

# How to Use History to Clarify Common Confusions in Geometry

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Most people judge the size of cities simply from their circumference. So that when one says that Megalopolis is fifty stades in contour and Sparta forty-eight, but that Sparta is twice as large as Megalopolis, what is said seems unbelievable to them. And when in order to puzzle them even more, one tells them that a city or camp with the circumference of forty stades may be twice as large as one of the circumference of which is one hundred stades, what is said seems to them absolutely astounding. The reason of this is that we have forgotten the lessons in geometry we learnt as children.

— Polybius, 2<sup>nd</sup> century B.C. [56]

## Introduction

We have found that students who take our senior/graduate level geometry course usually have very little background in geometry. From leading many workshops for professors on teaching geometry we found that even mathematicians are often confused about the history of geometry. In addition, many expository descriptions of geometry (especially non-Euclidean geometry) contain confusing and sometimes-incorrect statements — this is true even in expositions written by well-known research mathematicians. Therefore, we found it very important to give some historical perspective of the development of geometry, clearing up common misconceptions and increasing people's interest both in geometry and in the history of mathematics.

Questions that often have confusing or misleading (or incorrect) answers:

1. What non-Euclidean geometry appeared first?
2. Does Euclid's parallel postulate distinguish the non-Euclidean geometries from Euclidean geometry?
3. Is there a potentially infinite surface in 3-space whose intrinsic geometry is hyperbolic?
4. In what sense are the usual models of hyperbolic geometry "models"?
5. What does "straight" mean in geometry? How can we draw a straight line?

These questions are asked by our students and by mathematicians in our workshops. In the literature we often do not find clear answers.

In order to clarify these questions for ourselves and for our students, we find that it is important to talk about the development of different ideas in mathematics. It also gives us a way to bring the history of mathematics into the geometry course. Talking with our students, we noticed that most confusions related to the above questions come from not recognizing certain strands in the history of geometry. Thus, even though the topic of this volume is the use of more recent history of mathematics in our courses, we will start (as we do with our students) with a quick look back to various historical strands that can help us understand the issues surrounding these questions. More recent history will come later.

It is important to talk with students about deep roots of human experiences of mathematics and how it connects with modern theories. One of our student responses demonstrates how learning is motivated by history:

I found 451(Spherical and Euclidian Geometry) to be my favorite math class so far. I feel the way we are learning math is like collecting pieces of a puzzle through our education. We need lots of those pieces to form a picture (or I should say "part of a picture" here), but if we haven't collected "enough" of them, we won't be able to put them together and thus can't see anything. I think this is why my interest in mathematics is growing as I am learning more of it, and want to take [the history of mathematics class]. I want to know how and why people started to think about "mathematics".

## The Four Strands of Geometry

We introduced to our students the notion that the main aspects of geometry today emerged from four strands of early human activity, which seemed to have occurred in most cultures: art/patterns, navigation/stargazing, motion/machines, and building structures. These strands developed more or less independently into varying studies and practices that from the 18<sup>th</sup> and 19<sup>th</sup> century on were woven into what we now call *geometry*.

### *Art/Patterns Strand*

To produce decorations for their weaving, pottery, and other objects, early artists experimented with symmetries and repeating patterns. Later the study of symmetries of patterns led to tilings, group theory, crystallography, finite geometries, and in modern times to security codes and digital picture compression. Early artists also explored various methods of representing objects and living things. These explorations led to the study of perspective and then projective geometry and descriptive geometry, and (in 20th century) to computer-aided graphics, the study of computer vision in robotics, and computer-generated movies (for example, Toy Story). For more details, see [1], [3], [15], [18], [21], [23], [24], [26], [27], [40], [41].

### *Navigation/Stargazing Strand*

For astrological, religious, agricultural, and other purposes, ancient humans attempted to understand the movement of heavenly bodies (stars, planets, sun, and moon) in the apparently hemispherical sky. Early humans used the stars and planets as they navigated over long distances, thus they were able to solve problems in navigation and to understand the shape of the Earth. Ideas of trigonometry apparently were first developed by Babylonians in their studies of the motions of heavenly bodies. Even Euclid wrote an astronomical work, *Phaenomena*, [4], in which he studied properties of curves on a sphere. Navigation and large-scale surveying developed over the centuries around the world and along with it cartography, trigonometry, spherical geometry, differential geometry, Riemannian manifolds, and thence to many modern spatial theories in physics and cosmology. For more details, see [2], [4], [10], [20], [22], [53], [62].

### *Motion/Machines Strand*

Early human societies used the wheel for transportation, making pottery, and in pulleys. In ancient Greece, Archimedes and Hero used linkages and gears to solve geometrical problems – such as trisecting an angle, duplicating a cube, and squaring a circle. These solutions were not accepted in the Building Structures (Euclid) Strand, which leads to a common misconception that these problems are unsolvable and that Greeks did not allow motion in geometry. See [42] and [49]. We do not know much about the influence of machines on mathematics after Ancient Greece until the advent of machines in the Renaissance. In the 17<sup>th</sup> century, Descartes used linkages and the ideas of linkages to study curves. This study of curves led to the development of analytic geometry. Computing machines were also developed in the 17<sup>th</sup> century. As we will discuss below, there was an interaction between mathematics and mechanics that leads to the modern mathematics of rigidity and robotics. For more detail see [14], [16], [19], [52], [58], [65].

### *Building Structures Strand*

As humans built shelters, altars, bridges, and other structures, they discovered ways to make circles of various radii and various polygonal/polyhedral structures. In the process they devised systems of measurement and tools for measuring. The (2000-600 BC) *Sulbasutram* [60], written for altar builders, begins with a geometry handbook with proofs of some theorems and a clear general statement of the “Pythagorean” Theorem. Building upon geometric knowledge from Babylonian, Egyptian, and early Greek builders and scholars, Euclid (325-265 BC) wrote his *Elements*, which became the most used mathematics textbook in the world for the next 2300 years and codified what we now call Euclidean geometry. Using the *Elements* as a basis in the period 300 BC to about 1000 AD, Greek and Islamic mathematicians extended its results, refined its postulates, and developed the study of conic sections and geometric algebra. Within Euclidean geometry, analytic geometry, vector geometry (linear algebra and affine geometry), and algebraic geometry developed later. The *Elements* also started what became known as the axiomatic method in mathematics. For the next 2000 years mathematicians attempted to prove Euclid's Fifth (Parallel) Postulate as a theorem (based on the other postulates); these attempts culminated around 1825 with the discovery of hyperbolic geometry. (See below for discussion of the more recent history of parallel postulates.) Further developments with axiomatic methods in geometry led to the axiomatic theories of the real numbers and analysis, and to elliptic geometries and axiomatic projective geometry. For more detail see [7], [9], [10], [12], [14], [28], [59], [60].

We will now describe how these strands can be used to clarify issues surrounding the five questions identified in the introduction. We include those aspects of the recent history of mathematics that are either related to these issues or are difficult to find in the current literature.

## Spherical Geometry – the first non-Euclidean geometry

Answering the first question: Spherical geometry can be said to be the first non-Euclidean geometry. It developed early in the Navigation/Stargazing Strand. The connections of spherical geometry with other strands will be covered in the following sections. For at least 2000 years humans have known that the earth is (almost) a sphere and that the shortest distance between two points on the earth is along a great circle (the intersection of the sphere with a plane through the center of the sphere). For example:

...it will readily be seen how much space lies between the two places themselves on the circumference of the large circle which is drawn through them around the earth. ... [W]e grant that it has been demonstrated by mathematics that the surface of the land and water is in its entirety a sphere,... and that any plane which passes through the center makes at its surface, that is, at the surface of the earth and of the sky, great circles, and that the angles of the planes, which angles are at the center, cut the circumferences of the circles which they intercept proportionately,...

— Claudius Ptolemy, *Geographia* (ca. 150 AD), Book One, Chapter II

Spherical geometry is the geometry of a sphere. The great circles are intrinsically straight on a sphere in the sense that the shortest distances on a sphere are along arcs of great circle and because great circles have the same symmetries on a sphere as straight lines have on the Euclidean plane. The geometry on spheres of different radii is different; however, the difference is only one of scale. In Aristotle we can find evidence that spherical non-Euclidean geometry was studied even before Euclid. (See [30], p 57 and [64].) Even Euclid in his work on astronomy, *Phaenomena* [4], discusses propositions of spherical geometry. Menelaus, a Greek of the first century AD, published a book *Sphaerica*, which contains many theorems about spherical triangles and compares them to triangles on the Euclidean plane. (*Sphaerica* survives only in an Arabic version. For a discussion see [45], page 119–120.)

Up into the 19<sup>th</sup> century, spherical geometry occurred almost entirely in the Navigation/Stargazing Strand and was used by Brahe and Kepler in studying the motion of stars and planets and by navigators and surveyors. The popular book *Spherical Trigonometry: For the use of colleges and schools* (1886) by Todhunter [62] contains several discussions of the use of spherical geometry in surveying and was used in British schools before hyperbolic geometry was widely known in the British Isles. On the continent, C.F. Gauss (1777-1855) was using spherical geometry in various large-scale surveying

projects. This happened before the advent of hyperbolic geometry. Currently, 3-dimensional spherical geometry is being considered as one of the possible shapes for our physical universe. (See [64] and [34].)

## The Parallel Postulate(s)

Let us now discuss Question 2: Does Euclid's parallel postulate distinguish the non-Euclidean geometries from Euclidean geometry? After developing for some 2000 years, spherical geometry was related to the Euclidean postulates of the Building Structure Strand in the 19<sup>th</sup> and 20<sup>th</sup> centuries. There are many popular accounts (for example, Katz [42], p 782, and Davis & Hersh [13], pp 219-221) that attempt to distinguish between Euclidean and spherical geometries on the basis of the parallel postulate. In Euclid's *Elements*, the "parallel postulate" is the Fifth Postulate (here stated in Heath's translation [29], page 155, but in more modern English as given in [34], Appendix A):

(EFP) If a straight line intersecting two straight lines makes the interior angles on the same side less than two right angles, then the two lines (if extended indefinitely) will meet on that side on which the angles are less than two right angles.

The interested reader can check that EFP is not needed to assert the *existence* of parallel lines. In fact Euclid in I.31 proves (in modern wording):

Given a line  $l$  and a point  $P$  not on  $l$  there exists a line through  $P$  parallel to  $l$ .

In his proof, Euclid constructs first a line  $n$  through  $P$  perpendicular to  $l$  and then another line  $m$  through  $P$  and perpendicular to  $n$ . Then Euclid concludes (I.27) that  $m$  is parallel to  $l$  because the alternate interior angles are equal. In none of these constructions or conclusions does Euclid use his Fifth Postulate. Now, in this situation, the reader can easily see that EFP immediately implies that any line through  $P$ , other than  $m$ , will intersect  $l$  and thus  $m$  is the *unique* parallel line through  $P$  parallel to  $l$ .

Note that EFP is true in spherical geometry (because all lines intersect) and is even provable in the strong sense that the two lines meet closest on that side on which the angles are less than two right angles (see, for example, [34], Chapter 10). However, the *existence* of parallel lines is false in spherical geometry and thus it must be one of the other postulates that fails on the sphere. Thus it is not true that EFP distinguishes Euclidean from non-Euclidean geometries.

Over the years, EFP has been replaced by many other postulates (see for example, [29], p 220, [31], p 203, [13], p 219). Any of these alternate postulates plus Euclid's first four postulates imply EFP; and EFP plus the first four imply each of the alternate postulates. The most common of these alternate postulates used in modern high school textbooks and popular accounts is:

(HSP) Given a line and a point not on the line there is *one and only one* line through the point that is parallel to the given line.

We will call this the High School Postulate (HSP). Note that this postulate posits both the *existence* and *uniqueness* of parallel lines. Therefore, unlike EFP, the High School Postulate is not true in spherical geometry. Thus, in any context that includes Euclidean and spherical geometry, EFP and HSP are not equivalent.

Often, HSP is called by many authors (including us in [32, p 124]!) "Playfair's Parallel Postulate". However, Playfair actually stated ([55], p 11, 1839, and in the editions Philadelphia, 1806; Boston, 1814; Edinburgh, 1819; and New York, 1835):

(PP) Two straight lines which intersect one another, cannot be both parallel to the same straight line.

Many other authors (including David Hilbert in his axiomatization of Euclidean geometry [35]) state the Playfair postulate in its logically equivalent form:

(PP) Given a point  $P$  and a line  $l$ , there exists at most one line through  $P$  and parallel to  $l$ .

This postulate only posits *uniqueness* and thus it is true in both spherical and Euclidean geometries.

We see that **EFP** and **PP** are true in spherical geometry, but **HSP** is not true. Nevertheless, many writers are confused, for example: The excellent history text [42] (p 782) says that one of the “two possible negations of Euclid’s parallel postulate” leads to spherical geometry. In the useful paper [31] (p 203) it is stated (correctly) that **PP** is true on a sphere, but continues “so [**PP**] can only be equivalent to [**EFP**] under some additional condition ...” The delightful book [13] (pp 218-221) states **EFP** and then asserts (correctly) that it is “totally equivalent” to **HSP** (but calls it “Playfair’s Axiom” and “Playfair’s postulate”) in the sense that each plus the first four of Euclid’s postulates implies the other. However, the text then asserts that elliptic geometry (geometry on a sphere with diametrical opposite points identified) satisfies the first four postulates of Euclid but not **HSP**. It is easy to check that **EFP** holds in elliptic geometry, which contradicts the assertion of total equivalence.

All of these confusions seem to come from replacing **EFP** with **HSP**. We do not know in detail how this happened but there are two hints: Heath’s translation of Euclid has been for the past century the de facto standard and in his commentary he states [29] (p 220) “Playfair’s Axiom” as:

*Through a given point only one parallel can be drawn to a given straight line or, Two straight lines which intersect one another cannot both be parallel to one and the same straight line.*

The second statement here clearly posits *uniqueness* while the first statement could be mistaken for the **HSP**. And then Kline, in [45] (p 865) which was for many years the de facto standard history of mathematics, repeats the first statement of Heath’s and then adds:

This is the axiom used in modern books (which for simplicity usually say there is “one and only one line ...”).

Because of the 2000-year long history of investigations into parallel postulates within the Building Structures Strand, many books misleadingly call hyperbolic geometry “*the* non-Euclidean geometry” or “*the* first non-Euclidean geometry”. Spherical geometry has been studied since ancient times but it did not fit easily into this axiomatic approach and thus was (and still is) left out of many discussions of non-Euclidean geometry.

In the discussions that do include spherical geometry it is called by various names which causes more confusions: Riemannian, projective, elliptic, double elliptic, and spherical. These different labels for spherical geometry usually imply different settings and contexts as summarized in the following:

- Bernhard Riemann (1826–1866) pioneered the intrinsic (and analytic) view for surfaces and space; and, in particular, he introduced an intrinsic analytic view of the sphere that became known as the *Riemann Sphere*. The Riemann Sphere is usually studied in a course on complex analysis. Some writers use the term “Riemannian geometry” to describe spherical geometry (usually in parallel with “Lobachevskian geometry”), but this practice has led to confusion among students and mathematicians because the term “*Riemannian geometry*” is most often used to describe general manifolds as a part of differential geometry.
- *Double-elliptic geometry* usually indicates an axiomatic formalization of spherical geometry in which antipodal points are considered as separate points. This axiom system for spherical geometry is in the spirit of Euclid’s Axioms (as embellished by Hilbert [35]) for Euclidean geometry (see for example, [8]), in this context spherical geometry is usually called *double-elliptic geometry*. This axiomatization has seemed not to be very useful.
- *Elliptic geometry* usually indicates an axiomatic formalization of spherical geometry in which each pair of antipodal points is identified as one point.
- *Projective geometry* originally developed within the Art/Patterns Strand in the study of perspective drawings. If you project space (or a plane) onto a sphere from the center of the sphere (called a *gnomic projection*) then straight lines in space correspond to great circles on the sphere. Thus projective geometry can be thought of as a non-metric version of spherical geometry

- *Spherical geometry* often indicates the geometry of the sphere sitting extrinsically in Euclidean 3-space.

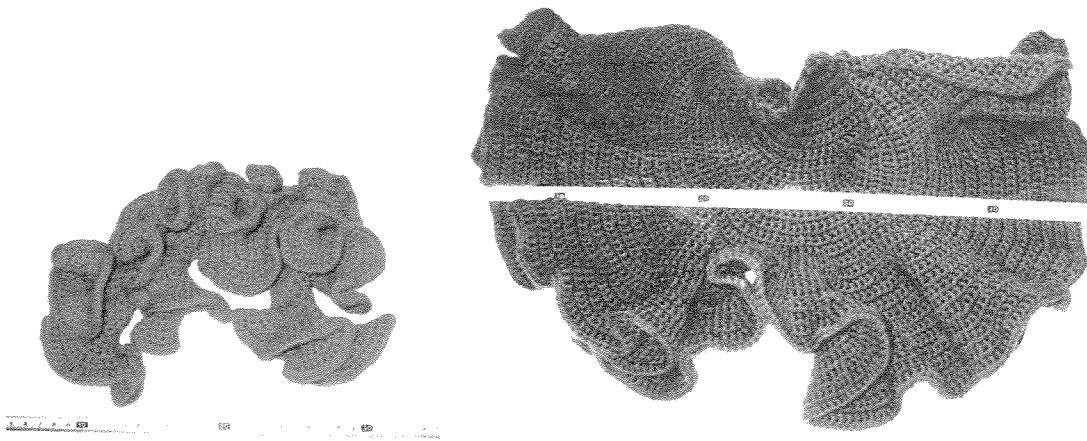
Starting soon after the *Elements* were written and continuing for the next 2000 years mathematicians attempted to either prove Euclid's Fifth Postulate as a theorem (based on the other postulates) or to modify it in various ways. These attempts culminated around 1825, when Nicolai Lobachevsky (1792-1850) and János Bolyai (1802-60) independently discovered a geometry that satisfies all of Euclid's Postulates and Common Notions except that the Fifth Postulate does not hold. It is this geometry that is called *hyperbolic geometry*.

## Hyperbolic Surfaces

Historically, one of the first open questions about hyperbolic geometry was whether it is the geometry of any surface in Euclidean space, in the same sense that spherical geometry is the geometry of the sphere in Euclidean 3-space. This is also our Question 3. In the mid-19th century beginnings of differential geometry it was shown that hyperbolic surfaces would be precisely surfaces with constant negative curvature. This aspect of hyperbolic geometry belongs in the Navigation/Star Gazing strand of geometry, in the sense that differential geometry of surfaces (and higher-dimensional manifolds) uses calculus to study the geometric properties that are *intrinsic* – properties of the surface that a bug crawling on the surface could detect. Intrinsic properties include geodesics (intrinsically straight lines), length of paths, shortest paths (which are almost always segments of geodesics), angles, surface area, and so forth.

Students (and mathematicians) desire to touch and feel a hyperbolic surface in order to experience its intrinsic properties. Many people have trouble with standard “models” and pictures of hyperbolic geometry in textbooks, because the intrinsic meanings of geodesics, lengths, angles, and areas cannot be directly seen. However, it is a common misconception that you cannot have a surface in 3-space whose intrinsic geometry is hyperbolic geometry.

Mathematicians looked for surfaces whose intrinsic geometry is complete hyperbolic geometry in the same sense that the intrinsic geometry of a sphere has the complete spherical geometry. In 1868, Eugenio Beltrami (1835-1900) described a surface, called the *pseudosphere*, whose local intrinsic geometry is hyperbolic geometry but is not complete in the sense that some geodesics (intrinsically straight lines) cannot be continued indefinitely. (See [25], p 218, for photos of the surfaces that Beltrami constructed and further discussion of the pseudosphere.) In 1901, David Hilbert (1862-1943) [36] proved that it is impossible to define by (real analytic) equations a complete hyperbolic surface. In those days, “surface” normally meant one defined by real analytic equations and so the search for a complete hyperbolic surface was abandoned and still today many works state that a complete hyperbolic surface is impossible. For popularly written examples, see [53] (p 158) and [37] (p 243). In 1964, N. V. Efimov [17] extended Hilbert’s result by proving that there is no isometric embedding defined by functions whose first and second derivatives are continuous. However, in 1955, Nicolas Kuiper [48] proved, without giving an explicit construction, the existence of complete hyperbolic surfaces defined by continuously differentiable functions. Then in some workshops in the 1970’s William Thurston described the construction of complete hyperbolic surfaces (that can be made out of paper), see [61] (p 49 and p 50).



**Figure 1.** Crocheted hyperbolic planes of radius 4 cm and 8 cm.  
(crocheted by Daina Taimina, photo by David Henderson)

Directions for constructing Thurston's surface out of paper can be found in [32] (p 48), [33], or [34], (p 58). These references also contain a description of the method (invented by the first author of this article) for crocheting these surfaces. See Figure 1 for crocheted hyperbolic planes with different radii. In addition, there is in these references a description of an easily constructible polyhedral hyperbolic surface, called the "hyperbolic soccer ball", that consists of heptagons (7-sided regular polygons) each surrounded by 7 hexagons (the usual spherical soccer ball consists of pentagons each surrounded by 5 hexagons) – this construction was discovered recently by the second author's son, Keith Henderson. The intrinsic straight lines ("geodesics") on a hyperbolic surface can be found by folding the surface (in the same way that folding a sheet of paper will produce a straight line on the paper). This folding also determines a reflection about the intrinsic straight line.

## Projections, Coordinate Systems, “Models” of Hyperbolic Plane

Question 4 asks: In what sense are the usual models of hyperbolic geometry “models”? To study the geometry of the sphere analytically we develop coordinate systems and projections (maps) of the sphere onto the plane. (Note that the usual spherical coordinates give a projection of portions of the sphere onto a rectangle in the plane.) For the same reason, coordinate systems and projections of the hyperbolic plane are useful for systematic analytic study. At the same time as he discovered the pseudosphere, Beltrami described (in 1868) a projection of the hyperbolic plane onto a disk in the plane, this projection was more fully developed in 1871 by Felix Klein (1849-1925). This projection, which can also be thought of as describing a coordinate system, is now called in the literature *projective disk model*, *Beltrami-Klein model*, or *Klein model*.

The Beltrami-Klein model (and other models) was called a “model” instead of a “coordinate system” or “projection” because of the absence of a known surface whose intrinsic geometry was complete hyperbolic geometry. But after Thurston's construction we can now describe this model (and the ones described below) as projections or coordinate systems of the complete hyperbolic surface – for details, see [34], Chapter 17.

The Beltrami-Klein model was based on projective geometry and projective transformations, which had their origins in the Art/Pattern Strand. The Beltrami-Klein model (thought of as a transformation) takes straight lines in the hyperbolic plane to straight lines in the Euclidean plane in the same way that gnomic projection takes straight lines (great circles) on the sphere to straight lines on the Euclidean plane. As with gnomic projection, the measure of angles is not preserved in the Beltrami-Klein model. (See Chapters 14 and 17 of [34].)

Other well known models of the hyperbolic plane were developed in 1882 by Henri Poincaré and were based on circles and inversions in circles. Both Poincaré models preserve the measure of angles but take straight lines in the hyperbolic plane to semicircles in the upper half-plane or disk. Reflection through a straight line on the hyperbolic plane corresponds in the Poincaré models to inversion in the semicircles. See [28] (Section 39), and [34] (Chapter 17).

Inversions in a circle are a necessary component of the analytic study of hyperbolic geometry. As we will see below they are also useful to understanding linkages that convert straight-line motion into circular motion. Our students ask questions about the origins of inversions, but there are not many sources on the history of inversions. We describe inversions as emerging from the Art/Pattern Strand and trace the ideas in the theory of inversions in circles back to Apollonius of Perga (225 B.C. - 190 B.C.), who investigated one particular family of circles and straight lines. We know about this from a commentary of Pappus of Alexandria (290-350). Apollonius defined the curve  $c_k(A,B)$  to be the locus of points  $P$  such that  $PA = kPB$ , where  $A$  and  $B$  are points in the Euclidean plane, and  $k$  is a positive constant. Now this curve is a straight line if  $k=1$  and a circle otherwise and is usually called an Apollonian Circle. Apollonius proved that a circle  $c$  (with center  $C$  and radius  $r$ ) belongs to the Apollonian family  $\{c_k(A,B)\}$  if and only if  $BC \cdot AC = r^2$  and  $A$  and  $B$  are on a same ray from  $C$ . [In modern terms, we say “if and only if  $B$  is the inversion of  $A$  with respect to  $c$  (and vice versa)”.] See Figure 2. Theory of inversions in circles can be developed purely geometrically from Euclid's Book III but was not done so in ancient times. (We would say that this is because it was part of a separate strand.) For discussion of properties of circle inversions, see [6], [11], [28] (Section 37), [34] (Chapter 16), [39], and [51] (Chapter 15).

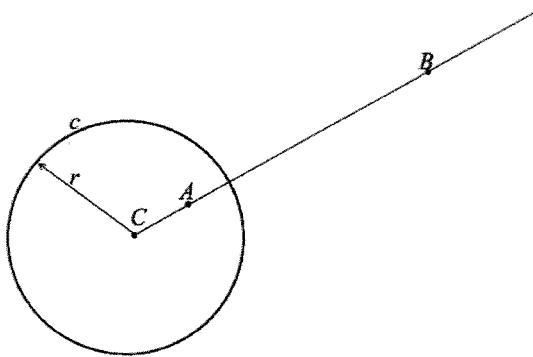


Figure 2.  $B$  is the inversion of  $A$  with respect to  $C$  (and vice versa).

A systematic study of inversions in circles started only in the 19th century. Jakob Steiner (1796-1863) was among the first to use extensively the technique of inversions in circles. Steiner had no early schooling and did not learn to read or write until he was age 14. Against the wishes of his parents, at age 18 he went to the Pestalozzi School at Yverdon, Switzerland, where his extraordinary geometric intuition was discovered. By age 28 he was making many geometric discoveries using inversions. At age 38 he occupied the chair of geometry established for him at the University of Berlin, a post he held until his death. We find these facts inspiring to students. For more details, see [47]. A complete recent history of inversions can be found in the book [50].

Inversions were also used in the 19<sup>th</sup> century to solve a long-standing engineering problem (from the Motion/Machines Strand) we describe in the next section.

## What is “Straight”? How Can We Draw a Straight Line?

This is Question 5. When using a compass to draw a circle, we are not starting with a model of a circle; instead we are using a fundamental property of circles that the points on a circle are a fixed distance from a center. Or we can say we use Euclid's definition of a circle. So, now what about drawing a straight line: Is there a tool (serving the role of a compass) that will draw a straight line? One could say: We can use a

straightedge for constructing a straight line. Well, how do you know that your straightedge is straight? How can you check that something is straight? What does "straight" mean? Think about it!

We can try to use Euclid's definition: "A straight line is a curve that lies symmetrically with the points on itself." (See [34], Appendix A, for a justification of this translation; and Chapter 1 for a discussion of "What is straight?") This leads to knowing that folding a flat piece of paper will produce a straight line. This use of symmetry, stretching, and folding can also be extended to spheres and hyperbolic planes (See [34], Chapters 2 and 5.) We can also use the usual high school definition, "A straight line is the shortest distance between two points." This leads to producing a straight line by stretching a string. Students are confused when reading in the literature that "straight line" is an undefined term, or that straight lines on the sphere are "defined to be arcs of great circles". We find that putting it in the context of the four strands helps clarify this: Symmetry comes from the Art/Pattern Strand, "undefined terms" come from the Building Structures Strand, and "shortest distance" from the Navigation/Star Gazing Strand.

But there is still left unanswered the question of whether there are mechanisms analogous to a compass that will draw an accurate straight line. Discussions about straightness in class had a different perspective after we learned ourselves and then presented to our students the following not widely known history of the problem. The students responded that it was important to learn this connection between a purely geometrical problem and a mechanical problem. It also shows that you can find a history of mathematical problems in the history of other sciences, in this case mechanics, which leads us into the Motion/Machines Strand.

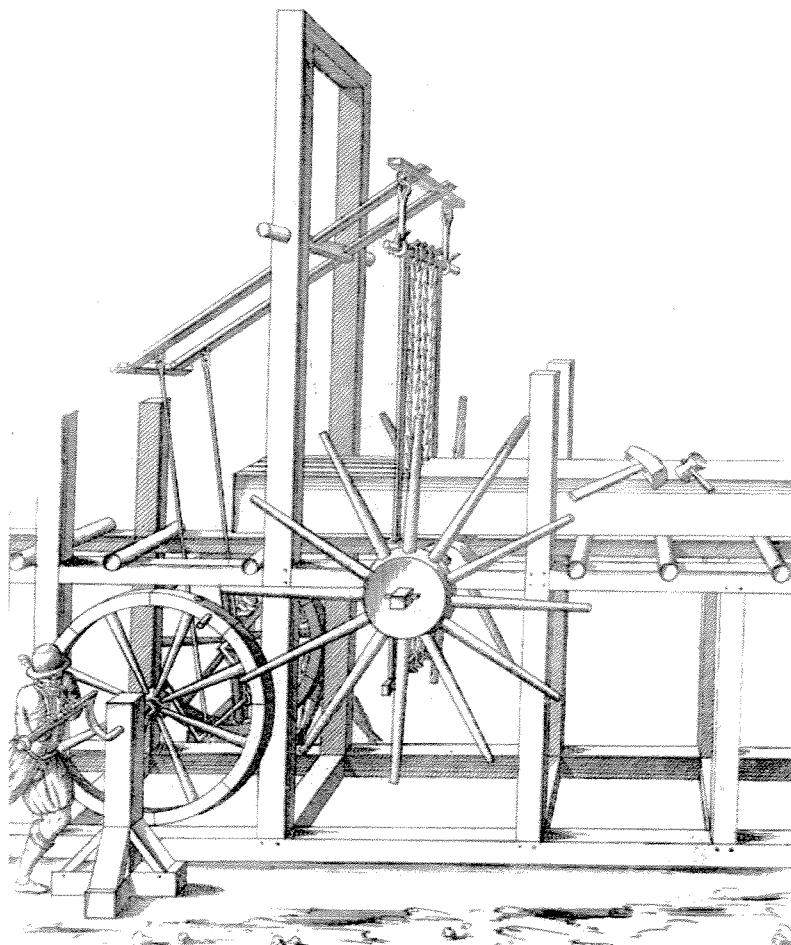
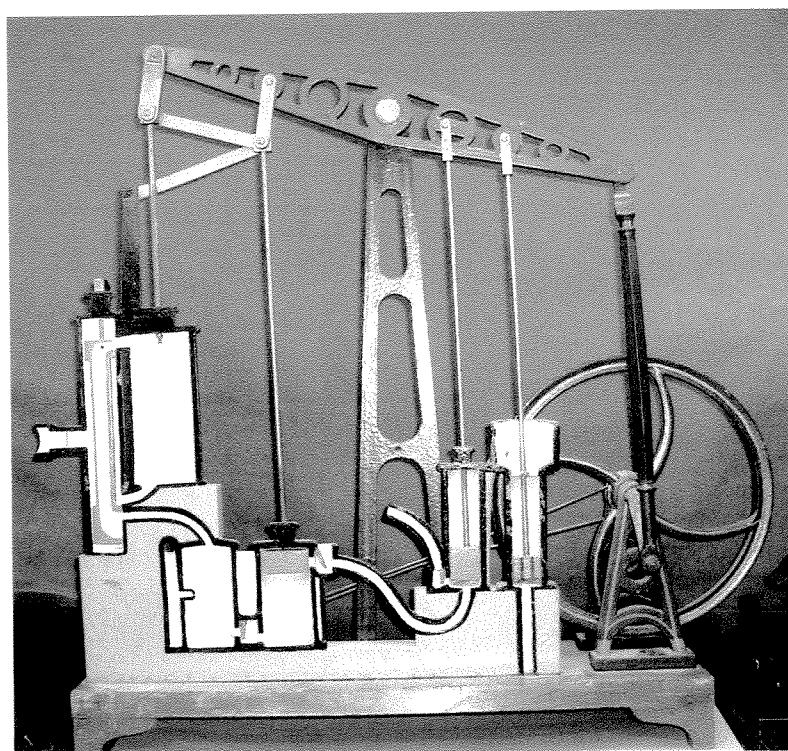


Figure 3. Hand cranked saw mill from before 1578, Plate 13 in [5]

Turning circular motion into straight line motion has been a practical engineering problem since at least the 13<sup>th</sup> century. As we can see in some 13<sup>th</sup> century drawings of a sawmill (see Figure 3) four bar linkage (rigid bars constrained to be near a plane and joined at their ends by rivets) was in use and probably was originated much earlier. In 1588, Agostino Ramelli published his book [58] on machines where linkages were widely used. In the late 18<sup>th</sup> century people started turning to steam engines for power. James Watt (1736-1819), a highly gifted designer of machines, worked on improving the efficiency and power of steam engines. In a steam engine the steam pressure pushes a piston down a straight cylinder. Watt's problem was how to turn this linear motion into the circular motion of a wheel (such as on steam locomotives).

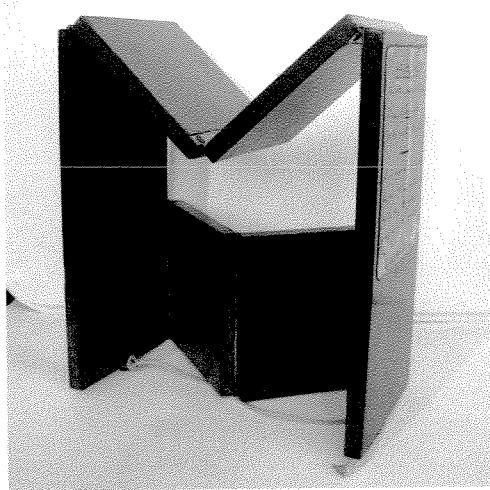
It took Watt several years to design the straight-line linkage that would change straight-line motion to circular motion. Years later Watt told his son:

Though I am not over anxious after fame, yet I am more proud of the parallel motion than of any other mechanical invention I have ever made. (quoted in [19], pp 197-198)



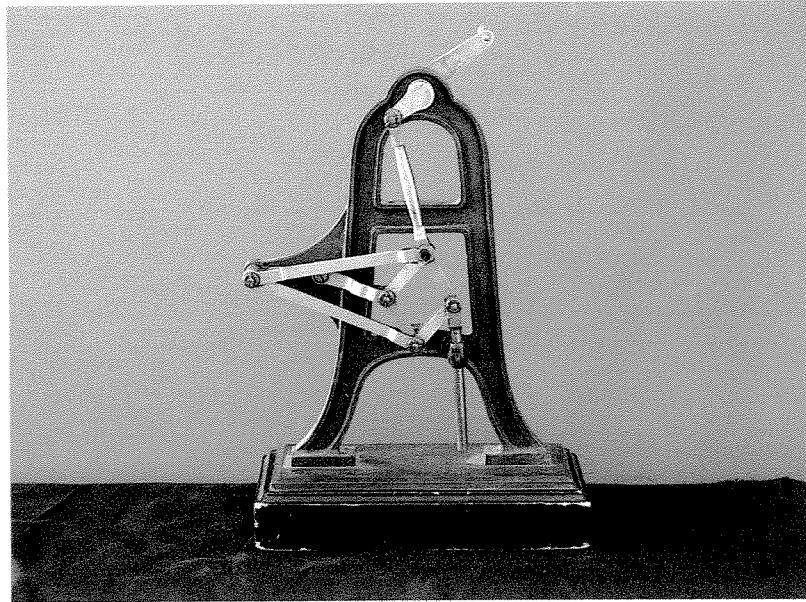
**Figure 4.** A steam engine with Watt's "parallel motion" linkage.  
(photo by the authors of a model in Cornell University's kinematic model collection)

"Parallel motion" is a name Watt used for his linkage, which was included in an extensive patent of 1784. Watt's linkage was a good solution to the practical engineering problem. See Figure 4 where the linkage is the parallelogram and associated links in the upper left hand corner. But Watt's solution did not satisfy mathematicians who knew that it can draw only approximate straight lines. In 1853, Pierre-Frederic Sarrus (1798-1861), a French professor of mathematics at Strasbourg, devised an accordion-like spatial linkage that traced exact straight lines (see Figure 5) if one end is held fixed.



**Figure 5.** Sarrus Spatial Linkage for drawing a straight line.  
(photo by Francis C. Moon, Cornell University, courtesy of the Cambridge University Engineering Department)

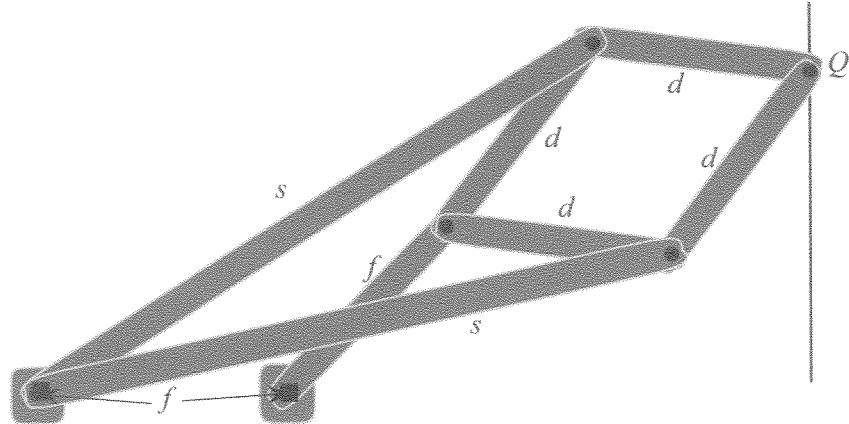
Mathematics continued to look for a planar straight-line linkage. An exact straight-line linkage in a plane was not known until 1864-1871 when a French army officer, Charles Nicolas Peaucellier (1832-1913), and a Russian graduate student, Lipmann I. Lipkin (1851-1875), independently developed a linkage that draws an exact straight line. See Figure 6. (There is not much known about Lipkin. Some sources say that he was born in Lithuania and was Chebyshev's student but died before completing his doctoral dissertation.)



**Figure 6.** Peaucellier-Lipkin linkage from Cornell University's F. Reuleaux kinematic model collection.  
(photo by Francis C. Moon, Cornell University)

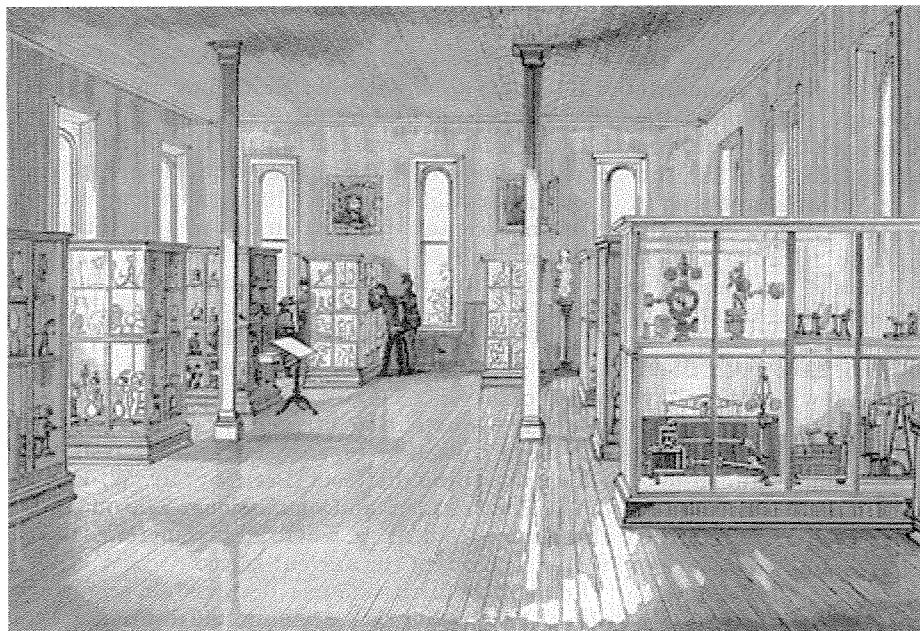
The drawing in Figure 7 depicts the working parts of the Peaucellier-Lipkin linkage. The linkage works because the point  $P$  is inverted to the point  $Q$  through a circle with center at  $C$  and a radius squared equal to  $s^2 - d^2$ . The point  $P$  is constrained to move on a circle that has center at  $D$  and that passes through  $C$  and thus  $Q$  must move along a straight line. If the distance between  $C$  and  $D$  is changed to  $g$  (not equal to  $f$ ), then  $Q$  instead of moving along a straight line will move along an arc of a circle of radius  $(s^2 - d^2)f / (g^2 - f^2)$ . This allows one to draw an arc of a large circle without using its center. Note that the definition

of a straight line used here is: “A straight line is a circle of infinite radius.” For a learning module on these topics, see [46].



**Figure 7.** Working parts of the Peaucellier-Lipkin linkage.

In the late 19<sup>th</sup> century Franz Reuleaux (1829-1905), an engineer with a strong mathematical background, collected Watt's and 38 other linkages that had been constructed to turn circular motion into straight-line motion or vice versa. These linkages were a part of a larger collection of about 800 models of mechanisms that he considered to be a basis of a theory of machines. (For Reuleaux, “mechanisms” are the simple component parts that are put together to form a machine that does useful work.) The first president of Cornell University, A. D. White, purchased in 1882 a large number of these models and there are today more than 220 remaining. A picture of this new collection was on a cover of *Scientific American* in 1885. See Figure 8. This Reuleaux model collection at Cornell University is now being digitized and put on the web as part of the National Digital Science Library complete with viewer controlled motion, simulations, learning modules, and amazingly the possibility for printing out 3-D working models of many of the mechanisms. See [52]. It is viewable at [46]. Among mathematicians, the most popular of these mechanisms is the Peaucellier-Lipkin linkage which is described above.



**Figure 8.** Reuleaux kinematic model collection in Cornell University Sibley School of Mechanical Engineering.  
(reproduced from *Scientific American*, 1885 cover, Cornell University Library)

We add here a few remarks that excited our students. In January 1874, James Joseph Sylvester (1814-1897) delivered a lecture "Recent Discoveries in Mechanical Conversion of Motion." Sylvester's aim was to bring the Peaucellier-Lipkin linkage to the notice of the English-speaking world. Sylvester learned about this problem from Chebyshev during a visit of the Russian to England. He observed:

The perfect parallel motion of Peaucellier looks so simple, and moves so easily that people who see it at work almost universally express astonishment that it waited so long to be discovered. (quoted in [19], p 206)

Later Mr. Prim, "engineer to the Houses" (the Houses of Parliament in London) was pleased to show his adaptation of the Peaucellier linkage in his new "blowing engines" for the ventilation and filtration of the Houses. Those engines proved to be exceptionally quiet in their operation. See [44]. Sylvester recalled his experience with a little mechanical model of the Peaucellier linkage at a dinner meeting of the Philosophical Club of the Royal Society. The Peaucellier model had been greeted by the members with lively expressions of admiration,

when it was brought in with the dessert, to be seen by them after dinner, as is the laudable custom among members of that eminent body in making known to each other the latest scientific novelties. (quoted in [19], p 207)

And Sylvester would never forget the reaction of his brilliant friend Sir William Thomson (later Lord Kelvin) upon being handed the same model in the Athenaeum Club. After Sir William had operated it for a time, Sylvester reached for the model, but he was rebuffed by the exclamation:

No! I have not had nearly enough of it -- it is the most beautiful thing I have ever seen in my life.  
(quoted in [19], p 207)

In summer of 1876 Alfred Bray Kempe (1849-1922), a barrister who pursued mathematics as a hobby, delivered at London's South Kensington Museum (now Science Museum) a lecture with the provocative title "How to Draw a Straight Line" which in the next year was published in a small book [44]. In this book, which is viewable on the web as part of the Cornell University Library's online book collection, you can find pictures of the linkages we have mentioned here. Kempe essentially knew that linkages are capable of drawing any algebraic curve. Other authors provided more complete proofs of this fact during the period 1877-1902. The Peaucellier-Lipkin linkage is also used in computer science to prove theorems about workspace topology in robotics [38].

## Conclusion

We described four strands in the history of geometry. We used an understanding of these strands as part of our search for answers to the five questions in the Introduction and for clarifications of the issues surrounding these questions. In summary:

1. *What non-Euclidean geometry appeared first?* Within the Building Structures Strand hyperbolic geometry was the first non-Euclidean geometry. However, spherical geometry has an ancient history that goes back more than 2000 years within the Navigation/Stargazing Strand. In addition, other geometries (for example, projective geometry) developed out of the Art/Patterns Strand before the discovery of hyperbolic geometry.
2. *Does Euclid's parallel postulate distinguish the non-Euclidean geometries from Euclidean geometry?* Looked at only from within the Building Structures Strand the answer is, Yes. However, in any context that includes spherical geometry the answer is clearly, No.
3. *Is there a potentially infinite surface in 3-space whose intrinsic geometry is hyperbolic?* As Hilbert proved there is no such surface that can be described by analytic equations. However, there are such surfaces that can be made from paper or can be crocheted.

4. *In what sense are the usual models of hyperbolic geometry “models”?* In light of the answers to Question 3, we can say that the Beltrami-Klein and Poincaré Models can be called “analytic models” because there is no analytic surface with complete hyperbolic geometry. However, in the context of the existence of a hyperbolic surface these analytic models serve as projections or coordinate systems.

5. *What does “straight” mean in geometry? How can we draw a straight line?* The meaning of “straight” seems to depend on the strand: In the Building Structures Strand, “straight” is an undefined term. “Straight” is described in terms of symmetries in the Art/Patterns Strand. “Shortest path” is the appropriate notion of “straightness” in the Navigation/Stargazing Strand. In the Motion/Machines Strand we obtain a straight line as a “circle of infinite radius”.

Using history in a mathematics class shows the human side of mathematics and explores connections within mathematics and with other sciences. There are questions, which students and mathematicians have, that can only be clarified by reference to an appropriate history.

Partially supported by National Science Foundation's National Science, Technology, Engineering, and Mathematics Education Digital Library (NSDL) Program under grant DUE-0226238.

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