Combining non-Euclidean Geometry and Fiber Arts:

Crocheting constructions of the hyperbolic plane in $\mathbb{R}^3$

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1 Curvature

1.1 Extrinsic curvature

In the $\mathbb{R}^2$ plane, straight lines have zero curvature and circles have constant curvature. For a circle of radius R, the curvature can be defined as $\frac{1}{R}$. This is fairly intuitive: small circles curve a lot, while arcs of very large circles appear to be nearly straight.[7] For smooth curves without constant curvature, the curvature must be defined at a point P rather than for the curve as a whole. If we choose points on either side of P, we can find the circle that passes through the three points. Choosing points closer and closer to P will produce better approximations of the curve. The curvature at P is defined by

$$\kappa(P) = \frac{1}{R} \tag{1}$$

where R is the radius of the osculating circle (the circle that is the best approximation of the curve). The sign of $\kappa(P)$ can be determined by choosing the direction of the normal vector at P. By convention, the curvature is positive if the curve lies on one side of the line tangent to P and the normal vector points to the same side. If the normal vector points to the opposite side of the tangent line as the curve, the curvature is negative. The curve will cross the tangent line at P if $\kappa(P)$ is zero.[5] This means of determining curvature requires us to observe how a one-dimensional curve is embedded in the $\mathbb{R}^2$ plane, so it is an *extrinsic* measure.

A similar approach can be used for the curvature of surfaces in $\mathbb{R}^3$, as shown by Euler. Given a smooth surface Σ and a point P on Σ, we define the normal line l as the line through P perpendicular to the tangent plane at P. We then form the intersection of Σ with a plane that contains l and compute the curvature κ of the intersection curve (i.e., the normal section) in that plane, given by

$$\kappa = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \tag{2}$$

where κ_1 is the largest curvature, κ_2 is the smallest curvature, and θ is the angle between the planes corresponding to κ_1 and κ. The planes corresponding to κ_1 and κ_2 are orthogonal. The sign of the normal curvature is dependent upon the choice of the normal. Replacing the normal line l with a normal vector to the surface allows us to fix the sign of the curvature.[5] This is also a measure of extrinsic curvature, as Euler's method depends on knowing how the surface is embedded in space.[7]

1.2 Gaussian curvature

Consider an ant on a sphere. As she crawls around the surface of the sphere, the ant can only observe the surface in two dimensions. The ant can still determine that she is living on a sphere without measuring extrinsic curvature. If she ties a short rope of length r to a post at the point P, holds the other end of the rope, and walks in a circle around the post with the rope held taut, she can measure the distance δ around the circle. She knows that the circumference of a Euclidean circle, so she can determine that she lives on a sphere if $\delta < 2\pi r$.[7] The ant observes the intrinsic properties of the surface to come to this conclusion, rather than external knowledge of the shape. The amount that the circumference of a circle differs from $2\pi r$ is called the angle defect.[1] This allows us to define the intrinsic curvature at a point P to be

$$K_P = \lim_{r \to 0} \frac{\theta_r}{A_r} \tag{3}$$

[1] Or the angle excess, if greater than $2\pi r$.

where θ_r is the angle defect of a circle centered at P and A_r is the area of the circle. If $K_P \neq 0$ (i.e., the surface is not flat), then $A_r \neq \pi r^2$.[1]

Gauss defined the intrinsic curvature of a surface at a point P as

$$K = \kappa_1 \kappa_2 \tag{4}$$

where κ_1 and κ_2 are the principal curvatures[2] at P.[2] While (3) is equivalent to (4), the Gaussian curvature as defined in (4) is rather more useful. The sign of K is not affected by how we choose the direction of the normal vector when finding κ_1, κ_2, as long as we are consistent; both will change signs if we reverse the normal vector.

Spheres, like circles, have constant curvature. K, the product of the principal curvatures, will always be positive since $\kappa_1 = \kappa_2$. A point P on a sphere Σ with radius r is determined by the intersection of two great circles, also with radius r. The curvature of each of the great circles is $\frac{1}{r}$ as defined above, so the curvature at P and other points on Σ is $\frac{1}{r^2}$. An intuitive interpretation of "positive curvature" is that all of the normal sections curve in the same direction, meaning the surface is convex.[5]

A hyperbolic plane $\mathcal{H}$ will also have constant curvature. Unlike spheres, however, one of the principal curvatures will be positive while the other is negative. By (4), $\mathcal{H}$ has *negative* constant curvature

$$K = \frac{-1}{\rho^2} \tag{5}$$

where ρ is the radius of the hyperbolic plane. This will be further discussed in 2.4.

The *Theorema Egregium* follows from the definition of curvature. It states that the Gaussian curvature is an intrinsic invariant of the surface.[2] The manner in which the surface is embedded in $\mathbb{R}^3$ does not affect the determination of curvature. Extending this to the Gauss-Bonnet Theorem means that a change in the embedding of a surface that changes the curvature at some point P_1 will also change the curvature at some other point P_2 such that there is no net change in the curvature of the surface.[1]

2 Constructing the hyperbolic plane

Why construct hyperbolic planes in $\mathbb{R}^3$? Discussions on the shape of the universe aside, we are living in a Euclidean world.[3] Representing the various models of the hyperbolic plane such as the Poincaré disc and the upper half-plane can be very difficult to do in an intuitive manner. Just as attempting to map the surface of a sphere to a plane leads to the sort of distortion that causes Greenland to appear as large as Africa, mapping the hyperbolic plane to $\mathbb{R}^2$ is also distorted.[1] This distortion has been used for artistic effect by M. C. Escher and Carlo Séquin, among others, but it tends to make explorations of hyperbolic geometry more difficult.[7]

As a consequence of the *Theorema Egregium*, it is impossible to isometrically embed the hyperbolic plane as a complete subset of $\mathbb{R}^3$.[2]

> Hilbert proved that there is no real analytic isometric embedding of the hyperbolic plane onto a complete subset of 3-space, and his arguments also work to show that there is

[2] The principal curvatures κ_1, κ_2 are the same as the curvatures of the intersection curves in Euler's definition of extrinsic curvature.

[3] And I am a Euclidean girl.[4]

> no isometric embedding whose derivatives up to order four are continuous. Moreover, in 1964, N. V. Efimov extended Hilbert's result by proving that there is no isometric embedding defined by functions whose first and second derivatives are continuous. However, in 1955, N. Kuiper proved that there is an isometric embedding with continuous first derivatives of the hyperbolic plane onto a closed subset of 3-space.[3]

Henderson and Taimina began with Thurston's work[8] to describe finite surfaces that can apparently be extended indefinitely, but and not always differentiably embedded.

Given a hyperbolic plane $\mathcal{H}$ and the Euclidean space $\mathbb{R}^3$, there exists a Gauss map γ such that

$$\gamma : \mathcal{H} \hookrightarrow \mathbb{R}^3 \tag{6}$$

is an embedding (an injective and structure-preserving map). We shall define a construction as an approximation of an isometric embedding (6) of the hyperbolic plane as a surface in $\mathbb{R}^3$[3], as opposed to compass-and-straightedge constructions.[4]

2.1 The annular hyperbolic plane

William Thurston first developed a construction of the hyperbolic plane by using paper annuli held together with tape.[8] An annulus is the region between two concentric circles; an annular strip is a portion of an annulus cut off by an angle from the center of the circles.[3] An approximation of the hyperbolic plane is constructed by cutting out of paper several identical annular strips of radius ρ and thickness δ.[5] The strips are attached together with the inner circle of one taped to the outer circle of another or with the straight ends together. Holding ρ fixed while letting $\delta \to 0$ allows us to obtain the actual annular hyperbolic plane.[3][7]

2.2 Crocheting the hyperbolic plane

Choose a yarn that is not very stretchy (cheap acrylic yarn works very well) and a crochet hook slightly smaller than the size recommended on the label.[6]

- Work 10-20 chain stitches.
- For each row, single crochet for the first n stitches.
- The $(n+1)^{\text{st}}$ stitch should be worked as an increase (single crochet into the same loop as the n^{th} stitch).
- Continue in this manner to the end of each row, work a chain stitch, and begin the next row.
- Repeat until the model is as large as desired.

This method is very easy to use, but it is definitely an approximation. It cannot be described by analytic equations. A more precise approximation can be produced by examining the geometry of the pseudosphere and performing some calculations based on the size of a single crochet stitch.

[4] These constructions could also be described as yarn-and-crochet-hook constructions.

[5] David Henderson's template can be downloaded here: http://www.math.cornell.edu/∼henderson/ExpGeom/annuli.jpg

[6] For those readers who do not already know how to crochet, a very good reference can be found in [6].

2.3 The pseudosphere and symmetric hyperbolic planes

A pseudosphere is the surface generated by rotating a tractrix about its asymptote. In 1868, Eugenio Beltrami proved that hyperbolic geometry holds locally on the pseudosphere. To crochet the pseudosphere, we would ideally start from a point. This is not physically possible, so instead we start with only a few chain stitches to make a tiny circle. We then crochet in a spiral, increasing with some constant ratio as described above. The project will initially form a cone shape, then it will flare out and form ruffles. The cone will be absorbed by the ruffles if the pseudosphere is made large enough.

A symmetric hyperbolic plane is produced by using the pseudosphere method and adjusting the rate of increase to be sure the surface will have constant negative curvature. We begin with the equation

$$C = 2\pi\rho \sinh\left(\frac{r}{\rho}\right) \tag{7}$$

which can also be written as

$$C = \pi\rho\left(e^{r/\rho} - e^{-r/\rho}\right) \tag{8}$$

where ρ is the radius of the hyperbolic plane to be crocheted, r is the intrinsic radius of a circle (intrinsic meaning measured along the surface of the hyperbolic plane; a symmetric hyperbolic plane will consist of crocheting "concentric" intrinsic circles), and C is the intrinsic circumference of a circle with intrinsic radius r on a hyperbolic plane with radius ρ. Since r depends on the height of a crocheted row h, the intrinsic radius of the nth row is $r_n = nh$. For each row, the intrinsic circumference $C(n)$ is

$$C(n) = \pi\rho\left(e^{nh/\rho} - e^{-nh/\rho}\right) \tag{9}$$

The ratio $\frac{C(n)}{C(n-1)}$ determines how to increase stitches. This needs to be a fraction of the form $\frac{(k+1)}{k}$, where k is an integer, to crochet the plane. The number of stitches in the n^{th} row is determined by $S(n) = \frac{C(n)}{w}$, where w is the width of one stitch. We can use (9) and the ratio $\frac{C(n)}{C(n-1)}$ to construct a table determining the increase ratio for each row. As the plane grows, the increase ratio will eventually stabilize. While r is small, the $e^{-r/\rho}$ term in (9) is significant. When r is large enough, we find the limit $\lim_{r\to\infty} = 0$ and (9) becomes

$$C(n) = \pi\rho e^{nh/\rho} \tag{10}$$

for sufficiently large n.

2.4 Radius and curvature of hyperbolic planes

The radius of an annular hyperbolic plane is defined as ρ, the radius of the annuli. We obtain different hyperbolic planes depending on the value of ρ; small values produce very "ruffly" planes while larger values produce less ruffly planes. As ρ increases, the plane has less curvature and hence becomes flatter. In fact, as ρ goes to infinity the plane becomes indistinguishable from the Euclidean plane. This also holds true for the sphere.[7]

3 Proof that we obtain a hyperbolic plane

The annular method of construction satisfies the following descriptions of the hyperbolic plane.

3.1 Defining geodesic coordinates

Proving that we obtain the hyperbolic plane is much easier with a system of natural coordinates. We begin by defining H_δ as the approximation of the annular hyperbolic plane constructed from annuli of fixed inner radius ρ and thickness δ. Pick as the base curve the inner curve of any annulus on H_δ and pick any point on the curve to call the origin O. Now we construct an intrinsic coordinate system $\mathbf{x}_\delta : \mathbb{R}^2 \to H_\delta$ by defining $\mathbf{x}_\delta(0,0) = O$ and $\mathbf{x}_\delta(w,0)$ to be the point on the base curve at a distance w from O. We further define $\mathbf{x}_\delta(w,s)$ to be the point at a distance s from $\mathbf{x}_\delta(w,0)$ along the radial curve[7] through $\mathbf{x}_\delta(w,0)$, choosing the positive direction to be from the outer curve to the inner curve of each annulus. If we were to crochet indefinitely, this coordinate map is one-to-one and onto.

Exploration of the crocheted hyperbolic plane allows us to easily observe several facts. First, the radial curves are geodesics[8] that have intrinsic reflection symmetry due to the symmetry of the annuli and the fact that the radial curves intersect the bounding curves at right angles. Reflection through these curves is an isometry. Second, the radial geodesics are asymptotic. If λ and μ are radial geodesics in H_δ, the distance between them changes by $\frac{\rho}{\rho+\delta}$ every time they cross an annulus. When we cross n annuli at a distance $c = n\delta$ from the base curve, the distance between λ and μ is

$$d\left(\frac{\rho}{\rho+\delta}\right)^n = \left(\frac{\rho}{\rho+\delta}\right)^{c/\delta}$$

Taking the limit as $\delta \to 0$ gives

$$d \exp -c/\rho \tag{11}$$

as the distance between λ and μ. Spheres and Euclidean planes never have asymptotic geodesics. Finally, there is an isometry that preserves the annuli. The composition of two reflections through radial geodesics must be an isometry that preserves each annulus, but it cannot be a rotation (as it would be in the Euclidean plane) because an annulus has no center. Furthermore, the radial geodesics do not intersect so the isometry has no fixed point. It also cannot be a translation because the isometry does no fixed geodesic. Instead, the isometry is called a *horolation* and can be thought of as shifting the annulus along itself or as a rotation about a point at infinity. The annular curves are called *horocycles* and can be described as circles with infinite radius.[3]

3.2 The hyperbolic plane as a pseudosphere

As previously discussed, the hyperbolic plane has the same intrinsic geometry as the pseudosphere. To prove this, take the annulus whose inner edge is the base curve and isometrically embed it in the $x-y$ plane as a complete annulus with center at the origin. Attach portions of the other annuli such that they form truncated cones. Let the vertical axis be the z-axis.[9] This leads to

$$\frac{\Delta R}{\Delta z} = \frac{-R(z)}{\sqrt{(\rho+\delta)^2 - R(z)^2}} \tag{12}$$

[7] Along the radii of each annulus.

[8] Intrinsically straight lines.

[9] See Figures 10 and 11 of [3], but be forewarned that the notation used in the figures is incorrect. With some intuition, the reader should be able to correct this.

Taking the limit as δ, ΔR, and Δz go to zero gives

$$\frac{dR}{dz} = \frac{-R(z)}{\sqrt{\rho^2 - R(z)^2}} \tag{13}$$

Using (11) implies that the circle at height z has circumference $2\pi\rho e^{-s/\rho}$, where s is the arc length along the surface from $(0, \rho)$ to $(z, R(z))$. If we explicitly solve for z, we get

$$z = \sqrt{\rho^2 - R^2} - \rho \ln \left| \frac{\rho + \sqrt{\rho^2 - R^2}}{R} \right| \tag{14}$$

Thus, z is a continuously differentiable function of R and the derivative (for $z \neq 0$) is never zero. This means R is also a continuously differentiable function of z and we conclude that this is the pseudosphere.[3]

3.3 Riemannian manifolds with constant negative Gaussian curvature

A surface is a Riemannian manifold if it is differentiably embedded into $\mathbb{R}^3$ by an isometry whose first and second derivatives are continuous (C^2). The *normal direction* at a given point P on the surface is chosen to be one of the two directions that are perpendicular to the surface at P. We define the *normal curvature* at a point P of a curve σ on the surface as the component of the curvature of σ in the normal direction. The principal curvatures at P are the maximal and minimal values of the collection of all normal curvatures of all smooth curves through P. As previously discussed, the Gaussian curvature of the surface at P is the product of the principal curvatures.

The pseudosphere is a Riemannian surface. The principal curvatures are the normal curvatures of the generating curves $z \mapsto R(z)$ and the circle $\theta \mapsto [R(z), \theta]$ at each point on the pseudosphere $[z, R(z), \theta]$. The curvature of the first curve, perpendicular to the surface, is

$$\frac{-R''(z)}{[1 + (R'(z))^2]^{3/2}}$$

The curvature of the circle is $1/R(z)$, defined above, and when projected perpendicular to the surface, gives the normal curvature as

$$\frac{1}{R(z)\sqrt{1 + (R'(z))^2}}$$

We can use (13) to find that the product of these normal curvatures is $\frac{-1}{\rho^2}$, thus proving that the pseudosphere has constant negative Gaussian curvature.[3]

3.4 The upper half-plane model

The coordinate map $\mathbf{x}$ defined in 3.1 preserves distances along the vertical second coordinate curves. The distances along the first coordinate curves at $\mathbf{x}(a, b)$ are distorted by a factor of $\exp(-b/\rho)$ as compared to $\mathbb{R}^2$ distances.

> *Let* $\mathbf{y} : A \to B$ *be a map from one metric space to another, and let* $t \mapsto \lambda(t)$ *be a curve in* A*. Then, the* distortion *of* $\mathbf{y}$ *along* λ *at the point* $p = \lambda(0)$ *is defined as:*
>
> $$\lim_{x \to 0} \frac{\text{arc length along } \mathbf{y}(\lambda) \text{ from } \mathbf{y}[\lambda(x)] \text{ to } \mathbf{y}[\lambda(0)]}{\text{arc length along } \lambda \text{ from } \lambda(x) \text{ to } \lambda(0)}$$

The map will be conformal (i.e., it will preserve angles) if distances are distorted to the same degree in both coordinate directions. To obtain this, we need a change of coordinates that will distort equally in both directions. We must try to make the distortion in the second coordinate direction the same as the distortion in the first, since zero distortion in both coordinate directions would be an isometry and as such is not possible. This change of coordinates is

$$\mathbf{z}(x,y) = \mathbf{x}\left[x, \rho \ln\left(\frac{y}{\rho}\right)\right]$$

where $\mathbf{x}$ is the geodesic coordinate map defined in 3.1. The domain of $\mathbf{z}$ is the upper half-plane

$$\mathbb{R}^{2+} \equiv \left\{(x,y) \in \mathbb{R}^2 | y > 0\right\}$$

This is the upper half-plane model of the hyperbolic plane.

References

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