane. We will show below that the Gaussian curvature of the hyperbolic plane is $-1/\rho^2$. So it makes sense to call this ρ the radius of the hyperbolic plane in agreement with spheres, where a sphere of radius ρ has Gaussian curvature $1/\rho^2$.

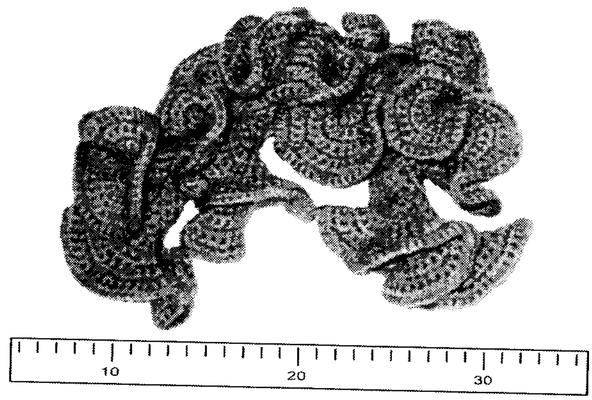


Fig. 3. Hyperbolic plane with $\rho \approx 4\,\mathrm{cm}$

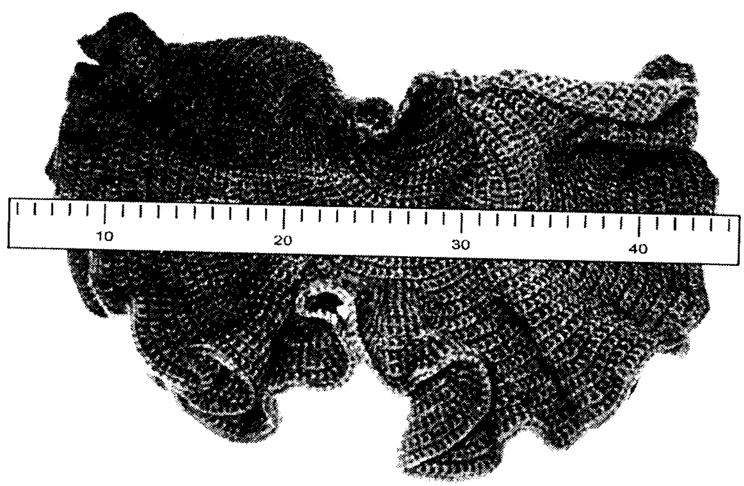


Fig. 4. Hyperbolic plane with $\rho \approx 8\,\mathrm{cm}$

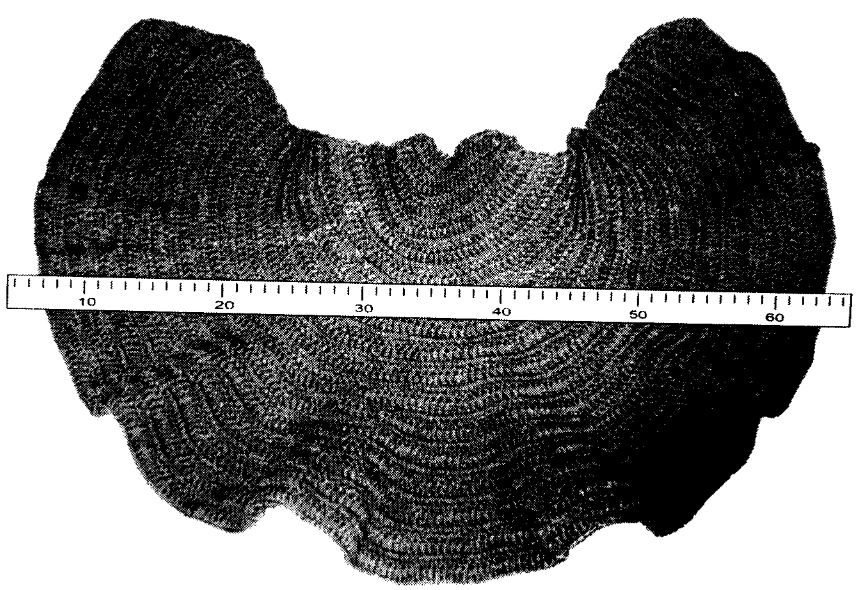


Fig. 5. Hyperbolic plane with $\rho \approx 16\,\mathrm{cm}$

HOW DO WE KNOW THAT WE OBTAIN THE HYPERBOLIC PLANE?

Why is it that the intrinsic geometry of an annular hyperbolic plane is a hyperbolic plane? The answer, of course, depends on what you mean by "hyperbolic plane." There are four main ways of describing the hyperbolic plane; hopefully, one of these is your favorite:

1. A hyperbolic plane is a simply connected complete space that has the same local (intrinsic) geometry as the *pseudosphere.*
2. A hyperbolic plane is a simply connected complete *Riemannian manifold with constant negative Gaussian curvature.*
3. A hyperbolic plane is described by the *upper half plane model.*
4. A hyperbolic plane satisfies all the postulates of Euclidean geometry except for *Euclid's Fifth (or Parallel) Postulate.*

The italicized terms will be explained below as we deal with each description in the sections that follow. But first we need to consider natural coordinates that we will find useful.

INTRINSIC GEODESIC COORDINATES

Let ρ be the fixed inner radius of the annuli and let H_δ be the approximation of the annular hyperbolic plane constructed, as above, from annuli of radius ρ and thickness δ. On H_δ, pick the inner curve of any annulus and, calling it the base curve, pick a positive direction on this curve, pick any point on this curve, and call it the origin O. We can now construct an (intrinsic) coordinate system $\mathbf{x}_\delta \colon \mathbb{R}^2 \to H_\delta$ by defining $\mathbf{x}_\delta(0,0) = O$, $\mathbf{x}_\delta(w,0)$ to be the point on the base curve at a distance w from O, and $\mathbf{x}_\delta(w,s)$ to be the point at a distance s from $\mathbf{x}_\delta(w,0)$ along the radial (along the radii of each annulus) curve through $\mathbf{x}_\delta(w,0)$, where the positive direction is chosen to be in the direction from outer- to inner-curve of each annulus (see Fig. 6). The reader can easily check that this coordinate map is one-to-one and onto (if you were to add annuli indefinitely).

Note that each coordinate map $\mathbf{x}_\delta$ induces a metric, d_δ, on $\mathbb{R}^2$ by defining $d_\delta(p,q)$ to be the (intrinsic) distance between $\mathbf{x}_\delta(p)$ and $\mathbf{x}_\delta(q)$ in H_δ. Those readers who desire a more formal

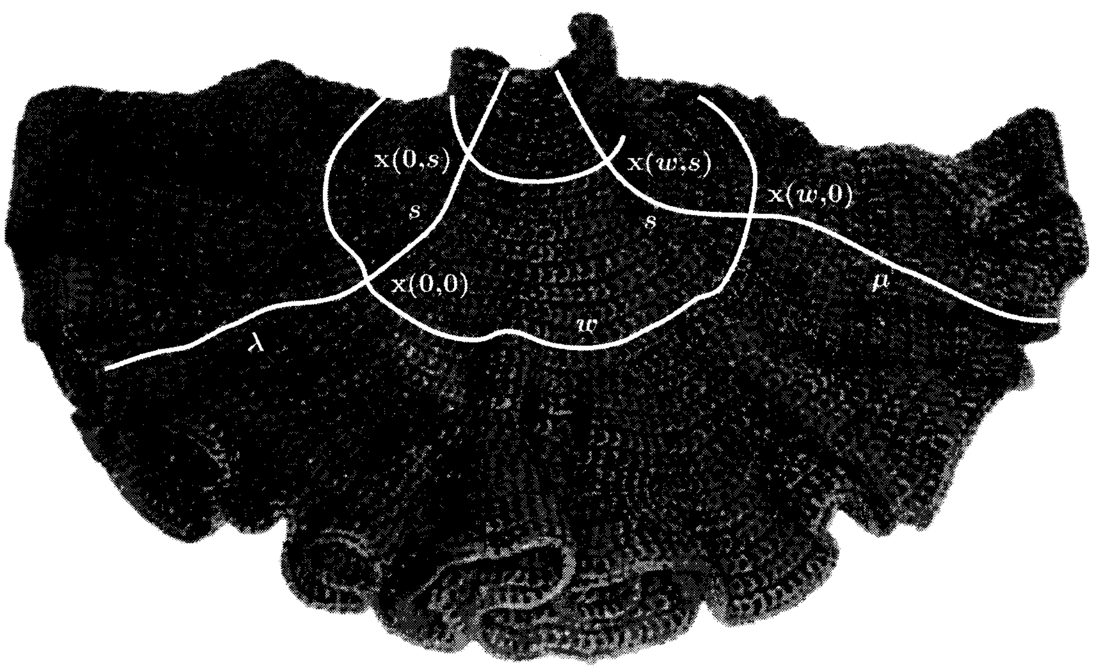


Fig. 6. Geodesic rectangular coordinates on annular hyperbolic plane

description of the limit can check that, in the limit as $\delta \to 0$, the metrics d_δ converge to a metric d on $\mathbb{R}^2$ and thus define the annular hyperbolic plane as $\mathbb{R}^2$ with a special metric. In fact, this process also defines a Riemannian metric, but this will be easier to see after we show the connections with the upper half plane model.

WHAT CAN WE EXPERIENCE ABOUT HYPERBOLIC GEODESICS AND ISOMETRIES?

The following facts were observed by our students during one class period in which they explored for the first time the crocheted hyperbolic plane in small groups.

The radial curves are geodesics with reflection symmetry. The radial curves (curves that run radially across each annulus) have intrinsic reflection symmetry in each H_δ because of the symmetry in each annulus and the fact that the radial curves intersect the bounding curves at right angles. These reflection symmetries carry over in the limit to the annular hyperbolic plane. Such bilateral symmetry is the basis of our intuitive notion of straightness (see [4, Ch. 1] and [5, Ch. 1] for more details), and thus we can conclude that these radial curves are geodesics (intrinsically straight curves) on the annular hyperbolic plane and that reflection through these curves is an isometry.

The radial geodesics are asymptotic. Looking at our hyperbolic surfaces, we see the radial geodesics getting closer and closer in one direction and diverging in the other direction. In fact, let λ and μ be two of the radial geodesics in H_δ. The distance between these radial geodesics changes by $\rho/(\rho+\delta)$ every time they cross one annulus. (Remember that the annuli all have the same radii.) If we cross n strips, then the distance in H_δ between λ and μ at a distance $c = n\delta$ from the base curve is

$$d\left(\frac{r}{r+\delta}\right)^n = l\left(\frac{r}{r+\delta}\right)^{c/\delta}.$$

Now take the limit as $\delta \to 0$ to show that the distance between λ and μ on the annular hyperbolic plane is

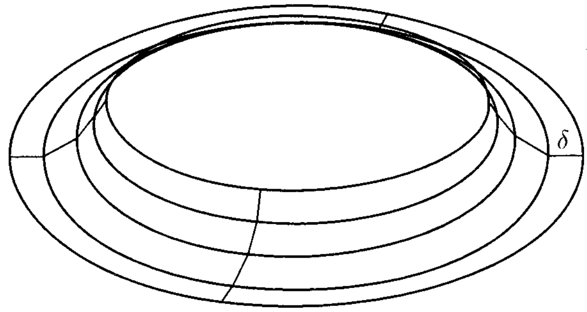
$$l\exp(-c/r). \tag{1}$$

Asymptotic geodesics never happen on a Euclidean plane or on a sphere.

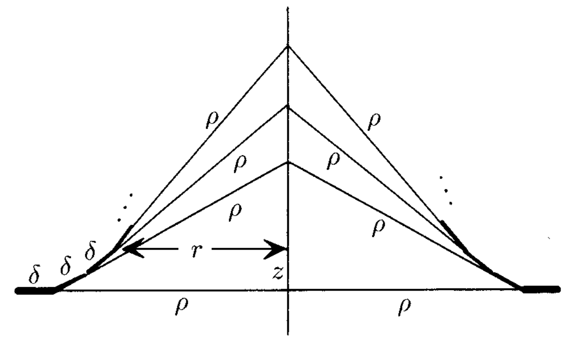
There is an isometry that preserves the annuli. Since reflections through radial geodesics are isometries that preserve each annulus, the composition of two such reflections must also be an isometry that preserves each annulus. A brief consideration of what happens on a given annulus should convince you that this isometry shifts the annulus along itself. In the plane, we would call such an isometry a rotation (about the center of the annulus). But intrinsically, on the annular hyperbolic plane, an annulus has no center, and the isometry has no fixed point because the radial geodesics (which are perpendicular to the annulus) do not intersect. Also, we do not want to call this isometry a translation because there is no geodesic that is preserved by the isometry. So, this is a type of isometry that we have not met before on the plane. Such isometries are traditionally called *horolations*, and annular curves are traditionally called *horocycles*. Horolations can be thought of as rotations about a point at infinity (since the radial geodesics are asymptotic), and the horocycles can be though of as circles with infinite radius.

CONNECTION TO THE PSEUDOSPHERE

Take the annulus whose inner edge is the base curve and embed it isometrically in the $x-y$ plane as a complete annulus with center at the origin. Now attach to this annulus portions of the other annuli as indicated in Fig. 7. Note that the second and subsequent annuli form truncated cones.



Surface in perspective



Cross-section of surface

Fig. 7. Hyperbolic surface of revolution—pseudosphere

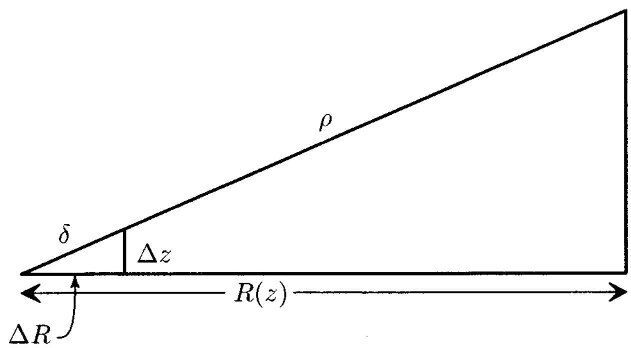


Fig. 8. Relating $R(z)$, ρ, Δz, and ΔR

Let the vertical axis be the z-axis, and then, at each z, we have the picture shown in Fig. 8. Thus,

$$\frac{\Delta R}{\Delta z} = \frac{-R(z)}{\sqrt{(\rho+\delta)^2 - (R(z))^2}}.$$

In the limit as δ (and ΔR and Δz) go to zero, we get

$$\frac{dR}{dz} = \frac{-R(z)}{\sqrt{\rho^2 - (R(z))^2}}. \tag{2}$$

We can get the same differential equation by using (2) above, which implies that the circle at height z has circumference $2\pi\rho e^{-s/\rho}$, where s is the arc length along the surface from $(0, r)$ to $(z, R(z))$. We can solve this differential equation explicitly for z and obtain the solution

$$z = \sqrt{\rho^2 - R^2} - \rho \ln \left| \frac{\rho + \sqrt{\rho^2 - R^2}}{R} \right|.$$

Here z is a continuously differentiable function of R, and the derivative (for $z \neq 0$) is never zero; hence, R is also a continuously differentiable function of z. Since R is never zero, we can conclude that this hyperbolic surface of revolution is a smooth surface (traditionally called the *pseudosphere*). Thus,

Theorem. *The pseudosphere is locally isometric to the annular hyperbolic plane.*

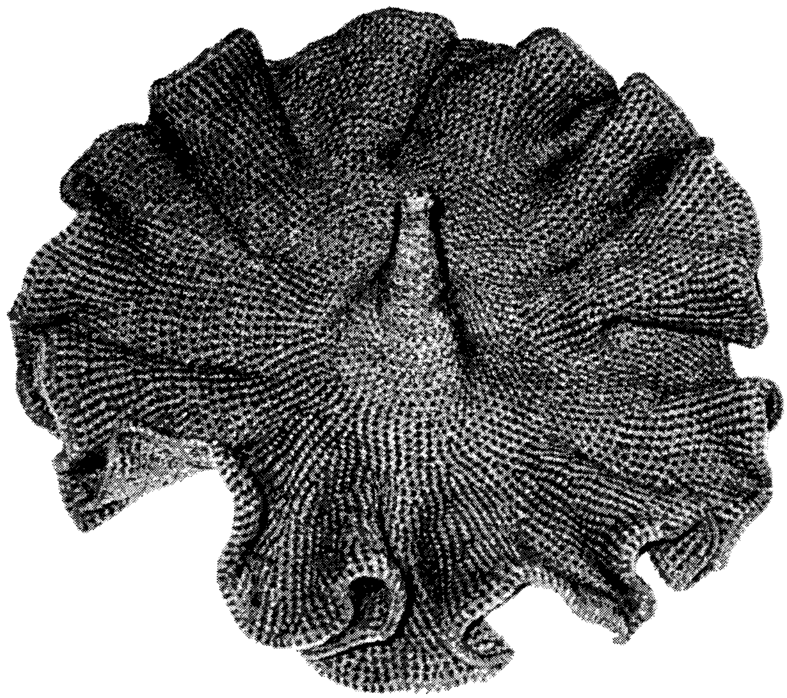


Fig. 9. Crocheted pseudosphere

Note that, when you add paper annuli (or crochet) beyond the annular strip that lays flat and forms a complete annulus, the surface begins to form ruffles and is no longer a surface of revolution. In fact, it appears that it is not even differentiable where the ruffles start since the "top ridge" of the ruffles (see Fig. 9) appear to be straight and thus not tangent to the plane of the complete annulus.

CONNECTIONS TO RIEMANNIAN MANIFOLDS WITH CONSTANT NEGATIVE GAUSSIAN CURVATURE

If a surface is differentiably embedded into 3-space by an isometry whose first and second derivatives are continuous (C^2), then the surface is said to be a Riemannian manifold. At a given point P on the surface, we call the *normal direction* one of the two directions that are perpendicular to the surface at that point. The *normal curvature* at a point of a curve on the surface is defined to be the component of the curvature of the curve that is in the normal direction. The collection of all normal curvatures of all the (smooth) curves through P has a maximum and a minimum value. These extremal values of the normal curvature are the principal curvatures (and can be shown to be the normal curvatures of two curves that are perpendicular at P). The *Gaussian curvature* of the surface at P is defined to be the product of these two principal curvatures.

The pseudosphere is a Riemannian surface, and at each point $(z, R(z), \theta)$, the principal curvatures are the normal curvatures of generating curves, $z \mapsto R(z)$ and the circle $\theta \mapsto (R(z), \theta)$. The curvature of the first curve is

$$\frac{-R''(z)}{[1+(R'(z))^2]^{3/2}}$$

and is perpendicular to the surface; thus, it is also ($\pm$) normal curvature. The curvature of the circle is $1/R(z)$, which must be projected onto the direction perpendicular to the surface, giving the normal curvature as

$$\frac{1}{R(z)\sqrt{1+(R'(z))^2}}.$$

We do not have a formula for R, but we do have formula (2) for $R'(z)$. The Gaussian curvature is then the product of these two normal curvatures, which, as you can check (using (2)), is $-1/\rho^2$, where the minus sign occurs because the two normal curvatures are in opposite directions. Thus, the pseudosphere has constant negative Gaussian curvature.

Gauss' famous *Theorema Egregium* states that the Gaussian curvature is independent of the (C^2) embedding and thus is an intrinsic property of the surface. Thus, since the annular hyperbolic plane is locally isometric to the pseudosphere, we can say that it also has constant negative Gaussian curvature. Most differential geometry texts (see, for example, [4]) give intrinsic methods for determining the Gaussian curvature, which also can be applied directly to the annular hyperbolic plane (see [4, Problem 7.7] for two such methods). The result of N.V. Efimov [2] discussed at the beginning of this article shows that, no matter how the annular hyperbolic plane is placed in 3-space, if it is extended enough, it cannot be C^2 embedded (and thus cannot have principal curvatures at all points).

CONNECTION TO THE UPPER HALF PLANE MODEL

As shown above, the coordinate chart $\mathbf{x}$ preserves (does not distort) distances along the (vertical) 2nd coordinate curves, but at $\mathbf{x}(a,b)$, the distances along the 1st coordinate curve are distorted by the factor of $\exp(-b/\rho)$ when compared to the distances in $\mathbb{R}^2$. To be more precise,

Definition. Let $\mathbf{y}\colon A \to B$ be a map from one metric space to another, and let $t \mapsto \lambda(t)$ be a curve in A. Then, the *distortion* of $\mathbf{y}$ along λ at the point $p = \lambda(0)$ is defined as

$$\lim_{x\to 0} \frac{\text{The arc length along } \mathbf{y}(\lambda) \text{ from } \mathbf{y}(\lambda(x)) \text{ to } \mathbf{y}(\lambda(0))}{\text{The arc length along } \lambda \text{ from } \lambda(x) \text{ to } \lambda(0)}.$$

We seek a change of coordinates that will distort distances equally in both directions. The reason for seeking this change is that if distances are distorted equally in both coordinate directions, then the chart will preserve angles. (We call such a chart *conformal.*)

We cannot hope to have no distortion in both coordinate directions (if there were no distortion, then the chart would be an isometry); so we try to make the distortion in the 2nd coordinate direction the same as the distortion in the 1st coordinate direction. After a little experimentation, we find that the desired change is

$$\mathbf{z}(x,y) = \mathbf{x}(x, \rho\ln(y/\rho)),$$

with the domain of $\mathbf{z}$ being the upper half plane

$$\mathbb{R}^{2+} \equiv \{(x,y) \in \mathbb{R}^2 \mid y > 0\},$$

where $\mathbf{x}$ is the geodesic rectangular coordinates defined above. This is the usual *upper half plane model* of the hyperbolic plane thought of as a map of the hyperbolic plane in the same way that we use planar maps of the spherical surface of the earth.

Lemma. *The distortion of* $\mathbf{z}$ *along both coordinate curves*

$$x \to \mathbf{z}(x,b) \qquad \textit{and} \qquad y \to \mathbf{z}(a,y)$$

at the point $\mathbf{z}(a,b)$ *is* ρ/b.

Proof. We now focus on the point $\mathbf{z}(a,b)=\mathbf{x}(a,\rho\ln(b/\rho))$. Along the first coordinate curve, $x\to\mathbf{z}(x,b)=\mathbf{x}(x,\rho\ln(b/\rho))$, the arc length from $\mathbf{x}(a,c)$ to $\mathbf{x}(x,c)$ is $|x-a|\exp(-c/\rho)$ by (1) above. Thus, we can calculate the distortion:

$$\lim_{x\to a}\frac{|x-a|\exp[-(\rho\ln(b/\rho))/\rho]}{|x-a|}=\rho/b.$$

Now, look at the second coordinate curve, $y\to\mathbf{z}(a,y)=\mathbf{x}(a,\rho\ln(y/\rho))$. Along this coordinate curve (a radial geodesic), the speed is not constant; but since the second coordinate of $\mathbf{x}$ measures arc length, the arc length from $\mathbf{z}(a,y)=\mathbf{x}(a,\rho\ln(y/\rho))$ to $\mathbf{z}(a,b)=\mathbf{x}(a,\rho\ln(b/\rho))$ is

$$|\rho\ln(y/\rho)-\rho\ln(b/\rho)|,$$

and the distortion is

$$\lim_{y\to b}\frac{|\rho\ln(y/\rho)-\rho\ln(b/\rho)|}{|y-b|}=\rho\left|\lim_{y\to b}\frac{\ln(y/\rho)-\ln(b/\rho)}{y-b}\right|=\rho\left|\frac{d}{dy}\ln(y/\rho)\Big|_b\right|=\rho/b.$$

In the above situation, we call these distortions the *distortion of the map* $\mathbf{z}$ *at the point* p and denote it $\operatorname{dist}(\mathbf{z})(p)$. Thus,

$$\operatorname{dist}(\mathbf{z})((a,b))=\rho/b. \tag{3}$$

HYPERBOLIC ISOMETRIES AND GEODESICS

We have seen that there are reflections in the annular hyperbolic plane about the radial geodesics, but we saw the existence of other reflections and geodesics only approximately. However, we were able to see that nonradial geodesics appear to be tangent to one annulus and then in both directions from that point to become more and more perpendicular to the annuli. To assist us in looking at transformations of the annular hyperbolic space, we use the upper half plane model. Since the annuli correspond to horizontal lines in the upper half plane model, geodesics should then be curves that start and end perpendicular to the boundary x-axis. Semicircles with centers on the x-axis are such curves, and we can show directly that they are geodesics with bilateral symmetry. In particular, we can show directly that inversion in a semicircle corresponds to a reflection isometry in the annular hyperbolic plane.

Definition. An *inversion with respect to a circle* Γ is a transformation from the extended plane (the plane with ∞, the point at infinity, added) to itself that takes C, the center of the circle, to ∞ and vice versa and that takes a point at a distance s from the center to the point on the same ray (from the center) that is at a distance of r^2/s from the center, where r is the radius of the circle (see Fig. 10). We call (P,P') an *inversive pair* because (as the reader can check) they are taken to each other by the inversion. The circle Γ is called the *circle of inversion.*

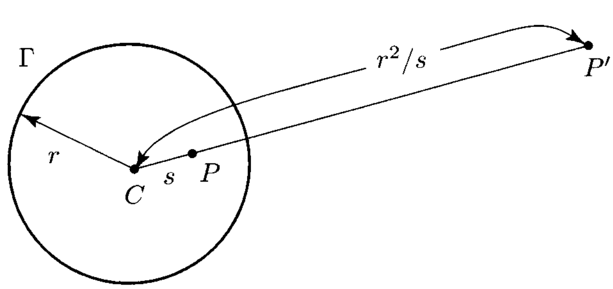


Fig. 10. Inversion with respect to a circle

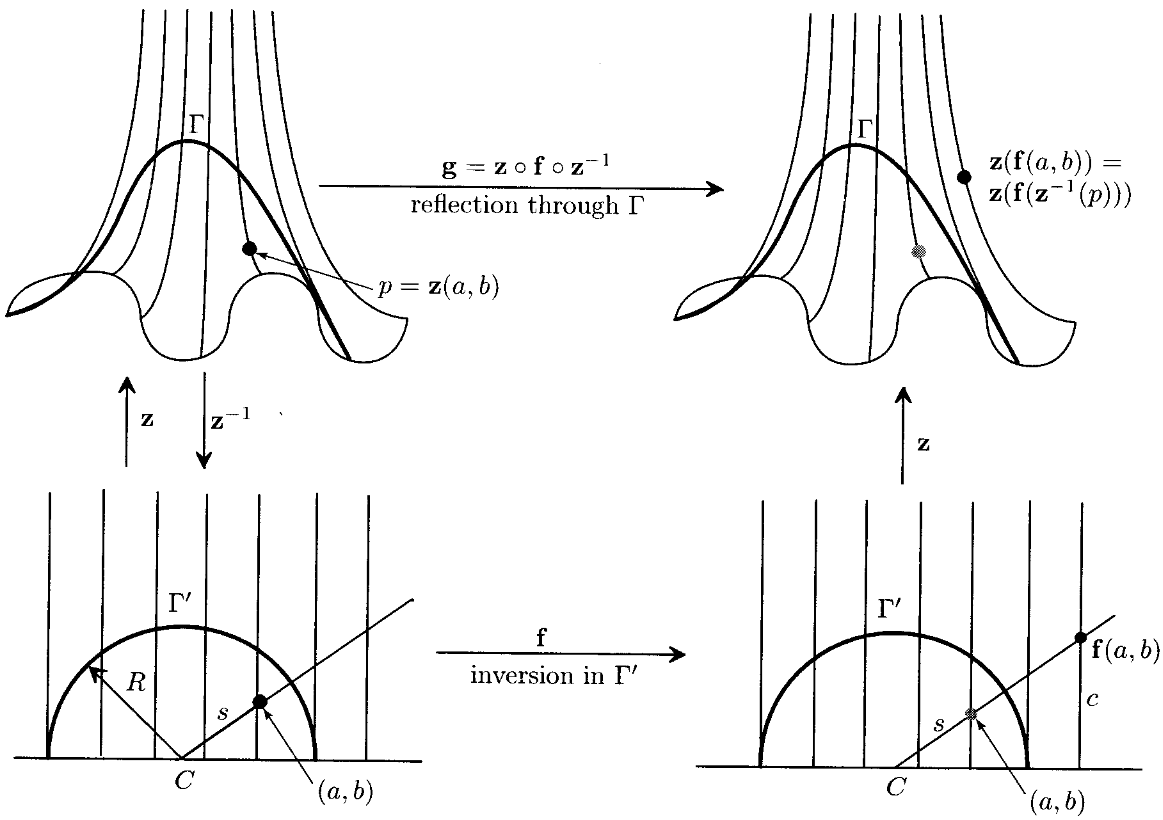


Fig. 11. Hyperbolic reflections correspond to inversions

Inversions have the following well-known properties (see [1, Ch. 5] and [5, Ch. 14]):

- Inversions are conformal.
- Inversions take circles that do not pass through the center of inversion to circles.
- Inversions take circles passing through the center of inversion to straight lines that do not pass through the center of inversion.

If $\mathbf{f}$ is a transformation taking the upper half plane $\mathbb{R}^{2+}$ to itself, then consider the diagram

$$\begin{tikzcd} H^2 \arrow[r, "\mathbf{g}"] \arrow[d, "\mathbf{z}^{-1}"'] & H^2 \\ \mathbb{R}^{2+} \arrow[r, "\mathbf{f}"'] & \mathbb{R}^{2+} \arrow[u, "\mathbf{z}"'] \end{tikzcd}$$

We call $\mathbf{g} = \mathbf{z} \circ \mathbf{f} \circ \mathbf{z}^{-1}$ the transformation of H^2 that corresponds to $\mathbf{f}$. We will call $\mathbf{f}$ an *isometry of the upper half plane model* if the corresponding $\mathbf{g}$ is an isometry of the annular hyperbolic plane.

Theorem. *Let* $\mathbf{f}$ *be the inversion in a circle whose center is on the x-axis. Then, the corresponding* $\mathbf{g} = \mathbf{z} \circ \mathbf{f} \circ \mathbf{z}^{-1}$ *has distortion* 1 *at every point and is thus an isometry.*

Proof (refer to Fig. 11). 1. Note that each of the maps $\mathbf{z}$, $\mathbf{z}^{-1}$, and $\mathbf{f}$ is conformal and has at each point a (nonzero) distortion that is the same for all curves at that point. Using the definition of distortion, the reader can easily check that

$$\operatorname{dist}(\mathbf{g})(p) = \operatorname{dist}(\mathbf{z})((\mathbf{f} \circ \mathbf{z}^{-1})(p)) \cdot \operatorname{dist}(\mathbf{f})(\mathbf{z}^{-1}(p)) \cdot \operatorname{dist}(\mathbf{z}^{-1})(p).$$

2. If $\mathbf{z}(a, b) = p$, then (using (1))

$$\operatorname{dist}(\mathbf{z}^{-1})(p) = 1/[\operatorname{dist}(\mathbf{z})(\mathbf{z}^{-1}(p))] = b/\rho.$$

3. Let r be the radius of the circle C that defines $\mathbf{f}$, and let s be the distance from the center of C to $(a,b) = \mathbf{z}^{-1}(p)$. Since the inversion is conformal, the distortion is the same in all directions. Thus, we need only check the distortion along the ray from C, the center of circle, through p. Then, the reader can verify that this distortion is

$$\mathrm{dist}(\mathbf{f})((a,b)) = r^2/s^2.$$

One way to do this is to note that, in this case, the distortion is the speed (at s) of the curve $t \mapsto r^2/t$.

4. By (1), $\mathrm{dist}(\mathbf{z})(\mathbf{f}(\mathbf{z}^{-1}(p))) = \rho/c$, where c is the y-coordinate of $\mathbf{f}(\mathbf{z}^{-1}(p)) = \mathbf{f}(a,b)$. To determine c, look at Fig. 11. By similar triangles, $s/b = (r^2/s)/c$. Thus, $c = b(r^2/s^2)$ and

$$\mathrm{dist}(\mathbf{z})((\mathbf{f} \circ \mathbf{z}^{-1})(p)) = \frac{\rho s^2}{br^2}.$$

5. Putting everything together, we now have

$$\mathrm{dist}(\mathbf{g})(p) = \frac{\rho s^2}{br^2}\frac{r^2}{s^2}\frac{b}{\rho} = 1.$$

Since this is true at all points p, the map $\mathbf{g}$ must be an isometry of the annular hyperbolic plane.

We call these inversions through semicircles with center on the x-axis (or the corresponding transformations in the annular hyperbolic plane) *hyperbolic reflections.* Thus, the images of the semicircles in the upper half plane have bilateral symmetry and are intrinsically straight (geodesics).

Thus, we have established that the annular hyperbolic plane is the same as the usual upper half plane model of the hyperbolic plane. The usual analysis of the hyperbolic plane can now be considered as analysis of the intrinsic geometry of the annular hyperbolic plane. We give only one example here because it results in the interesting formula πr^2.

AREA OF IDEAL TRIANGLES

It is impossible to picture the whole of an ideal triangle in an annular hyperbolic plane, but it is easy to picture ideal triangles in the upper half plane model. In the upper half plane model, an *ideal triangle* is a triangle with all three vertices either on the x-axis or at infinity (see Fig. 12).

At first glance, it appears that there must be many different ideal triangles; however,

Theorem. *All ideal triangles on the same hyperbolic plane are congruent.*

Proof outline. Perform an inversion (hyperbolic reflection) that takes one of the vertices (on the x-axis) to infinity and thus takes the two sides from that vertex to vertical lines. Then, apply a similarity to the upper half plane taking this to the standard ideal triangle with vertices $(-1,0)$, $(0,1)$, and ∞ (see Fig. 12).

Theorem. *The area of an ideal triangle is* $\pi\rho^2$. (Remember that this ρ is the radius of the annuli and is equal to $\sqrt{-1/K}$, where K is the Gaussian curvature.)

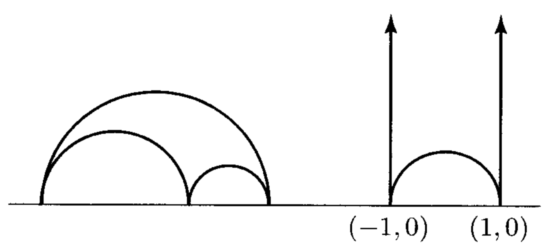


Fig. 12. Ideal triangles in the upper half plane model

Proof. By (3), the distortion $\mathrm{dist}(\mathbf{z})(a,b)$ is ρ/b, and thus the desired area is

$$\int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\infty}\left(\frac{\rho}{y}\right)^2 dy\,dx = \pi\rho^2.$$

Theorem. *The area of a hyperbolic triangle is*

$$(\pi - \Sigma\alpha_i)\rho^2.$$

For more details and extensions of the ideas in this article, see [6] and [5], especially Chapters 5 and 16 in [5].

This paper is in fond memories of the author's membership in the seminar of L.V. Keldysh at the Steklov Institute for four months in 1970.

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