Grafty Geometry
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CRAFTY GEOMETRY

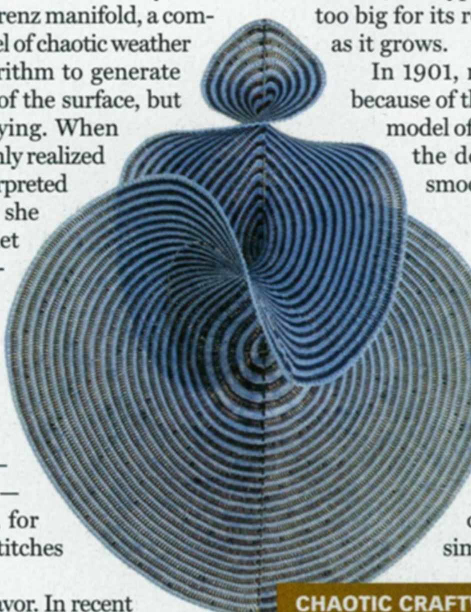
Mathematicians are knitting and crocheting to visualize complex surfaces

BY ERICA KLARREICH

During the 2002 winter holidays, mathematician Hinke Osinga was relaxing with some lace crochet work when her partner and mathematical collaborator Bernd Krauskopf asked, "Why don't you crochet something useful?" Some crocheters might bridle at the suggestion that lace is useless, but for Osinga, Krauskopf's question sparked an exciting idea. "I looked at him, and we thought the same thing at the same moment," Osinga recalls. "We realized that you could crochet the Lorenz manifold."

For years, Osinga and Krauskopf, both of the University of Bristol in England, had been studying the Lorenz manifold, a complicated surface that emerges from a model of chaotic weather systems. The pair had created an algorithm to generate 2-dimensional computer visualizations of the surface, but Osinga found the flat images unsatisfying. When Krauskopf asked his question, she suddenly realized that the computer algorithm could be interpreted as crochet instructions. "I had to try it," she says. Eighty-five hours and 25,511 crochet stitches later, Osinga had a Lorenz manifold almost a meter tall and about 25 centimeters in diameter, which now hangs in the pair's house as a decoration.

Mathematics has long been an essential tool for the fiber arts. Knitters and crocheters use mathematical principles—often without recognizing them as such—to map the pattern of a cable sweater, for instance, or figure out how to space the stitches when adding a sleeve onto a jacket.



CHAOTIC CRAFTWORK — A crocheted Lorenz manifold brings the shape's swirls into sharp relief.

UNIV. BRISTOL

Now, the two crafts are returning the favor. In recent years, mathematicians such as Osinga have started knitting and crocheting concrete physical models of hard-to-visualize mathematical objects. One mathematician's crocheted models of a counterintuitive shape called a hyperbolic plane are enabling her students and fellow mathematicians to gain new insight into startling properties. Other mathematicians have knitted or crocheted fractal objects, surfaces that have no inside or outside, and shapes whose patterns display mathematical theorems.

"Knitting and crocheting are helping us think about math we already know in a different light," says Carolyn Yackel, a mathematician at Mercer University in Macon, Ga.

A HYPERBOLIC YARN In 1997, as Daina Taimina geared up to teach an undergraduate-geometry class, she faced a challenge. As a visiting mathematician at Cornell University, she planned to cover the basic geometries of three types of surfaces: planar, or Euclidean; spherical; and hyperbolic. She knew that everyone can use intuition to conceive of the first two geometries, which are the realms of, say, sheets of paper and basketballs. The hyperbolic plane, however, lies outside of daily experience of the physical world.

Geometry teachers usually try to explain the hyperbolic plane via flat models that wildly distort its geometry—making lines look like semicircles, for instance. How, Taimina wondered, could she give her students a feel for hyperbolic geometry's counterintuitive properties? While attending a workshop, the answer came to her: Crochet a piece of hyperbolic fabric.

In a flat plane or a sphere, the circumference of a circle grows at most linearly as the radius increases. By contrast, in the hyperbolic plane, the circumference of a circle grows exponentially. As a result, the hyperbolic plane is somewhat like a carpet that, too big for its room, buckles and flares out more and more as it grows.

In 1901, mathematician David Hilbert proved that because of this buckling, it's impossible to build a smooth model of the hyperbolic plane. His result, however, left the door open for models that are not perfectly smooth.

In the 1970s, William Thurston, now also at Cornell, described a way to build an approximate physical model of the hyperbolic plane by taping together paper arcs into rings whose circumferences grow exponentially. However, these models take many hours to build and are so fragile that they generally need to be protected from much rough-and-tumble hands-on study.

Taimina realized that she could crochet a durable model of the hyperbolic plane using a simple rule: Increase the number of stitches in each row by a fixed factor, by adding a new stitch after, for instance, every two (or three or four or n) stitches. In 2001, Taimina and her Cornell colleague David Henderson proved that the crocheted objects indeed capture the geometry of the hyperbolic plane. Over the past decade, Taimina has crocheted dozens of these models.

Taimina's models have made it easy to study hyperbolic lines—the shortest paths between two points on the hyperbolic plane. Given two points, all that's necessary is to grab each point and gently pull tight the fabric between them. The line can then be marked, for future reference, by sewing yarn along it.

Taimina has used these sewn lines in the classroom to illustrate the hyperbolic plane's most famous property. The plane violates Euclid's parallel postulate, which states that given a line and a point off the line, there is just one line through the point that never

HYPERBOLIC FABRIC — Many of the lines that could be inscribed on this crocheted hyperbolic plane curve away from each other, defying Euclid's parallel postulate.

meets the given line. By sewing lines with yarn, Taimina's students have observed that in the hyperbolic plane there are, in fact, infinitely many lines through a given point that never meet a given line. Loosely speaking, this happens because the hyperbolic plane's extreme flaring makes certain lines veer away from each other instead of intersecting as they would in a flat plane.

Because the hyperbolic plane is so hard to visualize, Taimina's crocheted models are helping even seasoned mathematicians develop a better intuition for its properties. Taimina recalls that one mathematician, upon examining one of her hyperbolic planes, exclaimed, "So that's what they look like!"

Taimina has crocheted models for many mathematics departments and for the Smithsonian Institution as examples of math teaching tools, but she now thinks twice before agreeing to make someone a model. Because of the exponential growth, crocheting a hyperbolic plane takes a long time. For instance, one of Taimina's models started with a 1.5-inch row, but the 20th row was already more than 30 feet long. What's more, the crochet work is hard on the hands, Taimina says, since the stitches must be tight to prevent the fabric from stretching out of its characteristic hyperbolic shape. Luckily for Taimina, many mathematicians "are now enthusiastically making their own models," she says.

Taimina's hyperbolic planes have also attracted interest from art lovers. Her models have appeared in art shows all over the United States, and some are currently on display in Latvia and Italy.

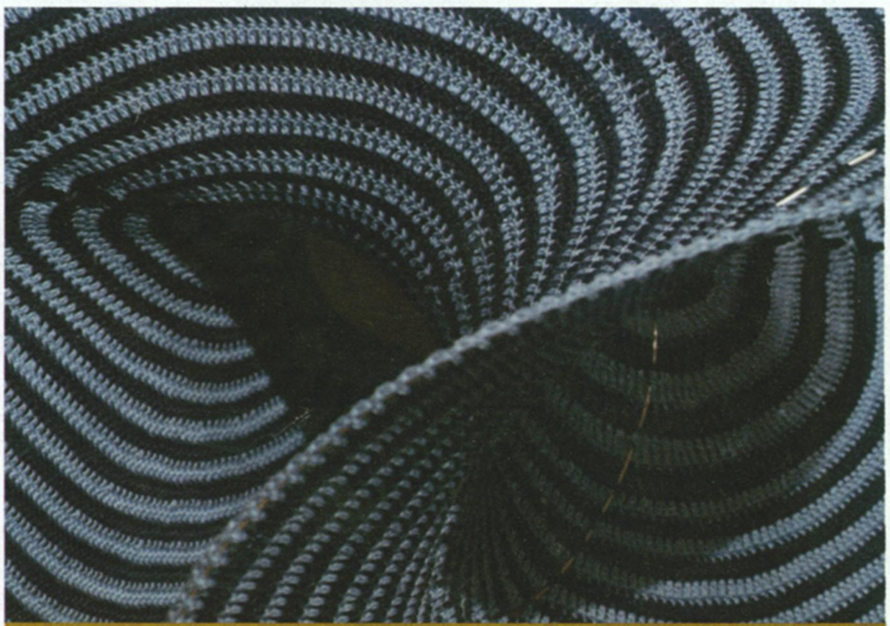
"I have met so many people now who don't have a math background, but who want to understand what these hyperbolic planes mean," Taimina says. "It makes me happy that people can learn beautiful geometry and not be intimidated."

CROCHETED CHAOS Osinga launched her crochet project in the hopes of finally getting her hands on a Lorenz manifold, a mathematical object that she had been studying theoretically for years. Meteorologist Edward Lorenz, now an emeritus professor at the Massachusetts Institute of Technology, had set down three equations in 1963 as a highly simplified description of weather dynamics. These Lorenz equations have tremendous mathematical and historical significance. While simulating the equations' dynamics on a computer, Lorenz found that tiny round-off errors result in hugely different outcomes, a discovery that launched the field of chaos theory.

Osinga explains that Lorenz' equations describe a flow in three-dimensional space, and the Lorenz manifold corresponds to a certain specific part of a river. "If you throw a leaf in the water and watch it flow downstream toward a rock, the leaf might go to the right or left of the rock," she says. "But there are particular points where, if you drop the leaf exactly there, it will flow down and get stuck on the rock." The Lorenz manifold is the two-dimensional surface consisting of all the points where you can drop a leaf and it will flow to the rock, which is represented by the central point, or origin, in a three-dimensional coordinate space.

Since the system is chaotic, the Lorenz manifold twists around with many changes in curvature. To build a computer image of the surface, Osinga and Krauskopf devised an algorithm that starts at the origin and works its way outward in concentric rings. For each ring, the algorithm looks for points from which an object would flow to the origin. The algorithm can't find all such points, since there are infinitely many, so instead it identifies a collection of prototypical points that are about evenly spaced along the surface and then connects neighboring points by links so that the resulting mesh will resemble the Lorenz surface. In areas where the surface has floppy, hyperbolic geometry, the algorithm will identify many mesh points; where the surface has more tightly curved geometry, the algorithm will identify fewer points.

Osinga realized that the mesh instructions could be read as a cro-



OFF THE HOOK — Line-by-line crocheting instructions that tell where to increase or decrease numbers of stitches create the global shape of the Lorenz manifold.

TAIMINA; UNIV. BRISTOL

chet pattern: Crochet outward in rings and simply add or remove stitches to suit the mesh pattern. As the fabric grew under her nimble fingers—Osinga has been crocheting since age 7—it automatically took on the curvature of the Lorenz manifold.

"Just local information about where to increase stitches created the entire global shape," Osinga says. When Osinga had finished crocheting, she and Krauskopf mounted the fabric on garden wire, and it indeed took the shape of the Lorenz manifold, Osinga says.

Unlike Taimina's hyperbolic planes, whose crochet instructions can be summed up in a single sentence, the instructions for the Lorenz surface fill two pages of a paper that Osinga and Krauskopf published in 2004. "An expert needleworker will be able to [crochet a hyperbolic plane] while having a nice conversation or watching TV," the pair say in the paper. "Crocheting the Lorenz manifold, on the other hand, requires continuous attention to the instructions in order not to miss when to add or indeed remove an extra crochet stitch."

Despite the difficulty of making a Lorenz manifold, Osinga hears regularly from crocheters trying to follow her pattern, which is available at a link from her Web site. "I get emails from crafters who are not at all scientifically inclined but want to understand what they are making," she says. "They ask very intelligent math questions."

Like Taimina's hyperbolic planes, Osinga's Lorenz manifold has taken to the road frequently since its construction, making appearances at mathematical conferences, at art shows, and even on television news. "In my teaching, the students take me way more seriously now," she says. "This complicated math I do, which seems so useless, gets you on TV."

A MENAGERIE OF MODELS While Taimina's and Osinga's models have achieved the most fame, a host of other mathematicians in recent years has started crocheting and knitting mathematical shapes. An exhibit of mathematically inspired fiber arts at the 2005 annual Joint Mathematics Meeting in Atlanta boasted an impressive array of such models. In addition to Taimina's hyperbolic planes and a Lorenz surface crocheted by Yackel, the exhibit featured Möbius strips, which are twisted rings that have only one side, and Klein bottles, which are closed surfaces that have no inside. There were also crocheted versions of the five Platonic solids—the cube, the tetrahedron, the octahedron, the dodecahedron, and the icosahedron—as well as a bricklike fractal object called Menger's sponge.

It's not clear just why mathematical craftwork has suddenly taken off, says sarah-marie belcastro, a mathematician at Smith College in Northampton (Mass.), who organized the exhibit with Yackel. "Part of me says it's because there are so many more women in math now," she says. "But every time we give talks, there are men in the audience who say they knit or crochet."

For a gathering last March in Atlanta to honor mathematics writer Martin Gardner, belcastro and Yackel created doughnut-shaped surfaces, called tori. The patterns on their tori illustrate two well-known mathematical ideas about maps and networks on a torus.

Given a map showing several countries, consider the ways to color each country so that no neighboring countries have the same color. In 1976, mathematicians famously proved that in the flat plane, no such map would require more than four colors. On a torus, however, where there are more ways for a country to wrap around and touch another country, mathematicians showed as long ago as 1890 that as many as seven colors can be required. Yackel's crocheted torus displays one seven-color map that, remarkably, has only seven countries on it—every country touches every other.

DOUGHNUT MATH — The two tori at top display a network (left) and a colored map of countries (right) that can't be depicted on a flat sheet of paper without crossings and overlaps.

Belcastro's knitted torus, which can be seen as a companion piece to Yackel's, displays an intriguing fact about networks on the torus. The torus depicts a collection of points connected by paths. This network is derived from the map on Yackel's torus by marking one point inside each country and then connecting each pair of points by a path, like a railroad line, that crosses the boundary between their respective countries. Such a network of seven points, each connected to every other by a path, can't be drawn in the flat plane without some paths crossing. On the torus, however, as belcastro's knitting demonstrates, the paths can snake around the hole and avoid each other.

Belcastro and Yackel thought that making the tori would be a simple matter since pictures of the seven-color map and the corresponding network on the torus are readily available. However, it turned out to be "a nightmare," belcastro says. The challenge was figuring out how to make lines and boundaries look smooth despite the discrete nature of the stitching.

Yackel and belcastro are now editing a book to be called *Making Mathematics with Needlework*. It will feature patterns and mathematical discussions of 10 craft projects, including knitting, crocheting, embroidery, and quilting. The book isn't due out until spring. Nevertheless, this holiday season, instead of the ubiquitous gift sweater, you might want to consider knitting a Möbius scarf or a Klein bottle hat, or crocheting some hyperbolic Christmas tree ornaments. ■

HYPER GROWTH — Because the hyperbolic plane grows exponentially, the violet outer boundary consumes as much yarn as the deep-purple center section does.

C. COLEMAN; TAIMINA