Brunnian Clothes on the Runway: Not for the Bashful

Colin Adams, Thomas Fleming, and Christopher Koegel

Figure 1. Ralf's models strut their stuff.

Heads turned on the fashion runways in Milan this week when Ralf Laurent displayed his new line. Mouths fell open as models draped in Brunnian clothes strutted their stuff.

"Yes, it is risqué," said Laurent, "but the purpose of fashion is to challenge, to flaunt unconventionality, to confront us with the evanescence of material goods."

Vogue managing editor Danielle Witherspoon explained. "Brunnian clothes are made of a fabric with a rather unusual property. It is woven out of many, many tiny individually unknotted loops. It works just fine and has a wonderful drape. But the moment any one loop is broken, the entire piece of clothing falls apart into thousands of unlinked trivial loops lying at your feet. It's the hottest thing since Saran Wrap ponchos."

Fashion aficionados were caught off guard, not ever having heard of Brunnian clothing before. Carol Pantaloon, a representative of Laurent Industries provided the details. "You see, a link is just a set of closed loops that can be individually knotted and tangled together. Each loop is called a component. Figure 2 shows a random link of three components.

"The very simplest link of n components is just a set of unknotted untangled rings, called the unlink of n components. Figure 3 displays the unlink of five components.

"We call a link Brunnian if the disappearance of any single component leaves an unlink of $n - 1$ components. This has to hold no matter which component is removed. The simplest such link is the famous Borromean rings in Figure 4, named for the Borromeo family of the Italian Renaissance, who used it on their family crest.

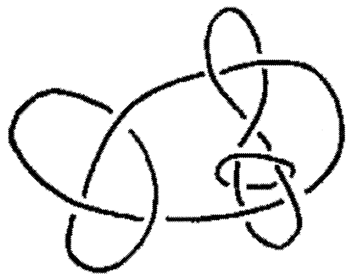

Figure 2. A link.

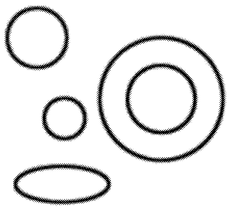

Figure 3. A trivial link of five components.

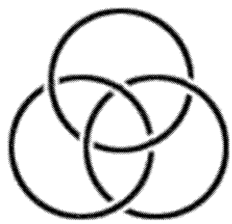

Figure 4. The Borromean links are Brunnian.

"An easy way to make a Brunnian link with arbitrarily many components is to create a long linked chain of rubber bands, and then to hook the first to the last as in Figure 5.

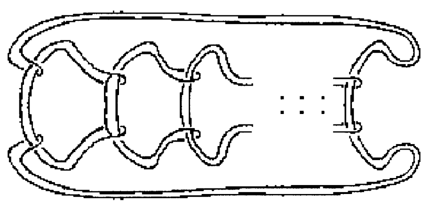

Figure 5. A Brunnian link of arbitrarily many components.

"The idea of a Brunnian link was introduced by Hermann Brunn in an 1892 paper '*Über Verkettung*' (see [**1**]). As Ralf often perused old math journals for fashion ideas, he came across Brunn's work, and it immediately got him to thinking. What if this idea for links could be extending to fabric? What if there were a way to make Brunnian clothing? What a statement for the wearer: 'I am so self-confident I wear clothes where even the slightest friction might tear a loop, causing the clothes to disintegrate into a million tiny unknots. But do I care? Am I nervous? Not a bit. I almost hope it happens.'

"Ralf first developed a simple cylindrical piece of Brunnian cloth. The square piece of cloth that you see in Figure 6 is not Brunnian, since it immediately comes apart as is. But if the square is made into a cylinder by curling the left side around to the right and then hooking the loops on the left through the loops on the right in a manner similar to the way the rest of the loops hook around one another, the result is a Brunnian tube, as in Figure 7. This can be worn in place of a halter top or T-shirt.

"Not satisfied to stop at tube tops, Ralf also designed a Brunnian muscle shirt. The trick is to build up two bridges from the front across to the back, as in Figure 8. This relies on Ralf's ability to taper the cloth by changing the number of components in adjacent rows in the link. This tapering can be seen near the bottoms of the arm holes

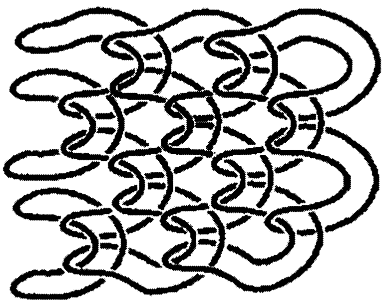

Figure 6. A square piece of cloth.

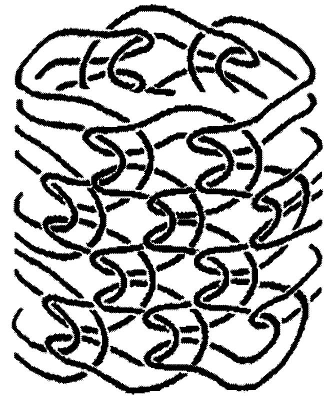

Figure 7. Brunnian cylinder from square piece of cloth.

as well as at the base of the neck line. Sealing the vertical edges of the cloth is also necessary along shoulder straps.

"Ultimately Ralf wanted to be able to make Brunnian clothes for dogs, fish, you name it. He wanted to be able to clothe species, no matter what shape, no matter how many holes or appendages. This meant being able to cover objects like a sphere or a doughnut, and perhaps objects with additional handles as well. And he needed to be able to put in as many arm, leg, or head holes as necessary. Maybe a few extra holes just as a statement.

"For the sphere, he shrunk the holes at the top and bottom of the cylinder as in Figure 9.

"For the doughnut, he took the cylinder, pulled the top edge around to the bottom edge and hooked them together as in Figure 10. This particular design is nice in that each loop in the fabric is woven in exactly the same pattern as every other loop.

"To add a handle, one makes an additional cylinder and then attaches it to a hole in the cloth. The hole is pictured in Figure 11. Some of the components around the hole, the ones labeled A, do not have loose ends. Those that do are labeled B. The top end of the cylinder is pictured in Figure 12.

"Unlike the cylinder in Figure 8, the ends of the last row of components are left loose and are labeled A' or B'. To attach the handle, A' loops are attached to A loops

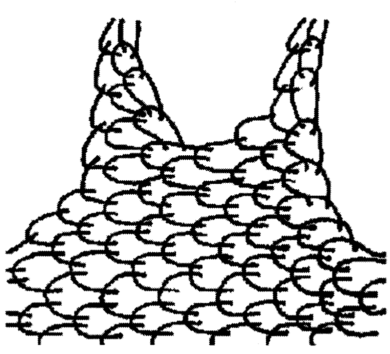

Figure 8. A Brunnian muscle shirt. Each "C" represents a circle linked with its neighbors, as in Figure 6.

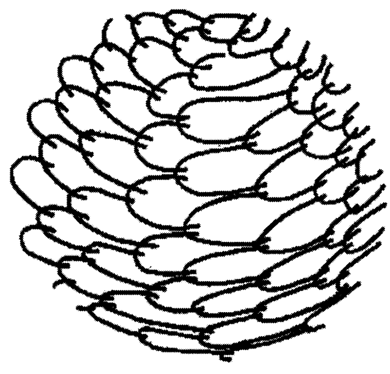

Figure 9. A snug sphere warmer.

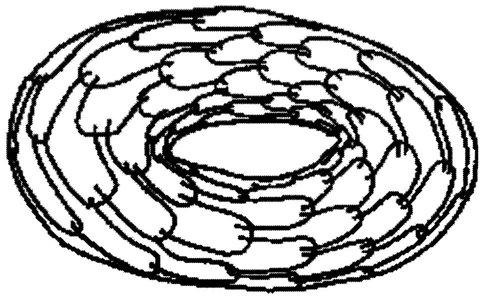

Figure 10. A doughnut cozy.

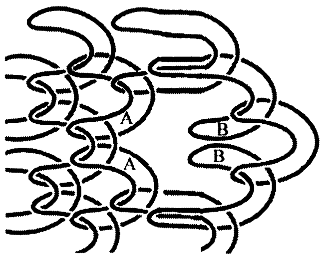

Figure 11. Labeling the hole.

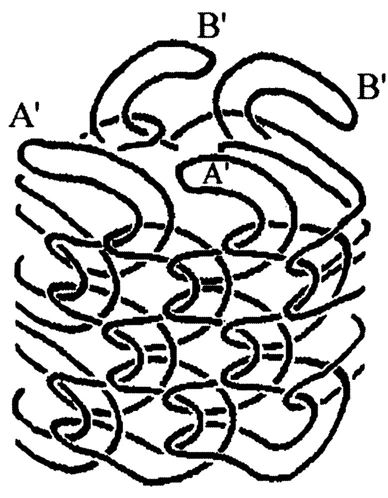

Figure 12. The top of the cylinder.

and B' loops to B loops in the manner depicted in Figure 13. When done on both ends of the tube, the result is a new handle.

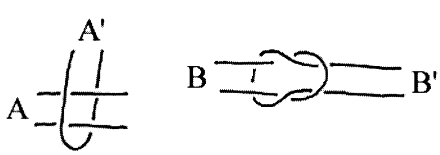

Figure 13. Attaching components.

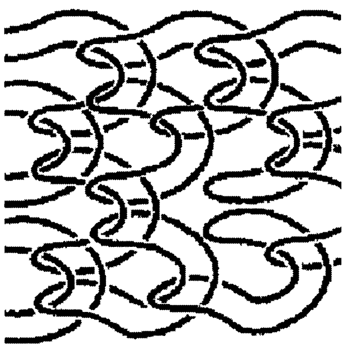

Figure 14. Removing a component.

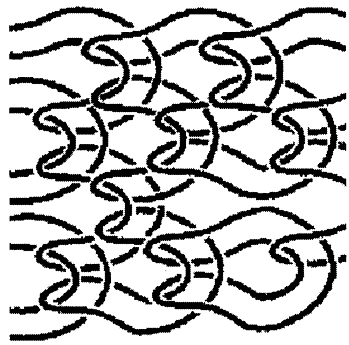

Figure 15. Reattaching the loose ends.

"To make a hole, we remove a single component as in Figures 14 and 15. By adding handles and holes to the spherical Brunnian clothes, Ralf is able to cover any shape you might come across in three-space, even something like the model in Figure 16."

Figure 16. We can cover this.

Besides Ralf Laurent, Donna Karin is also showing Brunnian clothes this season. Her method is to create a long cyclic linked chain as in Figure 5, and then to weave the chain through itself like a potholder to create cloth in various styles.

Laurent scoffed at his competition. "When a component is broken in her cloth, unraveling proceeds only along the chain, propagating in one dimension. When a link is broken in my cloth, unraveling propagates both vertically and horizontally. Which will leave you naked faster?"

I asked Ralf whether the individual components can be made perfectly round. "Good grief no," he answered. "Haven't you read Freedman and Skora [**3**]? There is no Brunnian link made from perfectly round circle components. However, if you make them from a springy material that starts round, like some kind of small rubber washers that have been deformed to make the Brunnian cloth, the disintegration process is dramatically accelerated. One loop breaks—and boing, boing, boing, no clothes."

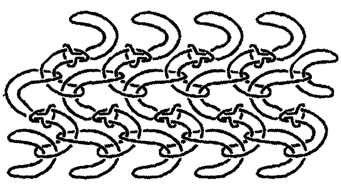

Figure 17. This cloth unravels even faster.

Ralf's design team has been working on modifications of his Brunninan cloth that unravel even faster. The latest prototype is shown in Figure 17. "The goal is to break the three-second barrier," said Ralf. "We should be able to achieve a new world record."

In addition, thinking of Donna's chain as one-dimensional Brunnian cloth, and his current design as two-dimensional Brunnian cloth, Ralf plans to make three-, four-, or even n-dimensional cloth. (The dimension refers to the number of independent directions in which the unraveling propagates. All cloth is made of one-dimensional components.) Ralf demonstrated his idea by showing me how to create his two-dimensional cloth from many copies of one-dimensional cloth. First he put several chains side by side, as in Figure 18, then he detached two loops in neighboring chains as in Figure 19, and connected the two chains with a new component to get Figure 20. Repeating the process Ralf arrived at his two-dimensional cloth.

Thus, to create n-dimensional cloth, one first fills n-space with layers of $(n-1)$-dimensional cloth. Detaching neighboring components and connecting their sheets with the same move that Ralf used earlier, the n-dimensional link can be built up. When a component is cut, the $(n-1)$-dimensional cloth still unravels in all $n-1$ directions, and the new components carry the unraveling in one additional direction to the other sheets. Tapering, holes, and handles may be added to higher dimensional cloth in a manner similar to those in use for two-dimensional cloth. The beginning

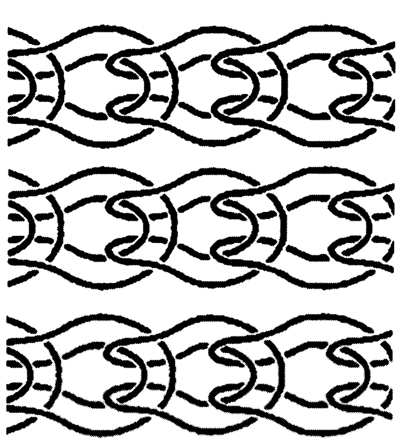

Figure 18. Several Brunnian chains.

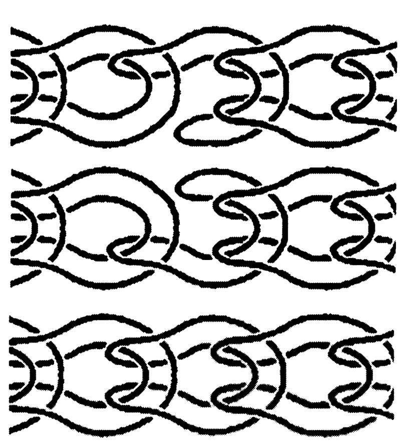

Figure 19. Freeing neighboring components.

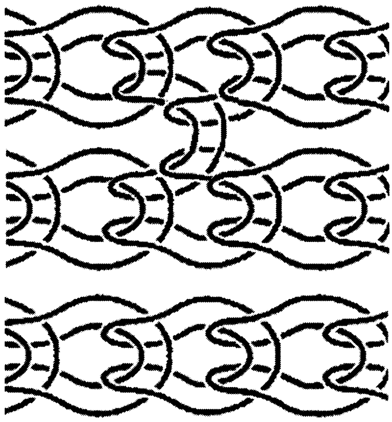

Figure 20. Adding a new component.

moves of forming three-dimensional cloth can be seen in Figure 21, where the grey component connects adjacent layers.

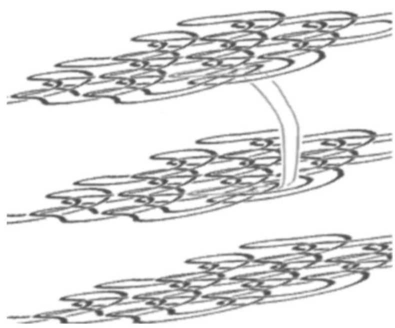

Figure 21. Forming three-dimensional Brunnian cloth.

While current textile technology can only produce Brunnian links of dimension at most three, Laurent is undaunted. “Fashion is the realm of the mind,” he said, “who cares if technology can’t keep up?”

Ralf explained that his designs can be generalized in other ways as well. A link is said to be (k, n)-Brunnian if it consists of n components and k is the smallest integer such that the removal of any k components makes it trivial. With this terminology a Brunnian link becomes a $(1, n)$-Brunnian link. We saw in Figure 5 that there exist $(1, n)$-Brunnian links for all positive integers n.

Notice that if each individual component of a given link is unknotted when considered by itself, then that link is automatically $(n - 1, n)$-Brunnian. So we are left to consider the (k, n)-Brunnian links with $n \geq k + 2$. Indeed, I asked Ralf, “Is it true that there is a (k, n)-Brunnian link for any k no smaller than two and any positive integer n such that $n \geq k + 2$?”

“Oh, yes,” he replied. “You must read Debrunner [**2**] and Penney [**4**]. But it would be interesting to find other constructions of such links.” “Will we see (k, n)-Brunnian clothes in the near future?” I asked. Laurent smiled. “It is only a matter of time.”

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